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PMATH 351 Course Notes

Real Analysis

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. These notes are my interpretation and transcription of the content covered in lectures. The instructor has not verified or confirmed the accuracy of these notes, and any discrepancies, misunderstandings, typos, etc. as these notes relate to course's content is not the responsibility of the instructor. If you spot any errors or would like to contribute, please contact me directly.

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1.1 Basic notation

We denote

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \ldots\} \\
\mathbb{Z} &= \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
\mathbb{Q} &= \{\frac{n}{m} \mid n \in \mathbb{Z}, m \in \mathbb{N}\} \\
\mathbb{R} &= \text{real numbers}
\end{aligned}$$

We use $\subset$ and $\subseteq$ interchangeably, and use $\subsetneq$ for strict subsets.

1.2 Basic set theory

We denote X as our universal set. If $\{A_\alpha\}_{alpha \in I}$ is such that $A_\alpha \subset X$ for all $\alpha \in I$ (index set), then

$$\bigcup_{\alpha \in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for some } \alpha \in I\} \qquad \text{(union)}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for all } \alpha \in I\} \qquad \text{(intersection)}$$

Define for $A, B \subseteq X$

$$\begin{aligned}
A \setminus B &= \{x \in X \mid x \in A, x \notin B\} && \text{(set difference)} \\
A \Delta B &= \{x \in X \mid x \in A \text{ and } x \notin B\} \text{ OR } x \in B \text{ and } x \notin A\} && \text{(semantic difference)} \\
A^c &= X \setminus A = \{x \in X \mid x \notin A\} && \text{(complement)} \\
&\varnothing && \text{(empty set)} \\
P(X) &= \{A \mid A \subset X\} \quad \varnothing \in P(X), X \in P(X) && \text{(power set)}
\end{aligned}$$

1.3 De Morgan's laws

De Morgan's laws states that given $\{A_\alpha\}_{\alpha \in I} \subset P(X)$

$$\left(\bigcup_{\alpha \in I} A_\alpha\right)^c = \bigcap_{\alpha \in I} A_\alpha^c$$

$$\left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \bigcup_{\alpha \in I} A_\alpha^c$$

Question: what if $I = \varnothing$, what is $\bigcup_{\alpha\in\varnothing} A_\alpha$? It is in fact $\bigcup_{\alpha\in\varnothing} A_\alpha = \varnothing$.
Note that $\bigcap_{\alpha\in\varnothing} A_\alpha = X$ (from De Morgan's Law, and also $A_\alpha = A_\alpha^c$).

1.4 Products of sets, relations, and functions

Given X, Y define the product

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

If $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$ then $X \times Y = \{(x_i, y_j) \mid i = 1, \ldots, n \quad j = 1, \ldots, m\}$ containing nm elements.

Definition 1.1. A **relation** on X, Y is a subset R of the product $X \times Y$.
We write xRy if $(x, y) \in R$. The **domain** of R is

$$\{x \in X \mid \exists y \in Y \text{ with } (x, y) \in R\}$$

which need not cover our universal set.
The **range** of R is

$$\{y \in Y \mid \exists x \in X \text{ with } (x, y) \in R\}$$

Definition 1.2. A **function** from X into Y is a relation R such that for every $x \in X$, there exists exactly one $y \in Y$ with $(x, y) \in R$.

Suppose that we have $X_1, X_2, \ldots, X_n$ non-empty sets. Define

$$X_1 \times X_2 \times \ldots \times X_n = \prod_{i=1}^{n} X_i = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i\}$$

or a set of *n-tuples*.
If $X_i = X_j = X$ for all $i, j = 1, \ldots, n$, then

$$\prod_{i=1}^{n} X_i = \prod_{i=1}^{n} X = X^n$$

Problem 1.1. Given a collection $\{X_\alpha\}_{\alpha\in I}$ of non-empty sets, what do we mean by $\prod_{\alpha\in I} X_\alpha$?
Motivation: consider $X_1 \times \ldots \times X_n = \{(x_1, \ldots, x_n) \mid x_i \in X_i\}$. We choose some $(x_1, \ldots, x_n) \in \prod_{i\in\{1,\ldots,n\}} = I$. This point induces *a function*

$$f_{(x_1,\ldots,x_n)} : \{1, \ldots, n\} \to \bigcup_{i=1}^{n} X_i$$

with $f(1) = x_1 \in X_1$, $f(i) = x_i \in X_i$, $f(n) = x_n \in X_n$, etc. Assume we have have $f : \{1, \ldots, n\} \to \bigcup_{i=1}^{n} X_i$ such that $f(i) \in X_i$. Then

$$(f(1), f(2), \ldots, f(n)) = \prod_{i=\{1,\ldots,n\}} X_i$$

Definition 1.3. Given a collection $\{X_\alpha\}_{\alpha\in I}$ of non-empty sets we let

$$\prod_{\alpha\in I} X_\alpha = \{f : I \to \bigcup_{\alpha\in I} X_\alpha\}$$

such that $f(\alpha) \in X_\alpha$ (it is a "set of functions"). f is called a **choice function**.
Question: If $X_\alpha \neq \varnothing$, is $\prod_{\alpha\in I} X_\alpha \neq \varnothing$?