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STAT 433/833 Course Notes

Stochastic Processes

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Last Revision: September 20, 2018

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. These notes are my interpretation and transcription of the content covered in lectures. The instructor has not verified or confirmed the accuracy of these notes, and any discrepancies, misunderstandings, typos, etc. as these notes relate to course's content is not the responsibility of the instructor. If you spot any errors or would like to contribute, please contact me directly.

1 September 6, 2018

1.1 Example 1.2 solution

Use the definition of the Markov property to show that

$$\begin{aligned}P(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_{n-k+1} = x_{n-k+1}, X_{n-k-1} = x_{n-k-1}, \ldots, X_0 = x_0) \\ = P(X_{n+1} = x_{n+1} \mid X_n = x_n), \quad k = 1, 2, \ldots, n\end{aligned}$$

(i.e. we are missing one past observation).

Solution. Applying the definition of conditional probability, our expression is equivalent to

$$\frac{P(X_{n+1} = x_{n+1}, X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_{n-k+1} = x_{n-k+1}, X_{n-k-1} = x_{n-k-1}, \ldots, X_0 = x_0)}{P(X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_{n-k+1} = x_{n-k+1}, X_{n-k-1} = x_{n-k-1}, \ldots, X_0 = x_0)} = \frac{N}{D}$$

By the law of total probability

$$\begin{aligned}N &= \sum_{x_{n-k} \in S} P(X_{n+1} = x_{n+1}, \ldots, X_{n-k} = x_{n-k}, \ldots, X_0 = x_0) \\ &= \sum_{x_{n-k} \in S} P(X_{n+1} = x_{n+1} \mid X_n = x_n, \ldots, X_{n-k} = x_{n-k}, \ldots, X_0 = x_0) \times P(X_n = x_n, \ldots, X_{n-k} = x_{n-k}, \ldots, X_0 = x_0)\end{aligned}$$

By the Markov property

$$\begin{aligned}&= P(X_{n+1} = x_{n+1} \mid X_n = x_n) \sum_{x_{n-k} \in S} P(X_n = x_n, \ldots, X_{n-k} = x_{n-k}, \ldots, X_0 = x_0) \\ &= P(X_{n+1} = x_{n+1} \mid X_n = x_n) P(X_n = x_n, \ldots, X_{n-k} \in S, \ldots, X_0 = x_0)\end{aligned}$$

Since $X_{n-k} \in S$ is an event with probability 1

$$\begin{aligned}&= P(X_{n+1} = x_{n+1} \mid X_n = x_n) P(X_n = x_n, \ldots, X_{n-k+1} = x_{n-k+1}, X_{n-k-1} = x_{n-k-1}, \ldots, X_0 = x_0) \\ &= P(X_{n+1} = x_{n+1} \mid X_n = x_n) \cdot D\end{aligned}$$

The result follow.

2 September 11, 2018

2.1 Section 1.2: Transitivity of communication relation

Prove that if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$ (and thus the communication relation "$\leftrightarrow$" is an equivalence relation).

Proof. $\exists n, m \in \mathbb{N}$ such that $P_{i,j}^{(n)} > 0$ and $P_{j,k}^{(m)} > 0$.
Note that

$$P_{i,k}^{(n+m)} = \sum_{l \in S} P_{i,l}^{(n)} P_{l,k}^{(m)} \geq P_{i,j}^{(n)} P_{j,k}^{(m)} > 0$$

Similarly we can show $k \to i$, thus $i \leftrightarrow k$. □

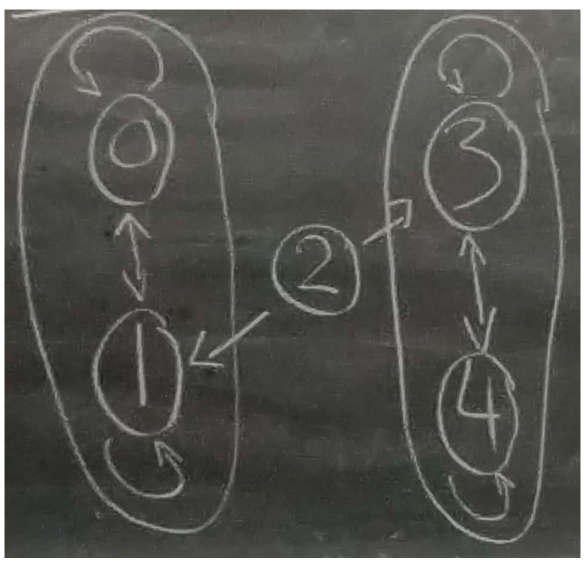
2.2 Example 1.3 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 0.1 & 0.9 \end{bmatrix} \end{array}$$

