richardwu.ca

PMATH 351 Course Notes

Real Analysis

Brian Forrest • Fall 2018 • University of Waterloo

Last Revision: September 19, 2018

Table of Contents

Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. These notes are my interpretation and transcription of the content covered in lectures. The instructor has not verified or confirmed the accuracy of these notes, and any discrepancies, misunderstandings, typos, etc. as these notes relate to course's content is not the responsibility of the instructor. If you spot any errors or would like to contribute, please contact me directly.

1 September 10, 2018

1.1 Basic notation

We denote

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \ldots\} \\
\mathbb{Z} &= \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
\mathbb{Q} &= \{\frac{n}{m} \mid n \in \mathbb{Z}, m \in \mathbb{N}\} \\
\mathbb{R} &= \text{real numbers}
\end{aligned}$$

We use $\subset$ and $\subseteq$ interchangeably, and use $\subsetneq$ for strict subsets. $\subset$ or $\subseteq$ is called "inclusion", and $\supset$ or $\supseteq$ is called "containment".

1.2 Basic set theory

We denote X as our universal set. If $\{A_\alpha\}_{alpha \in I}$ is such that $A_\alpha \subset X$ for all $\alpha \in I$ (index set), then

$$\bigcup_{\alpha \in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for some } \alpha \in I\} \qquad \text{(union)}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for all } \alpha \in I\} \qquad \text{(intersection)}$$

Define for $A, B \subseteq X$

$$\begin{aligned}
A \setminus B &= \{x \in X \mid x \in A, x \notin B\} && \text{(set difference)} \\
A \Delta B &= \{x \in X \mid x \in A \text{ and } x \notin B\} \text{ OR } x \in B \text{ and } x \notin A\} && \text{(semantic difference)} \\
A^c &= X \setminus A = \{x \in X \mid x \notin A\} && \text{(complement)} \\
&\varnothing && \text{(empty set)} \\
P(X) &= \{A \mid A \subset X\} \quad \varnothing \in P(X), X \in P(X) && \text{(power set)}
\end{aligned}$$

1.3 De Morgan's laws

De Morgan's laws states that given $\{A_\alpha\}_{\alpha\in I} \subset P(X)$

$$\left(\bigcup_{\alpha\in I} A_\alpha\right)^c = \bigcap_{\alpha\in I} A_\alpha^c$$
$$\left(\bigcap_{\alpha\in I} A_\alpha\right)^c = \bigcup_{\alpha\in I} A_\alpha^c$$

Question: what if $I = \varnothing$, what is $\bigcup_{\alpha\in\varnothing} A_\alpha$? It is in fact $\bigcup_{\alpha\in\varnothing} A_\alpha = \varnothing$.
Note that $\bigcap_{\alpha\in\varnothing} A_\alpha = X$ (from De Morgan's Law, and also $A_\alpha = A_\alpha^c$).

1.4 Products of sets, relations, and functions

Given X, Y define the product
$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

If $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$ then $X \times Y = \{(x_i, y_j) \mid i = 1, \ldots, n \quad j = 1, \ldots, m\}$ containing nm elements.

Definition 1.1 (Relation)**.** A **relation** on X, Y is a subset R of the product $X \times Y$.
We write xRy if $(x, y) \in R$. The **domain** of R is

$$\{x \in X \mid \exists y \in Y \text{ with } (x, y) \in R\}$$

which need not cover our universal set.
The **range** of R is
$$\{y \in Y \mid \exists x \in X \text{ with } (x, y) \in R\}$$

Definition 1.2 (Function (as a relation))**.** A **function** from X into Y is a relation R such that for every $x \in X$, there exists exactly one $y \in Y$ with $(x, y) \in R$.

