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PMATH 351 Course Notes

Real Analysis

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Table of Contents

1 September 10, 2018 **1**

1.1 Basic notation . . . 1
1.2 Basic set theory . . . 1
1.3 De Morgan's laws . . . 2
1.4 Products of sets, relations, and functions . . . 2

2 September 12, 2018 **3**

2.1 Zermelo's Axiom of Choice . . . 3
2.2 Properties of relations . . . 3
2.3 Partially and totally ordered sets . . . 3
2.4 Bounds on posets . . . 5

3 September 14, 2018 **5**

3.1 Maximal . . . 5
3.2 Zorn's Lemma . . . 6
3.3 Well-ordered . . . 6
3.4 Equivalence relations and partitions . . . 7

4 September 17, 2018 **8**

4.1 Cardinality . . . 8
4.2 Pigeonhole Principle . . . 8
4.3 Countable . . . 9
4.4 Infinite sets has countably infinite subset . . . 9
4.5 1-1 and onto duality . . . 9
4.6 Partial order on cardinalities . . . 10

5 September 19, 2018 **10**

5.1 Cantor-Schröder-Bernstein theorem (*****) . . . 10
5.2 Proving countability . . . 11
5.3 Uncountability and Cantor's diagonal proof . . . 11

6 September 21, 2018 **12**

6.1 Comparibility of cardinals . . . 12
6.2 Cardinal arithmetic . . . 13

7 September 24, 2018 **14**
7.1 $2^{\aleph_0} = c$ . . . 14
7.2 Countable union of countable sets . . . 15
7.3 Cardinality of power sets . . . 15
7.4 Russell's Paradox . . . 15
7.5 Continuum Hypothesis . . . 16

8 September 25, 2018 **16**
8.1 Metric spaces . . . 16
8.2 Norms . . . 16

9 September 28, 2018 **18**
9.1 l_p norm . . . 18
9.2 Young's Inequality . . . 18
9.3 Holder's Inequality (*****) . . . 19
9.4 Minkowski's Inequality . . . 20
9.5 Sequence spaces . . . 21

10 October 1, 2018 **22**
10.1 Normed linear spaces on arbitrary spaces Γ . . . 22
10.2 Normed linear spaces on continuous closed intervals . . . 22

11 October 3, 2018 **23**
11.1 Normed linear spaces on linear maps . . . 23
11.2 Topology on metric spaces . . . 24
11.3 Topology . . . 25

12 October 5, 2018 **25**
12.1 Metric space properties . . . 25
12.2 Equivalence class and decomposition of open sets . . . 26
12.3 Decomposition of closed sets and the Cantor set . . . 26

13 October 12, 2018 **27**
13.1 Closures and interiors . . . 27
13.2 Boundary and closed sets . . . 27
13.3 Closure and boundary . . . 28
13.4 Separable . . . 28

14 October 15, 2018 **28**
14.1 Limit points . . . 28
14.2 Properties of interiors, closures, and boundaries . . . 29
14.3 Convergence of sequences . . . 30

15 October 17, 2018 **30**
15.1 Convergence of sequences in $\mathbb{R}^n$ . . . 30

16 October 19, 2018 **32**
16.1 Induced metrics and topologies . . . 32
16.2 Continuity in metric spaces . . . 33
16.3 Sequential characterization of continuity . . . 33

17 October 22, 2018 **33**
17.1 Continuity on a set . . . 33
17.2 Homeomorphism . . . 34

18 October 24, 2018 **35**
18.1 Completeness of metric spaces: Cauchy sequences . . . 35
18.2 Properties of Cauchy sequences . . . 35
18.3 Homeomorphism and completeness . . . 37

19 October 26, 2018 **37**
19.1 Completeness of l_p . . . 37
19.2 Uniform and pointwise convergence . . . 38
19.3 Uniform convergence carries continuity (*********) . . . 38

20 October 31, 2018 **38**
20.1 Banach space . . . 38
20.2 Completeness theorem for $C_b(X)$ (*****) . . . 38
20.3 Characterization of completeness . . . 39

21 November 2, 2018 **40**
21.1 Cantor's Intersection Principle . . . 40
21.2 Series and partial sums . . . 40

22 November 5, 2018 **41**
22.1 Weierstrass M-Test . . . 41

23 November 7, 2018 **44**
23.1 Isometry . . . 44
23.2 Completion of a metric space . . . 44
23.3 Uniform continuity . . . 44
23.4 Completion theorem . . . 44

24 November 9, 2018 **45**
24.1 Lipschitz and contractions . . . 45
24.2 Banach Contractive Mapping Theorem . . . 45

25 November 12, 2018 **48**
25.1 Motivation and definitions for categories . . . 48
25.2 Baire Category Theorem . . . 49

26 November 14, 2018 **50**
26.1 Uniform convergence at one point . . . 50
26.2 Compactness . . . 51

27 November 16, 2018 **51**
27.1 Closed sets are compact in a closed metric space 51
27.2 Sequentially compact and Bolzano-Weierstrass Property 52
27.3 Finite Intersection Property 52
27.4 ϵ-net 53

28 November 19, 2018 **53**
28.1 Compactness implies Bolzano-Weierstrass Property 53
28.2 Sequentially compact implies complete and totally bounded 53
28.3 Sequential compactness is preserved under continuous functions 53
28.4 Lebesgue's theorem 54
28.5 Complete and totally bounded implies compactness 54

Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. These notes are my interpretation and transcription of the content covered in lectures. The instructor has not verified or confirmed the accuracy of these notes, and any discrepancies, misunderstandings, typos, etc. as these notes relate to course's content is not the responsibility of the instructor. If you spot any errors or would like to contribute, please contact me directly.

1 September 10, 2018

1.1 Basic notation

We denote

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \ldots\} \\
\mathbb{Z} &= \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
\mathbb{Q} &= \{\frac{n}{m} \mid n \in \mathbb{Z}, m \in \mathbb{N}\} \\
\mathbb{R} &= \text{real numbers}
\end{aligned}$$

We use $\subset$ and $\subseteq$ interchangeably, and use $\subsetneq$ for strict subsets. $\subset$ or $\subseteq$ is called "inclusion", and $\supset$ or $\supseteq$ is called "containment".

1.2 Basic set theory

We denote X as our universal set. If $\{A_\alpha\}_{alpha \in I}$ is such that $A_\alpha \subset X$ for all $\alpha \in I$ (index set), then

$$\bigcup_{\alpha \in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for some } \alpha \in I\} \qquad \text{(union)}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for all } \alpha \in I\} \qquad \text{(intersection)}$$

Define for $A, B \subseteq X$

$$\begin{aligned}
A \setminus B &= \{x \in X \mid x \in A, x \notin B\} && \text{(set difference)} \\
A \Delta B &= \{x \in X \mid x \in A \text{ and } x \notin B\} \text{ OR } x \in B \text{ and } x \notin A\} && \text{(semantic difference)} \\
A^c &= X \setminus A = \{x \in X \mid x \notin A\} && \text{(complement)} \\
&\varnothing && \text{(empty set)} \\
P(X) &= \{A \mid A \subset X\} \quad \varnothing \in P(X), X \in P(X) && \text{(power set)}
\end{aligned}$$

1.3 De Morgan's laws

De Morgan's laws states that given $\{A_\alpha\}_{\alpha\in I} \subset P(X)$

$$\left(\bigcup_{\alpha\in I} A_\alpha\right)^c = \bigcap_{\alpha\in I} A_\alpha^c$$
$$\left(\bigcap_{\alpha\in I} A_\alpha\right)^c = \bigcup_{\alpha\in I} A_\alpha^c$$

Question: what if $I = \varnothing$, what is $\bigcup_{\alpha\in\varnothing} A_\alpha$? It is in fact $\bigcup_{\alpha\in\varnothing} A_\alpha = \varnothing$.
Note that $\bigcap_{\alpha\in\varnothing} A_\alpha = X$ (from De Morgan's Law, and also $A_\alpha = A_\alpha^c$).

1.4 Products of sets, relations, and functions

Given X, Y define the product

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

If $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$ then $X \times Y = \{(x_i, y_j) \mid i = 1, \ldots, n \quad j = 1, \ldots, m\}$ containing nm elements.

Definition 1.1 (Relation)**.** A **relation** on X, Y is a subset R of the product $X \times Y$.
We write xRy if $(x, y) \in R$. The **domain** of R is

$$\{x \in X \mid \exists y \in Y \text{ with } (x, y) \in R\}$$

which need not cover our universal set.
The **range** of R is

$$\{y \in Y \mid \exists x \in X \text{ with } (x, y) \in R\}$$

Definition 1.2 (Function (as a relation))**.** A **function** from X into Y is a relation R such that for every $x \in X$, there exists exactly one $y \in Y$ with $(x, y) \in R$.

Suppose that we have $X_1, X_2, \ldots, X_n$ non-empty sets. Define

$$X_1 \times X_2 \times \ldots \times X_n = \prod_{i=1}^n X_i = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i\}$$

or a set of *n-tuples*.
If $X_i = X_j = X$ for all $i, j = 1, \ldots, n$, then

$$\prod_{i=1}^n X_i = \prod_{i=1}^n X = X^n$$

Problem 1.1. Given a collection $\{X_\alpha\}_{\alpha\in I}$ of non-empty sets, what do we mean by $\prod_{\alpha\in I} X_\alpha$?
Motivation: consider $X_1 \times \ldots \times X_n = \{(x_1, \ldots, x_n) \mid x_i \in X_i\}$. We choose some $(x_1, \ldots, x_n) \in \prod_{i\in\{1,\ldots,n\}} = I$.
This point induces *a function*

$$f_{(x_1,\ldots,x_n)} : \{1, \ldots, n\} \to \bigcup_{i=1}^n X_i$$

with $f(1) = x_1 \in X_1$, $f(i) = x_i \in X_i$, $f(n) = x_n \in X_n$, etc. Assume we have have $f : \{1, \dots, n\} \to \bigcup_{i=1}^{n} X_i$ such that $f(i) \in X_i$. Then

$$(f(1), f(2), \dots, f(n)) = \prod_{i=\{1,\dots,n\}} X_i$$

Definition 1.3 (Product of sets)**.** Given a collection $\{X_\alpha\}_{\alpha \in I}$ of non-empty sets we let

$$\prod_{\alpha \in I} X_\alpha = \{f : I \to \bigcup_{\alpha \in I} X_\alpha\}$$

such that $f(\alpha) \in X_\alpha$ (i.e $\prod_{\alpha \in I} X_\alpha$ is a "set of functions"). f is called a **choice function**.
Question: If $X_\alpha \neq \varnothing$, is $\prod_{\alpha \in I} X_\alpha \neq \varnothing$?

2 September 12, 2018

2.1 Zermelo's Axiom of Choice

Question: If $\{X_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty sets is

$$\prod_{\alpha \in I} X_\alpha \neq \varnothing$$

This is analogous to saying: given a collection of non-empty sets in $\mathbb{R}$, how would you choose an element from each subset of $\mathbb{R}$? This is easy if they were subsets of $\mathbb{N}$ (take the least element which exists by the *well-ordering principle*) but much more difficult in $\mathbb{R}$.

Axiom 2.1 (Zermelo's Axiom of Choice)**.** If $\{X_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty sets, then $\prod_{\alpha \in I} \neq \varnothing$.

Equivalently we have an analogous version:

Axiom 2.2 (Axiom of Choice V2)**.** If $X \neq \varnothing$, then there exists a function

$$f : P(X) \setminus \{\varnothing\} \to X$$

such that $f(A) \in A$ for all $A \in P(X) \setminus \{\varnothing\}$ (we can always pick out a subset ($e \in P(X)$) from a non-empty set A).

2.2 Properties of relations

Definition 2.1 (Relation properties)**.** A relation R on X (i.e. $R \subseteq X \times X$) is

1. **reflexive** if $x \; R \; x$ for all $x \in X$
2. **symmetric** if $x \; R \; y \Rightarrow y \; R \; x$
3. **anti-symmetric** if $x \; R \; y$ and $y \; R \; x$, then $x = y$
4. **transitive** if $x \; R \; y$ and $y \; R \; z$ implies $x\mathbb{R}z$

2.3 Partially and totally ordered sets

Example 2.1. Let $X = \mathbb{R}$. We have $x \; R \; y$ iff $x \leq y$.
Note that $\leq$ is reflexive, anti-symmetric, and transitive.

Example 2.2. Let $Y \neq \varnothing$ and $X = P(Y)$.
We write $A \; R \; B$ iff $A \subseteq B$.
Note that $\subseteq$ is reflexive, anti-symmetric, and transitive.

Example 2.3. Let $Y \neq \varnothing$ and $X = P(Y)$.
We write $A \; R \; B$ iff $B \subseteq A$.
Note that $\subseteq$ is reflexive, anti-symmetric, and transitive.