Use a state transition diagram to determine the equivalence classes.

Solution. We draw the following state transition diagram and note that there are three communication classes: $\{0, 1\}, \{2\}, \{3, 4\}$.



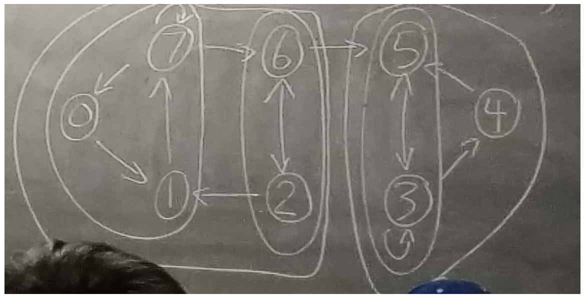
2.3 Example 1.4 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0.3 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

Use a state transition diagram to determine the equivalence classes.

Solution. We draw the following state transition diagram and note that there are two communication classes: $\{0, 1, 2, 6, 7\}, \{3, 4, 5\}$.



2.4 Example 1.5 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0.3 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

Use sample paths to prove that all states within the communication classes found in Example 1.4 communicate.

Solution. class $\{3, 4, 5\}$ Note that $P_{3,4}P_{4,5}P_{5,3} > 0$ i.e. the sample path $3 \to 4 \to 5 \to 3$ has positive probability, thus states 3,4, and 5 communicate since for any pair of states $i, j \in \{3, 4, 5\}$, $\exists n_{i,j} \leq 3$ such that $P_{i,j}^{(n_{i,j})} > 0$.

class $\{0, 1, 2, 6, 7\}$ We have sample path $0 \to 1 \to 7 \to 6 \to 2 \to 1 \to 7 \to 0$ with positive probability.

By a similar argument as above the five states communicate.

2.5 Theorem 1.1 proof: periodicity is a class property

Theorem 2.1. If $i \leftrightarrow j$ then $d(i) = d(j)$ (equal periods).

Proof. Since $i \leftrightarrow j$, then $\exists n, m \in \mathbb{N}$ such that $P_{i,j}^{(n)} > 0$ and $P_{j,i}^{(m)} > 0$.
$\forall L \in \mathbb{Z}^+$ s.t. $P_{j,j}^{(L)} > 0$, we have

$$\begin{aligned} P_{i,i}^{(m+n+L)} &= \sum_{k \in S} P_{i,k}^{(n)} P_{k,i}^{(m+L)} \\ &= \sum_{k \in S} \sum_{l \in S} P_{i,k}^{(n)} P_{k,l}^{(L)} P_{l,i}^{(m)} \\ &\geq P_{i,j}^{(n)} P_{j,j}^{(L)} P_{j,i}^{(m)} \\ &> 0 \end{aligned}$$

Thus $d(i)$ divides $n+m+L$.
Note that $P_{i,i}^{(n+m)} = \sum_{k\in S} P_{i,k}^{(n)} P_{k,i}^{(m)} \geq P_{i,j}^{(n)} P_{j,i}^{(m)} > 0$, thus $d(i)$ divides $n+m$.
Therefore $d(i)$ divides $(n+m+L)-(n+m) = L\ \forall L$ s.t. $P_{j,j}^{(L)} > 0$, thus $d(i)$ divides $gcd\{L \in \mathbb{Z}^+ \mid P_{j,j}^{(L)} > 0\} = d(j)$.
Similarly, $d(j)$ divides $d(i)$, thus $d(i) = d(j)$. □

3 September 13, 2018

3.1 Example 1.6 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} \end{array}$$

Show that $d(i) = 1$ despite the fact that $P_{i,i}^{(1)} = P_{i,i} = 0$ for $i = 0, 1, 2$.

