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# MATH 247 Course Notes

Calculus 3 (Advanced)

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**Abstract**

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

# 1 January 3, 2018

## 1.1 Euclidean space $\mathbb{R}^n$

Most postulates and theorems apply to any $n$-dimensional real vector space with a positive-definite inner product.

$$\mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n); x_j \in \mathbb{R}, j = 1, \ldots, n\}$$

Some properties of vectors in $\mathbb{R}^n$ where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, and $t \in \mathbb{R}$:

$$\begin{aligned}
&x + y = (x_1 + y_1, \ldots, x_n + y_n) \\
&tx = (tx_1, \ldots, tx_n) \\
&x + y = y + x \\
&(x + y) + z = x + (y + z) \\
&s(tx) = (st)x \\
&t\vec{0} = \vec{0} \\
&\vec{0}x = \vec{0} \\
&(t + s)x = tx + sx \\
&t(x + y) = tx + ty
\end{aligned}$$

## 1.2 Euclidean inner product

An important additional structure on $\mathbb{R}^n$ is the natural **Euclidean inner product** (aka the *dot product*).

$$\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

which can be written as $x \cdot y \in \mathbb{R}$.
Dot products are **billinear**, **symmetric**, and **positive-definite**. **Bilinear forms** satisfy

$$\begin{aligned}
(x + y) \cdot z &= x \cdot z + y \cdot z \\
x \cdot (y + z) &= x \cdot y + x \cdot z \\
(tx) \cdot y = x \cdot (ty) &= t(x \cdot y)
\end{aligned}$$

**symmetric** denotes

$$x \cdot y = y \cdot x$$

and **positive-definiteness** means $x \cdot x \geq 0$ with equality $\iff x = \vec{0}$.

**Definition 1.1.** The dot product is defined for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$

$$x \cdot y = \sum_{k=1}^{n} x_k y_k$$

**Definition 1.2.** The norm $\|x\|$ of $x \in \mathbb{R}^n$ (induced by some inner product $\langle x, x\rangle = x \cdot x$) is defined as

$$\begin{aligned}\|x\|^2 &= x \cdot x \\ \|x\| &= \sqrt{x \cdot x}\end{aligned}$$

### 1.3 Triangle inequality

**Proposition 1.1.** Triangle inequality states

$$\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^n$$

To prove the above, we need the **Cauchy-Schwarz Inequality**.

**Theorem 1.1.** The Cauchy-Schwarz inequality states that

$$|x \cdot y| \leq \|x\|\|y\|$$

with equality iff $x = ty$ or $y = tx$ for some $t \in \mathbb{R}$.

*Proof.* For the equality case, WLOG if $x = ty$

$$\begin{aligned}x \cdot y = ty \cdot y &= t\|y\|^2 \\ &= |t|\|y\|^2 \\ &= \|x\|\|y\|\end{aligned}$$

Let $t \in \mathbb{R}$. Note for all $t$

$$\begin{aligned}0 \leq \|x - ty\|^2 &= (x - ty) \cdot (x - ty) \\ &= x \cdot x - ty \cdot x - tx \cdot y + t^2 y \cdot y \\ &= \|x\|^2 + t^2\|y\|^2 - 2t(x \cdot y)\end{aligned}$$

Thus we have

$$at^2 + bt + c \geq 0 \quad \forall t \in \mathbb{R}$$

where $a = \|y\|^2$, $b = -2x \cdot y$ and $c = \|x\|^2$. Note there can exist at most one root (positive parabola where all values are non-negative). For $at^2 + bt + c = 0$ to have at most one real root (such that $t$ exists), we need $b^2 - 4ac \leq 0$ (from the quadratic formula).

$$\begin{aligned}4(x \cdot y)^2 &\leq 4\|x\|^2\|y\|^2 \\ |x \cdot y| &\leq \|x\|\|y\|\end{aligned}$$

If we have equality $\exists \quad t_0$ such that $at_0^2 + bt_0 + c = 0$ or $\|x - t_0 y\|^2 = 0$ so $x = t_0 y$. $\square$

**Corollary 1.1.** The triangle inequality

$$\begin{aligned}\|x+y\|^2 &= (x+y)\cdot(x+y)\\ &= \|x\|^2 + 2x\cdot y + \|y\|^2\\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\|+\|y\|)^2\end{aligned}$$

where the last line follows from the Cauchy-Schwarz inequality.

**Definition 1.3.** The **distance** between two points $x, y \in \mathbb{R}^n$ is defined to be

$$d(x,y) = \|x-y\|$$

which satisfies the properties

$$\begin{aligned}&d(x,y) = d(y,x)\\ &d(x,x) = 0\\ &d(x,y) \geq 0 \quad \text{with equality iff} \quad x = y\end{aligned}$$

so we can restate the triangle inequality as $d(x,y) \leq d(x,z) + d(z,x) \quad \forall x,y,z \in \mathbb{R}^n$.

## 1.4 Norms

There exists different "natural" norms on $\mathbb{R}^n$

**Definition 1.4.** A norm $\|\cdot\|$ on $\mathbb{R}^n$ is a map

$$\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^{\geq 0}$$

such that

1. $\|x\| = 0 \iff x = \vec{0}$
2. $\|tx\| = |t|\|x\|$
3. $\|x+y\| \leq \|x\| + \|y\|$

All inner products determine a norm but not all norms are from inner products. We saw that the dot product determines a norm called the Euclidean norm.

$l^1$ norm $\|x\|_1 = \sum_{k=1}^n |x_k|$

$l^p$ norm $\|x\|_p = (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}}$

sup norm (aka $l^\infty$ norm) $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$

One can see that $l^\infty$ norm is a "limit" of $l^p$ norms as $p \to \infty$.
Note the $l^2$ norm is the Euclidean norm.

Why are norms important? **A norm determines a distance**. For example

$$d(x,y) = \|x-y\|$$

(all norms determine a distance but not all distances are from norms).
Distance is important to define a **limit** which is crucial for differentiability/integrability.

### 1.5 Angle between two vectors

A corollary to C-S for $x, y \neq \vec{0}$

$$-1 \leq \frac{x \cdot y}{\|x\|\|y\|} \leq 1$$

Define the angle $\theta \in [0, \pi]$ between $x$ and $y$ to be

$$\cos\theta = \frac{x \cdot y}{\|x\|\|y\|}$$

so we have another definition of the dot product

$$x \cdot y = \|x\|\|y\| \cos\theta$$

We say $x, y$ are **orthogonal** if $\theta = \frac{\pi}{2} \iff x \cdot y = 0$.
Why is this the correct definition?

$$\begin{aligned}\|y - x\|^2 &= (y - x) \cdot (y - x) \\ &= \|x\|^2 + \|y\|^2 - 2x \cdot y \\ &= \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos\theta\end{aligned}$$

This aligns with the Law of Cosines $c^2 = a^2 + b^2 - 2ab\cos\theta$.

# 2 January 5, 2018

### 2.1 Linear maps

**Definition 2.1.** A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if $T$ takes linear combinations to linear combinations i.e.

$$T(\sum_{k=1}^{N} t_k v_k) = \sum_{k=1}^{N} t_k T(v_k) \quad t_i \in \mathbb{R} \quad v_j \in \mathbb{R}^n$$

We will see linear maps are closely related to **differentiability**.
Some facts about linear maps: let $e_1, \ldots, e_n$ be the standard basis.

$$x \in \mathbb{R}^n = (x_1, \ldots, x_n) = \sum_{k=1}^{n} x_k e_k$$

Let $f_1, \ldots, f_m$ be the standard basis of $\mathbb{R}^m$ where $f_j = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^m$.

$$y \in \mathbb{R}^m = (y_1, \ldots, y_n) = \sum_{k=1}^{m} y_k f_k$$

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear and let

$$\begin{aligned} y = \sum_{l=1}^{m} y_l f_l = T(x) &= T(\sum_{k=1}^{n} x_k e_k) \\ &= \sum_{k=1}^{n} x_k T(e_k) \\ &= \sum_{k=1}^{n} x_k (\sum_{l=1}^{m} A_{lk} f_l) \\ &= \sum_{k=1}^{n} (\sum_{l=1}^{m} A_{lk} x_k) f_l \end{aligned}$$

By uniqueness of the expansion of a vector in terms of a basis ($f_j$s) we conclude that

$$y_l = \sum_{k=1}^{n} A_{lk} x_k \quad l = 1, \ldots, m$$

or in matrix form

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \ldots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

We've shown that any linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is necessarily **matrix multiplication**

$$y = T(x) = A \cdot x$$

for some unique $m \times n$ matrix $A$ (with respect to some bases in $\mathbb{R}^n$ and $\mathbb{R}^m$).
The rule of matrix multiplication is automatic from the composition of linear maps. Let

$$\begin{aligned} &T : \mathbb{R}^n \to \mathbb{R}^m \\ &S : \mathbb{R}^m \to \mathbb{R}^p \\ &y = T(x) = A \cdot x \quad m \times n \\ &z = S(y) = B \cdot y \quad p \times m \end{aligned}$$

Therefore $S \circ T : \mathbb{R}^n \to \mathbb{R}^p$ is linear.

$$\begin{aligned} (S \circ T)(\sum t_k v_k) &= S(T(\sum_k t_k v_k)) \\ &= S(\sum_k x_k T(v_k)) \\ &= \sum_k x_k S(T(v_k)) \\ &= \sum_k t_k (S \circ T)(v_k) \end{aligned}$$

So we have

$$\begin{aligned} z_l = \sum_{j=1}^{m} B_{lj} y_j &= \sum_{j=1}^{m} B_{lj} (\sum_{i=1}^{n} A_{ji} x_i) \\ &= \sum_{i=1}^{n} (\sum_{j=1}^{m} B_{lj} A_{ji}) x_i \\ &= \sum_{i=1}^{n} C_{li} x_i \end{aligned}$$

where

$$z = (S \circ T)(x) = C \cdot x \quad p \times n$$

Recall the space $L(\mathbb{R}^n, \mathbb{R}^m)$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ is itself a finite dimensional real vector space of dimension $nm$ (isomorphic to $\mathbb{R}^{nm}$).

$$T \in L(\mathbb{R}^n, \mathbb{R}^m) \iff A \in M_{m \times n}(\mathbb{R})$$

where $M_{m\times n}(\mathbb{R})$ is the space of real $m \times n$ matrices. There is a unique 1-1 correspondence between $T$ and $A$ (as shown before).

## 2.2 Operator norm

Note one can define norm on matrices. The natural Euclidean norm for matrix $A$ can be defined as

$$\|A\|_2 = \sqrt{\sum_{i=1,\dots,m; j=1,\dots,n} (A_{ij})^2}$$

**Definition 2.2.** The **operator norm** is defined for a $T : \mathbb{R}^n \to \mathbb{R}^m$ linear map as

$$\|T\|_{op} = \inf\{C > 0, \|T(x)\| \leq C\|x\| \quad \forall x \in \mathbb{R}^n\}$$

We need to show this norm is

1. Well-defined
2. $\|\cdot\|_{op}$ is a norm

1. Show well-defined

$$T(x) = A \cdot x \quad A \quad m \times n$$

$$\begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} A_1 \cdot x \\ \vdots \\ A_m \cdot x \end{bmatrix} = T(x)$$

So the norm is

$$\begin{aligned} \|T(x)\|^2 &= (A_1 \cdot x)^2 + \dots + (A_m \cdot x)^2 \\ &\leq \|A_1\|^2 \|x\|^2 + \dots + \|A_m\|^2 \|x\|^2 && \text{C-S} \\ &= (\|A_1\|^2 + \dots + \|A_m\|^2) \|x\|^2 \end{aligned}$$

**Case 1** Assume $\|A_1\|^2 + \ldots + \|A_m\|^2 = 0$.

$$\begin{aligned} \|A_1\|^2 + \ldots + \|A_m\|^2 = 0 &\iff A = 0_{m\times n} \\ &\iff T = 0 \in L(\mathbb{R}^n, \mathbb{R}^m) \end{aligned}$$

Then $T(x) = 0 \quad \forall x$ so $\|T(x)\| \leq C\|x\|$ holds $\forall C > 0$, thus the infimum of positive real numbers (0) implies $\|T\|_{op} = 0$.

**Case 2** Assume $\|A_1\|^2 + \ldots + \|A_m\|^2 > 0$.
$\{C > 0, \|T(x)\| \leq C\|x\| \quad \forall x \in \mathbb{R}^n\}$ is non-empty because $\sqrt{\|A_1\|^2 + \ldots + \|A_m\|^2}$ is in there. By the completeness of $\mathbb{R}$, $\|T\|_{op}$ exists and is $\geq 0$.

2. We've shown $\|T\|_{op}$ exists and is $\geq 0$ for all $T \in L(\mathbb{R}^n, \mathbb{R}^m)$. It remains to shown $\|T\|_{op}$ is a norm:

   (a) $\|T\|_{op} = 0$ only for the zero map
   (b) $\|\lambda T\|_{op} = |\lambda|\|T\|_{op} \quad \forall \lambda \in \mathbb{R}$
   (c) $\|T + S\|_{op} \leq \|T\|_{op} + \|S\|_{op}$

   To see this, we note that since

$$\|T\|_{op} = \inf\{C > 0, \|T(x)\| \leq C\|x\| \quad \forall x \in \mathbb{R}^n\}$$

   $\exists$ a **decreasing sequence** $c_k \geq 0$ such that $\|T(x)\| \leq c_k\|x\| \quad \forall x \in \mathbb{R}^n$ and $\lim_{k\to\infty} c_k = \|T\|_{op}$.
   Take limit as $k \to \infty$ of the predicate in $\|T\|_{op}$.

$$\begin{aligned} \|T(x)\| &\leq (\lim_{k\to\infty} c_k)\|x\| \\ \|T(x)\| &\leq \|T\|_{op}\|x\| \end{aligned}$$

   So we have

$$\begin{aligned} \|T\|_{op} = 0 &\Rightarrow \|T(x)\| \leq 0 \quad \forall x \\ &\Rightarrow T(x) = 0 \quad \forall x \\ &\Rightarrow T = 0 \in L(\mathbb{R}^n, \mathbb{R}^m) \end{aligned}$$

   which proves (a).

$$\|\lambda T\|_{op} = |\lambda|\|T\|_{op}$$

   follows from

$$\begin{aligned} \|(\lambda T)(x)\| &= \|\lambda(T(x))\| \\ &= |\lambda|\|T(x)\| \quad \forall x \end{aligned}$$

   If $\lambda = 0$, $\lambda T = 0 \Rightarrow \|\lambda T\|_{op} = 0 = |\lambda|\|T\|_{op}$.

If $\lambda \neq 0$

$$\begin{aligned}
\|\lambda T\|_{op} &= \inf\{C > 0, \|(\lambda T)(x)\| \leq C\|x\|\} \\
&= \inf\{C > 0, |\lambda|\|T(x)\| \leq C\|x\|\} \\
&= \inf\{C > 0, \|T(x)\| \leq \frac{C}{|\lambda|}\|x\|\} \\
&= |\lambda| \inf\{\tilde{C} > 0, \|T(x)\| \leq \tilde{C}\|x\|\} && \tilde{C} = \frac{C}{\lambda} \\
&= |\lambda|\|T\|_{op}
\end{aligned}$$

which proves (b). (c) is similar.

# 3 January 8, 2018

## 3.1 Topology of $\mathbb{R}^n$

Topology is the study of **closeness** in a space.

## 3.2 Open and closed balls

**Definition 3.1.** Let $x \in \mathbb{R}^n$ and $r > 0$. The **open ball** at radius $r$ centred at $x$ is denoted

$$B_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\| < r\}$$

It consists of all points in $\mathbb{R}^n$ whose distance from $x$ is *strictly less than* $r$.

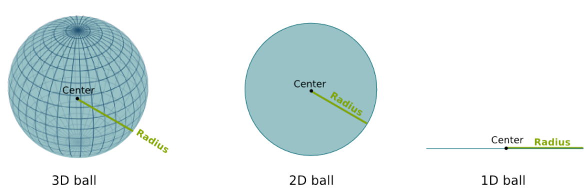


**Figure 3.1:** Open balls in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

In $\mathbb{R}$, $B_r(x) = (x - r, x + r)$. In $\mathbb{R}^3$, $B_r(x)$ is the *interior* of a sphere of radius $r$ centred at $x$.

**Definition 3.2.** Let $x \in \mathbb{R}^n$, $r > 0$. The **closed ball** of radius $r > 0$ centred at $x$ is denoted

$$\overline{B_r(x)} = \{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$$

**Remark 3.1.** The notation will be explained in the following class/section. Note that

$$\overline{B_r(x)} = B_r(x) \cup \{\text{points exactly at distance } r\}$$

For $n = 1$, $\overline{B_r(x)} = [x - r, x + r]$.

## 3.3 Open sets

**Definition 3.3.** A subset $U \subseteq \mathbb{R}^n$ is called an **open set** (or open) iff $\forall x \in U$, $\exists r > 0$ ($r$ depends on $x$) such that $B_r(x) \subseteq U$.

(Informally: a subset $U$ is open if for every $x \in U$, all points sufficiently close to $x$ are *also* in $U$).

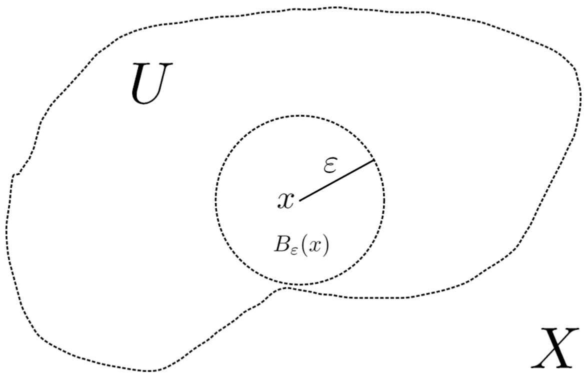


**Figure 3.2:** One can form an open ball for every point $x$ in an open set $U$.

**Example 3.1.** Set that is not open

- $[0,1] \subseteq \mathbb{R}$. Note: $\nexists r > 0$ for $x = 1$ such that $B_r(x) \subseteq [0,1]$.

Sets that are open

- $\mathbb{R}^n$ since $x + \epsilon \in \mathbb{R}^n$ by definition.
- $\varnothing$ (vacuous: satisfied trivially $\varnothing$ has no points).

**Proposition 3.1.** An **open ball** is an **open set**.

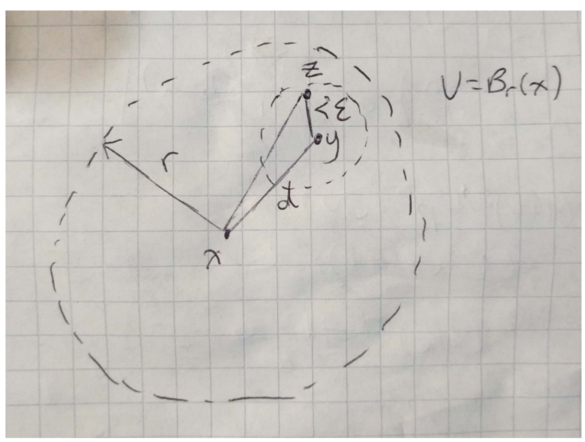


**Figure 3.3:** An open ball is an open set (see proof below).

*Proof.* Let $U = B_r(x)$ and $y \in U = B_r(x)$. We need to find some $\epsilon > 0$ such that $B_\epsilon(y) \subseteq U$.
Let $d = \|x - y\| < r$ since $y \in U = B_r(x)$.
Let $\epsilon = r - d > 0$.
Suppose $z \in B_\epsilon(y)$ thus $\|y - z\| < \epsilon$.
We thus have

$$\|z - x\| \overset{\triangle}{\leq} \|z - y\| + \|y - x\| < \epsilon + d = r$$

So $B_\epsilon(y) \subseteq U$ hence $U$ is open. □

We can construct more from open sets.

### 3.4 Properties of open sets

**Lemma 3.1.** 1. Let $U_\alpha \subseteq \mathbb{R}^n$ be open $\forall \alpha \in A$ (countably or uncountably many), then

$$\bigcup_{\alpha \in A} U_\alpha$$

is open.

2. Let $U_1, \ldots, U_k$ be open (**must be finite** number of sets). Then

$$\bigcap_{j=1}^{k} U_j$$

is open.

Informally, *arbitrary unions* of open sets are open. *Finite intersections* of open sets are open.

*Proof.*

1. We want to show $\bigcup_{\alpha \in A} U_\alpha$ is open.

   Let $x \in \bigcup_{\alpha \in A} U_\alpha$ so $\exists$ some $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$ (holds since union of sets).

   But $U_{\alpha_0}$ is open so $\exists r > 0$ such that $B_r(x) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in A} U_\alpha$.

2. Show $x \in \bigcap_{j=1}^{k} U_j$ so $x \in U_j$ for all $j = 1, \ldots, k$. Each $U_j$ is open so $\forall j, \exists \epsilon_j > 0$ such that $B_{\epsilon_j}(x) \subseteq U_j$.

   Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_k\} > 0$. $\forall j$ we have $B_\epsilon(x) \subseteq B_{\epsilon_j}(x) \subseteq U_j$ hence $B_\epsilon(x) \subseteq \bigcap_{j=1}^{k} U_j$.

   **Remark 3.2.** Arbitrary (e.g. nonfinite) intersections of open sets need not be open (the min. of infinite numbers is not well defined. An infimum of positive numbers need not be $> 0$ i.e. it could be 0).

   Even intersection of countably infinite sets may not be open. Suppose $U_k = (0, 1 + \frac{1}{k}) \subseteq \mathbb{R} \quad \forall k \in \mathbb{N}$. Note that $\bigcap_{k=1}^{\infty} U_k = (0, 1]$ is not open.

$\square$

### 3.5 Closed sets

**Definition 3.4.** A subset $F \subseteq \mathbb{R}^n$ is called **closed** if $F^c = \mathbb{R} \setminus F$ is open (note: this definition is based on open's definition).

