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STAT 433/833 Course Notes

Stochastic Processes

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. These notes are my interpretation and transcription of the content covered in lectures. The instructor has not verified or confirmed the accuracy of these notes, and any discrepancies, misunderstandings, typos, etc. as these notes relate to course's content is not the responsibility of the instructor. If you spot any errors or would like to contribute, please contact me directly.

1 September 6, 2018

1.1 Example 1.2 solution

Use the definition of the Markov property to show that

$$\begin{aligned}&P(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_{n-k+1} = x_{n-k+1}, X_{n-k-1} = x_{n-k-1}, \ldots, X_0 = x_0)\\ &\qquad = P(X_{n+1} = x_{n+1} \mid X_n = x_n), \quad k = 1, 2, \ldots, n\end{aligned}$$

(i.e. we are missing one past observation).

Solution. Applying the definition of conditional probability, our expression is equivalent to

$$\frac{P(X_{n+1} = x_{n+1}, X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_{n-k+1} = x_{n-k+1}, X_{n-k-1} = x_{n-k-1}, \ldots, X_0 = x_0)}{P(X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_{n-k+1} = x_{n-k+1}, X_{n-k-1} = x_{n-k-1}, \ldots, X_0 = x_0)} = \frac{N}{D}$$

By the law of total probability

$$\begin{aligned}N &= \sum_{x_{n-k} \in S} P(X_{n+1} = x_{n+1}, \ldots, X_{n-k} = x_{n-k}, \ldots, X_0 = x_0)\\ &= \sum_{x_{n-k} \in S} P(X_{n+1} = x_{n+1} \mid X_n = x_n, \ldots, X_{n-k} = x_{n-k}, \ldots, X_0 = x_0) \times P(X_n = x_n, \ldots, X_{n-k} = x_{n-k}, \ldots, X_0 = x_0)\end{aligned}$$

By the Markov property

$$\begin{aligned}&= P(X_{n+1} = x_{n+1} \mid X_n = x_n) \sum_{x_{n-k} \in S} P(X_n = x_n, \ldots, X_{n-k} = x_{n-k}, \ldots, X_0 = x_0)\\ &= P(X_{n+1} = x_{n+1} \mid X_n = x_n) P(X_n = x_n, \ldots, X_{n-k} \in S, \ldots, X_0 = x_0)\end{aligned}$$

Since $X_{n-k} \in S$ is an event with probability 1

$$\begin{aligned}&= P(X_{n+1} = x_{n+1} \mid X_n = x_n) P(X_n = x_n, \ldots, X_{n-k+1} = x_{n-k+1}, X_{n-k-1} = x_{n-k-1}, \ldots, X_0 = x_0)\\ &= P(X_{n+1} = x_{n+1} \mid X_n = x_n) \cdot D\end{aligned}$$

The result follow.

2 September 11, 2018

2.1 Section 1.2: Transitivity of communication relation

Prove that if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$ (and thus the communication relation "$\leftrightarrow$" is an equivalence relation).

Proof. $\exists n, m \in \mathbb{N}$ such that $P_{i,j}^{(n)} > 0$ and $P_{j,k}^{(m)} > 0$.
Note that
$$P_{i,k}^{(n+m)} = \sum_{l \in S} P_{i,l}^{(n)} P_{l,k}^{(m)} \geq P_{i,j}^{(n)} P_{j,k}^{(m)} > 0$$
Similarly we can show $k \to i$, thus $i \leftrightarrow k$. □

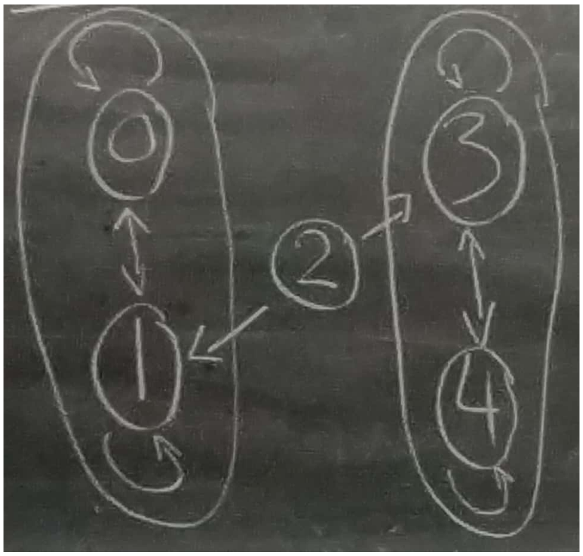
2.2 Example 1.3 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 0.1 & 0.9 \end{bmatrix} \end{array}$$

Use a state transition diagram to determine the equivalence classes.

Solution. We draw the following state transition diagram and note that there are three communication classes: $\{0, 1\}, \{2\}, \{3, 4\}$.



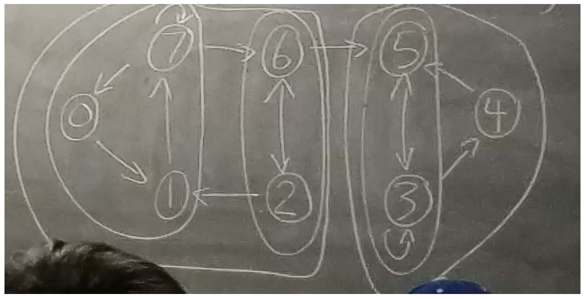
2.3 Example 1.4 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0.3 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

Use a state transition diagram to determine the equivalence classes.

Solution. We draw the following state transition diagram and note that there are two communication classes: $\{0, 1, 2, 6, 7\}, \{3, 4, 5\}$.



2.4 Example 1.5 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0.3 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

Use sample paths to prove that all states within the communication classes found in Example 1.4 communicate.

Solution. class $\{3, 4, 5\}$ Note that $P_{3,4}P_{4,5}P_{5,3} > 0$ i.e. the sample path $3 \to 4 \to 5 \to 3$ has positive probability, thus states 3,4, and 5 communicate since for any pair of states $i, j \in \{3, 4, 5\}$, $\exists n_{i,j} \leq 3$ such that $P_{i,j}^{(n_{i,j})} > 0$.

class $\{0, 1, 2, 6, 7\}$ We have sample path $0 \to 1 \to 7 \to 6 \to 2 \to 1 \to 7 \to 0$ with positive probability.

By a similar argument as above the five states communicate.

2.5 Theorem 1.1 proof: periodicity is a class property

Theorem 2.1. If $i \leftrightarrow j$ then $d(i) = d(j)$ (equal periods).

Proof. Since $i \leftrightarrow j$, then $\exists n, m \in \mathbb{N}$ such that $P_{i,j}^{(n)} > 0$ and $P_{j,i}^{(m)} > 0$.
$\forall L \in \mathbb{Z}^+$ s.t. $P_{j,j}^{(L)} > 0$, we have

$$\begin{aligned} P_{i,i}^{(m+n+L)} &= \sum_{k \in S} P_{i,k}^{(n)} P_{k,i}^{(m+L)} \\ &= \sum_{k \in S} \sum_{l \in S} P_{i,k}^{(n)} P_{k,l}^{(L)} P_{l,i}^{(m)} \\ &\geq P_{i,j}^{(n)} P_{j,j}^{(L)} P_{j,i}^{(m)} \\ &> 0 \end{aligned}$$

Thus $d(i)$ divides $n+m+L$.
Note that $P_{i,i}^{(n+m)} = \sum_{k \in S} P_{i,k}^{(n)} P_{k,i}^{(m)} \geq P_{i,j}^{(n)} P_{j,i}^{(m)} > 0$, thus $d(i)$ divides $n+m$.
Therefore $d(i)$ divides $(n+m+L)-(n+m) = L$ $\forall L$ s.t. $P_{j,j}^{(L)} > 0$, thus $d(i)$ divides $gcd\{L \in \mathbb{Z}^+ \mid P_{j,j}^{(L)} > 0\} = d(j)$.
Similarly, $d(j)$ divides $d(i)$, thus $d(i) = d(j)$. □

3 September 13, 2018

3.1 Example 1.6 solution

Given the DTMC with TPM

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} \end{array}$$

Show that $d(i) = 1$ despite the fact that $P_{i,i}^{(1)} = P_{i,i} = 0$ for $i = 0, 1, 2$.

Solution. Consider state 0 where we have

$$\begin{aligned}
P_{0,0}^{(1)} &= 0 \\
P_{0,0}^{(2)} &= \sum_{k \in S} P_{0,k} P_{k,0} \geq P_{0,1} P_{0,1} = \frac{1}{4} > 0 \\
P_{0,0}^{(3)} &= \sum_{k \in S} P_{0,k} P_{k,l} P_{l,0} \geq P_{0,1} P_{1,2} P_{2,0} = \frac{1}{8} > 0
\end{aligned}$$

Therefore $d(0) = \gcd\{2, 3, \ldots\} = 1$.

Since the sample path $0 \to 1 \to 2 \to 0$ has positive prob., all of the states communicate and the DTMC is irreducible, thus $d(2) = d(1) = d(0) = 1$ as well.

3.2 Theorem 1.2 proof: transience/recurrence are class properties

Theorem 3.1. Transience and recurrence are class properties i.e. if $i \leftrightarrow j$ and i is recurrent, then j is recurrent.

Proof. It clearly holds if $i = j$, so assume $i \neq j$. $i \leftrightarrow j$ so $\exists m, n \in \mathbb{Z}^+$ s.t. $P_{j,i}^{(m)} > 0$ and $P_{i,j}^{(n)} > 0$.
Note that

$$\begin{aligned}
\sum_{n=1}^{\infty} P_{j,j}^{(n)} &\geq \sum_{l=m+n+1}^{\infty} P_{j,j}^{(l)} \\
&\geq \sum_{l=m+n+1}^{\infty} P_{j,i}^{(m)} P_{i,i}^{(l-m-n)} P_{i,j}^{(n)} \\
&= P_{j,i}^{(m)} P_{i,j}^{(n)} \sum_{l=m+n+1}^{\infty} P_{i,i}^{(l-m-n)} \\
&= P_{j,i}^{(m)} P_{i,j}^{(n)} \sum_{L=1}^{\infty} P_{i,i}^{(L)} \\
&= \infty
\end{aligned}$$

since i is recurrent thus $\sum_{L=1}^{\infty} P_{i,i}^{(L)} = \infty$, thus state j is recurrent.
Transience is proven similarly. □

3.3 Theorem 1.3 proof: recurrent classes with states i, j imply $f_{i,j} = 1$

Theorem 3.2. If $i \leftrightarrow j$ and state i is recurrent, then

$$f_{i,j} = P(\text{DTMC ever makes future visit to state } j \mid X_0 = i) = 1$$

Proof. If $i = j$, then result follows by definition of recurrence.
Let $i \neq j$. Since $i \leftrightarrow j$, then $\exists n \in \mathbb{Z}^+$ s.t. $P_{j,i}^{(n)} > 0$.
State j is recurrent by theorem 1.2 so $f_{j,j} = 1$.
Assume that $f_{i,j} < 1$ for a contradiction.

Method 1 Note that

$$\begin{aligned} f_{j,j} &= P(\text{DTMC ever makes future visit to j} \mid X_0 = j) \\ &= 1 - P(\text{never visits j} \mid X_0 = j) \\ &\leq 1 - P_{j,i}^{(n)}(1 - f_{i,j}) && P(\text{never visits j} \mid X_0 = j) \geq P_{j,i}^{(n)}(1 - f_{i,j}) \\ &< 1 \end{aligned}$$

which is a contradiction, so $f_{i,j} < 1$.

Method 2 Note that

$$\begin{aligned} &\{X_n = i, \text{never visits j after i}\} \subseteq \{\text{never returns to state j}\} \\ \Rightarrow &P(X_n = i, \text{never visits j after i } \mid X_0 = j) \leq P(\text{never returns to j} \mid X_0 = j) \\ \Rightarrow &P_{j,i}^{(n)}(1 - f_{i,j}) \leq 1 - f_{j,j} \\ \Rightarrow &P_{j,i}^{(n)}(1 - f_{i,j}) \leq 0 \end{aligned}$$

which is a contradiction since $P_{j,i}^{(n)} > 0$ and $1 - f_{i,j} > 0$, so we must have $f_{i,j} = 1$.

□

3.4 Theorem 1.4 proof

Theorem 3.3. If state i is recurrent and state i does not communicate with state j, then $P_{i,j} = 0$.

Proof. Assume $i \neq j$. State i is recurrent so $f_{i,i} = 1$.
Assume that $P_{i,j} > 0$ for a contradiction so $i \to j$. Since i and j don't communicate and $i \to j$, then i is not accessible from j ($j \not\to i$).