Suppose that we have $X_1, X_2, \ldots, X_n$ non-empty sets. Define

$$X_1 \times X_2 \times \ldots \times X_n = \prod_{i=1}^{n} X_i = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i\}$$

or a set of *n-tuples*.
If $X_i = X_j = X$ for all $i, j = 1, \ldots, n$, then

$$\prod_{i=1}^{n} X_i = \prod_{i=1}^{n} X = X^n$$

Problem 1.1. Given a collection $\{X_\alpha\}_{\alpha\in I}$ of non-empty sets, what do we mean by $\prod_{\alpha\in I} X_\alpha$?
Motivation: consider $X_1 \times \ldots \times X_n = \{(x_1, \ldots, x_n) \mid x_i \in X_i\}$. We choose some $(x_1, \ldots, x_n) \in \prod_{i\in\{1,\ldots,n\}} = I$. This point induces *a function*
$$f_{(x_1,\ldots,x_n)} : \{1, \ldots, n\} \to \bigcup_{i=1}^{n} X_i$$

with $f(1) = x_1 \in X_1$, $f(i) = x_i \in X_i$, $f(n) = x_n \in X_n$, etc. Assume we have have $f : \{1, \dots, n\} \to \bigcup_{i=1}^n X_i$ such that $f(i) \in X_i$. Then

$$(f(1), f(2), \dots, f(n)) = \prod_{i=\{1,\dots,n\}} X_i$$

Definition 1.3 (Product of sets). Given a collection $\{X_\alpha\}_{\alpha\in I}$ of non-empty sets we let

$$\prod_{\alpha\in I} X_\alpha = \{f : I \to \bigcup_{\alpha\in I} X_\alpha\}$$

such that $f(\alpha) \in X_\alpha$ (i.e $\prod_{\alpha\in I} X_\alpha$ is a "set of functions"). f is called a **choice function**.
Question: If $X_\alpha \neq \varnothing$, is $\prod_{\alpha\in I} X_\alpha \neq \varnothing$?

2 September 12, 2018

2.1 Zermelo's Axiom of Choice

Question: If $\{X_\alpha\}_{\alpha\in I}$ is a non-empty collection of non-empty sets is

$$\prod_{\alpha\in I} X_\alpha \neq \varnothing$$

This is analogous to saying: given a collection of non-empty sets in $\mathbb{R}$, how would you choose an element from each subset of $\mathbb{R}$? This is easy if they were subsets of $\mathbb{N}$ (take the least element which exists by the *well-ordering principle*) but much more difficult in $\mathbb{R}$.

Axiom 2.1 (Zermelo's Axiom of Choice). If $\{X_\alpha\}_{\alpha\in I}$ is a non-empty collection of non-empty sets, then $\prod_{\alpha\in I} \neq \varnothing$.

Equivalently we have an analogous version:

Axiom 2.2 (Axiom of Choice V2). If $X \neq \varnothing$, then there exists a function

$$f : P(X) \setminus \{\varnothing\} \to X$$

such that $f(A) \in A$ for all $A \in P(X) \setminus \{\varnothing\}$ (we can always pick out a subset ($e \in P(X)$) from a non-empty set A).

2.2 Properties of relations

Definition 2.1 (Relation properties). A relation R on X (i.e. $R \subseteq X \times X$) is

1. **reflexive** if $x \ R \ x$ for all $x \in X$
2. **symmetric** if $x \ R \ y \Rightarrow y \ R \ x$
3. **anti-symmetric** if $x \ R \ y$ and $y \ R \ x$, then $x = y$
4. **transitive** if $x \ R \ y$ and $y \ R \ z$ implies $x\mathbb{R}z$

2.3 Partially and totally ordered sets

Example 2.1. Let $X = \mathbb{R}$. We have $x \ R \ y$ iff $x \leq y$.
Note that $\leq$ is reflexive, anti-symmetric, and transitive.

Example 2.2. Let $Y \neq \varnothing$ and $X = P(Y)$.
We write $A\ R\ B$ iff $A \subseteq B$.
Note that $\subseteq$ is reflexive, anti-symmetric, and transitive.