Definition 2.2 (Partially ordered sets)**.** A set X with a relation R on X is called a **partially ordered set** if R is

1. reflexive
2. anti-symmetric
3. transitive

(R is a **partial order** on X if it satisfies these three conditions).
We write (X, R) and call this a **poset**.

Definition 2.3 (Totally ordered sets)**.** If (X, R) is a poset, then if $A \subseteq X$ and $R_1 = R_{|A\times A}$ then (A, R_1) is a poset. We say (A, R_1) is **totally ordered** if for each $x, y \in A$ *either* $x \; R \; y$ or $y \; R \; x$. We also call totally ordered sets **chains**.

How many partial orderings can we have for a given set X (i.e. the number of ways to define partial order relations)?

Example 2.4. Let $X = \{x\}$. We have one relation $R = \{(x, x)\}$ (from $X \times X$) and thus 1 partial ordering.

Example 2.5. Let $X = \{x, y\}$. We know posets $(X, \preceq)$ must be reflexive, thus we have one relation where $x \preceq x$ and $y \preceq y$.
We can also have a poset with the reflexive relations above as well as $x \preceq y$. Similarly we can have a poset with $y \preceq x$.

Example 2.6. Let $X = \{x, y, z\}$.

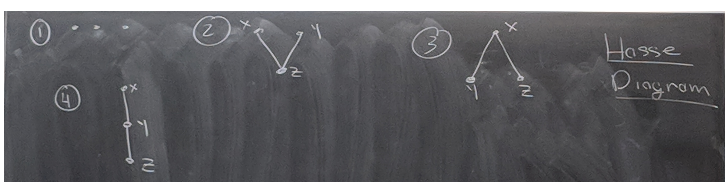

Figure 2.1: Hasse diagrams for the possible $(X, \preceq)$ posets (an edge downwards from a to b denotes $a \preceq b$; note reflexive $a \preceq a$ is assumed automatically).

We have the poset with just the reflexive relations $e \preceq e$ for $e \in X$.
We have the poset with the reflexive relations and $x \preceq z$ and $y \preceq z$ (3 posets with permutations).
We have the poset with the reflexive relations and $x \preceq y$ and $x \preceq z$ (3 posets with permutations).
We have the poset with the reflexive relations and $x \preceq y$ and $y \preceq z$ (6 posets with permutations).
We have the poset with the reflexive relations and $y \preceq z$ (6 posets with permutations, not shown in diagram above).

Note that when identifying these posets isomorphisms, we should not draw lines between two elements $a \leq b$ if the transitive property already implies that. For example if we had the chain $a \leq b \leq c$, the diagram with a line from a to c would be redundant (thus we will end up double counting).

2.4 Bounds on posets

Definition 2.4 (Upper and lower bounds)**.** Let $(X, \preceq)$ be a partially ordered set.
Let $A \subset X$. We say that $x_0 \in X$ is an **upper bound** for A if $x \preceq x_0$ for all $x \in A$.
If A has an upper bound, we say it is **bounded above**.
If A is bounded above then x_0 is the **least upper bound** if

1. x_0 is an upper bound of A
2. If y is an upper bound of A then $x_0 \preceq y$.

We write $x_0 = \text{lub}(A)$ or $x_0 \sup(A)$ (supremum).
If $x_0 = \text{lub}(A) \in A$, then x_0 is the *maximum* in A.
Similarly we define the same for lower bounds (infimum).

Example 2.7. Let $X = \mathbb{R}$ and $\preceq$ the usual ordering.

Fact 2.1. Every non-empty subset that is bounded above has a least upper bound (LUBP (lub property) for $\mathbb{R}$).

Example 2.8. Let $Y \neq \varnothing$, $X = P(Y)$, and $\preceq$ be $\subseteq$ (ordering by inclusion).
Y is the maximum element of $(X, \subseteq)$.
If $\{A_\alpha\}_{\alpha \in I} \subset P(X)$ is bounded above by Y, but note that

$$\text{lub}(\{A_\alpha\}_{\alpha \in I}) = \bigcup_{\alpha \in I} A_\alpha$$
$$\text{glb}(\{A_\alpha\}_{\alpha \in I}) = \bigcap_{\alpha \in I} A_\alpha$$

Recall that if $I = \varnothing$, then the glb is all of $\mathbb{R}$: this is in fact correct (it's the greatest set that is a lower bound for relation $\subseteq$).

3 September 14, 2018

3.1 Maximal

Definition 3.1. Let $(X, \preceq)$ be a partially ordered set. An element $x \in X$ is **maximal** if whenever $y \in X$ such that $x \preceq y$, we must have $x = y$.

Example 3.1. Suppose we have $x \preceq x$, $y \preceq y$, and $z \preceq z$. Then all of x, y, z are maximal.
Suppose we have $x \preceq z$ and $y \preceq z$ (as well as the reflexive relations). Then only z is maximal.
Suppose we have $x \preceq y$ and $x \preceq z$ (as well as the reflexive relations). Then y and z are maximal.
Suppose $x \preceq y \preceq z$ (and transitives). Only z is maximal.
Suppose $x \preceq y$ (and transitives). Then both y and z are maximal.
For $X \neq \varnothing$ and $(P(X), \subseteq)$, X is maximal.
For $X \neq \varnothing$ and $(P(X), \supseteq)$, $\varnothing$ is maximal.
For $(\mathbb{R}, \leq)$ has no maximal element.

3.2 Zorn's Lemma

Axiom 3.1 (Zorn's Lemma)**.** If $(X, \preceq)$ is a non-empty partially ordered set such that every chain $S \subset X$ has an upper bound. Then $(X, \preceq)$ has a maximal element.

We can apply Zorn's Lemma to prove a fundamental linear algebra theorem:

Theorem 3.1. Every non-zero vector space V has a basis.

Proof. Let $= \{A \subset X \mid A \text{ is linear indep.}\}$. Note $\neq \varnothing$ because $V \neq \{0\}$.
Order with $\subseteq$.
A basis is a maximal element in $(, \subseteq)$ (if we add vector to this basis, it would be a linear combination of the basis vectors by definition of a basis).
Let $S = \{A_\alpha\}_{\alpha \in I}$ be a chain in . Let $A_0 = \bigcup_{\alpha \in I} A_\alpha$.
Choose $x_1, \dots, x_n \in A_0$ distinct elements where $x_i \in A_{\alpha_i}$. Assume that $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$. But $x_i \in A_{\alpha_i}$ and we can assume that

$$A_{\alpha_1} \subseteq A_{\alpha_2} \subseteq \ldots \subseteq A_{\alpha_n} \Rightarrow \{x_1, \dots, x_n\} \subset A_{\alpha_n}$$

So $\alpha_i = 0$ for all $i = 1, \dots, n$, thus A_0 is an upper bound of S. By Zorn's Lemma we have a maximal element which will be a basis. □

3.3 Well-ordered

Definition 3.2 (Well-ordered)**.** We say that a partially ordered set $(X, \preceq)$ is **well-ordered** if every non-empty subset A of X has a least element in A.
For example, $(\mathbb{N}, \preceq)$ is well-ordered.
Note that if a set is well-ordered it must also be totally ordered (how would you compare some arbitrary element to the least element if the set was not well-ordered?)

Axiom 3.2 (Well-Ordering Principle)**.** Every non-empty set of $\mathbb{Z}^+$ can be well-ordered.

Theorem 3.2. The following are equivalent:

1. Axiom of Choice
2. Zorn's Lemma
3. Well-Ordering Principle

Example 3.2. Let $X = \mathbb{Q}$. Define the function ϕ

$$\phi(\frac{m}{n}) = \begin{cases} 2^m 5^n & \text{if } m > 0 \\ 1 & \text{if } m = 0 \\ 3^{-m} 7^n & \text{if } m < 0 \end{cases}$$

Note that $\phi : \mathbb{Q} \to \mathbb{N}$ is 1-1. (we could have used any combination of unique primes, as long as we ensure there is a 1-1 mapping).
Note that we can map the rationals to a subset of $\mathbb{N}$, thus the rationals are well-ordered by the Well-Ordering Principle.
Note that we also have $r \leq s \iff \phi(r) \leq \phi(s)$ (ϕ is an order isomorphism).

3.4 Equivalence relations and partitions

Definition 3.3 (Equivalence relation)**.** Let X be non-empty. A relation $\sim$ on X is an **equivalence relation** if the relation is

1. reflexive
2. symmetric
3. transitive

Observation 3.1. Let $[x] = \{y \in X \mid x \sim y\}$ or the **equivalence class** of X.
Then

1. Either $[x] = [y]$ or $[x] \cap [y] = \varnothing$
2. $X = \bigcup_{x \in X} [x]$

Definition 3.4. Let $X \neq \varnothing$. A **partition** of X is a collection $\{A_\alpha\}_{\alpha \in I} \subset P(X)$ such that

1. $A_\alpha \neq \varnothing$
2. $A_\alpha \cap A_\beta = \varnothing$ if $\alpha \neq \beta$
3. $X = \bigcup_{\alpha \in I} A_\alpha$

Observation 3.2. If $\{A_\alpha\}_{\alpha \in I}$ is a partition of X and $x \sim y$ *iff* $x, y \in A_\alpha$, then $\sim$ is an equivalence relation (i.e. if we start with a partition based on some relation $\sim$, we can show $\sim$ is an equivalence relation).

Example 3.3. How many equivalence relations are there on $X = \{1, 2, 3\}$? We can count the number of partitions:

1. $\{\{1\}, \{2\}, \{3\}\}$
2. $\{\{1, 2, 3\}\}$
3. $\{\{1, 2\}\{3\}\}$ (3 permutations since $\binom{3}{2}$)

Example 3.4. Let X be any set (empty or non-empty). Define $\sim$ on $P(X)$ by $A \sim B$ *iff* there exists $f : A \to B$ that is $1 - 1$ and onto.
$\sim$ has properties:

reflexive Take $\mathrm{id} : A \to A$ where $\mathrm{id}(x) = x$

symmetric If we have $f : A \to B$ then we have $f^{-1} : B \to A$ since f is bijective.

transitive If we have $f : A \to B$ and $g : B \to C$, then we have $g \circ f : A \to B$

thus $\sim$ is an equivalence relation.
For $X = \{1, 2, 3\}$, we have four equivalence classes on $P(X)$: one for every possible subset size $(0, \ldots, 3)$.

4 September 17, 2018

4.1 Cardinality

Definition 4.1 (Equivalence of sets)**.** We say that two sets X and Y are **equivalent** if there exists a *1-1 and onto* function $f : X \to Y$. We write $X \sim Y$.

Definition 4.2 (Cardinality)**.** If $X \sim Y$, we say that the two sets have the same **cardinality** and write $|X| = |Y|$.

Definition 4.3 (Finite sets)**.** X is **finite** if $X = \varnothing$ or if $X \sim \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. If $X \sim \{1, \ldots, n\}$ we say X has cardinality n and write $|X| = n$. We let $|\varnothing| = 0$.

Definition 4.4 (Infinite sets)**.** X is **infinite** if it is not finite.

Example 4.1. We know $\mathbb{N}$ is infinite. We claim $\{2, 3, \ldots\}$ is also infinite.
Note that $f : \mathbb{N} \to \{2, 3, \ldots\}$ where $f(n) = f(n + 1)$ is a 1-1 and onto map, thus $\mathbb{N} \sim \{2, 3, \ldots\}$ so $\{2, 3, \ldots\}$ is infinite as well.

4.2 Pigeonhole Principle

Question 4.1. If $n \neq m$, can $\{1, \ldots, n\} \sim \{1, \ldots, m\}$?

Theorem 4.1 (Pigeonhole Principle)**.** The set $\{1, \ldots, n\}$ is **not** equivalent to any proper subset.

Proof. We prove this by induction on n.

Base case Note that $\{1\} \not\sim \varnothing$.

Inductive step Assume the statement holds for $\{1, \ldots, k\}$ for some k.

Suppose that we had a 1-1 function $f : \{1, 2, \ldots, k, k+1\} \to \{1, 2, \ldots, k, k+1\} \setminus \{m\}$ for some $m \in \{1, \ldots, k+1\}$. We have one of two possibilities:

$m = k + 1$ Then

$$f_{|\{1,\ldots,k\}} : \{1, \ldots, k\} \overset{1-1}{\to} \{1, \ldots, k\} \setminus \{f(k + 1)\}$$

where $f_{|A}$ is restrict of f to A.

Thus $f_{|\{1,\ldots,k\}}$ is a 1-1 onto function to a proper subset of $\{1, \ldots, k\}$ (since $f(k + 1)$ must map to one of $\{1, 2, \ldots, k, k + 1\} \setminus \{m\} = \{1, \ldots, k\}$), which is a contradiction of inductive hypothesis.