Solution. Consider state 0 where we have

$$\begin{aligned} P_{0,0}^{(1)} &= 0 \\ P_{0,0}^{(2)} &= \sum_{k\in S} P_{0,k} P_{k,0} \geq P_{0,1} P_{0,1} = \frac{1}{4} > 0 \\ P_{0,0}^{(3)} &= \sum_{k\in S} P_{0,k} P_{k,l} P_{l,0} \geq P_{0,1} P_{1,2} P_{2,0} = \frac{1}{8} > 0 \end{aligned}$$

Therefore $d(0) = \gcd\{2, 3, \ldots\} = 1$.

Since the sample path $0 \to 1 \to 2 \to 0$ has positive prob., all of the states communicate and the DTMC is irreducible, thus $d(2) = d(1) = d(0) = 1$ as well.

3.2 Theorem 1.2 proof: transience/recurrence are class properties

Theorem 3.1. Transience and recurrence are class properties i.e. if $i \leftrightarrow j$ and i is recurrent, then j is recurrent.

Proof. It clearly holds if $i = j$, so assume $i \neq j$. $i \leftrightarrow j$ so $\exists m, n \in \mathbb{Z}^+$ s.t. $P_{j,i}^{(m)} > 0$ and $P_{i,j}^{(n)} > 0$.
Note that

$$\begin{aligned} \sum_{n=1}^{\infty} P_{j,j}^{(n)} &\geq \sum_{l=m+n+1}^{\infty} P_{j,j}^{(l)} \\ &\geq \sum_{l=m+n+1}^{\infty} P_{j,i}^{(m)} P_{i,i}^{(l-m-n)} P_{i,j}^{(n)} \\ &= P_{j,i}^{(m)} P_{i,j}^{(n)} \sum_{l=m+n+1}^{\infty} P_{i,i}^{(l-m-n)} \\ &= P_{j,i}^{(m)} P_{i,j}^{(n)} \sum_{L=1}^{\infty} P_{i,i}^{(L)} \\ &= \infty \end{aligned}$$

since i is recurrent thus $\sum_{L=1}^{\infty} P_{i,i}^{(L)} = \infty$, thus state j is recurrent.
Transience is proven similarly. $\square$

3.3 Theorem 1.3 proof: recurrent classes with states i, j imply $f_{i,j} = 1$

Theorem 3.2. If $i \leftrightarrow j$ and state i is recurrent, then

$$f_{i,j} = P(\text{DTMC ever makes future visit to state } j \mid X_0 = i) = 1$$

Proof. If $i = j$, then result follows by definition of recurrence.
Let $i \neq j$. Since $i \leftrightarrow j$, then $\exists n \in \mathbb{Z}^+$ s.t. $P_{j,i}^{(n)} > 0$.
State j is recurrent by theorem 1.2 so $f_{j,j} = 1$.
Assume that $f_{i,j} < 1$ for a contradiction.

Method 1 Note that

$$\begin{aligned}
f_{j,j} &= P(\text{DTMC ever makes future visit to j} \mid X_0 = j) \\
&= 1 - P(\text{never visits j} \mid X_0 = j) \\
&\leq 1 - P_{j,i}^{(n)}(1 - f_{i,j}) && P(\text{never visits j} \mid X_0 = j) \geq P_{j,i}^{(n)}(1 - f_{i,j}) \\
&< 1
\end{aligned}$$

which is a contradiction, so $f_{i,j} < 1$.

Method 2 Note that

$$\begin{aligned}
&\{X_n = i, \text{never visits j after i}\} \subseteq \{\text{never returns to state j}\} \\
\Rightarrow &P(X_n = i, \text{never visits j after i } \mid X_0 = j) \leq P(\text{never returns to j} \mid X_0 = j) \\
\Rightarrow &P_{j,i}^{(n)}(1 - f_{i,j}) \leq 1 - f_{j,j} \\
\Rightarrow &P_{j,i}^{(n)}(1 - f_{i,j}) \leq 0
\end{aligned}$$

which is a contradiction since $P_{j,i}^{(n)} > 0$ and $1 - f_{i,j} > 0$, so we must have $f_{i,j} = 1$.