Example 2.3. Let $Y \neq \varnothing$ and $X = P(Y)$.
We write $A\ R\ B$ iff $B \subseteq A$.
Note that $\subseteq$ is reflexive, anti-symmetric, and transitive.

Definition 2.2 (Partially ordered sets)**.** A set X with a relation R on X is called a **partially ordered set** if R is

1. reflexive
2. anti-symmetric
3. transitive

(R is a **partial order** on X if it satisfies these three conditions).
We write (X, R) and call this a **poset**.

Definition 2.3 (Totally ordered sets)**.** If (X, R) is a poset, then if $A \subseteq X$ and $R_1 = R_{|A\times A}$ then (A, R_1) is a poset. We say (A, R_1) is **totally ordered** if for each $x, y \in A$ *either* $x\ R\ y$ or $y\ R\ x$. We also call totally ordered sets **chains**.

How many partial orderings can we have for a given set X (i.e. the number of ways to define partial order relations)?

Example 2.4. Let $X = \{x\}$. We have one relation $R = \{(x, x)\}$ (from $X \times X$) and thus 1 partial ordering.

Example 2.5. Let $X = \{x, y\}$. We know posets $(X, \preceq)$ must be reflexive, thus we have one relation where $x \preceq x$ and $y \preceq y$.
We can also have a poset with the reflexive relations above as well as $x \preceq y$. Similarly we can have a poset with $y \preceq x$.

Example 2.6. Let $X = \{x, y, z\}$.

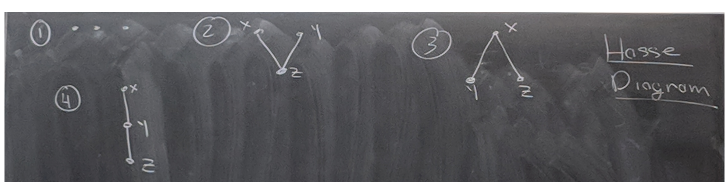

Figure 2.1: Hasse diagrams for the possible $(X, \preceq)$ posets (an edge downwards from a to b denotes $a \preceq b$; note reflexive $a \preceq a$ is assumed automatically).

We have the poset with just the reflexive relations $e \preceq e$ for $e \in X$.
We have the poset with the reflexive relations and $x \preceq z$ and $y \preceq z$ (3 posets with permutations).
We have the poset with the reflexive relations and $x \preceq y$ and $x \preceq z$ (3 posets with permutations).
We have the poset with the reflexive relations and $x \preceq y$ and $y \preceq z$ (6 posets with permutations).
We have the poset with the reflexive relations and $y \preceq z$ (6 posets with permutations, not shown in diagram above).

Note that when identifying these posets isomorphisms, we should not draw lines between two elements $a \leq b$ if the transitive property already implies that. For example if we had the chain $a \leq b \leq c$, the diagram with a line from a to c would be redundant (thus we will end up double counting).

2.4 Bounds on posets

Definition 2.4 (Upper and lower bounds)**.** Let $(X, \preceq)$ be a partially ordered set.
Let $A \subset X$. We say that $x_0 \in X$ is an **upper bound** for A if $x \preceq x_0$ for all $x \in A$.
If A has an upper bound, we say it is **bounded above**.
If A is bounded above then x_0 is the **least upper bound** if

1. x_0 is an upper bound of A
2. If y is an upper bound of A then $x_0 \preceq y$.

We write $x_0 = \text{lub}(A)$ or $x_0 \sup(A)$ (supremum).
If $x_0 = \text{lub}(A) \in A$, then x_0 is the *maximum* in A.
Similarly we define the same for lower bounds (infimum).

Example 2.7. Let $X = \mathbb{R}$ and $\preceq$ the usual ordering.

Fact 2.1. Every non-empty subset that is bounded above has a least upper bound (LUBP (lub property) for $\mathbb{R}$).

