Multiplicities: Adding a Vertex to a Graph

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Abstract Given an Hermitian matrix A whose graph G is a simple undirected graph and its eigenvalues, we suppose the status of each vertex in the graph is known for each eigenvalue of A. We investigate the change of the multiplicity of each eigenvalue, when we add a pendent vertex with given value to a particular vertex in the graph via an edge with given weight. It is shown how each multiplicity changes based on this information. The results are applied to show that more than one eigenvalue may increase in multiplicity with the addition of just one vertex. The intended focus is trees, but the analysis is given for general graphs.

Keywords Eigenvalues · Graph · Matrix · Multiplicities · Symmetric

1 Introduction

If G is a simple, undirected graph on n vertices, denote by $\mathscr{H}(G)$ the set of all n-by-n Hermitian matrices, the graph of whose off-diagonal entries is G. There is long-standing interest in the possible lists of multiplicities for the eigenvalues of matrices in $\mathscr{H}(G)$, especially when G is a tree T. There are several papers on the subject, including ones relating the structure of T to eigenvalue multiplicity, Refs. [2, 4, 5, 7–9]. In many papers, the multiplicity of eigenvalues in a tree is considered when a slight change occurs. Here, we deal with a general graph and consider the new,

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but natural issue of adding a single vertex. As all necessary information, particularly multiplicities may be updated, the results could be applied sequentially.

If A is Hermitian, denote the multiplicity of an eigenvalue λ of A by $m_A(\lambda)$. When we remove a vertex u from G, the remaining graph is denoted by $G(u)$. Then we denote by $A(u)$ the $(n-1)$-by-$(n-1)$ principal submatrix of $A \in \mathscr{H}(G)$, resulting from deletion of the row and column corresponding to u. $A[S]$ denotes the principal submatrix of A corresponding to the subgraph S of G. For an identified $A \in \mathscr{H}(G)$, we often speak interchangeably about the graph and the matrix, for convenience.

Our interest here is in precisely what happens to the multiplicities when we add a (pendent) vertex v to a tree T at an identified vertex u. Specifically, we show what happens, for each $A \in \mathscr{H}(T)$, to the multiplicities $m_A(\lambda)$, when we pass to the new tree $\widetilde{T}$, for $\widetilde{A} \in \mathscr{H}(\widetilde{T})$ with $\widetilde{A}(v) = A$, eigenvalue by eigenvalue. Since the analysis is only slightly more complicated when G is a general graph, we present our results at that level of generality.

Because of the interlacing inequalities for an Hermitian matrix and a principal submatrix of it [1], a multiplicity may change by at most 1 when we pass from G to $\widetilde{G}$. For trees, the theory of what may happen, when a particular vertex is deleted, was summarized and further developed in [4], but the basic definitions are the same for general graphs G. A vertex u of G is called "*Parter*" (respectively "*neutral*" or "*downer*") for an eigenvalue λ of $A \in \mathscr{H}(G)$ if

$$m_{A(u)}(\lambda) = m_A(\lambda) + 1 \text{ (resp. } m_A(\lambda),\ m_A(\lambda) - 1).$$

The "*status*" of a vertex u is discussed in [4]. It refers to which of these eventualities occurs, and why.

2 Main Results

We denote the characteristic polynomial of a square matrix A by $p_A(x)$. Suppose that G is a graph on n vertices, that $A \in \mathscr{H}(G)$ is given, and that a new vertex v is appended to G at the vertex u of G, resulting in the graph $\widetilde{G}$ with pendent vertex v. If the weight $\alpha \in \mathbb{R}$ is placed on v and the weight $\tilde{a}_{uv} \in \mathbb{C}$ is placed on the new edge, a new matrix $\widetilde{A} \in \mathscr{H}(\widetilde{G})$ results. Of courese $\widetilde{A}(v) = A$, and, we mean that the u, v entry of $\widetilde{A}$ is $\tilde{a}_{uv}$ and $\tilde{a}_{vv} = \alpha$.