$\square$

3.4 Theorem 1.4 proof

Theorem 3.3. If state i is recurrent and state i does not communicate with state j, then $P_{i,j} = 0$.

Proof. Assume $i \neq j$. State i is recurrent so $f_{i,i} = 1$.
Assume that $P_{i,j} > 0$ for a contradiction so $i \to j$. Since i and j don't communicate and $i \to j$, then i is not accessible from j ($j \not\to i$).

Method 1 Note that

$$\begin{aligned}
f_{i,i} &= P(\text{DTMC ever makes future visit to i} \mid X_0 = i) \\
&= 1 - P(\text{never visits i} \mid X_0 = i) \\
&\leq 1 - P_{i,j} && P(\text{never visits i} \mid X_0 = i) \geq P_{i,j} \text{ since } j \not\to i \\
&< 1
\end{aligned}$$

which is a contradiction so $P_{i,j} = 0$.

Method 2 Note that

$$\begin{aligned}
&\{X_1 = j, \text{never returns to i after j}\} \subseteq \{\text{never returns to state i}\} \\
\Rightarrow &P(X_1 = j, \text{never visits j after i } \mid X_0 = i) \leq P(\text{never return to i} \mid X_0 = i) \\
\Rightarrow &P_{i,j} \leq 1 - f_{i,i}
\end{aligned}$$

where the last line follows since i is not accessible from j.

Since $f_{i,i} = 1$, we have $P_{i,j} \leq 0$ which is a contradiction, thus $P_{i,j} = 0$.

□

4 September 18, 2018

4.1 Example 1.7 solution

Consider the DTMC with one-step transition probabilities

$$\begin{aligned}
P_{1,j} = \frac{1}{2^j} \quad j = 2^n \quad n \in Z^+ \\
P_{i,i-1}1 \quad i = 2, 3, 4, \ldots
\end{aligned}$$

Show that all states are null recurrent and check that a stationary distribution does not exist.

Solution. It is clear that evey state communicates an the DTMC is irreduiclb.e By Theorem 1.5, we only need to check one state for null reucrrence.
For state 1, note that

$$\begin{aligned}
f_{1,1} = \sum_{n=1}^{\infty} f_{1,1}^{(n)} &= \sum_{n=1}^{\infty} P_{1,n} \\
&= \sum_{m=1}^{\infty} P_{1,2^m} \\
&= \sum_{m=1}^{\infty} \frac{1}{2^m} \\
&= \frac{\frac{1}{2}}{1 - \frac{1}{2}} \\
&= 1
\end{aligned}$$

So state 1 is indeed recurrent. To show it is null recurrent, we look at its mean recurrent time m_1

$$\begin{aligned} m_1 = \sum_{n=1}^{\infty} n f_{1,1}^{(n)} \\ &= \sum_{n=1}^{\infty} n P_{1,n} \\ &= \sum_{m=1}^{\infty} 2^m P_{1,2^m} \\ &= \sum_{m=1}^{\infty} 2^m \frac{1}{2^m} \\ &= \sum_{m=1}^{\infty} 1 \\ &= \infty \end{aligned}$$

State 1 and hence the entire DTMC is null recurrent.
Does a stationary distribution exist? We observe $p = pP$ where $p = (p_1, p_2, \ldots)$ by vector-matrix multiplication

$$\begin{aligned} p_1 &= p_2 \\ p_2 &= \frac{1}{2}p_1 + p_3 \\ &\vdots \\ p_{2^m} &= \frac{1}{2^m}p_1 + p_{2^m+1} \qquad m \in \mathbb{Z}^+ \end{aligned}$$

