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PMATH 351 Course Notes

Real Analysis

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. These notes are my interpretation and transcription of the content covered in lectures. The instructor has not verified or confirmed the accuracy of these notes, and any discrepancies, misunderstandings, typos, etc. as these notes relate to course's content is not the responsibility of the instructor. If you spot any errors or would like to contribute, please contact me directly.

1 September 10, 2018

1.1 Basic notation

We denote

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \ldots\} \\
\mathbb{Z} &= \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
\mathbb{Q} &= \{\frac{n}{m} \mid n \in \mathbb{Z}, m \in \mathbb{N}\} \\
\mathbb{R} &= \text{real numbers}
\end{aligned}$$

We use $\subset$ and $\subseteq$ interchangeably, and use $\subsetneq$ for strict subsets. $\subset$ or $\subseteq$ is called "inclusion", and $\supset$ or $\supseteq$ is called "containment".

1.2 Basic set theory

We denote X as our universal set. If $\{A_\alpha\}_{alpha\in I}$ is such that $A_\alpha \subset X$ for all $\alpha \in I$ (index set), then

$$\bigcup_{\alpha\in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for some } \alpha \in I\} \qquad \text{(union)}$$

$$\bigcap_{\alpha\in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for all } \alpha \in I\} \qquad \text{(intersection)}$$

Define for $A, B \subseteq X$

$$\begin{aligned}
A \setminus B &= \{x \in X \mid x \in A, x \notin B\} && \text{(set difference)} \\
A \Delta B &= \{x \in X \mid x \in A \text{ and } x \notin B\} \text{ OR } x \in B \text{ and } x \notin A\} && \text{(semantic difference)} \\
A^c &= X \setminus A = \{x \in X \mid x \notin A\} && \text{(complement)} \\
&\varnothing && \text{(empty set)} \\
P(X) &= \{A \mid A \subset X\} \quad \varnothing \in P(X), X \in P(X) && \text{(power set)}
\end{aligned}$$

1.3 De Morgan's laws

De Morgan's laws states that given $\{A_\alpha\}_{\alpha\in I} \subset P(X)$

$$\left(\bigcup_{\alpha\in I} A_\alpha\right)^c = \bigcap_{\alpha\in I} A_\alpha^c$$
$$\left(\bigcap_{\alpha\in I} A_\alpha\right)^c = \bigcup_{\alpha\in I} A_\alpha^c$$

Question: what if $I = \varnothing$, what is $\bigcup_{\alpha\in\varnothing} A_\alpha$? It is in fact $\bigcup_{\alpha\in\varnothing} A_\alpha = \varnothing$.
Note that $\bigcap_{\alpha\in\varnothing} A_\alpha = X$ (from De Morgan's Law, and also $A_\alpha = A_\alpha^c$).

1.4 Products of sets, relations, and functions

Given X, Y define the product
$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

If $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$ then $X \times Y = \{(x_i, y_j) \mid i = 1, \ldots, n \quad j = 1, \ldots, m\}$ containing nm elements.

Definition 1.1 (Relation)**.** A **relation** on X, Y is a subset R of the product $X \times Y$.
We write xRy if $(x, y) \in R$. The **domain** of R is

$$\{x \in X \mid \exists y \in Y \text{ with } (x, y) \in R\}$$

which need not cover our universal set.
The **range** of R is
$$\{y \in Y \mid \exists x \in X \text{ with } (x, y) \in R\}$$

Definition 1.2 (Function (as a relation))**.** A **function** from X into Y is a relation R such that for every $x \in X$, there exists exactly one $y \in Y$ with $(x, y) \in R$.

Suppose that we have $X_1, X_2, \ldots, X_n$ non-empty sets. Define

$$X_1 \times X_2 \times \ldots \times X_n = \prod_{i=1}^{n} X_i = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i\}$$

or a set of *n-tuples*.
If $X_i = X_j = X$ for all $i, j = 1, \ldots, n$, then

$$\prod_{i=1}^{n} X_i = \prod_{i=1}^{n} X = X^n$$

Problem 1.1. Given a collection $\{X_\alpha\}_{\alpha\in I}$ of non-empty sets, what do we mean by $\prod_{\alpha\in I} X_\alpha$?
Motivation: consider $X_1 \times \ldots \times X_n = \{(x_1, \ldots, x_n) \mid x_i \in X_i\}$. We choose some $(x_1, \ldots, x_n) \in \prod_{i\in\{1,\ldots,n\}} = I$. This point induces *a function*
$$f_{(x_1,\ldots,x_n)} : \{1, \ldots, n\} \to \bigcup_{i=1}^{n} X_i$$