The function $f(x) = \frac{p_{A(u)}(x)}{p_A(x)}$ will be important to us. After cancellation of like terms in the numerator and denominator, because of interlacing, it will be a ratio of two products, each of distinct linear terms. In the numerator will be terms of the form $(x - \tau)$ for eigenvalues τ for which u is Parter, along with eigenvalues of $A(u)$ that do not occur in A. In the denominator will be such terms for eigenvalues μ for which u is a downer. The number of μ's is one more than the number of τ's, and the τ's strictly interlace μ's because of the interlacing inequalities. Important for us is that $f(x)$ will be well-defined and nonzero when evaluated at any eigenvalue for which u is neutral.

Lemma 1 *With the conventions mentioned above, we have for any* $\lambda \in \mathbb{R}$*:*

(a) If u is a Parter vertex for λ *in G,*

$$m_{\widetilde{A}}(\lambda) = \begin{cases} m_A(\lambda) + 1 & \text{if } \alpha = \lambda \\ m_A(\lambda) & \text{if } \alpha \neq \lambda \end{cases} ;$$

(b) If u is a neutral vertex for λ *in G,*

$$m_{\widetilde{A}}(\lambda) = \begin{cases} m_A(\lambda) + 1 & \text{if } \alpha = \lambda - |\tilde{a}_{uv}|^2 f(\lambda) \\ m_A(\lambda) & \text{otherwise} \end{cases} ;$$

and

(c) If u is a downer vertex for λ *in G,*

$$m_{\widetilde{A}}(\lambda) = m_A(\lambda) - 1.$$

Proof Given $A \in \mathscr{H}(G)$, let $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ be the distinct eigenvalues of A, and their multiplicities in A be $\{m_1, m_2, \ldots, m_l\}$. We focus on a specified eigenvalue λ_k, $(1 \leq k \leq l)$, and now we put $\lambda_k = \lambda$ and $m_k = m$. Then, the characteristic polynomial of $\widetilde{A} = (\tilde{a}_{ij})$ can be represented as follows (cf. [8]).

$$p_{\widetilde{A}}(x) = (x - \alpha)p_A(x) - |\tilde{a}_{uv}|^2 p_{A(u)}(x), \tag{1}$$

We further let the distinct eigenvalues of $A(u)$ be $\sigma(A(u)) = \{\mu_1, \mu_2, \ldots\}$, and their multiplicities be $\{m'_1, m'_2, \ldots\}$. As we focus upon one eigenvalue $\lambda = \lambda_k$, $p_{\widetilde{A}}(x)$ can be written,

$$p_{\widetilde{A}}(x) = (x - \alpha)(x - \lambda)^m f_1(x) - |\tilde{a}_{uv}|^2 (x - \lambda)^{m'} f_2(x), \tag{2}$$

in which $f_1(x) = \prod\limits_{i \neq k} (x - \lambda_i)^{m_i}$, $f_2(x) = \prod\limits_{\mu_i \neq \lambda} (x - \mu_i)^{m'_i}$

In (2), if λ is not an eigenvalue of A or $A(u)$, then m or m' is 0.

If u is a Parter vertex for λ in A, then $m' = m + 1$ in (2). Then,

$$p_{\widetilde{A}}(x) = (x - \lambda)^m \{(x - \alpha) f_1(x) - |\tilde{a}_{uv}|^2 (x - \lambda) f_2(x)\}.$$

Here we set $g_1(x) = (x - \alpha) f_1(x) - |\tilde{a}_{uv}|^2 (x - \lambda) f_2(x)$. When $\alpha = \lambda$, $g(\lambda) = 0$, thus $m_{\widetilde{A}}(\lambda) = m_A(\lambda) + 1$. However when $\alpha \neq \lambda$, $g(\lambda) \neq 0$, then $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$.

If u is a neutral vertex for λ in A, then $m' = m$ in (2). Then,

$$p_{\widetilde{A}}(x) = (x - \lambda)^m \{(x - \alpha) f_1(x) - |\tilde{a}_{uv}|^2 f_2(x)\}.$$

When we set $g_2(x) = (x - \alpha) f_1(x) - |\tilde{a}_{uv}|^2 f_2(x)$, if α and $\tilde{a}_{uv}$ has the relation such that $\alpha = \lambda - |\tilde{a}_{uv}|^2 \frac{f_2(\lambda)}{f_1(\lambda)} = \lambda^*$, then $g_2(\lambda) = 0$, so $m_{\widetilde{A}}(\lambda) = m_A(\lambda) + 1$. Since

$\frac{f_2(\lambda)}{f_1(\lambda)} = [\frac{p_{A(u)}(x)}{p_A(x)}]_\lambda$ holds, if we put $f(x) = \frac{f_2(x)}{f_1(x)}$, then the assertion holds. If $\alpha \neq \lambda^*$, then $g_2(\lambda) \neq 0$, so $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$.

