CO250 Notes

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1 May 8, 2017

1.1 IP: Discrete values

How would you represent a variable $x_C \in \{0, 2\}$ (This is not a valid LP constraint)?

Introduce a binary variable z_C to indicate whether item C is used or not

$$\begin{aligned} 0 \leq z_C \leq 1, z_C &\text{ integer} \\ x_C &= 2z_C \end{aligned}$$

Note in IP we *are allowed* to use $z_C \in \{0, 1\}$ to represent this constraint.

Similarly, for more than 2 discrete values e.g. $x \in \{A, B, C\}$, define binary variables $y_A, y_B, y_C \in \{0, 1\}$ for each value in $\{A, B, C\}$.

$$\begin{aligned} x &= Ay_A + By_B + Cy_C \\ y_A + y_B + y_C &= 1 \end{aligned}$$

2 May 10, 2017

2.1 Incident Edges Notation

To indicate all incident edges of a vertex v

$$S(v) = \{e_1, e_2, \ldots e_n\}$$

where e_i are all edges that have v as one of its vertices.

2.2 Matching Edges

A set of edges M is matching in a graph G with vertices V if and only if

$$|S(v) \cap M| \leq 1 \qquad \forall v \in V$$

3 May 12, 2017

3.1 IP is LP?

Note Integer Programming (IP) is **not** a subset of Linear Programming (LP). It however uses linear constraints with an integral restrition.

4 May 15, 2017

4.1 Non-linear Programming (NLP)

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ for all $i \in \{1, \dots, m\}$.

Non-linear programming (NLP) problem is an optimization problem of the form

$$min \qquad f(x)$$

subject to

$$\begin{gathered} g_i(x) \leq 0 \\ \vdots \\ g_m(x) \leq 0 \end{gathered}$$

Example. We have three different figures (or boundaries on 2D Cartesian plane) $F_1, F_2, F_3 \subseteq \mathbb{R}^2$ and a point $x = (x_1, x_2)^\intercal$. Determine a point from $F_1 \cup F_2 \cup F_3$ which is closest to the point x.

F_1 is a circle of radius 3 centered at $(-4, 0)^\intercal$. F_2 is a rectangle with bottom-left corner $(1, 2)^\intercal$ and top-right corner $(6, 5)^\intercal$. F_2 is a triangle with corners $(2, -1)^\intercal, (4, -5)_\intercal, (2, -5)^\intercal$.

For two points $x, y \in \mathbb{R}^2$ the distance between them is

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Define the variables

$$\begin{aligned} t &= (t_1, t_2)^\intercal \in F_1 \\ y &= (y_1, y_2)^\intercal \in F_2 \\ p &= (p_1, p_2)^\intercal \in F_3 \end{aligned}$$

z_1, z_2, z_3 indicates the figure in which the closest point to x lies

subject to

$$(t_1 + 4)^2 + t_2^2 \leq 9 \qquad (t_1, t_2)^\intercal \in F_1$$

$$\left.\begin{array}{ll} y_1 & \geq 1 \\ y_1 & \leq 6 \\ y_2 & \geq 2 \\ y_2 & \leq 5 \end{array}\right\} (y_1, y_2)^\intercal \in F_2$$

$$\left.\begin{array}{ll} p_1 & \geq 2 \\ p_2 & \geq -3 \\ p_1 + p_2 & \leq 1 \end{array}\right\} (p_1, p_2)^\intercal \in F_3$$

$$\begin{gathered} z_1, z_2, z_3 \in \{0, 1\} \\ z_1 + z_2 + z_3 = 1 \end{gathered}$$

Finally, the objective function is

$$min \qquad z_1\sqrt{(x_1-t_1)^2+(x_2-t_2)^2}+z_2\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}+z_3\sqrt{(x_1-p_1)^2+(x_2-p_2)^2}$$

Note in order for our constraints to fit our definition of NLP (i.e. $g_i(x) \le 0$, we must reformulate them

$$(t_1+4)^2+t_2^2 \le 9 \to (t_1+4)^2+t_2^2-9 \le 0$$
$$y_1 \ge 1 \to -y_1 \le -1 \to -y_1+1 \le 0$$
$$\vdots$$
$$z_1+z_2+z_3 = 1 \to z_1+z_2+z_3 \le 1 \to z_1+z_2+z_3-1 \le 0$$

5 May 17, 2017

5.1 Feasibility, Optimal, and Unbounded

Example. Given an (LP) problem denoted by (P) where (P) is a maximization (LP) problem. Does (P) have a feasible solution?

$$max \qquad 7x_1+3x_2$$

subject to

$$\begin{aligned} 2x_1 - x_2 &= 4 \\ 4x_1 - 2x^2 &= 9 \\ x_1, x_2 &\ge 0 \end{aligned}$$

This is clearly infeasible (divide 2nd equation by 2, same LHS as equation 1 but different RHS) thus (P) is infeasible. We can fix this by changing equation 2 to

$$4x_1 + 2x^2 = 8$$

Note that $x = (2,0)^\intercal$ is now a feasible solution.

Optimal solution $\bar{x}$ for *(P)* if for every feasible solution x' the value of $\bar{x}$ is not not smaller than the value of x'. Note the value of x' is the value of the objective function for (P). The complement is true for minimization problems.

Note for (P), we can describe a feasible solution as

$$x' = (2,0)^\intercal + t(1,2)^\intercal$$

for every $t \ge 0$. Note

$$\begin{aligned} x'_1 &= 2+t \\ x'_2 &= 0+2t \end{aligned}$$

So the *value* of x' is $14+13t$. There is no maximum for a line.

If for every *alpha* there is a feasible solution which has value larger than α, then (P) is **unbounded**.

5.2 Certificate of Infeasibility

Example. Let our LP problem be

$$max \qquad x_1 + 2x_2 + x_3$$

subject to

$$\begin{aligned}
&(1) x_1 - x_2 - x_3 = 1\\
&(2) 2x_1 - x_2 - x_3 = 5\\
&(3) x_1 + x_2 + 2x_3 = 2\\
&(4) x_1, x_2, x_3 \geq 0
\end{aligned}$$

Let's simplify the three equations. Take $(1) + -2(2) + 3(3)$ (arbitrary LC) which results in

$$0x_1 + 4x_2 + 7x_3 = -3$$

Note this is obviously infeasible since one cannot produce a negative RHS given $x_2, x_3 \geq 0$. Therefore this LP problem is **infeasible**.

Proposition. *(Prop 2.1, page 46)*
Let A be a matrix and b be a vector. Then the system

$$Ax = b \qquad x \geq 0$$

has no solution if there is a vector y such that

1. $y^\intercal A \geq 0$ *AND*
2. $y^\intercal b < 0$

*where in the above example $y = (1, -2, 3)^\intercal$. Such a y that satisfies (1) and (2) is called a **certificate of infeasibility** for the given system (or constraints).*

Proof. Assume there is a solution for this system denoted by $\bar{x}$. We have

$$A\bar{x} = b \qquad \bar{x} \geq 0$$

note the constraint is because our quantities must be non-negative. Then

$$y^\intercal A\bar{x} = y^\intercal b$$

Let $z = y^\intercal A$, which we know $z \geq 0$ by (1). So $z = (z_1, \ldots, z_n)$ where $z_i \geq 0$ for all $1 \leq i \leq n$. Note

$$y^\intercal A\bar{x} = z\bar{x} = z_1\bar{x}_1 + z_2\bar{x}_2 + \ldots + z_n\bar{x}_n \geq 0$$

since $x_i \geq 0$ for all $1 \leq i \leq n$. Therefore the RHS ≥ 0 and the LHS < 0 which is a contradiction. $\square$

6 May 19, 2017

6.1 Other Ways to Show Infeasibility

Example. Let there be an optimization problem such that our objective function is

$$max \qquad x_1 + x_2 + x_3$$

subject to

$$\begin{aligned}
&2x_1 + x_2 + 3x_3 = 5\\
&x_1 + x_2 + 2x_3 = 4\\
&x_1 \geq 0, x_2 \leq 0
\end{aligned}$$

Let us sum the equalities multiplied by the corresponding coefficients in $y = (2, -3)^\intercal$. We get

$$\begin{aligned}
1x_1 - 1x_2 + 0x_3 &= -2\\
1x_1 + (-1)x_2 + 0x_3 &= -2
\end{aligned}$$

Let us assume that there is a feasible solution denoted by $\bar{x}$. Then

$$1\bar{x}_1 + (-1)\bar{x}_2 + 0\bar{x}_3 = -2$$

Note $\bar{x}_1 \geq 0$, $(-1)\bar{x}_2 \geq 0$ and $0\bar{x}_3 = 0$ (from our constraint $x_1 \geq 0, x_2 \leq 0$. The LHS is therefore ≥ 0 but the RHS is < 0. We obtain a contradiction to the assumption that there is a feasible solution, therefore this problem is *infeasible*.

6.2 Unbounded Solutions

Example. Recall from a previous example

$$max \qquad 7x_1 + 3x_2$$

subject to

$$\begin{aligned}
2x_1 - x_2 &= 4\\
4x_1 - 2x_2 &= 8\\
x_1 \geq 0, x_2 \geq 0
\end{aligned}$$

Note that we found $\bar{x} = (2, 0)^\intercal$ is a feasible solution. More generally, we note $\tilde{x} = (2, 0)^\intercal + t(1, 2)^\intercal$ is feasible for every $t \geq 0$ with value $14 + 13t$.

6.3 Certificate of Unboundeness

Proposition. *Given the optimization problem*

$$max \qquad c^\intercal x$$

subject to

$$Ax = b \qquad x \geq 0$$

Let $\bar{x}$ be a feasible solution for the above system. Let d be a vector such that

$$\begin{aligned} &(1)\ Ad = 0 \\ &(2)\ d \geq 0 \\ &(3)\ c^\intercal d > 0 \end{aligned}$$

Then the system is unbounded*. Such a pair $\bar{x}, d$ is called a* ***certificate of unboundedness****.*

Example. In our previous example

$$A = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

and $c = (7, 3)^\intercal$. Then in this example $d = (1, 2)^\intercal$ or the "slope" of our general equation for all feasible solutions.

Proof. Let us assume the contrary, that $Ax = b$ is not unbounded.

In other words, there exists α such that for every feasible solution $\tilde{x}$ the value of $\tilde{x}$ is at most α.

Define $\tilde{x}$ such that

$$\tilde{x} = \bar{x} + td \text{ for some } t \geq 0$$

Then $\tilde{x}$ is a feasible solution for $Ax = b$. Indeed this is true when we check this

$$\begin{aligned} A\tilde{x} &= A(\bar{x} + td) \\ &= A\bar{x} + A(td) \\ &= A\bar{x} + tAd \\ &= b \end{aligned}$$

Remember that for any particular feasible $\bar{x}$, $A\bar{x} = b$ and from (1) $Ad = 0$. So $A\tilde{x} = b$ thus $\tilde{x}$ is a valid solution set.

Going back our definition $\tilde{x} = \bar{x} + td$, note that $\bar{x} \geq 0$ (part of problem $x \geq 0$, $t \geq 0$ (by definition), and $d \geq 0$ from (2). Inserting $\tilde{x}$ into our objective function

$$\begin{aligned} c^\intercal \tilde{x} &= c^\intercal (\bar{x} + td) \\ &= c^\intercal \bar{x} + c^\intercal (td) \\ &= c^\intercal \bar{x} + tc^\intercal d \end{aligned}$$

where $c^\intercal d > 0$ from (3).

If we choose $t \geq 0$ so that

$$c^\intercal \tilde{x} = c^\intercal \bar{x} + tc^\intercal d > \alpha$$

for our chosen "fixed" α value, then we obtain a contradiction. Solving for this t

$$t > \frac{\alpha - c^\intercal \bar{x}}{c^\intercal d}$$

□

7 May 23, 2017

7.1 More Unbounded Examples

Example. Let there be a **minimization** optimization problem

$$min \qquad x_1 + x_2 + x_3$$

subject to

$$\begin{aligned}
&6x_1 - x_2 + 4x_3 = 39 \\
&2x_1 - x_2 + 2x_3 = 15 \\
&x_1 \geq 0, x_2 \leq 0
\end{aligned}$$

A feasible solution is $\bar{x} = (3, -7, 1)_\intercal$ where $d = (1, -2, -2)^\intercal$. Note other feasible solutions can then be written as

$$\tilde{x} = \bar{x} + td \text{ for some } t \geq 0$$

Define $\tilde{x} = \bar{x} + td$ for some $t \geq 0$. Note that $\tilde{x}$ is a feasible solution. We can check this

$$\begin{aligned}
&\tilde{x}_1 = 3 + t \cdot 1 \\
&\tilde{x}_2 = -7 + t \cdot (-2) \\
&\tilde{x}_3 = 1 + t \cdot (-2)
\end{aligned}$$

Plugging into our constraints

$$\begin{aligned}
6\tilde{x}_1 - \tilde{x}_2 + 4\tilde{x}_3 = 6(3 + t) - (-7 - 2t) + 4(1 - 2t) = 29 + 0t = 29 \\
2\tilde{x}_1 - \tilde{x}_2 + 2\tilde{x}_3 = 2(3 + t) - (-7 - 2t) + 2(1 - 2t) = 15 + 0t = 15
\end{aligned}$$

moreover note our constraint $x_1 \geq 0, x_2 \leq 0$, so

$$\begin{aligned}
&\tilde{x}_1 = 3 + t \geq 0 \text{ for every } t \geq 0 \\
&\tilde{x}_2 = -7 - 2t \leq 0 \text{ for every } t \geq 0
\end{aligned}$$

therefore $\tilde{x}$ is a feasible solution for every $t \geq 0$. The value of $\tilde{x}$ is therefore $\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = -3 - t$ which becomes arbitrarily small as t increases.