$m \neq k + 1$ Assume that $f(j_0) = k + 1$ and also $m \in \{1, \ldots, k\}$.

Note if $j_0 = k + 1$, then $f_{|\{1,\ldots,k\}} : \{1, \ldots, k\} \to \{1, \ldots, k\} \setminus \{m\}$, which is a contradiction of the inductive hypothesis. Thus $j_0 \neq k + 1$ so $f(k + 1) \neq k + 1$.

Let $g : \{1, \ldots, k + 1\} \to \{1, \ldots, k + 1\} \setminus \{m\}$ where

$$g(i) = \begin{cases} k + 1 & \text{if } i = k + 1 \\ f(k + 1) & \text{if } i = j_0 \\ f(i) & \text{if } i \neq k + 1, j_0 \end{cases}$$

so g is a 1-1 function where $g(k + 1) = k + 1$, but we already know that such a function cannot exist thus this is impossible.

□

Corollary 4.1. If X is finite, then X is not equivalent to any proper subset.

Proof. If we assume there is a 1-1 and onto $g : X \to A \subsetneq X$, then for some $m \neq n$ we could apply $f(\{1, \dots, m\}) = X$ and $f^{-1}(A) = \{1, \dots, n\}$, thus

$$\{1, \dots, m\} \xrightarrow{f} X \to g \to A \xrightarrow{f^{-1}} \{1, \dots, n\}$$

which would contradict the Pigeonhole principle since $n < m$. □

4.3 Countable

Definition 4.5 (Countable)**.** We say that X is **countable** if either X is finite or if $X \sim \mathbb{N}$.
If $X \sim \mathbb{N}$ we can say that X is **countably infinite** and we write $|X| = |N| = \aleph_0$ or **aleph naught**.

4.4 Infinite sets has countably infinite subset

Proposition 4.1 (Infinite set has countably infinite subset)**.** Every infinite set contains a subset $A \sim \mathbb{N}$.

Proof. Assume X is infinite. Let $f : P(X) \setminus \{\varnothing\} \to X$ where for every $A \subset X$ the Axiom of Choice permits $f(A) \in A$.
Let $x_1 = f(X)$. We define recursively

$$x_{n+1} = f(X \setminus \{x_1, \dots, x_n\})$$

This gives us a sequence $\{x_n\} = \{x_1, x_2, \dots, x_n, \dots\} = A \sim \mathbb{N}$. □

Corollary 4.2. Every infinite set X is equivalent to a proper subset.

Proof. Given X construct $\{x_n\}$ as above. Define $f : X \to X \setminus \{x_1\}$ by

$$f(x) = \begin{cases} x_{n+1} & \text{if } x = x_n \\ x & \text{if } x \notin \{x_n\} \end{cases}$$

thus we have a 1-1 and onto function to a proper subset of X. □

4.5 1-1 and onto duality

Proposition 4.2. The follow are equivalent (TFAE):

1. There exists $f : X \to Y$ that is 1-1
2. There exists $g : Y \to X$ that is onto

Proof. $1 \to 2$ Assume $f : X \to Y$ is 1-1. Define $g : Y \to X$ by

$$g(y) = \begin{cases} x & \text{if } y = f(x) \text{ for some } x \\ x_0 & \text{for some arbitrary } x_0 \in X \end{cases}$$

$2 \to 1$ Let $g : Y \to X$ be onto and let $h : P(Y) \setminus \{\varnothing\}$ be a choice function.
For each $x \in X$ define

$$f(x) = h(g^{-1}(\{x\}))$$

where $g^{-1}(\{x\}) = \{y \in Y_{|g}\}$.

□

4.6 Partial order on cardinalities

Definition 4.6 ($\leq$ relation on cardinalities)**.** Given X, Y we write $|X| \leq |Y|$ if there exists a **1-1 function** $f : X \to Y$ ($f(X) \sim X$).

Observation 4.1. Note that $|\mathbb{N}| \leq |\mathbb{Q}|$ since $f(n) = \frac{n}{1}$ is a 1-1 function $f : \mathbb{N} \to \mathbb{Q}$.
Also $|\mathbb{Q}| \to |\mathbb{N}|$ since

$$g\left(\frac{m}{n}\right) = \begin{cases} 2^m 3^n & \text{if } m > 0 \\ 1 & \text{if } m = 0 \\ 5^{-m} 7^n & \text{if } m < 0 \end{cases}$$

where g is still a function by unique prime factorization of the integers.
Does this imply $|\mathbb{N}| = |\mathbb{Q}|$, that is does $|X| \leq |Y|$ and $|Y| \leq |X|$ imply $|X| = |Y|$?

5 September 19, 2018

Note: theorems marked with (*****) are important and one should be familiar with the proof.

5.1 Cantor-Schröder-Bernstein theorem (*****)

Theorem 5.1 (Cantor-Schröder-Bernstein theorem)**.** Let $A_2 \subset A_1 \subset A_0 = A$. Assume that $A_2 \sim A_0$. Then $A_0 \sim A_1$.
(aside: support $f : X \to Y$ is 1-1 and onto. Let $A \subset B$, then $f(B \setminus A) = f(B) \setminus f(A)$).

Proof. Let $\phi : A_0 \to A_2$ be 1-1 and onto. Let $A_3 = \phi(A_1)$ and $A_4 = \phi(A_2)$.
In fact, we let $A_{n+2} = \phi(A_n)$.
Notice that $A_{n+1} \subset A_n$ for all $n \in Z^+$.
Key observation:

$$A_0 = (A_0 \setminus A_1) \cup (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup \ldots \cup \bigcap_{n=0}^{\infty} A_n$$

Similarly, we have

$$A_1 = (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup \ldots \cup \bigcap_{n=1}^{\infty} A_n$$

We want to show there is a 1-1 and onto mapping between the two expressions for A_0 and A_1.
Notice that the two $\bigcap A_n$ are equivalent since $A_0 \cap \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n$.
We can map $(A_1 \setminus A_2)$ in A_0 to $(A_1 \setminus A_2)$ in A_1, $(A_3 \setminus A_4)$ in both, etc. Note ϕ maps $A_0 \setminus A_1$ to $\phi(A_0) \setminus \phi(A_1) = A_2 \setminus A_3$ (from aside before).
More formally, we define $f : A_0 \to A_1$ by

$$f(x) = \begin{cases} x & \text{if } x \in \bigcap_{n=0}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n \\ x & \text{if } x \in A_{2k+1} \setminus A_{2k+2} \text{ for } k = 0, 1, \ldots \\ \phi(x) & \text{if } x \in A_{2k} \setminus A_{2k+1} \text{ for } k = 0, 1, \ldots \end{cases}$$

Clearly f is 1-1 and onto, thus $A_0 \sim A_1$. □

Corollary 5.1. If $A_1 \subset A$, $B_1 \subset B$ and $A \sim B_1$ (i.e. $|A| \leq |B|$) $B \sim A_1$ (i.e. $|B| \leq |A|$), then $A \sim B$.

Proof. Let $f : A \to B_1$ and $g : B \to A_1$ be 1-1 and onto functions.
Let $A_2 = g(B_1)$, then $A_2 \subseteq A_1 \subseteq A$. Then $g \circ f : A \to A_2$ is 1-1 and onto so $A \sim A_2$, thus by CSB we have $A \sim A_1 \sim B$. □

Example 5.1. Back to the example where we have $|\mathbb{Q}| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |\mathbb{Q}|$, by CSB we have $|\mathbb{Q}| = |\mathbb{N}|$.

5.2 Proving countability

Proposition 5.1. If X is infinite then $|X| = |\mathbb{N}| = \aleph_0$ **if and only if** there is a 1-1 function $f : X \to \mathbb{N}$.

Proof. If $|X| = |N|$, then there is a 1-1 and onto function from $f : X \to \mathbb{N}$ by definition.
Assume there exists a 1-1 $f : X \to \mathbb{N}$. Then $|X| \leq |\mathbb{N}|$.
Since X is infinite, there exists a countably infinite subset of cardinality $|\mathbb{N}|$, thus $|\mathbb{N}| \leq |X|$. By CSB we have $|X| = |\mathbb{N}|$. □

Example 5.2. Show that $\mathbb{N} \times \mathbb{N}$ is countable.
Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined as $f(n, m) = 2^n 3^m$. Thus we have a 1-1 function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, thus by the previous proposition $\mathbb{N} \times \mathbb{N}$ is countable.

5.3 Uncountability and Cantor's diagonal proof

Definition 5.1 (Uncountable sets)**.** A set X is **uncountable** if X is not countable.

Theorem 5.2 (Cantor)**.** $(0, 1)$ is uncountable.

Proof. Assume that $(0, 1)$ is countable.
We can write

$$\begin{aligned} a_1 &= 0.a_{11}a_{12}a_{13}\dots \\ a_2 &= 0.a_{21}a_{22}a_{23}\dots \\ &\vdots \\ a_n &= 0.a_{n1}a_{n2}a_{n3}\dots \end{aligned}$$

and these representations are unique if we do not allow the representations to end in a string of 9's.
We want to construct some number $b \in (0, 1)$ that is not within our countable set.
Let $b = 0.b_1b_2\dots$ where

$$b_n = \begin{cases} 7 & \text{if } a_{nn} \neq 7 \\ 3 & \text{if } a_{nn} = 7 \end{cases}$$

Thus $b \notin$ our set. □

Corollary 5.2. $\mathbb{R}$ is uncountable.
Note that $(0, 1) \sim \mathbb{R}$ since we have $f : (0, 1) \to \mathbb{R}$ where

$$f(x) = \tan(\pi x - \frac{\pi}{2})$$

which is a 1-1 and onto function.
We denote $|\mathbb{R}|$ by c.

Question 5.1. Given X, Y: is it always true that either

1. $|X| = |Y|$
2. $|X| < |Y|$
3. $|Y| < |X|$

If we accept AC, the answer is yes.
If we do not accept AC, the answer could be no.

6 September 21, 2018

6.1 Comparibility of cardinals

Theorem 6.1 (Comparibility of cardinals)**.** If X, Y are non-empty then either $|X| \preceq |Y|$ or $|Y| \preceq |X|$.

Proof. Let $S = \{(A, B, f) \mid A \subseteq X, B \subseteq Y, f : A \to B \text{ is 1-1 and onto}\}$ (note $S \neq \varnothing$; take singletons from each X,Y with trivial f).
Order S as follows: $(A_1, B_1, f_1) \preceq (A_2, B_2, f_2)$ if $A_1 \subseteq A_2$, $B_1 \subseteq B_2$ and $f_1 = f_{2|A_1}$ (this is possible since $A_1 \subseteq A_2$ so restriction exists) (if any of the three conditions fail, we cannot order the two triples: this is fine since we are only looking for a partial order).
Let $C = \{(A_\alpha, B_\alpha, f_\alpha)\}_{\alpha\in I}$ be a chain in $(S, \preceq)$.
Let $A_0 = \bigcup_{\alpha\in I} A_\alpha$, $B_0 = \bigcup_{\alpha\in I} B_\alpha$, and $f_0 : A_0 \to B_0$ by $f_0(x) = f_{\alpha_0}(x)$ if $x \in A_{\alpha_0}$ (we find the subset A_{α_0} which the point x we pick out from A_0 is found in: then we take the corresponding function f_{α_0} as our function for that point).
Note: if $x \in A_{\alpha_1}$ and $x \in A_{\alpha_2}$ we can assume taht $(A_{\alpha_1}, B_{\alpha_1}, f_{\alpha_1}) \preceq (A_{\alpha_2}, B_{\alpha_2}, f_{\alpha_2})$ then

$$f_{\alpha_1}(x) = f_{\alpha_2|A_{\alpha_1}}(x) = f_{\alpha_2}(x)$$

thus f is well-defined (it doesn't really matter which f_{α_0} we choose since they're all the same for a given point x).
We need to show

$f_0 : A_0 \to B_0$ **is 1-1** Let $x_1, x_2 \in A_0$, $x_1 \neq x_2$. We may assume that $x_1 \in A_{\alpha_1}$, $x_2 \in A_{\alpha_2}$ with $A_{\alpha_1} \subseteq A_{\alpha_2}$ thus $x_1, x_2 \in A_{\alpha_2}$. Since f_{α_2} is 1-1 ($f_{\alpha_2}(x_1) \neq f_{\alpha_2}(x_2)$) then $f_0(x_1) \neq f_0(x_2)$.

f_0 **is onto** Let $y_0 \in B_0 \Rightarrow y_0 \in B_{\alpha_0}$ for some α_0.
Then there exists $x_0 \in A_{\alpha_0}$ with $f_{\alpha_0}(x_0) = y_0$ (since f_{α_0} is onto), thus $f_0(x_0) = y_0$.

thus (A_0, B_0, f_0) belongs to our set S (since f_0 is 1-1 and onto) and it is an **upper bound** for C (A_0, B_0 are the unions so they're upper bounds for all A_α, B_α, and f restricted to any subset is equivalent to the function on that subset).
Let (A, B, f) be maximal in S by Zorn's Lemma: we have three cases

1. if $A = X$, then $|X| \leq |Y|$ (since we have a 1-1, onto function from X onto $B \subseteq Y$).
2. if $B = Y$, then $|Y| = |A||X|$
3. Suppose $X \setminus A \neq \varnothing$ and $Y \setminus B \neq \varnothing$. Let $x_0 \in X \setminus A$, $y_0 \in Y \setminus B$.
   Let $A^* = A \cup \{x_0\}$, $B^* = B \cup \{y_0\}$, and $f^* : A^* \to B^*$ by

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in A \\ y_0 & \text{if } x = x_0 \end{cases}$$

   Thus $(A, B, f) \not\preceq (A^*, B^*, f^*)$ which is impossible (i.e. this case is impossible).