Lastly, If u is a downer vertex for λ in A, then $m' = m - 1$ in (2). Then

$$p_{\widetilde{A}}(x) = (x-\lambda)^{m-1}\{(x-\alpha)(x-\lambda)f_1(x) - |\tilde{a}_{uv}|^2 f_2(x)\}.$$

If we set $g_3(x) = (x-\alpha)(x-\lambda)f_1(x) - |\tilde{a}_{uv}|^2 f_2(x)$, then $g(\lambda) \neq 0$, thus $m_{\widetilde{A}}(\lambda) = m_A(\lambda) - 1$ for any real number α. $\square$

When we focus on an identified real number λ, if a vertex is appended to a Parter vertex for λ, the multiplicity of λ in $\widetilde{A}$ depends only on the value on the pendent vertex. If it is appended to a neutral vertex for λ, the multiplicity of λ depends only on the relation between the value on the pendent vertex and the weight on the new edge. If the relation $\alpha = \lambda - |\tilde{a}_{uv}|^2 f(\lambda)$ holds, then $|\tilde{a}_{uv}|^2 = \frac{\lambda-\alpha}{f(\lambda)}$ must be positive. So, if $f(\lambda) > 0$, then α must be less than λ, and if $f(\lambda) < 0$, then α must be greater than λ.

If a pendent vertex is appended to a downer vertex, the multiplicity of λ decreases whatever the value on the pendent vertex and the weight on the new edge are.

We note that it follows from the lemma that any eigenvalue of multiplicity 1 in A, for which u is a downer, disappears when we pass to $\widetilde{A}$. In particular, any multiplicity 1 eigenvalue, for which every vertex is a downer, disappears. In the case of trees, for every eigenvalue of multiplicity 1 that has no Parter vertex(equivalently, no neutral vertex), every vertex will be a downer [4] and, so, will disappear. Most of these will be replaced by new eigenvalues in $\widetilde{A}$ that also have multiplicity 1 and no Parter vertex. From the above lemma, we can deduce the next theorem.

Theorem 1 *Let G be a general graph, $A \in \mathscr{H}(G)$ and $\lambda \in \mathbb{R}$. Let u be a vertex in G, and $\widetilde{G}$ be a graph obtained by adding a vertex v valued α to the vertex u of G. Let $\widetilde{A} \in \mathscr{H}(\widetilde{G})$, such that $\widetilde{A}(v) = A$. Let m be the multiplicity of λ as an eigenvalue in A, and let n be the multiplicity of λ in $\widetilde{A}$. Then,*

(a) $m - n = -1$ if and only if u is a Parter vertex for λ in A and $\alpha = \lambda$, or u is a neutral vertex in A and $\alpha = \lambda - |\tilde{a}_{uv}|^2 f(\lambda)$.
(b) $m - n = 0$ if and only if u is a Parter vertex for λ in A and $\alpha \neq \lambda$, or u is a neutral vertex for λ in A and $\alpha \neq \lambda - |\tilde{a}_{uv}|^2 f(\lambda)$.
(c) $m - n = 1$ if and only if u is a downer vertex for λ in A.

In Lemma 1, the status of vertex u in A changes to that in $\widetilde{A}$ as follows.

Corollary 1 *Let G be a general graph, $A \in \mathscr{H}(G)$ and $\lambda \in \mathbb{R}$. Let u be a vertex in G and $\widetilde{G}$ be the graph obtained by adding a vertex v valued α to the vertex u in G. Let $\widetilde{A} \in \mathscr{H}(\widetilde{G})$, such that $\widetilde{A}(v) = A$.*

(a) In case u is Parter for λ in A, the status of u for λ in $\widetilde{A}$ is Parter.
(b) In case u is neutral for λ in A, if $\alpha = \lambda - |\tilde{a}_{uv}|^2 f(\lambda)$, then the status of u for λ in $\widetilde{A}$ becomes downer, if $\alpha = \lambda$, then Parter, and, otherwise, neutral.