7.2 Optimal Solutions

Example. Suppose we have

$$max \qquad 5x_2 - 9x_3 + 7x_4$$

subject to

$$\begin{aligned} &(1)\ x_1 + 2x_2 - 3x_3 + x_4 = 6 \\ &(2)\ 2x_1 - x_2 + x_3 - 5x_4 = -5 \\ &\ \ x \geq 0 \end{aligned}$$

Note that x_1 is missing from the objective function. We want to combine our constraints such that it looks similar to the object function. Eliminate x_1 by summation of our two constraints i.e. $2(1) - 1(2)$ we get

$$0x_1 + 5x_2 - 7x_3 + 7x_4 = 17$$

Let us suppose $\bar{x} = (1, 2, 0, 1)^{\intercal}$ is the most optimal solution. Suppose that (from the objective function)

$$5\bar{x}_2 - 9\bar{x}_3 + 7\bar{x}_4 = 17$$

To show that $\bar{x}$ is the optimal solution we need to prove that for every feasible solution $\tilde{x}$

$$5\tilde{x}_2 - 9\tilde{x}_3 + 7\tilde{x}_4 \leq 17$$

since this is a maximization problem and we supposed $\bar{x}$ is the most optimal.

From our constraint we have

$$5\tilde{x}_2 - 7\tilde{x}_3 + 7\tilde{x}_4 = 17$$

Since $\tilde{x}$ is feasible, then from our constraint $x \geq 0 \rightarrow \tilde{x}_3 \geq 0$. Thus

$$5\tilde{x}_2 - 7\tilde{x}_3 + 7\tilde{x}_4 - 2\tilde{x}_3 \leq 17$$

since $-2\tilde{x}_3 \leq 0$.

7.3 Certificate of Optimality

Proposition. *Suppose we are given an LP problem*

$$max \qquad c^{\intercal}x$$

subject to

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

and a feasible solution $\bar{x}$. If there is a vector y such that

$$\begin{aligned} &(1)\ A^{\intercal}y \geq c \\ &(2)\ c^{\intercal}\bar{x} = y^{\intercal}b \end{aligned}$$

then $\bar{x}$ is an optimal solution for the LP problem.

Proof. We need to prove that for every feasible solution $\tilde{x}$, we have

$$c^\intercal \tilde{x} \leq c^\intercal \bar{x}$$

Let us compute $c^\intercal \bar{x}$

$$\begin{aligned} c^\intercal \bar{x} &= y^\intercal b && \text{from (2)} \\ &= y^\intercal A \tilde{x} \\ &= c^\intercal \tilde{x} + y^\intercal A \tilde{x} - c^\intercal \tilde{x} \\ &= c^\intercal \tilde{x} + (y^\intercal A - c^\intercal) \tilde{x} \end{aligned}$$

Note that $y^\intercal A - c^\intercal \geq 0$ from (1) and $\tilde{x}$ from $x \geq 0$. Thus the right sumamnd is always ≥ 0 so

$$c^\intercal \bar{x} = c^\intercal \tilde{x} + (A^\intercal y - c)^\intercal \tilde{x} \geq c^\intercal \tilde{x}$$

such a vector y is called a **certificate of optimality** for $\bar{x}$. □

7.4 Harder Example for Optimality

Example. Let there be

$$max \qquad 14x_1 + x_2 + 10x_3 + 3x_4$$

subject to

$$\begin{aligned} &(1)\ 2x_1 + x_2 + 3x_3 - x_4 = 3 \\ &(2)\ 4x_1 - x_2 + x_3 + 2x_4 = 9 \\ &\quad\ \ x_1, x_3 \geq 0 \\ &\quad\ \ x_2, x_4 \leq 0 \end{aligned}$$

taking $y = (3,2)^\intercal$ or $3(1) + 2(2)$ we get

$$14x_1 + x_2 + 11x_3 + x_4 = 27$$

we claim $\bar{x} = (2, -1, 0, 0)^\intercal$ is a feasible solution. Let us show that $\bar{x}$ is also the **optimal solution**. Going back to our combined constraint, every feasible solution $\tilde{x}$ we have

$$\begin{aligned} &14\tilde{x}_1 + \tilde{x}_2 + 11\tilde{x}_3 + \tilde{x}_4 = 27 \\ &\to 14\tilde{x}_1 + \tilde{x}_2 + 10\tilde{x}_3 + 3\tilde{x}_4 + \tilde{x}_3 - 2\tilde{x}_4 \end{aligned}$$

Note that $\tilde{x}_3 \geq 0$ and $-2\tilde{x}_4 \geq 0$. So from the objective function

$$14\tilde{x}_1 + \tilde{x}_2 + 10\tilde{x}_3 + 3\tilde{x}_4 \leq 14\tilde{x}_1 + \tilde{x}_2 + 11\tilde{x}_3 + \tilde{x}_4 = 27$$

Thus every feasible solution $\tilde{x} \leq 27$ thus $\bar{x}$ is the most optimal since $c^\intercal \bar{x} = 27$.

8 May 24, 2017

8.1 Standard Equality Form

An LP problem is in **Standard Equality Form** if it is of the form

$$max \qquad c^{\intercal}x + \bar{z}$$

subject to

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

where $c^{\intercal}, \bar{z}, A, b$ are fixed numbers, matrices, or vectors. That is it must satisfy

(1) Maximization problem

(2) *Every* variable has a *non-negativity* constraint

(3) Except for non-negativity constraints, all constraints are *equations*

8.2 Equivalent LP problems

What does it mean for two LP problems (P) and (Q) to be **equivalent**?

(1) (P) is infeasible $\iff$ (Q) is infeasible

(2) (P) is unbounded $\iff$ (Q) is unbounded

(3) from an optimal solution for (P), we can construct an optimal solution for (Q) (and vice versa)

Example. Given

$$min \qquad 7x_1 + 3x_2 - 2x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\geq 7 \\ 2x_1 - x_2 + 7x_3 &\leq -1 \\ -x_1 + x_2 - x_3 &= 0 \\ x_1 &\leq 0 \\ x_2 &\geq 0 \\ x_3 & free \end{aligned}$$

convert this to Standard Equality Form.

Step 1 Change minimization to maximization (if necessary). That is negate every term

$$min \qquad 7x_1 + 3x_2 - 2x_3 \to max \qquad -7x_1 - 3x_2 + 2x_3$$

Step 2 Convert all inequalities to equations. In the example, for every feasible solution we have

$$x_1 + x_2 + x_3 \geq 7$$

create a temp variable $x_4 \geq 0$ such that

$$x_1 + x_2 + x_3 - x_4 = 7$$

At the end of step 2, we have for our example

$$max \qquad -7x_1 - 3x_2 + 2x_3$$

subject to

$$\begin{aligned}
&x_1 + x_2 + x_3 - x_4 = 7 \\
&2x_1 - x_2 + 7x_3 + x_5 = -1 \\
&-x_1 + x_2 - x_3 = 0 \\
&x_1 \leq 0, x_2 \geq 0, x_3 \text{ free }, x_4 \geq 0, x_5 \geq 0
\end{aligned}$$

Step 3 Replace all non-positive variables by non-negative variables. In the example, $x_1 \to -x_1$

$$max \qquad 7x_1 - 3x_2 + 2x_3$$

subject to

$$\begin{aligned}
&-x_1 + x_2 + x_3 - x_4 = 7 \\
&-2x_1 - x_2 + 7x_3 + x_5 = -1 \\
&x_1 + x_2 - x_3 = 0 \\
&x_1 \geq 0, x_2 \geq 0, x_3 \text{ free }, x_4 \geq 0, x_5 \geq 0
\end{aligned}$$

Step 4 Replace all free variables by a difference of two non-negative variables. In our example

$$x_3 = x_3^+ - x_3^-$$

where $x_3^+, x_3^- \geq 0$. Thus at the end of step 4 we have for our example

$$max \qquad 7x_1 - 3x_2 + 2x_3^+ - 2x_3^-$$

subject to

$$\begin{aligned}
&-x_1 + x_2 + x_3^+ - x_3^- - x_4 = 7 \\
&-2x_1 - x_2 + 7x_3^+ - 7x_3^- + x_5 = -1 \\
&x_1 + x_2 - x_3^+ + x_3^- = 0 \\
&x_1, x_2, x_3^+, x_3^-, x_4, x_5 \geq 0
\end{aligned}$$

9 May 26, 2017

9.1 Simplex Iteration

Example. Given

$$min \qquad 5x_1 - x_2 - 4x_3 - 4x_4$$

subject to

$$\begin{aligned}
&-4x_1 + x_2 + 3x_3 + 3x_4 \leq 17 \\
&-x_1 + x_3 + x_4 \leq 4 \\
&-2x_1 + x_2 + 2x_3 + 3x_4 \leq 10 \\
&x_1 \leq 0, x_2, x_3, x_4 \geq 0
\end{aligned}$$

First we must transform this LP problem to Standard Equality Form (SEF).

$$max \qquad -5x_1 + x_2 + 4x_3 + 4x_4$$

subject to

$$\begin{aligned}
&-4x_1 + x_2 + 3x_3 + 3x_4 + x_5 = 17 \\
&-x_1 + x_3 + x_4 + x_6 = 4 \\
&-2x_1 + x_2 + 2x_3 + 3x_4 + x_7 = 10 \\
&x_1 \leq 0, x_2, \ldots, x_7 \geq 0
\end{aligned}$$

Then we modify $x_1 \to -x_1$ and get

$$max \qquad 5x_1 + x_2 + 4x_3 + 4x_4$$

subject to

$$\begin{aligned}
&(1)\ 4x_1 + x_2 + 3x_3 + 3x_4 + x_5 = 17 \\
&(2)\ x_1 + x_3 + x_4 + x_6 = 4 \\
&(3)\ 2x_1 + x_2 + 2x_3 + 3x_4 + x_7 = 10 \\
&\quad\ x \geq 0
\end{aligned}$$

which is our **1st iteration**.

An obvious solution is $x = (0,0,0,0,17,4,10)^\intercal$ (just look at the free variables in each equation). Is this optimal? Probably not (no). This is because from our objective function, we see that $x_1, \ldots, x_4$ contribute to the optimal value, but we have set them to 0 in favor of $x_5, \ldots, x_7$. What if we try to find $x_1, \ldots, x_4 > 0$ that fits within our constraint?

In practice, we choose the first variable with a coefficient > 0 to optimize (**Bland's Rule**). Let's try to increase (and maximize) $t = x_1, t \geq 0$ *first* and modify x_5, x_6, x_7 accordingly hwhile leaving $x_2, x_3, x_4 = 0$. Solving for x_5 in each of the constraints

$$
\begin{gathered}
4x_1 + x_5 = 17 \to x_5 = 17 - 4t \\
x_1 + x_6 = 4 \to x_6 = 4 - t \\
2x_1 + x_7 = 10 \to x_7 = 10 - 2t
\end{gathered}
$$

Note we cannot simply set $t = 100$ for example since $x_5, x_6, x_7 \geq 0$.

Solving for t in each of the new constraints ≥ 0, we get $t \geq \frac{17}{4}, t \geq 4, t \geq 5$, respectively. Since ALL of them have to be non-negative, we take the *min.* of these bounds ie. $t \leq min\{\frac{17}{4}, 4, 5\} = 4$.

Letting $t = 4$ we get the new solution $x = (4, 0, 0, 0, 1, 0, 2)^\intercal$ with the value of 20.

Note that our t was bounded by equation (2). Let us eliminate x_1 from the other equations (1) and (3) by subtracting each with (2). For the objective function, we can write (2) in terms of 0 then subtract/add 0 from/to the objective function. So we get

$$max \qquad x_2 - x_3 - x_4 - 5x_6 + 20$$

subject to

$$
\begin{aligned}
&(1)\ x_2 - x_3 - x_4 + x_5 - 4x_6 = 1 \\
&(2)\ x_1 + x_3 + x_4 + x_6 = 4 \\
&(3)\ x_2 + x_4 - 2x_6 + x_7 = 2 \\
&\quad\ \ x \geq 0
\end{aligned}
$$

which is our **2nd iteration**.

This allows us to keep $x_1 = 4$ but now optimize for x_2. Setting $x_3, x_4 = 0$ and solving for the free variables

$$
\begin{aligned}
x_2 + x_5 = 1 \to x_5 = 1 - t \to 1 - t \geq 0 \to t \leq 1 \\
x_1 = 4 \geq 0 \\
x_2 + x_7 = 2 \to x_7 = 2 - t \to 2 - t \geq 0 \to t \leq 2
\end{aligned}
$$

Thus we have $t \leq min\{1, 2\} = 1$. Increasing x_2 in the current solution to the value $t, t \geq 0$ (leaving $x_3, x_4, x_6 = 0$) we obtain the more optimal solution $x = (4, 1, 0, 0, 0, 0, 1)^\intercal$ with value equal to 21.

By subtracting all functions by (1) to eliminate x_2

$$max \qquad -x_5 - 5x_6 + 21$$

subject to

$$
\begin{aligned}
&(1)\ x_2 - x_3 - x_4 + x_5 - 4x_6 = 1 \\
&(2)\ x_1 + x_3 + x_4 + x_6 = 4 \\
&(3)\ x_3 + 2x_4 - x_5 + 2x_6 + x_7 = 1 \\
&\quad\ \ x \geq 0
\end{aligned}
$$

which is our **3rd iteration**.

Observe our last solution $x = (4, 1, 0, 0, 0, 0, 1)^\intercal$ with value 21 is optimal since $x \geq 0$ and $\mathbb{Z}_3(x) \leq 21$ ($\mathbb{Z}_3$ is the objective function of the 3rd iteration). To prove this, we can find a certifiate of optimality for x from the final LP: that is $y \in \mathbb{R}^3$ s.t.