Example 2.8. Let $Y \neq \varnothing$, $X = P(Y)$, and $\preceq$ be $\subseteq$ (ordering by inclusion).
Y is the maximum element of $(X, \subseteq)$.
If $\{A_\alpha\}_{\alpha \in I} \subset P(X)$ is bounded above by Y, but note that

$$\text{lub}(\{A_\alpha\}_{\alpha \in I}) = \bigcup_{\alpha \in I} A_\alpha$$
$$\text{glb}(\{A_\alpha\}_{\alpha \in I}) = \bigcap_{\alpha \in I} A_\alpha$$

Recall that if $I = \varnothing$, then the glb is all of $\mathbb{R}$: this is in fact correct (it's the greatest set that is a lower bound for relation $\subseteq$).

3 September 14, 2018

3.1 Maximal

Definition 3.1. Let $(X, \preceq)$ be a partially ordered set. An element $x \in X$ is **maximal** if whenver $y \in X$ such that $x \preceq y$, we must have $x = y$.

Example 3.1. Suppose we have $x \preceq x$, $y \preceq y$, and $z \preceq z$. Then all of x, y, z are maximal.
Suppose we have $x \preceq z$ and $y \preceq z$ (as well as the reflexive relations). Then only z is maximal.
Suppose we have $x \preceq y$ and $x \preceq z$ (as well as the reflexive relations). Then y and z are maximal.
Suppose $x \preceq y \preceq z$ (and transitives). Only z is maximal.
Suppose $x \preceq y$ (and transitives). Then both y and z are maximal.
For $X \neq \varnothing$ and $(P(X), \subseteq)$, X is maximal.
For $X \neq \varnothing$ and $(P(X), \supseteq)$, $\varnothing$ is maximal.
For $(\mathbb{R}, \leq)$ has no maximal element.

3.2 Zorn's Lemma

Axiom 3.1 (Zorn's Lemma)**.** If $(X, \preceq)$ is a non-empty partially ordered set such that every chain $S \subset X$ has an upper bound. Then $(X, \preceq)$ has a maximal element.

We can apply Zorn's Lemma to prove a fundamental linear algebra theorem:

Theorem 3.1. Every non-zero vector space V has a basis.

Proof. Let $= \{A \subset X \mid A \text{ is linear indep.}\}$. Note $\neq \varnothing$ because $V \neq \{0\}$.
Order with $\subseteq$.
A basis is a maximal element in $(, \subseteq)$ (if we add vector to this basis, it would be a linear combination of the basis vectors by definition of a basis).
Let $S = \{A_\alpha\}_{\alpha \in I}$ be a chain in . Let $A_0 = \bigcup_{\alpha \in I} A_\alpha$.
Choose $x_1, \ldots, x_n \in A_0$ distinct elements. Assume that $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$. But $x_i \in A_{\alpha_i}$ and we can assume that

$$A_{\alpha_1} \subseteq A_{\alpha_2} \subseteq \ldots \subseteq A_{\alpha_n} \Rightarrow \{x_1, \ldots, x_n\} \subset A_{\alpha_n}$$

So $\alpha_i = 0$ for all $i = 1, \ldots, n$, thus A_0 is an upper bound of S. By Zorn's Lemma we have a basis. □

3.3 Well-ordered

Definition 3.2 (Well-ordered)**.** We say that a partially ordered set $(X, \preceq)$ is **well-ordered** if every non-empty subset A of X has a least element in A.
For example, $(\mathbb{N}, \preceq)$ is well-ordered.
Note that if a set is well-ordered it must also be totally ordered (how would you compare some arbitrary element to the least element if the set was not well-ordered?)

Axiom 3.2 (Well-Ordering Principle)**.** Every non-empty set of $\mathbb{Z}^+$ can be well-ordered.