(c) *In case u is downer for λ in A, if $\alpha = \lambda$, then the status of u for λ in $\widetilde{A}$ becomes Parter, and, otherwise, neutral.*

Proof (a) If $\alpha = \lambda$, then $m_{\widetilde{A}}(\lambda) = m_A(\lambda) + 1$ from Lemma 1. When u is removed from $\widetilde{G}$, $m_{\widetilde{A}(u)}(\lambda) = m_A(\lambda) + 2$, since u is Parter for λ in G and $\alpha = \lambda$, so that u is Parter in $\widetilde{A}$.

If $\alpha \neq \lambda$, then $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$. When u is removed from $\widetilde{G}$,

$$m_{\widetilde{A}(u)}(\lambda) = m_A(\lambda) + 1,$$

so that u is Parter in $\widetilde{A}$.

(b) If $\alpha = \lambda - |\tilde{a}_{uv}|^2 f(\lambda)$, $m_{\widetilde{A}}(\lambda) = m_A(\lambda) + 1$. When u is removed from $\widetilde{G}$, $m_{\widetilde{A}(u)}(\lambda) = m_A(\lambda)$, so that u is downer in $\widetilde{A}$. If $\alpha = \lambda$, then $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$, and $m_{\widetilde{A}(u)}(\lambda) = m_A(\lambda) + 1$, so that u is Parter in $\widetilde{A}$. If otherwise, $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$, and $m_{\widetilde{A}(u)}(\lambda) = m_A(\lambda)$, so that u is neutral in $\widetilde{A}$.

(c) $m_{\widetilde{A}}(\lambda) = m_A(\lambda) - 1$. If $\alpha = \lambda$, then when u is removed from $\widetilde{G}$, $m_{\widetilde{A}(u)}(\lambda) = m_A(\lambda)$, so that $m_{\widetilde{A}(u)}(\lambda) = m_{\widetilde{A}}(\lambda) + 1$ and u is Parter in $\widetilde{A}$. If $\alpha \neq \lambda$, then $m_{\widetilde{A}(u)}(\lambda) = m_A(\lambda) - 1$, so that u is neutral in $\widetilde{A}$. □

Let T_0 be a branch at vertex v in tree T, and let $A_0 \in \mathscr{H}(T_0)$. Let u be the vertex adjacent to v in T_0. If $m_{A_0(u)}(\lambda) = m_{A_0}(\lambda) - 1$, then T_0 is called a downer branch at v for λ in T relative to A. If a downer branch has eigenvalue λ with multiplicity 1, then we call it a *simple downer branch* for λ. Next we consider the change of multiplicity of λ when we add a simple downer branch for λ to a tree T.

Let b be a simple downer branch for λ. Let $\widehat{T}$ be a tree obtained by adding b to the vertex u in T inserting an edge between u and a downer vertex in b. Let $A \in \mathscr{H}(T)$, $\widehat{A} \in \mathscr{H}(\widehat{T})$ in which A is a principal submatrix of $\widehat{A}$ corresponding to T, and $B \in \mathscr{H}(b)$. Since b is a downer branch for λ at u in $\widehat{A}$, and u is a Parter vertex in $\widehat{A}$, if we set $m_{\widehat{A}}(\lambda) = k$, then $m_{\widehat{A}(u)}(\lambda) = k + 1$. Since $m_B(\lambda) = 1$, $m_{A(u)}(\lambda) = k + 1 - 1 = k$. Thus, $m_{\widehat{A}}(\lambda) = m_{A(u)}(\lambda)$. From this argument, the next Corollary follows.

Corollary 2 *Let $\widehat{T}$ be the tree obtained by adding a simple downer branch for λ to the vertex u of a tree T connecting with an edge. Let $A \in \mathscr{H}(T)$, $\widehat{A} \in \mathscr{H}(\widehat{T})$ in which A is a principal submatrix of $\widehat{A}$ corresponding to T, Then if u is a Parter vertex for λ in A, then $m_{\widehat{A}}(\lambda) = m_A(\lambda) + 1$. If u is a neutral vertex for λ in A, then $m_{\widehat{A}}(\lambda) = m_A(\lambda)$. If u is a downer vertex for λ in A, then $m_{\widehat{A}}(\lambda) = m_A(\lambda) - 1$.*

It is well known that when T is a path, either pendent vertex is a downer for every eigenvalue, all of which are multiplicity 1. Thus, when an end vertex is removed, every eigenvalue disappears and all interlacing inequalities are strict. So a path is a simple downer branch for each eigenvalue. Thus, the previous corollary is applicable to the case that a path is appended to G. Furthermore, by Theorem 1, addition of a new vertex at a pendent vertex also makes every original eigenvalue disappear. This is actually a special case of something much more general that also follows from the theorem.