**Proposition 3.2.** A closed ball $\overline{B_r(x)} = \{y \in \mathbb{R}^n \mid \|y - x\| \leq r\}$ is a closed set.

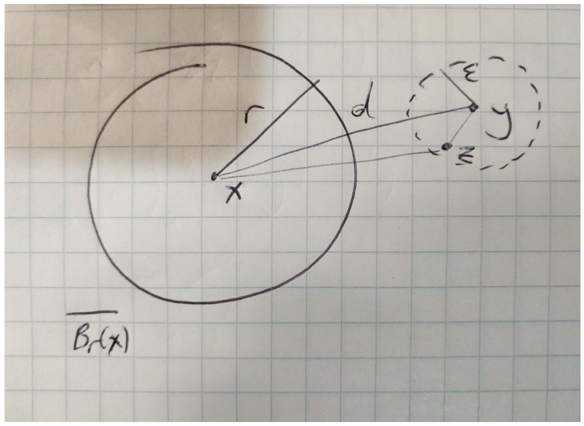


**Figure 3.4:** A closed ball is a closed set (see proof below).

*Proof.* Let $F = \overline{B_r(x)}$ and

$$F^c = (\overline{B_r(x)})^c = \{y \in \mathbb{R}^n \mid \|y - x\| > r\}$$

Let $y \in \overline{B_r(x)}^c$: need to find $\epsilon > 0$ such that $B_\epsilon(y) \subseteq F^c$.
Let $d = \|x - y\| > r$ and let $\epsilon = d - r > 0$.
If $z \in B_\epsilon(y)$, then

$$\begin{aligned} \|x - y\| &\overset{\triangle}{\leq} \|x - z\| + \|z - y\| \\ d &\leq \|x - z\| + \|z - y\| \\ \|x - z\| \geq d - \|z - y\| & \\ &> d - \epsilon = r \end{aligned}$$

Hence $z \in F^c$ so $B_\epsilon(y) \subseteq F^c$, thus $F^c$ is open and by definition $F$ is closed. □

## 3.6 Properties of closed sets

**Lemma 3.2.** Note: this lemma is the inverse of the equivalent for open sets.

1. If $F_1, \dots, F_k$ is closed, then $\bigcup_{j=1}^k F_j$ is closed.
2. If $F_\alpha$ is closed $\forall \alpha \in A$, then $\bigcap_{\alpha \in A} F_\alpha$ is closed.

Finite unions of closed sets are closed. Arbitrary intersections of closed sets are closed.

*Proof.* By De Morgan's laws

$$(\bigcup_{j=1}^k F_j)^c = \bigcap_{j=1}^k (F_j)^c$$
$$(\bigcap_{\alpha \in A} F_\alpha)^c = \bigcup_{\alpha \in A} (F_\alpha)^c$$

□

### 3.7 Neither open nor closed

A subset $V$ of $\mathbb{R}^n$ need not be either open or closed. It can be open, closed, neither or both!

**Example 3.2.** Examples of non-exclusive open or closed sets are

- $(0,1] \subseteq \mathbb{R}$ - neither
- $\mathbb{R}^n, \varnothing$ are *open and closed*

### 3.8 Interior

Sometimes a set is neither open nor closed, but there are always **natural open (interior) and closed (closure) sets** which can be associated to any subset of $\mathbb{R}^n$.

**Definition 3.5.** Let $A \subseteq \mathbb{R}^n$ (could be $\varnothing$).

$$\begin{aligned} A^o &= int(A) && \text{interior of } A \\ &= \bigcup_{\substack{V \subseteq A \\ V \text{open in } \mathbb{R}^n}} V && \text{union of \textbf{all} open subsets of } \mathbb{R}^n \text{ that are contained in } A \end{aligned}$$

**Remark 3.3.** 1. $A^o$ is open (arbitrary union of open sets) and $A^0 \subseteq A$

2. if $V$ is any open subset of $R^n$ that is contained in $A$, then $V \subseteq A^o$ ($A^o$ is the largest open subset of $R^n$ that is contained in $A$)

3. $A$ is open iff $A^o = A$

*Proof.* **Forwards:**

$A$ is open and $A \subseteq A$ thus $A$ must be a $V$ in the union, but since all $V \subseteq A^o \subseteq A$ (where $A$ is a $V$) then $A^o = A$.

**Backwards:**

$A^o = A$. Since $A^o$ is open, $A$ is open. □

### 3.9 Closure

**Definition 3.6.**

$$\begin{aligned} \overline{A} &= cl(A) && \text{closure of } A \\ &= \bigcap_{\substack{F \supseteq A \\ F \text{closed in } \mathbb{R}^n}} F && \text{intersection of \textbf{all} closed subsets of } \mathbb{R}^n \text{ that contains } A \end{aligned}$$

**Remark 3.4.** 1. $\overline{A}$ is closed (arbitrary intersection of closed sets) and $\overline{A} \supseteq A$

2. if $F$ is any closed subset of $R^n$ that contains $A$, then $F \supseteq \overline{A}$ ($\overline{A}$ is the smallest closed set of $R^n$ containing $A$)

3. $A$ is closed iff $\overline{A} = A$

# 4 January 10, 2018

## 4.1 Closure of open ball is closed ball

**Proposition 4.1.** The closure of the open ball $B_\epsilon(x)$ is the closed ball $\overline{B_\epsilon(x)}$ (hence the notation).

*Proof.* Remember
$$\overline{B_\epsilon(x)} = \{y \in \mathbb{R}^n \mid \|y - x\| \leq \epsilon\}$$
Let $A =$ closure of $B_\epsilon(x)$.
Let $F = \{y \in \mathbb{R}^n \mid \|x - y\| \leq \epsilon\}$.
We want to show $A = F$.
We know $F$ is closed and $F \supset B_\epsilon(x)$, so $F$ contains $A$: the closure of $B_\epsilon(x)$ (any closed set containing another set is in the intersection of the closure) or
$$F \supset A \supset B_\epsilon(x)$$
Suppose $F \neq A$, then $\exists y \in F$ with $y \notin A \Rightarrow y \notin B_\epsilon(x)$ so
$$\|x - y\| = \epsilon$$
(it's sandwiched between the closed ball ($\leq \epsilon$) and the open ball ($< \epsilon$), so it must hold with equality with $\epsilon$ by the Trichotomy property).

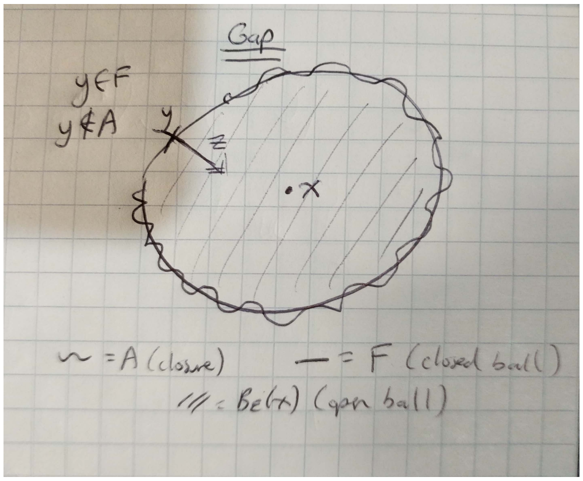


**Figure 4.1:** The closure of an open ball is the corresponding closed ball.

$A$ is closed and $y \notin A$ so $A^c$ is open and $y \in A^c$. So $\exists \delta > 0$ such that $B_\delta(y) \subseteq A^c$.
Let $t > 0$ with $t < \min\{\delta, \epsilon\}$.
Let
$$z = y + t\frac{(x - y)}{\|x - y\|}$$
(add $t$ unit vectors from $y$ to $x$). Note that
$$\|z - y\| = t < \delta$$
so $z \in B_\delta(y) \subseteq A^c$.

Also

$$\begin{aligned} x - z &= x - y - t\frac{(x-y)}{\|x-y\|} \\ &= (\|x-y\| - t)\frac{(x-y)}{\|x-y\|} \end{aligned}$$

where the left term is the norm of the vector and the right term is the unit vector.
Thus

$$\|x-z\| = |\|x-y\| - t| = |\epsilon - t| = \epsilon - t < \epsilon$$

So $z \in B_\epsilon(x) \subseteq A$, but we assumed $z \in A^c$ which is a contradiction.
So we must have $F = A$.

**Remark 4.1.** There is a much simpler proof of this using sequences and limit points.

□

## 4.2 Boundary

**Definition 4.1.** Let $A \subseteq \mathbb{R}^n$. We define the **boundary** of $A$ denoted $\partial A = bd(A)$ to be

$$\partial A = bd(A) = \{x \in \mathbb{R}^n \mid B_\epsilon(x) \cap A \neq \varnothing, B_\epsilon(x) \cap A^c \neq \varnothing \quad \forall \epsilon > 0\}$$

That is, $x \in \partial A$ iff every open ball centred at $x$ contains a point in $A$ **and** a point in $A^c$.
Clearly

$$\begin{aligned} \partial B_\epsilon(x) &= \{y \in \mathbb{R}^n \mid \|y-x\| = \epsilon\} \\ &= \partial(\overline{B_\epsilon(x)}) \end{aligned}$$

## 4.3 Characterization of boundary

**Proposition 4.2.** Let $A \subseteq \mathbb{R}^n$: then

$$\begin{aligned} \partial A &= \overline{A} \setminus A^o \\ &= cl(A) \setminus int(A) \end{aligned}$$

*Proof.* The following two claims and proofs revolve around complements of sets and how if set $A$ intersect a set $B$ is the empty set, then $A$ is a subset of $B^c$.

**Claim 1**

$$x \in \overline{A} \iff B_\epsilon(x) \cap A \neq \varnothing \quad \forall \epsilon > 0$$

*Proof.* **Forwards:**

Suppose $x \in \overline{A}$ but $\exists \epsilon_0 > 0$ $B_\epsilon(x) \cap A = \varnothing$.

So $B_\epsilon(x) \subseteq A^c \Rightarrow (B_\epsilon(x))^c \supset A$.

Since $(B_\epsilon(x))^c$ is closed, then $(B_\epsilon(x))^c \supset \overline{A}$ (by remark (2) after closure definition).

So $\overline{A} \cap B_\epsilon(x) = \varnothing$, but $x \in B_\epsilon(x) \Rightarrow x \notin \overline{A}$, which is a contradiction.

**Backwards:**

We prove the contrapositive

$$x \notin \overline{A} \Rightarrow B_\epsilon(x) \cap A = \varnothing \quad \exists \epsilon > 0$$

Assume $x \notin \overline{A} \Rightarrow x \in (\overline{A})^c$ which is open, so $\exists \epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \subseteq (\overline{A})^c$. Therefore $B_{\epsilon_0}(x) \cap \overline{A} = \varnothing$ (where $\overline{A} \supset A$), which proves our claim). □

**Claim 2**

$$x \notin A^o \iff B_\epsilon(x) \cap A^c \neq \varnothing \quad \forall \epsilon > 0$$

*Proof.* **Forwards:**

Suppose $x \notin A^o$. Assume (for contradiction) $\exists \epsilon_0 > 0$ such that

$$B_{\epsilon_0}(x) \cap A^c = \varnothing \Rightarrow B_{\epsilon_0}(x) \subseteq A$$

(nothing in $A^c$, thus all in $A$).

Ergo $x \in (A^o)^c$ and $B_{\epsilon_0}(x) \subseteq A^o$ (since $B_{\epsilon_0}(x)$ is an open set contained in $A$ - remark (2) after interior definition).

So $B_{\epsilon_0}(x) \cap (A^o)^c = \varnothing$ but $x \in B_{\epsilon_0}(x) \cap (A^o)^c$ which is a contradiction.

**Backwards:**

(Contrapositive): suppose $x \in A^o$. $A^o$ is open so $\exists \epsilon > 0$ such that

$$B_{\epsilon_0}(x) \subseteq A^o \subseteq A$$

so $B_{\epsilon_0}(x) \cap A^c = \varnothing$ for some $\epsilon_0 > 0$. □

Putting the claims together:

$$\begin{aligned} x \in \overline{A} &\iff B_\epsilon(x) \cap A \neq \varnothing \quad \forall \epsilon > 0 & (1) \\ x \in (A^o)^c &\iff B_\epsilon(x) \cap A \neq \varnothing \quad \forall \epsilon > 0 & (2) \\ x \in \partial A &\iff (1) + (2) \\ &\iff x \in \overline{A} \cap (A^o)^c = \overline{A} \setminus A^o \end{aligned}$$

□

## 4.4 Sequential characterization of limits

**Definition 4.2.** Let $(x_k)$ be a sequence of points in $\mathbb{R}^n, k \in \mathbb{N}$. We say $(x_k)$ **converges** to a point $x \in \mathbb{R}^n$ **iff** for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ ($N$ depends on $\epsilon$ in general)

$$k \geq N \Rightarrow \|x_k - x\| < \epsilon$$

(i.e. for any $\epsilon > 0$, **all** the elements of sequence $x_k$ after some $k = N$ are within $\epsilon$ of $x$).

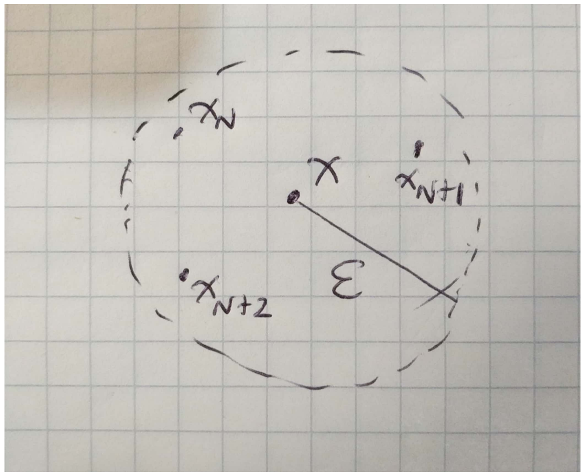


**Figure 4.2:** All points after $k = N$ for a converging sequence is within $\epsilon$.

If $(x_k)$ converges to $x$, we denote

$$\lim_{k\to\infty} x_k = x$$

where $x$ is **the limit** of $x_k$.

## 4.5 Uniqueness of limits

**Lemma 4.1.** Suppose $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} x_k = y$. Then $x = y$ (i.e. a sequence may not converge, but if it does the limit is unique).

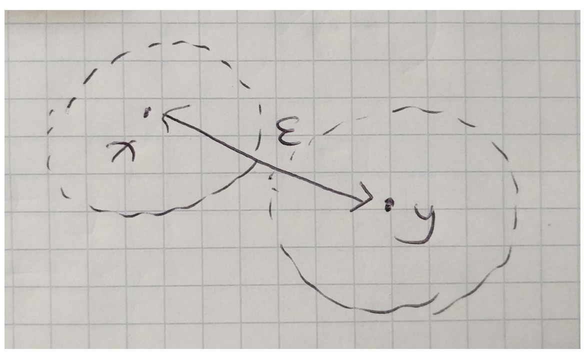


**Figure 4.3:** Sketch of proof with $x \neq y$ (see below).

*Proof.* Suppose $x \neq y$, so $\|x - y\| = \epsilon > 0$.
Since $(x_k)$ converges to $x$, $\exists N_1 \in N$ such that $k \geq N_1$ and

$$\|x_k - x\| < \frac{\epsilon}{2}$$

Similarly for $y$ $\exists k \geq N_2$.

Suppose $k \geq \max\{N_1, N_2\}$. Then

$$\begin{aligned}\|x-y\| &\overset{\triangle}{\leq} \|x-x_k\| + \|x_k - y\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

So $x = y$ by contradiction. □

### 4.6 Neighbourhood

**Definition 4.3.** Let $x \in \mathbb{R}^n$. A subset $U \in \mathbb{R}^n$ is called a **neighbourhood (n'h'd)** of $x$ if $\exists \epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \subseteq U$.

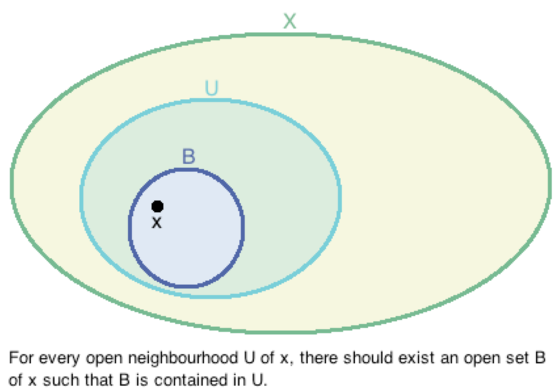


**Figure 4.4:** $U$ is a neighbourhood of $x$ since there exists an open set $B$ of $x$ contained in $U$.

(Equivalently, $U$ is a n'h'd of $x \iff U$ contains an open set containing $x$.)

**Definition 4.4.** An open n'h'd of $x$ is any open set containing $x$. (A set is an open n'h'd of $x$ if it contains $x$ **and** all points sufficiently close to $x$).

**Lemma 4.2.** Let $(x_k)$ be a sequence in $\mathbb{R}^n$. Suppose $\lim_{k\to\infty} x_k$ exists and equal $x \in \mathbb{R}^n$. Then any n'h'd of $x$ contains all $x_k$'s for $k$ sufficiently large, i.e. if $U$ is a n'h'd of $x$, $\exists N \in \mathbb{N}$ ($N$ depends on $U$) such that

$$k \geq N \Rightarrow x_k \in U$$

*Proof.* $U$ is a n'h'd of $x$ so $\exists \epsilon_0 > 0$ such that $B_\epsilon(x) \subseteq U$.
Since $\lim_{k\to\infty} x_k = x$, $\exists N \in N$ such that $k \geq N \Rightarrow \|x_k - x\| < \epsilon_0$ so $x_k \in B_\epsilon(x) \subseteq U \quad \forall k \geq N$. □

# 5 January 12, 2018

## 5.1 Relations between convergent sequences and open/closed sets

**Recall:** $x \in \overline{A} \iff B_\epsilon(x) \cap A \neq \varnothing \quad \forall \epsilon > 0$.

**Proposition 5.1.** Suppose $x \in \overline{A}$. Take $\frac{1}{k} > 0$. From above: $\exists x_k \in A$ such that $\|x_k - x\| < \frac{1}{k}$, then $\lim_{k\to\infty} x_k = x$.

*Proof.* Let $\epsilon > 0$ so $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$ (Archimedean Principle). $\forall k \geq N$, $\frac{1}{k} \leq \frac{1}{N} < \epsilon$ so $\|x_k - x\| < \frac{1}{k} < \epsilon$. □

To summarize, if $x \in \overline{A}$, then $\exists$ a sequence $(x_k)$ such that $\lim_{k\to\infty} x_k = x$ **and** $x_k \in A \quad \forall \in \mathbb{N}$.

What about the converse?

**Proposition 5.2.** Suppose $x_k \in A \quad \forall k$ and $\lim_{k\to\infty} x_k = x$ **and** $x_k \in A \quad \forall k \in \mathbb{N}$. Then $x \in \overline{A}$.

*Proof.* If not, $x \in (\overline{A})^c$ so $\exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq (\overline{A})^c$. But $\exists N \in \mathbb{N}$ such that

$$k \geq N \Rightarrow x_k \in B_\epsilon(x)$$

and by hypothesis $x_k \in A \subseteq \overline{A}$.
So $k \geq N \Rightarrow x_k \in \overline{A}$ but we assumed $x_k \in (\overline{A})^c$ which is a contradiction. □

(i.e. whenever $(x_k)$ is a convergent sequence of points all of whose elements are in $A$, then the limit is in $\overline{A}$).
**Special case:** If $A$ is closed ($\overline{A} = A$) then if $(x_k) \to x$ and $x_k \in A \forall k$ then $x \in A$; this is **not** true for open sets $A$.

## 5.2 Bounded and Cauchy sequences

**Definition 5.1.** A sequence $(x_k)$ in $\mathbb{R}^n$ is called **bounded** if $\exists M > 0$ such that

$$\|x_k\| \leq M \quad \forall k \in \mathbb{N}$$

(that is: all the $x_k$'s lie in some closed ball $\overline{B_M(x)}$ centred at 0).

**Definition 5.2.** A sequence $(x_k)$ is called **Cauchy** if for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$k, l \geq N \Rightarrow \|x_k - x_l\| < \epsilon$$

(eventually all points in the sequence are close to each other).

## 5.3 Convergent $\iff$ Cauchy

**Proposition 5.3.** Let $(x_k)$ be a convergent sequence. Then $(x_k)$ is Cauchy.

*Proof.* Let $x = \lim_{k\to\infty} x_k$. Let $\epsilon > 0$, then $\exists N$ such that

$$\|x_k - x\| < \frac{\epsilon}{2}$$

If $k, l \geq N$ then

$$\|x_k - x_l\| \overset{\triangle}{\leq} \|x_k - x\| + \|x - x_l\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

**Recall from MATH 147:** In $\mathbb{R}$ every Cauchy sequence converges (equivalent to **completeness** of $\mathbb{R}$ or the real line). We show Cauchy converges in $\mathbb{R}^n$ in assignment 2 by showing that each j-th component $x^{(j)}$ converges then by the completeness of $\mathbb{R}$ this follows for $\mathbb{R}^n$.

## 5.4 Convergence implies bounded

**Lemma 5.1.** Every convergent sequence is bounded.

*Proof.* Let $x = \lim_{k\to\infty} x_k$. Let $M_0 = \|x\| + \epsilon$ for $\epsilon > 0$. Then $\exists N$ such that

$$k \geq N \Rightarrow \|x_k - x\| < \epsilon$$

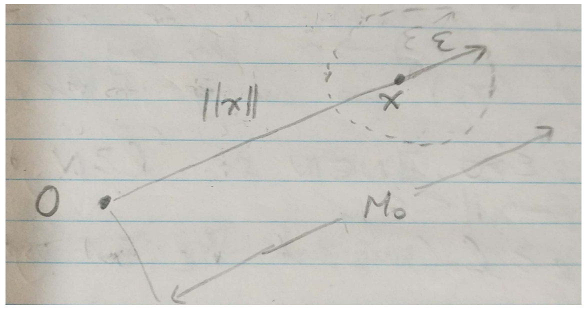


**Figure 5.1:** Convergent sequences can be bounded by the limit and $\epsilon$ and finite points in the sequence.

Note that

$$k \geq N \Rightarrow \|x_k\| \overset{\triangle}{\leq} \|x_k - x\| + \|x\| < \epsilon + \|x\| = M_0$$

Thus we let $M = \max\{\|x_1\|, \ldots, \|x_{N-1}\|, M_0\}$ then $\|x_k\| \leq M \quad \forall k \in \mathbb{N}$. $\square$

Note: not every bounded sequence is Cauchy. Consider $x_k = (-1)^{k+1}$ is $\mathbb{R}$, which is bounded but not convergent. Can we find a weaker statement that's true i.e. given a bounded sequence, can we somehow obtain from it a convergent sequence?