Theorem 3.2. The following are equivalent:

1. Axiom of Choice
2. Zorn's Lemma
3. Well-Ordering Principle

Example 3.2. Let $X = \mathbb{Q}$. Define the function ϕ

$$\phi(\frac{m}{n}) = \begin{cases} 2^m 5^n & \text{if } m > 0 \\ 1 & \text{if } m = 0 \\ 3^{-m} 7^n & \text{if } m < 0 \end{cases}$$

Note that $\phi : \mathbb{Q} \to \mathbb{N}$ is 1-1. (we could have used any combination of unique primes, as long as we ensure there is a 1-1 mapping).
Note that we can map the rationals to a subset of $\mathbb{N}$, thus the rationals are well-ordered by the Well-Ordering Principle.
Note that we also have $r \leq s \iff \phi(r) \leq \phi(s)$ (ϕ is an order isomorphism).

3.4 Equivalence relations and partitions

Definition 3.3 (Equivalence relation)**.** Let X be non-empty. A relation $\sim$ on X is an **equivalence relation** if the relation is

1. reflexive
2. symmetric
3. transitive

Observation 3.1. Let $[x] = \{y \in X \mid x \sim y\}$ or the **equivalence class** of X.
Then

1. Either $[x] = [y]$ or $[x] \cap [y] = \varnothing$
2. $X = \bigcup_{x \in X} [x]$

Definition 3.4. Let $X \neq \varnothing$. A **partition** of X is a collection $\{A_\alpha\}_{\alpha \in I} \subset P(X)$ such that

1. $A_\alpha \neq \varnothing$
2. $A_\alpha \cap A_\beta = \varnothing$ if $\alpha \neq \beta$
3. $X = \bigcup_{\alpha \in I} A_\alpha$

Observation 3.2. If $\{A_\alpha\}_{\alpha \in I}$ is a partition of X and $x \sim y$ *iff* $x, y \in A_\alpha$, then $\sim$ is an equivalence relation (i.e. if we start with a partition based on some relation $\sim$, we can show $\sim$ is an equivalence relation).

Example 3.3. How many equivalence relations are there on $X = \{1, 2, 3\}$? We can count the number of partitions:

1. $\{\{1\}, \{2\}, \{3\}\}$
2. $\{\{1, 2, 3\}\}$
3. $\{\{1, 2\}\{3\}\}$ (3 permutations since $\binom{3}{2}$)

Example 3.4. Let X be any set (empty or non-empty). Define $\sim$ on $P(X)$ by $A \sim B$ *iff* there exists $f : A \to B$ that is $1 - 1$ and onto.
$\sim$ has properties:

reflexive Take $\mathrm{id} : A \to A$ where $\mathrm{id}(x) = x$

symmetric If we have $f : A \to B$ then we have $f^{-1} : B \to A$ since f is bijective.

transitive If we have $f : A \to B$ and $g : B \to C$, then we have $g \circ f : A \to B$

thus $\sim$ is an equivalence relation.
For $X = \{1, 2, 3\}$, we have four equivalence classes on $P(X)$: one for every possible subset size $(0, \ldots, 3)$.

4 September 17, 2018

4.1 Cardinality

Definition 4.1 (Equivalence of sets)**.** We say that two sets X and Y are **equivalent** if there exists a *1-1 and onto* function $f : X \to Y$. We write $X \sim Y$.

Definition 4.2 (Cardinality)**.** If $X \sim Y$, we say that the two sets have the same **cardinality** and write $|X| = |Y|$.

Definition 4.3 (Finite sets)**.** X is **finite** if $X = \varnothing$ or if $X \sim \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. If $X \sim \{1, \dots, n\}$ we say X has cardinality n and write $|X| = n$. We let $|\varnothing| = 0$.

Definition 4.4 (Infinite sets)**.** X is **infinite** if it is not finite.