Method 1 Note that

$$\begin{aligned} f_{i,i} &= P(\text{DTMC ever makes future visit to i} \mid X_0 = i) \\ &= 1 - P(\text{never visits i} \mid X_0 = i) \\ &\leq 1 - P_{i,j} && P(\text{never visits i} \mid X_0 = i) \geq P_{i,j} \text{ since } j \not\to i \\ &< 1 \end{aligned}$$

which is a contradiction so $P_{i,j} = 0$.

Method 2 Note that

$$
\begin{aligned}
&\{X_1 = j, \text{never returns to i after j}\} \subseteq \{\text{never returns to state i}\} \\
\Rightarrow &P(X_1 = j, \text{never visits i after j } \mid X_0 = i) \leq P(\text{never return to i} \mid X_0 = i) \\
\Rightarrow &P_{i,j} \leq 1 - f_{i,i}
\end{aligned}
$$

where the last line follows since i is not accessible from j.

Since $f_{i,i} = 1$, we have $P_{i,j} \leq 0$ which is a contradiction, thus $P_{i,j} = 0$.

□

4 September 18, 2018

4.1 Example 1.7 solution

Consider the DTMC with one-step transition probabilities

$$
\begin{aligned}
P_{1,j} &= \frac{1}{2^j} \quad j = 2^n \quad n \in Z^+ \\
P_{i,i-1} &= 1 \quad i = 2, 3, 4, \ldots
\end{aligned}
$$

Show that all states are null recurrent and check that a stationary distribution does not exist.

Solution. It is clear that evey state communicates an the DTMC is irreducible. By Theorem 1.5, we only need to check one state for null reucrrence.
For state 1, note that

$$
\begin{aligned}
f_{1,1} = \sum_{n=1}^{\infty} f_{1,1}^{(n)} &= \sum_{n=1}^{\infty} P_{1,n} \\
&= \sum_{m=1}^{\infty} P_{1,2^m} \\
&= \sum_{m=1}^{\infty} \frac{1}{2^m} \\
&= \frac{\frac{1}{2}}{1 - \frac{1}{2}} \\
&= 1
\end{aligned}
$$

So state 1 is indeed recurrent. To show it is null recurrent, we look at its mean recurrent time m_1

$$\begin{aligned}
m_1 &= \sum_{n=1}^{\infty} n f_{1,1}^{(n)} \\
&= \sum_{n=1}^{\infty} n P_{1,n} \\
&= \sum_{m=1}^{\infty} 2^m P_{1,2^m} \\
&= \sum_{m=1}^{\infty} 2^m \frac{1}{2^m} \\
&= \sum_{m=1}^{\infty} 1 \\
&= \infty
\end{aligned}$$

State 1 and hence the entire DTMC is null recurrent.
Does a stationary distribution exist? We observe $p = pP$ where $p = (p_1, p_2, \ldots)$ by vector-matrix multiplication

$$\begin{aligned}
p_1 &= p_2 \\
p_2 &= \frac{1}{2} p_1 + p_3 \\
&\vdots \\
p_{2^m} &= \frac{1}{2^m} p_1 + p_{2^m+1} \qquad m \in \mathbb{Z}^+
\end{aligned}$$

Also note that

$$p_i = p_{i+1} \qquad i \neq 2^m \text{ for some } m \in \mathbb{Z}^+$$

thus we have

$$p_{2^m+1} = p_{2^m+2} = \ldots = p_{2^{m+1}-2} = p_{2^{m+1}-1} = p_{2^{m+1}}$$

So our p vector is now

$$\begin{aligned}
p &= (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, \ldots) \\
&= (p_1, p_2, p_4, p_4, p_8, p_8, p_8, p_8, p_{16}, \ldots)
\end{aligned}$$

If we expand out our p_{2^m}

$$\begin{aligned}
p_{2^m} &= \frac{1}{2^m}p_1 + p_{2^m+1} \\
&= \frac{1}{2^m}p_1 + \frac{1}{2^{m+1}}p_1 + p_{2^{m+2}} \\
&= p_1 \sum_{l=m}^{\infty} \big(\frac{1}{2}\big)^l \\
&= p_1 \big(\frac{1}{2}\big)^m \sum_{n=0}^{\infty} \big(\frac{1}{2}\big)^n \\
&= p_1 \big(\frac{1}{2}\big)^{m-1}
\end{aligned}$$

We need $pe' = \sum_{n=1}^{\infty} p_n = 1$. Note that

$$\begin{aligned}
\sum_{n=1}^{\infty} p_n &= p_1 + \sum_{m=1}^{\infty} \sum_{l=2^{m-1}+1}^{2^m} p_l && \text{recall we have } 2^{m-1} \text{ of each } p_{2^m} \\
&= p_1 + \sum_{m=1}^{\infty} \sum_{l=2^{m-1}+1}^{2^m} p_{2^m} \\
&= p_1 + \sum_{m=1}^{\infty} 2^{m-1} \frac{1}{2^{m-1}} p_1 && 2^m - 2^{m-1} = 2^{m-1}(2-1) = 2^{m-1} \\
&= p_1 + \sum_{m=1}^{\infty} p_1 \\
&= \sum_{m=0}^{\infty} p_1
\end{aligned}$$

which is 0 if $p_1 = 0$ or ∞ if $p_1 > 0$. It can't hold that $pe' = 1$ while satisfying $p = pP$, thus a stationary distribution does not exist.

4.2 Example 1.8

Consider a DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Show that more than one stationary distribution exists.

Solution. We have two equivalence classes: $\{0, 2\}$ and $\{1\}$ and they are positive recurrent:

$$\begin{aligned}
&P(N_1 = 1 \mid X_0 = 1) = 1 \Rightarrow m_1 = 1 < \infty \\
&P(N_j = 2 \mid X_0 = j) = 2 \Rightarrow m_1 = 2 < \infty \qquad j = 0, 2
\end{aligned}$$

Consider $p = (\frac{1}{2}, 0, \frac{1}{2})$ and $q = (0, 1, 0)$.

For the former:

$$pP = (\frac{1}{2}, 0, \frac{1}{2}) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = (\frac{1}{2}, 0, \frac{1}{2}) = p$$

and

$$pe' = \frac{1}{2} + \frac{1}{2} = 1$$

Similarly for q, thus both p and q are both stationary.
In fact, any convex combination $\alpha p + (1-\alpha)q$, $\alpha \in [0,1]$ is a stationary distribution.
That is any $(\frac{\alpha}{2}, 1-\alpha, \frac{\alpha}{2})$ is stationary, thus there are infinitely many stationary distributions.

4.3 Theorem 1.7: Irreducible DTMC positive recurrent iff stationary distribution

Theorem 4.1. An irreducible DTMC is positive recurrent iff a stationary distribution exists.

Proof. Proof deferred. □

4.4 Uniqueness of stationary distributions

Theorem 4.2. Once we have theorem 1.7 we can prove uniqueness of stationary distributions i.e. the stationary distribution will not be unique if the DTMC has more than one positive recurrent equivalence class.

Proof. Consider a DTMC with two positive recurrent classes c_1, c_2. We can write the TPM as

$$P = \begin{array}{c c} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} c_1 \\ c_2 \end{array} & \left[\begin{array}{cc} P_1 & 0 \\ 0 & P_2 \end{array} \right] \end{array}$$

where P_1 and P_2 are irreducible TPMs when considered in isolation.
So if we had A DTMC $\{y_n, n \in \mathbb{N}\}$ with TPM P_1 then $\{y_n, n \in \mathbb{N}\}$ would be irreducible and positive recurrent i.e. $\exists p_1$ such that $p_1 P_1 = p_1$ and $p_1 e' = 1$.
Similarly $\exists p_2$ for P_2.
Consider

$$[\alpha p_1, (1-\alpha)p_2] = [(\alpha p_{1,1}, \ldots, \alpha p_{1,n}), ((1-\alpha)p_{2,1}, \ldots, (1-\alpha)p_{2,n})]$$

thus we have

$$\begin{aligned} pP &= [\alpha p_1, (1-\alpha)p_2] \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \\ &= [\alpha p_1 P_1, (1-\alpha)p_2 P_2] \\ &= [\alpha p_1, (1-\alpha)p_2] \\ &= p \end{aligned}$$

And note $pe' = \alpha p_1 e' + (1-\alpha)p_2 e' = \alpha + (1-\alpha) = 1$.
Thus we do not have a unique stationary distribution. □

5 September 20, 2018

5.1 Example 1.11 solution

Consider a DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{2}{3} & \frac{1}{12} \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Solve the limit TPM $\lim_{n\to\infty} P^{(n)}$. Does the limiting distribution of X_n depend on the initial distribution?

Solution. Clearly the equivalence classes are $\{0\}$ and $\{2\}$ (recurrent) and $\{1\}$ (transient).
$P_{0,0} = P_{2,2} = 1$ so states 0 and 2 are *absorbing states*.
We'd like to build up our matrix $\lim_{n\to\infty} P^{(n)}$.
Note $P^{(n)}_{0,0} = P^{(n)}_{2,2} = 1$ for all $n \in \mathbb{N}$ thus

$$\lim_{n\to\infty} P^{(n)}_{0,0} = \lim_{n\to\infty} P^{(n)}_{2,2} = 1$$

Thus

$$\lim_{n\to\infty} P^{(n)}_{0,1} = \lim_{n\to\infty} P^{(n)}_{0,2} = \lim_{n\to\infty} P^{(n)}_{2,0} = \lim_{n\to\infty} P^{(n)}_{2,1} = 0$$

Note that

$$\lim_{n\to\infty} P^{(n)}_{1,1} = \lim_{n\to\infty} (\frac{2}{3})^n = 0$$

Also

$$\begin{aligned}
P^{(n)}_{1,0} &= P(X_n = 0 \mid X_0 = 1) \\
&= \sum_{m=1}^{n} P(\text{DTMC first visits state 0 at time m} \mid X_0 = 1) \\
&= \sum_{m=1}^{n} P(X_n = 0, X_{n-1} = 0, \ldots, X_m = 0 \mid X_0 = 1) \\
&= \sum_{m=1}^{n} P(X_n = 0, X_{n-1} = 0, \ldots, X_m = 0, X_{m-1} = 1, \ldots, X_1 = 1 \mid X_0 = 1) \\
&= \sum_{m=1}^{n} P(X_n = 0 \mid X_m = 0) P(X_m = 0 \mid X_{m-1} = 1) P(X_{m-1} = 1 \mid X_{m-2} = 1) \ldots P(X_1 = 1 \mid X_0 = 1) \\
&= \sum_{m=1}^{n} 1 \cdot \frac{1}{4} \cdot (\frac{2}{3})^{m-1} \\
&= \frac{1}{4} \sum_{l=0}^{n-1} (\frac{2}{3})^l \\
&= \frac{1}{4} (\frac{1 - (\frac{2}{3})^n}{1 - \frac{2}{3}}) \\
&= \frac{3}{4} (1 - (\frac{2}{3})^n)
\end{aligned}$$

Similarly $P_{1,2}^{(n)} = \frac{1}{4}(1 - \left(\frac{2}{3}\right)^n)$.
Taking the limit of either, we get $\lim_{n\to\infty} P_{1,0}^{(n)} = \frac{3}{4}$ and $\lim_{n\to\infty} P_{1,2}^{(n)} = \frac{1}{4}$, thus we have

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

5.2 Theorem 1.8: Limiting probability of transient states

Theorem 5.1. For any state $i \in S$ and transient state $j \in S$ of a DTMC, $\lim_{n\to\infty} P_{i,j}^{(n)} = 0$.

Proof. Note that

$$\begin{aligned}
\sum_{n=1}^{\infty} P_{i,j}^{(n)} &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} f_{i,j}^{(k)} P_{j,j}^{(n-k)} \\
&= \sum_{k=1}^{\infty} f_{i,j}^{(k)} \sum_{n=k}^{\infty} P_{j,j}^{(n-k)} \\
&= \sum_{k=1}^{\infty} f_{i,j}^{(k)} \sum_{l=0}^{\infty} P_{j,j}^{(l)} \\
&= f_{i,j}(1 + \sum_{l=1}^{\infty} P_{j,j}^{(l)}) \\
&\leq 1 + \sum_{l=1}^{\infty} P_{j,j}^{(l)} && f_{i,j} \leq 1 \\
&< \infty && \text{since } j \text{ is transient } \sum_{l=1}^{\infty} P_{j,j}^{(l)} < \infty
\end{aligned}$$

Therefore $\lim_{n\to\infty} P_{j,j}^{(n)} = 0$ by the nth term test for infinite series (i.e. otherwise the sum above will be infinite). ☐

6 September 25, 2018

6.1 Example 1.12 solution

Consider the DTMC with TPM

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

which clearly has period $d = d(0) = d(1) = 2$. Confirm that the extended BLT for periodic DTMCs holds true for this DTMC.