### 5.5 Subsequences

Let $(x_k)$ be a sequence in $\mathbb{R}^n$. Let $1 \leq k_1 < k_2 < \ldots < k_e < k_{e+1} < \ldots$ be a sequence of $1, 2, 3, 4, \ldots$. Then the corresponding sequence $(y_l)$ (or $(x_{k_l})$) in $\mathbb{R}^n$ given by $y_l = x_{k_l}$ is called a **subsequence** of $(x_k)$.

**Example 5.1.** The subsequence given by $k_l = 2l - 1$ (odd numbers) is

$$(x_{2l-1}) = x_1, x_3, x_5, \ldots$$

**Proposition 5.4.** Suppose $(x_k) \to x$. Then any subsequence $(x_{k_l})$ of $(x_k)$ also converges to the same limit $x$.

*Proof.* Let $\epsilon > 0$. $\exists N \in \mathbb{N}$ such that $l \geq N$ then $\|x_l - x\| < \epsilon$, but $k_l \geq l$ (since each $k_e$ must be strictly larger $> k_{e-1}$), so $\|x_{k_l} - x\| < \epsilon \quad \forall l \geq N$ hence $\lim_{k_l \to \infty} x_{k_l} = x$. $\square$

**Note:** A sequence $(x_k)$ that does not converge can have

1. Subsequences that don't converge (e.g. $k_l = l$ so $x_{k_l} = x_l$).
2. Distinct subsequences with different limits.

   For example, $x_k = (-1)^{k+1}$ which is $1, -1, 1, -1, \ldots$, we can have two subsequences

$$x_{2l-1} = (-1)^{2l} = 1, 1, 1, \ldots \qquad (x_{2l-1}) \to 1$$
$$x_{2l} = (-1)^{2l-1} = -1, -1, -1, \ldots \qquad (x_{2l}) \to -1$$

### 5.6 Bolzano-Weierstrass (B-W) Theorem

**Theorem 5.1.** In $\mathbb{R}^n$, every bounded sequence has a convergent subsequence.

**Remark 5.1.** This convergent subsequence is **not** unique. We'll see in the proof that we make many arbitrary choices.

*Proof.* By induction on $n$.
**Case $n = 1$:** Let $(x_k)$ be a sequence in $\mathbb{R}$ that is **bounded**. So $\exists M > 0$ such that $|x_k| \le M \quad \forall k \in \mathbb{N} \iff x_k \in [-M, M]$.

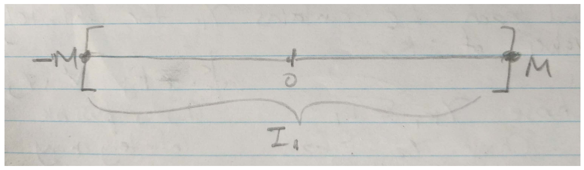


**Figure 5.2:** $I_1$ is the interval of our bounded sequence in $\mathbb{R}$.

Define

$$I_1 = [-M, M] = [-M, 0] \cup [0, M]$$

At least one (maybe both) of $[-M, 0]$ and $[0, M]$ contains $x_k$ for infinite many values of $k$ (the $x_k$'s could initially be all in one side then infinitely many in the other, or the $x_k$'s could jump back and forth so both would have infinitely many).
Let $I_2$ denote the one with infinitely many. That is $x_k \in I_2$ for infinitely many $x_k$'s. Note that

$$\begin{aligned} &I_2 \subseteq I_1 \\ &I_2 = [a, b] = [a, \frac{a+b}{2}] \cup [\frac{a+b}{2}, b] \end{aligned}$$

Again, at least one of these halves contains infinitely many $x_k$'s. Let $I_3$ be that one.
Keep subdividing in this way and choosing a half which contains $x_k$ for infinitely many $k$'s. We have

$$\begin{aligned} \text{length } I_1 &= 2M \\ \text{length } I_2 &= M \\ \text{length } I_3 &= \frac{M}{2} \\ &\vdots \\ \text{length } I_l &= \frac{2M}{2^{l-1}} \end{aligned}$$

moreover,

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_e \supseteq I_{e+1} \supseteq \ldots$$

and each $I_l$ contains $x_k$ for infinitely many values of $k$.
We can thus choose some $x_{k_1} \in I_1, x_{k_2} \in I_2, \ldots, x_{k_l} \in I_l \quad \forall l \in \mathbb{N}$ where $1 \le k_1 < k_2 < \ldots < k_e < k_{e+1} < \ldots$. This is possible since there are infinitely many $x_k$'s in each interval.
We claim:

1. $$\bigcap_{l=1}^{\infty} I_l \neq \varnothing$$

   and in fact contains **exactly one point** $x$.

   Note that

   $$I_l = [a_l, b_l] \quad \text{for some } a_l < b_l$$

and also

$$I_l \supset I_{l+1} \Rightarrow a_1 \le a_l \le a_{l+1} < b_{l+1} \le b_l \le b_1 \quad \forall l$$

(i.e. either endpoints move inwards for each successive interval).

So $(a_l)$ is an increasing sequence bounded by $b_1$, therefore $\exists a$ such that $\lim_{l\to\infty} a_l = a$ and $a_l \le a \le b_1 \quad \forall l$.

Similarly $(b_l)$ is a decreasing sequence bounded by $a_1$, so $\exists b$ such that $\lim_{l\to\infty} b_l = b$ and $a_1 \le b \le b_l \quad \forall l$.

We have $a_l < b_l \quad \forall l$. Taking the limit we have $a \le b$ (limit can only be guaranteed with potential for equality).

$$a_1 \le a_l \le a_{l+1} \le a \le b \le b_{l+1} \le b_l \le b_1$$

Note that

$$\begin{aligned} 0 \le b - a \le b_l - a_l &= length(I_l) \\ &= \frac{2M}{2^{l-1}} \to 0 \text{ as } l \to \infty \end{aligned}$$

hence $a = b$ (call this $x$).

By construction $x = a = b \in [a_l, b_l] = I_l \quad \forall l$ so

$$x \in \bigcap_{l=1}^{\infty} I_l$$

so there exists an element. Suppose $y \in \bigcap_{l=1}^{\infty} I_i$ then $x, y \in I_l \quad \forall l$ and

$$|x - y| \le \frac{2M}{2^{l-1}} \quad \forall l \Rightarrow x = y \quad (\text{as } l \to \infty)$$

2.

$$\lim_{l\to\infty} x_{k_l} = x$$

Assume $x_{k_l} \in I_l$ and $x \in I_l \quad \forall l$ (from claim 1). So

$$|x_{k_l} - x| \le \frac{2M}{2^{l-1}} \to 0 \text{ as } l \to \infty$$

thus $\lim_{l\to\infty} x_{k_l} = x$.

The above two claims prove the theorem for $n = 1$.
Now suppose the thoerem is true for $n$, we show it is true for $n + 1$.
Let $(x_k)$ be a bounded sequence in $\mathbb{R}^{n+1}$, so $\exists M$ such that $\|x_k\| \le M \quad \forall k$.
We write $x_k = (x_k^1, x_k^2, \ldots, x_k^{n+1})$ where $x_k^j$ is the j-th component of vector $x_k \in \mathbb{R}^{n+1}$.
So

$$\|x_k\|^2 = |x_k^1|^2 + |x_k^2|^2 + \ldots + |x_k^n|^2 + |x_k^{n+1}|^2 \le M^2 \tag{5.1}$$

Define a sequence $(y_k)$ in $\mathbb{R}^n$ as the first $n$ components of $x_k$

$$y_k = (x_k^1, \ldots, x_k^n)$$

therefore $\|y_k\| \le M \quad \forall k$ by equation 5.1.
By the inductive hypothesis, $\exists$ a subsequence $(y_{k_l})$ of $(y_k)$ that converges to some point $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$.

Consider the sequence $(x_{k_l}^{n+1})$ in $\mathbb{R}^1$ **(TODO(richardwu): why can't we just use $(x_k^{n+1})$ here instead?)**. By equation 5.1, $|x_{k_l}^{n+1}| \leq M \quad \forall l$, so $(x_{k_l}^{n+1})$ is a bounded sequence in $\mathbb{R}$. By B-W for $n = 1$, $\exists$ subsequence $(x_{k_{l_j}}^{n+1})$ that converges to some $x^{n+1} \in \mathbb{R}$.
Consider the subsequence $(y_{k_{l_j}})$ of $(y_{k_l})$, which also converges to $(x^1, \ldots, x^n) \in \mathbb{R}^n$.
So $x_{k_{l_j}}^a \to x^a$ as $j \to \infty$ for $a = 1, \ldots, n$ and $a = n+1$.
Thus the sequence $x_{k_{l_j}} \to x$ as $j \to \infty$.

**Remark 5.2.** We used the IH/B-W on the first $n$ components and then the $n+1$ component to find corresponding convergent subsequences. In order to "meld" them together, we needed to take the subsequence of either subsequence (to have a 2-level subsequence) to ensure it converges for the same $k_{l_j}$'s as the other 1-level subsequence.
**TODO(richardwu): see the above TODO for why we don't just take $k_l$'s instead of $k_{l_j}$'s.**

□

# 6 Janaury 15, 2017

## 6.1 Connectedness

**Definition 6.1.** Let $E$ be a non-empty subset of $\mathbb{R}^n$.
We say $E$ is **disconnected** if there exists a **separation** for $E$. A separation of $E$ is a pair $U, V$ open sets in $\mathbb{R}^n$ such that

1. $E \cap U \neq \varnothing$
2. $E \cap V \neq \varnothing$
3. $E \cap U \cap V = \varnothing$
4. $E \subseteq U \cup V$

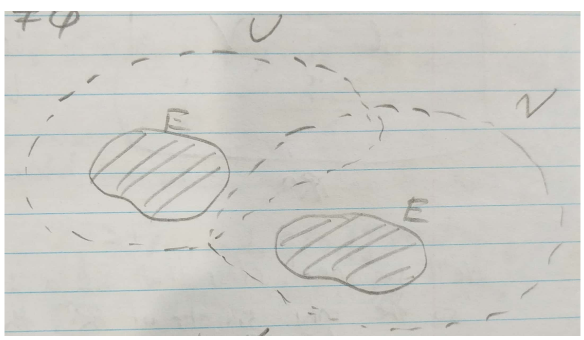


**Figure 6.1:** $E$ is disconnected since there are open sets $U, V$ that form a separation.

Note that $U \cap V$ need not be empty, but it must be disjoint from $E$.
(intuitively a set is disconnected if it is more than one piece).

**Definition 6.2.** $E$ is **connected** if $\nexists$ any separation of $E$.

**Remark 6.1.** Connectedness and disconnectedness do not apply to $\varnothing$.

## 6.2 Is $R^n$ connected?

(Yes it is).
Suppose $\exists$ a separation of $\mathbb{R}^n$ of open sets $U, V$ such that

1. $$\varnothing \neq U \cap \mathbb{R}^n = U$$
$$\varnothing \neq V \cap \mathbb{R}^n = V$$
which implies $U, V$ both non-empty. Furthermore
2. $$U \cap V \cap \mathbb{R}^n = U \cap V = \varnothing$$
which implies $U, V$ are disjoint.
3. $$\mathbb{R}^n \subseteq U \cup V \subseteq \mathbb{R}^n$$
so $\mathbb{R}^n = U \cup V$. Since $U \cap V = \varnothing$, then $U^c = V$ and $V^c = U$.

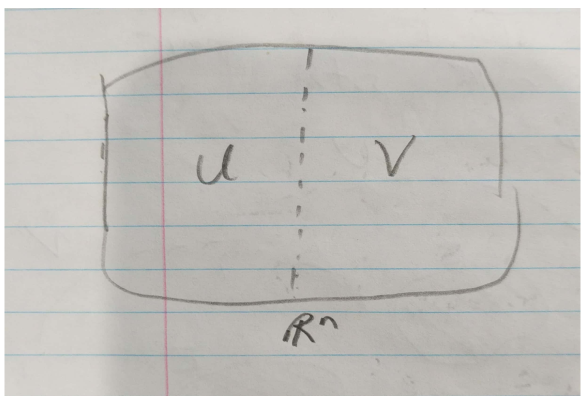


**Figure 6.2:** Sketch of what disconnected $\mathbb{R}^n$ would look like.

This would mean $U, V$ are both **non-empty** subsets that are **both open and closed** and $U, V \neq \mathbb{R}^n$ (since they are non-empty disjoint).
In other words, if $\exists U$ such that $U \neq 0, U \neq \mathbb{R}^n$ and $U$ is both open and closed, then $U, V = U^c$ gives a separation of $\mathbb{R}^n$.
We'll see (next class) that $\nexists$ a separation of $\mathbb{R}^n$ for any $n$, so the only subsets of $\mathbb{R}^n$ that are both open and closed are $\varnothing, \mathbb{R}^n$.

## 6.3 $[0, 1]$ is connected

This is an example of a connected subset in $\mathbb{R}$ and will be used next time to prove $\mathbb{R}^n$ is connected and *more*.

**Theorem 6.1.** Let $E = [0, 1] \subseteq \mathbb{R}$. Then $E$ is connected.
(Aside: in fact: *any* interval $[a, b], [a, b), (a, b], (a, b)$ in $\mathbb{R}$ is connected and these are the **only** connected subsets in $\mathbb{R}$ i.e. connectedness $\Rightarrow$ interval).

*Proof.* By contradiction.
Suppose $[0,1]$ is not connected. $\exists$ a separation $U, V$ open subsets of $[0,1]$ where

1. $U \cap [0,1] \neq \varnothing$
2. $V \cap [0,1] \neq \varnothing$
3. $U \cap V \cap [0,1] = \varnothing$
4. $[0,1] \subseteq U \cup V$

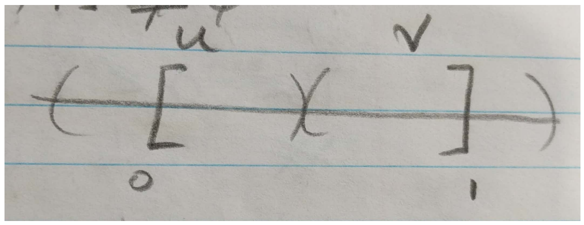


**Figure 6.3:** Sketch of $U, V$ open sets as (potential) separation for $[0,1]$.

WLOG $0 \in U$. Since $U$ is open and $0 \in U$, $\exists \epsilon_0 > 0$ such that $B_{\epsilon_0}(0) = (-\epsilon_0, \epsilon_0) \subseteq U$.
WLOG, $\epsilon_0 < 1$ so $[0, \epsilon_0) \subseteq U \cap [0,1]$.
Define $t_0$ as

$$\sup\{\epsilon \in (0,1) \mid [0,\epsilon) \subseteq U \cap [0,1]\}$$

note: the above is a non-empty subset of $\mathbb{R}$ since $\epsilon_0$ is in the set. It's bounded above by 1, so the supremum or $t_0$ must exist.
We have $0 < \epsilon_0 \leq t_0 \leq 1$ so $t_0 \in (0,1]$, thus $t_0 \in U$ or $t_0 \in V$.

**Case 1:** $t_0 \in U$ Since $U$ is open (all open sets have some open ball around every point) $\exists \delta > 0$ such that

$$(t_0 - \delta, t_0 + \delta) \subseteq U \tag{6.1}$$

WLOG $\delta < t_0$ but $0 < t_0 - \delta < t_0$ so by definition of $t_0$ (as supremum), $\exists \hat{\epsilon} > 0$ with $t_0 - \delta < \hat{\epsilon} < t_0$ such that

$$[0, \hat{\epsilon}) \subseteq U \cap [0,1] \tag{6.2}$$

Combining equation 6.1 and 6.2 (joining the two intervals together since we do not know if either separately are in $U$), we have

$$[0, t_0 + \delta) \subseteq U \cap [0,1] \tag{6.3}$$

We have two subcases:

$t_0 < 1$ Then we can shrink $\delta > 0$ further to ensure $t_0 + \delta < 1$ ($\delta < 1 - t_0$).
Then $0 < t_0 + \delta < 1$ and $[0, t_0 + \delta) \subseteq U \cap [0,1]$ which contradicts $t_0$ as the supremum.

$t_0 = 1$ This implies $U \cap [0,1] = [0,1]$ by equation 6.3 but then $V \cap [0,1] = \varnothing$ (since $U \cap V \cap [0,1] = \varnothing$), which is a contradiction since $V$ must be non-empty.

**Case 2:** $t_0 \in V$ Since $V$ is open $\exists \zeta > 0$ such that

$$(t_0 - \zeta, t_0 + \zeta) \subseteq V \tag{6.4}$$

WLOG $\zeta < t_0$ but $0 < t_0 - \zeta < t_0$ so by definition of $t_0$ (as supremum) $\exists \tilde{\epsilon} > 0$ with $t_0 - \zeta < \tilde{\epsilon} \leq t_0$ such that

$$[0, \tilde{\epsilon}) \subseteq U \cap [0, 1] \tag{6.5}$$

(it's $U$ since that was the set $t_0$ was defined with).

Pick $s \in (t_0 - \zeta, \tilde{\epsilon})$. Then $s \in U \cap [0, 1]$ by equation 6.5 but also $s \in V \cap [0, 1]$ by equation 6.4, which is a contradiction.

By the contradiction of the two cases above, $[0, 1]$ is connected. □

# 7 January 17, 2017

## 7.1 Convex sets

**Definition 7.1.** A **non-empty** subset $E$ of $\mathbb{R}^n$ is called **convex** if whenever $x, y \in E$ then

$$tx + (1 - t)y \in E \quad \forall t \in [0, 1]$$

i.e. the line segment between any 2 points in $E$ is contained in $E$.

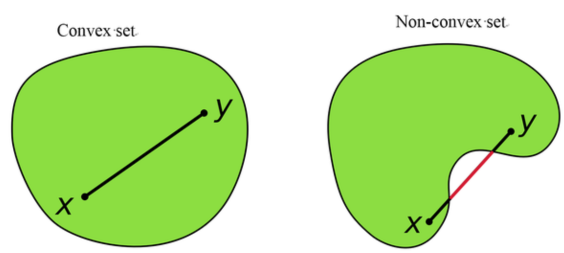


**Figure 7.1:** Convex and non-convex sets in $\mathbb{R}^2$.

## 7.2 Convex ⇒ connected

**Corollary 7.1.** Any convex subset $E$ of $\mathbb{R}^n$ is connected. This implies two corollaries:

**Corollary 7.2.** $\mathbb{R}^n$ is connected $\forall n$ since $\mathbb{R}^n$ is trivially convex.

**Corollary 7.3.** The only subsets of $\mathbb{R}^n$ that are both open and closed are $\varnothing, \mathbb{R}^n$ (see the remark about $\mathbb{R}^n$ connectedness from above).

*Proof.* Let $E$ be convex and suppose $E$ is *not* connected. $\exists$ open subsets $U, V$ such that

1. $U \cap E \neq \varnothing$
2. $V \cap E \neq \varnothing$
3. $U \cap V \cap E = \varnothing$
4. $E \subseteq U \cup V$

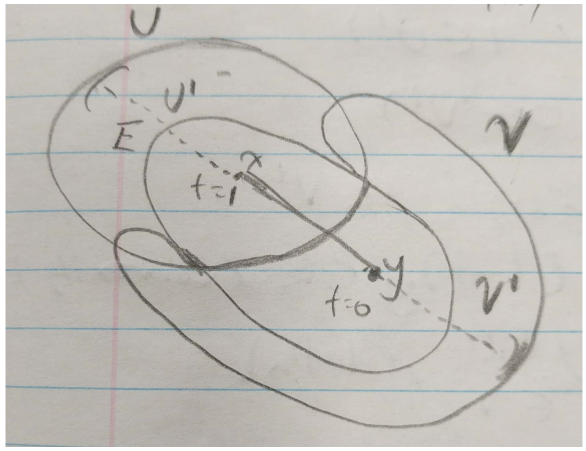


**Figure 7.2:** Suppose convex $E$ is not connected and there exists a separation $U, V$.

Let $x \in U \cap E$ and $y \in V \cap E$ (therefore $x \neq y$ since $U \cap V \cap E = \varnothing$). Since $E$ is convex,

$$tx + (1-t)y \in E \quad \forall t \in [0,1]$$

Define $U', V'$ subsets of $\mathbb{R}^n$ by

$$U' = \{t \in \mathbb{R} : tx + (1-t)y \in U\} \qquad V' = \{t \in \mathbb{R} : tx + (1-t)y \in V\}$$

(note: $U', V'$ is not restricted to elements $[0,1]$: $t$ could extend arbitrarily into $E^c$).
**Claim:** $U', V'$ are open subsets of $\mathbb{R}$. Let $t_0 \in U'$ so $x_0 = t_0 + (1-t_0)y \in U$. Since $U$ is open in $\mathbb{R}^n$ $\exists \epsilon_0 > 0$ such that $B_{\epsilon_0}(x_0) \in U$. We pick $t \in \mathbb{R}$ such that

$$|t - t_0| < \frac{\epsilon_0}{\|x\| + \|y\|}$$

then

$$\begin{aligned} B_{\epsilon_0}(x_0) \Rightarrow \|(tx + (1-t)y) - x_0\| &= \|tx + (1-t)y - t_0x - (1-t_0)y\| \\ &= \|(t-t_0)x + (t_0 - t)y\| \\ &\overset{\triangle}{\leq} |t - t_0|(\|x\| + \|y\|) \\ &< \epsilon_0 \end{aligned}$$

But $B_{\epsilon_0}(x_0) \subseteq U$ so if $|t - t_0| < \frac{\epsilon_0}{\|x\|+\|y\|}$ then $t \in U'$ (we want our choice of $t$ to imply $t \in U'$).
So $\frac{B_{\epsilon_0}(t_0)}{\|x\|+\|y\|} \subseteq U'$ so $U'$ is open.
Similarly, $V'$ is open.
Thus here are the properties of $U', V'$. They are both open in $\mathbb{R}$ and

1. $U' \cap [0,1] \neq \varnothing$ (since $1 \in U'$)

2. $V' \cap [0,1] \neq \varnothing$ (since $0 \in V'$)

3. $U' \cap V' \cap [0,1] = \varnothing$

   Given some $t \in [0,1]$ (since $tx \in (1-t)y \in E$ from convexity), note that either $t \in U'$ from $tx + (1-t)y \in U$ or $t \in V'$ from $tx + (1-t)y \in V$ (we know from before that $U \cap V \cap E = \varnothing$ thus this must hold for the subsets

$U', V'$).