1. $y^\intercal A \geq c^\intercal$
2. $y^\intercal b = c^\intercal x$

we get y by taking the *negative vector of coefficients* of (x_5, x_6, x_7) in the **final objective function** so $y = -(-1, -1, 0)^\intercal = (1, 1, 0)^\intercal$. Multiplying out y with our initial LP:

$$
\begin{aligned}
(1,1,0)(17,4,10)^\intercal = 17 + 4 = 21 &= Z_3(x) \\
&= c^\intercal x = (5,1,4,4,0,0,0)(4,1,0,0,0,0,1)^\intercal = 20 + 1 = 21
\end{aligned}
$$

Note that in the 3 iterations in our example, the LPs were in a special form: The columns of the constraint matrix corresponding to the non-zero entries of the current solution (for a given iteration) are LI (linearly independent).

9.2 Basis

Definition. Suppose A is an $m \times n$ constraint matrix of an LP in SEF, such that $rank(A) = m$. Then any subset of m column indices that are LI are called a *basis*.

Example. $\{1, 5, 7\}$ is clearly a basis for the LP in the 2nd iteration. Also $\{1, 5, 6\}$.

10 May 31, 2017

10.1 Basic and Non-Basic

We also say that x_j for $j \in B$ is a **basic variable** and x_j with $j \notin B$ is called **non-basic**. Let us denote $N = \{1, 2, \ldots, n\} - B$.

We'll also consider restrictions of A, x, c to B, N respectively, denoted as A_B, x_B, c_B, and A_N, x_N, c_N respectively for the basic and non-basic parts.

Example. In the previous example, one basis is $\{5, 6, 7\}$.

Note that wiht $B = \{5, 6, 7\}$, then $N = \{1, 2, 3, 4\}$. Thus we have (from the

1st iteration)

$$A_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad c_b = (0,0,0)^\intercal$$

$$A_N = \begin{bmatrix} 4 & 1 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 2 & 3 \end{bmatrix} \qquad c_N = (5,1,4,4)^\intercal$$

Note that in our 1st iteration we had the naive solution $\bar{x} = (0,0,0,0,17,4,10)^\intercal$. Thus $\bar{x}_B = (17,4,10)^\intercal$ and $\bar{x}_N = (0,0,0,0)^\intercal$. This is a *basic* solution.

Definition. We say a solution $\bar{x} \in \mathbb{R}^n$ is *basic* for basis B if $\bar{x}_N = 0$, $A\bar{x} = b$.

Note that

$$A = \begin{bmatrix} A_B & A_N \end{bmatrix}$$

by permuting columns if necessary. A_B has m columns and A_N has $m - n$ columns. Note we can show that $\bar{x}_B$ always exists given a basis and A_B

$$\begin{aligned} & A\bar{x} = b \\ \iff & \begin{bmatrix} A_B & A_N \end{bmatrix} \begin{bmatrix} \bar{x}_B \\ \bar{x}_N \end{bmatrix} = b \\ \iff & A_B \cdot \bar{x}_B + A_N \cdot \bar{x}_N = b \end{aligned}$$

Note that $A_N \cdot \bar{x}_N = 0$ since $\bar{x}_N = 0$ so

$$A_B \bar{x}_B = b \iff \bar{x}_B = A_B^{-1} b$$

where A_B is invertible since its columns are LI.

So given a basis, we can always find a basic solution corresponding to it. Suppose we take $B = \{1,2,5\}$ then by permuting columns we write them in order $(1,2,5,3,4,6,7)$ or

$$A = \begin{bmatrix} 4 & 1 & 1 & | & 3 & 3 & 0 & 0 \\ 1 & 0 & 0 & | & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & | & 2 & 3 & 0 & 1 \end{bmatrix}$$

Example. $x \in \mathbb{R}^6$, $x \geq 0$ and

$$\begin{bmatrix} 0 & -2 & 1 & 0 & 2 & 6 \\ 1 & 1 & -1 & 1 & 3 & -3 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Are the following bases?

1. $\{2,3\}$ yes

2. $\{2,6\}$ no

3. $\{1\}$ no, similarly for $\{3,4,5\}$ not a basis

A basis solution for $B = \{2,3\}$ is

$$A_B \bar{x}_B = b$$
$$\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

which gives us

$$-2\bar{x}_2 + \bar{x}_3 = 2\bar{x}_2 - \bar{x}_3 = 4$$

Solving gives us $\bar{x}_2 = -6$ and $\bar{x}_3 = -10$. Thus the solution would be $\bar{x} = (0, -6, -10, 0, 0, 0)^\intercal$.

Note however that $x \geq 0$ thus this solution is *infeasible*.

Definition. We say that a basic solution $\bar{x}$ (for basis B) is *feasible* (a basic *feasible* solution) if $\bar{x} \geq 0$.

Note that if we chose $B = \{3,4\}$ then we get $\bar{x} = (0,0,2,6,0,0)$ which is a basic feasible solution for B.

10.2 Canonical Form

Note that in our example from the day before in the 1st iteration, the LP satisfies

1. $A_B = I$
2. $c_B = 0$

Definition. Whenever an LP

$$max \qquad c^\intercal x$$

s.t.

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

satisfies $A_B = I$, $c_B = 0$ for a basis B, then we say that the LP is in **canonical form** for B.

In the 2nd iteration, we had a solution $(4,0,0,0,1,0,2)$ a basic feasible solution for basis $\{1,5,7\}$.

$$A_B = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \neq I$$

(even up to permutations of columns). This is not canonical.

11 June 2, 2017

11.1 Converting to Canoncial Form

To bring an LP in SEF to canonical form for a basis B

1. Multiply the constraints $Ax = b$ by A_B^{-1} on both sides to get $A_B^{-1}Ax = A_B{}^{-}1b$ (note these are equivalent to each other) Therefore on the LHS we have
$$A_B^{-1}A = A_B^{-1}\begin{bmatrix} A_B & | & A_N \end{bmatrix} = \begin{bmatrix} A_B^{-1}A_B & | & A_N \end{bmatrix}$$
so in the constraint matrix $\tilde{A} = A_B^{-1}A$, thus $\tilde{A_B} = I$.

2. We multiply $\tilde{A}x = \tilde{b}$ (where $\tilde{b} = A_B^{-1}b$) by $c_B^\intercal$ (objective coefficient restricted to B) to get
$$\begin{aligned} c_B^\intercal \tilde{A}x &= c^\intercal \tilde{b} \\ \iff c_B^\intercal \tilde{b} - c_B^\intercal \tilde{A}x &= 0 \end{aligned}$$
If we add the LHS to the objective function $z(x) = c^\intercal x$ (adding 0) we get
$$z(x) = c^\intercal x + c_B^\intercal \tilde{b} - c_B^\intercal \tilde{A}x$$
for every feasible solution x $\tilde{Z}(x) = Z(x)$. Observe that
$$\begin{aligned} \tilde{Z}(x) &= \begin{bmatrix} c_B^\intercal & | & c_N^\intercal \end{bmatrix}\begin{bmatrix} x_B \\ x_N \end{bmatrix} - c_B^\intercal \begin{bmatrix} I & | & \tilde{A}_N \end{bmatrix}\begin{bmatrix} x_B \\ x_N \end{bmatrix} + c_B^\intercal \tilde{b} \\ &= c_B^\intercal x_B + c_N^\intercal x_N - c_B^\intercal x_B - c_B^\intercal \tilde{A}x_n + c_B^\intercal \tilde{b} \\ &= c_N^\intercal x_N - c_B^\intercal \tilde{A}x_n + c_B^\intercal \tilde{b} \end{aligned}$$
So the coefficients of the basic variables in $\tilde{c}$, where $\tilde{c}^\intercal = c^\intercal - c_B^\intercal \tilde{A} = c^\intercal - c_B^\intercal A_B^{-1}A$, are all 0.

   So in summary, starting with
$$max \qquad Z(x) = c^\intercal x$$
s.t.
$$\begin{aligned} &Ax = b \\ &x \geq 0 \end{aligned}$$
we get
$$max \qquad (c^\intercal - c_B^\intercal A_B^{-1}A)x + c_B^\intercal b$$
s.t.
$$\begin{aligned} &A_B^{-1}Ax = A_B^{-1}b \\ &x \geq 0 \end{aligned}$$
This is in canonical form for B.

Example. Given the first iteration of our previous example

$$max \qquad 5x_1 + x_2 + 4x_3 + 4x_4$$

subject to

$$\begin{aligned}
&4x_1 + x_2 + 3x_3 + 3x_4 + x_5 = 17\\
&x_1 + x_3 + x_4 + x_6 = 4\\
&2x_1 + x_2 + 2x_3 + 3x_4 + x_7 = 10\\
&x \geq 0
\end{aligned}$$

our solution is $(4, 0, 0, 0, 1, 0, 2)^\intercal$.

1. Let

   $$A_B = A_{(5,1,7)} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

   thus A_B^{-1} would be

   $$A_B^{-1} = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

   Multiplying out $A_B^{-1}A$

   $$A_B^{-1}A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & -4 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -2 & 1 \end{bmatrix}$$

   where we have $(A_B^{-1}A)_{(5,1,7)} = I$ as desired.

2. Note that $c^\intercal = (5, 1, 4, 4, 0, 0, 0)$ thus $c^\intercal_{(5,1,7)} = (0, 5, 0)$ (mind the order, must match A_B).

   So that gives us

   $$\begin{aligned}
   \tilde{c}^\intercal &= c^\intercal - (0, 5, 0)\tilde{A}\\
   &= c^\intercal - (5, 0, 5, 5, 0, 5, 0)\\
   &= (0, 1, -1, -1, 0, -5, 0)
   \end{aligned}$$

3. With $\tilde{c}^\intercal$ and $\tilde{A}$, we can rewrite our LP problem in canonical form

   $$max \qquad x_2 - x_3 + x_4 - 5x_6 + 20$$

   subject to

   $$\begin{aligned}
   &x_2 + x_3 + x_5 - 4x_6 = 1\\
   &x_1 + x_3 + x_4 + x_6 = 4\\
   &x_2 + x_4 - 2x_6 + x_7 = 2\\
   &x \geq 0
   \end{aligned}$$

   where the RHS of the constraints is $\tilde{b} = A_B^{-1}b$.

11.2 Simplex: Better Feasible Solution

1. Start with LP in *canonical form* for a basis B and basic feasible solution $\bar{x}$ for B.

2. If $c_j > 0$ for some non-basic variable x_j, then we try to find a new feasible solution with $x_j > 0$ (say $x_j = t$) with all the remaining non-basic variables $= 0$.

   So for a given constraint $Ax = b$ that is

$$\iff \begin{bmatrix} I & | & A_N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b$$

   note the I contains A_B and the jth column A_j is in A_N.

   Setting $x_j = t_j$ and all other non-basic variables $= 0$ we get

$$x_B + tA_j = b$$

   we take the max t such that $x_B = b - tA_j \geq 0$ (and $t \geq 0$).

12 June 5, 2017

12.1 Simplex from Canonical Form

Example.

$$max \qquad 5x_1 + 3x_2$$

subject to

$$\begin{aligned} &x_1 + x_2 + x_3 = 4 \\ &5x_1 + 2x_2 + x_4 = 10 \\ &x \geq 0 \end{aligned}$$

Note that $B = 3, 4$ (canonical form) with the corresponding basis solution $(0, 0, 4, 10)^\intercal$.

Since the corresponding basic solution is feasible, B is a feasible basis. The value of the basic solution is 0.

Increasing x_1 to the value $t \geq 0$ while keeping x_2 equal to 0 Then

$$\begin{aligned} x_3 &= 4 - t \\ x_4 &= 10 - 5t \end{aligned}$$

So for x_3 and x_4 to be non-negative, we get $t \leq 4$ and $t \leq 2$. Thus we need $t \leq 2$.

Setting $t = 2$ we get a new solution $(2, 0, 2, 0)^\intercal$ with value 10. Let us update the basis, where x_1 enters the basis and x_4 leaves the basis. Thus the new

$B = \{1, 3\}$. We need to update our LP such that it is in canonical form again. Thus for $B = \{1, 3\}$ we have

$$max \qquad x_2 - x_4 + 10$$

subject to

$$\begin{aligned}
&\frac{3}{5}x_2 + x_3 - \frac{1}{5}x_4 = 2 \\
&x_1 + \frac{2}{5}x_2 + \frac{1}{5}x_4 = 2 \\
&x \geq 0
\end{aligned}$$

This was derived by noting that $5x_1 + 2x_2 + x_4 - 10 = 0$ (from the second constraint), subtracting this (0) from $z(x) = 5x_1 + 3x_2$ to get rid of x_1, and similarly subtracting one fifth of that equation from the first constraint to get rid of x_1 from the first constraint. Finally, dividing the second constraint by 5 such that $A_B = I$.

Now we can optimize for $x_2 = t \geq 0$ similar to what we did before, fixing x_1 and x_3 and setting everything else (x_4) to 0.

$$\begin{aligned}
x_3 + \frac{3}{5}t = 2 &\iff x_3 = 2 - \frac{3}{5}t \\
x_1 + \frac{2}{5}t = 2 &\iff x_1 = 2 - \frac{2}{5}t
\end{aligned}$$

where $t = min\{\frac{10}{3}, 5\}$ we obtain solution $(\frac{2}{3}, \frac{10}{3}, 0, 0)$ with value $\frac{40}{3}$ (note we "overrode" the previous value of x_1 where it's lower than before but gives us a better value).

Let us update the basis: x_2 enters B and x_3 leaves B. The new basis is $B = \{1, 2\}$. Computing the canonical form for B.