□

6.2 Cardinal arithmetic

Sum If $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$ and $X \cap Y = \varnothing$, then $|X| = n$, $|Y| = m$, and $|X \cup Y| = n + m$ obviously.

Definition 6.1. Assume that X and Y are such that $X \cap Y \neq \varnothing$, we define

$$|X| + |Y| = |X \cup Y|$$

(note if we had $X_1 \sim X_2$, $Y_1 \sim Y_2$ and $X_1 \cap Y_1 = \varnothing$ and $X_2 \cap Y_2 = \varnothing$. We have $X_1 \cup Y_1 \sim X_2 \cup Y_2$ since we always have a 1-1 and onto mapping: simply partition points in $X_1 \cup Y_1$ into $x_1 \in X_1$ and $x_1 \in Y_1$: we have 1-1 mappings to each of X_2 and Y_2, respectively).

Question 6.1. What is $\aleph_0 + \aleph_0$?

Let $X = \{2, 4, 6, \ldots\}$ and $Y = \{1, 3, 5, \ldots\}$ then $X \cup Y = \{1, 2, 3, \ldots\}$ thus $\aleph_0 + \aleph_0 = \aleph_0$ by definition.

Question 6.2. What is $c + c$ ($|\mathbb{R}| = c$)?

Let $X = (0, 1) \Rightarrow |X| = c$ and $Y = (1, 2) \Rightarrow |Y| = c$, then

$$c \leq |X| \leq |X| + |Y| \leq |\mathbb{R}| = c$$

thus by CSB we have $c + c = c$.

Theorem 6.2. Given X, Y if X is infinite, then

1. $|X| + |X| = |X|$
2. $|X| + |Y| = \max\{|X|, |Y|\}$

Proof. 1. Exercise. (Hint: for countably infinite, we can create two countably infinite sets indexed by even and odd numbers. For infinite sets, we simply take out a countably infinite set (by theorem) A_1. If $X \setminus A_1$ is finite, then we are done. Otherwise we keep taking out countably infinite sets to form a collection of disjoint countably infinite sets).

□

Multiplication Let $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$, and $X \times Y = \{(x_i, y_j) \mid i = 1, \ldots, n, j = 1, \ldots, m\}$. Then $|X \times Y| = n \times n$.

Definition 6.2. Given X, Y define

$$|X| \cdot |Y| = |X \times Y|$$

Example 6.1. $|\mathbb{N} \times \mathbb{N}| = \aleph_0$, where define $f(n, m) = 2^n 3^m$ and $g(n) = (n, n)$ (1-1 and onto functions).

Question 6.3. What is $c \cdot c$?

Theorem 6.3. If X is infinite and $Y \neq 0$, then

1. $|X \times X| = |X| \Rightarrow |X||X| = |X|$
2. $|X \times Y| = \max(|X|, |Y|)$

Exponentiation Recall if $\{Y_x\}_{x\in X}$ is a collection of non-empty sets, then

$$\Pi_{x\in X} Y_x = \{f : X \to \bigcup_{x\in X} Y_x \mid f(x) \in Y_x\}$$

If $Y = Y_x$ for all $x \in X$ we have

$$Y^x = \Pi_{x\in X} Y = \{f : X \to Y\}$$

Example 6.2. Let $Y = \{1, \dots, m\}$, $X = \{1, \dots, n\}$, then

$$Y^X = \{f : \{1, \dots, n\} \to \{1, \dots, m\}\}$$

What is $|Y^X|$? It is m^n (for each $1, \dots, m$, you have n choices thus we have $m \cdot \dots \cdot m$ or m^n).

Definition 6.3. We define

$$|Y|^|X| = |Y^X|$$

Theorem 6.4. If X, Y are non-empty then

1. $|Y|^{|X|} \cdot |Y|^|Z| = |Y|^{|X|+|Z|}$
2. $(|Y|^{|X|})^{|Z|} = |Y|^{(|X|\cdot|Z|)}$

7 September 24, 2018

7.1 $2^{\aleph_0} = c$

Theorem 7.1. $2^{\aleph_0} = c$.

Proof. Observation:

$$2^{\aleph_0} = |\{0,1\}^{\mathbb{N}}| = |\{f : \mathbb{N} \to \{0,1\}\}| = |\{\{a_n\}_{n=1}^{\infty} \mid a_n = 0,1\}|$$

Given a sequence $\{a_n\} \in \{0,1\}^{\mathbb{N}}$, define

$$\phi(\{a_n\}) = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

where $\phi : \{0,1\}^{\mathbb{N}} \to (0,1)$ is 1-1 (note that we cannot map to two of the same real numbers in base 3 **unless** we had trailing 2s: but in this case we can't have 2s).
So $2^{\aleph_0} \le c$.
Given $\alpha \in (0,1)$ let

$$\alpha = \sum_{n=1}^{\infty} \frac{b_n}{2^n} \qquad b_n = 0,1$$

i.e. the binary representation of our α (there may be *multiple binary representations*, but we could just pick one).
Let $\psi : (0,1) \to \{0,1\}^{\mathbb{N}}$, where

$$\psi(\alpha) = \psi(\sum_{n=1}^{\infty} \frac{b_n}{2^n}) = \{b_n\}$$

so ψ is 1-1 which means $c \le 2^{\aleph_0}$. ☐

7.2 Countable union of countable sets

Observation 7.1. Suppose that $\{X_\alpha\}_{\alpha \in I}$ is a countable collection of countable sets.

Claim. $\bigcup_{\alpha \in I} X_i$ is countable.

Note: we can assume that $X_i \cap X_j = \varnothing$ if $i \neq j$. Why? Otherwise we can let

$$\begin{aligned} &E_1 = X_1 \\ &E_2 = X_2 \setminus E_1 \\ &E_3 = X_3 \setminus (E_1 \cup E_2) \\ &\vdots \\ &E_{n+1} = X_{n+1} \setminus (\bigcup_{i=1}^{n} E_i) \end{aligned}$$

Note $\bigcup_{i=1}^{\infty} X_i = \bigcup_{i=1}^{\infty} E_i$.
Let $E_n = \{x_{n,1}, x_{n,2}, \ldots\}$ (we need to use the Axiom of Choice here to pick out an enumeration of our set). Define $f : \bigcup_{i=1}^{\infty} E_i \to \mathbb{N}$

$$f(x_{n,j}) = 2^n 3^j$$

7.3 Cardinality of power sets

Question 7.1. Show that $|P(X)| = 2^{|X|} = |2^X|$.

Solution. Given $f : X \to \{0,1\}$ let $A = \{x \in X \mid f(x) = 1\} \subset X$.
Define $\Gamma : 2^X \to P(X)$ by $\Gamma(f) = f^{-1}(\{1\})$. Γ is 1-1 (since two functions f differ only if they map something differently, one to 0 and one to 1, thus they will map to different sets in $P(X)$).
Convsersely, given $A \subset X$ define

$$X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$X_A \in 2^X$ (X_A is the characteristic function of A). We define $\Phi(A) = X_A$, thus $\Phi : P(X) \to 2^X$ is 1-1.
By CSB we have $|P(X)| = |2^X|$.

7.4 Russell's Paradox

Theorem 7.2 (Russsell's Paradox)**.** For any X, $|X| < 2^{|X|}$.

Proof. Let $f : X \to P(X)$ defined by $f(x) = \{x\}$ (1-1) thus $|X| \leq |P(X)|$.

Claim. There is no onto function $g : X \to P(X)$.
Suppose we had such a g. Let $A = \{x \in X \mid x \notin g(x)\}$.
Since $A \subseteq X$ and g is onto, there exists some $x_0 \in X$ with $g(x_0) = A$.
If $x_0 \in A$, then $x_0 \in g(x_0) \Rightarrow x_0 \notin A$ by definition of A, a contradiction.
If $x_0 \notin A$, then $x_0 \notin g(x_0)$ so $x_0 \in A$ by definition of A, another contradiction.
Hence there must not be an onto function.

$\square$

7.5 Continuum Hypothesis

Axiom 7.1 (Continuum Hypothesis)**.** If $\aleph_0 \leq \gamma \leq c = 2^{\aleph_0}$, then either $\gamma = \aleph_0$ or $\gamma = c = 2^{\aleph_0}$.

Axiom 7.2 (Generalized Contnuum Hypothesis)**.** If $|X| \leq \gamma \leq 2^{|X|}$ then either $\gamma = |X|$ or $\gamma = 2^{|X|}$.

8 September 25, 2018

8.1 Metric spaces

Definition 8.1 (Metric and metric space)**.** Given X: a **metric** on X is a function $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ **iff** $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

The pair (X, d) is a called a **metric space**.

Example 8.1. If $X = \mathbb{R}$, let $d(x, y) = |x - y|$ (standard metric on $\mathbb{R}$).

Question 8.1. Can we define a metric on any X?
Yes: we have the **discrete metric** where

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

We can verify that all conditions for a metric is satisfied by d.

Example 8.2. Let $X = \mathbb{R}^n$ and $d_2(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$. This is the **Euclidean** or **2-metric** on $\mathbb{R}^n$.

8.2 Norms

Definition 8.2 (Norm)**.** Given a vector space V (over $\mathbb{R}$), a norm on V is a function $\|\cdot\| : V \to \mathbb{R}$ such that

1. $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$
2. $\|\alpha \cdot v\| = |\alpha| \|v\|$
3. $\|v + w\| \leq \|v\| + \|w\|$

Example 8.3. Define $\|\cdot\|_2$ on $\mathbb{R}^2$ by

$$\|(x_1, \ldots, x_n)\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

then $\|\cdot\|_2$ is a norm. Note if $n = 1$ we have $\|x\| = |x|$ or the absolute value.
Furthermore note that $d_2(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_2$.

Definition 8.3 (Normed linear/vector space)**.** A pair $(V, \|\cdot\|)$ is called a **normed linear space** (nls) or normed vector space.

Remark 8.1. Given a nls $(V, \|\cdot\|)$ we can define a *metric* $d_{\|\cdot\|}$ on V by $d_{\|\cdot\|}(x,y) = \|x-y\|$.
Note that this is a well-defined metric. Positive definiteness and symmetry properties are obvious. Let $x, y, z \in V$

$$\begin{aligned} d_{\|\cdot\|}(x,y) = \|x-y\| &= \|(x-z)+(z-y)\| \\ &\le \|x-z\| + \|z-y\| \\ &= \|x-z\| + \|y-z\| \\ &= d_{\|\cdot\|}(x,z) + d_{\|\cdot\|}(y,z) \end{aligned}$$

Other norms on $\mathbb{R}^n$:

1. $\|\cdot\|_1$ on $\mathbb{R}^n$ where $\|(x_1, \ldots, x_n)\|_1 = \sum_{i=1}^n |x_i|$. Note that

$$\begin{aligned} \|\vec{x} + \vec{y}\|_1 &= \sum_{i=1}^n |x_i + y_i| \\ &\le \sum_{i=1}^n |x_i| + |y_i| \\ &= \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i| \\ &= \|\vec{x}\|_1 + \|\vec{y}\|_1 \end{aligned}$$

   So we define $d_1(\vec{x}, \vec{y}) = \sum_{i=1}^n |x_i - y_i|$.

2. Let $\|\cdot\|_\infty$ (infinity norm) by $\|\vec{x}\|_\infty = \max\{|x_i|\}$.

   Positive definiteness is straightforward. Scalar multiple is obvious too.

   Note for any i, $|x_i + y_i| \le |x_i| + |y_i|$, thus $\max\{|x_i + y_i|\} \le \max\{|x_i|\} + \max\{|y_i|\}$.

   We can thus define the metric $d_\infty(\vec{x}, \vec{y}) = \|_{\|} \infty(\vec{x} - \vec{y}) = \max\{|x_i - y_i|\}$.