with $f(1) = x_1 \in X_1$, $f(i) = x_i \in X_i$, $f(n) = x_n \in X_n$, etc. Assume we have have $f : \{1, \dots, n\} \to \bigcup_{i=1}^n X_i$ such that $f(i) \in X_i$. Then

$$(f(1), f(2), \dots, f(n)) = \prod_{i=\{1,\dots,n\}} X_i$$

Definition 1.3 (Product of sets)**.** Given a collection $\{X_\alpha\}_{\alpha \in I}$ of non-empty sets we let

$$\prod_{\alpha \in I} X_\alpha = \{f : I \to \bigcup_{\alpha \in I} X_\alpha\}$$

such that $f(\alpha) \in X_\alpha$ (i.e $\prod_{\alpha \in I} X_\alpha$ is a "set of functions"). f is called a **choice function**.
Question: If $X_\alpha \neq \varnothing$, is $\prod_{\alpha \in I} X_\alpha \neq \varnothing$?

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2.1 Zermelo's Axiom of Choice

Question: If $\{X_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty sets is

$$\prod_{\alpha \in I} X_\alpha \neq \varnothing$$

This is analogous to saying: given a collection of non-empty sets in $\mathbb{R}$, how would you choose an element from each subset of $\mathbb{R}$? This is easy if they were subsets of $\mathbb{N}$ (take the least element which exists by the *well-ordering principle*) but much more difficult in $\mathbb{R}$.

Axiom 2.1 (Zermelo's Axiom of Choice)**.** If $\{X_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty sets, then $\prod_{\alpha \in I} \neq \varnothing$.

Equivalently we have an analogous version:

Axiom 2.2 (Axiom of Choice V2)**.** If $X \neq \varnothing$, then there exists a function

$$f : P(X) \setminus \{\varnothing\} \to X$$

such that $f(A) \in A$ for all $A \in P(X) \setminus \{\varnothing\}$ (we can always pick out a subset ($e \in P(X)$) from a non-empty set A).

2.2 Properties of relations

Definition 2.1 (Relation properties)**.** A relation R on X (i.e. $R \subseteq X \times X$) is

1. **reflexive** if $x\ R\ x$ for all $x \in X$
2. **symmetric** if $x\ R\ y \Rightarrow y\ R\ x$
3. **anti-symmetric** if $x\ R\ y$ and $y\ R\ x$, then $x = y$
4. **transitive** if $x\ R\ y$ and $y\ R\ z$ implies $x\mathbb{R}z$

2.3 Partially and totally ordered sets

Example 2.1. Let $X = \mathbb{R}$. We have $x\ R\ y$ iff $x \leq y$.
Note that $\leq$ is reflexive, anti-symmetric, and transitive.

Example 2.2. Let $Y \neq \varnothing$ and $X = P(Y)$.
We write $A \; R \; B$ iff $A \subseteq B$.
Note that $\subseteq$ is reflexive, anti-symmetric, and transitive.

Example 2.3. Let $Y \neq \varnothing$ and $X = P(Y)$.
We write $A \; R \; B$ iff $B \subseteq A$.
Note that $\subseteq$ is reflexive, anti-symmetric, and transitive.

Definition 2.2 (Partially ordered sets)**.** A set X with a relation R on X is called a **partially ordered set** if R is

1. reflexive
2. anti-symmetric
3. transitive

(R is a **partial order** on X if it satisfies these three conditions).
We write (X, R) and call this a **poset**.

Definition 2.3 (Totally ordered sets)**.** If (X, R) is a poset, then if $A \subseteq X$ and $R_1 = R_{|A \times A}$ then (A, R_1) is a poset. We say (A, R_1) is **totally ordered** if for each $x, y \in A$ *either* $x \; R \; y$ or $y \; R \; x$. We also call totally ordered sets **chains**.

How many partial orderings can we have for a given set X (i.e. the number of ways to define partial order relations)?

Example 2.4. Let $X = \{x\}$. We have one relation $R = \{(x, x)\}$ (from $X \times X$) and thus 1 partial ordering.