If T is a tree and λ is a multiplicity 1 eigenvalue for which exactly one vertex is Parter, and that vertex is degree 2, then upon appending a new vertex anywhere in T, except at the Parter vertex, will make the multiplicity 1 eigenvalue disappear. Of course any non-upward multiplicity 1 eigenvalue will disappear, as well. For an eigenvalue λ of A, if there is a vertex such that $m_{A(v)}(\lambda) = m_A(\lambda) + 1$, then λ is called an *upward* eigenvalue, and otherwise *non-upward*. Here is the formal statement.

Corollary 3 *Suppose that T is a tree, that $A \in \mathscr{H}(T)$, that $\lambda \in \sigma(A)$ satisfies $m_A(\lambda) = 1$ and that either λ is upward with exactly one Parter vertex that is degree 2, or that λ is non-upward. Then, if $\widetilde{T}$ is the result of appending a new vertex v at any vertex of T (or any vertex other than u in the upward case), then $\lambda \notin \sigma(\widetilde{A})$ for any $\widetilde{A} \in \mathscr{H}(\widetilde{T})$ such that $\widetilde{A}(v) = A$.*

The multiplicity of an eigenvalue λ of A is changeable by adding a pendent vertex to a graph G as Lemma 1 and Theorem 1 show. However, by perturbing some diagonal entries in $\widetilde{A}$, the multiplicity of the eigenvalue can be preserved as it was in A. Before showing that, we need the next lemma from [5, Theorem 5].

The lemma shows how the multiplicity of an eigenvalue λ changes as a result of perturbing the value on a vertex.

Lemma 2 ([5]) *Let G be a graph, and i a vertex in G. For $A \in \mathscr{H}(G)$, let $A' = A + tE_{ii}$, $t \neq 0$, where E_{ii} denote the same size matrix with A such that (i, i) element is 1 and zeros elsewhere, then*

(a) $m_{A'}(\lambda) = m_A(\lambda)$ if and only if i is Parter in A or i is neutral in A and t is a unique t_0.
(b) $m_{A'}(\lambda) = m_A(\lambda) + 1$ if and only if i is neutral in A, and $t = t_0$.
(c) $m_{A'}(\lambda) = m_A(\lambda) - 1$ if and only if i is downer in A.

From Lemmas 1 and 2, we can observe the next proposition.

Proposition 1 *Let G be a graph. We suppose that $A \in \mathscr{H}(G)$ has an eigenvalue λ with multiplicity m. Let $\widetilde{G}$ be the graph obtained by adding a pendent vertex v valued α to the vertex u of G connecting with an edge weighted $\tilde{a}_{uv}$. Let the matrix $\widetilde{A} \in \mathscr{H}(\widetilde{G})$, such that $\widetilde{A}(v) = A$. Then there is a $\widetilde{B} \in \mathscr{H}(\widetilde{G})$ such that $\widetilde{B}$ has eigenvalue λ with multiplicity m, and it can be obtained by changing the value on v or u in $\widetilde{A}$.*

Proof First, we suppose that a pendent vertex is added to a Parter vertex for λ in A. If $\alpha \neq \lambda$, then the multiplicity of λ stay same, so it does not matter. If $\alpha = \lambda$, then multiplicity of λ is $m + 1$ in $\widetilde{A}$. In $\widetilde{A}$, the status of vertex v is downer for λ. So, if we perturb the value on v slightly and let the matrix B, the multiplicity of λ will go down, then $m_B(\lambda) = m$.