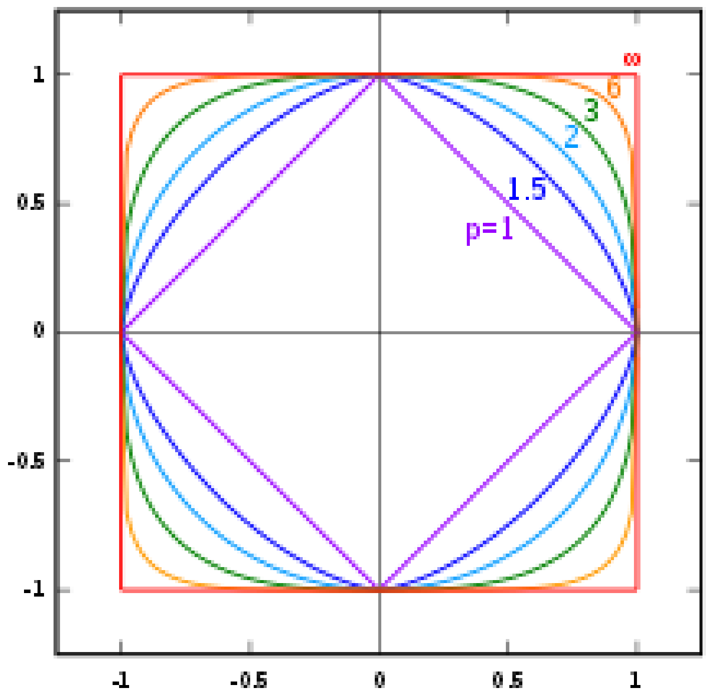

Figure 8.1: Diagrams for the l_p norms "balls" where $S_p = \{\vec{x} \in \mathbb{R}^2 \mid \|\vec{x}\|_p = 1\}$. In the diagram we have $p = 1, 1.5, 3, 6, \infty$.

We observe that $d_\infty \leq d_2 \leq d_1$: the number of points with distance ≤ 1 (inside their respective S_p balls) is the smallest for d_1, thus distances are "larger" for points in $\mathbb{R}^2$.

9 September 28, 2018

9.1 l_p norm

Definition 9.1 (l_p norm)**.** For $1 < p < \infty$, define on $\mathbb{R}^n$

$$\|\vec{x}\|_p = \Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}}$$

we can also define the metric

$$d_p(\vec{x}, \vec{y}) = \Big(\sum_{i=1}^n |x_i - y_i|^p\Big)^{\frac{1}{p}}$$

Note that l_p for $0 < p < 1$ results in a non-convex ball: this means any convex combination of two points may result in a point outside the ball. This implies that the triangle inequality does not hold.

We can show that $\|\cdot\|_p$ is a norm.

9.2 Young's Inequality

Lemma 9.1 (Young's Inequality)**.** If $1 < p < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and if $\alpha, \beta > 0$ then $\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$.

Proof. Let us draw $u = t^{p-1}$ where u is the y-axis and t is the y-axis.
We bound the area with $t = \alpha$ and $u = \beta$. Note that the inverse becomes $t = u^{\frac{1}{p-1}} = u^{q-1}$ (where $\frac{1}{p-1} = q - 1$ after a bit of algebraic manipulation).
We clearly see that the area above and below the curve is greater than the box, thus

$$\begin{aligned}\alpha\beta &\leq \int_0^\alpha t^{p-1}\,\mathrm{d}t + \int_0^\beta u^{q-1}\,\mathrm{d}u \\ &= \frac{t^p}{p}\ |_0^\alpha + \frac{u^q}{q}\ |_0^\beta \\ &= \frac{\alpha^p}{p} + \frac{\beta^q}{q}\end{aligned}$$

□

9.3 Holder's Inequality (*****)

Theorem 9.1 (Holder's Inequality)**.** Let $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$. Let $\vec{x} = (x_1, \ldots, x_n)$ $\vec{y} = (y_1, \ldots, y_n)$. Then

$$\sum_{i=1}^n |x_i y_i| \leq \big(\sum_{i=1}^n |x_i|^p\big)^{\frac{1}{p}} \cdot \big(\sum_{i=1}^n |y_i|^q\big)^{\frac{1}{q}}$$

Note $p = 2$ is the Cauchy-Schwarz Inequality (i.e. Holder's is a generalization of Cauchy Schwarz).

Proof. WLOG we may assume that $\vec{x}, \vec{y} \neq \vec{0}$.
Note if $\alpha, \beta \neq 0$ then

$$\sum_{i=1}^n |x_i y_i| \leq \big(\sum_{i=1}^n |x_i|^p\big)^{\frac{1}{p}} \cdot \big(\sum_{i=1}^n |y_i|^q\big)^{\frac{1}{q}}$$

holds **if and only if**

$$\sum_{i=1}^n |(\alpha x_i)\beta y_i| \leq \big(\sum_{i=1}^n |\alpha x_i|^p\big)^{\frac{1}{p}} \cdot \big(\sum_{i=1}^n |\beta y_i|^q\big)^{\frac{1}{q}}$$

(we can arbitrarily scale our vectors $\vec{x}, \vec{y}$). Hence we can assume that

$$\big(\sum_{i=1}^n |x_i|^p\big)^{\frac{1}{p}} = 1 = \big(\sum_{i=1}^n |y_i|^q\big)^{\frac{1}{q}}$$

(that is we scale our vectors so that the above equality holds). By Jensen's inequality we have

$$|x_i y_i| \leq \frac{|x_i|^p}{p} + \frac{|y_i|^q}{q}$$

Thus the sume over all $i = 1, \dots, n$ is

$$
\begin{aligned}
\sum_{i=1}^{n} |x_i y_i| &\leq \frac{\sum_{i=1}^{n} |x_i|^p}{p} + \frac{\sum_{i=1}^{n} |y_i|^q}{q} \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1 \\
&= \big(\sum_{i=1}^{n} |\alpha x_i|^p\big)^{\frac{1}{p}} \cdot \big(\sum_{i=1}^{n} |\beta y_i|^q\big)^{\frac{1}{q}}
\end{aligned}
$$

□

9.4 Minkowski's Inequality

Theorem 9.2 (Minkowski's Inequality). Let $1 < p < \infty$. If $\vec{x} = (x_1, \dots, x_n), \vec{y} = (y_1, \dots, y_n)$ then

$$\big(\sum_{i=1}^{n} |x_i + y_i|^p\big)^{\frac{1}{p}} \leq \big(\sum_{i=1}^{n} |x_i|^p\big)^{\frac{1}{p}} + \big(\sum_{i=1}^{n} |y_i|^p\big)^{\frac{1}{p}}$$

i.e. the *triangle inequality* for l_p norm holds

$$\|\vec{x} + \vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p$$

Proof. We can assume that $\vec{x} + \vec{y} \neq 0$. We have

$$
\begin{aligned}
\sum_{i=1}^{n} |x_i + y_i|^p &= \sum_{i=1}^{n} |x_i + y_i| |x_i + y_i|^{p-1} \\
&\overset{\triangle}{\leq} \sum_{i=1}^{n} |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| |x_i + y_i|^{p-1} \\
&\leq \big(\sum_{i=1}^{n} |x_i|^p\big)^{\frac{1}{p}} \cdot \big(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q}\big)^{\frac{1}{q}} + \big(\sum_{i=1}^{n} |y_i|^p\big)^{\frac{1}{p}} \cdot \big(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q}\big)^{\frac{1}{q}}
\end{aligned}
$$

where the last line follows from Holder's inequality. Thus we have

$$
\begin{aligned}
&\sum_{i=1}^{n} |x_i + y_i|^p \leq \Big(\big(\sum_{i=1}^{n} |x_i|^p\big)^{\frac{1}{p}} + \big(\sum_{i=1}^{n} |y_i|^p\big)^{\frac{1}{p}}\Big) \cdot \big(\sum_{i=1}^{n} |x_i + y_i|^p\big)^{\frac{1}{q}} \\
&\Rightarrow \sum_{i=1}^{n} |x_i + y_i|^{1-\frac{1}{q}} \leq \big(\sum_{i=1}^{n} |x_i|^p\big)^{\frac{1}{p}} + \big(\sum_{i=1}^{n} |y_i|^p\big)^{\frac{1}{p}} \\
&\Rightarrow \sum_{i=1}^{n} |x_i + y_i|^{\frac{1}{p}} \leq \big(\sum_{i=1}^{n} |x_i|^p\big)^{\frac{1}{p}} + \big(\sum_{i=1}^{n} |y_i|^p\big)^{\frac{1}{p}}
\end{aligned}
$$

as desired. □

Remark 9.1. This shows that $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$.

Observation 9.1. Given $1 \le p \le q \le \infty$ we have $\|\cdot\|_\infty \le \|\cdot\|_q \le \|\cdot\|_p \le \|\cdot\|_1$.

9.5 Sequence spaces

Definition 9.2 (Sequence space)**.**

1. Let the l_1 space be defined as

$$l_1(\mathbb{N}) = l_1 = \{\{x_n\} \mid x_n = \mathbb{R}, \sum_{n=1}^{\infty} |x_n| < \infty\}$$

(i.e. sequences that converge).

We define a norm on l_1

$$\|\{x_n\}\|_1 = \sum_{n=1}^{\infty} |x_n|$$

Let $\{x_i\}, \{y_i\} \in l_1$. For all $n \in \mathbb{N}$

$$\begin{aligned}\sum_{i=1}^{n} |x_i + y_i| &\overset{\triangle}{\le} \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| \\ &= \|\{x_i\}\|_1 \|\{y_i\}\|_1\end{aligned}$$

hence $\sum_{i=1}^{\infty} |x_i + y_i| \le \|\{x_i\}\|_1 + \|\{y_i\}\|_1$, thus $\{x_i + y_i\} \in l_1$ (finite sum) and the triangle inequality holds i.e. $\|\{x_i + y_i\}\|_1 \le \|\{x_i\}\|_1 + \|\{y_i\}\|_1$.

Let $\{x_i\} \in l_1$, $\alpha \in \mathbb{R}$. We know for a convergent sequence

$$\sum_{i=1}^{\infty} |\alpha x_i| = |\alpha| \sum_{i=1}^{\infty} |x_i|$$

thus $\{\alpha x_i\} \in l_1$ and $\|\{\alpha x_i\}\|_1 = |\alpha| \|\{x_i\}\|_l$.

Positive definiteness is trivial, thus l_1 is a vector space and $(l_1, \|\cdot\|_1)$ is a normed linear space.

2. Let

$$l_\infty(\mathbb{N}) = l_\infty = \{\{x_i\} \mid \{x_i\} \text{ is bounded}\}$$

Define the norm on l_∞

$$\|\{x_i\}\|_\infty = \text{lub}\{|x_i|\} \qquad i \in \mathbb{N}$$

If $\{x_i\}, \{y_i\} \in l_\infty$ then

$$|x_i + y_i| \le |x_i| + |y_i| \le \|\{x_i\}\|_\infty + \|\{y_i\}\|_\infty$$

for all $i \in \mathbb{N}$. So $\{x_i + y_i\} \in l_\infty$ and $\|\{x_i + y_i\}\|_\infty \le \|\{x_i\}\|_\infty + \|\{y_i\}\|_\infty$.

Similarly $\{\alpha x_i\} \in l_\infty$ and $\|\{\alpha x_i\}\|_\infty = |\alpha| \|\{x_i\}\|_\infty$. Therefore l_∞ is a vector space and $(l_\infty, \|\cdot\|_\infty)$ is a normed linear space.

10 October 1, 2018

10.1 Normed linear spaces on arbitrary spaces Γ

Question 10.1. Can we define $l_p(\Gamma)$ for any $s \in \Gamma$ (i.e. can we define our normed spaces and norms on an arbitrary set)?

Example 10.1. Let

$$l_\infty(\Gamma) = \{f : \Gamma \to \mathbb{R} \mid f(\Gamma) \text{ is bounded}\}$$

If $f \in l_\infty(\Gamma)$ define

$$\|f\|_\infty = \text{lub}(\{|f(x)| \mid x \in \Gamma\})$$

Note if $f, g \in l_\infty(\Gamma)$ and if $\alpha \in \mathbb{R}$ then $f + g \in l_\infty(\Gamma)$ where $(f + g)(x) = f(x) + g(x)$ and we see that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. Moreover if $(\alpha f)(x) = \alpha f(x)$ (definition), then $\alpha f \in l_\infty(\Gamma)$ and $\|\alpha f\|_\infty = |\alpha|\|f\|_\infty$. Therefore $(l_\infty, \|\cdot\|_\infty)$ is a normed linear space.