Solution. Clearly $m_0 = m_1 = 2$. We will check the LHS and RHS of the extended BLT equation.

Note that

$$\begin{aligned}
\lim_{n\to\infty} P^{(2n)} &= \lim_{n\to 0} \Pi_{i=1}^{n} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \lim_{n\to 0} \Pi_{i=1}^{n} I_{2\times 2} \\
&= I_{2\times 2}
\end{aligned}$$

so $\lim_{n\to\infty} P_{j,j}^{(2n)} = 1 = \frac{d}{m} = 1$.

6.2 Theorem 1.10: null recurrence has limiting probability of 0

Theorem 6.1. If state i is null recurrent, then $\lim_{n\to\infty} P_{i,j}^{(n)} = 0$ for any state j.

Proof. **Case 1:** $i = j$ By the extended BLT $\lim_{n\to\infty} P_{i,i}^{(nd)} = \frac{d}{m_i} = 0$ since i is null recurrent so $m_i = \infty$.

Also $P_{i,i}^{(k)} = 0$ if k is not divisible by d, thus $\lim_{n\to\infty} P_{i,i}^{(n)} = 0$.

Case 2: $i \neq j$ $i \not\leftrightarrow j$ Since i is recurrent and it does not communicate with state j, then $P_{i,j}^{(n)} = 0$ $\forall n \in Z^+$ so the statement holds.

$i \leftrightarrow j$ Since i, j communicate, j is also null recurrent.

$$\begin{aligned}
\lim_{n\to\infty} P_{i,j}^{(n)} &= \lim_{n\to\infty} \sum_{k=1}^{n} f_{i,j}^{(k)} P_{j,j}^{(n-k)} \\
&= \lim_{n\to\infty} \sum_{k=1}^{\infty} f_{i,j}^{(k)} P_{j,j}^{(n-k)} && P_{j,j}^{(s)} = 0 \text{ if } s < 0 \\
&= \sum_{k=1}^{\infty} f_{i,j}^{(k)} (\lim_{n\to\infty} P_{j,j}^{(n-k)}) \\
&= 0 && \lim_{n\to\infty} P_{j,j}^{(n-k)} = 0 \text{ by case 1 since j is null recurrent}
\end{aligned}$$

Remark 6.1. Note that case 2(b) implies that $\lim_{n\to\infty} P_{i,j}^{(n)} = 0$ if j is null recurrent (regardless of i).

Remark 6.2. We can exchange order of limit and infinite summation by applying the dominated convergence theorem (DCT) $Y_n = P_{j,j}^{(n-k)}$ with probability $f_{i,j}^{(k)}$, $Y = 0$ and $Z = 1$.

□

6.3 Theorem 1.5: positive/null recurrence is a class property

Theorem 6.2. If $i \leftrightarrow j$ and state i is positive recurrent, then state j is also positive recurrent (null/positive recurrence is a class property).

Proof. Since $i \leftrightarrow j$, $d(i) = d(j) = d$. Since i is positive recurrent ($m_i < \infty$) by the extended LT $\lim_{n\to\infty} P_{i,i}^{(nd)} = \frac{d}{m_i} > 0$.

Since $i \leftrightarrow j$, $\exists a, b \in \mathbb{Z}^+$ such that $P_{j,i}^{(a)} > 0$ and $P_{i,j}^{(b)} > 0$, thus $P_{j,j}^{(a+b)} \geq P_{j,i}^{(a)} \cdot P_{i,j}^{(b)} > 0$, so d must divide $a + b$ i.e. $\exists k \in Z^+$ s.t. $a + b = kd$.

Let $l = n - k \to n = l + k$ thus

$$\begin{aligned}\lim_{n\to\infty} P_{j,j}^{(nd)} &= \lim_{l\to\infty} P_{j,j}^{((l+k)d)} \\ &= \lim_{l\to\infty} P_{j,j}^{(a+ld+b)} \\ &\geq \lim_{\to\infty} P_{j,i}^{(a)} P_{i,i}^{(ld)} P_{i,j}^{(b)} \\ &= P_{j,i}^{(a)} (\lim_{\to\infty} P_{i,i}^{(ld)}) P_{i,j}^{(b)} \\ &> 0\end{aligned}$$

Thus we have $\frac{d}{m_j} = \lim_{n\to\infty} P_{j,j}^{(nd)} > 0$ so $m_j < \infty$ thus j is positive recurrent. □

7 September 27, 2018

7.1 Theorem 1.6: Finite DTMCs have no null recurrent states

Theorem 7.1. In a finite-state DTMC, there can never be any null recurrent states.

Proof. Assume there exists a null recurrent state i. By theorem 1.10, $\lim_{n\to\infty} P_{i,j}^{(n)} = 0$ for all $j \in S$. We have $1 = \sum_{j\in S} P_{i,j}^{(n)}$. Take the limiting of both sides

$$\begin{aligned}1 &= \lim_{n\to\infty} \sum_{j\in S} P_{i,j}^{(n)} \\ &= \sum_{j\in S} \lim_{n\to\infty} P_{i,j}^{(n)} \\ &= \sum_{j\in S} 0 \\ &= 0\end{aligned}$$

a contradiction. □

7.2 Theorem 1.7: Irreduicle DTMC positive recurrent iff stationary distribution

Theorem 7.2. An irreducible DTMC is positive recurrent iff a stationary distribution exists.

Proof. **Forwards** $\Rightarrow$ Assume the DTMC is positive recurrent. For some state i, define $\gamma = (\gamma_0, \gamma_1, \ldots)$ where $\gamma_j = \lim_{n\to\infty} \gamma_{i,j}^{(n)}$ and

$$\begin{aligned}\gamma_{i,j}^{(n)} &= E[\frac{1}{n}\sum_{k=1}^{n} 1_{\{X_k=j\}} \mid X_0 = i] \\ &= \frac{1}{n}\sum_{k=1}^{n} P_{i,j}^{(k)}\end{aligned}$$

Suppose that $S = \mathbb{N}$ (state space). For any $m = 1, 2, \ldots$ (so up to some state m) we have

$$\begin{aligned}
\sum_{j=0}^{m} \gamma_j &= \sum_{j=0}^{m} \lim_{n\to\infty} \gamma_{i,j}^{(n)} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \sum_{j=0}^{m} P_{i,j}^{(k)} \\
&\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}(1) && \text{row sum of n-step TPM up to state m, so } \leq 1 \\
&= \lim_{n\to\infty} 1 \\
&= 1
\end{aligned}$$

So $\sum_{j=0}^{\infty} \gamma_j \leq 1$ In fact, $\sum_{j=0}^{\infty} \gamma_j = 1$:

$$\begin{aligned}
\sum_{j=0}^{\infty} \gamma_j &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \sum_{j=0}^{\infty} P_{i,j}^{(k)} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}(1) \\
&= 1
\end{aligned}$$

We want to show that $\gamma = \gamma P$. Note that the RHS is

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} P^{(k)} P &= \frac{1}{n}\sum_{k=1}^{n} P^{(k)}(I + P - I) \\
&= \frac{1}{n}\sum_{k=1}^{n} P^{(k)} + \frac{1}{n}\sum_{k=1}^{n}(P^{(k+1)} - P^{(k)}) \\
&= \frac{1}{n}\sum_{k=1}^{n} P^{(k)} + \frac{1}{n}(P^{(n+1)} - P^{(1)}) && \text{telescoping}
\end{aligned}$$

Note that the ith row of $\frac{1}{n}\sum_{k=1}^{n} P^{(k)}$ is $(\gamma_{i,0}^{(n)}, \gamma_{i,1}^{(n)}, \ldots)$. Taking the limit of the ith row of the RHS

$$\begin{aligned}
\lim_{n\to\infty} \big[\frac{1}{n}\sum_{k=1}^{n} P^{(k)} + \frac{1}{n}(P^{(n+1)} - P^{(1)})\big]_{(i,\cdot)} &= \gamma + 0 \\
&= \gamma
\end{aligned}$$

where the second line follows since every element of a TPM is $\in [0,1]$ where

$$\big|[P^{(n+1)} - P^{(1)}]_{(i,j)}\big| \leq 1 \Rightarrow \lim_{n\to\infty} \frac{1}{n}(P^{(n+1)} - P^{(1)})_{(i,\cdot)} = 0$$

Now taking the limit of the (i,l)th element of the LHS

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[P^{(k)}P\right]_{(i,l)} &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\sum_{j\in S}P_{i,j}^{(k)}P_{j,l} \\
&= \sum_{j\in S}\left[\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}P_{i,j}^{(k)}\right]P_{j,l} && \text{DCT} \\
&= \sum_{j\in S}\gamma_j P_{j,l} \\
&= [\gamma P]_l
\end{aligned}$$

where DCT is applied where $Y_n = P_{j,l}$ with probability $\frac{1}{n}\sum_{k=1}^{n}P_{i,j}^{(k)}$ i.e. $Y = P_{j,l}$ with probability γ_j, $Z = 1$.

So the jth row of the LHS converges to γP.

$\gamma = \gamma P$ and so γ satisfies the stationary condition and is a stationary distribution (actually: we let $\pi = \frac{\gamma}{\gamma e'}$ to make it a true distribution).

Backwards $\Leftarrow$ Assume there exists a stationary distribution. Assume the DTMC is null recurrent or transient.

In either case, $\lim_{n\to\infty}P_{i,j}^{(n)} = 0$ for all $j \in S$.

We have $\pi = \pi P^{(n)} \Rightarrow \pi_j = \sum_{i\in S}\pi_i P_{i,j}^{(n)}$ for all $j \in S$, for all $n \in \mathbb{N}$.

Taking the limit of both sides as $n \to \infty$

$$\begin{aligned}
\pi_j &= \lim_{n\to\infty}\sum_{j\in S}\pi_i P_{i,j}^{(n)} \\
&= \sum_{j\in S}\pi_i(\lim_{n\to\infty}P_{i,j}^{(n)}) && \text{DCT} \\
&= \sum_{j\in S}\pi_i 0 \\
&= 0 \qquad \forall j \in S
\end{aligned}$$

where the second line follows from applying DCT where $Y_n = P_{i,j}^{(n)}$ with probability π_j, where $Y = 0$ and $Z = 1$.

Thus $\pi = (0, 0, \ldots)$ which is not a distribution, thus we have a contradiction.

$\square$

8 October 2, 2018

8.1 Example 1.13 solution

Consider a DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} p & 0 & 1-p & 0 \\ 0 & r & 0 & 1-r \\ q & 0 & 1-q & 0 \\ 0 & s & 0 & 1-s \end{bmatrix} \end{array}$$

with $o < p, q, r, s < 1$ so that it has two positive recurrent communication classes $C_1 = \{0, 2\}$ and $C_2 = \{1, 3\}$. Rewrite P according to its canonical decomposition and solve for $\lim_{n\to\infty} P^n$.

Solution. We can rearrange the TPM as

$$P^* = \begin{array}{c} \\ 0 \\ 2 \\ 1 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 2 & 1 & 3 \end{array} \\ \begin{bmatrix} p & 1-p & 0 & 0 \\ q & 1-q & 0 & 0 \\ 0 & 0 & r & 1-r \\ 0 & 0 & s & 1-s \end{bmatrix} \end{array} = \begin{array}{c} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \end{array} \Rightarrow (P^*)^n = \begin{array}{c} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} P_1^n & 0 \\ 0 & P_2^n \end{bmatrix} \end{array}$$

We can find the limiting probabilities within each class in isolation.

C_1 The conditions of the BLT are satisfied so $\exists \tilde{\pi}_1$ such that $\tilde{\pi}_1 P_1 = \tilde{\pi}_1$ and $\tilde{\pi}_1 e' = 1$ ($\tilde{\pi}_1 = (\pi_0, \pi_2)$). We have

$$\begin{aligned} \pi_0 &= p\pi_0 + q\pi_2 \\ \Rightarrow \pi_2 &= \frac{1-p}{q}\pi_0 \end{aligned}$$

and also

$$\begin{aligned} 1 &= \pi_0 + \pi_2 \\ \Rightarrow \pi_0 &= \frac{q}{q+1-p}, \pi_2 = \frac{1-p}{q+1-p} \end{aligned}$$

C_2 Similarly (as above) $\pi_1 = \frac{s}{s+1-r}, \pi_3 = \frac{1-r}{s+1-r}$.