Example 2.5. Let $X = \{x, y\}$. We know posets $(X, \preceq)$ must be reflexive, thus we have one relation where $x \preceq x$ and $y \preceq y$.
We can also have a poset with the reflexive relations above as well as $x \preceq y$. Similarly we can have a poset with $y \preceq x$.

Example 2.6. Let $X = \{x, y, z\}$. We have the poset with just $e \preceq e$ for $e \in X$.
We have the poset with the reflexive relations and $x \preceq y$ and $y \preceq z$ (3 posets with permutations).
We have the poset with the reflexive relations and $z \preceq x$ and $z \preceq y$ (3 posets with permutations).
We have the poset with the reflexive relations and $y \preceq z$ (6 posets with permutations).
We have the poset with the reflexive relations and $x \preceq y$ and $y \preceq z$ (6 posets with permutations).

2.4 Bounds on posets

Definition 2.4 (Upper and lower bounds)**.** Let $(X, \preceq)$ be a partially ordered set.
Let $A \subset X$. We say that $x_0 \in X$ is an **upper bound** for A if $x \preceq x_0$ for all $x \in A$.
If A has an upper bound, we say it is **bounded above**.
If A is bounded above then x_0 is the **least upper bound** if

1. x_0 is an upper bound of A
2. If y is an upper bound of A then $x_0 \preceq y$.

We write $x_0 = \text{lub}(A)$ or $x_0 \sup(A)$ (supremum).
If $x_0 = \text{lub}(A) \in A$, then x_0 is the *maximum* in A.
Similarly we define the same for lower bounds (infimum).

Example 2.7. Let $X = \mathbb{R}$ and $\preceq$ the usual ordering.

Fact 2.1. Every non-empty subset that is bounded above has a least upper bound (LUBP (lub property) for $\mathbb{R}$).

Example 2.8. Let $Y \neq \varnothing$, $X = P(Y)$, and $\preceq$ be $\subseteq$ (ordering by inclusion).
Y is the maximum element of $(X, \subseteq)$.
If $\{A_\alpha\}_{\alpha \in I} \subset P(X)$ is bounded above by Y, but note that

$$\text{lub}(\{A_\alpha\}_{\alpha\in I}) = \bigcup_{\alpha \in I} A_\alpha$$
$$\text{glb}(\{A_\alpha\}_{\alpha\in I}) = \bigcap_{\alpha \in I} A_\alpha$$

Recall that if $I = \varnothing$, then the glb is all of $\mathbb{R}$: this is in fact correct (it's the greatest set that is a lower bound for relation $\subseteq$).

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3.1 Maximal

Definition 3.1. Let $(X, \preceq)$ be a partially ordered set. An element $x \in X$ is **maximal** if whenver $y \in X$ such that $x \preceq y$, we must have $x = y$.

Example 3.1. Suppose we have $x \preceq x$, $y \preceq y$, and $z \preceq z$. Then all of x, y, z are maximal.
Suppose we have $x \preceq z$ and $y \preceq z$ (as well as the reflexive relations). Then only z is maximal.
Suppose we have $x \preceq y$ and $x \preceq z$ (as well as the reflexive relations). Then y and z are maximal.
Suppose $x \preceq y \preceq z$ (and transitives). Only z is maximal.
Suppose $x \preceq y$ (and transitives). Then both y and z are maximal.
For $X \neq \varnothing$ and $(P(X), \subseteq)$, X is maximal.
For $X \neq \varnothing$ and $(P(X), \supseteq)$, $\varnothing$ is maximal.
For $(\mathbb{R}, \leq)$ has no maximal element.

3.2 Zorn's Lemma

Axiom 3.1 (Zorn's Lemma)**.** If $(X, \preceq)$ is a non-empty partially ordered set such that every chain $S \subset X$ has an upper bound. Then $(X, \preceq)$ has a maximal element.

We can apply Zorn's Lemma to prove a fundamental linear algebra theorem:

Theorem 3.1. Every non-zero vector space V has a basis.