4. $[0,1] \subseteq U' \cup V'$

   If $t \in [0,1]$, then $z = tx + (1-t)y \in E$ so $z \in U \cup V$ from before, so $z \in U$ or $z \in V$, thus by their definitions $t \in U'$ or $t \in V'$.

Then $U', V'$ is a separation for $[0,1]$, which is a contradiction. Thus $E$ is connected. □

**Remark 7.1.** In general, to prove a set $E$ is connected it is generally easier to assume it is *not* connected and there exists a separation, then derive a contradiction.

### 7.3 Open cover and compactedness

**Definition 7.2.** Let $E$ be a subset of $\mathbb{R}^n$. An **open cover** of $E$ is a collection of open subsets $U_\alpha$ $\alpha \in A$ of $\mathbb{R}^n$ such that

$$E \subseteq \bigcup_{\alpha \in A} U_\alpha$$

(finite or infinite union of open subsets).

**Definition 7.3.** The subset $E$ is called **compact** *iff* every open cover of $E$ admits a **finite subcover**.
That is: if $\bigcup U_\alpha$ $\alpha \in A$ is an open cover of $E$, then $\exists$ a finite subset $A_0$ of $A$ such that

$$E \subseteq \bigcup_{\alpha \in A_0} U_\alpha$$

Informally, whenever a compact $E$ is covered by a collection of open sets, it is actually covered by just finitely many of those open sets.

**Remark 7.2.** This definition is not very useful for checking if a subset is compact (because you would have to check every open cover of $E$).

### 7.4 Bounded sets

**Definition 7.4.** A subset $E \subseteq \mathbb{R}^n$ is called **bounded** if $\exists M > 0$ such that $E \subseteq \overline{B_M(0)}$. That is $\|x\| \leq M$ $\forall x \in E$.

## 8 January 19, 2018

### 8.1 Heine-Borel theorem

**Theorem 8.1.** Let $E$ be a subset of $\mathbb{R}^n$. $E$ is **compact** *iff* $E$ is both **closed and bounded**.
The following proof uses the *density of rationals*.

*Proof.* **Step 1:** Suppose $E$ is *compact*. We want to show that $E$ is *bounded*.
Let $U_k = B_k(0) = \{x \in \mathbb{R}^n \mid \|x\| < k\}$. $U_k$ is open $\forall k$, thus $U_k \subseteq U_{k+1}$ $\forall k \in \mathbb{N}$. Therefore

$$E \subseteq \bigcup_{k=1}^{\infty} U_k = \mathbb{R}^n$$

$\{U_k, k \in \mathbb{N}\}$ is an open cover of $E$. Since $E$ is compact, $\exists$ a finite subcover so $\exists\ k_1 < k_2 < \ldots < k_N \in \mathbb{N}$ such that

$$E \subseteq \bigcup_{j=1}^{N} U_{k_j} = U_{k_N} = B_{k_N}(0)$$

since they're nested so $E$ is bounded.

**Corollary 8.1.** $\mathbb{R}^n$ is not compact because it is not bounded.

**Step 2:** Let $E$ be *compact*, we show it is *closed*.
To do this: we show $E^c$ is open (aside if $E^c = \varnothing$ then we are done. This never happens since $E$ is not $\mathbb{R}^n$).
Let $x \in E^c$. We need to find an open ball centred at $x$ lying completely in $E^c$. Note $E \subseteq \mathbb{R}^n \setminus \{x\}$ since $x \notin E$.
Let (different $U_k$ from before)

$$U_k = (\overline{B_{\frac{1}{k}}(x)})^c = \{x \in \mathbb{R}^n \mid \|x - y\| > \frac{1}{k}\}$$

which is open (complement of closed ball). We can use this as covers.

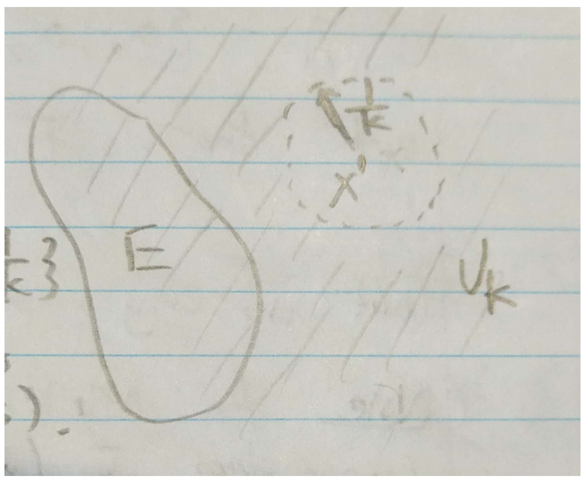


**Figure 8.1:** $U_k$ is the complement of the closed ball centred at $x \in E^c$ with radius $\frac{1}{k}$ for some $k \in \mathbb{N}$.

If $l > k$, then $\frac{1}{l} < \frac{1}{k}$. Thus if $y \in U_k$, then $\|y - x\| > \frac{1}{k} > \frac{1}{l}$ so $y \in U_l$. That is

$$U_k \subseteq U_l \quad k < l \tag{8.1}$$

Note that we have

$$E \subseteq \mathbb{R}^n \setminus \{x\} = \bigcup_{k=1}^{\infty} U_k$$

where the infinite union of $U_k$ is an open cover of $E$. Since $E$ is compact, we have a finite subcover $U_{k_1}, \ldots, U_{k_N}$ such that

$$\begin{aligned} E &\subseteq \bigcup_{j=1}^{N} U_{k_j} \\ &= U_{k_N} && \text{equation 8.1} \\ &= (\overline{B_{\frac{1}{k_N}}(0)})^c \end{aligned}$$

Take complements (from $A \subseteq B \Rightarrow B^c \subseteq A^c$)

$$x \in B_{\frac{1}{k_N}}(x) \subseteq \overline{B_{\frac{1}{k_N}}(x)} \subseteq E^c$$

So $\exists$ an open ball for $x$ thus $E^c$ is open and $E$ is closed.
Before we prove the converse:

**Lemma 8.1.** Let $E$ be any subset of $\mathbb{R}^n$. Let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of $E$ (so $U_\alpha$ open $\forall \alpha \in A$). That is

$$E \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Thus $\exists$ a countable subset $\tilde{A}$ of $A$

$$\tilde{A} = \{\alpha_1, \alpha_2, \ldots\} = \{\alpha_k \mid k \in \mathbb{N}\}$$

such that $E \subseteq \bigcup_{k=1}^{\infty} U_{\alpha_k}$.
That is: **every open cover admits a countable subcover**. (Note: an infinite set is countable **iff** $\exists$ bijective correspondence with $\mathbb{N}$. Rational numbers are countable whereas $\mathbb{R}$ is not).

*Proof.* Assume $E \subseteq \bigcup_{\alpha \in A} U_\alpha$.
Let $x \in E$. Then $\exists \alpha(x) \in A$ such that $x \in U_{\alpha(x)}$.
Since $U_{\alpha(x)}$ is open $\exists \epsilon(x) > 0$ such that

$$B_{\epsilon(x)}(x) \subseteq U_{\alpha(x)} \tag{8.2}$$

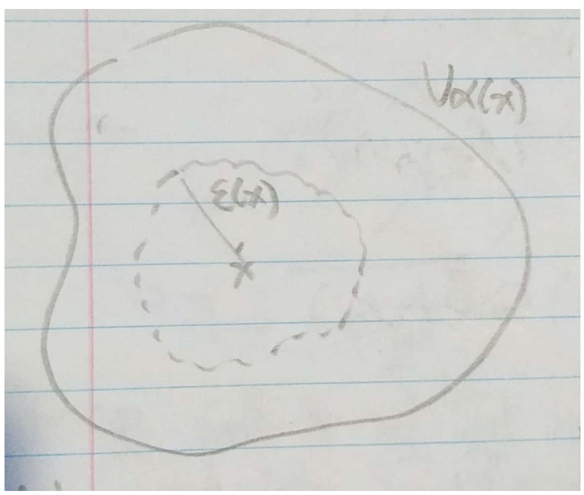


**Figure 8.2:** We can construct an open ball $B_{\epsilon(x)}(x)$ within some $U_{\alpha(x)}$ for every $x \in E$.

Then $E \subseteq \bigcup_{x \in E} B_{\epsilon(x)}(x)$ (all $x$). Since the rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$, $\exists q(x) \in \mathbb{Q}^n$ and $\zeta(x) \in \mathbb{Q}$ such that

$$\|x - q(x)\| < \frac{\epsilon(x)}{4}$$
$$\frac{\epsilon(x)}{4} < \zeta(x) < \frac{\epsilon(x)}{2}$$

Therefore

$$\|x - q(x)\| < \frac{\epsilon(x)}{4} < \zeta(x)$$

So $x \in B_{\zeta(x)}(q(x))$.

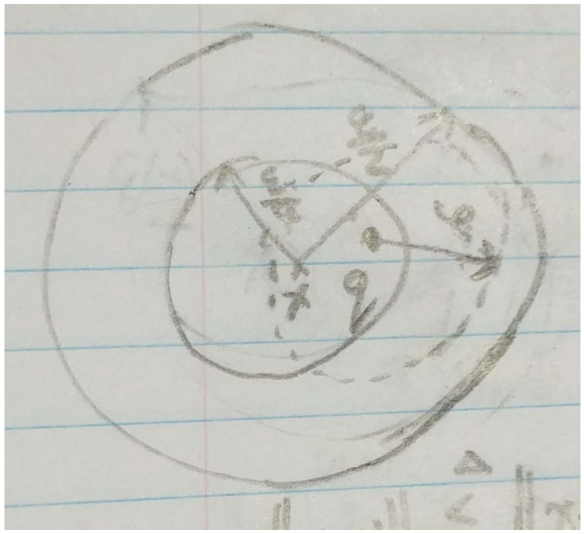

**Figure 8.3:** We find some open ball centred at $q(x) \in \mathbb{Q}^n$ with radius $\zeta(x) \in \mathbb{Q}$ that contains $x$.

Suppose $y \in B_{\zeta(x)}(q(x))$, then we have

$$\begin{aligned} \|x - y\| &\overset{\triangle}{\leq} \|x - q(x)\| + \|q(x) - y\| \\ &< \frac{\epsilon(x)}{4} + \zeta(x) \\ &< \frac{\epsilon(x)}{4} + \frac{\epsilon(x)}{2} \\ &< \epsilon(x) \end{aligned}$$

So $y \in B_{\epsilon(x)}(x)$ therefore $B_{\zeta(x)}(q(x)) \subseteq B_{\epsilon(x)}(x)$.
We have shown for every $x \in E$, $\exists q(x) \in \mathbb{Q}^n$ and $\zeta(x) \in \mathbb{Q}$ such that

$$x \in B_{\zeta(x)}(q(x)) \subseteq B_{\epsilon(x)}(x)$$

So $E \subseteq_{x \in E} B_{\zeta(x)}(q(x))$ but $\mathbb{Q}$ and $\mathbb{Q}^n$ are countable so $\exists q_j \in \mathbb{Q}^n$ and $\zeta_j \in \mathbb{Q}$ where $j \in \mathbb{N}$ such that

$$\begin{aligned} E &\subseteq \bigcup_{j=1}^{\infty} B_{\zeta_j}(q_j) \\ &\subseteq \bigcup_{j=1}^{\infty} B_{\epsilon(x_j)}(x_j) \\ &\subseteq \bigcup_{j=1}^{\infty} U_\alpha(x_j) && \text{by equation 8.2} \end{aligned}$$

so we have a countable subcover. □

Back to proving the converse: **Step 3:** Let $E$ be *closed and bounded.* We want to show $E$ is *compact.*

Let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of $E$. We showed in the above lemma that $\exists q_j \in \mathbb{Q}^n$, $\zeta_j \in \mathbb{Q}$ where $j \in \mathbb{N}$ such that

$$E \subseteq \bigcup_{j=1}^{\infty} B_{\zeta_j}(q_j)$$

**Claim:** $\exists N \in \mathbb{N}$ such that $E =\subseteq \bigcup_{j=1}^{N} B_{\zeta_j}(q_j)$. (i.e. we need only need finitely many of these balls).
If the claim is true, then

$$\begin{aligned} E &\subseteq \bigcup_{j=1}^{N} B_{\zeta_j}(q_j) \\ &\subseteq \bigcup_{j=1}^{N} B_{\epsilon(x_j)}(x_j) \\ &\subseteq \bigcup_{j=1}^{N} U_\alpha(x_j) \end{aligned}$$

so we would have a finite subcover.
It remains to prove the claim. Suppose the claim is false (proof by contradiction). Then for every $k \in \mathbb{N}$, we have $E \setminus \bigcup_{j=1}^{k} B_{\zeta_j}(q_j) \neq \varnothing$.
We choose $x_k \in E \setminus \bigcup_{j=1}^{k} B_{\zeta_j}(q_j)$. Note that $(x_k)$ is a sequence in $E$ and $E$ is bounded, so $(x_k)$ is a bounded sequence.
By Bolzano-Weierstrass, there exists a subsequence $(x_{k_l})$ that converges.
$(x_{k_l}) \in E\ \forall l \in \mathbb{N}$ and $E$ is closed so $x = \lim_{l\to\infty} x_{k_l} \in E$ as well.
$x \in E$, so $\exists$ some $J \in \mathbb{N}$ such that

$$x \in B_{\zeta_J}(q_J) \tag{8.3}$$

from our lemma.
But since $\lim_{l\to\infty} x_{k_l} = x$ then $\exists N \in \mathbb{N}$ such that $\forall l \geq N$

$$x_{k_l} \in B_{\zeta_j}(q_j)$$

(definition of convergent sequence).
But by construction of our sequence

$$x_{k_l} \notin \bigcup_{j=1}^{N} B_{\zeta_j}(q_j)$$

for any $k_l \geq N$.
So if $k_l > J$, then

$$x_{k_l} \notin B_{\zeta_J}(q_J) \tag{8.4}$$

for $l \geq \max(N, J) \Rightarrow k_l \geq \max(N, J)$.
From equation 8.3 and equation 8.4, we have a contradiction.
(The idea of this proof revolves around showing that all $x \in E$ must be in some open ball with rational parameters. By assuming the contrary of the claim that there is a finite subcover, we choose some sequence outside of all finite subcovers (made of the rational parameters) and show that it is not in an open ball with rational parameters. Thus we have a contradiction so there must be some finite subcover with the open balls of rational parameters). $\square$

# 9 January 22, 2018

## 9.1 Limits of functions

Let $V \subseteq \mathbb{R}^n$ be an *open set* with $x_0 \in V$. Let $f : V \setminus \{x_0\} \to \mathbb{R}^m$ for some $m$ (i.e. $f$ is defined at all points of $V$ except *possibly* at $x_0$).

**Definition 9.1.** We say $\lim_{x \to x_0} f(x)$ exists and equals $L \in \mathbb{R}^m$ **iff** $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$0 < \|x - x_0\| < \delta \Rightarrow \|f(x) - L\| < \epsilon$$

(note that $B_\delta(x_0) \subseteq V$ must hold). In general, $\delta$ depends on both $\epsilon$ and $x_0$ (and on $f$ as well if suppose it has a different domain).

**Remark 9.1.** When $n > 1$, things get more complicated since in $n = 1$, there exists only 2 ways to approach $x_0$: from left or right (i.e. $\lim_{x \to \infty} f(x)$ exists *iff* both left and right limits exist and are equal in $n = 1$).
In $n > 1$, $\exists$ *infinitely many ways* to approach $x_0$. This is what makes establishment of the existence of limits harder for $n > 1$.

**Example 9.1.** Example where different linear paths result in a different limit:
Let $n = 2, m = 1$ ($f : \mathbb{R}^2 \to \mathbb{R}$) where we denote $(x, y) \in \mathbb{R}^2$.
Suppose we wish to find

$$\lim_{(x,y)\to(2,3)} \frac{(x-2)^2}{(x-2)^2 + (y-3)^2}$$

where $f(x, y)$ defined everywhere except $(2, 3)$.

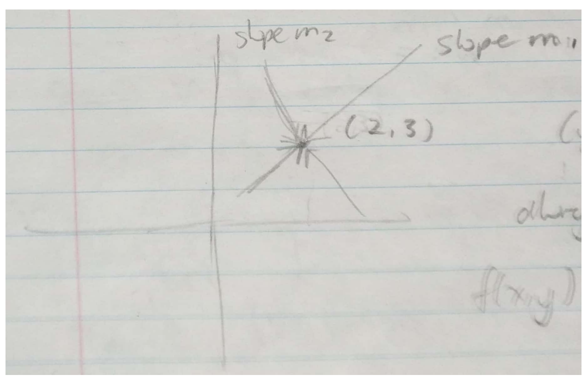


**Figure 9.1:** There exists many paths to approach $x_0 = (2, 3)$ in $f$, $m_1 \neq m_2$.

Suppose we have paths/lines with slope $m$ where $(y - 3) = m(x - 2)$. Along this line we have

$$\begin{aligned} f(x, y) &= \frac{(x-2)^2}{(x-2)^2 + (y-3)^2} \\ &= \frac{1}{1 + m^2} \end{aligned}$$

So $f$ is a constant function which depends on the slope of the line/path (it depends on $m$). Since we found at least 2 paths towards $(2, 3)$ along which $f$ approaches different limiting values, then the limit **DNE**.

**Example 9.2.** Example where linear paths converge but quadratic paths do not:

We wish to find

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

where the domain is $\mathbb{R}^2 \setminus \{0,0\}$.

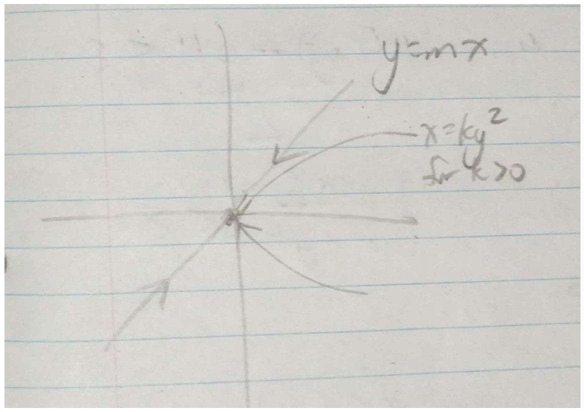


**Figure 9.2:** There are linear and non-linear paths that approaches $x_0 = (0,0)$.

Note that unlike previously, linear paths $y = mx$ do converge

$$\begin{aligned} f(x,y) &= \frac{x(mx)^2}{x^2+(mx)^4} \\ &= \frac{m^2x^3}{x^2+m^4x^4} \\ &= \frac{m^2x}{1+m^4x^2} \end{aligned}$$

So as $x \to 0$, then $\frac{m^2x}{1+m^4x^2} \to 0$ for any $m$.
Along $x = 0$ we still have

$$f(x,y) = \frac{0 \cdot y^2}{0^2+y^4} = 0 \quad \forall y \neq 0$$

(this is important since a vertical line is not explicit). So $f$ approaches 0 as $(x,y) \to (0,0)$ for linear paths.
We must consider other non-linear paths as well e.g. along $x = ky^2$ we have

$$f(x,y) = \frac{(ky^2)y^2}{(ky^2)^2+y^4} = \frac{k}{k^2+1}$$

which is a constant that depends on $k$, thus the limit **DNE**.

**Example 9.3.** Example where the limit does exist in $n > 1$ space:
We wish to find

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2+y^2}$$

We *expect the limit to converge* since the degree of the numerator is > degree of denominator, thus numerator $\to 0$ "much faster" than the denominator so the quotient should go to zero.
Let $\epsilon > 0$. We want to find $\delta > 0$ (depends on $\epsilon$) such that if

$$\|(x,y)-(0,0)\| < \delta \Rightarrow \|f(x,y)-0\| < \epsilon$$

where $L = 0$.
Rewriting the above: we have if $x^2 + y^2 < \delta^2$, then

$$|\frac{x^4}{x^2+y^2}| < \epsilon$$

Observe that $x^2 \leq x^2 + y^2$ so

$$\frac{x^2}{x^2+y^2} \leq 1 \quad (x,y) \neq (0,0)$$

Furthermore note

$$\begin{aligned}
|\frac{x^4}{x^2+y^2}| = \frac{x^4}{x^2+y^2} &= x^2\left(\frac{x^2}{x^2+y^2}\right) \\
&\leq x^2 && \frac{x^2}{x^2+y^2} \leq 1 \\
&\leq x^2 + y^2 \\
&< \delta^2 = \epsilon
\end{aligned}$$

Thus we can take $\delta = \sqrt{\epsilon}$ such that

$$x^2 + y^2 < \delta^2 = \epsilon \Rightarrow |f(x,y) - 0| < \epsilon$$

## 9.2 Uniqueness of limits

**Remark 9.2.** A given limit may not exist, but if it does it's **unique** (same proof as uniqueness of limits of sequences).