Example 10.2. How would we define $l_1(\Gamma)$?
We say that f belongs to $l_1(\Gamma)$ if

$$\|f\|_1 = \text{lub}\{\sum_{i=1}^{n} |f(x_i)| \mid x_1, \ldots, x_n \in \Gamma\}$$

where $n \in \{1, 2, \ldots\}$ (i.e. a finite collection). Note that f must be bounded (otherwise we could choose some element that contradicts our convergent series) thus $l_1(\Gamma) \subseteq l_\infty(\Gamma)$.
We do get that $(l_1(\Gamma), \|\cdot\|_1)$ is a nls.

Observation 10.1. If $f \in l_1(\Gamma)$ then for every $n \in \mathbb{N}$ $A_n = \{x \in |gamma \mid |f(x)| \geq \frac{1}{n}\}$ is finite.
Note that $A_0 = \bigcup_{n=1}^{\infty} A_n$ is countable where

$$A_0 = \{x \in \Gamma \mid |f(x)| \neq 0\}$$

i.e. f must be defined on a set with at most countably many non-zero elements.

10.2 Normed linear spaces on continuous closed intervals

Example 10.3. Let $X = C[a, b] = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuous}\}$.
Note that

$$\|f\|_\infty = \text{lub}\{|f(x)| \mid x \in [a, b]\} = \max\{|f(x)| \mid x \in [a, b]\}$$

$f(x)$ is bounded since f is continuous and is defined on a closed interval.
Note that $(C[a, b], \|\cdot\|_\infty)$ is a nls. Furthermore $C[a, b] \subset l_\infty([a, b])$.

Example 10.4. Let $X = C[a, b]$. Note

$$\|f\|_1 = \int_a^b |f(x)| \, \mathrm{d}x \leq (b - a)\|f\|_\infty$$

Note that since f is continuous, the integral cannot be 0 unless f is zero so positive definiteness holds (note integral not being 0 does not hold in general: e.g. if one had a function that is non-zero at only a single point x in the interval $[a, b]$).

The scalar multiple condition is trivial. Furthermore

$$\begin{aligned}\|f+g\|_1 &= \int_a^b |f(x)+g(x)|\,\mathrm{d}x \\ &\le \int_a^b |f(x)|\,\mathrm{d}x + \int_a^b |g(x)|\,\mathrm{d}x \\ &= \|f\|_1 + \|g\|_1\end{aligned}$$

THus $(C[a,b], \|\cdot\|_1)$ is a nls.

Example 10.5. Let $X = C[a,b]$, $1 < p < \infty$. Define

$$\|f\|_p = \big(\int_a^b |f(x)|^p\,\mathrm{d}x\big)^{\frac{1}{p}}$$

We claim $(C[a,b], \|\cdot\|_p)$ is nls.
The proof requires the use of Holder's inequality where if $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_a^b |f(x)g(x)|\,\mathrm{d}x \le \big(\int_a^b |f(x)|^p\big)^{\frac{1}{p}}\big(\int_a^b |f(x)|^q\big)^{\frac{1}{q}}$$

We later see that $(C[a,b], \|\cdot\|_2)$ has a 1-1 mapping to $l_2(\mathbb{N})$(?)

11 October 3, 2018

11.1 Normed linear spaces on linear maps

Example 11.1. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be nls. Let $T : X \to Y$ be linear. Define

$$\|T\| = \text{lub}\{\|Tx\|_Y \mid \|x\|_X \le 1\}$$

We say that T is bounded if $\|T\| < \infty$. We define

$$B(X,Y) = \{T : X \to Y \mid T \text{ is bounded}\}$$

Claim. We claim that $(B(X,Y), \|\cdot\|)$ is a nls.
Let $S, T \in B(X,Y)$. Let $\|x\|_X \le 1$.
Note

$$\begin{aligned}\|(S+T(x)\|_Y &= \|S(x)+T(x)\|_Y \\ &\le \|S(x)\|_Y + \|T(x)\|_Y \\ &\le \|S\| + \|T\|\end{aligned}$$

Thus $S+T \in B(X,Y)$ and $\|S+T\| \le \|S\| + \|T\|$.
If $\alpha \in \mathbb{R}$, then (note that $T(\alpha x) = \alpha T(x)$)

$$\|(\alpha S)\|_Y = \|(S(\alpha x)\|_Y = |\alpha|\|S(x)\|_Y \le |\alpha|\|S(x)\|_Y$$

In fact

$$\text{lub}\{\|(\alpha S)(x)\|_Y \mid \|x\|_X = 1\} = |\alpha|\text{lub}\{\|S(x)\|_Y \mid \|x\|_X = 1\}$$

Therefore $(\alpha S) \in B(X, Y)$ and $\|\alpha S\| = |\alpha| \|S\|$.

11.2 Topology on metric spaces

Definition 11.1 (Open/closed balls and sets)**.** Let (X, d) be a metric space.

open ball If $x_0 \in X$, $\epsilon > 0$

$$B(x_0, \epsilon) = \{y \in X \mid d(x_0, y) < \epsilon\}$$

is called the **open ball** centered at x_0 with radius ϵ.

closed ball If $x_0 \in X$, $\epsilon > 0$

$$B[x_0, \epsilon] = \{y \in X \mid d(x_0, y) \leq \epsilon\}$$

is called the **closed ball** centered at x_0 with radius ϵ.

open set We say that $U \in X$ is open if for each $x_0 \in U$ there exists $\epsilon_0 > 0$ such that $B(x_0, \epsilon_0) \subset U$.

Remark 11.1. Note that our definition of an open set hinges on the metric defined for open balls, hence a set is open relative to the metric d specified.

closed set We say that $F \subset X$ is closed if F^c is open.

Proposition 11.1 (Unions and intersections on open sets)**.** Let (X, d) be a metric space:

1. $X, \varnothing$ are open
2. If $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets then $U = \bigcup_{\alpha \in I} U_\alpha$ is open.
3. If $\{U_1, \ldots, U_n\}$ are open, then $\bigcap_{i=1}^n U_i = U$ is open.

Proof. 1. If $x_0 \in X$ then clearly $B(x_0, 1) \subseteq X$ thus X is open.
$\varnothing$ is open vacuously.

2. Let $U = \bigcup_{\alpha \in I} U_\alpha$. Let $x_0 \in U$. There exists α_0 with $x_0 \in U_{\alpha_0}$. There exists $\epsilon_0 > 0$ such that $B(x_0, \epsilon_0) \subset U_{\alpha_0} \subset U$.
3. Let $x_0 \in \bigcap_{i=1}^n U_i$. For each $i = 1, \ldots, n$, we can find $\epsilon_i > 0$ such that $B(x_0, \epsilon_i) \subset U_i$.
Let $\epsilon_0 = \min\{\epsilon_1, \ldots, \epsilon_n\}$ then $\epsilon_0 \leq \epsilon_i$ for all i thus $B(x_0, \epsilon_0) \subset B(x_0, \epsilon_i) \subset U_i$ for all i. Hence $B(x_0, \epsilon_0) \subset \bigcap_{i=1}^n U_i$.

□

Proposition 11.2 (Unions and intersections on closed sets)**.** Let (X, d) be a metric space:

1. $X, \varnothing$ are closed
2. If $\{F_\alpha\}_{\alpha \in I}$ is a collection of closed sets then $F = \bigcap_{\alpha \in I} F_\alpha$ is closed.
3. If $\{F_1, \ldots, F_n\}$ are open, then $\bigcup_{i=1}^n F_i = F$ is closed.

Proof. This follows from the fact that F is closed iff $U = F^c$ is open.
The rest follows from the previous proposition with open sets and De Morgan's Law. □

Example 11.2. Let X any set, d be the discrete metric where $d(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ if $x \neq y$.

Question 11.1. What sets are open in (X,d)?
$X, \varnothing$ is open.

Claim. $\{x_0\}$ is open since $B(x_0, \frac{1}{2}) \subseteq \{x_0\}$.

Thus if $A \subset (X,d)$ then $A = \bigcup x \in A\{x\}$ thus A is open.

11.3 Topology

Definition 11.2 (Topology)**.** Given any X a set $\Im \subset P(X)$ is called a **topology** on X if

1. $X, \varnothing \in \Im$
2. If $\{U_\alpha\}_{\alpha \in I}$ such that $U_\alpha \in \Im$ for all $\alpha \in I$, then $U = \bigcup_{\alpha \in I} U_\alpha$ is such that $U \in \Im$.
3. If $\{U_1, \ldots, U_n\} \subset \Im$, then $U = \bigcap_{i=1}^n U_i \in \Im$

If (X,d) is a metric space then
$$\Im_d = \{U \subseteq X \mid U \text{ is open in } (X,d)\}$$
is the d**-topology** associated with metric d.
$(X, \Im)$ is called a **topological space**.

Example 11.3. Given X:

1. $P(X)$ is a topology on X.

   This topology $\Im$ is called the **discrete topology** (i.e. this topology works when d is the discrete metric).

2. $\{\varnothing, X\}$ is called the **indiscrete topology**.

12 October 5, 2018

12.1 Metric space properties

Theorem 12.1. Given (X,d) a metric space

1. $B(x_0, \epsilon)$ is open

   Proof. Let $x \in B(x_0, \epsilon)$. Let $r = d(x, x_0)$.
   Let $\alpha = \epsilon - r$. Assume that $y \in B(x, \alpha)$ then
   $$\begin{aligned} d(x_0, y) &\overset{\triangle}{\leq} d(x_0, x) + d(x,y) \\ &< r + \alpha \\ \epsilon \end{aligned}$$
   □

2. $B[x_0, \epsilon]$ is closed

Proof. Let $y \in B[x_0, \epsilon]^c$. Let $r = d(x_0, y)$. Let $\alpha = r - \epsilon$.
Assume that $z \in B(y, \alpha)$. Suppose for contradiction that $z \in B[x_0, \epsilon]$. Then

$$\begin{aligned} r = d(x_0, y) &\overset{\triangle}{\leq} d(x_0, z) + d(z, y) \\ &< \epsilon + \alpha \\ &= r \end{aligned}$$

which is a contradiction hence $z \in B[x_0, \epsilon]^c$. □

3. Every open set is the union of open balls

Proof. Let $U \subset X$ be open. For each $x \in U$ let ϵ_x be such that $B(x, \epsilon_x) \subset U$. Then

$$\bigcup_{x \in U} B(x, \epsilon_x) = U$$

□

4. For each $x \in X$, $\{x\}$ is closed

Proof. Let $y \in X$, $y \neq x$. Let $r = d(y, x)$, Then $x \notin B(y, \frac{r}{2})$,
thus $B(y, \frac{r}{2}) \subset \{x\}^c$ hence $\{x\}$ is closed. □

Example 12.1. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$.

1. Every open interval is open.

Proof. Let $I = (a, b)$ and $a, b \in \mathbb{R} \cup \{\pm\infty\}$ be an open interval.
Let $x \in I$. If $\epsilon = \min\{1, x - a, b - x\}$ (we need the 1 for unbounded case) then $B(x, \epsilon) \subset I$. □

12.2 Equivalence class and decomposition of open sets

if $U \subset \mathbb{R}$ is open we can define $\sim$ on U by $x \sim y$ iff (x, y) (or (y, x)) $\subset U$: $\sim$ is an equivalence relation.
Note that the equivalence class for x: $I_x = [x]$ is an **open interval**.
Furthermore if U is open in $\mathbb{R}$ then U is a union of a collection $\{I_\alpha\}_{\alpha \in I}$ of open intervals which are pairwise disjoint.

12.3 Decomposition of closed sets and the Cantor set

Question 12.1. Can every closed set in $\mathbb{R}$ be written as a countable union of closed intervals? No!

Example 12.2 (Cantor set)**.** The cantor set is defined as such:
Let $P_0 = [0, 1]$. Let P_1 be P_0 with the middle open $\frac{1}{3}$ removed i.e.

$$P_1 = [0, 1] \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

which is closed (verify via its complement). Similarly P_2 remove open middle $\frac{1}{3}$ of each of the two closed interval in P_1

$$P_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, \frac{9}{9}]$$

In general, P_{n+1} is obtained by removing the open middle intervals of length $\frac{1}{3^{n+1}}$ from each of the 2^n closed intervals in P_n.
Let $P = \bigcap_{n=0} P_n$ the **Cantor (ternary) set**.
Properties of P:

1. P is closed since P_n is closed (and it is an arbitrary intersection of closed sets).
2. $x \in P$ iff $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ where $a_n = 0, 2$ (i.e. the end points of our intervals i.e. when x has a base 3 expansion).
3. Note that $|P| = 2^{\aleph_0} = c$ (since every element can be mapped to a sequence of $\{0, 2\}$ which has cardinality $2^{\aleph_0}$).
4. P_n does not contain any intervals of length $\geq \frac{1}{3^n}$ so the interval $\to 0$ in P.