Proof. Let $= \{A \subset X \mid A \text{ is linear indep.}\}$. Note $\neq \varnothing$ because $V \neq \{0\}$.
Order with $\subseteq$.
A basis is a maximal element in $(, \subseteq)$ (if we add vector to this basis, it would be a linear combination of the basis vectors by definition of a basis).
Let $S = \{A_\alpha\}_{\alpha \in I}$ be a chain in . Let $A_0 = \bigcup_{\alpha \in I} A_\alpha$.
Choose $x_1, \ldots, x_n \in A_0$ distinct elements. Assume that $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$. But $x_i \in A_{\alpha_i}$ and we can assume that

$$A_{\alpha_1} \subseteq A_{\alpha_2} \subseteq \ldots \subseteq A_{\alpha_n} \Rightarrow \{x_1, \ldots, x_n\} \subset A_{\alpha_n}$$

So $\alpha_i = 0$ for all $i = 1, \ldots, n$, thus A_0 is an upper bound of S. By Zorn's Lemma we have a basis. $\square$

3.3 Well-ordered

Definition 3.2 (Well-ordered)**.** We say that a partially ordered set $(X, \preceq)$ is **well-ordered** if every non-empty subset A of X has a least element in A.
For example, $(\mathbb{N}, \preceq)$ is well-ordered.
Note that if a set is well-ordered it must also be totally ordered (how would you compare some arbitrary element to the least element if the set was not well-ordered?)

Axiom 3.2 (Well-Ordering Principle)**.** Every non-empty set of $\mathbb{Z}^+$ can be well-ordered.

Theorem 3.2. The following are equivalent:

1. Axiom of Choice
2. Zorn's Lemma
3. Well-Ordering Principle

Example 3.2. Let $X = \mathbb{Q}$. Define the function ϕ

$$\phi(\frac{m}{n}) = \begin{cases} 2^m 5^n & \text{if } m > 0 \\ 1 & \text{if } m = 0 \\ 3^{-m} 7^n & \text{if } m < 0 \end{cases}$$

Note that $\phi : \mathbb{Q} \to \mathbb{N}$ is 1-1. (we could have used any combination of unique primes, as long as we ensure there is a 1-1 mapping).
Note that we can map the rationals to a subset of $\mathbb{N}$, thus the rationals are well-ordered by the Well-Ordering Principle.
Note that we also have $r \leq s \iff \phi(r) \leq \phi(s)$ (ϕ is an order isomorphism).

3.4 Equivalence relations and partitions

Definition 3.3 (Equivalence relation)**.** Let X be non-empty. A relation $\sim$ on X is an **equivalence relation** if the relation is

1. reflexive
2. symmetric
3. transitive

Observation 3.1. Let $[x] = \{y \in X \mid x \sim y\}$ or the **equivalence class** of X.
Then

1. Either $[x] = [y]$ or $[x] \cap [y] = \varnothing$
2. $X = \bigcup_{x \in X} [x]$

Definition 3.4. Let $X \neq \varnothing$. A **partition** of X is a collection $\{A_\alpha\}_{\alpha \in I} \subset P(X)$ such that

1. $A_\alpha \neq \varnothing$
2. $A_\alpha \cap A_\beta = \varnothing$ if $\alpha \neq \beta$

3. $X = \bigcup_{\alpha \in I} A_\alpha$

Observation 3.2. If $\{A_\alpha\}_{\alpha \in I}$ is a partition of X and $x \sim y$ *iff* $x, y \in A_\alpha$, then $\sim$ is an equivalence relation (i.e. if we start with a partition based on some relation $\sim$, we can show $\sim$ is an equivalence relation).

Example 3.3. How many equivalence relations are there on $X = \{1, 2, 3\}$? We can count the number of partitions:

1. $\{\{1\}, \{2\}, \{3\}\}$
2. $\{\{1, 2, 3\}\}$
3. $\{\{1, 2\}\{3\}\}$ (3 permutations since $\binom{3}{2}$)

Example 3.4. Let X be any set (empty or non-empty). Define $\sim$ on $P(X)$ by $A \sim B$ *iff* there exists $f : A \to B$ that is $1 - 1$ and onto.
$\sim$ has properties:

reflexive Take $\text{id} : A \to A$ where $\text{id}(x) = x$

symmetric If we have $f : A \to B$ then we have $f^{-1} : B \to A$ since f is bijective.

transitive If we have $f : A \to B$ and $g : B \to C$, then we have $g \circ f : A \to B$

thus $\sim$ is an equivalence relation.
For $X = \{1, 2, 3\}$, we have four equivalence classes on $P(X)$: one for every possible subset size $(0, \ldots, 3)$.