Example 4.1. We know $\mathbb{N}$ is infinite. We claim $\{2, 3, \dots\}$ is also infinite.
Note that $f : \mathbb{N} \to \{2, 3, \dots\}$ where $f(n) = f(n + 1)$ is a 1-1 and ontoo map, thus $\mathbb{N} \sim \{2, 3, \dots\}$ so $\{2, 3, \dots\}$ is infinite as well.

4.2 Pigeonhole Principle

Question 4.1. If $n \neq m$, can $\{1, \dots, n\} \sim \{1, \dots, m\}$?

Theorem 4.1 (Pigeonhole Principle)**.** The set $\{1, \dots, n\}$ is **not** equivalent to any proper subset.

Proof. We prove this by induction on n.

Base case Note that $\{1\} \not\sim \varnothing$.

Inductive step Assume the statement holds for $\{1, \dots, k\}$ for some k.

Suppose that we had a 1-1 function $f : \{1, 2, \dots, k, k+1\} \to \{1, 2, \dots, k, k+1\} \setminus \{m\}$ for some $m \in \{1, \dots, k+1\}$. We have one of two possibilities:

$m = k + 1$ Then

$$f_{|\{1,\dots,k\}} : \{1, \dots, k\} \overset{1-1}{\to} \{1, \dots, k\} \setminus \{f(k + 1)\}$$

where $f_{|A}$ is restrict of f to A.

Thus $f_{|\{1,\dots,k\}}$ is a 1-1 onto function to a proper subset of $\{1, \dots, k\}$ (since $f(k + 1)$ must map to one of $\{1, 2, \dots, k, k + 1\} \setminus \{m\} = \{1, \dots, k\}$), which is a contradiction of inductive hypothesis.

$m \neq k + 1$ Assume that $f(j_0) = k + 1$ and also $m \in \{1, \dots, k\}$.

Note if $j_0 = k + 1$, then $f_{|\{1,\dots,k\}} : \{1, \dots, k\} \to \{1, \dots, k\} \setminus \{m\}$, which is a contradiction of the inductive hypothesis. Thus $j_0 \neq k + 1$ so $f(k + 1) \neq k + 1$.

Let $g : \{1, \dots, k + 1\} \to \{1, \dots, k + 1\} \setminus \{m\}$ where

$$g(i) = \begin{cases} k + 1 & \text{if } i = k + 1 \\ f(k + 1) & \text{if } i = j_0 \\ f(i) & \text{if } i \neq k + 1, j_0 \end{cases}$$

so g is a 1-1 function where $g(k + 1) = k + 1$, but we already know that such a function cannot exist thus this is impossible.

∎

Corollary 4.1. If X is finite, then X is not equivalent to any proper subset.

Proof. If we assume there is a 1-1 and onto $g : X \to A \subsetneq X$, then for some $m \neq n$ we could apply $f(\{1, \ldots, m\}) = X$ and $f^{-1}(A) = \{1, \ldots, n\}$, thus

$$\{1, \ldots, m\} \xrightarrow{f} X \to g \to A \xrightarrow{f^{-1}} \{1, \ldots, n\}$$

which would contradict the Pigeonhole principle since $n < m$. □

4.3 Countable

Definition 4.5 (Countable)**.** We say that X is **countable** if either X is finite or if $X \sim \mathbb{N}$.
If $X \sim \mathbb{N}$ we can say that X is **countably infinite** and we write $|X| = |N| = \aleph_0$ or **aleph naught**.

4.4 Infinite sets has countably infinite subset

Proposition 4.1 (Infinite set has countably infinite subset)**.** Every infinite set contains a subset $A \sim \mathbb{N}$.

Proof. Assume X is infinite. Let $f : P(X) \setminus \{\varnothing\} \to X$ where for every $A \subset X$ the Axiom of Choice permits $f(A) \in A$.
Let $x_1 = f(X)$. We define recursively

$$x_{n+1} = f(X \setminus \{x_1, \ldots, x_n\})$$

This gives us a sequence $\{x_n\} = \{x_1, x_2, \ldots, x_n, \ldots\} = A \sim \mathbb{N}$. □

Corollary 4.2. Every infinite set X is equivalent to a proper subset.