Secondly we suppose that a pendent vertex is added to a neutral vertex for λ in A. If the relation between α and $\tilde{a}_{uv}$ such as $\alpha = \lambda - |\tilde{a}_{uv}|^2 f(\lambda)$ holds, then multiplicity of λ is $m + 1$ in $\widetilde{A}$. Then the status of vertex u is downer in $\widetilde{A}$. So by perturbing the value on u in $\widetilde{A}$ slightly, we get B such that $m_B(\lambda) = m$. If $\alpha \neq \lambda - |\tilde{a}_{uv}|^2 f(\lambda)$, then $m_{\widetilde{A}}(\lambda) = m_A(\lambda)$. So we can set $\widetilde{A} = B$.

Next we suppose that a pendent vertex is added to a downer vertex for λ in A. Then $m_{\widetilde{A}}(\lambda) = m - 1$. If $\alpha \neq \lambda$, then u is neutral in $\widetilde{A}$. So from Lemma 2, by perturbing the value on u, we can get B such that $m_B(\lambda) = m$.

If $\alpha = \lambda$, then u and v are Parter in $\widetilde{A}$. So we can not make the multiplicity of λ increase only by perturbing the value on u. Then we perturb the value on v slightly from λ, then the status of u is neutral. So, similarly by perturbing the value on u, we can get B such that $m_B(\lambda) = m$. $\square$

From the above proposition, we can observe that when we add a pendent vertex v to the vertex u in G, even if the multiplicity of an eigenvalue changes in $\widetilde{A}$, by further perturbing the value on u or v, we can keep the multiplicity of the eigenvalue as it was in A.

Let $m_1, m_2, \ldots, m_k$ be the multiplicities of the distinct eigenvalues of $A \in \mathscr{H}(T)$. Then we order them as $m_1 \geq m_2 \geq \cdots \geq m_k$. This is called the *unordered multiplicity list* for A, because when the eigenvalues corresponding to this multiplicity list are put in order, their multiplicities are not generally in descending order or increasing order. Let $\mathscr{L}(T)$ be the set of unordered multiplicity lists for all $A \in \mathscr{H}(T)$. There are some papers studying $\mathscr{L}(T)$ [3, 6] etc.; however, for trees with many vertices, not all multiplicity lists have yet been determined. Let $M(T)$ be the maximum multiplicity of an eigenvalue of $A \in \mathscr{H}(T)$. $M(T)$ is equal to the path cover number. (cf. [7]).

Theorem 2 *Let T be a tree, and suppose $(m, 1, 1, \ldots, 1) \in L(T)$ for $m \geq 2$. When we add a pendent vertex to a certain vertex in T and construct $\widetilde{T}$, then there is an Hermitian matrix such that $(m + 1, 1, 1, \ldots, 1) \in L(\widetilde{T})$.*

Proof Let A be an Hermitian matrix with unordered multiplicity list $(m, 1, 1, \ldots, 1)$. We suppose $\sigma(A)$ is ordered as

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots < \lambda_{n-m+1}$$

Let the multiplicity of λ_i be m_i. We suppose $m_k = m$ for the eigenvalue λ_k, $2 \leq k \leq n - m$. Now we shift A as $A - \lambda_k I = B$. B also has an unordered multiplicity list $(m, 1, 1, \ldots, 1)$ in which m represents the multiplicity of the eigenvalue 0. Here we order $\sigma(B)$ as $\mu_1 < \mu_2 < \cdots < \mu_k = 0 < \cdots < \mu_{n-m+1}$.

Next we add a pendent vertex v with value 0 to a Parter vertex u for 0 in B. Then we assign the weight of edge $\tilde{b}_{uv}$ and $\tilde{b}_{vu}$ to be ε such that $0 < \varepsilon < \min_i\{\frac{\mu_{i+1} - \mu_i}{2}\}$. Then we get the tree $\widetilde{T}$ and corresponding matrix $\widetilde{B} \in \mathscr{H}(\widetilde{T})$, in which B is a principal submatrix of $\widetilde{B}$. If the eigenvalues of $\widetilde{B}$ are ordered as $\tilde{\mu}_1 \leq \tilde{\mu}_2 \leq \cdots \leq \tilde{\mu}_k = 0 \leq \cdots \tilde{\mu}_{n-m+1}$, then $\mu_i - \varepsilon \leq \tilde{\mu}_i \leq \mu_i + \varepsilon$, because spectral radius $\rho(\widetilde{B} - B) = \varepsilon$ and $|\tilde{\mu}_i - \mu_i| \leq \varepsilon$. So, $m_{\widetilde{B}}(\tilde{u}_j) = 1$, $j \neq k$, and $m_{\widetilde{B}}(\tilde{u}_k) = m_B(\mu_k) + 1$, because the pendent vertex is added at a Parter vertex in B. From these, the assertion of the theorem holds. $\square$