Also note that

$$p_i = p_{i+1} \qquad i \neq 2^m \text{ for some } m \in \mathbb{Z}^+$$

thus we have

$$p_{2^m+1} = p_{2^m+2} = \ldots = p_{2^{m+1}-2} = p_{2^{m+1}-1} = p_{2^{m+1}}$$

So our p vector is now

$$\begin{aligned} p &= (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, \ldots) \\ &= (p_1, p_2, p_4, p_4, p_8, p_8, p_8, p_8, p_{16}, \ldots) \end{aligned}$$

If we expand out our p_{2^m}

$$\begin{aligned}
p_{2^m} &= \frac{1}{2^m}p_1 + p_{2^m+1} \\
&= \frac{1}{2^m}p_1 + \frac{1}{2^{m+1}}p_1 + p_{2^{m+2}} \\
&= p_1 \sum_{l=m}^{\infty} \left(\frac{1}{2}\right)^l \\
&= p_1 \left(\frac{1}{2}\right)^m \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \\
&= p_1 \left(\frac{1}{2}\right)^{m-1}
\end{aligned}$$

We need $pe' = \sum_{n=1}^{\infty} p_n = 1$. Note that

$$\begin{aligned}
\sum_{n=1}^{\infty} p_n &= p_1 + \sum_{m=1}^{\infty} \sum_{l=2^{m-1}+1}^{2^m} p_l && \text{recall we have } 2^{m-1} \text{ of each } p_{2^m} \\
&= p_1 + \sum_{m=1}^{\infty} \sum_{l=2^{m-1}+1}^{2^m} p_{2^m} \\
&= p_1 + \sum_{m=1}^{\infty} 2^{m-1} \frac{1}{2^{m-1}} p_1 && 2^m - 2^{m-1} = 2^{m-1}(2-1) = 2^{m-1} \\
&= p_1 + \sum_{m=1}^{\infty} p_1 \\
&= \sum_{m=0}^{\infty} p_1
\end{aligned}$$

which is 0 if $p_1 = 0$ or ∞ if $p_1 > 0$. It can't hold that $pe' = 1$ while satisfying $p = pP$, thus a stationary distribution does not exist.

4.2 Example 1.8

Consider a DTMC with TPM

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Show that more than one stationary distribution exists.

Solution. We have two equivalence classes: $\{0, 2\}$ and $\{1\}$ and they are positive recurrent:

$$\begin{aligned}
&P(N_1 = 1 \mid X_0 = 1) = 1 \Rightarrow m_1 = 1 < \infty \\
&P(N_j = 2 \mid X_0 = j) = 2 \Rightarrow m_1 = 2 < \infty \qquad j = 0, 2
\end{aligned}$$

Consider $p = (\frac{1}{2}, 0, \frac{1}{2})$ and $q = (0, 1, 0)$.

For the former:

$$pP = (\frac{1}{2}, 0, \frac{1}{2}) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = (\frac{1}{2}, 0, \frac{1}{2}) = p$$

and

$$pe' = \frac{1}{2} + \frac{1}{2} = 1$$

Similarly for q, thus both p and q are both stationary.
In fact, any convex combination $\alpha p + (1-\alpha)q$, $\alpha \in [0,1]$ is a stationary distribution.
That is any $(\frac{\alpha}{2}, 1-\alpha, \frac{\alpha}{2})$ is stationary, thus there are infinitely many stationary distributions.

4.3 Theorem 1.7: Irreducible DTMC positive recurrent iff stationary distribution

Theorem 4.1. An irreducible DTMC is positive recurrent iff a stationary distribution exists.

Proof. Proof deferred. □

4.4 Uniqueness of stationary distributions

Theorem 4.2. Once we have theorem 1.7 we can prove uniqueness of stationary distributions i.e. the stationary distribution will not be unique if the DTMC has more than one positive recurrent equivalence class.