## 9.3 Sequential characterization of limits of functions

**Proposition 9.1.** For $f : V \setminus \{x_0\} \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $\lim_{x\to x_0} f(x) = L$ **iff** the sequence $f(x_k)$ converges to $L$ for every sequence $(x_k)$ in $V \setminus \{x_0\}$ converging to $x_0$.
i.e. this states the path heuristic from before works formally with sequences too.

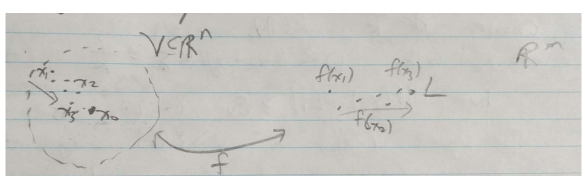


**Figure 9.3:** Any given sequence $(x_k)$ in $V$ that converges to $x_0$ must have $f(x_k)$ converge to $l$ in $\mathbb{R}^m$.

*Proof.* **Forwards:** Suppose $\lim_{x\to x_0} f(x) = L$.
Let $(x_k)$ be a sequence in $\mathbb{R}^n$ with $x_k \in V \setminus \{x_0\}$ $\forall k$ and $\lim_{k\to\infty} x_k = x_0$.
We need to show $\lim_{k\to\infty} f(x_k) = L$.
Let $\epsilon > 0$. From our premise $\exists \delta > 0$ such that

$$\|x - x_0\| < \delta \Rightarrow \|f(x) - L\| < \epsilon$$

Since $x_k \to x_0$ as $k \to \infty$, $\exists N \in \mathbb{N}$ such that

$$k \geq N \Rightarrow 0 < \|x_k - x_0\| < \delta$$

(definition of convergence).
So $k \geq N \Rightarrow \|f(x_k) - L\| < \epsilon$ so the forwards direction holds.
**Backwards:** Conversely, suppose the sequence $f(x_k)$ converges to $L$ for every $(x_k)$ in $V \setminus \{x_0\}$ converging to $x_0$. We want to show $\lim_{x\to x_0} f(x_0) = L$.
Suppose the limit does not converge to $L$ (contradiction). Negation of the statement is: $\exists \epsilon_0 > 0$ such that $\forall \delta > 0$, $\exists x_\delta$ such that

$$0 < \|x_\delta - x_0\| < \delta \text{ but } \|f(x_\delta) - L\| \geq \epsilon_0$$

(this is the negation of 1) $\forall \epsilon > 0$, 2) $\exists \delta > 0$, and 3) $\forall x$ $\|x - x_0\| < \delta \Rightarrow \|f(x) - L\| < \epsilon$).
Choose $\delta = \frac{1}{k}$, $k \in \mathbb{N}$. $\exists x_k$ with

$$0 < \|x_k - x_0\| < \delta = \frac{1}{k} \tag{9.1}$$

but $\|f(x_k) - L\| \geq \epsilon_0$ (4).
The sequence $(x_k)$ in $V \setminus \{x_0\}$ converges to $x_0$ by the premise but $f(x_k) \not\to L$ by equation 9.1. This contradicts the premise. $\square$

### 9.4 Properties of limits of functions

Let $f, g : V \setminus \{x_0\} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and suppose

$$\lim_{x\to x_0} f(x) = L \qquad \lim_{x\to x_0} g(x) = M$$

Then (the above limits **must exist**)

$$\begin{aligned}
&\lim_{x\to x_0} (f(x) + g(x)) = L + M && \text{(additive)}\\
&\lim_{x\to x_0} cf(x) = cL && \text{(scalar multiplicative)}\\
&\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M} && \text{if } m = 1, M \neq 0\\
&\lim_{x\to x_0} (f(x)g(x)) = LM && \text{if } m = 1 \text{ (same proof as for n = 1)}
\end{aligned}$$

Proofs are left as exercises.

## 10 January 24, 2018

### 10.1 Component functions

**Definition 10.1.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $U$ is open. Then for $x \in U$

$$f(x) = (f_1(x), \ldots, f_m(x)) \in \mathbb{R}^m$$

$f_i : U \to \mathbb{R}$, $1 \leq i \leq m$ are the **component functions** of $f$ (real-valued).

**Lemma 10.1.** $x_0 \in V$ open in $\mathbb{R}^n$. Let $f : V \setminus \{x_0\} \to \mathbb{R}^m$. Then $\lim_{x\to x_0} f(x) = L = (L_1, \ldots, L_m)$ **iff** $\lim_{x\to x_0} f_i(x) = L_i$ $\forall i = 1, 2, \ldots, m$.

*Proof.* By property of convergence of limits of sequences in assignment 3 and sequence characterization of limits of functions. That is

$$\lim_{x \to x_0} f(x) = L \overset{seq.char.}{\iff} \lim_{k \to \infty} (x_k) = L \overset{a3\#2}{\iff} \lim_{k \to \infty} f_i(x_k) = L_i \overset{seq.char.}{\iff} \lim_{x \to x_0} f_i(x) = L_i$$

We can also prove this using $\epsilon - \delta$. $\square$

## 10.2 Squeeze theorem

**Theorem 10.1.** Suppose $f, g, h : V \setminus \{x_0\} \to \mathbb{R}$ $(m = 1!)$. If $f(x) \leq g(x) \leq h(x)$ $\forall x \in V \setminus \{x_0\}$ (this only really needs to hold in the n'h'd of $x_0$) and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L \in \mathbb{R}$, then

$$\lim_{x \to x_0} g(x) = L$$

*Proof.* Same as proof in $n = 1$ case. $\square$

## 10.3 Norm properties of limits

**Proposition 10.1.** Suppose $f : V \setminus \{x_0\} \to \mathbb{R}^m$ and $\lim_{x \to x_0} f(x) = L$ then

$$\lim_{x \to x_0} \|f(x)\| = \|\lim_{x \to x_0} f(x)\| = \|L\|$$

*Proof.* Let $\epsilon > 0$, $\exists \delta > 0$ such that

$$0 < \|x - x_0\| < \delta \Rightarrow \|f(x) - L\| < \epsilon$$

Note that

$$\|f(x)\| \overset{\triangle}{\leq} \|f(x) - L\| + \|L\|$$
$$\|L\| \overset{\triangle}{\leq} \|L - f(x)\| + \|f(x)\|$$

So rearranging each of the inequalities above and using the premise we see that

$$|\|f(x)\| - \|L\|| < \epsilon$$

if $0 < \|x - x_0\| < \delta$ so $\lim_{x \to x_0} \|f(x)\| = \|L\|$. $\square$

## 10.4 Continuity

**Definition 10.2.** Let $U \subseteq \mathbb{R}^n$ be *open* and $f : U \to \mathbb{R}^m$.
Let $x_0 \in U$. We say $f$ is **continuous at** $x_0$ if

1. $\lim_{x \to x_0} f(x)$ exists
2. The limit equals $f(x_0)$

Explicitly, $f$ is **cts (continuous)** at $x_0$ **iff** $\forall \epsilon > 0$ $\exists \delta > 0$ such that

$$\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon$$

Equivalently, by the sequential characterization of limits, $f$ is cts at $x_0$ iff whenever $(x_k)$ is a sequence in $U$ converging to $x_0$, then $f(x_k)$ is a sequence in $\mathbb{R}^m$ converging to $f(x_0)$.

## 10.5 Continuity on a set

**Definition 10.3.** $f$ is **continuous on** $U$ (an open set) if it is continuous at every $x \in U$.

**Example 10.1.** $n = m$ and $U = \mathbb{R}^n$ and $f(x) = x$ (identity map). Then $\forall \epsilon > 0$, let $\delta = \epsilon > 0$ such that

$$\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| = \|x - x_0\| < \delta = \epsilon$$

**Example 10.2.** Let $c \in \mathbb{R}^m$ be fixed, $U = \mathbb{R}^n$. Then $f(x) = c$ is a constant function and is cts on $\mathbb{R}^n$ since $\forall \epsilon > 0$, take *any* $\delta > 0$ we have

$$\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| = \|c - c\| = 0 < \epsilon$$

**Example 10.3.** Let $m = 1$, $U = \mathbb{R}^n$, and $f(x, y) = xy$ for $x = (x_1, x_2) = (x, y) \in \mathbb{R}^2$.
We claim $f(x)$ is cts on $\mathbb{R}^n$.
Before we prove this example, for the *component functions*:

**Remark 10.1.** If $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $f$ is cts at $x_0 \in U$ **iff** $f_i : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is cts at $x_0$ for all $i = 1, \ldots, n$.

*Proof.* Let $h(x, y) = (x, y)$ (identity map, cts. on $\mathbb{R}^2$ by example 10.1).
So $h_1(x, y) = x$ and $h_2(x, y) = y$ are cts on $\mathbb{R}^2$.
$f(x, y) = xy = h_1(x, y)h_2(x, y)$ is cts on $\mathbb{R}^2$ because limits of product equals products of limits. □

## 10.6 Composition of continuous functions is continuous

**Proposition 10.2.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be cts on $U$. Let $g : V \subseteq \mathbb{R}^m \to \mathbb{R}^p$ be cts on $V$. Suppose $f(U) = \{f(x) \mid x \in U\} \subseteq V$ so the composition

$$h = g \circ f : U \subseteq \mathbb{R}^n \to \mathbb{R}^p$$

is defined $g(f(x))$. Then $h = g \circ f$ is cts on $U$.

*Proof.* Assignment 4. □

More generally, if $f$ is cts at $x_0 \in U$, $f(x_0) \in V$ and $g$ is cts at $f(x_0)$ then $h = g \circ f$ is cts at $x_0$.

## 10.7 Dot product of continuous functions is continuous

Suppose $f, g : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Define $f \cdot g : U \subseteq \mathbb{R}^n \to \mathbb{R}$ by

$$(f \cdot g)(x) = f(x) \cdot g(x) = f_1(x)g_1(x) + f_2(x)g_2(x) + \ldots + f_m(x)g_m(x)$$

If $f, g$ cts at $x_0$, then $f \cdot g$ is cts at $x_0$ (by addition and product of cts functions on $\mathbb{R}$).

## 10.8 Inverse image

**Definition 10.4.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $U$ is open. Let $A \subseteq \mathbb{R}^m$.
The **inverse image** of $A$ under $f$ is denoted $f^{-1}(A)$ and is defined to be

$$f^{-1}(A) = \{x \in U \mid f(x) \in A\}$$

(i.e. the domain of $f$ which maps to the image $A$).

**Remark 10.2.** The notation is a bit ambiguous. Suppose we *restrict* $f$ to be a proper subset $V \subset U$ that is still open.
Call this $f_{|V} : V \subseteq \mathbb{R}^n \to \mathbb{R}^m$. So $f_{|V}(x) = f(x)\ \forall x \in V$ then if $A \subseteq \mathbb{R}^m$

$$f_{|V}^{-1}(A) = \{x \in V \mid f(x) \in A\} = f^{-1}(A) \cap V$$

**Remark 10.3. Note:** $f^{-1}(A)$ could be **empty**.

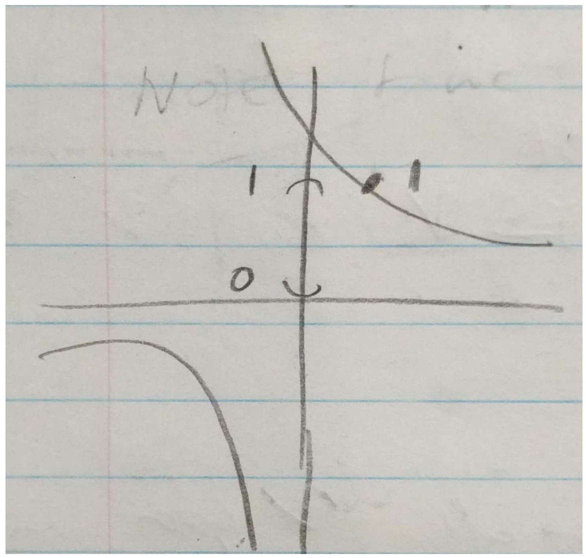

**Figure 10.1:** Inverse images of $f(x) = \frac{1}{x}$ for $A = (0, 1)$ correspond to $f^{-1}(A) = (1, \infty)$. It may be empty however (e.g. for $A = \{0\}$).

Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Note that

$$f^{-1}((0, 1)) = (1, \infty)$$
$$f^{-1}(\{0\}) = \varnothing$$

# 11 January 26, 2018

## 11.1 Continuity and open/closed sets

**Proposition 11.1.** $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $U$ is open. Then $f$ is continuous on $U$ **iff** $f^{-1}(V)$ is open in $\mathbb{R}^n$ whenever $V$ is open in $\mathbb{R}^m$.
(that is: cts iff the inverse image of any open set is open).

*Proof.* **Forwards:** Suppose $f$ is cts on $U$. Let $V \subseteq \mathbb{R}^m$ be open. WLOG $f^{-1}(V) \neq \varnothing$. Let $x_0 \in f^{-1}(V) \subseteq U \Rightarrow f(x_0) \in V$.
Since $V$ is open, $\exists \epsilon > 0$ such that $B_\epsilon(f(x_0)) \subseteq V$. But $f$ is cts at $x_0$ so $\exists \delta > 0$ such that

$$\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0\| < \epsilon$$

(we take $\delta$ small enough such that $B_\delta(x_0) \subseteq U$).
Thus $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0)) \subseteq V$. So $B_\delta(x_0) \subseteq f^{-1}(V)$ hence $\forall x_0 \in f^{-1}(V)\ \exists \delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(V)$ so $f^{-1}(V)$ is open.

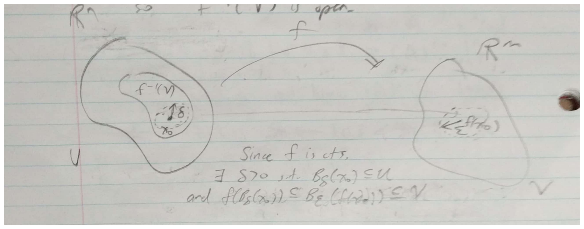


**Figure 11.1:** $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is continuous **iff** the inverse image of any open set $V \in \mathbb{R}^m$ is open.

**Backwards:** Suppose $f^{-1}(V)$ is open in $\mathbb{R}^n$ for all $V$ open in $\mathbb{R}^m$. We need to show that $f$ is cts on $U$.
Let $x_0 \in U$. Let $\epsilon > 0$, $B_\epsilon(f(x_0))$ is open in $\mathbb{R}^m$ so by our assumption

$$f^{-1}(V) = f^{-1}(B_\epsilon(f(x_0)))$$

is open in $\mathbb{R}^n$.
Also $x_0 \in f^{-1}(V)$ since $f(x_0) \in V = B_\epsilon(f(x_0))$ so $\exists \delta > 0$ such that

$$B_\delta(x_0) \subseteq f^{-1}(V)$$

(since $f^{-1}(V)$ is open).
Hence $f(B_\delta(x_0)) \subseteq V = B_\epsilon(f(x_0))$ so $f$ is cts at $x_0$. $\square$

**Remark 11.1.** One can also show that $f^{-1}$(*closed*) = *closed* is also equivalent to continuity (on assignment 4).

**Remark 11.2. Question:** Is the reverse the open set property true? That is, suppose $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ $U$ open, $f$ cts on $U$. Let $V \subseteq U$ defined $f(V) = \{f(x) \mid x \in V\}$ the image of $V$ under $f$. If $V$ is open in $\mathbb{R}^n$, is $f(V)$ necessarily open in $\mathbb{R}^m$?
**No:** here's a counter-example.

**Example 11.1.** $n = m = 1$, $U = \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ (cts on $\mathbb{R}$).
Take $V = (-1, 1)$ open in $\mathbb{R}$. Then $f(V) = [0, 1)$ which is not open in $\mathbb{R}$.

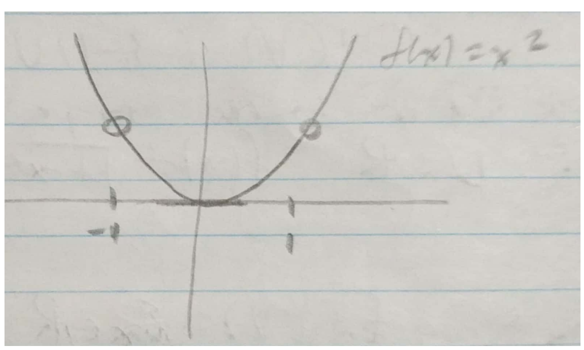


**Figure 11.2:** An open domain on a cts $f(x) = x^2$ may not admit an open image.

Similarly, if $V$ is closed in $\mathbb{R}^n$, $f(V)$ *need not* be closed in $\mathbb{R}^m$.

**Example 11.2.** $n = m = 1$, $U = \mathbb{R} \setminus \{0\}$, and $f(x) = \frac{1}{x}$.
Let $V = [1, \infty)$ which is closed on $\mathbb{R}$ (although this is unbounded there is a closed boundary so this is still closed).
Then $f(V) = (0, 1]$ is not closed.

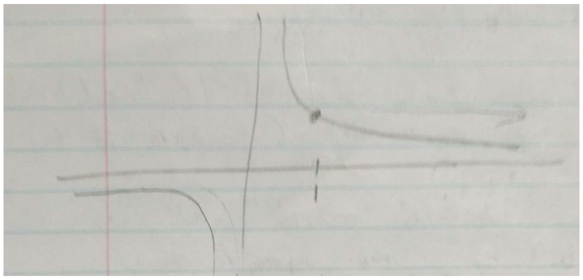

Figure 11.3: A closed domain on a cts $f(x) = \frac{1}{x}$ may not admit a closed image.

Two other types of subsets were **compact and connected**.

## 11.2 Continuity and compact sets

Suppose $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ cts on $U$, $U$ open.
Does the same property hold for compact/connected sets as it did for open/closed sets?
That is, if $V \subseteq \mathbb{R}^m$ is **compact**, is $f^{-1}(V)$ **compact** on $\mathbb{R}^n$? If $V \subseteq \mathbb{R}^m$ is **connected**, is $f^{-1}(V)$ **connected** on $\mathbb{R}^n$?
**No to both!**

**Example 11.3.** Counter-example for compact set:
$n = m = 1$, $U = \mathbb{R}$, and $f(x) = \frac{1}{1+x^2}$. Let $V = [0, 1]$ which is compact. Then $f^{-1}(V) = \{x \in \mathbb{R} \mid \frac{1}{1+x^2} \in [0, 1]\} = \mathbb{R}$ is not compact.

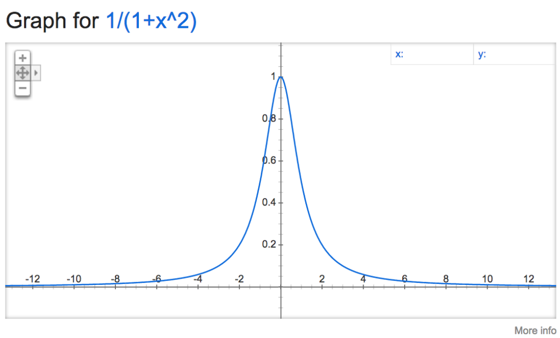


Figure 11.4: A compact image on a cts $f(x) = \frac{1}{1+x^2}$ may not admit a compact inverse image.

**Example 11.4.** Counter-example for connected set:
$n = m = 1$, $U = \mathbb{R}$, and $f(x) = x^2$. Let $V = (1, 9)$ which is connected. Then $f^{-1}(V) = (-3, -1) \cup (1, 3)$ is not connected.

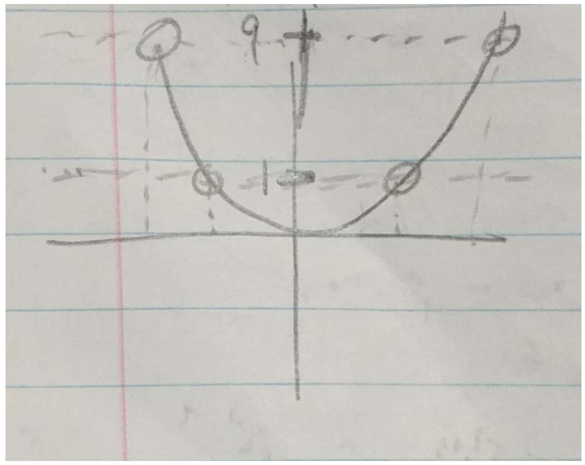


**Figure 11.5:** A connected image on a cts $f(x) = x^2$ may not admit a connected inverse image.

**Proposition 11.2.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $U$ is open. Let $K \subseteq U$ be **compact**. Then $f(K) = \{f(x) \mid x \in K\}$ is **compact** in $\mathbb{R}^m$.

*Proof.* Let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of $f(K)$ i.e. $U_\alpha \subseteq \mathbb{R}^m$ is open $\forall \alpha \in A$ and

$$f(K) \subseteq \bigcup_{\alpha \in A} U_\alpha$$

**Claim:** $K \subseteq \bigcup_{\alpha \in A} f^{-1}(U_\alpha)$.
If $x \in K$, then $f(x) \in f(K)$ so $f(x) \in U_\alpha$ for some $\alpha \Rightarrow x \in f^{-1}(U_\alpha)$.
Since $f$ is cts, $U_\alpha$ open $\Rightarrow f^{-1}(U_\alpha)$ open in $\mathbb{R}^n$ $\forall \alpha$ (by previous propositions).
So $\{f^{-1}(U_\alpha) \mid \alpha \in A\}$ is an open cover of $K$ which is compact.
So $\exists \alpha_1, \ldots, \alpha_n \in A$ such that

$$K \subseteq \bigcup_{j=1}^{N} f^{-1}(U_{\alpha_j})$$

Then $f(K) \subseteq \bigcup_{j=1}^{N} U_{\alpha_j}$ because if $y \in f(K)$ where $y = f(x)$ for some $x \in K$ where $x \in f^{-1}(U_{\alpha_j})$ for some $j \in \{1, \ldots, N\}$ so $f(x) \in U_{\alpha_j}$.
So $f(K)$ is compact. $\square$

**Remark 11.3.** By Heine-Borel, compact $\iff$ closed and bounded. But we've seen that if $f$ is cts $f(\textit{closed}) \neq \textit{closed}$ in general. This implies the additional bounded property makes it valid.
What about $f(\textit{bounded}) = \textit{bounded}$ for a cts $f$? **No:** see this counter-example:

**Example 11.5.** $U = (0, \infty) \subseteq \mathbb{R}$ for $n = m = 1$ and $f(x) = \log x$.
Let $V = (0, 1)$ which is bounded. Then $f(V) = (-\infty, 0)$ which is not bounded.