3 Examples

Example 1 Let A be an Hermitiam matrix as below,

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 \end{bmatrix}$$

The graph of A is represented in Fig. 1. The circled numbers correspond to the index of the vertex. And the numbers outside of circles represent the values assigned on the vertices. The matrix A has eigenvalues 0 and 3 with multiplicity 2 each, among others. When we remove vertex 1 from T, the multiplicities of eigenvalues 0 and 3 become 3 and 2 in $A(1) \in \mathscr{H}(T(1))$, respectively. So vertex 1 is a Parter vertex for 0 and neutral vertex for 3 in A. When we add a pendent vertex at vertex 1, we consider the case in which the multiplicities of the eigenvalues 0 and 3 each go up in the new graph $\widetilde{A}$. To make the multiplicity of 0 go up in $\widetilde{A}$, the value on the added vertex 9 must be 0, because vertex 1 is Parter for 0.

Furthermore, to make the multiplicity of 3 go up, we must set the weight of the edge $\tilde{a}_{19}, \tilde{a}_{91}$ as the next equation dictates by Lemma 1 or Theorem 1.

$$3 - |\tilde{a}_{19}|^2 f(3) = 0, \tag{3}$$

in which $f(3)$ is the value of $f(x) = \frac{p_{A(1)}(x)}{p_A(x)}$ at 3. Since $p_A(x) = x^2(x-3)^2(x^4 - 3x^3 - 7x^2 + 12x + 9)$, and $p_{A(1)}(x) = x^3(x-3)^2(x^2 - 3x - 3)$, the value of $\tilde{a}_{19}$ is $\sqrt{6}e^{i\theta}$. Then, $\widetilde{A}$ is as follows, and $\widetilde{A}$ has eigenvalues 0 and 3 with multiplicity 3 each.

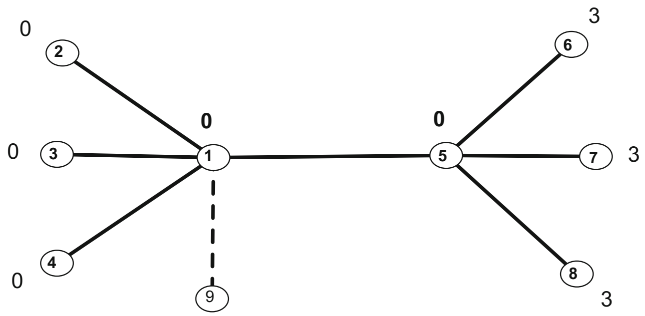

Fig. 1 Example 1

$$\widetilde{A} = \begin{bmatrix} 0 & 1\ 1\ 1\ 1\ 0\ 0\ 0 & \sqrt{6}e^{i\theta} \\ 1 & 0\ 0\ 0\ 0\ 0\ 0\ 0 & 0 \\ 1 & 0\ 0\ 0\ 0\ 0\ 0\ 0 & 0 \\ 1 & 0\ 0\ 0\ 0\ 0\ 0\ 0 & 0 \\ 1 & 0\ 0\ 0\ 0\ 1\ 1\ 1 & 0 \\ 0 & 0\ 0\ 0\ 1\ 3\ 0\ 0 & 0 \\ 0 & 0\ 0\ 0\ 1\ 0\ 3\ 0 & 0 \\ 0 & 0\ 0\ 0\ 1\ 0\ 0\ 3 & 0 \\ \sqrt{6}e^{-i\theta} & 0\ 0\ 0\ 0\ 0\ 0\ 0 & 0 \end{bmatrix}$$

Example 2 Let B be an Hermitian matrix as below,

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

The graph of B is represented in Fig. 2. The values assigned to vertices are placed outside the circles. B has eigenvalues 1 and 3 with multiplicity 2 each. And $B(1)$ also has eigenvalues 1 and 3 with multiplicity 2 respectively. So, vertex 1 is neutral for both eigenvalues 1 and 3. In this example, we show that the multiplicities of 1 and 3 increase simultaneously by adding one pendent vertex to a vertex in T that is neutral vertex for the two eigenvalues.