Returning to our original TPM

$$\lim_{n\to\infty} P^n = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \pi_0 & 0 & \pi_2 & 0 \\ 0 & \pi_1 & 0 & \pi_3 \\ \pi_0 & 0 & \pi_2 & 0 \\ 0 & \pi_1 & 0 & \pi_3 \end{bmatrix} \end{array} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{q}{q+1-p} & 0 & \frac{1-p}{q+1-p} & 0 \\ 0 & \frac{s}{s+1-r} & 0 & \frac{1-r}{s+1-r} \\ \frac{q}{q+1-p} & 0 & \frac{1-p}{q+1-p} & 0 \\ 0 & \frac{s}{s+1-r} & 0 & \frac{1-r}{s+1-r} \end{bmatrix} \end{array}$$

8.2 Random walk transience/recurrence

Theorem 8.1. The simple random walk is transient if $p \neq q$, and null recurrent if $p = q = \frac{1}{2}$.

Proof. **Case 1:** $p \neq q$ Without loss of generality, $p > q$. By the strong law of large numbers

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = E[X_1] = p - q > 0$$

thus we have

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \sum_{i=1}^{n} X_i \\
&= \lim_{n\to\infty} n\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \infty \cdot (p-q) \\
&= \infty && p-q>0
\end{aligned}$$

There is a last visit to state 0 (since we may go off to infinity) thus 0 is transient hence the DTMC is transient (class property).

Case 2: $p = q = \frac{1}{2}$ We want to show $\sum_{n=1}^{\infty} P_{0,0}^{(n)} = \infty$ to show that state 0 is null recurrent.

We know $P_{0,0}^{(2n+1)} = 0$, $n \in \mathbb{N}$ and

$$P_{0,0}^{(2n)} = P(\text{n steps to the right and n steps to the left})$$

This is in fact $BIN(2n, \frac{1}{2})$ thus

$$P_{0,0}^{(2n)} = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} = \frac{(2n)!}{n!n!}\left(\frac{1}{4}\right)^{n}$$

By Stirling's Formula for large n: $n! = \sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}$, thus

$$\begin{aligned}
\frac{(2n)!}{n!n!} &= \frac{\sqrt{2\pi}e^{-2n}(2n)^{2n+\frac{1}{2}}}{(\sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}})^2} \\
&= \frac{2^{2n+\frac{1}{2}}}{\sqrt{2\pi}\sqrt{n}} \\
&= \frac{4^n}{\sqrt{n\pi}}
\end{aligned}$$

Thus for large n, $P_{0,0}^{(2n)} = \frac{1}{\sqrt{n\pi}}$, therefore

$$\begin{aligned}
\sum_{n=1}^{\infty} P_{0,0}^{(n)} &= \sum_{m=1}^{\infty} P_{0,0}^{(2m)} \\
&\sim \sum_{m=1}^{\infty} \frac{1}{\sqrt{n\pi}} \\
&\geq \frac{1}{\sqrt{\pi}} \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \\
&= \infty
\end{aligned}$$

Thus state 0 is recurrent.

We consider $\pi = \pi P$ and $\pi e' = 1$ for this DTMC i.e.

$$\begin{aligned} \pi_i &= P_{i-1,i}\pi_{i-1} + P_{i+1,i}\pi_{i+1} \\ \Rightarrow \pi_i &= \frac{1}{2}\pi_{i-1} + \frac{1}{2}\pi_{i+1} \\ \Rightarrow \pi_{i+1} - \pi_i &= \pi_i - \pi_{i-1} && \forall i \in \mathbb{Z} \\ \Rightarrow \pi_i &= \pi_0 + id && d = \pi_1 - \pi_0 \end{aligned}$$

Since $\pi_i \in [0,1]$ for all i we must have $d = 0$, so $\pi_i = \pi_0$, but $\pi e' = \sum_{i=-\infty}^{\infty} \pi_i = \sum_{i=-\infty}^{\infty} \pi_0$ which is 0 if $\pi_0 = 0$ or ∞ if $\pi_0 > 0$.

Therefore there is no stationary distribution and hence state 0 is not positive recurrent.

□

9 October 4, 2018

9.1 Section 1.7.2: conditions for positive recurrent G/M/1 queue

Question 9.1. What are the conditions for a G/M/1 queue and its associated DTMC where $b_0 > 0$ (can transition to the right) and $b_0 + b_1 < 1$ (can transition to the left) to be positive recurrent?
Note under the above two conditions the DTMC is irreducible and aperiodic.

Claim. The DTMC with the above two conditions is positive recurrent iff $E[B] = \sum_{k=1}^{\infty} kb_k > 1$. That is, the expected number of potential service completions during a single interarrival time is greater than 1.
The stationary distribution $p = (p_0, p_1, \ldots)$ when it exists, satisfies $p_k = r_0^k(1 - r_0)$ for $k \in \mathbb{N}$ where $r_0 \in (0,1)$ is the solution to $r_0 = \Phi_B(r_0)$ and $\Phi_B(z) = E[z^B] = \sum_{k=0}^{\infty} z^k b_k$ is the probability generating function of B.

Proof. Recall from theorem 1.7 that a DTMC is positive recurrent iff a stationary distribution exists.
We will confirm that stationary distribution exists to prove the claim. Note that we want $p = pP$, thus

$$\begin{aligned} p_0 &= p_0(1 - b_0) + p_1(1 - b_0 - b_1) + \ldots \\ &= \sum_{i=0}^{\infty} p_i\left(1 - \sum_{j=0}^{i} b_j\right) \end{aligned}$$

Also

$$\begin{aligned} p_1 &= p_0 b_0 + p_1 b_1 + p_2 b_2 + \ldots \\ p_2 &= p_1 b_0 + p_2 b_1 + p_3 b_2 + \ldots \\ &\vdots \\ p_k &= \sum_{i=0}^{\infty} p_{k-1+i} b_i \end{aligned}$$

We also want $pe' = 1$ i.e. $\sum_{i=0}^{\infty} p_i = 1$.
Assume $p_k = r_0^k(1 - r_0)$, $k \in \mathbb{N}$ (geometric distribution), where $r_0 \in (0,1)$.
We want to check under what conditions this equation for p_k satisfies our three equations for p_0, p_k and $pe' = 1$.

Note that

$$1 = \sum_{i=0}^{\infty} p_i = \sum_{i=0}^{\infty} r_0^k (1 - r_0) = \frac{1 - r_0}{1 - r_0} = 1$$

For our p_k we have

$$\begin{aligned}
p_k &= \sum_{i=0}^{\infty} p_{k-1+i} b_i \\
\Rightarrow r_0^k (1 - r_0) &= \sum_{i=0}^{\infty} r_0^{k-1+i} (1 - r_0) b_i \\
\Rightarrow r_0 &= \sum_{i=0}^{\infty} r_0^i b_i \\
\Rightarrow r_0 &= \Phi_B(r_0)
\end{aligned}$$

since $\Phi_B(z) = E[z^B]$ is the pgf of B.
When does $r_0 = \Phi_B(r_0)$ have a solution for $r_0 \in (0, 1)$? If r_0 is a solution to $z = \Phi_B(z)$ then it is the intersection of the lines $y = z$ and $y = \Phi_B(z)$.
Properties of $\Phi_B(z) = \sum_{i=0}^{\infty} z^i b_i$:

1. $\Phi_B(z)$ is continuous
2. $$\Phi_B(0) = \sum_{i=0}^{\infty} z^i b_i \mid_{z=0} = b_0 > 0$$
3. $$\Phi_B(1) = \sum_{i=0}^{\infty} z^i b_i \mid_{z=1} = \sum_{i=0}^{\infty} b_i = 1$$
4. $$\Phi'_B(z) = \frac{d}{dz} \Big(\sum_{i=0}^{\infty} z^i b_i \Big) = \sum_{i=0}^{\infty} i z^{i-1} b_i > 0 \qquad \forall z \in (0, 1)$$
   Note that
   $$\Phi'_B(1) = \sum_{i=0}^{\infty} i 1^{i-1} b_i = E[B]$$
5. $$\Phi''_B(z) = \sum_{i=0}^{\infty} i(i-1) z^{i-2} b_i > 0 \qquad \forall z \in (0, 1)$$
   therefore $y = \Phi_B(z)$ is convex.

We have two cases from $E[B]$:

Case 1 $E[B] > 1$ We are guaranteed an intersection at $z \in (0, 1)$ since the slope $y = \Phi_B(z)$ is greater than that of $y = z$ at $z = 1$ (from $\Phi'_B(1) = E[B]$).

Case 2 $E[B] \leq 1$ There is not intersection before $z = 1$.

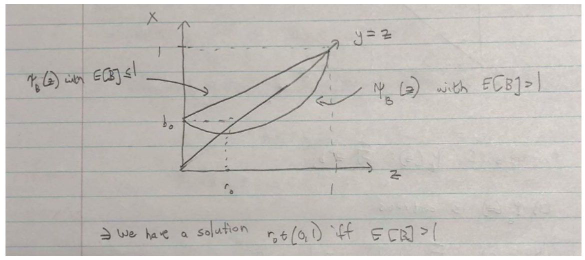


Figure 9.1: Diagram of $y = z$ and $y = \Phi_B(z)$ when $E[B] \leq 1$ and $E[B] > 1$. Note that $E[B]$ is exactly the derivative $\Phi'_B(z)$ at $z = 1$.

Thus we have a solution for $r_0 \in (0, 1)$ iff $E[B] > 1$.
We verifying that our equation for p_0 is satisfied

$$
\begin{aligned}
& p_0 = \sum_{i=0}^{\infty} p_i (1 - \sum_{j=0}^{i} b_j) \\
\iff & r_0^0 (1 - r_0) = \sum_{i=0}^{\infty} p_i - \sum_{i=0}^{\infty} \sum_{j=0}^{i} p_i b_j \\
\iff & 1 - r_0 = 1 - \sum_{j=0}^{\infty} b_j \sum_{i=j}^{\infty} r_0^i (1 - r_0) && pe' = 1 \\
\iff & 1 - r_0 = 1 - \sum_{j=0}^{\infty} b_j r_0^j && \sum_{i=j}^{\infty} r_0^i (1 - r_0) = r_0^j \text{ (geometric series)} \\
\iff & 1 - r_0 = 1 - \Phi_B(r_0) \\
\iff & 1 - r_0 = 1 - r_0
\end{aligned}
$$

So when $E[B] > 1$ $p_k = r_0^k (1 - r_0)$ for $k \in \mathbb{N}$ is a stationary distribution hence the DTMC is positive recurrent. □

10 October 11, 2018

10.1 Example 1.14 solution

Consider a DTMC with the following TPM

$$
P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{bmatrix} \end{array}
$$

where $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$. Clearly state 1 is transient and 0 and 2 are absorbing. Define

$$
T = \min\{n \in \mathbb{N} \mid X_n = 0 \text{ or } X_n = 2\}
$$

as the time until absorption (a random variable). Solve for

(a) $u = P(X_T = 0 \mid X_0 = 1)$, the absorption probability to state 0

(b) $w = E[T \mid X_0 = 1]$

(c) the distribution of $T \mid (X_0 = 1)$

Solution. We solve for these using first-step analysis.

(a) Note that

$$\begin{aligned} u = P(X_T = 0 \mid X_0 = 1) &= \sum_{i=0}^{2} P(X_T = 0 \mid X_1 = i, X_0 = 1) P(X_1 = i \mid X_0 = 1) \\ &= 1 \cdot \alpha + \beta P(X_T = 0 \mid X_1 = 1, X_0 = 0)\beta + 0\gamma \\ &= \alpha + \beta u \end{aligned}$$

So $(1-\beta)u = \alpha$ thus $u = \frac{\alpha}{1-\beta} = \frac{\alpha}{\alpha+\gamma}$.

(b) Note that

$$\begin{aligned} w = E[T \mid X_0 = 1] &= \sum_{i=0}^{2} E[T \mid X_1 = i, X_0 = 1] P(X_1 = i \mid X_0 = 1) \\ &= 1 \cdot \alpha + (1 + E[T \mid X_0 = 1])\beta + 1 \cdot \gamma \\ &= \alpha + \beta + \gamma + \beta w \\ &= 1 + \beta w \end{aligned}$$

So $w = \frac{1}{1-\beta} = \frac{1}{\alpha+\gamma}$.

(c) Note that

$$\begin{aligned} P(T = k \mid X_0 = 1) &= P(X_k \in \{0, 2\}, X_{k-1} = 1, \ldots, X_1 = 1 \mid X_0 = 1) \\ &= P(X_1 = 1 \mid X_0 = 1) P(X_2 = 1 \mid X_1 = 1) \ldots P(X_k \in \{0, 2\} \mid X_{k-1} = 1) \\ &= \beta^{k-1}(\alpha + \gamma) \\ &= \beta^{k-1}(1 - \beta) \qquad k \in \mathbb{Z}^+ \end{aligned}$$

Thus $T \sim GEO(1-\beta)$: tracking the number of trials until 1st success.

