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PMATH 351 Course Notes

Real Analysis

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. These notes are my interpretation and transcription of the content covered in lectures. The instructor has not verified or confirmed the accuracy of these notes, and any discrepancies, misunderstandings, typos, etc. as these notes relate to course's content is not the responsibility of the instructor. If you spot any errors or would like to contribute, please contact me directly.

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1.1 Basic notation

We denote

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \ldots\} \\
\mathbb{Z} &= \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
\mathbb{Q} &= \{\frac{n}{m} \mid n \in \mathbb{Z}, m \in \mathbb{N}\} \\
\mathbb{R} &= \text{real numbers}
\end{aligned}$$

We use $\subset$ and $\subseteq$ interchangeably, and use $\subsetneq$ for strict subsets.

1.2 Basic set theory

We denote X as our universal set. If $\{A_\alpha\}_{alpha \in I}$ is such that $A_\alpha \subset X$ for all $\alpha \in I$ (index set), then

$$\bigcup_{\alpha \in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for some } \alpha \in I\} \qquad \text{(union)}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for all } \alpha \in I\} \qquad \text{(intersection)}$$

Define for $A, B \subseteq X$

$$\begin{aligned}
A \setminus B &= \{x \in X \mid x \in A, x \notin B\} && \text{(set difference)} \\
A \Delta B &= \{x \in X \mid x \in A \text{ and } x \notin B\} \text{ OR } x \in B \text{ and } x \notin A\} && \text{(semantic difference)} \\
A^c &= X \setminus A = \{x \in X \mid x \notin A\} && \text{(complement)} \\
&\varnothing && \text{(empty set)} \\
P(X) &= \{A \mid A \subset X\} \quad \varnothing \in P(X), X \in P(X) && \text{(power set)}
\end{aligned}$$

1.3 De Morgan's laws

De Morgan's laws states that given $\{A_\alpha\}_{\alpha \in I} \subset P(X)$

$$\left(\bigcup_{\alpha \in I} A_\alpha\right)^c = \bigcap_{\alpha \in I} A_\alpha^c$$

$$\left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \bigcup_{\alpha \in I} A_\alpha^c$$

Question: what if $I = \varnothing$, what is $\bigcup_{\alpha\in\varnothing} A_\alpha$? It is in fact $\bigcup_{\alpha\in\varnothing} A_\alpha = \varnothing$.
Note that $\bigcap_{\alpha\in\varnothing} A_\alpha = X$ (from De Morgan's Law, and also $A_\alpha = A_\alpha^c$).

1.4 Products of sets, relations, and functions

Given X, Y define the product

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

If $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$ then $X \times Y = \{(x_i, y_j) \mid i = 1, \ldots, n \quad j = 1, \ldots, m\}$ containing nm elements.

Definition 1.1 (Relation)**.** A **relation** on X, Y is a subset R of the product $X \times Y$.
We write xRy if $(x, y) \in R$. The **domain** of R is

$$\{x \in X \mid \exists y \in Y \text{ with } (x, y) \in R\}$$

which need not cover our universal set.
The **range** of R is

$$\{y \in Y \mid \exists x \in X \text{ with } (x, y) \in R\}$$

Definition 1.2 (Function (as a relation))**.** A **function** from X into Y is a relation R such that for every $x \in X$, there exists exactly one $y \in Y$ with $(x, y) \in R$.

Suppose that we have $X_1, X_2, \ldots, X_n$ non-empty sets. Define

$$X_1 \times X_2 \times \ldots \times X_n = \prod_{i=1}^{n} X_i = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i\}$$

or a set of *n-tuples*.
If $X_i = X_j = X$ for all $i, j = 1, \ldots, n$, then

$$\prod_{i=1}^{n} X_i = \prod_{i=1}^{n} X = X^n$$

Problem 1.1. Given a collection $\{X_\alpha\}_{\alpha\in I}$ of non-empty sets, what do we mean by $\prod_{\alpha\in I} X_\alpha$?
Motivation: consider $X_1 \times \ldots \times X_n = \{(x_1, \ldots, x_n) \mid x_i \in X_i\}$. We choose some $(x_1, \ldots, x_n) \in \prod_{i\in\{1,\ldots,n\}} = I$. This point induces *a function*

$$f_{(x_1,\ldots,x_n)} : \{1, \ldots, n\} \to \bigcup_{i=1}^{n} X_i$$

with $f(1) = x_1 \in X_1$, $f(i) = x_i \in X_i$, $f(n) = x_n \in X_n$, etc. Assume we have have $f : \{1, \ldots, n\} \to \bigcup_{i=1}^{n} X_i$ such that $f(i) \in X_i$. Then

$$(f(1), f(2), \ldots, f(n)) = \prod_{i=\{1,\ldots,n\}} X_i$$

Definition 1.3 (Product of sets)**.** Given a collection $\{X_\alpha\}_{\alpha\in I}$ of non-empty sets we let

$$\prod_{\alpha\in I} X_\alpha = \{f : I \to \bigcup_{\alpha\in I} X_\alpha\}$$

such that $f(\alpha) \in X_\alpha$ (i.e $\prod_{\alpha\in I} X_\alpha$ is a "set of functions"). f is called a **choice function**.
Question: If $X_\alpha \neq \varnothing$, is $\prod_{\alpha\in I} X_\alpha \neq \varnothing$?