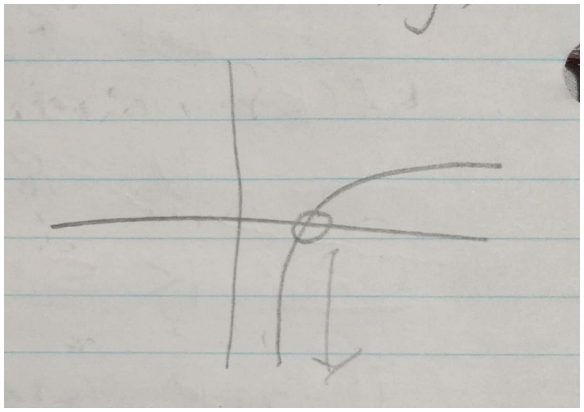

**Figure 11.6:** A bounded domain on a cts $f(x) = \log x$ may not admit a bounded image.

But as we proved before, $f(\textit{closed} + \textit{bound}) = \textit{closed} + \textit{bounded}$ so both conditions are sufficient.

### 11.3 Extreme value theorem (EVT)

**Corollary 11.1.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $U$ is open ($m = 1!$) and $f$ is cts on $U$.
Let $K \subseteq U$ be **compact**. Then $\exists x_1, x_2$ in $K$

$$f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in K$$

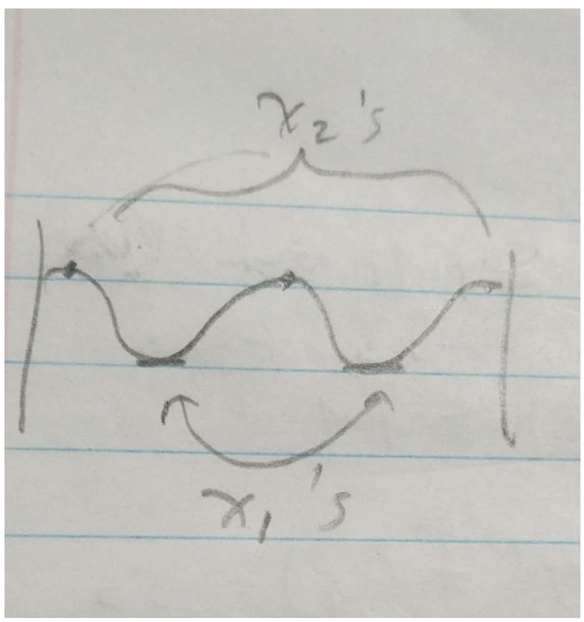


**Figure 11.7:** $x_1, x_2$ may not be unique in the Extreme Value Theorem.

This means a cts. real-valued function on a compact set attains a global maximum value and global minimum value. Clearly this wouldn't work in $\mathbb{R}^m, m > 1$ since there is no notion of min/max of vectors).

*Proof.* Assume $K$ compact, $f$ is cts on $U$ so $f(K)$ is a compact subset of $\mathbb{R}$ (by previous proposition).

By Heine-Borel $f(K)$ is **closed and bounded** in $\mathbb{R}$, so

$$M = \sup_{x \in K} f(x)$$
$$m = \inf_{x \in K} f(x)$$

both exists and is finite (bounded intervals has an infima and suprema).
Let $k \in \mathbb{N}$ such that $M - \frac{1}{k} < M$ so $\exists x_k \in K$ such that

$$M - \frac{1}{k} < f(x_k) \leq M$$

$K$ is bounded so $(x_k)$ is a bounded sequence in $\mathbb{R}$, so by Bolzano-Weierstrass $\exists$ convergent subsequence $(x_{k_l}) \to x$ as $l \to \infty$.
But $K$ is closed so $\lim_{l \to \infty} x_{k_l} = x \in K$.
Since $f$ is cts, so

$$\lim_{l \to \infty} f(x_{k_l}) = f(\lim_{l \to \infty}(x_{k_l})) = f(x) \in f(K)$$

since $x \in K$. Thus

$$M - \frac{1}{k_l} < f(x_{k_l}) \leq M$$

then as $l \to \infty$, we have $M \leq f(x) \leq M$.
So $x \in K$ and $f(x) = \sup_{y \in K} f(y)$ (some $y$) so global max is attained.
For global min, similarly $m \leq f(x_k) < m + \frac{1}{k}$.

**Remark 11.4.** The non-uniqueness of extreme values come from choosing an arbitrary convergent subsequence from $(x_k)$.

□

This generalizes EVT from 147: if $f$ is cts on $[a, b]$ then $f$ attains a global max/min.

**Remark 11.5.** Note that $f$ must be cts for this to hold. Otherwise we could have

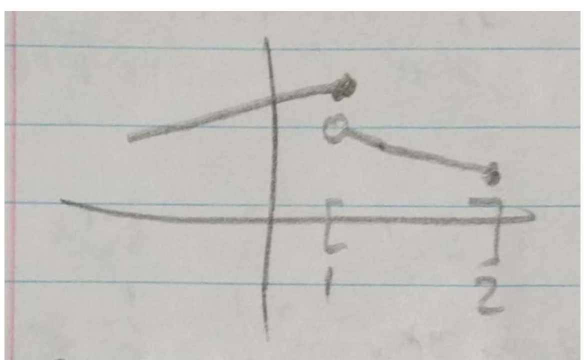


**Figure 11.8:** There is a global min on $[1, 2]$ but no global max.

# 12 January 29, 2018

## 12.1 Continuity and connected sets

(See above for connected image does not imply connected inverse image example.)

**Proposition 12.1.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ cts on $U$ which is open.
Let $E \subseteq U$ be **connected** on $\mathbb{R}^n$. Then $f(E)$ is **connected** in $\mathbb{R}^m$ (i.e. cts image of connected set is connected).

*Proof.* Suppose $f(E)$ is not connected. Let $V_1, V_2 \in \mathbb{R}^m$ open sets be a separation of $f(E)$

1. $f(E) \subseteq V_1 \cup V_2$
2. $f(E) \cap V_1 \neq \varnothing$
3. $f(E) \cap V_2 \neq \varnothing$
4. $f(E) \cap V_1 \cap V_2 = \varnothing$

Since $f$ is cts, $f^{-1}(V_1), f^{-1}(V_2)$ are open in $\mathbb{R}^n$. If $x \in E$, $f(x) \in f(E) \subseteq V_1 \cup V_2$. So $f(x) \in V_1$ or $f(x) \in V_2$ which implies $x \in f^{-1}(V_1) \cup f^{-1}(V_2)$, that is

$$E \subseteq f^{-1}(V_1) \cup f^{-1}(V_2)$$

Let $y \in f(E) \cap V_1 \neq \varnothing$. So $\exists x \in E$ such that $y = f(x) \in V$, that is

$$x \in f^{-1}(V) \cap E \neq \varnothing$$

Similarly $f^{-1}(V_2) \cap E \neq \varnothing$.
If $x \in E \cap f^{-1}(V_1) \cap f^{-1}(V_2)$ then $f(x) \in f(E) \cap V_1 \cap V_2 \neq \varnothing$ which is a contradiction of our initial assumptions that $V_1, V_2$ is a separation.
So $f^{-1}(V_1) \cap f^{-1}(V_2) \cap E = \varnothing$, which means $\{f^{-1}(V_1), f^{-1}(V_2)\}$ is a separation of $R$ which is a contradiction since $E$ is connected.
Thus $f(E)$ must be connected. □

## 12.2 Intermediate value theorem (IVT)

**Corollary 12.1.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, where $U$ open ($m = 1!$).
Suppose $f$ is cts on $U$ and let $E \subseteq U$ be connected. Let $x, y \in E$ such that $f(x) < f(y)$. Then for **each** $w \in (f(x), f(y))$, $\exists z \in E$ such that $f(z) = w$.
(i.e. a cts real-valued fn on a connected set admits all values between any two of its values).

*Proof.* Assume the contrary: that is $\exists w_0 \in (f(x), f(y))$ such that $w_0 \notin f(E)$.
Let

$$V_1 = \{w \in \mathbb{R} \mid w < w_0\} = (-\infty, w_0)$$
$$V_2 = \{w \in \mathbb{R} \mid w > w_0\} = (w_0, \infty)$$

then $V_1, V_2$ is a spearation of $f(E)$ but $f(E)$ is connected by previous proposition, which is a contradiction. □

**Aside:** If $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ for $U$ open is cts on $U$ then

$$\|f\| : U \to \mathbb{R}$$

where $\|f\|(x) = \|f(x)\|$ is cts. real-valued so we can apply EVT or IVT to $\|f\|$.

### 12.3 Uniform continuity

**Definition 12.1.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $U$ is open, and let $D \subseteq U$.
We say that $f$ is **uniformly continuous on** $D$ **iff** $\forall \epsilon > 0\ \exists \delta > 0$ such that

$$x, y \in D \quad \|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \epsilon$$

**Remark 12.1.**

1. Uniformly continuous only makes sense with respect to a particular subset $D$ of $U$. $f$ may be unif. cts on $D_1 \subseteq U$ but not unif. cts on $D_2 \subseteq U$.
2. If $f$ is unif cts on $D$, this means given any $\epsilon > 0$, we can find **a single** $\delta > 0$ depending only on $\epsilon$ that works to establish continuity of $f_{|D}$ at $x \in D$ for all $x \in D$.
3. If $f$ is unif cts on $D$, then $f_{|D}$ is cts at $x \in D\ \forall x \in D$ (but the converse is not necessarily true).

**Example 12.1.** Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ and $f(x) = \frac{1}{x}$ ($f$ is cts).
Let $D = (0, 1]$.

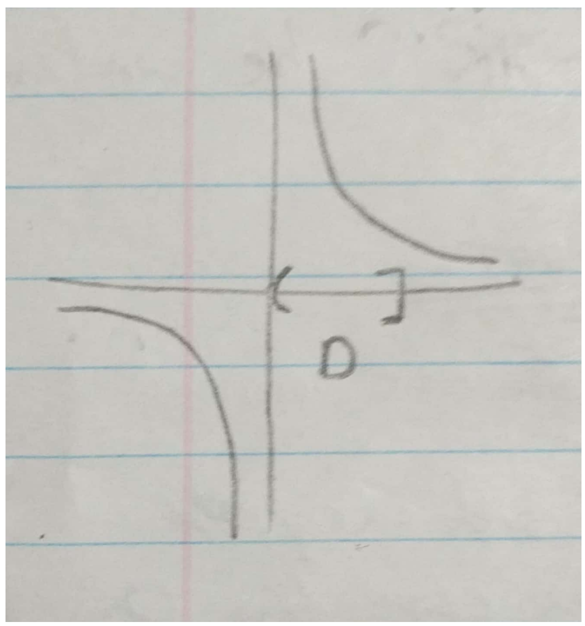


**Figure 12.1:** For the subdomain $D = (0, 1]$, $f(x) = \frac{1}{x}$ is not uniformly continuous on $D$. We can find $x, y$ arbitrary close to 0 such that for a given $\epsilon > 0$, there is not single $\delta > 0$.

**Claim:** $f$ is NOT unif cts on $D$ (so we find an $\epsilon$ where no single $\delta$ works).
Let $\epsilon = \frac{1}{2}$. Let $\delta > 0$ be arbitrary. Let $n \in \mathbb{N}$ be large enough so that

$$\frac{1}{n(n+1)} < \delta$$

Let $x = \frac{1}{n}$ and $y = \frac{1}{n+1} \in D$. So we have

$$|x - y| = |\frac{1}{n} - \frac{1}{n+1}| = \frac{1}{n(n+1)} < \delta$$

but we have

$$|f(x) - f(y)| = |n - (n+1)| = 1 > \epsilon$$

**Example 12.2.** Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = x^2$ ($f$ is cts).
Let $D = [0, \infty)$.
**Claim:** $f$ is not unif cts on $D$.
Let $\epsilon = 1$, let $\delta > 0$ be arbitrary. Let $x = \frac{2}{\delta}$ and $y = \frac{2}{\delta} + \frac{\delta}{2} \in D$ (close to each other).
Note

$$|x - y| = |\frac{2}{\delta} - (\frac{2}{\delta} + \frac{\delta}{2})| = \frac{\delta}{2} < \delta$$

and

$$|f(x) - f(y)| = |(x-y)(x+y)| = \frac{\delta}{2}(\frac{4}{\delta} + \frac{\delta}{2}) = 2 + \frac{\delta^2}{4} > 2 > \epsilon$$

### 12.4 Uniform continuity and compact sets

**Theorem 12.1.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be cts on $U$ open. Let $K \subseteq U$ be **compact**.
Then $f$ is unif cts on $K$.

**Remark 12.2. Aside:** In example 12.1, we chose $D = (0, 1]$ and a proof with a counter $n$ when $x, y$ gets arbitrarily close to 0.
If $D = [\zeta, 1]$ such that is compact, our counter argument with $n$ would fail because we can't get arbitrarily close to 0.

*Proof.* Let $\epsilon > 0$ and let $x \in K$. Since $f$ is cts at $x$, $\exists \delta(x) > 0$ such that

$$\|y - x\| < \delta(x) \Rightarrow \|f(y) - f(x)\| < \frac{\epsilon}{2}$$

i.e. $f(B_{\delta(x)}(x)) \subseteq B_{\frac{\epsilon}{2}}(f(x))$.
Also $K \subseteq \bigcup_{x \in K} B_{\frac{\delta(x)}{2}}(x)$ (this is an arbitrary union for every point in $K$).
By compactness of $K$, $\exists$ finite set $x_1, \ldots, x_N \in K$ such that

$$K \subseteq \bigcup_{j=1}^{N} B_{\frac{\delta(x_j)}{2}}(x_j) \tag{12.1}$$

Let $\delta = \min\{\frac{\delta(x_1)}{2}, \ldots, \frac{\delta(x_N)}{2}\} > 0$.
Suppose $x, y \in K$ and $\|x - y\| < \delta$, and $x \in B_{\frac{\delta(x_j)}{2}}(x_j)$ for some $j \in [1, \ldots, N]$ by equation 12.1.
Then we have

$$\begin{aligned}
\|y - x_j\| &\overset{\triangle}{\leq} \|y - x\| + \|x - x_j\| \\
&< \delta + \frac{\delta(x_j)}{2} \\
&< \frac{\delta(x_j)}{2} + \frac{\delta(x_j)}{2} \\
&= \delta(x_j)
\end{aligned}$$

That is $\|y - x_j\| < \delta_(x_j)$ so $y \in B_{\delta(x_j)}(x_j)$, thus we have

$$\begin{aligned}\|f(x) - f(y)\| &\overset{\triangle}{\leq} \|f(x) - f(x_j)\| + \|f(x_j) - f(y)\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

so we found a $\delta > 0$ (single $\delta$) such that $x, y \in K$ where $\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \epsilon$. □

### 12.5 Differentiability

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $U$ open and let $a \in U$. We want to define what it means for $f$ to be differentiable at $a$ and what is the derivative of $f$ at $a$ (denoted $Df(a)$).
We'll see soon that if $f$ is differentiable at $a$, then $Df(a)$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ ($m \times n$ matrix).

**Remark 12.3.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Choose basis $\beta$ of $\mathbb{R}^n$, basis $\gamma$ of $\mathbb{R}^m$. Then $T$ is represented wrt these two bases by an $m \times n$ basis $[T]_{\gamma,\beta}$.
If $\tilde{\beta}$ is another basis of $\mathbb{R}^n$ and $\tilde{\gamma}$ is another basis of $\mathbb{R}^m$, then let $P$ (invertible $n \times n$) and $Q$ (invertible $m \times m$) be the change of bases matrices from basis $\beta$ to $\tilde{\beta}$ and from basis $\gamma$ to $\tilde{\gamma}$, respectively. Thus

$$[T]_{\tilde{\gamma},\tilde{\beta}} = Q^{-1}[T]_{\gamma,\beta}P$$

If $n = m$ and we choose $\beta = \gamma$ and $\tilde{\beta} = \tilde{\gamma}$ then

$$[T]_{\tilde{\beta}} = P^{-1}[T]_{\beta}P$$

If $n = m = 1$ such that $\beta = \gamma, \tilde{\beta} = \tilde{\gamma}$ we have

$$[T]_{\tilde{\beta}} = [T]_{\beta} \quad \text{since the } 1 \times 1 \text{ matrices commute}$$

i.e. the matrices representing a linear map $\mathbb{R} \to \mathbb{R}$ is **unique** (doesn't depend on $\beta$).

## 13 January 31, 2018

### 13.1 Single variable differentiability

**Definition 13.1.** Let $f : U \subseteq \mathbb{R} \to \mathbb{R}$, $U$ open, and $a \in U$. We say $f$ is **differentiable** at $a$ **iff**

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. If so, we call the limit the **derivative** of $f$ at $a$ and we denote it

$$f'(a) = \frac{df(a)}{dx} = Df(a)$$

**Remark 13.1. Claim:** If $f$ is differentiable at $a$ then $f$ is continuous at $a$.
We have

$$f(a+h) - a = \frac{f(a+h) - a}{h} \cdot h$$

Taking the limit of both sides we get

$$\lim_{h\to 0} f(a+h) - a = 0 = \lim_{h\to 0} \frac{f(a+h)-a}{h} \cdot h$$

So we get (from the right equality)

$$\begin{aligned} &\lim_{h\to 0} f(a+h) = f(a) \\ \Rightarrow &\lim_{x\to a} f(x) = f(a) \end{aligned}$$

so $f$ is cts at $a$ (limit exists and $= f(a)$).
The converse is false: e.g. $f(x) = |x|$.

When we choose our definition of differentiable in general, we'll want the property that "differentiable at $a$" $\Rightarrow$ "continuous at $a$".

## 13.2 Partial derivatives

**Definition 13.2.** Let $i \in \{1, \ldots, n\}$. The **partial derivative** of $f$ *in the $x_i$-direction* at the point $a$ is defined to be

$$\lim_{h\to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_n)}{h}$$

*if it exists.*
This is the ordinary derivative at $x_i = a_i$ of $f$ thought of as only a function of $x_i$ with all other $x_j = a_j$ constant.

**Notation:** when the partial derivative exists, it is denoted

$$\frac{\partial f}{\partial x_i}(a) = f_{x_i}(a)$$

The shorthand definition: let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Then

$$\frac{\partial f}{\partial x_i}(a) = \lim_{h\to 0} \frac{f(a + he_i) - f(a)}{h}$$

e.g. in $\mathbb{R}^1$, $e_1 = (1) \Rightarrow a + he_1 = a + h \in \mathbb{R}^1$.

**Example 13.1.**

1. $f(x, y) = \sin(xy)$
2. $g(x, y, z) = e^{x^2 z} \log(y + z)$
3. $h(x, y, z) = y^3 \sin(xz) + e^{13z + x^3 \log(z^5+1)}$
4. $\lambda(x, y, z) = x^2 \sin(y) - \frac{yz}{x}$

Then we have

$$\frac{\partial f}{\partial x} = y\cos(xy) \qquad \frac{\partial f}{\partial y} = x\cos(xy)$$
$$\frac{\partial g}{\partial x} = 2xze^{x^2z}\log(y+z) \qquad \frac{\partial g}{\partial y} = \frac{e^{x^2z}}{y+z}$$
$$\frac{\partial g}{\partial z} = \frac{e^{x^2z}}{y+z} + x^2e^{x^2z}\log(y+z)$$
$$\frac{\partial h}{\partial y} = 3y^2\sin(xz)$$
$$\frac{\partial \lambda}{\partial x} = 2x\sin(y) + \frac{yz}{x^2}$$

## 13.3 Wrong definitions of differentiability

**Remark 13.2.** A reasonable guess for the definition of differentiability of $f$ at $a$ is to say all the partial derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ all exist at $a$.
This is **WRONG!** because there exists examples where $\frac{\partial f}{\partial x_i}$ all exist at $a$ but $f$ is not cts at $a$ (see assignment 5)!
From intuition: $f$ may not be continuous *in between* $x_i$-directions.

**Definition 13.3.** You can also consider the rate of change of $f$ at $a$ in the direction of *any* unit vector $u$ (i.e. in between the standard vectors $e_i$).
This is called the **directional derivative** of $f$ at $a$ in the $u$-direction and is denoted

$$(D_u f)(a)$$

(On assignment 5: show $(D_{e_i} f)(a) = \frac{\partial f}{\partial x_i}(a)$.)

**Remark 13.3.** Another reasonable guess; $f$ is differentiable at $a$ if **all** the directional derivatives $(D_u f)(a)$ exists at $a$ for all unit vectors $u$.
This is **also WRONG!**, since there exists examples where all $(D_u f)(a)$ exists at $a$ but $f$ is not continuous at a!
From intuition: one can take more complicated paths to $a$ where contunity may not hold, e.g. as before with $x^2$ or $x^3$ paths.

## 13.4 Second partial derivatives

Suppose $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, $U$ open, and suppose $\frac{\partial f}{\partial x_i}$ exists everywhere on $U$.
So $\frac{\partial f}{\partial x_i} : U \subseteq \mathbb{R}^n \to \mathbb{R}$ (it's a function on $U$).
So we can ask about the existence of $\frac{\partial}{\partial x_j}(\frac{\partial f}{\partial x_i})$ or

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = f_{x_i x_j}$$

(remark the order of the notation).

