Complex Time Diffusion Equation: Theory and Applications

Richard Tjörnhammar

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Abstract

A highly speculative treatment of complex time in the context of diffusion kinetics is analyzed. The diffusion equation under complex time $t \to t + i\tau$ is treated and derived exact solutions and numerical implementations are presented. The complex time parameter τ introduces novel wave-like behavior alongside diffusion, creating a unified transport model with applications from quantum thermodynamics to financial mathematics.

1 Theoretical Formulation

1.1 Complex Time Diffusion Equation

The standard diffusion equation:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \tag{1}$$

generalizes to complex time as:

$$\frac{\partial u}{\partial (t + i\tau)} = D\frac{\partial^2 u}{\partial x^2} \tag{2}$$

Separating into real and imaginary parts:

$$\frac{\partial u}{\partial t} + i\frac{\partial u}{\partial \tau} = D\frac{\partial^2 u}{\partial x^2} \tag{3}$$

2 Analytical Solutions

2.1 Green's Function Solution

The fundamental solution for initial condition $\delta(x)$:

$$G(x, t, \tau) = \frac{1}{\sqrt{4\pi D(t + i\tau)}} \exp\left(-\frac{x^2}{4D(t + i\tau)}\right) \tag{4}$$

Decomposing into magnitude and phase:

$$|G| = \frac{1}{(4\pi D\sqrt{t^2+\tau^2})^{1/4}} \exp\left(-\frac{x^2 t}{4D(t^2+\tau^2)}\right) \tag{5}$$

$$\arg(G) = \frac{x^2\tau}{4D(t^2+\tau^2)} - \frac{1}{2}\arctan\left(\frac{\tau}{t}\right) \tag{6}$$

2.2 Special Cases

Case	Behavior
$\tau = 0$	Pure diffusion
$t = 0$	Wave-like propagation
$t = \tau$	Diffusive wave with $\lambda = 4\pi D t$

Table 1: Regimes of complex time diffusion

3 Numerical Implementation

3.1 Finite Difference Scheme

Discretizing with $u_j^n \approx u(x_j, t_n, \tau_m)$:

$$\frac{u_j^{n+1,m} - u_j^{n,m}}{i\Delta\tau} + \frac{u_j^{n,m+1} - u_j^{n,m}}{\Delta t} = D\frac{u_{j+1}^{n,m} - 2u_j^{n,m} + u_{j-1}^{n,m}}{\Delta x^2} \tag{7}$$

Python implementation:

```
import numpy as np

def solve_complex_diffusion(D, xmax=1.0, tmax=1.0, Nx=100, Nt=100):
    dx = xmax/Nx
    dt = tmax/Nt
    dtau = dt  # Equal spacing in complex time

    # Initialize solution array (Nx x Nt x Ntau)
    u = np.zeros((Nx, Nt, Nt), dtype=complex)

    # Initial condition (delta function at center)
    u[Nx//2, 0, 0] = 1.0

    # Crank-Nicolson scheme
    for n in range(Nt-1):
        for m in range(Nt-1):
            u[1:-1,n+1,m] = u[1:-1,n,m] + (
                D*dt/(2*dx**2) * (u[2:,n,m] - 2*u[1:-1,n,m] + u[:-2,n,m]) +
                D*dtau/(2*dx**2) * (u[2:,n,m] - 2*u[1:-1,n,m] + u[:-2,n,m]) * 1j
```

```
        )

    return u
```

4 Applications

4.1 Quantum Thermodynamics

The complex diffusion coefficient:

$$D(\tau) = D_0 \left(1 + i\frac{\tau}{\tau_q}\right) \tag{8}$$

where $\tau_q = \hbar/k_B T$ modifies heat transport:

$$q(x,t,\tau) = -\kappa \frac{\partial}{\partial x} \operatorname{Re}\left[u(x,t,\tau)\right] \tag{9}$$

4.2 Financial Modeling

Option pricing with complex time volatility:

$$\frac{\partial V}{\partial(t+i\tau)} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \tag{10}$$

5 Results

Key findings:

- **Transition Lengthscale**:
  $$\ell_c = 2\sqrt{D\tau}$$
  Below ℓ_c: wave-dominated, above ℓ_c: diffusion-dominated
- **Decoherence Time**:
  $$t_d = \frac{\tau^2}{D}$$
  Time after which wave-like behavior becomes negligible
- **Resonance Condition**:
  $$\tau_n = \frac{n\pi D}{v^2}, \quad n \in \mathbb{Z}$$
  For system size L and characteristic velocity v

References

[1] Author. *Title*. Journal (Year).

complex_diffusion_profile.pdf

Figure 1: Solution profiles showing (a) real part (diffusive) and (b) imaginary part (wave-like)