Subtract $\frac{5}{3}$ of (2) (second constraint) from $z(x)$, multiply $\frac{5}{3}$ with (1) to get, and subtracting $\frac{2}{3}$ of (1) from (2) to get

$$max \qquad -\frac{5}{3}x_3 - \frac{2}{3}x_4 + \frac{40}{3}$$

subject to

$$\begin{aligned}
&x_2 + \frac{5}{3}x_3 - \frac{1}{3}x_4 = \frac{10}{3} \\
&x_1 - \frac{2}{3}x_3 + \frac{1}{3}x_4 = \frac{2}{3} \\
&x \geq 0
\end{aligned}$$

Note that in our objective function, $-\frac{5}{3}x_3$ and $-\frac{2}{3}x_4$ are both ≤ 0, thus $z(x) \leq \frac{40}{3}$. Thus $(\frac{2}{3}, \frac{10}{3}, 0, 0)^\intercal$ is an optimal solution with value $\frac{40}{3}$. (We only stop once the coefficients are negative since $x \geq 0$.

Note that the certificate of optimality is $y = (\frac{5}{3}, \frac{2}{3})^\intercal$ (derived from negative coefficient of final objective function with respect to the initial LP (where x_3, x_4 had coefficients of 1). Note if we plug y into our initial LP, it satisfies $y^\intercal A \geq c^\intercal$ and $c^\intercal \bar{x} = y^\intercal b$.

13 June 7, 2017

13.1 Bland's Rule

Increase the variable with the smallest index that has a non-zero coefficient.

13.2 Unboundedness from Simplex

Example.

$$max \qquad x_1 + x_2 = z(x)$$

subject to

$$\begin{aligned} -2x + 3x_2 + x_3 &= 9 \\ x_1 - 2x_2 + x_4 &= 2 \\ x &\geq 0 \end{aligned}$$

where $B = \{3, 4\}$ with the corresponding basis solution $(0, 0, 9, 2)$ with value 0.

Increase x_1 (Bland's Rule) to the value $t \geq 0$ keeping x_2 equal to 0. Then

$$\begin{aligned} -2t + x_3 = 9 &\iff x_3 = 9 + 2t \\ t + x_4 = 2 &\iff x_4 = 2 - t \end{aligned}$$

Note x_3 has no role in the constraint (it will always be non-negative for all $t \geq 0$ since the coefficient of x_1 in the first constraint is negative). Thus $t = 2$ from $x_4 = 2 - t \geq 0$. x_1 enters B and x_4 leaves B, thus the basis is now $B = \{1, 3\}$.

The canonical form for this basis is as follows

$$max \qquad 3x_2 - x_4 + 2$$

subject to

$$\begin{aligned} -x_2 + x_3 + 2x_4 &= 13 \\ x_1 - 2x_2 + x_4 &= 2 \\ x &\geq 0 \end{aligned}$$

Increase x_2 to value $t \geq 0$ keeping $x_4 = 0$

$$\begin{aligned} x_3 &= 13 + t \geq 0 \\ x_1 &= 2 + 2t \geq 0 \end{aligned}$$

for all $t \geq 0$. Since t is unbounded, then this tells us the problem is unbounded.

We must thus find a certificate of unboundedness. Note that a feasible solution is $\bar{x} = (2, 0, 13, 0)^\intercal$ (the constant in the above $f(t)$). Note for for every increase of x_2 by 1, x_3 increases by 1 and x_1 increases by 2 (from the coefficient of t). Thus $d = (2, 1, 1, 0)^\intercal$.

We must check

1. $c^\intercal d > 0$: $c^\intercal d = 3 > 0$.
2. $d \geq 0$ which holds.
3. $Ad = 0$ which also holds.

14 June 9, 2017

14.1 From SEF to Simplex (Given a Feasible Basis)

Given

$$max \qquad (3,2,1,1,0)x$$

subject to

$$\begin{bmatrix} 8 & 3 & 4 & 1 & 1 \\ 10 & 4 & 4 & 2 & 1 \end{bmatrix} x = \begin{bmatrix} 16 \\ 20 \end{bmatrix}$$
$$x \geq 0$$

Note that we need to convert this to canonical form for a feasible basis B. We are *given a hint* that $B = \{1,5\}$ is feasible (**It is important that the initial basis is feasible**). Thus the canonical form for B is

$$max \qquad (0, \frac{1}{2}, 1, -\frac{1}{2}, 0)x + 6$$

subject to

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 4 & -3 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$
$$x \geq 0$$

The current solution is $x = (2,0,0,0,0)^\intercal$. Bland's rule and solving for $x_2 = t$, we get $x_2 = 4$. x_2 enters and x_1 leaves the basis leaving us with $B = \{2,5\}$ and the corresponding canonical form

$$max \qquad (-1,0,1,-1,0)x + 8$$

subject to

$$\begin{bmatrix} 2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 4 & -2 & 1 \end{bmatrix} x = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$
$$x \geq 0$$

Bland's rule implies $x_3 = t$, and solving we get $x_3 = 1$. x_3 enters and x_5 exits the basis where $B = \{2,3\}$.

$$max \qquad (-\frac{3}{2}, 0, 0, -\frac{1}{2}, -\frac{1}{4})x + 9$$

subject to

$$\begin{bmatrix} 2 & 1 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 & -\frac{1}{2} & \frac{1}{4} \end{bmatrix} x = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$
$$x \geq 0$$

Since all coefficients are negative, $x = (0, 4, 1, 0, 0)^\intercal$ is an optimal solution.

What is the certificate of optimality? It is $y = A_B^{-T} c_B$ from proposition 2.4, where B is the final basis $B = \{2, 3\}$. So note that

$$A_B = \begin{bmatrix} 3 & 4 \\ 4 & 4 \end{bmatrix}, c_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus

$$y = A_B^{-T} c_B = -\frac{1}{4} \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{5}{4} \end{bmatrix}$$

We can check that $y^\intercal A \geq c^\intercal$ and $c^\intercal \bar{x} = y^\intercal b$ where $\bar{x} = (0, 4, 1, 0, 0)^\intercal$.

14.2 Two-Phase Simplex

How do we use Simplex iteration if we are not given the feasible basis B to iterate on?

Phase I we create an auxiliary LP by adding two dummy variables x_5, x_6

$$max \qquad -x_5 - x_6$$

subject to

$$\begin{bmatrix} -3 & 8 & -6 & 2 & 1 & 0 \\ -2 & 2 & -4 & 1 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$
$$x \geq 0$$

Note that the negative signs bound the LP to have a max value 0. So we can either have an optimal value of 0 or < 0. Note that if b is negative, we need to first convert it to positive by multiplying rows by -1 since $x_5, x_6 \geq 0$ and the basic feasible solution must be positive.

If the optimal value of the auxliary LP is < 0, then the original LP is infeasible.

Proof. Assume that the original LP has a feasible solution $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)^\intercal$, then $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, 0, 0)^\intercal$ is feasible too for the auxiliary LP which will have an optimal value of 0. So an optimal value of 0 in the auxiliary LP implies a feasible solution. $\square$

For the auxiliary LP, the basis $B = \{5, 6\}$ is feasible. Thus we convert to the canonical form

$$max \qquad (-5, 10, -1, 3, 0, 0)x - 14$$

subject to

$$\begin{bmatrix} -3 & 8 & -6 & 2 & 1 & 0 \\ -2 & 2 & -4 & 1 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$
$$x \geq 0$$

15 June 12, 2017

15.1 Infeasible LP from Auxiliary LP

Example. From last time, we have the auxiliary LP in canonical form for $B = \{5, 6\}$

$$max \qquad (-5, 10, -1, 3, 0, 0)x - 14$$

subject to

$$\begin{bmatrix} -3 & 8 & -6 & 2 & 1 & 0 \\ -2 & 2 & -4 & 1 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$
$$x \geq 0$$

Note x_2 enters and x_5 exits yielding us the new basis $B = \{2, 6\}$. Converting to its canonical form we get

$$max \qquad (-\frac{5}{4}, 0, -\frac{5}{2}, \frac{1}{2}, -\frac{5}{4}, 0)x - \frac{11}{4}$$

subject to

$$\begin{bmatrix} -\frac{3}{8} & 1 & -\frac{6}{8} & \frac{2}{8} & \frac{1}{8} & 0 \\ -\frac{5}{4} & 0 & -\frac{5}{2} & \frac{1}{2} & -\frac{1}{4} & 1 \end{bmatrix} x = \begin{bmatrix} \frac{9}{8} \\ \frac{11}{4} \end{bmatrix}$$
$$x \geq 0$$

Now x_4 enters and x_2 leaves. So we get $B = \{4, 6\}$ with the canonical form

$$max \qquad (-\frac{1}{2}, -2, -1, 0, -\frac{3}{2}, 0)x - \frac{1}{2}$$

subject to

$$\begin{bmatrix} -\frac{3}{2} & 4 & -3 & 1 & \frac{1}{2} & 0 \\ -\frac{1}{2} & -2 & -1 & 0 & -\frac{1}{2} & 1 \end{bmatrix} x = \begin{bmatrix} \frac{9}{2} \\ \frac{1}{2} \end{bmatrix}$$
$$x \geq 0$$

Note the optimal value for this auxiliary LP is $-\frac{1}{2} < 0$ which by our previous theorem means that the original LP is infeasible. **How do we find a certificate of infeasibility** for the original LP?

Note the *certifiate of optimality* of the *auxliary LP* is a *certificate of infeasibility* for the *original LP*!

16 June 14, 2017

Example. From last time, we can find the certificate of infeasibility from $y = A_B^{-T} c_B$, the certificate of optimality of the auxiliary LP. That is

$$A_B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, c_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore

$$y = A_B^{-T} c_B = \frac{1}{2}\begin{bmatrix}1 & -1 \\ 0 & 2\end{bmatrix}\begin{bmatrix}0 \\ -1\end{bmatrix} = \begin{bmatrix}\frac{1}{2} \\ -1\end{bmatrix}$$

which is the certificate of infeasibility for the original LP.

16.1 Prove Certificate of Optimality of Auxiliary is Certificate of Infeasibility of Original LP

The original LP is

$$max \qquad c^\intercal x$$

subject to

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

And the auxiliary LP is

$$max \qquad -x_{n+1} - x_{n+2} - \ldots - x_m$$

subject to

$$\begin{aligned} \begin{bmatrix}A & I_m\end{bmatrix} x &= b \\ x &\geq 0 \end{aligned}$$

where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, b \geq 0, c \in \mathbb{R}^n$ and

$$I_m = \begin{bmatrix}1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ 0 & 0 & \ldots & 1\end{bmatrix} \in \mathbb{R}^{m\times m}$$

Let $\bar{x}$ be an optimal solution for the auxiliary LP with negative objective value. Let y be the certificate of optimality for $\bar{x}$. Then y is also a certificate of infeasibility for the original LP.

Proof. Note $c' = (0, 0, 0, \ldots, 0, -1, -1, \ldots, -1)^\intercal$ where there are n 0s and m -1s.

Note a certifiate of optimality has $y^\intercal A' \geq c'$. Note

$$y^\intercal \begin{bmatrix}A & I_m\end{bmatrix} = \begin{bmatrix}y^\intercal A & y^\intercal I_m\end{bmatrix}$$

so we know $y^\intercal A \geq c_A = 0$ and $y^\intercal b = c'\bar{x} < 0$. Thus this will be the certificate of infeasibility for the original LP. □

16.2 Feasible Two-Phase Simplex

Example.

$$max \qquad (2,-1,2)x$$

subject to

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$x \geq 0$$

Phase I: Construct the auxiliary LP

$$max \qquad -x_4 - x_5$$

subject to

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$x \geq 0$$

where $B = \{4, 5\}$ is feasible for the auxiliary LP.

$$max \qquad (2,1,0,0,0)x - 4$$

subject to

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$x \geq 0$$

Note we when solve this using Simplex, we'll get $B = \{1, 5\}$ then $B = \{1, 3\}$.
Note the same objective function can be used from before!!!

$$max \qquad -x_4 - x_5$$

subject to

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{3}{2} & 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$x \geq 0$$

Phase II: Use the feasible basis $B = \{1, 3\}$ for the original LP.
The canonical form correspond to $B = \{1, 3\}$

$$max \qquad 6 + x_2$$

subject to

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$x \geq 0$$

x_2 enters and x_1 leaves thus we get $B = \{2, 3\}$. So we get

$$max \qquad -2x_1 + 10$$

subject to

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$
$$x \geq 0$$

thus the optimal basis for the original LP is $\bar{x} = (0, 4, 7)^\intercal$ is an optimal solution.

16.3 Summary of Two-Phase Simplex

1. Phase I: Construct and solve for basic feasible solution of auxiliary LP.
   (a) If optimal value is 0, continue to phase II.
   (b) Otherwise, the original LP is infeasible and the certificate of infeasibility is $y = A_B^{-T} c_B$.
2. Phase II: Use the feasible basis from Phase I to solve for the solution of the original LP.

17 June 16, 2017

17.1 Fundamental Theorem of LP (SEF)

If (P) is an SEF has does not have an optimal solution, then (P) is either infeasible or unbounded. If (P) is feasible, then it has a *basic* feasible solution. If (P) has an optimal solution, then (P) has a basic feasible solution that is optimal.

17.2 Fundamental Theorem of LP

Exactly one of the following holds for an LP (P):

1. (P) is infeasible
2. (P) is unbounded
3. (P) has an optimal solution

which follows from the fundamental theorem of LP (SEF).