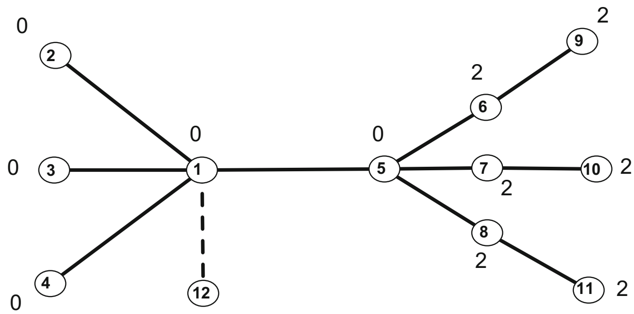

Fig. 2 Example 2

To make the multiplicity of 1 and 3 increase simultaneously, the next equations must hold, in which α is the value assigned to the pendent vertex,

$$\alpha = 3 - |\tilde{b}_{1,12}|^2 f(3) = 1 - |\tilde{b}_{1,12}|^2 f(1),$$

in which $f(x) = \frac{p_{B(1)}(x)}{p_B(x)}$. We have $f(x) = \frac{p_{B(1)}(x)}{p_B(x)} = \frac{x(x^3-4x^2+6)}{(x-2)(x^4-2x^3-8x^2+6x+9)}$, then $f(3) = 0.5$, $f(1) = -0.5$. Then $\tilde{b}_{1,12} = \sqrt{2}e^{i\theta}$. Therefore, $\widetilde{B}$ is as follows with $\alpha = 2$ and $\tilde{b}_{1,12} = \tilde{b}_{12,1} = \sqrt{2}$, then the multiplicity of each eigenvalue 1 and 3 simultaneously goes up to multiplicity 3.

$$\widetilde{B} = \begin{bmatrix} 0&1&1&1&1&0&0&0&0&0&0&\sqrt{2} \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&1&1&1&0&0&0&0 \\ 0&0&0&0&1&2&0&0&1&0&0&0 \\ 0&0&0&0&1&0&2&0&0&1&0&0 \\ 0&0&0&0&1&0&0&2&0&0&1&0 \\ 0&0&0&0&0&1&0&0&2&0&0&0 \\ 0&0&0&0&0&0&1&0&0&2&0&0 \\ 0&0&0&0&0&0&0&1&0&0&2&0 \\ \sqrt{2}&0&0&0&0&0&0&0&0&0&0&2 \end{bmatrix}$$

References

1. Horn, R., Johnson, C.R.: Matrix Analysis, 2nd edn. Cambridge University Press, New York (2013)
2. Johnson, C.R.: Leal Duarte, A.: The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree. Linear Multilinear Algebra **46**, 139–144 (1999)
3. Johnson, C.R.: Leal Duarte, A.: On the possible multiplicities of the eigenvalues of a Hermitian matrix whose graph is a tree. Linear Algebra Appl. **348**, 7–21 (2002)
4. Johnson, C.R., Leal Duarte, A., Saiago, C.M.: The Parter-Wiener theorem: refinement and generalization. SIAM J. Matrix Anal. Appl. **25**(2), 352–361 (2003)
5. Johnson, C.R., Leal Duarte, A., Saiago, C.M.: The change in eigenvalue multiplicity associated with perturbation of a diagonal entry. Linear and Multilinear Algebra **60**(5), 525–532 (2012)
6. Johnson, C.R., Li Andrew, A., Walker Andrew, J.: Ordered multiplicity lists for eigenvalues of symmetric matrices whose graph is a linear tree. Discret. Math. **333**, 39–55 (2014)
7. Leal Duarte, A., Johnson, C.R.: On the minimum number of distinct eigenvalues for a symmetric matrix whose graph is a given tree. Math. Inequal. Appl. **5**(2), 175–180 (2002)
8. Parter, S.: On the eigenvalues and eigenvectors of a class of matrices. J. Soc. Indust. Appl. Math. **8**, 376–388 (1960)
9. Wiener, G.: Spectral multiplicity and splitting results for a class of qualitative matrices. Linear Algebra Appl. **61**, 15–29 (1984)