For example if $n = 2$, there are 4 ($n^2$) second partial derivatives

$$f_{xx} = \frac{\partial^2 f}{\partial x \partial x} = \frac{\partial^2 f}{\partial x^2}$$
$$f_{yy} = \frac{\partial^2 f}{\partial y^2}$$
$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$$
$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$$

**Example 13.2.** For $f(x, y) = \sin(xy)$, we have

$$f_x = y\cos(xy) \quad f_y = x\cos(xy)$$

thus we have

$$f_{xx} = -y^2 \sin(xy) \qquad f_{xy} = \cos(xy) - xy\sin(xy)$$
$$f_{yy} = -x^2 \sin(xy) \qquad f_{yx} = \cos(xy) - xy\sin(xy)$$

Notice that $f_{xy} = f_{yx}$ everywhere!
For $\lambda(x, y, z) = x^2 \sin(y) - \frac{yz}{x}$.

$$\lambda_x = 2x\sin(y) + \frac{yz}{x^2} \quad \lambda_y = x^2\cos(y) - \frac{z}{x} \quad \lambda_z = \frac{-y}{x}$$

Thus we have

$$\lambda_{xy} = 2x\cos(y) + \frac{z}{x^2} \qquad \lambda_{yx} = 2x\cos(y) + \frac{z}{x^2}$$
$$\lambda_{xz} = \frac{y}{x^2} \qquad \lambda_{zx} = \frac{y}{x^2}$$
$$\lambda_{yz} = \frac{-1}{x} \qquad \lambda_{zy} = \frac{-1}{x}$$

So again

$$\frac{\partial^2 \lambda}{\partial x_i \partial x_j} = \frac{\partial^2 \lambda}{\partial x_j \partial x_i} \quad \forall i, j$$

**Question:** is this always true? **NO!** There exists examples where $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ ($n > 1$) such that

$$\frac{\partial^2 \lambda}{\partial x_i \partial x_j}(a) \neq \frac{\partial^2 \lambda}{\partial x_j \partial x_i}(a)$$

for certain $i, j$ and $a$ (for a certain point $a$ usually).

## 13.5 $C^k(U)$ (partial derivative class)

**Definition 13.4.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, $U$ open. We say $f$ is in $C^0(U)$ if $f$ is continuous on $U$.
We say $f$ is in $C^1(U)$ if $f$ is in $C^0(U)$ and all $\frac{\partial f}{\partial x_i}$'s exist and are continuous on $U$.
We say $f$ is in $C^2(U)$ if $f$ is in $C^1(U)$ and all $\frac{\partial^2 f}{\partial x_i \partial x_j}$'s exist and are continuous on $U$.

In general, for $k \in \mathbb{N}$, $f$ is in $C^k(U)$ if $f$ is in $C^{k-1}(U)$ and all $\frac{\partial^k f}{\partial x_{i_k} \ldots \partial x_{i_1}}$ exist and are continuous on $U$.

**Definition 13.5.** $f$ is in $C^\infty(U)$ if $f \in \bigcap_{k=0}^{\infty} C^k(U)$ i.e. if $f \in C^k(U)$ $\forall k \in \mathbb{N}$.

**Remark 13.4.**

$$C^0(U) \supset C^1(U) \supset \ldots \supset C^k(U) \supset C^{k+1}(U) \supset \ldots \supset C^\infty(U)$$

# 14 February 2, 2018

## 14.1 Mean Value Theorem (MVT)

**Theorem 14.1.** Let $f : U \subseteq \mathbb{R} \to \mathbb{R}$, $U$ open, be continuous on $[a, b] \in U$ and differentiable on $(a, b)$. There $\exists c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

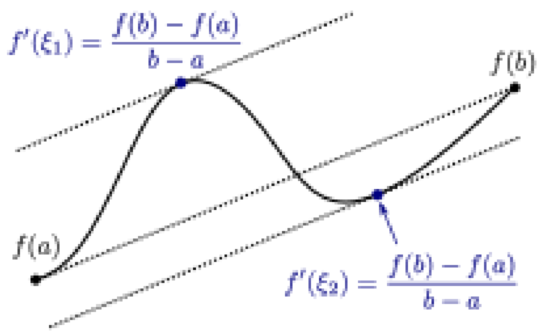


**Figure 14.1:** There may be multiple $c \in (a, b)$ that satisfy the MVT property.

## 14.2 "Commutativity" of mixed partial derivatives

**Theorem 14.2.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, $U$ open. Let $a \in U$. Suppose $\frac{\partial f}{\partial x_j}$, $\frac{\partial f}{\partial x_k}$ **exist and are continuous** ($j \neq k, j, k \in \{1, \ldots, n\}$) on a *neighbourhood of* $a$.
Furthermore, suppose that $\frac{\partial^2 f}{\partial x_j \partial x_k}$ exists in a *neighbourhood of* $a$ and is continuous on $a$.
Then $\frac{\partial^2 f}{\partial x_k \partial x_j}$ exists at $a$ and

$$\frac{\partial^2 f}{\partial x_k \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_k}(a)$$

**Remark 14.1.** In the examples above: all the first and second partial derivatives existed and were continuous everywhere on the domain of $f$. So we have much more than we need (we only require they exist and cts at $a$) to apply the above theorem and conclude that the mixed partials are equal.

**Remark 14.2.** The partial derivatives need only be continuous on *a neighbourhood of* $a$: nothing needs said about the space away from $a$.

*Proof.* We will require 3 applications of the single variable Mean Value Theorem (MVT).
First we'll show we can reduce the problem to $n = 2$ ($x_j = x, x_k = y$) and $a = (0, 0)$.
Let $s, t \in \mathbb{R}$ be small enough such that

$$h(s, t) = f(a + se_j + te_k)$$

is defined (this is possible since $a \in U$ and $U$ is open so open ball).
Let's compute $\frac{\partial h}{\partial s}(s_0, t_0)$

$$\begin{aligned}
\frac{\partial h}{\partial s}(s_0, t_0) & \lim_{\epsilon \to 0} \frac{h(s_0 + \epsilon, t_0) - h(s_0, t_0)}{\epsilon} \\
& \lim_{\epsilon \to 0} \frac{f(a + (s_0 + \epsilon)e_j + t_0 e_k) - f(a + s_0 e_j + t_0 e_k)}{\epsilon} \\
& \frac{\partial f}{\partial x_j}(a + s_0 e_j + t_0 e_k) & \text{definition of } \frac{\partial f}{\partial x_j}
\end{aligned}$$

Similarly,

$$\frac{\partial h}{\partial t}(s_0, t_0) = \frac{\partial f}{\partial x_k}(a + s_0 e_j + t_0 e_k)$$

Note that these hold for **any** $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, $U$ open therefore (recall the partial derivatives are themselves functions on $U \to \mathbb{R}$, so we can apply the above recursively on itself)

$$\begin{aligned}
\frac{\partial^2 f}{\partial s \partial t}(s_0, t_0) &= \frac{\partial^2 f}{\partial x_j x_k}(a + s_0 e_j + t_0 e_k) \\
\frac{\partial^2 f}{\partial t \partial s}(s_0, t_0) &= \frac{\partial^2 f}{\partial x_k x_j}(a + s_0 e_j + t_0 e_k)
\end{aligned}$$

So

$$\begin{aligned}
\frac{\partial^2 f}{\partial s \partial t}(0, 0) &= \frac{\partial^2 f}{\partial x_j x_k}(a) \\
\frac{\partial^2 f}{\partial t \partial s}(0, 0) &= \frac{\partial^2 f}{\partial x_k x_j}(a)
\end{aligned}$$

(assuming these all exist). Therefore it is enough to consider the case when $n = 2$, $a = (0, 0)$ (we reduced our problem into a $\mathbb{R}^2 \to \mathbb{R}$ problem using arbitrary $s, t$).
Let's prove our thoerem with $h : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, $U$ open, and $0 \in U$.
Note that $\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}$ exists and are continuous on a neighbourhood of 0 (these are from our initial premise after converting to our reduced problem). Also, $\frac{\partial^2 h}{\partial t \partial s}$ exists on a n'h'd of 0 and is cts. at 0.
We want to show $\frac{\partial^2 h}{\partial s \partial t}$ exists at 0 and equals $\frac{\partial^2 h}{\partial t \partial s}(0)$ at 0.
Let us define

$$H(s, t) = (h(s, t) - h(s, 0)) - (h(0, t) - h(0, 0))$$

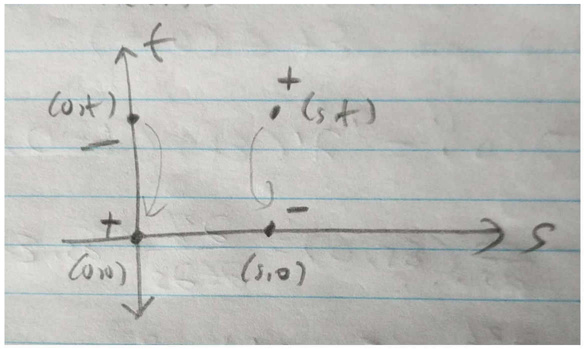


**Figure 14.2:** Sketch of how we set up $H(s,t)$.

Fix $t$ sufficiently close to 0. Define $k(s) = h(s,t) - h(s,0)$. So we have

$$H(s,t) = k(s) - k(0)$$

Also $k'(s) = \frac{\partial h}{\partial s}(s,t) - \frac{\partial h}{\partial s}(s,0)$ exists and cts on n'h'd of 0 (by hypothesis). So we can apply the MVT to $k$ on $[0,s]$, then

$$\exists \delta \in (0,1) \Rightarrow \delta s \in (0,s)$$

such that

$$k(s) - k(0) = k'(\delta s)(s-0)$$

So we have

$$H(s,t) = s[\frac{\partial h}{\partial s}(\delta s, t) - \frac{\partial h}{\partial s}(\delta s, 0)]$$

Fix $s$ (and hence $\delta$). We define

$$\lambda(t) = \frac{\partial h}{\partial s}(\delta s, t)$$
$$\lambda'(t) = \frac{\partial^2 h}{\partial t \partial s}(\delta s, t)$$

exists near $t = 0$ (by hypothesis) so $\lambda(t)$ is cts. and diffable (differentiable) on $[0,t]$ for $t$ small enough. Applying the MVT to $\lambda$ on $[0,t]$

$$\exists \epsilon \in (0,1) \Rightarrow \epsilon t \in (0,t)$$

such that

$$\lambda(t) - \lambda(0) = \lambda'(\epsilon t)(t-0)$$

So we have (from definition of $\lambda$)

$$\frac{\partial h}{\partial s}(\delta s, t) - \frac{\partial h}{\partial s}(\delta s, 0) = \frac{\partial^2 h}{\partial t \partial s}(\delta s, \epsilon t)(t)$$

Substituting this into $H(s,t)$, we get

$$H(s,t) = st\frac{\partial^2}{\partial t \partial s}(\delta s, \epsilon t)$$

for some $\delta, \epsilon \in (0,1)$. Therefore when we take the limit

$$\lim_{(s,t)\to(0,0)} \frac{1}{st} H(s,t) = \frac{\partial^2 h}{\partial t \partial s}(0,0)$$

since $\frac{\partial^2 h}{\partial t \partial s}$ is assumed to be cts. at $(0,0)$. If we can show the LHS is equivalent to $\frac{\partial^2 h}{\partial s \partial t}(0,0)$, then we are done.
Recall that

$$H(s,t) = h(s,t) - h(s,0) - h(0,t) + h(0,0)$$

We can also write

$$H(s,t) = (h(s,t) - h(0,t)) - (h(s,0) - h(0,0))$$

(notice the regrouping: in the graph, we are now subtracting in the other direction).
We define $\mu(t) = h(s,t) - h(0,t)$ so $H(s,t) = \mu(t) - \mu(0)$. Therefore

$$\mu'(t) = \frac{\partial h}{\partial t}(s,t) - \frac{\partial h}{\partial t}(0,t)$$

exists $\forall t$ sufficiently close to 0 (from hypothesis where $a = 0$). So $\mu(t)$ is cts on $[0,t]$ and is diffable on $(0,t)$ for $t$ small.
Applying the MVT to $\mu$ on $[0,t]$, then

$$\exists \theta \in (0,1) \Rightarrow \theta t \in (0,t)$$

such that

$$\mu(t) - \mu(0) = \mu'(\theta t)(t - 0)$$

Thus we have

$$H(s,t) = t[\frac{\partial h}{\partial t}(s,\theta t) - \frac{\partial h}{\partial t}(0,\theta t)]$$

We can rewrite this as

$$\frac{H(s,t)}{st} = \frac{1}{s}[\frac{\partial h}{\partial t}(s,\theta t) - \frac{\partial h}{\partial t}(0,\theta t)]$$

Since $\frac{\partial h}{\partial t}$ is cts. on a n'h'd of $(0,0)$ (hypothesis), then we let $t \to 0$ **first** so that we get

$$\frac{\partial h}{\partial t}(s,\theta t) \to \frac{\partial h}{\partial t}(s,0)$$
$$\frac{\partial h}{\partial t}(0,\theta t) \to \frac{\partial h}{\partial t}(0,0)$$

Thus we have

$$\lim_{(s,t)\to(0,0)} = \lim_{s\to 0}[\frac{1}{s}(\frac{\partial h}{\partial t}(s,0) - \frac{\partial h}{\partial t}(0,0))]$$

This is precisely the definition of the partial derivative with respect to $s$, thus we have

$$\lim_{(s,t)\to(0,0)} \frac{H(s,t)}{st} = \frac{\partial^2 h}{\partial s \partial t}(0,0) = \frac{\partial^2 h}{\partial t \partial s}(0,0)$$

□

### 14.3 Defining multivariable differentiability

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ (general $m$), $U$ open. Let $a \in U$. We want to define what it means for $f$ to be differentiable at $a$.
We *expect* that if $f$ is differentiable at $a$, then

1. $f$ should be **continuous on** $a$
2. all the partial derivatives of $f$ (if $m = 1$) should exist at $a$

**Example 14.1.** Example of where partial derivatives exist but is not continuous at given $a = (0, 0)$:
Let $n = 2$ where $U = \mathbb{R}^2$ and

$$f(x, y) = \begin{cases} x + y & \text{if } x = 0 \text{ or } y = 0 \\ 1 & \text{otherwise} \end{cases}$$

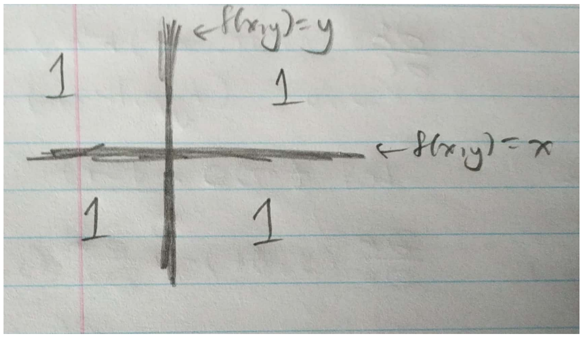


**Figure 14.3:** A function where the partial derivatives exist but are not continuous at a given point $(0, 0)$.

Note that

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{h-0}{h} = 1 \end{aligned}$$

and similarly, $\frac{\partial f}{\partial y}(0, 0) = 1$.
Both partials exist at $(0, 0)$ but clearly $f$ is not continuous at $(0, 0)$ ($\lim_{x\to 0} f(x) \neq f(x)$).

**Remark 14.3.** This shows that our previous bad definition that if partials exist, then differentiable is *wrong*.

## 14.4 Differentiability with linear maps

Going back to the $n = m = 1$ simple case, where $f : U \subseteq \mathbb{R} \to \mathbb{R}$ for $U$ open and $x_0 \in U$. Then $f$ is **differentiable at** $x_0 \iff \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists (previously defined).
Note that $f'(x_0) \in \mathbb{R}$ are $1 \times 1$ matrices i.e. a linear map from $\mathbb{R}$ to $\mathbb{R}$.
Assuming $f$ is differentiable at $x_0$, we define the linear map $T : \mathbb{R} \to \mathbb{R}$ by

$$T(h) = f'(x_0)h$$

such that

$$\begin{aligned} \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} &= f'(x_0) \\ &= \lim_{h\to 0} \frac{f'(x_0)h}{h} \\ &= \lim_{h\to 0} \frac{T(h)}{h} \end{aligned}$$

So (rewriting the above)

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0) - T(h)}{h} = 0$$

Note that for any limits on an arbitrary $g(x)$ that approach to 0

$$\lim_{h\to 0} g(x) = 0 \iff \lim_{h\to 0} |g(x)| = 0$$

so we have

$$\lim_{h\to 0} \frac{|f(x_0+h) - f(x_0) - T(h)|}{|h|} = 0$$

We've shown that if $f$ is differentiable at $x_0$, $\exists$ a linear map $T : \mathbb{R} \to \mathbb{R}$ such that the above holds.
We've thus motivated the general definition of differentiability:

**Definition 14.1.** For $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $U$ open, let $x_0 \in U$.
We say $f$ is **differentiable** at $x_0$ if $\exists$ a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{\|f(x_0+h) - f(x_0) - T(h)\|}{\|h\|} = 0 \tag{14.1}$$

where we take the norm of an $\mathbb{R}^m$ vector in the numerator and the norm of an $\mathbb{R}^n$ vector in the denominator (this is also the reason why we needed to take the norm to be able to divide the two).

# 15 February 5, 2018

## 15.1 Differential (Jacobian matrix) $(Df)_a$

**Remark 15.1.** We'll show soon that if such a $T$ linear map exists, it is necessarily **unique** and it's called the *derivative of $f$ at $a$* and is denoted $(Df)_a$ (i.e. $(Df)a$ is an $m \times n$ matrix of real numbers).
$(Df)_a$ is also called the **differential** of $f$ at $a$, or the **linearization** of $f$ at $a$, or the **Jacobian matrix**.

**Notice:** If $(Df)_a = T$ exists satisfying equation 14.1 then

1. $T : \mathbb{R}^n \to \mathbb{R}^m$ is linaer

2. $T$ is a very good approximation to the map $h \mapsto f(a+h) - f(a)$ near $h = 0$ in the following sense (recall $T(h) = f'(a)h = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \cdot h$).

$$h \mapsto f(a+h) - f(a)$$
$$h \mapsto T(h)$$

Both agree at $h = \vec{0}$ (i.e. both send $\vec{0} \to \vec{0}$) and moreover the difference

$$\|f(a+h) - f(a) - T(h)\| \to 0 \quad \text{as } h \to 0$$

**so fast** that the quotient in equation 14.1 still goes to 0 as $h \to \vec{0}$ (sends numerator to 0 faster than $h \to \vec{0}$).

## 15.2 Differentiability implies continuity with $T(h)$

**Proposition 15.1.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $U$ open, and $a \in U$. Suppose $f$ is diffable at $a$. Then $f$ is **cts** at $a$ (we show this is true for the linear map definition now).

*Proof.* We need to show that $\lim_{x\to a} f(x) = f(a)$ or $\lim_{h\to 0} f(a+h) = f(a)$ (where $x = a+h$).

Note that

$$\begin{aligned}\lim_{h\to 0}\|f(a+h)-f(a)-T(h)\| &= \lim_{h\to 0}\frac{\|f(a+h)-f(a)-T(h)\|}{\|h\|}\cdot\|h\| \\ &= 0\cdot 0 \\ &= 0\end{aligned}$$

where the second equality holds from products of existing limits. So we have

$$\|f(a+h)-f(a)\| \overset{\triangle}{\le} \|f(a+h)-f(a)-T(h)\| + \|T(h)\|$$

Since $T$ is linear, we have $\|T(h)\| \le \|T\|_{op}\|h\|$ which $\to 0$ as $h\to\vec{0}$. Thus combining the above we have

$$\|f(a+h)-f(a)\| \le \|f(a+h)-f(a)-T(h)\| + \|T\|_{op}\|h\|$$

which by the squeeze theorem we have

$$\lim_{h\to 0} f(a+h) = f(a)$$

and thus $f$ is cts at $a$. □

### 15.3 Differential is matrix of partial derivatives

**Theorem 15.1.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $a\in U$. Suppose $f$ is diffable at $a$.
We have

$$f(x)\in\mathbb{R}^m = (f_1(x),\dots,f_m(x))$$

where $f_j : U\subseteq\mathbb{R}^n\to\mathbb{R}$ are the component functions of $f$, $1\le j\le m$.
Then all the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exists at $a$ for $1\le i\le m$, $1\le j\le n$. Moreover,

$$T = (Df)_a$$

is the $m\times n$ matrix whose $(i,j)$-entry is $\frac{\partial f_i}{\partial x_j}(a)$. This shows $(Df)_a$ is unique if it exists.

*Proof.* By assumption, $\exists$ $m\times n$ matrix $T$ such that

$$\lim_{h\to 0}\frac{\|f(x_0+h)-f(x_0)-T(h)\|}{\|h\|} = 0$$

Recall from linear algebra we have

$$\begin{bmatrix} T(e_1) & T(e_2) & \vdots & T(e_n)\end{bmatrix}$$

which is an $m\times n$ matrix (each column is the image of $e_i$, the standard basis vector).
Since the above limit exists and is zero, we get 0 if $h\to\vec{0}$ along *any path*.
Choose the path

$$h = te_j \quad j\in\{1,\dots,n\}$$

as $t\in\mathbb{R}$ goes to 0, then $h\to\vec{0}$.

We have

$$\lim_{t\to 0} \frac{\|f(a+te_j) - f(a) - T(te_j)\|}{\|te_j\|} = 0$$

$$\iff \lim_{t\to 0} \|\frac{f(a+te_j) - f(a) - T(te_j)}{t}\| = 0 \qquad \therefore \|e_j\| = 1$$

$T$ is linear so $T(te_j) = tT(e_j)$ thus

$$\lim_{t\to 0} \|\frac{f(a+te_j) - f(a)}{t} - T(e_j)\| = 0$$

Recall that $\lim_{x\to x_0} \|g(x) - L\| = 0 \iff \lim_{x\to x_0} g(x) = L$ (trivial by epsilon-delta, true for all $\epsilon > 0$). Thus

$$\lim_{t\to 0} \frac{f(a+te_j) - f(a)}{t} = T(e_j)$$

so the $i$-th component of the quotient above is $\frac{\partial f_i}{\partial x_j}(a)$. Therefore we've shown

$$T_{ij} = \frac{\partial f_i}{\partial x_j}(a)$$

exists and holds. □

**Remark 15.2.** If $f$ is diffable at $a$, then all $\frac{\partial f_i}{\partial x_j}$ exist at $a$ (as above). So if *even one* $\frac{\partial f_i}{\partial x_j}$ does not exist at $a$, then $f$ is *not diffable* at $a$.