Note that the Cantor set is an uncountable set that cannot be represented as the union of countable close intervals. What is the length of the Cantor set? Note that the length of $P_n = \left(\frac{2}{3}\right)^n$ (sum of all the individual intervals, which we take away $\frac{1}{3}$ each iteration), thus the length of the Cantor set should be 0.

13 October 12, 2018

13.1 Closures and interiors

Definition 13.1 (Closure)**.** Let $A \subseteq (X, d)$. We define the **closure** $\bar{A}$ of A to be $\bar{A} = \bigcap\{F \subset X \mid F \text{ is closed and } A \subset F\}$.
Note: $\bar{A}$ is the smallest closed set that contains A.

Definition 13.2 (Interior)**.** We define the interior A^o of A by $A^o = \bigcup\{U \subset X \mid U \text{ is open and } U \subset A\}$.

Definition 13.3 (Neighborhood)**.** We say that a set A is a **neighborhood** of a point $x \in X$ if $x \in A^o$.
Note a neighborhood of $x \in X$ if and only if there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset A$.

Definition 13.4 (Boundary)**.** Given $A \subset (X, d)$ a point x is called a **boundary point** for A if for any $\epsilon > 0$, $B(x, \epsilon) \cap A \neq \varnothing$ and $B(x, \epsilon) \cap A^c \neq \varnothing$.
We denote the **boundary** or the collection of all boundary points of A by $\text{bdy}(A)$.

13.2 Boundary and closed sets

Proposition 13.1. Let (X, d) be a metric space and $A \subset X$. TFAE:

1. A is closed
2. $\text{bdy}(A) \subset A$

Proof. Suppose A is closed and $x \in A^c$. Then $\exists \epsilon > 0$ such that $B(x, \epsilon) \subseteq A^c \Rightarrow x \notin \text{bdy}(A)$ so $\text{bdy}(A) \subset A$.
Suppose $\text{bdy}(A) \subset A$. Let $x \in A^c$, so $x \notin \text{bdy}(A)$. Hence there exists $\epsilon > 0$ such that either $B(x, \epsilon) \subset A$ or $B(x, \epsilon \subset A^c$, but $x \notin A$ thus $B(x, \epsilon) \subset A^c$ hence A^c is open so A is closed. ☐

13.3 Closure and boundary

Proposition 13.2. We claim $\bar{A} = A \cup \text{bdy}(A)$.

Proof. We claim $\text{bdy}(A) \subseteq \bar{A}$. Let $x \in \bar{A}^c$. Since $\bar{A}^c$ is open, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset \bar{A}^c$. Thus $x \notin \text{bdy}(A)$, so $\text{bdy}(A) \subseteq \bar{A}$. Therefore $A \cup \text{bdy}(A) \subseteq \bar{A}$.
We claim $A \cup \text{bdy}(A)$ is closed. Let $x \in \text{bdy}(A \cup \text{bdy}(A))$. Given $\epsilon > 0$, we have $B(x, \epsilon) \cap (A \cup \text{bdy}(A)) \neq \varnothing$ and $B(x, \epsilon) \cap (A \cup \text{bdy}(A))^c \neq \varnothing$.
If $B(x, \epsilon) \cap A \neq \varnothing$, we are done. So we can assume that $B(x, \epsilon) \cap \text{bdy}(A) \neq \varnothing$ (from the first $\neq \varnothing$).
Let $z \in B(x, \epsilon) \cap \text{bdy}(A)$. Let $r = d(x, z)$ and let $\alpha = \epsilon - r > 0$.
By the triangle inequality we have $B(z, \alpha) \subset B(x, \epsilon)$.
Since $z \in \text{bdy}(A)$ we have $B(z, \alpha) \cap A \neq \varnothing$ so $B(x, \epsilon) \cap A \neq \varnothing$.
Since $B(x, \epsilon) \cap A \neq \varnothing$ and $B(x, \epsilon) \cap A^c \neq \varnothing$ (froms second $\neq \varnothing$ above), then $x \in \text{bdy}(A)$.
Hence $A \cup \text{bdy}(A)$ is closed so $\bar{A} \subseteq A \cup \text{bdy}(A)$ since $\bar{A}$ is the smallest closed set containing A.
The result follows. ☐

Some examples of boundaries, interiors, and closures

Example 13.1. If $X = \mathbb{R}$ and $A = [0, 1)$, then $\text{bdy}(A) = \{0, 1\}$, $A^o = (0, 1)$, and $\bar{A} = [0, 1]$.

Example 13.2. If $X = \mathbb{R}$ and $A = \mathbb{Q}$, then $\text{bdy}(A) = \mathbb{R}$, $A^o = \varnothing$, and $\bar{A} = \mathbb{R}$.

13.4 Separable

Definition 13.5 (Separable metric space)**.** A metric space (X, d) is **separable** if there exists a *countable set* $A \subset X$ such that $\bar{A} = X$.
It is non-separable otherwise.

1. Every finite metric space (X, d) is separable
2. $\mathbb{R}$ is separable since $\bar{\mathbb{Q}} = \mathbb{R}$
3. $\mathbb{R}^n$ is separable if d_p for all $1 \leq p \leq \infty$ (p metric)

   Claim. We claim $\overline{\mathbb{Q}^n} = \mathbb{R}^n$.

   That is: we can approximate any point $(x_1, \ldots, x_n)$ in $(\mathbb{R}^n, d_p)$ with points $(r_1, \ldots, r_n) \in \mathbb{Q}^n$ as closely as we like.

Remark 13.1. $\bar{A} = X$ if and only if for every $x \in X$ and $\epsilon > 0$ we have $B(x, \epsilon) \cap A \neq \varnothing$.

Definition 13.6 (Dense sets)**.** A is **dense** in (X, d) if $\bar{A} = X$.

Question 13.1. Is $(l_1, \|\cdot\|_1)$ separable? Yes.
Is $(l_\infty, \|\cdot\|_\infty)$ separable? No.

14 October 15, 2018

14.1 Limit points

Definition 14.1 (Limit point)**.** Let (X, d) be a metric space, $A \subset X$. We say that x_0 is a **limit point** for A if for every neighbourhood N of x_0 we have that $N \cap (A \setminus \{x_0\}) \neq \varnothing$.

Remark 14.1. $N \cap (A \setminus \{x_0\})$ must be uncountable since if it was countable we could take the minimum of the neighbourhood radius and create a smaller neighbourhood which must contain a point in A.

Equivalent for each $\epsilon > 0$, $B(x_0, \epsilon)$ contains a point in A other than x.
We often call limit points *cluster points* of the set A.
Let $\text{Lim}(A) = \{x_0 \in X \mid x_0 \text{ is a limit point of } A\}$.

Example 14.1. Let $X = \mathbb{R}$ and $A = [0, 1)$. Note that $\text{Lim}(A) = [0, 1]$.

Example 14.2. Let $X = \mathbb{R}$ and $A = \mathbb{N}$. Note that $\text{Lim}(\mathbb{N}) = \varnothing$.

Proposition 14.1. Let $A \subset (X, d)$.

1. A is closed if and only if $\text{Lim}(A) \subset A$.
2. $\bar{A} = A \cup \text{Lim}(A)$

Proof.

1. Forwards direction: if A is closed and $x_0 \in A^c$ which is open. $\exists \epsilon > 0$ such that $B(x_0, \epsilon) \subseteq A^c$. Thus $x_0 \notin \text{Lim}(A) \Rightarrow \text{Lim}(A) \subseteq A$.

   Backwards direction: Assume that $\text{Lim}(A) \subseteq A$. Let $x_0 \in A^c$. Since $x_0 \notin \text{Lim}(A)$, then there exists $\epsilon > 0$ such that $B(x_0, \epsilon) \cap A$ could only contain x_0, which A does not, so $B(x_0, \epsilon) \subseteq A^c$ thus A is closed.

2. We know $A \subset \bar{A}$. If $x_0 \in \bar{A}^c$ open, then there exists $\epsilon > 0$ such that $B(x_0, \epsilon) \subset \bar{A}^c$ which implies $B(x_0, \epsilon) \cap A = \varnothing$, so $x_0 \notin \text{Lim}(A) \Rightarrow \text{Lim}(A) \subseteq \bar{A}$ and thus $A \cup \text{Lim}(A) \subseteq \bar{A}$.

   Claim. $A \cup \text{Lim}(A)$ is closed.

   Assume that $x_0 \in (A \cup \text{Lim}(A))^c$. Then there exists $\epsilon > 0$ such that $B(x_0, \epsilon) \cap A = \varnothing$. Suppose for contradiction that $z \in \text{Lim}(A)$ and $z \in B(x_0, \epsilon)$ then since $B(x_0, \epsilon)$ is a neighbourhood of z then $B(x_0, \epsilon) \cap A \neq \varnothing$ which is a contradiction, thus $(A \cup \text{Lim}(A))^c$ is open so $A \cup \text{Lim}(A)$ is closed, thus $\bar{A} \subseteq A \cup \text{Lim}(A)$.

   Therefore $\bar{A} = A \cup \text{Lim}(A)$.

□

14.2 Properties of interiors, closures, and boundaries

Proposition 14.2. Let $A \subseteq B \subseteq (X, d)$.

1. $\bar{A} \subseteq \bar{B}$
2. $int(A) \subset int(B)$
3. $int(A) = A \setminus \text{bdy}(A)$
4. $\text{bdy}(A) = \text{bdy}(A^c)$
5. $int(A) = (\overline{A^c})^c$

Proposition 14.3. Let $A, B \subset (X, d)$

1. $\overline{A \cup B} = \bar{A} \cup \bar{B}$
2. $int(A \cap B) = int(A) \cap int(B)$

Proof. 1. Note that

$$\begin{aligned} A \subset \bar{A}, B \subseteq \bar{B} &\Rightarrow A \cup B \subset \bar{A} \cup \bar{B} \\ &\Rightarrow \overline{A \cup B} \subset \bar{A} \cup \bar{B} \qquad \bar{A} \cup \bar{B} \text{ is closed, closure is smallest containing closed set} \end{aligned}$$

Similarly $A \subset A \cup B \Rightarrow \bar{A} \subset \overline{A \cup B}$ and similarly $\bar{B} \subset \overline{A \cup B}$ thus $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$. The result follows.

2. Exercise.

□

Question 14.1. Is $\overline{A \cap B} = \bar{A} \cap \bar{B}$?

Example 14.3. Let $X = \mathbb{R}$, $A = \mathbb{Q}$, $B = \mathbb{R} \setminus Q$. Note that $\bar{A} = \bar{B} = \mathbb{R}$, thus $\overline{A \cap B} = \varnothing$ but $\bar{A} \cap \bar{B} = \mathbb{R}$.

Question 14.2. Is $\overline{B(x_0, \epsilon)} = B[x_0, \epsilon]$?
Yes under the Euclidean metric but consider the discrete metric:

Example 14.4. Let X any set with 2 or more elements and d the discrete metric.
$B(x_0, 1) = \{x_0\}$ but $B[x_0, 1] = X$.

14.3 Convergence of sequences

Definition 14.2 (Sequence convergence)**.** Given a sequence $\{x_n\} \subset (X, d)$ and $x_0 \in X$, we say that $\{x_n\}$ **converges** to x_0 if for every $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that if $n \geq N_0$ then $d(x_n, x_0) < \epsilon$.
This is equivalent to saying that $\{d(x_n, x_0)\}$ converges to 0 in $\mathbb{R}$.
We write

$$x_0 = \lim_{n \to \infty} x_n$$

or $x_n \to x_0$.
If there is no such x_0 we say that the sequence **diverges**.

Theorem 14.1 (Uniqueness of limits of sequences)**.** If $\{x_n\} \subset (X, d)$ with $x_n \to x_0$ and $y_n \to y_0$, then $x_0 = y_0$.

Proof. Assume $x_0 \neq y_0$. Let $\epsilon = d(x_0, y_0)$. Then $B(x_0, \frac{\epsilon}{2}) \cap B(y_0, \frac{\epsilon}{2}) = \varnothing$ (follows from triangle inequality) but there exists $N_0 \in N$ so that $n \geq N_0$, $x_n ß B(x_0, \frac{\epsilon}{2}) \cap B(y_0, \frac{\epsilon}{2})$ which is impossible. □

15 October 17, 2018

15.1 Convergence of sequences in $\mathbb{R}^n$

Example 15.1. Suppose $X = \mathbb{R}^n$, $d = d_p$ for $1 < p \leq \infty$.
Let $\vec{x}_k = \{(x_{k,1}, x_{k,2}, \ldots, x_{k,n})\}$ (sequence in $\mathbb{R}^n$).