17.3 Geometry

Linear problems (or systems of equations) can be interpreted geometrically. Note in the follow example

Example.

$$max \qquad (c_1, c_2)x$$

s.t.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} x \leq \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

The set of points $(x_1, x_2)^\intercal$ satisfying the second row *with equality* is the line in row 2. Set of points that satisfies the constraint is all points to the left of the line in row 2. A similar argument holds for the rest of the constraints.

These constraints will form a shaded region of feasible solutions.

Recall that the dot product $a^\intercal b = \|a\|\|b\|cos(\theta)$ where θ is the angle between a and b. Note that $a^\intercal b = 0$ when a, b are orthogonal, > 0 when the angle is less than 90 degrees, and < 0 when the angle is larger than 90 degrees.

18 June 19, 2017

18.1 Hyperplane and Halfspace

Note that for linear constraints, we can either have $=$ (equality) or $\leq$ (inequality). Let $\beta \in \mathbb{R}$, a a non-zero vector with n components. Then

Hyperplane $H = \{x \in \mathbb{R}^n : a^\intercal x = \beta\}$

Halfspace $F = \{x \in \mathbb{R}^n : a^\intercal x \leq \beta\}$

In the above example, H is the set of points satisfying the constraints with equality and F is the set of points satisfying the constraints in general.

Note if $\bar{x} \in H$, then $a^\intercal \bar{x} = \beta$. Thus $a^\intercal (x - \bar{x}) = 0$. Thus we get the following remarks. Let $\bar{x} \in H$

- H is the set of points x for which a and $x - \bar{x}$ are orthogonal
- F is the set of points x for which a and $x - \bar{x}$ form an angle of at least 90 degrees (since $a^\intercal \bar{x} = \beta$, subtracting β from a value $< \beta$ would result in a negative number, which translates to an angle greater than 90 degrees).

Hyperplanes are all $n - 1$ dimension. Note that the halfspace is a **polyhedron**.

18.2 Converting to Halfspaces (a polyhedron)

Note we can convert $a^\intercal x \geq \beta$ constraints to $-a^\intercal x \leq -\beta$ and for equalities $a^\intercal x = \beta$ to $a^\intercal x \leq \beta$ and $-a^\intercal x \leq -\beta$. Hence any set of linear constraints can be written as $a^\intercal x \leq b$ for some A and some b. Thus all linear constraints can be mapped to a polyhedron.

18.3 Convexity

Note we define the line through $x^{(1)}$ and $x^{(2)}$ to be the set of points

$$\{\lambda x^{(1)} + (1-\lambda)x^{(2)} : \lambda \in \mathbb{R}\}$$

If we would like to bound the line to a line segment with ends $x^{(1)}$ and $x^{(2)}$, then we get

$$\{\lambda x^{(1)} + (1-\lambda)x^{(2)} : 0 \leq \lambda \leq 1\}$$

We define a set C of $\mathbb{R}^n$ to be **convex** if for *every* pair of points $x^{(1)}$ and $x^{(2)}$ in C the line segment with ends $x^{(1)}, x^{(2)}$ is included in C.

Theorem. *All halfspaces are convex.*

Proof. Let x_1 and x_2 be two arbitrary points in the half space $H = \{x : a^\intercal x \leq \beta\}$. Let $\bar{x}$ be an arbitrary point on the line segment between x_1 and x_2, such that (from our definition) $\bar{x} = \lambda x_1 + (1-\lambda)x_2$ where $0 \leq \lambda \leq 1$. So we get

$$a^\intercal \bar{x} = a^\intercal(\lambda x_1 + (1-\lambda)x_2) = \lambda a^\intercal x_1 + (1-\lambda)a^\intercal x_2 = \lambda\beta + (1-\lambda)\beta = \beta$$

where $\bar{x} \in H$. $\square$

Theorem. *For every $j \in J$ let C_j denote a convex set. Then the intersection*

$$C = \bigcap\{C_j : j \in J\}$$

is convex. Note that J can be infinite.

Proof. Let x_1 and x_2 be two arbitrary points in C. For every $j \in J$ where $x_1, x_2 \in C_j$ since C_j is convex the line segment between x_1 and x_2 is in C_j. It follows that the line segment between x_1 and x_2 is in C. Hence C is convex. $\square$

Polyhedra are convex.

19 June 21, 2017

19.1 Extreme Points

A point is *properly contained* in a line segment if it is in a line segment but it is *distinct from its ends* (not one of the end points).

A point x is an **extreme point** of C if there are no line segments in C that properly contains x. That is

Definition. $x \in C$ is **not** an extreme point of C if and only if

$$x = \lambda x_1 + (1-\lambda)x_2$$

for distinct points $x_1, x_2 \in C$ and λ with $0 < \lambda < 1$ (Note non-equality).

How do we isolate for the boundaries of the half space for a given point x on the boundary? We know for some rows of A, there we have $a^\intercal x = \beta$ (equality). We call these constraints tight (or active) for x. These rows of A can be represented by $A^=$. That is

Theorem. *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron and let $\bar{x} \in P$. Then $A^= x = b^=$ be the set of tight constraints for $\bar{x}$. Then $\bar{x}$ is an extreme point of P if and only if $rank(A^=) = n$ (Intuition: in $\mathbb{R}^n$, there must be n vectors from A that bounds a point by equality, sort of like a "corner").*

Proof. Backwards direction:

Suppose $rank(A^=) = n$. Suppose $\bar{x}$ is not an extreme point. Thus there exists $x_1 \neq x_2 \in P$ such that $0 < \lambda < 1$ and $\bar{x} = \lambda x_1 + (1-\lambda)x_2$, Thus

$$b^= = A^=\bar{x} = A^=(\lambda x_1 + (1-\lambda)x_2) = \lambda A^= x_1 + (1-\lambda)A^= x_2 = \lambda b^= + (1-\lambda)b^= = b^=$$

Hence we have an equality throughout which means $A^= x_1 = A^= x_2 = b^=$. As $rank(A^=) = n$, there is a unique solution to $A^= x = b^=$, this is a contradiction thus $\bar{x}$ is an extreme point.

Forwards direction (Contrapositive):

Suppose $rank(A^=) < n$, then we will show $\bar{x}$ is not an extreme point. Note since columns of $A^=$ are linearly dependent, there is a non-zero vector d such that $A^= d = 0$. We pick $\epsilon > 0$ small and define

$$x_1 = \bar{x} + \epsilon d$$
$$x_2 = \bar{x} - \epsilon d$$

Note that $\bar{x} = \frac{1}{2}x_1 + \frac{1}{2}x_2$ and x_1 and x_2 are distinct. Thus $\bar{x}$ is in the line segment between x_1 and x_2. We need to show that $x_1, x_2 \in P$ for $\epsilon > 0$ small enough. Note

$$A^= x_1 = A^=(\bar{x} + \epsilon d) = A^=\bar{x} + \epsilon A^= d = b^=$$

Similarly for $A^= x_2 = b^=$. Let $a^\intercal x \leq \beta$ be any inequalities of $Ax \leq b$ that is not in $A^= x \leq b^=$. It follows for $\epsilon > 0$ small enough

$$a^\intercal x_1 = a^\intercal(\bar{x} + \epsilon d) = a^\intercal \bar{x} + \epsilon a^\intercal d \leq \beta$$

since $a^\intercal \bar{x} \leq \beta$ and ϵ is small. Thus $x_1 \in P$ and similarly $x_2 \in P$. □

19.2 Linear Constraints to Polyhedron

We can convert linear constraints (or a polyhedron) to an intersection of halfspaces for which we can solve for the extreme points.

Example. Consider the polyhedron P

$$P = \left\{ x \in \mathbb{R}^4 : \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, x \geq 0 \right\}$$

We claim $\bar{x} = (0, 0, 2, 1)^\intercal$ the basic feasible solution is an extreme point. Note from before equalities can be converted to $\leq$ and $x \geq 0$ can be represented as $-x_i \leq 0$. Thus

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ -1 & -3 & -1 & 0 \\ 2 & 2 & 0 & 1 \\ -2 & -2 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} x \leq \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \}$$

Thus the set of tight constraints $A^= x \leq b^=$ for $\bar{x}$ is

$$A^= = \begin{bmatrix} 1 & 3 & 1 & 0 \\ -1 & -3 & -1 & 0 \\ 2 & 2 & 0 & 1 \\ -2 & -2 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

(This is derived by taking the two constraints, and ensuring that $x_1, x_2 \geq 0$). Obviously $rank(A^=) = 4 = n$. By the theorem that $rank(A^=) = n \to \bar{x}$ is an extreme point, we have that the basic feasible solution is an extreme point.

Theorem. *Let A be a matrix where the rows are linear independent and let b be a vector. Let $P = \{x : Ax = b, x \geq 0\}$ and let $\bar{x} \in P$. Then $\bar{x}$ is an extreme point if and only if $\bar{x}$ is a basic feasible solution to $Ax = b$.*

Proof. TODO (was in lectures, ask Boshen?) □

20 June 23, 2017

20.1 Non-Polyhedron Example

Example. If $Q = \{x \in \mathbb{R}^2, sin(\alpha)x_1 + cos(\alpha)x_2 \leq 1\}$ for all $\alpha \in \mathbb{R}\}$. $\bar{x} = (1, 0)^\intercal$ is an extreme point of Q. Note however that Q is not a polyhedron so all our theorems before breaks.

20.2 Duality

Example.

$$max \qquad (1,2,0,0)x$$

subject to

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$
$$x \geq 0$$

Solving this LP with simplex method, we will get an optimal solution $\bar{x} = (0,5,3,0)^\intercal$ and a certificate of optimatlity $\bar{y} = (0,2)^\intercal$.

We can check that

1. $\bar{y}^\intercal A \geq c^\intercal \rightarrow (2,2,0,2) \geq (1,2,0,0)$
2. $\bar{y}^\intercal b = c^\intercal \bar{x} \rightarrow 10 = 10$

Another vector that satisfies (1) is $\bar{\bar{y}} = (\frac{1}{2}, 2)^\intercal$ satisfies (1) then for every x feasible for the original LP we have

$$c^\intercal x = (c^\intercal - \bar{\bar{y}}^\intercal A)x + \bar{\bar{y}}^\intercal Ax$$

where $c^\intercal - \bar{\bar{y}}^\intercal A) \leq 0$ (from (1)) and $x \geq 0$. Thus $c^\intercal x \leq \bar{\bar{y}}^\intercal x = \bar{\bar{y}}^\intercal b$.

So if there's a certificate of optimality $\bar{y}$ in the original LP (**primal**) then $y^\intercal A \geq c^\intercal$ implies that the upper bound is $y^\intercal b$. Let us construct a **dual** LP where we minimize that upper bound. For a given **Primal LP (P)** with respect to x

$$max \qquad c^\intercal x$$

subject to

$$Ax = b$$
$$x \geq 0$$

then the **Dual LP (D)** with respect to y is

$$min \qquad b^\intercal y$$

subject to

$$A^\intercal y \geq c$$

20.3 Weak Duality in SEF

Theorem. *Consider the primal-dual pair (P), (D). Let $\bar{x}$ be a feasible solution for (P) and let $\bar{y}$ be a feasible solution for (D).*

Then $c^\intercal \bar{x} \leq b^\intercal \bar{y}$.

Proof. Note from our (P) we have

$$c^\intercal x = (c^\intercal - \bar{y}^\intercal A)\bar{x} + \bar{y}^\intercal A\bar{x}$$

where $c^\intercal - \bar{y}^\intercal A \leq 0$ from the constraints of D, $\bar{x} \geq 0$, so

$$c^\intercal x \leq \bar{y}^\intercal A\bar{x} = \bar{y}^\intercal b = b^\intercal \bar{y}$$

□

20.4 Primal vs Duality

(D) is unbounded (the upper bound $b^\intercal \bar{y}$ is unbounded downwards) if and only if (P) must be infeasible since it would be pushed down indefinitely.

(P) is unbounded if and only if (D) is infeasible (by similar logic).

21 June 26, 2017

21.1 Optimality of Primal and Dual

From last time using weak duality for SEF, we have $c^\intercal \bar{x} \leq b^\intercal \bar{y}$ for some feasible solutions $\bar{x}$ for (P) and $\bar{y}$ for (D).

Theorem. *If $c^\intercal \bar{x} = b^\intercal \bar{y}$, then $\bar{x}$ is an optimal solution for (P) and similarly $\bar{y}$ is optimal for (D).*

Proof. Indeed for every $\tilde{x}$ feasible for (P), we have

$$c^\intercal \tilde{x} \leq b^\intercal \bar{y} = c^\intercal \bar{x}$$

by weak duality, thus $\bar{x}$ is optimal.

Similarly for every $\tilde{y}$ feasible for (D), we have

$$b^\intercal \tilde{y} \geq c^\intercal \bar{x} = b^\intercal \bar{y}$$

again by weak duality, so $\bar{y}$ is optimal. □

21.2 Infeasibility of Primal and Dual

It might happen that both (P) and (D) are infeasible, for example, let (P) be

$$max \qquad (0,1)x$$

subject to

$$\begin{aligned} &(1,0)x = -1 \\ &x \geq 0 \end{aligned}$$

and let (D) be

$$min \qquad -y$$

subject to

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} y \geq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

21.3 Strong Duality in SEF

Let the primal LP (P) be

$$\mathit{max} \qquad c^\intercal x$$

subject to

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

then the dual LP (D) is

$$\mathit{min} \qquad b^\intercal y$$

subject to

$$A^\intercal y \geq c$$

Theorem. *Let (P) and (D) be the LPs above. If (P) has an optimal solution then (D) has an optimal solution. Moreover, the optimal value of (P) equals the optimal value of (D).*

Proof. (We need the correctness of Simplex Method: the simplex with Bland's rule terminates) So we assume that the Simplex Method applied to (P) terminates with an optimal solution $\bar{x}$. This means that $\bar{x}$ is a basic feasible solution for an optimal basis B. That is, the *objective function corresponding* to the canonical from for B has only non-positive coefficients (Simplex method)

$$\bar{y}^\intercal b + (c^\intercal - \bar{y}^\intercal A)x$$

where $\bar{y} = A_B^{-\intercal} c_B$. Since B is an optimal basis $c^\intercal - \bar{y}^\intercal A \leq 0^\intercal \iff A^\intercal \bar{y} \geq c$. So $\bar{y}$ is feasible for (D).