**Warning:** Just because all $\frac{\partial f_i}{\partial x_j}(a)$ exist **DOES NOT** necessarily imply that $f$ is diffable at $a$, because with $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_{ij} = \frac{\partial f_i}{\partial x_j}(a)$$

it may not be true that

$$\lim_{h\to 0} \frac{\|f(x_0+h) - f(x_0) - T(h)\|}{\|h\|} = 0$$

## 15.4 Gradient notation

For $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ (note $m = 1$!), $a \in U$, and $f$ diffable at $a$, then $(Df)_a$ is a $1 \times n$ matrix

$$(Df)_a = \begin{bmatrix} \frac{\partial f}{\partial x_1}(a) & \dots & \frac{\partial f}{\partial x_n}(a) \end{bmatrix}$$

This is called the **gradient** of $f$ at $a$ and is also denoted

$$(\nabla f)_a = (Df)_a = \begin{bmatrix} \frac{\partial f}{\partial x_1}(a) & \dots & \frac{\partial f}{\partial x_n}(a) \end{bmatrix}$$

Now for general $m$, if $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is diffable at $a \in U$ then

$$(Df)_a = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \dots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \dots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \dots & \frac{\partial f_m}{\partial x_n}(a) \end{bmatrix} = \begin{bmatrix} (\nabla f_1)(a) \\ (\nabla f_2)(a) \\ \vdots \\ (\nabla f_m)(a) \end{bmatrix}$$

### 15.5 Differentiable $\iff$ all components are differentiable

**Lemma 15.1.** Let $f : U : \mathbb{R}^n \to \mathbb{R}^m$, $a \in U$. Then $f$ is diffable at $a$ **iff** each component function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is diffable at $a$ $\forall i = 1, \dots, m$.

*Proof.* $f$ is diffable at $a$ iff

$$\begin{aligned} &\lim_{h \to 0} \frac{\|f(a+h) - f(a) - T(h)\|}{\|h\|} = 0 \\ \iff &\lim_{h \to 0} \left\| \frac{f(a+h) - f(a) - T(h)}{\|h\|} \right\| = 0 \\ \iff &\lim_{h \to 0} \left\| \frac{f_i(a+h) - f_i(a) - T_i(h)}{\|h\|} \right\| = 0 \quad \forall i = 1, \dots, m \end{aligned}$$

where the last $\iff$ follows from the fact that the vector inside the outer $\|\cdot\|$ is an $\mathbb{R}^m$ vector, and any vector converges $\iff$ its components converges (shown before). $\square$

### 15.6 Linear combination is differentiable

**Proposition 15.2.** Let $f, g : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Suppose $f, g$ both diffable at $a \in U$. Let $\lambda, \mu \in \mathbb{R}$. Then $\lambda f + \mu g : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ or

$$(\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x)$$

is diffable at $a$ and

$$(D(\lambda f + \mu g))_a = \lambda (Df)_a + \mu (Dg)_a$$

*Proof.* Assignment 6 (use triangle inequality and the fact that $(Dh)_a$ is linear, then squeeze theorem). $\square$

## 16 February 7, 2018

### 16.1 Partial derivatives exist and continuous implies differentiability

**Theorem 16.1.** (Sufficient but *NOT NECESSARY* condition for differentiability).
Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $a \in U$. Suppose all $\frac{\partial f_i}{\partial x_j}$ **exists on a n'h'd of** $a$ and are **continuous at** $a$.
Then $f$ is diffable at $a$.

*Proof.* From last time $f$ is diffable $\iff$ every $f_i$ is also diffable at $a$.

Hence it's enough to prove the theorem for $m = 1$. Let

$$\begin{aligned} a &= (a_1, \dots, a_n) \\ h &= (h_1, \dots, h_n) \end{aligned}$$

Define for $j = 1, \dots, n$

$$v_j = \sum_{k=1}^{j} h_k e_k = (h_1, h_2, \dots, h_j, 0, \dots, 0)$$

Therefore $v_n = h$. We set $v_0 = \vec{0} = (0, \dots, 0)$.
Note that

$$\begin{aligned} f(a+h) - f(a) &= f(a+v_n) - f(a+v_0) && (16.1) \\ &= \sum_{k=1}^{n} [f(a+v_k) - f(a+v_{k-1})] && (16.2) \end{aligned}$$

□

Note $v_k = v_{k-1} + h_k e_k$. By hypothesis, $\frac{\partial f}{\partial x_k}$ exists in a n'h'd of $a$ so for $h$ sufficient close to 0 the function

$$\begin{aligned} \mu_k(t) &= f(a + v_{k-1} + te_k) \\ &= f(a + h_1, \dots, a_{k-1} + h_{k-1}, a_k + t, \dots, a_{k+1}, \dots, a_n) \end{aligned}$$

is a diffable function of $t$ on $[0, h_k)$ for $h_k$ sufficient small. Thus

$$\mu_k(t) = \frac{\partial f}{\partial x_k}(a + v_{k-1} + te_k)$$

We apply MVT to $\mu_k$, $\exists \epsilon_k \in (0,1)$ so $e_k h_k = (0, h_k)$ such that

$$\begin{aligned} &\mu_k'(\epsilon_k h_k)(h_k - 0) = \mu_k(h_k) - \mu_k(0) \\ \Rightarrow &h_k[\frac{\partial f}{\partial x_k}(a + v_{k-1} + \epsilon_k h_k e_k)] = f(a+v_k) - f(a+v_{k-1}) \end{aligned}$$

Hence equation 16.1 becomes

$$f(a+h) - f(a) = \sum_{k=1}^{n} h_k \cdot \frac{\partial f}{\partial x_k}(a + v_{k-1} + \epsilon_k h_k e_k) \tag{16.3}$$

For

$$(Df)_a = \begin{bmatrix} \frac{\partial f}{\partial x_1}(a) & \dots & \frac{\partial f}{\partial x_n}(a) \end{bmatrix}$$

this exists (by hypothesis). We need to show

$$\frac{\|f(a+h) - f(a) - (Df)_a(h)\|}{\|h\|} \to 0$$

as $h \to 0$.

Recall that

$$(Df)_a(h) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(a) h_k$$

(where LHS is a $1 \times n$ matrix multiplied by an $n \times 1$ vector). So from equation 16.3

$$\begin{aligned} f(a+h) - f(a) - (Df)_a(h) &= \sum_{k=1}^{n} [\frac{\partial f}{\partial x_k}(a + v_{k-1} + \epsilon_k h_k e_k) - \frac{\partial f}{\partial x_k}(a)] h_k \\ &= L \cdot h \end{aligned}$$

where $L_k$ is the stuff inside the summation and $L = (L_1, \dots, L_n)$.
Therefore we have

$$\begin{aligned} \frac{|f(a+h) - f(a) - (Df)_a(h)|}{\|h\|} &= \frac{\|L \cdot h\|}{\|h\|} \\ &\leq \frac{\|L\|\|h\|}{\|h\|} & \text{Cauchy-Schwarz} \\ &= \|L\| \end{aligned}$$

so enough to show that

$$\lim_{h \to 0} L = 0$$

(where we then apply squeeze theorem).
So we've reduced the problem to show

$$\lim_{h \to \vec{0}} L_k = 0 \quad \forall k = 1, \dots, n$$

or

$$\lim_{h \to \vec{0}} \frac{\partial f}{\partial x_k}(a + v_{k-1} + \epsilon_k h_k e_k) - \frac{\partial f}{\partial x_k}(a) = 0$$

Note that $v_{k-1} = \sum_{j=1}^{k-1} h_j e_j \to 0$ as $h \to \vec{0}$. Furthermore for $0 < \epsilon_k < 1$ we have $\epsilon_k h_k e_k \to 0$ as $h \to \vec{0}$ thus

$$a + v_{k-1} + \epsilon_k h_k e_k \to a$$

as $h \to \vec{0}$. Note that $\frac{\partial f}{\partial x_k}$ is assumed to be *continuous at a*, so $\lim_{h \to L_k} \to 0$ as desired.

### 16.2 Summary about differentiability

To check if $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is diffable at $a \in U$

1. If $f$ is **not cts at** $a$, then $f$ is **not diffable** at $a$
2. If any of $\frac{\partial f_i}{\partial x_j}$ do not exist at $a$, $f$ is **not diffable** at $a$
3. Let $(Df)_a$ be the $m \times n$ matrix whose $i, j$ entry is $\frac{\partial f_i}{\partial x_j}(a)$. Then $f$ is diffable at $a \iff$

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - T(h)\|}{\|h\|} = 0$$

4. We can avoid step 3 if we know all $\frac{\partial f_i}{\partial x_j}$ exist on a n'h'd of $a$ and are cts at $a$ (this implies $f$ is diffable at $a$ by previous theorem).

### 16.3 Differentiability and $C^1$

Let $U \subseteq \mathbb{R}^n$ open. We say $f$ is in $C^1(U)$ if all $\frac{\partial f_i}{\partial x_j}$ exist and are cts everywhere on $U$. by the previous theorem, if $f \in C^1(U)$ then $f$ is diffable at any point in $U$.
Also $C^0(U)$ implies continuous function on $U$. Note from before

$$C^1(U) \subseteq C^0(U)$$

So we have the desired property that $C^1 \Rightarrow$ diffable $\Rightarrow$ continuous . Functions in $C^1$ are sometimes called **continuously differentiable**.

**Example 16.1.** To show conditions of the theorem are sufficient but not necessary, let $n = 2$, $U \subseteq \mathbb{R}^2$

$$f(x,y) = (x^2+y^2)\sin(\frac{1}{\sqrt{x^2+y^2}})$$

for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

**Step 1** $f$ is cts on at $(0,0)$ (by squeeze).

**Step 2** Compute $f_x(0,0)$ and $f_y(0,0)$

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{h^2}{h}\sin(\frac{1}{\sqrt{h^2}}) \\ &= 0 \end{aligned}$$

by squeeze. Similarly $f_y(0,0) = 0$. Thus we have $(Df)_{(0,0)} = [0,0]$.

**Step 3** Need to check

$$\lim_{(h_1,h_2)\to(0,0)} \frac{|f((0,0)+(h_1,h_2)) - f(0,0) - (Df)_{(0,0)}((h_1,h_2))|}{\sqrt{h_1^2+h_2^2}} = 0$$

Thus we have

$$\begin{aligned} &\lim_{(h_1,h_2)\to(0,0)} \frac{(h_1^2+h_2^2)\sin(\frac{1}{\sqrt{h_1^2+h_2^2}})}{\sqrt{h_1^2+h_2^2}} \\ =&\lim_{(h_1,h_2)\to(0,0)} \sqrt{h_1^2+h_2^2}\sin(\frac{1}{\sqrt{h_1^2+h_2^2}}) \\ =&0 \end{aligned}$$

by squeeze.

So $f$ is diffable at $(0,0)$.

**Follow-up:** we show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ (which exists everywhere) are not necessarily continuous at $(0,0)$ (to show that our previous conditions are sufficient but not necessary)

$$f(x,y) = (x^2+y^2)\sin(\frac{1}{\sqrt{x^2+y^2}})$$

Recall that $f_x(0,0) = f_y(0,0) = 0$. So at a point $(x,y) \neq (0,0)$

$$\begin{aligned} f_x &= 2x\sin(\frac{1}{\sqrt{x^2+y^2}}) + (x^2+y^2)\cos(\frac{1}{\sqrt{x^2+y^2}}) \cdot (\frac{-1}{2})(x^2+y^2)^{\frac{-3}{2}} \cdot 2x \\ &= 2x\sin(\frac{1}{\sqrt{x^2+y^2}}) - \frac{x}{\sqrt{x^2+y^2}}\cos(\frac{1}{\sqrt{x^2+y^2}}) \end{aligned}$$

We want to check if

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0) = 0$$

and similarly for $f_y$. Note the first term $\to 0$ by squeeze. We thus want to show (to show it's not continuous)

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}\cos(\frac{1}{\sqrt{x^2+y^2}}) \quad \text{DNE}$$

**Remark 16.1.** One can imagine approaching 0 from the y-axis (fix $x = 0$) which obviously goes to 0, but one can also approach from the x-axis (where we have $\frac{x}{|x|}\cos(\frac{1}{|x|})$). Although $\cos(\frac{1}{|x|})$ is bounded we do not know what happens when the two terms are put together so we can't say it obviously exists.

By sequential characterization of limits

$$lim_{(x,y)\to(0,0)} h(x,y) = 0 \iff \lim_{k\to\infty} h(x_k,y_k) = 0$$

for all sequences $(x_k,y_k) \in \mathbb{R}^2$ converging to $(0,0)$.
Thus consider $(x_k,y_k) = (\frac{(-1)^k}{k\pi}, 0)$, so we have

$$\begin{aligned} h(x_k,y_k) &= \frac{(-1)^k\frac{1}{k\pi}}{\sqrt{\frac{1}{k^2\pi^2}}}\cos(\frac{1}{\sqrt{\frac{1}{k^2\pi^2}}}) \\ &= (-1)^k\cos(k\pi) \\ &= 1 \quad \forall k \end{aligned}$$

Similarly when $(x_k,y_k) = (\frac{(-1)^{k+1}}{k\pi}, 0)$, we have the limit approaching to $-1$. Since they have different limits, then the limit DNE so $f_x$ is not cts at $(0,0)$.

**Upshot:** We have

$$\text{cts} \supset \text{diffable} \supset C^1$$

where the right set inequality where the condition that $\frac{\partial f_i}{\partial x_j}$ exists and cts is not necessary.

# 17 February 9, 2018

## 17.1 Product rule for differentiability

**Proposition 17.1.** Let $U \subseteq \mathbb{R}^n$, $f, g : U \to \mathbb{R}^m$, $a \in U$.
Suppose $f, g$ are both differentiable at $a$. Then we claim $f \cdot g : U \to \mathbb{R}$, where $(f \cdot g)(x) = f(x) \cdot g(x)$ is diffable at $a$ and

$$D(f \cdot g)_a = f(a)^T (Dg)_a + g(a)^T (Df)_a \tag{17.1}$$

where we have $1 \times n$ matrix on the LHS and $1 \times m$, $m \times n$, $1 \times m$, and $m \times n$ matrices on the right.

**Remark 17.1.** Let $h = f \cdot g$ so $h = \sum_{k=1}^{m} f_k g_k$. If $h$ is diffable at $a$, its derivative $(Dh)_g$ would be

$$(Dh)_a = \begin{bmatrix} \frac{\partial h}{\partial x_1}(a) & \dots & \frac{\partial h}{\partial x_n}(a) \end{bmatrix}$$

But

$$\frac{\partial h}{\partial x_i} = \frac{\partial}{\partial x_i}(\sum_k f_k g_k) = \sum_k \frac{\partial f_k}{\partial x_i} g_k + f_k \frac{\partial g_k}{\partial x_i}$$

So the above two equations are just equation 17.1 in components.

*Proof.* We need to prove that

$$\lim_{t \to 0} \frac{\|f(x_0 + t) - f(x_0) - T(t)\|}{\|t\|} = 0 \quad t \in \mathbb{R}$$

Note that

$$h(a+t) - h(a) - (Dh)_a(t) = (f \cdot g)(a+t) - (f \cdot g)(a) - f(a)^T (Dy)_a(t) - g(a)^T (Df)_a(t)$$

(so we assume the product rule and show it implies differentiability since our theorem is an $\iff$ ). Note the above can be rewritten as

$$\begin{aligned}
&= (f(a+t) - f(a) - (Df)_a(t)) \cdot g(a+t) && \text{name this } t_1 \\
&= f(a) \cdot (g(a+t) - g(a) - (Dg)_a(t)) && \text{name this } t_2 \\
&= (Df)_a(t) \cdot (g(a+t) - g(a)) && \text{name this } t_3
\end{aligned}$$

By triangle inequality we have

$$|h(a+t) - h(a) - (Dh)_a(t)| \le |t_1| + |t_2| + |t_3|$$

We thus show that as $t \to 0$, then $frac|T_i|\|t\| \to 0$ for $i = 1, 2, 3$.
$f, g$ diffable at $a$ so $f, g$ are continuous at $a$. Therefore we have

$$\begin{aligned}
\frac{|T_1(t)|}{\|t\|} &\le \frac{\|f(a+t) - f(a) - (Df)_a(t)\|}{\|t\|} \cdot \|g(a+t)\| \\
\frac{|T_2(t)|}{\|t\|} &\le \|f(a)\| \cdot \frac{\|g(a+t) - g(a) - (Dg)_a(t)\|}{\|t\|} \\
\frac{|T_3(t)|}{\|t\|} &\le \frac{\|(Df)_a(t)\| \|g(a+t) - g(a)\|}{\|t\|} \\
&\le \frac{\|(Df)_a\|_{op} \|t\|}{\|t\|} \cdot \|g(a+t) - g(a)\|
\end{aligned}$$

where the inequalities are from Cauchy-Schwarz. These all $\to 0$ as $\|t\| \to 0$ (since they are all products of existing limits). $\square$

**Special case when $m = 1$:** We have $f, g : U \subseteq \mathbb{R}^n \to \mathbb{R}$ and $f \cdot g = fg$. Then

$$D(fg)_a^T = \nabla(fg)(a) = f(a) \cdot (\nabla g)(a) + g(a) \cdot (\nabla f)(a)$$

Informally, $\nabla(fg) = f \nabla g + g \nabla f$.

### 17.2 Chain rule

**Theorem 17.1.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be diffable at $a \in U$. Let $g : V \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be diffable at $b = f(a) \in V$. Assume $f(U) \subseteq V$.
Then $g \circ f : U \subseteq \mathbb{R}^n \to \mathbb{R}^p$ is diffable at $a$ and

$$D(g \circ f)_a = (Dg)_{f(a)}(Df)_a$$

where we have matrices of size $p \times n$ on the left and $p \times m$ and $m \times n$ on the right (note that the linear map is a composition of linear maps: that is the derivative of a composition is the composition of the derivatives).

*Proof.* Let $Q_1(h) = f(a+h) - f(a) - (Df)_a(h)$ (defined for $h$ small). Similarly, let $Q_2(k) = g(b+k) - g(b) - (Dg)_b(k)$ ($k$ small). By hypothesis we have

$$\lim_{h\to 0} \frac{\|Q_1(h)\|}{\|h\|} = 0$$
$$\lim_{k\to 0} \frac{\|Q_2(k)\|}{\|k\|} = 0$$

For $k$ small, set $k = f(a+h) - f(a) = f(a+h) - b$ (small by continuity). So we have

$$\begin{aligned}
g(f(a+h)) - g(f(a)) &= g(b+k) - g(b) \\
&= (Dg)_b(k) + Q_2(k) \\
&= (Dg)_b(f(a+h) - f(a)) + Q_2(k) \\
&= (Dg)_b((Df)_a(h) + Q_1(h)) + Q_2(k) \\
&= (Dg)_b((Df)_a(h)) + (Dg)_b(Q_1(h)) + Q_2(k) && \text{linearity}
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\frac{\|g(f(a+h)) - g(f(a)) - (Dg)_{f(a)}(Df)_a(h)\|}{\|h\|} \\
=&\frac{(Dg)_{f(a)}(Q_1(h)) + Q_2(k)}{\|h\|} \\
\leq&\|(Dg)_b\|_{op}\frac{\|Q_1(h)\|}{\|h\|} + \frac{\|Q_2(k)\|}{\|h\|} && \text{triangle inequality and op norm}
\end{aligned}$$

where the left term $\to 0$ as $h \to 0$ by hypothesis. We want o prove that

$$\lim_{h\to 0} \frac{\|Q_2(k)\|}{\|h\|} = 0$$

to finish the proof.
Let $\epsilon_1 > 0$ be arbitrary, since $\lim_{h\to 0} \frac{\|Q_1(h)\|}{\|h\|} = 0$. Then $\exists \delta_1 > 0$ such that

$$0 < \|h\| < \delta_1 \Rightarrow \frac{\|Q_1(h)\|}{\|h\|} < \epsilon_1$$

by definitions of limits. Thus $\|Q_1(h)\| \leq \epsilon_1 \|h\|$ $\forall h$ where $0 < \|h\| < \delta_1$.

Since $\lim_{k\to 0} \frac{\|Q_2(k)\|}{\|k\|} = 0$ for any arbitrary $\epsilon_2 > 0$, then $\exists \delta_2 > 0$ such that

$$0 < \|k\| < \delta_2 \Rightarrow \frac{\|Q_1(k)\|}{\|k\|} < \epsilon_2$$

We claim that

$$\|h\| < \delta_1 \Rightarrow \|Q_2(k)\| \leq \epsilon_2\|k\| \leq \epsilon_2\|h\|$$

Note that

$$\begin{aligned}
\|k\| &= normf(a+h) - f(a) \\
&= \|(Df)_a(h) + Q_1(h)\| \\
&\leq \|(Df)_a\|_{op}\|h\| + \|Q_1(h)\| && \text{triangle and op norm} \\
&(\|(Df)_a\|_{op} + \epsilon_1)\|h\|
\end{aligned}$$

for $\|h\| < \delta_1$. So $\exists C > 0$ such that $\|k\| \leq C\|h\|$ for $\|h\| < \delta_1$ thus our claim holds.
From our claim, we have

$$\frac{\|Q_2(k)\|}{\|h\|} \leq \epsilon_2 C$$

and since $\epsilon_2 > 0$ is arbitrary we have

$$\lim_{h\to 0} \frac{\|Q_2(k)\|}{\|h\|} = 0$$

□