11 October 16, 2018

11.1 Example 1.15 solution

Consider a DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/2 & 1/4 & 1/4 & 0 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 & 0 \end{bmatrix} \end{array}$$

Assuming that $X_0 = 0$, calculate the expected number of transitions until the DTMC makes its first visit to state 3, the expected number of visits to state 2 prior to its visit to state 3, and the probability of visiting state 2 before state 3.

Solution. We treat state 3 as an absorbing state

$$P^* = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1/2 & 1/4 & 1/4 & 0 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The expected number of transitions until absorption into state 3 is simply $w_i = 1 + \sum_{j=0}^{M-1} P_{i,j} w_j$, and we want to solve for w_0.
We thus have

$$\begin{aligned} w_0 &= 1 + \frac{1}{2}w_0 + \frac{1}{4}w_1 + \frac{1}{4}w_2 \\ w_1 &= 1 + \frac{1}{2}w_0 + \frac{1}{4}w_2 + \frac{1}{4}w_3 \\ w_2 &= 1 + \frac{1}{2}w_0 + + \frac{1}{2}w_3 \end{aligned}$$

Solving we get

$$w_0 = \big(\frac{5}{4}\big)^2 (1 + \frac{1}{2}w_0) = \frac{25}{16} + \frac{25}{32}w_0$$

Thus $w_0 = \frac{50}{7}$.
So the expected number of transitions until the first visit to state 3 is $\frac{50}{7}$.
Note that the expected number of visits to state k from i before absorption is given by $S_{i,k} = \delta_{i,k} + \sum_{j=0}^{M-1} P_{i,j} S_{j,k}$, and we want to find $S_{0,2}$

$$\begin{aligned} S_{0,2} &= 0 + \frac{1}{2}S_{0,2} + \frac{1}{4}S_{1,2} + \frac{1}{4}S_{2,2} \\ S_{1,2} &= 0 + \frac{1}{2}S_{0,2} + \frac{1}{4}S_{2,2} \\ S_{2,2} &= 1 + \frac{1}{2}S_{0,2} \end{aligned}$$

Solving we get

$$S_{0,2} = \frac{25}{16}(1 + \frac{1}{2}S_{0,2})$$

so $S_{0,2} = \frac{10}{7}$ the expected number of visits to state 2 before visiting state 3.
Finally, the probability of visiting state 2 before 3 is $f_{0,2}$ or

$$f_{0,2} = \frac{S_{0,2} - \delta_{0,2}}{S_{2,2}} = \frac{10/7}{12/7} = \frac{5}{6}$$

We can also calculate this probability by making 2 also an absorbing state and find $U_{0,2}$ or the probability of being absorbed into state 2 starting from state 1 where

$$P^{**} = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1/2 & 1/4 & 1/4 & 0 \\ 1/2 & 0 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Note that

$$\begin{aligned}
U &= (I-Q)^{-1}R \\
&= \frac{8}{3}\begin{bmatrix} 1 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1/4 & 0 \\ 1/4 & 1/4 \end{bmatrix} \\
&= \begin{bmatrix} 5/6 & 1/6 \\ 2/3 & 1/3 \end{bmatrix}
\end{aligned}$$

Thus $U_{0,2}$ agrees with our previous answer.

12 October 18, 2018

12.1 Limiting transition probability from transient to absorbing states

Adjusted derivation for $\lim_{n\to\infty} R_K(n)$ from slides:

$$\begin{aligned}
\lim_{n\to\infty} R_k(n) &= \lim_{n\to\infty} \sum_{m=1}^{n} Q_T^{m-1} Q_K P_K^{n-m} \\
&= \lim_{n\to\infty} \sum_{m=1}^{\infty} Q_T^{m-1} Q_K P_K^{n-m} && \text{let } P_K^S = 0, S < 0 \\
&= \sum_{m=1}^{\infty} \lim_{n\to\infty} Q_T^{m-1} Q_K P_K^{n-m} && DCT \\
&= \sum_{m=1}^{\infty} \lim_{n\to\infty} Q_T^{m-1} Q_K P_K^{n} \\
&= \sum_{m=1}^{\infty} Q_T^{m-1} Q_K \big(\lim_{n\to\infty} P_K^{n}\big) \\
&= \big(\sum_{m=0}^{\infty} Q_T^{m}\big) Q_K \big(\lim_{n\to\infty} P_K^{n}\big) \\
&= (I - Q_T)^{-1} Q_K \lim_{n\to\infty} P_K^{n}
\end{aligned}$$

12.2 Example 1.16 Discrete Phase-Type Distribution (DPH) solution

Consider again a DTMC with the following TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Solution. We can rearrange the TPM as

$$P = \begin{array}{c} \\ 1 \\ 0 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 0 & 2 \end{array} \\ \begin{bmatrix} \beta & \alpha & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Note that $Q = \beta$, $q' = \alpha + \gamma = 1 - \beta$ (where $\alpha_0^* = p$). Thus for $T \sim DPH(p, \beta)$ we have

$$P(T = 0) = 1 - \alpha_0^* e' = 1 - p$$

and we have

$$P(T = k) = \alpha_0^* Q^{k-1} q' = p\beta^{k-1}(1 - \beta) \qquad k \in Z^+$$

thus T is a zero modified geometric distribution where if $p = \beta$, we have $T \sim GEO(1 - \beta)$ (tracking number of failures) which is also $T \sim GEO(1 - p) \Rightarrow DPH(p, p)$.
If $p = 1$ then $T \sim GEO(1 - \beta)$ (tracking number of trials until success is observed). This means $T \sim DPH(1, \beta)$.

12.3 Sum of two independent DPH distributions

Let $X \sim DPH_M(\alpha_0^*, Q)$ and $Y \sim DPH_N(\beta_0^*, S)$ be independent DPHs. Then $Z = X + Y$ is also a DPH with corresponding DTMC with TPM

$$P = \begin{bmatrix} Q & q'\beta_0^* & (1 - \beta_0^* e')q' \\ 0 & S & s' \\ 0 & 0 & 1 \end{bmatrix}$$

where $q' = (I - Q)^{-1}e'$ and $s' = (I - S)^{-1}e'$.
Z is essentially the time it takes for both X and Y to be absorbed.
What do these entries mean? Intuitively, suppose we start at some transient point in X with probability α_0^*.

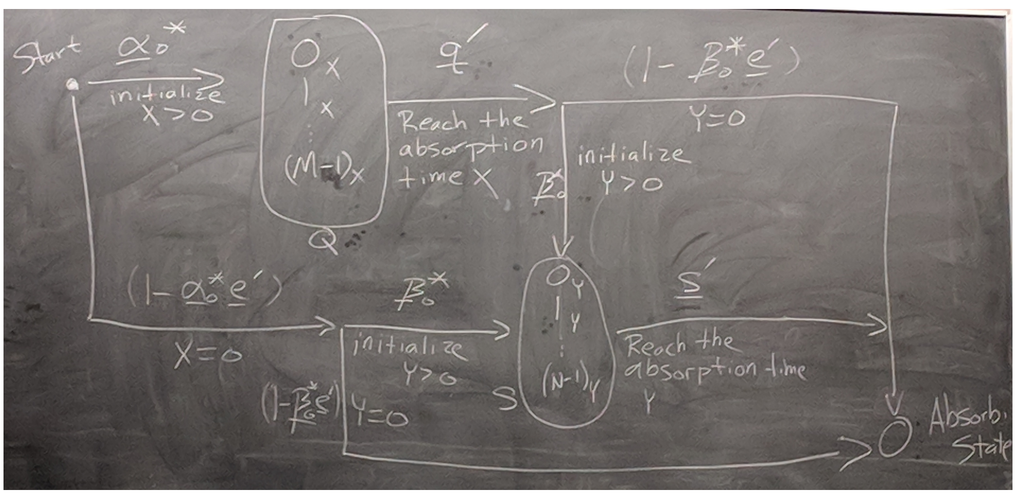


Figure 12.1: Diagram of the transitions (and their probabilities) we get for $Z = X + Y$, where we initially begin in some transient state of X with probability α_0^*.

The probability we stay in the transient states of X is Q (top-left entry). The probability we leave X is precisely the absorption probability q': we then either transition to a transient state of Y (with probability β_0^*) or to an absorption state of Y (with probability $1 - \beta_0^* e'$), hence we get our top-middle and top-right entries.
If we end up in a transient state of Y, we have S probability to stay within Y and s' probability of finally being absorbed (hence the middle-middle and middle-right entries).

13 October 23, 2018

13.1 Example 1.17 solution

Let $X, Y \sim DPH_1(\beta, \beta)$ be independent $GEO(1-\beta)$ random variables. Find the DPH representation of $Z = X + Y$. If we let $\{X_i\}_{i=1}^k$ denote a sequence of iid random variables with distribution $DPH_1(\beta, \beta)$ what is the DPH representation of $\sum_{i=1}^k X_j$?

Solution. Note that $Z = X + Y \sim DPH_2(\delta_0^*, C)$ where we have

$$\delta_0^* = [\beta, (1-\beta)\beta]$$

and

$$C = \begin{bmatrix} \beta & (1-\beta)\beta \\ 0 & \beta \end{bmatrix}$$

Since Z is the sum of iid $GEO(1-\beta)$ random variables (tracking # of failures before first success), this is the DPH representation of a $NB(2, 1-\beta)$ (tracking # of failures before two successes).
Thus $Z = \sum_{i=1}^k X_i \sim NB(k, 1-\beta) \sim DPH_k(\gamma_0^*, D)$ where

$$\gamma_0^* = [\beta, (1-\beta)\beta, (1-\beta)^2\beta, \ldots, (1-\beta)^{k-1}\beta]$$

and

$$D = \begin{bmatrix} \beta & (1-\beta)\beta & (1-\beta)^2\beta & \ldots & (1-\beta)^{k-2}\beta & (1-\beta)^{k-1}\beta \\ 0 & \beta & (1-\beta)\beta & \ldots & (1-\beta)^{k-3}\beta & (1-\beta)^{k-2}\beta \\ 0 & 0 & \beta & \ldots & (1-\beta)^{k-4}\beta & (1-\beta)^{k-3}\beta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \beta & (1-\beta)\beta \\ 0 & 0 & 0 & \ldots & 0 & \beta \end{bmatrix}$$

which is the DPH form of a $NB(k, 1-\beta)$.

13.2 Order statistics on DPH distributions

For $X \sim DPH_M(\alpha_0^*, Q)$ and $Y \sim DPH_N(\beta_0^*, S)$ be independent random variables. Consider $\min\{X, Y\}$ and $\max\{X, Y\}$.
We employ the concept of coupled DTMCs again $\{(X_n, Y_n), n \in \mathbb{N}\}$. That is $\min\{X, Y\} = \min\{n \in \mathbb{N} \mid X_n \geq M \text{ and/or } Y_n \geq N\}$.
Note that if

$$P_X = \begin{bmatrix} Q & q' \\ 0 & 1 \end{bmatrix}$$

$$P_Y = \begin{bmatrix} S & s' \\ 0 & 1 \end{bmatrix}$$

Then we have $P_X \otimes P_Y$ as

$$\begin{bmatrix} Q & q' \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} S & s' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} Q \otimes S & Q \otimes s' & q' \otimes S & q' \otimes s' \\ Q \otimes 0 & Q \otimes 1 & q' \otimes 0 & q' \otimes 1 \\ 0 & 0 & S & s' \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} Q \otimes S & Q \otimes s' & q' \otimes S & q' \otimes s' \\ 0 & Q & 0 & q' \\ 0 & 0 & S & s' \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that since we only care about the transient portion $(Q \otimes S)$ for $\min\{X, Y\}$ (since once it reaches any absorption state for either X or Y we are done), we can collapse the 3x3 matrix in the top-right into one term. First note that

$$q' = (I - Q)e' = e' - Qe' \Rightarrow Qe' = e' - q'$$

Secondly note that

$$\begin{aligned} (Q \otimes S)e' &= (Q \otimes S)(e' \otimes e') \\ &= (Qe') \otimes (Se') \\ &= (e' - q') \otimes (e' - s') \\ &= (e' \otimes e') - (e' \otimes s') - (q' \otimes e') + (q' \otimes s') \end{aligned}$$

Thus we have in the top-right entry

$$\begin{aligned} (I - Q \otimes S)e' &= e' - (e' \otimes e') - (e' \otimes s') - (q' \otimes e') + (q' \otimes s') \\ &= q' \otimes e' + e' \otimes s' - q' \otimes s' \end{aligned}$$

thus we can find the time until absorption to find $\min\{X, Y\}$ from the TPM

$$P = \begin{bmatrix} Q \otimes S & q' \otimes e' + e' \otimes s' - q' \otimes s' \\ 0 & 1 \end{bmatrix}$$

14 October 25, 2018

14.1 Example 1.18 solution

Find two different DPH representations for a rv $X \sim BIN(3, p)$.