Now it is enough to show that $c^\intercal \bar{x} = b^\intercal \bar{y}$ because of weak duality ($c^\intercal \bar{x} \leq b^\intercal \bar{y}$). That is

$$\begin{aligned} c^\intercal \bar{x} &= (c^\intercal - \bar{y}^\intercal A + \bar{y}^\intercal A)\bar{x} \\ &= (c^\intercal - \bar{y}^\intercal A)\bar{x} + \bar{y}^\intercal b \end{aligned}$$

where $\bar{y}^\intercal A\bar{x} = y^\intercal b$ because $A\bar{x} = b$.

Note that $\bar{x}$ is a basic feasible solution thus $\bar{x}_N = 0$. Furthermore, from (B), $c^\intercal - \bar{y}^\intercal A$ is the objective function in basis B, thus it has $(c^\intercal - \bar{y}^\intercal A)_B = 0$. $\square$

22 June 28, 2017

22.1 Primal and Dual Solution Types Matrix

Our final chart of possible solution types between the primal and dual LP is

	Dual		
Primal	Infeasible	Unbounded	Optimal
Infeasible	Yes	Yes	No
Unbounded	Yes	No	No
Optimal	No	No	Yes

22.2 Deriving Dual from Primal (non-SEF)

Example. Given the primal

$$min \qquad (1,3,5)x$$

subject to

$$\begin{aligned}
&(4,0,1)x \geq 1\\
&(7,-2,3)x \leq 3\\
&(0,1,8)x = 4\\
&(2,6,1)x \leq 10\\
&x_1 \text{ free}\\
&x_2 \geq 0\\
&x_3 \leq 0
\end{aligned}$$

We want a y such that

$$y^\intercal Ax \geq y^\intercal b$$

so we see that $y_1 \geq 0$, $y_2, y_4 \geq 0$, and y_3 free (based on the signs in the primal).

So, when can we say that $y^\intercal Ax \geq y^\intercal b$ implies $c^\intercal x \geq y^\intercal b$ for every feasible solution x?

Let's look at an example $(1,1,6)x \geq 7$ (our $y^\intercal Ax \geq y^\intercal b$ for x such that x_1 free, $x_2 \geq 0$, and $x_3 \leq 0$. Does this imply $(1,3,5)x \geq 7$ ($c^\intercal x \geq y^\intercal b$?

Let us subtract the two LHS

$$(1,1,6)x - (1,3,5)x = (0,-2,1)x = 0x_1 + (-2)x_2 + x_3$$

Note the domain of our x_i implies that the above expression is ≤ 0.

So note for each variable x_i

free we need the corresponding $y_i^\intercal A_i$ to be equivalent to $c_i^\intercal$ (in the above, $1 = 1$)

≥ 0 $y_i^\intercal A_i \leq c_i^\intercal$ (in the above, $1 \leq 3$)

≤ 0 $y_i^\intercal A_i \geq c_i^\intercal$ (in the above, $6 \geq 5$)

So our dual LP and its objective function $b^\intercal y$

$$max \qquad (1,3,4,10)y$$

subject to ($A^\intercal y \geq c$)

$$\begin{aligned}
&(4,7,0,2)y = 1\\
&(0,-2,1,6)y \leq 3\\
&(1,3,8,1)y \geq 5\\
&y_1 \geq 0\\
&y_2, y_4 \leq 0\\
&y_3 \text{ free}
\end{aligned}$$

Note the signage comes from the domain of x_1, x_2, x_3 correspondingly (if x_i is free, then use $=$; $x_i \geq 0$, then $\leq$; $x_i \leq 0$, then $\geq$).

23 June 30, 2017

23.1 Another Example of Dual from Primal

Given primal (P)

$$min \qquad (7,13,-3,-5)x$$

subject to

$$\begin{aligned}
&(2,1,2,0)x \leq 4\\
&(3,5,-3,-2)x \geq -5\\
&(1,3,2,-1)x \geq -4\\
&(-1,2,-4,7)x \leq 18\\
&x_1 \geq 0\\
&x_2 \leq 0\\
&x_3 \geq 0\\
&x_4 \geq 0
\end{aligned}$$

where $\bar{x} = (2,-1,0,3)^\intercal$ is a feasible solution.

Thus our dual would be (P)

$$max \qquad (4,-5,-4,18)y$$

subject to

$$\begin{aligned}
&(2,3,1,-1)y \leq 7\\
&(1,5,3,2)y \geq 13\\
&(2,-3,2,-4)y \leq -3\\
&(0,-2,-1,7)y \leq -5\\
&y_1 \leq 0\\
&y_2 \geq 0\\
&y_3 \geq 0\\
&y_4 \leq 0
\end{aligned}$$

where $\bar{y} = (0, 2, 1, 0)^\intercal$ is a feasible solution.

We can show that $\bar{x}$ and $\bar{y}$ are optimal for (P) and (D) respectively since $\bar{x}$ is feasible for (P), $\bar{y}$ is feasible for (D) and

$$(7, 13, -3, -5)\bar{x} = 14 - 13 - 15 = -14$$

$$(4, -5, -4, 18)\bar{y} = -10 - 4 = -14$$

by Weak Duality, since $-14 = -14$ then they are optimal.

The constraints in (P) that are tight for $\bar{x}$ are the 2nd and 3rd row. This follows from the fact that the non-zero coordinates of $\bar{y}$ are the 2nd and 3rd.

Similarly, the constraints in (D) held tight for $\bar{y}$ is the 1st, 2nd and 4th.

23.2 Dual of Dual is Primal

Note that (P) from the previous example is also the dual of (D).

23.3 Complementary Slackness (Special Case)

Theorem. *Given primal (P)*

$$max \qquad c^\intercal x$$

subject to

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned}$$

(note the $\leq$ for the constraints) and the dual (D)

$$min \qquad b^\intercal y$$

subject to

$$A^\intercal y \geq c$$

Let $\bar{x}$ be feasible for (P) and $\bar{y}$ be feasible for (D). Then $\bar{x}$ and $\bar{y}$ are optimal for (P) and (D), respectively if and only if ($A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$)

(i) $\forall i, 1 \leq i \leq m$, either the i-th constraint in $Ax \leq b$ is tight for $\bar{x}$ or $\bar{y}_i = 0$.

(ii) $\forall j, 1 \leq j \leq n$, either the j-th constraint in $A^\intercal y \geq c$ is tight for $\bar{y}$ or $\bar{x}_i = 0$.

Proof. Note that (remember $\bar{x}, \bar{y} \geq 0$)

$$c^\intercal \bar{x} \leq A^\intercal A\bar{x} \leq \bar{y}^\intercal b$$

By Strong Duality, we need $c^\intercal \bar{x} = b^\intercal \bar{y} = y^\intercal b$, that is

$$c^\intercal \bar{x} = \bar{y}^\intercal A\bar{x}$$

and

$$\bar{y}^\intercal A\bar{x} = \bar{y}^\intercal b$$

Since $\bar{x} \geq 0$ and $A^\intercal \bar{y} \geq c$, we have

$$c^\intercal \bar{x} = \bar{y}^\intercal A\bar{x}$$

only if condition (2) holds. Since $\bar{y} \geq 0$ and $A\bar{x} \leq b$, we have

$$\bar{y}^\intercal A\bar{x} = \bar{y}^\intercal b$$

only if condition (1) holds. $\square$

24 July 5, 2017

24.1 Cone

Definition. The **cone** generated by $a^{(1)}, \ldots, a^{(m)} \in \mathbb{R}^n$ is the set

$$cone\{a^{(1)}, \ldots, a^{(m)}\} = \{\sum_{i=1}^{m} \lambda_i a^{(i)} : \lambda_i \geq 0 \forall i\}$$

That is the cone is the linear combinations of the vectors $a^{(i)}$.

Note: We define the cone generated by the empty set to be the set containing only the zero vector i.e. the set $\{0\}$.

Graphically, the cone looks like a literal cone bounded by the given vectors $a^{(i)}$.

Note $(5,6) \in cone\{(5,2),(0,1)\}$ where $\lambda_1 = 1, \lambda_2 = 4$.

At the same time, $(-1,3) \in cone\{(5,2),(0,1)\}$. To prove this, show that there are no coefficients $\lambda_1, \lambda_2 \geq 0$ such that $(-1,3) = \lambda_1(5,2) + \lambda_2(0,1)$.

24.2 Geometric Characterization of Optimality

Consider the LP (P)

$$max \qquad c_1x_1 + c_2x_2$$

subject to

$$\begin{bmatrix} 5 & 2 \\ -1 & 1 \\ 0 & 1 \\ -1 & 0 \\ 1 & -4 \end{bmatrix} x \leq \begin{bmatrix} 30 \\ 3 \\ 5 \\ -1 \\ -3 \end{bmatrix}$$

x free

Note c_1 and c_2 are constants.

Consider the feasible solution $\bar{x} = (4,5)^\intercal$. For what coefficients $(c_1, c_2) \in \mathbb{R}^2$ is $\bar{x}$ an optimal solution of (P)?

The dual of (P) is

$$min \qquad (30, 3, 5, -1, -3)y$$

subject to

$$\begin{bmatrix} 5 & -1 & 0 & -1 & 1 \\ 2 & 1 & 1 & 0 & -4 \end{bmatrix} y = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$
$$y \geq 0$$

By strong duality and complementary slackness, we have $\bar{x}$ is optimal if and only if there exists a $\bar{y}$ such our constraints are satisfied and $\bar{y_i} = 0$ whenever the i-th constraint is not tight at $\bar{x}$.

For $\bar{x} = (4,5)^\intercal$, the 2nd, 4th, and 5th constraint are not tight, which implies $\bar{y_2}, \bar{y_4}, \bar{y_5} = 0$.

That is $\bar{x}$ is optimal if and only if

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} \bar{y_1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bar{y_3} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

for some $\bar{y_1}, \bar{y_3} \geq 0$.

So for $c = (10, 7)^\intercal$, $\bar{x}$ is an optimal solution since

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} 2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 3 = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$$

And we can do this for arbitrarily many cs.

We can generalize this for an arbitrary LP using cones

Theorem. *Let $\bar{x}$ be a feasible solution to*

$$max\{c^\intercal x | Ax \leq b\}$$

$\bar{x}$ is an optimal solution if and only if c lies in the cone of tight constraints for $\bar{x}$ (i.e. the cone generated by rows of A corresponding to tight constraints).

For our example, the theorem gives us that $\bar{x}$ is optimal if and only if

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in cone\{\begin{bmatrix} 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$$

Thus $c = (-1, 3)^\intercal$ cannot be a c such that $\bar{x}$ is optimal.

25 July 7, 2017

25.1 Minimum Cost Perfect Matching

Definition. For a given graph $G(V, E)$, we have costs c_e for all edges $e \in E$.

Recall a set of edges is matching if a given vertex is incident to at most one edge in the set.

A **perfect matching** M is a subset of edges that for *every vertex* v exactly one edge of M is incident to v.

We want to find the perfect matching that has the minimum cost, i.e. min $\sum e \in M c_e$, or the **minimum cost perfect matching**.

25.2 Showing Perfect Matching is Minimum

Example. Note for a given graph, say 3 edges $\{ag, hb, cd\}$ in a minimum cost perfect matching have costs $3 + 2 + 1 = 6$. How do we show that it is minimal without going through all combinations?

Note in our example (see page 117 in textbook for drawing), if we reduce the cost of edges incident to vertex b by 3, our minimum cost perfect matching will still be the minimum cost perfect matching in the graph with the reduced cost.

Note we pick values to subtract such that all edges have *non-negative* cost and our minimum cost perfect matching has a total cost of 0. Since all edges are non-negative, we thus know that our perfect matching is indeed minimal.

25.3 Reduced Cost and Potential

Definition. Let us assign every vertex u a number y_u that we call the **potential of the vertex** u. The **reduced cost** edge uv is

$$\bar{c_{uv}} = c_{uv} - y_u - y_v$$

Note since a perfect matching only includes every vertex once, the original cost is exactly $\sum_{u \in V} y_u$ more than the reduced cost of the perfect matching. Thus a perfect matching with reduced cost $\bar{c}$ must have an original cost c (a priori the above equation).

If reduced cost (edge) is nonnegative, then every perfect matching will have nonnegative costs. Thus a perfect matching with $\bar{c} = 0$ it will be minimum with respect to the reduced costs, which is also with respect to the original costs.

Definition. An **equality edge** (or tight edge) is a edge with a reduced cost of 0.

25.4 Feasibility for Minimum Cost Perfect Matching

Thus for our LP a set of potentials y is feasible if for every edge $e \in E$:

$$\bar{c_{uv}} = c_{uv} - y_u - y_v \geq 0$$

25.5 Optimality for Perfect Matchings

Proposition. *If the potentials y are feasible and all edges of M are equality edges with respect to y, then M is a minimum cost perfect matching.*

The integer programming formulation (IP) is

$$min \qquad \sum_{e \in E} c_e x_e$$

subject to

$$\sum_{e \in \delta(v)} x_e = 1, \forall \in V$$
$$x_e \geq 0, \forall e \in E$$
$$x_e \text{ integer}, \forall e \in E$$

Note that the **linear programming relaxation** of the (IP) is (P). (P)'s constraints are a subset of (LP)'s constraints thus the optimal value of (IP) would be greater than or equal to that of (P) (that is (P) is a lower bound for (IP)). This makes sense since (IP) can only get "worse" than (P).