Proof. Consider a DTMC with two positive recurrent classes c_1, c_2. We can write the TPM as

$$P = \begin{array}{c c} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} c_1 \\ c_2 \end{array} & \left[\begin{array}{cc} P_1 & 0 \\ 0 & P_2 \end{array} \right] \end{array}$$

where P_1 and P_2 are irreducible TPMs when considered in isolation.
So if we had A DTMC $\{y_n, n \in \mathbb{N}\}$ with TPM P_1 then $\{y_n, n \in \mathbb{N}\}$ would be irreducible and positive recurrent i.e. $\exists p_1$ such that $p_1 P_1 = p_1$ and $p_1 e' = 1$.
Similarly $\exists p_2$ for P_2.
Consider

$$[\alpha p_1, (1-\alpha)p_2] = [(\alpha p_{1,1}, \ldots, \alpha p_{1,n}), ((1-\alpha)p_{2,1}, \ldots, (1-\alpha)p_{2,n})]$$

thus we have

$$\begin{aligned} pP &= [\alpha p_1, (1-\alpha)p_2] \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \\ &= [\alpha p_1 P_1, (1-\alpha)p_2 P_2] \\ &= [\alpha p_1, (1-\alpha)p_2] \\ &= p \end{aligned}$$

And note $pe' = \alpha p_1 e' + (1-\alpha)p_2 e' = \alpha + (1-\alpha) = 1$.
Thus we do not have a unique stationary distribution. □

5 September 20, 2018

5.1 Example 1.11 solution

Consider a DTMC with TPM

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{2}{3} & \frac{1}{12} \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Solve the limit TPM $\lim_{n\to\infty} P^{(n)}$. Does the limiting distribution of X_n depend on the initial distribution?

Solution. Clearly the equivalence classes are $\{0\}$ and $\{2\}$ (recurrent) and $\{1\}$ (transient).
$P_{0,0} = P_{2,2} = 1$ so states 0 and 2 are *absorbing states*.
We'd like to build up our matrix $\lim_{n\to\infty} P^{(n)}$.
Note $P_{0,0}^{(n)} = P_{2,2}^{(n)} = 1$ for all $n \in \mathbb{N}$ thus

$$\lim_{n\to\infty} P_{0,0}^{(n)} = \lim_{n\to\infty} P_{2,2}^{(n)} = 1$$

Thus

$$\lim_{n\to\infty} P_{0,1}^{(n)} = \lim_{n\to\infty} P_{0,2}^{(n)} = \lim_{n\to\infty} P_{2,0}^{(n)} = \lim_{n\to\infty} P_{2,1}^{(n)} = 0$$

Note that

$$\lim_{n\to\infty} P_{1,1}^{(n)} = \lim_{n\to\infty} (\frac{2}{3})^n = 0$$

Also

$$\begin{aligned}
P_{1,0}^{(n)} &= P(X_n = 1 \mid X_0 = 1) \\
&= \sum_{m=1}^{n} P(\text{DTMC first visits state 0 at time m} \mid X_0 = 1) \\
&= \sum_{m=1}^{n} P(X_n = 0, X_{n-1} = 0, \dots, X_m = 0 \mid X_0 = 1) \\
&= \sum_{m=1}^{n} P(X_n = 0, X_{n-1} = 0, \dots, X_m = 0, X_{m-1} = 1, \dots, X_1 = 1 \mid X_0 = 1) \\
&= \sum_{m=1}^{n} P(X_n = 0 \mid X_m = 0) P(X_m = 0 \mid X_{m-1} = 1) P(X_{m-1} = 1 \mid X_{m-2} = 1) \dots P(X_1 = 1 \mid X_0 = 1) \\
&= \sum_{m=1}^{n} 1 \cdot \frac{1}{4} \cdot (\frac{2}{3})^{m-1} \\
&= \frac{1}{4} \sum_{l=0}^{n-1} (\frac{2}{3})^l \\
&= \frac{1}{4} (\frac{1 - (\frac{2}{3})^n}{1 - \frac{2}{3}}) \\
&= \frac{3}{4} (1 - (\frac{2}{3})^n)
\end{aligned}$$

Similarly $P_{1,2}^{(n)} = \frac{1}{4}(1 - \left(\frac{2}{3}\right)^n)$.
Taking the limit of either, we get $\lim_{n\to\infty} P_{1,0}^{(n)} = \frac{3}{4}$ and $\lim_{n\to\infty} P_{1,2}^{(n)} = \frac{1}{4}$, thus we have

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$