Solution. Recall for $X \sim BIN(3, p)$ the pdf is $P(X = x) = \binom{3}{x} p^x (1-p)^{3-x}$ for $x = 0, 1, 2, 3$. Applying our known result $X \sim DPH_3(\alpha_0^*, Q)$, where

$$\alpha_0^* = (P(X = 1), P(X = 2), P(X = 3)) = (3p(1-p)^2, 3p^2(1-p), p^3)$$

and

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We can also represent $X \sim DPH_3(\beta_0^*, S)$ where

$$\beta_0^* = (P(X>0), 0, 0) = (1-(1-p)^3, 0, 0)$$

and

$$S = \begin{bmatrix} 0 & P(X>1 \mid X>0) & 0 \\ 0 & 0 & P(X>2 \mid X>1) \\ 0 & 0 & 0 \end{bmatrix}$$
$$s' = \begin{bmatrix} P(X=1 \mid X>0) \\ P(X=2 \mid X>1) \\ P(X=3 \mid X>2) = 1 \end{bmatrix}$$

Let's confirm the accuracy of the second representation

$$P(X=0) = 1 - \beta_0^* e' = 1 - P(X>0) = P(X=0)$$
$$P(X=1) = \beta_0^* s' = (P(X>0), 0, 0) \begin{bmatrix} P(X=1 \mid X>0) \\ P(X=2 \mid X>1) \\ 1 \end{bmatrix} = P(X>0)P(X=1 \mid X>0) = P(X=1)$$
$$P(X=2) = \beta_0^* S s' = (P(X>0), 0, 0) \begin{bmatrix} 0 & P(X>1 \mid X>0) & 0 \\ 0 & 0 & P(X>2 \mid X>1) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P(X=1 \mid X>0) \\ P(X=2 \mid X>1) \\ 1 \end{bmatrix} = (0, P(X>0)P$$
$$P(X=3) = \beta_0^* S^2 s' = (P(X>0), 0, 0) \begin{bmatrix} 0 & 0 & P(X>2 \mid X>0) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P(X=1 \mid X>0) \\ P(X=2 \mid X>1) \\ 1 \end{bmatrix} = (0, 0, P(X>0)P(X>2 \mid X >$$

14.2 Example 1.19 solution

Find three different DPH representations for r.v. $Z = X$ with probability p and $Z = Y$ with probability $1-p$ where $X \sim DPH_2(\alpha_0^*, Q)$ and $Y \sim DPH_3(\beta_0^*, S)$ for $0 < q \le 1$ and

$$\alpha_0^* = (1, 0), \quad Q = \begin{bmatrix} 1-q & q \\ 0 & 1-q \end{bmatrix}$$

and

$$\beta_0^* = (1, 0, 0), \quad S = \begin{bmatrix} 1-q & q & 0 \\ 0 & 1-q & q \\ 0 & 0 & 1-q \end{bmatrix}$$

Solution. Applying our known result for mixtures, we have $Z \sim DPH_5(\delta_0^*, C)$ where

$$\delta_0^* = (p, 0, 1-p, 0, 0)$$
$$C = \begin{bmatrix} Q & 0 \\ 0 & S \end{bmatrix} = \begin{bmatrix} 1-q & q & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 & 0 \\ 0 & 0 & 1-q & q & 0 \\ 0 & 0 & 0 & 1-q & q \\ 0 & 0 & 0 & 0 & 1-q \end{bmatrix}$$

Due to similar structures of these DPHs, we can find alternative representations using fewer states $Z \sim DPH_3(\gamma_0^*, D)$ where

1.

$$\gamma_0^* = ((1-p), p, 0)$$
$$D = \begin{bmatrix} 1-q & q & 0 \\ 0 & 1-q & q \\ 0 & 0 & 1-q \end{bmatrix}$$

2.

$$\gamma_0^* = (1, 0, 0)$$
$$D = \begin{bmatrix} 1-q & q & 0 \\ 0 & 1-q & q(1-p) \\ 0 & 0 & 1-q \end{bmatrix}$$

14.3 Example 1.20 solution

Calculate the pgf of a geometric random variable X with pmf $P(X = x) = (1-\beta)\beta^x$ where $x \in \mathbb{N}$, $0 < \beta < 1$ and confirm it matches the pgf obtained by applying the formula

$$\phi_T(z) = E[z^T] = 1 - \alpha_0^* e' + z\alpha_0^*(I - zQ)^{-1}(I - Q)e'$$

Solution. We calculate

$$\begin{aligned} E[z^X] &= \sum_{x=0}^{\infty} z^x (1-\beta)\beta^x \\ &= (1-\beta)\sum_{x=0}^{\infty}(z\beta)^x \\ &= (1-\beta)\frac{1}{1-z\beta} && |z\beta| < 1 \end{aligned}$$

so $|z| < \frac{1}{\beta}$. With $T \sim DPH_1(\beta, \beta)$

$$\begin{aligned} \phi_T(z) &= 1 - \alpha_0^* e' + z\alpha_0^*(I - zQ)^{-1}(I - Q)e' \\ &= 1 - \beta + z\beta(1 - z\beta)^{-1}(1-\beta) \\ &= \frac{(1-\beta)(1-z\beta) + z\beta(1-\beta)}{1-z\beta} \\ &= \frac{1-\beta}{1-z\beta} \end{aligned}$$

with $|z| < \frac{1}{\beta}$ as well.

15 October 30, 2018

15.1 Theorem 2.1: sum of two Poisson processes

Theorem 15.1. Let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ be two independent Poisson processes with intensities λ_1, λ_2, respectively. Then $N(t) = N_1(t) + N_2(t)$ is also a Poisson process with intensity $\lambda = \lambda_1 + \lambda_2$.

Proof. We confirm $N(t)$ satisfies the conditions of a Poisson process.

1. We show that $N(0) = N_1(0) + N_2(0) = 0$.

2. Independent increments: for $0 \leq t_1 < t_2 \leq t_3 < t_4$ we have

$$N(t_2) - N(t_1) = (N_1(t_2) - N_1(t_1)) + (N_2(t_2) - N_2(t_1))$$
$$N(t_4) - N(t_3) = (N_1(t_4) - N_1(t_3)) + (N_2(t_4) - N_2(t_3))$$

   Note that $N_i(t_2) - N_i(t_1)$ is independent of $N_i(t_4) - N_i(t_3)$ for $i = 1, 2$ since $N_1(t)$ is independent of $N_2(t)$, thus $N(t)$ also has independent increments.

3. Stationary increments: for $0 \leq s < t$

$$\begin{aligned} N(t) - N(s) &= (N_1(t) - N_1(s)) + (N_2(t) - N_2(s)) \\ &= POI(\lambda_1(t-s)) + POI(\lambda_2(t-s)) \\ &= POI((\lambda_1 + \lambda_2)(t-s)) \end{aligned}$$

   So $N(t)$ has stationary increments.

From 3) we see that $N(t)$ is a Poisson process with intensity $\lambda_1 + \lambda_2$. □

16 November 1, 2018

16.1 Theorem 2.1: sum of two Poisson processes (MGF method)

Theorem 16.1. $X + Y \sim POI(a + b)$ if $X \sim POI(a)$ and $Y \sim POI(b)$, independent.

Proof. Note

$$\phi_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX}]E[e^{tY}]$$

Note that

$$\phi_X(t) = E[e^{tX}] = \sum_{n=0}^{\infty} e^{tn} e^{-a} \frac{a^n}{n!} = e^{-a} \sum_{n=0}^{\infty} \frac{(e^t a)^n}{n!} = e^{a(e^t - 1)}$$

Thus

$$\phi_{X+Y}(t) = e^{(a+b)(e^t - 1)}$$

which is the MGF of $POIS(a + b)$, so my the uniqueness property our claim follows. □

16.2 Theorem 2.2: multi-type Poisson processes are independent

Theorem 16.2. For $N(t)$ a Poisson process with λ intensity, let type 1 events occur with probability p and type 2 with probability $1 - p$. Let $N_1(t)$ and $N_2(t)$ be the corresponding counting processes for each type.
We claim $N_1(t)$ and $N_2(t)$ are independent Poisson processes with intensities $p\lambda$ and $(1 - p)\lambda$.

Proof. First we show that $N_1(t)$ and $N_2(t)$ are Poisson processes with intensities $p\lambda$ and $(1 - p)\lambda$, respectively.

1. Since $N(0) = 0$ then $N_1(0) = N_2(0) = 0$.

2. Since $N(t)$ has independent increments, the number of events in interval $(t_1, t_2]$ and $(t_3, t_4]$ are independent. Also the classification process is independent of *everything*.

   Therefore the number of type 1 events inthe disjoint intervals are disjoint hence $N_1(t)$ has independent increments. Similarly for $N_2(t)$.

3. Since $N(t)$ has stationary increments, and the classification process does not change over time, then $N_1(t+h) - N_1(t) \stackrel{d}{=} N_1(h)$ and $N_1(t)$ has stationary increments. Similarly for $N_2(t)$.

Now to find their intensities: note that

$$\begin{aligned} P(N_1(h)=1) &= P(N_1(h)=1 \mid N(h)=1)P(N(h)=1) + P(N_1(h)=1 \mid N(h)\geq 2)P(N(h)\geq 2) \\ &= p(\lambda h + o(h)) + P(N_1(h)=1 \mid N(h)\geq 2)o(h) \\ &= p\lambda h + o(h) \end{aligned}$$

Also

$$P(N_1(h)\geq 2) \leq P(N(h) =\geq 2) = o(h) \Rightarrow P(N_1(h)\geq 2) = o(h)$$

Thus $N_1(t)$ is a Poisson process with intensity $p\lambda$ (and similarly for $N_2(t)$ with intensity $(1-p)\lambda$).
We now confirm $N_1(t)$ and $N_2(t)$ are independent for any $m, n \in \mathbb{N}$

$$\begin{aligned} P(N_1(t)=m, N_2(t)=n) &= P(N_1(t)=m, N_2(t)=n, N(t)=m+n) \\ &= P(N_1(t)=m, N_2(t)=n \mid N(t)=m+n)P(N(t)=m+n) \\ &= \binom{m+n}{n} p^m (1-p)^n e^{-\lambda t} \frac{\lambda^{m+n}}{(m+n)!} \\ &= e^{-p\lambda t}\frac{p^m}{m!} e^{-(1-p)\lambda t}\frac{(1-p)^n}{n!} \\ &= P(N_1(t)=m)P(N_2(t)=n) \end{aligned}$$

since $N_1(t)$ and $N_2(t)$ are Poisson processes with intensities $p\lambda$ and $(1-p)\lambda$, respectively.
Therefore they are independent. $\square$

17 November 6, 2018

17.1 Theorem 2.3: joint distribution of arrival times S_n's

Theorem 17.1. Let $\{N(t), t \geq 0\}$ be a Poisson process with intensity λ. Conditional on $N(t) = n$, the points of N in $[0, t]$ are distributed as the order statistics from a sample of size n from the uniform distribution $U(0, t)$. That is

$$(S_1, \ldots, S_n) \mid (N(t)=n) \sim (U_{(1)}, \ldots, U_{(n)})$$

Proof. Remark for any continuous random variables $X_1, \ldots, X_n$ we have

$$\begin{aligned} P(X_i \in [x_i, x_i + \Delta x_i], i = 1, \ldots, n) &= \int_{x_1}^{x_1+\Delta x_1}\int_{x_2}^{x_2+\Delta x_2}\cdots\int_{x_n}^{x_n+\Delta x_n} f_{X_1,\ldots,X_N}(x_1,\ldots,x_n)\,\mathrm{d}x_n\,\mathrm{d}x_{n-1}\ldots\mathrm{d}_1 \\ &\approx f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)\Delta x_1 \Delta x_2 \ldots \Delta x_n \end{aligned}$$

So we have

$$f_{X_1,\ldots,X_n} = \lim_{\Delta x_i \to 0} \frac{P(X_i \in [x_i, x_i+\Delta x_i], i=1,\ldots,n)}{\Delta x_1 \Delta x_2 \ldots \Delta x_n}$$

For

$$0 \leq s_1 < s_1 + \Delta s_1 < s_2 < s_2 + \Delta s_2 < s_3 < \ldots s_n < s_n + \Delta s_n \leq t$$