Proof. Given X construct $\{x_n\}$ as above. Define $f : X \to X \setminus \{x_1\}$ by

$$f(x) = \begin{cases} x_{n+1} & \text{if } x = x_n \\ x & \text{if } x \notin \{x_n\} \end{cases}$$

thus we have a 1-1 and onto function to a proper subset of X. □

4.5 1-1 and onto duality

Proposition 4.2. TFAE:

1. There exists $f : X \to Y$ that is 1-1
2. There exists $g : Y \to X$ that is onto

Proof. $1 \to 2$ Assume $f : X \to Y$ is 1-1. Define $g : Y \to X$ by

$$g(y) = \begin{cases} x & \text{if } y = f(x) \text{ for some } x \\ x_0 & \text{for some arbitrary } x_0 \in X \end{cases}$$

$2 \to 1$ Let $g : Y \to X$ be onto and let $h : P(Y) \setminus \{\varnothing\}$ be a choice function.
For each $x \in X$ define

$$f(x) = h(g^{-1}(\{x\}))$$

where $g^{-1}(\{x\}) = \{y \in Y_{|g}\}$.

□

4.6 Partial order on cardinalities

Definition 4.6 ($\leq$ relation on cardinalities)**.** Given X, Y we write $|X| \leq |Y|$ if there exists a **1-1 function** $f : X \to Y$ ($f(X) \sim X$).

Observation 4.1. Note that $|\mathbb{N}| \leq |\mathbb{Q}|$ since $f(n) = \frac{n}{1}$ is a 1-1 function $f : \mathbb{N} \to \mathbb{Q}$.
Also $|\mathbb{Q}| \to |\mathbb{N}|$ since

$$g\left(\frac{m}{n}\right) = \begin{cases} 2^m 3^n & \text{if } m > 0 \\ 1 & \text{if } m = 0 \\ 5^{-m} 7^n & \text{if } m < 0 \end{cases}$$

where g is still a function by unique prime factorization of the integers.
Does this imply $|\mathbb{N}| = |\mathbb{Q}|$, that is does $|X| \leq |Y|$ and $|Y| \leq |X|$ imply $|X| = |Y|$?

5 September 19, 2018

Note: theorems marked with (*****) are important and one should be familiar with the proof.

5.1 Cantor-Schröder-Bernstein theorem (*****)

Theorem 5.1 (Cantor-Schröder-Bernstein theorem)**.** Let $A_2 \subset A_1 \subset A_0 = A$. Assume that $A_2 \sim A_0$. Then $A_0 \sim A_1$.
(aside: support $f : X \to Y$ is 1-1 and onto. Let $A \subset B$, then $f(B \setminus A) = f(B) \setminus f(A)$).

Proof. Let $\phi : A_0 \to A_2$ be 1-1 and onto. Let $A_3 = \phi(A_1)$ and $A_4 = \phi(A_2)$.
In fact, we let $A_{n+2} = \phi(A_n)$.
Notice that $A_{n+1} \subset A_n$ for all $n \in Z^+$.
Key observation:

$$A_0 = (A_0 \setminus A_1) \cup (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup \ldots \cup \bigcap_{n=0}^{\infty} A_n$$

Similarly, we have

$$A_1 = (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup \ldots \cup \bigcap_{n=1}^{\infty} A_n$$

We want to show there is a 1-1 and onto mapping between the two expressions for A_0 and A_1.
Notice that the two $\bigcap A_n$ are equivalent since $A_0 \cap \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n$.
We can map $(A_1 \setminus A_2)$ in A_0 to $(A_1 \setminus A_2)$ in A_1, $(A_3 \setminus A_4)$ in both, etc. We can also map Note ϕ maps $A_0 \setminus A_1$ to $\phi(A_0) \setminus \phi(A_1) = A_2 \setminus A_3$ (from aside before).
More formally, we define $f : A_0 \to A_1$ by

$$f(x) = \begin{cases} x & \text{if } x \in \bigcap_{n=0}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n \\ x & \text{if } x \in A_{2k+1} \setminus A_{2k+2} \text{ for } k = 0, 1, \ldots \\ \phi(x) & \text{if } x \in A_{2k} \setminus A_{2k+1} \text{ for } k = 0, 1, \ldots \end{cases}$$