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2.1 Zermelo's Axiom of Choice

Question: If $\{X_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty sets is

$$\prod_{\alpha \in I} X_\alpha \neq \varnothing$$

This is analogous to saying: given a collection of non-empty sets in $\mathbb{R}$, how would you choose an element from each subset of $\mathbb{R}$? This is easy if they were subsets of $\mathbb{N}$ (take the least element which exists by the *well-ordering principle*) but much more difficult in $\mathbb{R}$.

Axiom 2.1 (Zermelo's Axiom of Choice)**.** If $\{X_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty sets, then $\prod_{\alpha \in I} \neq \varnothing$.

Equivalently we have an analogous version:

Axiom 2.2 (Axiom of Choice V2)**.** If $X \neq \varnothing$, then there exists a function

$$f : P(X) \setminus \{\varnothing\} \to X$$

such that $f(A) \in A$ for all $A \in P(X) \setminus \{\varnothing\}$ (we can always pick out a subset ($e \in P(X)$) from a non-empty set A).

2.2 Properties of relations

Definition 2.1 (Relation properties)**.** A relation R on X (i.e. $R \subseteq X \times X$) is

1. **reflexive** if $x\ R\ x$ for all $x \in X$
2. **symmetric** if $x\ R\ y \Rightarrow y\ R\ x$
3. **anti-symmetric** if $x\ R\ y$ and $y\ R\ x$, then $x = y$
4. **transitive** if $x\ R\ y$ and $y\ R\ z$ implies $x\mathbb{R}z$

2.3 Partially and totally ordered sets

Example 2.1. Let $X = \mathbb{R}$. We have $x\ R\ y$ iff $x \leq y$.
Note that $\leq$ is reflexive, anti-symmetric, and transitive.

Example 2.2. Let $Y \neq \varnothing$ and $X = P(Y)$.
We write $A\ R\ B$ iff $A \subseteq B$.
Note that $\subseteq$ is reflexive, anti-symmetric, and transitive.

Example 2.3. Let $Y \neq \varnothing$ and $X = P(Y)$.
We write $A\ R\ B$ iff $B \subseteq A$.
Note that $\subseteq$ is reflexive, anti-symmetric, and transitive.

Definition 2.2 (Partially ordered sets)**.** A set X with a relation R on X is called a **partially ordered set** if R is

1. reflexive
2. anti-symmetric
3. transitive

(R is a **partial order** on X if it satisfies these three conditions).
We write (X, R) and call this a **poset**.

Definition 2.3 (Totally ordered sets)**.** If (X, R) is a poset, then if $A \subseteq X$ and $R_1 = R_{|A\times A}$ then (A, R_1) is a poset. We say (A, R_1) is **totally ordered** if for each $x, y \in A$ *either* $x\ R\ y$ or $y\ R\ x$. We also call totally ordered sets **chains**.

How many partial orderings can we have for a given set X (i.e. the number of ways to define partial order relations)?

Example 2.4. Let $X = \{x\}$. We have one relation $R = \{(x, x)\}$ (from $X \times X$) and thus 1 partial ordering.

Example 2.5. Let $X = \{x, y\}$. We know posets $(X, \preceq)$ must be reflexive, thus we have one relation where $x \preceq x$ and $y \preceq y$.
We can also have a poset with the reflexive relations above as well as $x \preceq y$. Similarly we can have a poset with $y \preceq x$.

Example 2.6. Let $X = \{x, y, z\}$. We have the poset with just $e \preceq e$ for $e \in X$.
We have the poset with the reflexive relations and $x \preceq y$ and $y \preceq z$ (3 posets with permutations).
We have the poset with the reflexive relations and $z \preceq x$ and $z \preceq y$ (3 posets with permutations).
We have the poset with the reflexive relations and $y \preceq z$ (6 posets with permutations).
We have the poset with the reflexive relations and $x \preceq y$ and $y \preceq z$ (6 posets with permutations).

2.4 Bounds on posets

Definition 2.4 (Upper and lower bounds)**.** Let $(X, \preceq)$ be a partially ordered set.
Let $A \subset X$. We say that $x_0 \in X$ is an **upper bound** for A if $x \preceq x_0$ for all $x \in A$.
If A has an upper bound, we say it is **bounded above**.
If A is bounded above then x_0 is the **least upper bound** if

1. x_0 is an upper bound of A
2. If y is an upper bound of A then $x_0 \preceq y$.

We write $x_0 = \text{lub}(A)$ or $x_0 \sup(A)$ (supremum).
If $x_0 = \text{lub}(A) \in A$, then x_0 is the *maximum* in A.
Similarly we define the same for lower bounds (infimum).

Example 2.7. Let $X = \mathbb{R}$ and $\preceq$ the usual ordering.

Fact 2.1. Every non-empty subset that is bounded above has a least upper bound (LUBP (lub property) for $\mathbb{R}$).

Example 2.8. Let $Y \neq \varnothing$, $X = P(Y)$, and $\preceq$ be $\subseteq$ (ordering by inclusion).
Y is the maximum element of $(X, \subseteq)$.
If $\{A_\alpha\}_{\alpha\in I} \subset P(X)$ is bounded above by Y, but note that

$$\text{lub}(\{A_\alpha\}_{\alpha\in I}) = \bigcup_{\alpha\in I} A_\alpha$$
$$\text{glb}(\{A_\alpha\}_{\alpha\in I}) = \bigcap_{\alpha\in I} A_\alpha$$

Recall that if $I = \varnothing$, then the glb is all of $\mathbb{R}$: this is in fact correct (it's the greatest set that is a lower bound for relation $\subseteq$).