Thus we have

$$\begin{aligned}
&P(S_i \in [s_i, s_i + \Delta s_i], i = 1, \ldots, n \mid N(t) = n) \\
=&\frac{P(S_i \in [s_i, s_i + \Delta s_i], i = 1, \ldots, n, N(t) = n)}{P(N(t) = n)} \\
=&\frac{P(s_i < S_i \leq s_i + \Delta s_i, i = 1, \ldots, n, N(t) = n)}{P(N(t) = n)} && \text{continuous } \Rightarrow < \\
=&\frac{P(N(s_1) = 0, N(s_1 + \Delta s_1) - N(s_1) = 1, \ldots, N(s_n + \Delta s_n) - N(s_n) = 1, N(t) - N(s_n + \Delta s_n = 0))}{P(N(t) = n)} \\
=&e^{-\lambda s_1}(\lambda \Delta s_1 e^{-\lambda \Delta s_1}) e^{-\lambda(s_2 - (s_1 + \Delta s_1))} \cdot \ldots \cdot e^{-\lambda s_n}(\lambda \Delta s_n e^{-\lambda \Delta s_n}) e^{-\lambda(t - (s_n + \Delta s_n))} \cdot \big(e^{-\lambda t}\frac{(\lambda t)^n}{n!}\big)^{-1} \\
=&e^{-\lambda s_1}\big(\prod_{i=1}^{n} \lambda \Delta s_i e^{-\lambda \Delta s_i}\big) \cdot \big(\prod_{i=1}^{n-1} e^{-\lambda(s_{i+1} - (s_i + \Delta s_i))}\big) \frac{e^{-\lambda(t - (s_n + \Delta s_n))}}{e^{-\lambda t}\frac{(\lambda t)^n}{n!}} \\
=&\frac{n}{t^n} \prod_{i=1}^{n} \Delta s_i
\end{aligned}$$

So we have

$$f_{S_1, \ldots, S_n \mid N(t) = n}(s_1, \ldots, s_n) = \lim_{\Delta s_i \to 0} \frac{\frac{n!}{t^n} \prod_{i=1}^{n} \Delta s_i}{\Delta s_1 \ldots \Delta s_n} = \frac{n!}{t^n}$$

for $0 \leq s_1 < s_2 < \ldots < s_n \leq t$, which is the joint pdf of n iid $U(0, t)$ random variables. □

17.2 Example 2.1 solution

Apply Theorem 2.3 and use the joint pdf of n $U(0, t)$ order statistics to confirm the result that $P(N(s) = m \mid N(t) = n) \sim BIN(n, s/t)$.

Solution. Note that

$$\begin{aligned}
&P(N(s) = m \mid N(t) = n) \\
=&P(S_m \leq s, S_{m+1} > s \mid N(t) = n) \\
=&P(U_{(m)} \leq s, U_{(m+1)} > s) && \text{Theorem 2.3} \\
=&\int_s^t \int_{u_{m+1}}^t \cdots \int_{u_{n-2}}^t \int_{u_{n-1}}^t \int_0^s \int_0^{u_m} \cdots \int_0^{u_3} \int_0^{u_2} \frac{n!}{t^n} \, \mathrm{d}u_1 \, \mathrm{d}u_2 \ldots \mathrm{d}u_{m-1} \, \mathrm{d}u_m \, \mathrm{d}u_n \, \mathrm{d}u_{n-1} \ldots \mathrm{d}u_{m+2} \, \mathrm{d}u_{m+1}
\end{aligned}$$

Note that the following is constant and has no references to u_i

$$\begin{aligned}
&\int_0^s \int_0^{u_m} \dots \int_0^{u_3} \int_0^{u_2} \frac{n!}{t^n} \, \mathrm{d}u_1 \, \mathrm{d}u_2 \dots \mathrm{d}u_{m-1} \, \mathrm{d}u_m \\
=&\int_0^s \int_0^{u_m} \dots \int_0^{u_3} \frac{n!}{t^n} u_2 \, \mathrm{d}u_2 \dots \mathrm{d}u_{m-1} \, \mathrm{d}u_m \\
=&\int_0^s \int_0^{u_m} \dots \int_0^{u_4} \frac{n!}{t^n} \frac{u_3^2}{2!} \, \mathrm{d}u_3 \dots \mathrm{d}u_{m-1} \, \mathrm{d}u_m \\
&\vdots \\
=&\int_0^s \frac{n!}{t^n} \frac{u_m^{m-1}}{(m-1)!} \, \mathrm{d}u_m \\
=&\frac{n!}{t^n} \frac{s^m}{m!}
\end{aligned}$$

Thus we have

$$\begin{aligned}
&P(N(s) = m \mid N(t) = n) \\
=&\frac{n!}{t^n} \frac{s^m}{m!} \int_s^t \int_{u_{m+1}}^t \dots \int_{u_{n-2}}^t \int_{u_{n-1}}^t 1 \, \mathrm{d}u_n \, \mathrm{d}u_{n-1} \dots \mathrm{d}u_{m+2} \, \mathrm{d}u_{m+1} \\
=&\frac{n!}{t^n} \frac{s^m}{m!} \int_s^t \int_{u_{m+1}}^t \dots \int_{u_{n-2}}^t (t - u_{n-1}) \, \mathrm{d}u_{n-1} \dots \mathrm{d}u_{m+2} \, \mathrm{d}u_{m+1} \\
=&\frac{n!}{t^n} \frac{s^m}{m!} \int_s^t \int_{u_{m+1}}^t \dots \int_{u_{n-3}}^t t(t - u_{n-2}) - \frac{1}{2}(t^2 - u_{n-2}^2) \, \mathrm{d}u_{n-2} \dots \mathrm{d}u_{m+2} \, \mathrm{d}u_{m+1} \\
=&\frac{n!}{t^n} \frac{s^m}{m!} \int_s^t \int_{u_{m+1}}^t \dots \int_{u_{n-3}}^t \frac{1}{2!}(t - u_{n-2})^2 \, \mathrm{d}u_{n-2} \dots \mathrm{d}u_{m+2} \, \mathrm{d}u_{m+1} \\
&\vdots \\
=&\frac{n!}{t^n} \frac{s^m}{m!} \int_s^t \frac{1}{(n-m-1)!}(t - u_{m+1})^{n-m-1} \, \mathrm{d}u_{m+1} \\
=&\frac{n!}{m!(n-m)!} \frac{s^m (t-s)^{n-m}}{t^n} \\
=&\binom{n}{m} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m}
\end{aligned}$$

18 November 13, 2018

18.1 Proof: Poisson process is a birth and death process

Treating the Poisson process as a birth and death process, we can show

$$P_{i,i+k}(t) = e^{-\lambda t} \frac{(\lambda t)^k}{k!} \qquad k \in \mathbb{N}$$

That is the number of births (i.e. events) in the time interval $[0, t]$ follows a $POI(\lambda t)$ distribution.

Proof. We will prove this using induction.

Base case: $P(N(t)=0) = P_{i,i}(t) = e^{-\lambda t}$ which follows by the fact that this is a pure birth process (and which we derived using Kolmogorov's Forward Equations (KFE)).
Induction step: Assume that for some $k = 0, 1, \ldots$ we have

$$P_{i,i+k}(t) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

Applying the recursive equation derived from KFE we have

$$\begin{aligned}
P_{i,i+k+1}(t) &= \lambda_{i+k}e^{-\lambda_{i+k+1}t}\int_0^t e^{\lambda_{i+k+1}s}P_{i,i+k}(s)\,\mathrm{d}s \\
&= \lambda e^{-\lambda t}\int_0^t e^{\lambda s}\cdot e^{-\lambda s}\frac{(\lambda s)^k)}{k!}\,\mathrm{d}s \\
&= e^{-\lambda t}\frac{\lambda^{k+1}}{k!}\int_0^t s^k\,\mathrm{d}s \\
&= e^{-\lambda t}\frac{\lambda^{k+1}}{k!}\frac{t^{k+1}}{k+1} \\
&= e^{-\lambda t}\frac{(\lambda t)^{k+1}}{(k+1)!}
\end{aligned}$$

as desired. □

19 November 15, 2018

19.1 Proof uniformization DTMC is equivalent

Fact 19.1. Stochastic processes $\{X(t), t \geq 0\}$ and $\{X^*(t), t \geq 0\}$ (modified DTMC with uniformization) are probabilistically equivalent.

Proof. The time spent in state i is simply $T_i = \sum_{n=1}^{N_i} T^*_{i,n}$ where $\{T^*_{i,n}\}_{n=1}^{\infty}$ are iid with distribution $T^*_i \sim EXP(v)$ (from uniformization) and $N_i \sim (GEO(\frac{v_i}{v})$ (# of trials) where $P(N_i = n) = (1-\frac{v_i}{v})^{n-1}(\frac{v_i}{v})$, $n \in \mathbb{Z}^+$ (this is the number of transitions until we leave state i).

The mgf of T_i is

$$\begin{aligned}
E[e^{tT_i}] &= E[e^{t\sum_{n=1}^{N_i} T^*_{i,n}}] \\
&= \sum_{m=1}^{\infty} E[e^{t\sum_{n=1}^{N_i} T^*_{i,n}} \mid N_i = m]P(N_i = m) \\
&= \sum_{m=1}^{\infty} \prod_{n=1}^{N_i} E[e^{tT^*_{i,n}} \mid N(t) = m]P(N_i = m) \\
&= \sum_{m=1}^{\infty} \prod_{n=1}^{m} (\phi_{T^*_i}(t))P(N_i = m) \\
&= \sum_{m=1}^{\infty} (\phi_{T^*_1}(t))^m P(N_i = m) && T^*_i \sim T_i \text{ since uniformization} \\
&= \sum_{m=1}^{\infty} (\phi_{T^*_1}(t))^m (1 - \frac{v_i}{v})^{m-1} (\frac{v_i}{v}) \\
&= (\frac{v_i}{v})(\phi_{T^*_1}(t)) \sum_{m=1}^{\infty} \big[(\phi_{T^*_1}(t))(1 - \frac{v_i}{v})\big]^{m-1}
\end{aligned}$$

Note that

$$\begin{aligned}
\phi_{T^*_1}(t) &= \int_0^{\infty} e^{tx} v e^{-vx} \, \mathrm{d}x \\
&= v \int_0^{\infty} e^{-(v-t)x} \, \mathrm{d}x \\
&= \frac{v}{v-t}
\end{aligned}$$

and also

$$\sum_{m=1}^{\infty} \big[(\phi_{T^*_1}(t))(1 - \frac{v_i}{v})\big]^{m-1} = \frac{1}{1 - \phi_{T^*_1}(t)(1 - \frac{v_i}{v})}$$

so

$$\begin{aligned}
\phi_{T_i}(t) &= \frac{v_i}{v}\big(\frac{v}{v-t}\big)\big[1 - \big(\frac{v}{v-t}\big)\big(\frac{v - v_i}{v}\big)\big]^{-1} \\
&= \frac{v_1}{v_1 - t} \qquad t < v_1
\end{aligned}$$

so $T_i \sim EXP(v_i)$ as required.

Now we need to verify the probability of transitioning to a different state j from i are the same:

$$\begin{aligned}
&P(\text{transition to } j \text{ when } X^*(t) \text{ leaves } i)\\
=&P(X^*(t)\text{transition to } j \neq i \mid X^*(t) \text{ does not transition to } i)\\
=&\frac{P(X^*(t)\text{transition to } j \neq i)}{P(X^*(t) \text{ does not transition to } i)}\\
=&\frac{P^*_{i,j}}{1-P^*_{i,i}}\\
=&\frac{q_{i,j}/v}{1-(1-\frac{v_i}{v})}\\
=&\frac{q_{i,j}}{v_i}\\
=&P_{i,j}
\end{aligned}$$

(we need to condition on "does not transition to i" since $P^*_{i,i}$ is no longer 0 anymore).
So $X(t)$ and $X^*(t)$ are probabilistically equivalent. □

19.2 Example 3.1 solution

Consider a CTMC having infinitesimal generator

$$R = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} \end{array}$$

where $\lambda, \mu > 0$. Use uniformization to calculate $P(t)$.

Solution. Solution: let $v = \lambda + \mu \geq \max\{\lambda, \mu\}$.

$$P^* = I + \frac{1}{v}R = \begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{bmatrix}$$

note that $P^{*(n)} = P^*$ for all $n \in \mathbb{Z}^+$.
So we have

$$\begin{aligned}
P_{0,0}(t) &= \sum_{n=0}^{\infty} P^{*(n)}_{0,0} e^{-(\lambda+\mu)t} \frac{(\lambda+\mu)t)^n}{n!}\\
&= e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} \sum_{n=1}^{\infty} \frac{(\lambda+\mu)t^n}{n!}\\
&= e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}(e^{(\lambda+\mu)t} - 1)\\
&= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}
\end{aligned}$$

Similarly $P_{1,1}(t) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t}$. Now since $P_{0,1}(t) = 1 - P_{0,0}(t)$ and $P_{1,0}(t) = 1 - P_{1,1}(t)$ we have

$$P(t) = \begin{bmatrix} \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t} & \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t} \\ \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t} & \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t} \end{bmatrix}$$

20 November 20, 2018

20.1 Example 3.2 solution

Consider a CTMC whose embedded DTMC has a TPM given by

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \dots \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ q & 0 & p & 0 & \dots \\ q & 0 & 0 & p & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix} \end{array}$$

with $0 < p < 1$ and $q = 1 - p$. Show that while the embedded DTMC is always positive recurrent whether or not the corresponding CTMC is positive or null recurrent depends on $\{v_i\}_{i \in S}$.