Clearly f is 1-1 and onto, thus $A_0 \sim A_1$. ∎

Corollary 5.1. If $A_1 \subset A$, $B_1 \subset B$ and $A \sim B_1$ (i.e. $|A| \leq |B|$) $B \sim A_1$ (i.e. $|B| \leq |A|$), then $A \sim B$.

Proof. Let $f : A \to B_1$ and $g : B \to A_1$ be 1-1 and onto functions.
Let $A_2 = g(B_1)$, then $A_2 \subseteq A_1 \subseteq A$. Then $g \circ f : A \to A_2$ is 1-1 and onto so $A \sim A_2$, thus by CSB we have $A \sim A_1 \sim B$. ☐

Example 5.1. Back to the example where we have $|\mathbb{Q}| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |\mathbb{Q}|$, by CSB we have $|\mathbb{Q}| = |\mathbb{N}|$.

5.2 Proving countability

Proposition 5.1. If X is infinite then $|X| = |\mathbb{N}| = \aleph_0$ **if and only if** there is a 1-1 function $f : X \to \mathbb{N}$.

Proof. If $|X| = |N|$, then there is a 1-1 and onto function from $f : X \to \mathbb{N}$ by definition.
Assume there exists a 1-1 $f : X \to \mathbb{N}$. Then $|X| \leq |\mathbb{N}|$.
Since X is infinite, there exists a countably infinite subset of cardinality $|\mathbb{N}|$, thus $|\mathbb{N}| \leq |X|$. By CSB we have $|X| = |\mathbb{N}|$. ☐

Example 5.2. Show that $\mathbb{N} \times \mathbb{N}$ is countable.
Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined as $f(n, m) = 2^n 3^m$. Thus we have a 1-1 function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, thus by the previous proposition $\mathbb{N} \times \mathbb{N}$ is countable.

5.3 Uncountability and Cantor's diagonal proof

Definition 5.1 (Uncountable sets)**.** A set X is **uncountable** if X is not countable.

Theorem 5.2 (Cantor)**.** $(0, 1)$ is uncountable.

Proof. Assume that $(0, 1)$ is countable.
We can write

$$\begin{aligned} a_1 &= 0.a_{11}a_{12}a_{13}\ldots \\ a_2 &= 0.a_{21}a_{22}a_{23}\ldots \\ &\vdots \\ a_n &= 0.a_{n1}a_{n2}a_{n3}\ldots \end{aligned}$$

and these representations are unique if we do not allow the representations to end in a string of 9's.
We want to construct some number $b \in (0, 1)$ that is not within our countable set.
Let $b = 0.b_1b_2\ldots$ where

$$b_n = \begin{cases} 7 & \text{if } a_{nn} \neq 7 \\ 3 & \text{if } a_{nn} = 7 \end{cases}$$

Thus $b \notin$ our set. ☐

Corollary 5.2. $\mathbb{R}$ is uncountable.
Note that $(0, 1) \sim \mathbb{R}$ since we have $f : (0, 1) \to \mathbb{R}$ where

$$f(x) = \tan(\pi x - \frac{\pi}{2})$$

which is a 1-1 and onto function.
We denote $|\mathbb{R}|$ by c.

Question 5.1. Given X, Y: is it always true that either

1. $|X| = |Y|$
2. $|X| < |Y|$
3. $|Y| < |X|$

If we accept AC, the answer is yes.
If we do not accept AC, the answer could be no.