We can thus use duality theory and say that the optimal value of (P) is greater than or equal to the value of any feasible solution $\bar{y}$ to (D).

So any feasible solution of (D) is a lower bound to cost of any perfect matching.

25.6 General Primal-Dual for Minimum Cost Perfect Matching

In general the *LP relaxation* for a minimum cost perfect matching (P) is

$$min \qquad \{c^\intercal x : Ax = 1; x \geq 0\}$$

where c is a vector of edge costs, and matrix A is defined as:

(i) rows of A are indexed by vertices $v \in V$

(ii) columns indexed by every edge $e \in E$

(iii) every row U and every column e

$$A[v,e] = \begin{cases} 1 & \text{if v is an endpoint of e} \\ 0 & \text{otherwise} \end{cases}$$

That is in row v of A, entries with a 1 corresponds to edges incident to v; column e of A, entries with a 1 corresponds to an endpoint in e.

The dual (D) is thus

$$max \qquad \{1^\intercal y : A^\intercal \leq c : y \text{ free}\}$$

Note this maps to

$$max \qquad \sum_{v \in V} y_v$$

subject to

$$y_u + y_v \leq c_{uv}, \forall uv \in E$$

which intuitively makes sense (maximize the potential drops but ensure $\bar{c_{uv}} \geq 0$.

Note that if every $uv \in M$ is an equality edge, then $\bar{y_u} + \bar{y_v} = c_{uv}$. Thus our objective function in the dual (D)

$$1^\intercal \bar{y} = \sum_{v \in V} \bar{y_v} = \sum_{uv \in M} (\bar{y_u} + \bar{y_v}) = \sum_{uv} c_{uv} = c(M)$$

Since the value of $\bar{y}$ is a lower bound, it follows that $\bar{x}$ M is a minimum cost perfect matching.

25.7 Bipartite graphs

Definition. A graph $G = (V, E)$ is bipartite if we can color its vertices RED(U) and BLUE(W) so that every edge has one red endpoint and one blue endpoint.

That is every edge has one vertex in U and one vertex in W.

25.8 Perfect Matching and Bipartite Graphs

Note that in order for a bipartite graph to have a perfect matching, $|U| = |W|$. This however does not necessarily gaurantee a perfect matching exists.

Note the case where we have four vertices $\{a, b, c, d\}$ in partition U and four vertices in $\{e, f, g, h\}$.

Note that $S = \{a, b, c\}$ is only connected to vertices $N_G(S) = \{e, f\}$. We call $N_G(S)$ the **set of neighbours** of S the set of all vertices outside of S that are joined by an edge to some vertex of S. That is

$$N_G(S) = \{r \in V \setminus S : sr \in E \text{ and } s \in S\}$$

Suppose there is a set $S \subseteq U$ where $|S| = |N_G(S)|$ (as is the case above). Let there be an arbitrary matching M of G. All edges of M that have an endpoint of S have an endpoint in $N_G(S)$. However, at most $|N_G(S)|$ of the vertices of S can be an endpoint of some edge of M since M is a matching.

Since $|S| > |N_G(S)|$, M cannot be a perfect matching and S is a **deficient set**.

25.9 Hall's Theorem

Theorem. *Let $G = (V, E)$ be a bipartite graph with bipartition U, W where $|U| = |W|$.*

There exists a perfect matching M in G if and only if G has no deficient set $S \subseteq U$.

There also exists an efficient (polynomial time)[3] *that will either find a perfect matching M or a deficient set $S \subseteq U$.*

26 July 10, 2017

26.1 Hungarian Algorithm for Minimum Cost Perfect Matching for Bipartite Graphs

We start with $y_u = \frac{1}{2} min\{c_e : c \in E\}$ for all $u \in V$. Note that since every edge is incident to only two edges, all edges will have non-negative costs.

For every iteration:

We then take all equality/tight edges (all edges where the reduced cost $\bar{c_{uv}} = c_{uv} - \bar{y_u} - \bar{y_v} = 0$) with all vertices and call this graph H.

If there is a deficient set $S \subseteq U(H)$ where U is one partition ($|S| > N_H(S)$), let us consider edges in G that are incident to any vertex in S and another vertex NOT in $N_H(S)$. We take the minimum reduced cost $\bar{c_{min}}$ of these edges.

We then subtract $\bar{c_{min}}$ from y_s for $s \in S$ (increase potential decrease of vertices in S) and add $\bar{c_{min}}$ to vertices in $N_H(S)$. Thus vertices in S are now reducing the costs of their edges even more (but are non-negative since we took the minimum of their incident edges). We offset this decrease for edges between S and $N_H(S)$ with $N_H(S)$.

Eventually all edges in G with an endpoint in S have an endpoint in $N_H(S)$ (all neighbours of S are in $N_H(S)$), then if S is a deficient set we stop (no perfect matching).

If there is a perfect matching at this stage (equality edges that form a matching that includes all vertices) then the corresponding edges compose the minimum cost perfect matching in the original graph.

27 July 12, 2017

27.1 Integer Programming as LP Relaxation (with Geometry)

Example. Given a (IP)

$$max \qquad (1,1)x$$

subject to

$$\begin{bmatrix} 2 & 4 \\ 2 & 0 \\ 1 & -4 \\ -2 & 1 \\ -3 & -2 \end{bmatrix} x \leq \begin{bmatrix} 11 \\ 5 \\ 4.5 \\ 1.5 \\ 0.5 \end{bmatrix}$$

$$x \text{ integer}$$

Let us consider the polyhedron P defined by the linear constraints. If we run the Simplex Method for the LP relaxation of (IP), we obtain the optimal solution $(\frac{5}{2}, \frac{3}{2})^\intercal$ (an extreme point of P). But $(\frac{5}{2}, \frac{3}{2})^\intercal$ is a fractional vector. In fact, all extreme points of P are fractional (problematic).

Let us consider $S = P \cap \mathbb{Z}^2$ where $\mathbb{Z}^2$ are integral coordinates. Thus S are the integral points inside the polyhedron region P. That is

$$S = \{(0,0),(1,0),(2,0),(0,1),(1,1),(2,1),(1,2)\}$$

In general, S can be very large (order of 2^{n-1}).

If Q is a polyhedron, then we can write an LP to maximize $(1,1)x$ over Q.

The idea is to take our integral points S and draw a polyhedron around the points such that the extreme points of the new polyhedron are integral points in S.

That is, suppose we can find Q convex set such that $S \subseteq Q$ and all extreme points of Q are in S.

In our example, we have $Q = \{x \in \mathbb{R}^2\}$ subject to

$$\begin{aligned}
&x_1 \geq 0\\
&x_2 \geq 0\\
&x_1 + x_2 \leq 3\\
&-x_1 + x_2 \leq 1\\
&x_1 \leq 2
\end{aligned}$$

If we then use the Simplex method to maximize $(1,1)x$ over Q, we will obtain the solution $(2,1)^\intercal$.

27.2 Convex Hull

Definition. In this example we chose Q to be $conv(S)$ where $conv(S)$ is the **convex hull**: the smallest convex set containing S.

How do we find such a convex hull?

Theorem. *Consider $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ where A, b have rational entries. Let S be the set $P \cap \mathbb{Z}^n$. Then the convex hull $conv(S)$ is a polyhedron and can be descrbed as*

$$\{x \in \mathbb{R}^n : \tilde{A}x \leq \tilde{b}\}$$

where $\tilde{A}, \tilde{b}$ have rational entries.

27.3 IP to LP with Convex Hull

That is for a given (IP)

$$max \qquad c^\intercal x$$

subject to

$$\begin{aligned}
Ax &\leq b\\
x &\in \mathbb{Z}^n
\end{aligned}$$

We can reformulate it using the convex hull to an (LP)

$$max \qquad c^\intercal x$$

subject to

$$\tilde{A}x \leq \tilde{b}$$

Some properties between (IP) and (LP):

(i) (IP) infeasible $\iff$ (LP) infeasible

(ii) (IP) unbounded $\iff$ (LP) unbounded

(iii) every optimal solution of (IP) is optimal solution of (LP)

(iv) every optimal solution of (LP) **which is integral** is an optimal solution of (IP) (not all optimal solutions of (LP) are optimal for (IP), e.g. a rational solution on the same optimal boundary line)

28 July 14, 2017

28.1 Cutting Planes

Example. Given the IP

$$max \qquad (2,5)x$$

subject to

$$\begin{bmatrix}1 & 4\\ 1 & 1\end{bmatrix} x \leq \begin{bmatrix}8\\4\end{bmatrix}$$
$$x \geq 0$$
$$x \text{ integer}$$

The optimal for the LP relaxation is $\bar{x} = (\frac{8}{3}, \frac{4}{3})^\intercal$.

Suppose we find a linear constraint $a_1x_1 + a_2x_2 \leq \beta$ such that

(i) $a_1x_1 + a_2x_2 \leq \beta$ is valid for every feasible solution of the IP

(ii) $a_1\bar{x_1} + a_2\bar{x_2} > \beta$, so $\bar{x}$ violates our linear constraint

Definition. That is, we find a halfspace constraint that makes our fractional optimal solution infeasible. This linear constraint is called a **cutting plane** for $\bar{x}$.

Example. Let us verify that the linear constraint

$$(1,3)x \leq 6$$

holds for all feasible solutions of IP. We add 2 of the first row and 1 of the second row to get

$$(3,9)x \leq 20$$

holds for all feasible solutions. We divide this by 3 to see that

$$(1,3)x \leq \frac{20}{3}$$

Note the LHS is an integer number for every feasible solution of the IP, thus

$$(1,3)x \leq \lfloor \frac{20}{3} \rfloor = 6$$

as desired.

28.2 Obtaining the Cutting Plane

Example. Note that the Simplex Method for the LP relaxation of the IP above finishes with basis $\{1,2\}$ and canonical form

$$max \qquad 12 + (0,0,-1,-1)x$$

subject to

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} x = \begin{bmatrix} \frac{8}{3} \\ \frac{4}{3} \end{bmatrix}$$

From our first row (we could use the second row), note that $1x_1 + (-\frac{1}{3})x_3 + \frac{4}{3}x_4 = \frac{8}{3}$. Since $x_3 \geq 0$, then for every feasible solution

$$-\frac{1}{3}x_3 \geq \lfloor -\frac{1}{3}x_3 \rfloor = -1x_3$$

Similarly for $x_4 \geq 0$

$$\frac{4}{3}x_4 \geq \lfloor \frac{4}{3}x_4 \rfloor = x_4$$

Therefore we have

$$1x_1 - 1x_3 + 1x_4 \leq \frac{8}{3}$$

Note the LHS is an integer number for every feasible solution for the IP. Thus

$$1x_1 - 1x_3 + 1x_4 \leq \lfloor \frac{8}{3} \rfloor = 2$$

Recall that x_3 and x_4 are slack variables such that

$$x_3 = 8 - x_1 - 4x_2$$
$$x_4 = 4 - x_1 - x_2$$

Substituting this into the equation we derived, we get our linear constraint

$$x_1 + 3x_2 \leq 6$$

which holds for all feasible solutions of IP.

29 July 17, 2017

29.1 Solving LP with Added Cutting Plane

Example. From last time, we obtained an additional cutting plane constraint $(1,3)x \leq 6$. When we add this to our LP relaxation of our IP

$$max \qquad (2,5)x$$

subject to

$$\begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} x \leq \begin{bmatrix} 8 \\ 4 \\ 6 \end{bmatrix}$$
$$x \geq 0$$
$$x \text{ integer}$$

Solving this LP relaxation, we obtain an optimal solution that is integral $(3,1)^\intercal$, thus we are done.

29.2 Formalized Cutting Plane Algorithm

Given an IP problem

$$max \qquad c^\intercal x$$

subject to

$$Ax \leq b$$
$$x \geq 0$$
$$x \text{ integer}$$

1. Consider LP relaxation of the current IP problem (IP problem with no integer constraint)

2. Solve LP relaxation of current LP problem. Assuming we obtain an optimal solution $\bar{x}$

3. If $\bar{x}$ is integer, then $\bar{x}$ is optimal solution for original IP problem

4. If $\bar{x}$ is not integer, the last canonical form has a row with fractional RHS. Use this row to get a cutting plane (*) for $\bar{x}$.

5. Add the linear constraint (*) to the current IP problem and go to step (2).

Remarks:

1. If LP relaxation is infeasible, then IP is *infeasible*

2. If LP relaxation is unbounded (**subject to A, b having rational entries**), then IP is either *unbounded or infeasible*

   Note if A, b do not have rational entries, then it could be anything

   Example.

   $$max \qquad (0,1)x$$

   subject to

   $$\begin{aligned}&(1, -\sqrt{2})x = 0\\&x \geq 0,\ \text{integer}\end{aligned}$$

   Note LP is unbounded with $\bar{x} = (0,0)$ and $d = (\sqrt{2}, 1)$. However, IP has one feasible solution $(0,0)$.

29.3 Non-linear Programming - Convex Functions

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for every pair of points $x^{(1)}, x^{(2)} \in \mathbb{R}^n$ and $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$ we have

$$f(\lambda_1 x^{(1)} + \lambda_2 x^{(2)}) \leq \lambda_1 f(x^{(1)}) + \lambda_2 f(x^{(2)})$$

That is if we draw a line between two points on the function (this is our RHS, linear combination of the two function values), then all points of the function that lie in between the two points (our LHS) is below this line).

30 July 19, 2017

30.1 Proving a Function is Convex

Example. $f(x) = x^2$ is convex. Proof in textbook.