Solution. For the embedded DTMC we have

$$f_{0,0}^{(1)} = P(\text{DTMC revists state 0 for the first time at time 1} \mid X_0 = 0) = 0$$

also for $n \geq 2$

$$\begin{aligned} f_{0,0}^{(n)} &= P_{0,1}\big(\prod_{i=1}^{n-2} P_{i,i+1}\big)P_{n-1,0} \\ &= p^{n-2}q \end{aligned}$$

So we have

$$\begin{aligned} f_{0,0} &= \sum_{n=1}^{\infty} f_{0,0}^{(n)} \\ &= 0 + \sum_{n=2}^{\infty} p^{n-2}q \\ &= q\frac{1}{1-p} \\ &= \frac{q}{q} = 1 \end{aligned}$$

Therefore the DTMC is recurrent.

Also

$$\begin{aligned}
m_0 &= \sum_{n=1}^{\infty} n f_{0,0}^{(n)} \\
&= \sum_{n=2}^{\infty} n p^{n-2} q \\
&= \sum_{l=1}^{\infty} (l+1) p^{l-1} q && l = n-1 \\
&= \sum_{l=1}^{\infty} l p^{l-1} q + \sum_{l=1}^{n} p^{l-1} q \\
&= \frac{1}{q} + 1 < \infty
\end{aligned}$$

therefore state 0 is positive recurrent thus the entire DTMC is positive recurrent.
Consider a CTMC $\{X(t), t \geq 0\}$ having this embedded DTMC, and let N_0 denote the number of transitions to return to 0 (given $X_0 = 0$) therefore $N_{0,0} = \sum_{i=1}^{N_0 - 1} T_i$ (i.e. time until returning to state 0 is just the sum of the sojourn times of the 1st transition ($i = 1$) up until the $n-1$-th transition) where $T_i \sim EXP(v_i)$ are independent. Applying the law of total expectation

$$\begin{aligned}
E[N_{0,0}] &= E[E[N_{0,0} \mid N_0]] \\
&= \sum_{n=2}^{\infty} E[N_{0,0} \mid N_0 = n] P(N_0 = n \mid X_0 = 0)
\end{aligned}$$

where

$$\begin{aligned}
E[N_{0,0} \mid N_0 = n] &= E[\sum_{i=1}^{n-1} T_i \mid N_0 = n] \\
&= E[\sum_{i=1}^{n-1} T_i] \\
&= \sum_{i=1}^{n-1} E[T_i]
\end{aligned}$$

where we can drop the condition since individual sojourn times are independent of what happens in the future i.e.

independent of N_0, so we have

$$\begin{aligned}
E[N_{0,0}] &= \sum_{n=2}^{\infty}\sum_{i=1}^{n-1} E[T_i]p^{n-2}q \\
&= \sum_{i=0}^{\infty} E[T_i] \sum_{n=i+1}^{\infty} p^{n-2}q \\
&= \sum_{i=0}^{\infty} E[T_i]p^{i-1} \sum_{m=1}^{\infty} p^{m-1}q \\
&= \sum_{i=0}^{\infty} \frac{1}{v_i}p^{i-1} & \sum_{m=1}^{\infty} p^{m-1}q = 1
\end{aligned}$$

Depending on our choices for $\{v_i\}_{i\in S}$ we can make this sum finite or infinite.
For example if $v_i = p^{i-1}$ then $E[N_{0,0}] = \sum_{i=1}^{\infty} 1 = \infty$ (null recurrent). For example if $v_i = p^{-1}$ then $E[N_{0,0}] = \sum_{i=1}^{\infty} p^i = \frac{1}{1-p} < \infty$ (positive recurrent).

20.2 Theorem 3.3: irreducible positive recurrent CTMC implies positive recurrent embedded DTMC

Theorem 20.1. If an irreducible CTMC is positive recurrent, then its corresponding embedded DTMC must also be positive recurrent.

Proof. Assume state i is positive recurrent (i.e. $E[N_{i,i}] < \infty$).
Again let N_i denote the number of transitions to return to state i for the first time and let $v \geq \max\{v_j, j \in S\}$.

$$\begin{aligned}
E[N_{i,i}] &= E[E[N_{i,i} \mid N_i]] \\
&= \sum_{n=1}^{\infty} E[N_{0,0} \mid N_0 = n]P(N_0 = n \mid X_0 = 0) \\
&\geq \sum_{n=1}^{\infty} \frac{n \cdot P(N_0 = n \mid X_0 = 0)}{v} & E[N_{0,0} \mid N_0 = n] \geq \sum_{i=1}^{n} \frac{1}{v} = \frac{n}{v} \\
&= \frac{E[N_i \mid X_0 = i]}{v} \\
&= \frac{m_i}{v}
\end{aligned}$$

therefore $m_i < \infty$ so state i is positive recurrent in the embedded DTMC. □

Corollary 20.1. Since $E[N_{i,i}] \geq m_i/v$, if state i is null recurrent in the embedded DTMC (i.e. $m_i = \infty$) then $E[N_{i,i}] \geq \infty$ or $E[N_{i,i}] = \infty$, and i must be null recurrent in the CTMC.

21 November 22, 2018

21.1 Example 3.3 solution

Recall the CTMC having infinitesimal generator

$$R = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} \end{array}$$

with $\lambda, \mu > 0$. We already showed that its transition probability function matrix is

$$P(t) = \begin{bmatrix} \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t} & \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t} \\ \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t} & \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t} \end{bmatrix}$$

Show that this CTMC meets the conditions of the BLT for CTMCs, and find stationary distribution π using both methods. Does this staionary distribution agree with $\lim_{t\to\infty} P(t)$?

Solution. Method 1: BLT for CTMCs Note that since $P_{i,j}(t) > 0$ for all $t > 0$ $i \neq j$ we now that the CTMC is irreducible.

Clearly the embedded DTMC is recurrent so the CTMC is recurrent.

We have $E[T_0] = \frac{1}{\lambda}, E[T_1] = \frac{1}{\mu}$. Note that $N_{0,0} = T_0 + T_1, N_{1,1} = T_1 + T_0$ (since we only have two states, state i can only exit state i then re-enter state i again).

We thus have $E[N_{0,0}]E[N_{1,1}] = E[T_0] + E[T_1] = \frac{1}{\lambda} + \frac{1}{\mu}$.

So

$$\pi_0 = \frac{E[T_0]}{E[N_{0,0}]} = \frac{1/\lambda}{1/\lambda + 1/\mu} = \frac{\mu}{\lambda+\mu}$$

Similarly $\pi_1 = \frac{\lambda}{\lambda+\mu}$.

Method 2: stationary equations From $0 = \pi R$ and $1 = \pi e'$ we have

$$\begin{aligned} 0 &= -\lambda\pi_0 + \mu\pi_1 \\ 0 &= \lambda\pi_0 - \mu\pi_1 \\ 1 &= \pi_0 + \pi_1 \end{aligned}$$

which gives us

$$\begin{aligned} \pi_1 &= \frac{\lambda}{\mu}\pi_0 \\ 1 &= \frac{\lambda+\mu}{\mu}\pi_0 \end{aligned}$$

so $\pi_0 = \frac{\mu}{\lambda+\mu}$ and $\pi_1 = \frac{\lambda}{\lambda+\mu}$.
Finally taking the limit as $t \to \infty$

$$\lim_{t\to\infty} P(t) = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{bmatrix} \end{array}$$

which has identical rows that equal our stationary distribution π.

22 November 27, 2018

22.1 Example 3.4 solution

Determine the conditions for a $M/M/1$ queue to be transient, positive recurrent or null recurrent (i.e. a birth and death process with $\lambda_n = \lambda$ for $n \in \mathbb{N}$ and $\mu_n = \mu$ for $n \in \mathbb{Z}^+$).

Solution. Note that for

$$\sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\mu_i}{\lambda_i} = \sum_{n=1}^{\infty}(\frac{\mu}{\lambda})^n$$

the above is $< \infty$ iff $\mu < \lambda$ and $= \infty$ otherwise. Also

$$\sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i} = \sum_{n=1}^{\infty}(\frac{\lambda}{\mu})^n$$

the above is $< \infty$ iff $\lambda < \mu$ and $= \infty$ otherwise.
Therefore it is

transient iff the first sum is $< \infty$ or $\mu < \lambda$ i.e. positive drift to infinity, births more frequent than deaths.

positive recurrent iff the second sum is $< \infty$ or $\lambda < \mu$ i.e. negative drift to 0, deaths more frequent than births.

null recurrent iff the both sums are $= \infty$ or $\lambda = \mu$

22.2 Example 3.5 solution

Suppose a $M/M/1$ queue is positive recurrent. Confirm that its stationary distribution agrees with the distribution of customers at arrival instants obtained in Exercise 1.7.2,

$$P(X_n = k) = (\frac{\lambda}{\mu})^k\big(1 - \frac{\lambda}{\mu}\big) \quad k \in \mathbb{N}$$

Solution. Recall that for

$$\begin{aligned}
p_0 &= \big(1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}\big)^{-1} \\
&= \big(\sum_{n=0}^{\infty}\prod_{i=1}^{n}\frac{\lambda}{\mu}\big)^{-1} \\
&= \big(\frac{1}{1-\lambda/\mu}\big)^{-1} \\
&= \big(\frac{\mu}{\mu-\lambda}\big)^{-1} \\
&= \big(1 - \frac{\lambda}{\mu}\big)
\end{aligned}$$

also

$$p_n = p_0\prod_{i=1}^{k}\frac{\lambda_{i-1}}{\mu_i} = \big(1 - \frac{\lambda}{\mu}\big)\big(\frac{\lambda}{\mu}\big)^k \qquad k \in \mathbb{Z}^+$$

as required.

22.3 PASTA Property for $M/M/1$ queue

The PASTA (Poisson Arrivals See Time Averages) Property states that the long-run probability a queue observes a certain state (i.e. number of customers) is simply the same as the long-run probability of that state observed by a customer arriving. To show this for an $M/M/1$ queue:
Let

$$\begin{aligned}
&A_h = \{\text{observe an arrival in the next } h \text{ time units}\}\\
&B_j = \{X(t) = j \text{ customers for large } t\}\\
&P(B_j) = \pi_j
\end{aligned}$$

Then (where $\lambda_j = \lambda$ for $j \in \mathbb{N}$)

$$\begin{aligned}
&P(X(t) = i \text{ customers in queue } \textbf{immediately} \text{ prior to an arrival for large } t)\\
=&\lim_{h\to 0} P(B_i \mid A_h)\\
=&\lim_{h\to 0} \frac{P(A_h \mid B_i)P(B_i)}{\sum_{j=0}^{\infty} P(A_h \mid B_j)P(B_j)}\\
=&\lim_{h\to 0} \frac{(\lambda_i h + o(h))\pi_i}{\sum_{j=0}^{\infty}(\lambda_j h + o(h))\pi_j}\\
=&\lim_{h\to 0} \frac{\lambda_i \pi_i + \frac{o(h)}{h}}{\sum_{j=0}^{\infty} \lambda_j \pi_j + \frac{o(h)}{h}}\\
=&\frac{\lambda_i \pi_i}{\sum_{j=0}^{\infty} \lambda_j \pi_j}\\
=&\frac{\pi_i}{\sum_{j=0}^{\infty} \pi_j} && \lambda_k = \lambda\\
=&\pi_i
\end{aligned}$$

23 November 29, 2018

23.1 Example of infitessimal generator of coupled CTMC

Suppose $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$, independent CTMCs, with infinitesimal generators

$$R_X = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} \end{array}$$

$$R_Y = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} -5 & 5 \\ 1 & -1 \end{bmatrix} \end{array}$$

we have the coupled DTMC $\{(X(t), Y(t)), t \geq 0\}$ with infinitesimal generator $R_{X,Y} = R_X \otimes I_2 + I_2 otimes R_Y$ which is

$$\begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 3 & 0 & -3 & 0 \\ 0 & 3 & 0 & -3 \end{bmatrix} + \begin{bmatrix} -5 & 5 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -5 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{cccc} (0,0) & (0,1) & (1,0) & (1,1) \end{array} \\ \begin{bmatrix} -7 & 5 & 2 & 0 \\ 1 & -3 & 0 & 2 \\ 3 & 0 & -8 & 5 \\ 0 & 3 & 1 & -4 \end{bmatrix} \end{array}$$

(note the Kronecker products and how they're distributed).