Example. Is $f(x) = |x|$ convex? Yes.

Proof. Let $x^{(1)}, x^{(2)} \in \mathbb{R}$, $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$ then

$$\begin{aligned}&f(\lambda_1 x^{(1)} + \lambda_2 x^{(2)}) \leq f(\lambda_1 x^{(1)}) + f(\lambda_2 x^{(2)})\\ \iff &|\lambda_1 x^{(1)} + \lambda_2 x^{(2)}| \leq |\lambda_1 x^{(1)}| + |\lambda_2 x^{(2)}|\\ \iff &|\lambda_1 x^{(1)} + \lambda_2 x^{(2)}| \leq \lambda_1 |x^{(1)}| + \lambda_2 |x^{(2)}|\end{aligned}$$

by the triangle inequality, where for every $t_1, t_2 \in \mathbb{R}$, $|t_1 + t_2| \leq |t_1| + |t_2|$. □

Example. Is $f(x) = x^3$ convex? No.

Proof. Let $x^{(1)} = -2, x^{(2)} = 1$, and $\lambda_1 = \lambda_2 = \frac{1}{2}$. Note that

$$\begin{aligned} & f(\lambda_1 x^{(1)} + \lambda_2 x^{(2)}) > \lambda_1 f(x^{(1)}) + \lambda_2 f(x^{(2)}) \\ \iff & f(0.5(-2) + 0.5(1)) > 0.5(-2)^3 + 0.5(1)^3 \\ \iff & (-0.5)^3 > -4 + 0.5 \\ \iff & -\frac{1}{8} > -\frac{7}{2} \end{aligned}$$

□

30.2 Level Set

Definition. Let $g : \mathbb{R}^n \to \mathbb{R}$, the set

$$\{x \in \mathbb{R}^n : g(x) \leq \beta\}$$

is called a **level set** of the function g.

Example. Let $g(x) = x^2$, where $g : \mathbb{R} \to \mathbb{R}$.

Then the level set would be

$$\{x \in R : x^2 \leq \beta\} = \begin{cases} [-\sqrt{\beta}, \sqrt{\beta}] & \beta \geq 0 \\ \varnothing & \beta < 0 \end{cases}$$

Remark. If g is a convex function then the level set

$$S = \{x \in \mathbb{R}^n : g(x) \leq \beta\}$$

is a convex set.

Proof. Let $x^{(1)}, x^{(2)} \in S$, $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$.

We have to show that $\lambda_1 x^{(1)} + \lambda_2 x^{(2)} \in S$.

In other words, we have to show

$$g(\lambda_1 x^{(1)} + \lambda_2 x^{(2)}) \leq \beta$$

Note that $g(x^{(1)}), g(x^{(2)}) \leq \beta$, note that

$$\begin{aligned} g(\lambda_1 x^{(1)} + \lambda_2 x^{(2)}) &\leq \lambda_1 g(x^{(1)}) + \lambda_2 g(x^{(2)}) \\ &\leq \lambda_1 \beta + \lambda_2 \beta \\ &= \beta \end{aligned}$$

□

Remark. The inverse of the previous remark is not true.

That is if every level set is a convex set (or a given level set is a convex set), this DOES NOT imply that g is a convex function.

Proof. For example, let $g(x) = x^3$ is not a convex function. Note that its level set

$$\{x \in \mathbb{R} : x^3 \leq \beta\} = (-\infty, \sqrt[3]{\beta}]$$

is always convex. □

31 July 21, 2017

31.1 Epigraph

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Then the set

$$epi(f) = \{\begin{bmatrix} \alpha \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n : f(x) \leq \alpha\}$$

is called an epigraph of f. That is the epigraph is the shaded region above the the graph f(x).

Proposition. *(Proposition 7.2 in textbook).*
f is convex if and only if $epi(f)$ is convex.

31.2 Non-Linear Programming Revisited

An NLP is an optimization problem of the form

$$min \qquad f(x)$$

subject to

$$g_1(x) \leq 0$$
$$\vdots$$
$$g_m(x) \leq 0$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$.

Remark. NLP is a very general class of problems. That is, NLP is more general than IP problem:

$$x \in \mathbb{Z} \iff sin(\pi x) = 0$$

31.3 Convex NLPs

If all functions $f : \mathbb{R}^n \to \mathbb{R}$, $g_1, \ldots, g_n : \mathbb{R}^n \to \mathbb{R}$ are convex, then the NLP is also convex.

Remark. The feasible region of a convex NLP problem is a convex set.

Proof. The feasible region is $\bigcap_{i=1}^m C_i$ where

$$C_i = \{x \in \mathbb{R}^n : g_i(x) \leq 0\}$$

Note that C_i is a level set of the convex function g_i, hence C_i is also convex for all $i \in \{1, \ldots, m\}$.

The intersection of convex sets is convex, so the feasible region is convex. □

31.4 Convex NLP Optimality Example

Example. Given an NLP

$$min \qquad -x_1 - x_2$$

subject to

$$x_2^2 - x_1 \leq 0$$
$$x_1^2 - x_2 \leq 0$$
$$-x_1 + \frac{1}{2} \leq 0$$

Note that $f(x)$ and $g_i(x)$ are all convex (sum of convex functions are convex, affine functions are convex), thus the NLP is convex.

We can draw our NLP by drawing out each constraint.

C_1 is a parabola with boundary $x_2^2 = x_1$ or $x_2^2 = \pm\sqrt{x_1}$ (sideways parabola that is the combination of $\pm\sqrt{x_1}$ function). The epigraph is everything in between (convex) since $x_2 \leq \sqrt{x_1}$ and $x_2 \geq -\sqrt{x_1}$.

C_2 is a parabola with boundary $x_2 = x_1^2$ (your canonical upright parabola). Epigraph is everything above the parabola ($x_2 \geq x_1^2$).

C_3 is a vertical line at $x_1 = \frac{1}{2}$ and epigraph of everything to the right.

Note that our objective function is some $x_2 = -x_1 - V$ where V is our optimal solution. For example, $x_2 = -x_1 + 2$ is a line through the top-right extreme point of our feasible region. $V = -2$ and we can intuitively see that this is the optimal solution (since as V decreases, the line moves farther away from the feasible region towards the top-right).

Note that $\bar{x} = (1,1)^\intercal$ is feasible for the NLP with value -2.

To prove that $\bar{x}$ is an optimal solution, we will consider an LP relaxation of the NLP (such that every feasible solution to the NLP is also feasible for the constructed LP; the LP is an upper bound).

Let $h_i(x)$ denote the corresponding LP constraint for $g_i(x)$. Note we want:

(i) $h_i : \mathbb{R}^2 \to \mathbb{R}$ is an affine function

(ii) $\{x \in \mathbb{R}^2 : g_i(x) \leq 0\} \subseteq \{x \in \mathbb{R}^2 : h_i(x) \leq 0$ (that is $h_i(x)$ is a larger/upper bound for $g_i(x)$)

(iii) $h_i(\bar{x}) = g_i(\bar{x}) = 0$ (It is tangent to $g_i(x)$ and intersects $\bar{x}$)

In our example, we only need to generate $h_1(x)$ and $h_2(x)$ (we can drop $g_3(x)$).

Thus our LP relaxation would be something like

$$min \qquad -x_1 - x_2$$

subject to

$$h_1(x) \leq 0$$
$$h_2(x) \leq 0$$

where $\bar{x}$ is optimal for this LP.

How do we find our $h_1(x)$ and $h_2(x)$?

We can compute the gradient of g_1 and g_2 at point x. That is

$$\nabla g_i(x) = [\frac{\partial g_i(x)}{\partial x_1}, \frac{\partial g_i(x)}{\partial x_2}]^\intercal$$

Then we can define

$$h_i(x) = g_i(\bar{x}) + \nabla g_i(\bar{x})^\intercal (x - \bar{x})$$

In our example, we have

$$\begin{aligned} \nabla g_1(x) &= [-1, 2x_2]^\intercal \\ \nabla g_1(\bar{x}) &= [-1, 2]^\intercal \\ g_1(\bar{x}) &= 0 \end{aligned}$$

Thus

$$\begin{aligned} h_1(x) &= (-1, 2) \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \\ &= -x_1 + 1 + 2x_2 - 2 \\ &= -1 - x_1 + 2x_2 \end{aligned}$$

Similarly $h_2(x) = -x_1 + 2x_2 - 1$.

Remark. For each convex differentiable $g : \mathbb{R}^n \to \mathbb{R}$ the function $h_i(x)$ satisfying (i), (ii), (iii) can be computed this way.

32 July 24, 2017

Example. From last time, we derived the LP relaxation:

$$min \qquad -x_1 - x_2$$

subject to

$$\begin{aligned} -x_1 + 2x_2 &\le 1 \\ 2x_1 - x_2 &\le 1 \end{aligned}$$

How do we prove that $\bar{x} = (1, 1)^\intercal$ is optimal for the LP relaxation?

We find an equivalent maximum problem (multiply objective function by -1)

$$max \qquad x_1 + x_2$$

subject to

$$\begin{aligned} -x_1 + 2x_2 &\le 1 \\ 2x_1 - x_2 &\le 1 \end{aligned}$$

Note row 1 and 2 of the constraints is still tight for $\bar{x}$.

Theorem 4.7 tells us that for a given LP $max\{c^\intercal x : Ax \leq b\}$, $\bar{x}$ is optimal for the LP if and only if c is in the cone of the tight constraints or

$$\begin{bmatrix}1\\1\end{bmatrix} \in cone\{\begin{bmatrix}-1\\2\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}\} = \{\lambda_1 \begin{bmatrix}-1\\2\end{bmatrix} + \lambda_2 \begin{bmatrix}2\\-1\end{bmatrix}, \lambda_1, \lambda_2 \geq 0\}$$

which it does for $\lambda_1 = \lambda_2 = 1$.

So $\bar{x}$ is an optimal solution for the LP relaxation.

32.1 Another Convex NLP Optimality Example

Example.

$$min \qquad (-2, 3)x$$

subject to

$$\begin{aligned} x_2^2 - x_1 &\leq 0 \\ x_1^2 - x_2 &\leq 0 \\ -x_1 + \frac{1}{2} &\leq 0 \end{aligned}$$

Show that $\bar{x} = (\frac{1}{2}, \frac{1}{4})^\intercal$ is optimal. Note that $tight(\bar{x}) = \{2, 3\}$ (rows 2 and 3 are tight).

$$\begin{aligned} \nabla g_2(x) &= (2x_1, -1)^\intercal \\ \nabla g_2(\bar{x}) &= (1, -1)^\intercal \\ \nabla g_3(x) &= (-1, 0)^\intercal \\ \nabla g_3(\bar{x}) &= (-1, 0)^\intercal \end{aligned}$$

Thus we have

$$min \qquad (-2, 3)x$$

subject to

$$\begin{aligned} (1, -1)x &\leq \frac{1}{4} \\ (-1, 0)x &\leq -\frac{1}{2} \end{aligned}$$

We take $max\{(2, -3)x, Ax \leq b\}$ and apply Theorem 4.7

$$\begin{bmatrix}2\\-3\end{bmatrix} \in cone\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}\}$$

where $\lambda_1 = 3, \lambda_2 = 1$. Note if it was $(-2, 3)$ instead, note the cone alway has $x_2 \leq 0$ but $3 > 0$, thus $(-2, 3)$ would not be in the cone.

We can also solve using the the inverted matrix and check if the solution has non-negative entries.

32.2 Optimality for Convex NLP

Proposition. *Given a convex NLP*

$$\min \qquad c^{\intercal}x$$

subject to

$$\begin{aligned} &g_1(x) \leq 0 \\ &\vdots \\ &g_m(x) \leq 0 \end{aligned}$$

Let $\bar{x}$ be a feasible solution and assume that $g_1, \ldots, g_m$ differentiable. Then $\bar{x}$ is an optimal solution if

$$-c \in cone\{\nabla g_i(\bar{x}) : i \in tight(\bar{x})\}$$

(Note the converse is not necessarily true, see below).

32.3 Converse of Optimality Proposition

Is the converse true? That is, if $\bar{x}$ is optimal, then does $-c$ belong in the cone of the tight constraints? **Not necessarily**.

Example.

$$\min \qquad c^{\intercal}x$$

subject to

$$x_1^2 + x_2^2 + \ldots + x_n^2 \leq 0$$

Clearly $\bar{x}$ is a unique feasible solution and thus optimal for every c. The gradients are

$$\begin{aligned} \nabla g_1(x) &= (2x_1, 2x_2, \ldots, 2x_n)^{\intercal} \\ \nabla g_1(\bar{x}) &= 0 \end{aligned}$$

But $c \in cone\{0\}$ if and only if $c = 0$, thus c is not necessarily in the cone.

32.4 Slater Point

Consider the convex NLP

$$\min \qquad c^{\intercal}x$$

subject to

$$\begin{aligned} &g_1(x) \leq 0 \\ &\vdots \\ &g_m(x) \leq 0 \end{aligned}$$

Definition. Then $\tilde{x}$ is called a **Slater point** if $g_1(\tilde{x}) < 0, \ldots, g_m(\tilde{x}) < 0$.

32.5 KKT Theorem

The converse may hold if:

Theorem. *Consider the convex NLP*

$$\min \qquad c^\mathsf{T} x$$

subject to

$$\begin{aligned} &g_1(x) \le 0 \\ &\vdots \\ &g_m(x) \le 0 \end{aligned}$$

Let $\bar{x}$ *be a feasible solution,* $g_1, \dots, g_m$ *differentiable. Assume the NLP has a Slater point. Then* $\bar{x}$ *is optimal if and only if*

$$-c \in cone\{\nabla g_i(\bar{x}) : i \in tight(\bar{x})\}$$