Claim. $\vec{x}_k \to \vec{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,n})$ if and only if $x_{k,j} \to x_{0,j}$ for all $j = 1, \ldots, n$.
In general note that $|x_{k,j} - x_{0,j}| \leq \|\vec{x}_k - \vec{x}_0\|_p$.
So if $\vec{x}_k \to \vec{x}_0$ then $x_{k,j} \to x_{0,j}$ for all $j = 1, \ldots, n$ by the squeeze theorem.
Assume $x_{k,j} \to x_{0,j}$ for all j.
If $p = \infty$, since $x_{k,j} \to x_{0,j}$ for any $\epsilon > 0$ we can find k_0 such that if $k \geq k_0$ then $|x_{k,j} - x_{0,j}| < \epsilon$ for all $j = 1, \ldots, n$, which would imply $\|\vec{x}_k - \vec{x}_0\|_\infty < \epsilon$ (since $\|\cdot\|_\infty$ is the max over our js).
If $p = 1$, we repeat but with $|x_{k,j} - x_{0,j}| < \frac{\epsilon}{n}$.

For $1 < p < \infty$ repeat with $|x_{k,j} - x_{0,j}| < \frac{\epsilon}{n^{\frac{1}{p}}}$ since we have

$$\Big(\sum_{j=1}^{n} |x_{k,j} - x_{0,j}|^p\Big)^{\frac{1}{p}} < \Big(\sum_{j=1}^{n} \big(\frac{\epsilon}{n^{\frac{1}{p}}}\big)^p\Big)^{\frac{1}{p}}$$

so the result follows.

Example 15.2. Suppose $X = (C[a,b], \|\cdot\|_\infty)$ (set of continuous functions on $[a,b]$).
$f_n \to f$ iff $\|f_n - f\|_\infty = 0$.
That is given $\epsilon > 0$ we can find $N_0 \in \mathbb{N}$ such that if $n \geq N_0$ we have $\max|f_n(x) - f(x)| < \epsilon$. This implies uniform convergence and pointwise convergence.

Theorem 15.1. Given $A \subset (X, d)$

1. $x_0 \in \text{Lim}(A)$ if and only if there exists a sequence $\{x_n\} \subset A$ with $x_n \neq x_0$ and $x_n \to x_0$.

   Proof. Assume $x_0 \in \text{Lim}(A)$. We have that for each $n \in \mathbb{N}$ there exists $x_n \in B(x_0, \frac{1}{n}) \setminus \{x_0\}$. Then $d(x_n, x_0) < \frac{1}{n}$ which implies $x_n \to x_0$.

   Assume $x_n \to x_0$, $x_n \neq x_0$, $\{x_n\} \in A$. Let $\epsilon > 0$, for $n \geq N_0$ we have $x_n \in B(x_0, \epsilon)$ by definition of sequence convergence. Thus x_0 is a limit point. □

2. $x_0 \in \text{bdy}(A)$ if and only if there exists two sequences $\{x_n\} \subset A$ and $\{y_n\} \subset A^c$ with $x_n \to x_0$ and $y_n \to x_0$.

   Proof. Similary to proof above: if $x_0 \in \text{bdy}(A)$, given any $n \in \mathbb{N}$ we can find $x_n \in B(x_0, \frac{1}{n}) \cap A$ and $y_n \in B(x_0, \frac{1}{n}) \cap A^c$.

   So $\{x_n\} \subset A$, $x_n \to x_0$ and $\{y_n\} \subset A^c$ and $y_n \to x_0$.

   Assume $\{x_n\} \subset A$, $\{y_n\} \subset A^c$ and $x_n \to x_0$, $y_n \to x_0$. For a given $\epsilon > 0$ we have for any $n \geq N_0$ we have $x_n \in B(x_0, \epsilon)$ and $y_n \in B(x_0, \epsilon)$ thus $x_0 \in \text{bdy}(A)$. □

3. A is closed if and only if whenever $\{x_n\} \subset A$ is such that $x_n \to x_0 \in X$ then $x_0 \in A$.

   Proof. Forwards: Suppose A is closed and we have $\{x_n\} \subset A$ and $x_n \to x_0$.

   Suppose also that $x_0 \in A^c$ which is open. Then there exists $\epsilon > 0$ such that $B(x_0, \epsilon) \subset A^c \Rightarrow x_n \notin B(x_0, \epsilon)$ which is impossible.

   Backwards (contrapositive): Suppose A is not closed. Then there exists $x_0 \in \text{Lim}(A) \setminus A$. By (1), there exists $\{x_n\} \subset A$ with $x_n \to x_0 \notin A$. Our statement follows by contrapositive. □

Example 15.3. Suppose X is any set and d is the discrete metric.
$x_n \to x_0$ iff there exists $N_0 \in \mathbb{N}$ such that $x_n = x_0$ for all $n \geq N_0$.

Remark 15.1. Let $c_0 = \{\{x_n\} \mid \lim_{n\to\infty} x_n = 0\} \subset l_\infty$ (set of sequences).

Claim. c_0 is closed in l_∞.

Proof. Assume $\vec{x}_k = \{x_{k,j}\}_{j=1}^{\infty} \subset c_0$ (sequence of sequences: kth element is a sequence indexed by j).
Let $\vec{x}_k \overset{\|\cdot\|_\infty}{\to} \vec{x}_0$ where $\{x_{0,j}\}_{j=1}^{\infty} \subset c_0$ (sequence of sequences converges to a sequence $\vec{x}_0$).
Let $\epsilon > 0$. We can find $N_0 \in \mathbb{N}$ such that if $k \geq N_0$ $\|\vec{x}_k - \vec{x}_0\|_\infty < \frac{\epsilon}{2}$.
Let $k_0 > N_0$. Since $\vec{x}_{k_0} \in c_0$, there exist $J_0 \in \mathbb{N}$ such that if $j \geq J_0$, then $|x_{k_0,j}| < \frac{\epsilon}{2}$.
If $j \geq J_0$, then

$$\begin{aligned} |x_{0,j}| &\leq |x_{k_0,j} - x_{0,j}| + |x_{k_0,j}| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

So $\lim_{j\to\infty} x_{0,j} = 0 \Rightarrow \vec{x}_0 \in c_0$, thus c_0 is closed since our limit is in c_0. □

16 October 19, 2018

16.1 Induced metrics and topologies

Definition 16.1 (Induced metric)**.** Given (X, d) and $A \subseteq X$ we define the **induced metric** d_A on A by $d_A : A \times A \to \mathbb{R}$ where $d_A(x, y) = d(x, y)$ for all $x, y \in A$.
We also denote $d_A = d_{|A\times A}$.

Definition 16.2 (Induced topology)**.** We define τ_A the **induced topology** on A by

$$\tau_A = \{W \subset A \mid W = U \cap A \text{ for some open } U \subset X\}$$

Claim. τ_A is a topology on A (i.e. $\varnothing, A \in \tau_A$, closed under arbitrary unions, closed under finite intersections).
This follows clearly from the distributive property of unions and intersections.

Question 16.1. Is $\tau_A = \tau_{d_A}$ (our previous definite vs topology induced by the induced metric)?

Theorem 16.1. $\tau_A = \tau_{d_A}$.

Proof. Assume $W \in \tau_A$. There exist U open in X such that $W = U \cap A$.
Let $x_0 \in W$. There exists $\epsilon > 0$ such that $B_X(x_0, \epsilon) \subseteq U$. But then

$$\begin{aligned} B_A(x_0, \epsilon) &= B_X(x_0, \epsilon) \cap A \\ &\subseteq U \cap A \\ &= W \end{aligned}$$

Therefore all open sets in τ_A are open sets in τ_{d_A} thus $\tau_A \subseteq \tau_{d_A}$.
Suppose $W \in \tau_{d_A}$. For each $x_0 \in W$, there exists some $\epsilon_x > 0$ such that

$$B_A(x_0, \epsilon_x) \subseteq W \Rightarrow W = \bigcup_{x_0 \in W} B_A(x_0, \epsilon_x)$$

Let $U = \bigcup_{x_0 \in W} B_X(x_0, \epsilon)$, but $W = \left(\bigcup_{x_0 \in W} B_X(x_0, \epsilon)\right) \cap A$, so every $W \in \tau_{d_A}$ is definitely open in τ_A since it is the intersection of some open set $U \in \tau_A$, thus $\tau_{d_A} \subseteq \tau_A$.
Thus $\tau_A = \tau_{d_A}$. □

16.2 Continuity in metric spaces

Definition 16.3 (Continuity)**.** Given $(X, d_X), (Y, d_Y)$ and $f : X \to Y$ we say that f is continuous at x_0 if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon$$

Theorem 16.2. Given $(X, d_X), (Y, d_Y)$ and $f : X \to Y$ then TFAE:

1. f is continuous at $x_0 \in X$
2. If W is a neighborhood of $f(x)$ in Y, then $f^{-1}(W)$ (**pullback**) is a neighborhood of x_0 in X, where $f^{-1}(W) = \{x \in X \mid f(x) \in W\}$.

Proof. $1 \Rightarrow 2$ Assume f is cont. at x_0 and W is a neighborhood of $y_0 = f(x_0)$. Since $f(x_0) = y_0 \in int(W)$, there exists $\epsilon > 0$ such that $B_Y(y_0, \epsilon) \subset W$.

By continuity there exists some $\delta > 0$ such that if $x \in B(x_0, \delta)$ then $d_Y(f(x), f(x_0)) < \epsilon$, hence $f(x) \in B_Y(f(x_0), \epsilon) \subset W$, thus $x \in f^{-1}(W)$, therefore $x_0 \in int(f^{-1}(W))$ so we have a neighborhood.

$2 \Rightarrow 1$ Suppose $f^{-1}(W)$ is a neighborhood of x_0 for each neighborhood W of $y_0 = f(x_0)$. Let $\epsilon > 0$. Then $W = B_Y(f(x_0), \epsilon)$ is a neighborhood of $f(x_0)$ thus $U = f^{-1}(W)$ is a neighborhood of x_0 in X where $x_0 \in int(f^{-1}(W))$. Hence there exists some $\delta > 0$ such that $B_X(x_0, \delta) \subset U = f^{-1}(W)$, thus we have $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon$ so we have continuity.

□

16.3 Sequential characterization of continuity

Theorem 16.3 (Sequential characterization of continuity)**.** TFAE:

1. $f : X \to Y$ is continuous at $x_0 \in X$
2. If $\{x_n\} \subset X$ with $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

Proof. $1 \Rightarrow 2$ Assume f is cont. at x_0. Let $x_n \to x_0$. Let $\epsilon > 0$, then there exists $\delta > 0$ such that if $x \in B_X(x_0, \delta)$ then $f(x) \in B_Y(f(x_0), \epsilon)$. Since $x_n \to x_0$, there exists some $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, $x_n \in B_X(x_0, \delta) \Rightarrow f(x_n) \in B_Y(f(x_0), \epsilon)$.

$2 \Rightarrow 1$ We use the contrapositive.

Assume that f is not continuous at x_0. Then there exists an $\epsilon > 0$ such that for every $\delta > 0$, we can find $x_\delta \in B_X(x_0, \delta)$ such that $f(x_\delta) \notin B_Y(f(x_0), \epsilon)$.

In particular, for each $n \in \mathbb{N}$, there exists $x_n \in B_x(x_0, \frac{1}{n})$ with $f(x_n) \notin B_Y(f(x_0), \epsilon)$. Hence $x_n \to x_0$, but $f(x_n) \not\to f(x_0)$.

□

17 October 22, 2018

17.1 Continuity on a set

Definition 17.1 (Continuity on a set)**.** $f : (X, d_x) \to (Y, d_Y)$ is continuous on X if f is continuous at each $x_0 \in X$. Let

$$C(X, Y) = \{f : X \to Y \mid f \text{ is continuous on } X\}$$

In the case where $Y = \mathbb{R}$ we will write $C(X)$. Let the **set of bounded continuous functions** be

$$C_b(X) = \{f \in C(X, \mathbb{R}) \mid f \text{ is bounded}\}$$

We can define $\|\cdot\|_\infty$ on $C_b(X, \mathbb{R})$ by $\|f\|_\infty = \text{lub}\{|f(x)| \mid x \in X\}$.

Theorem 17.1. Let $f : (X, d_x) \to (Y, d_y)$. Then TFAE

1. f is continuous
2. $f^{-1}(W)$ is open for every open set $W \subset Y$
3. If $x_n \to x_0 \in X$, then $f(x_n) \to f(x_0) \in Y$

Proof. $1 \Rightarrow 2$ Let $W \subset Y$ be open. Let $V = f^{-1}(W)$. Let $x_0 \in V$, then $y_0 = f(x_0) \in W$. Hence W is a neighborhood of $f(x_0)$ hence V is a neighborhood of x_0 which implies that $x_0 \in int(V)$ so V is open.

$2 \Rightarrow 3$ Assume that $x_n \to x_0$. Let $\epsilon > 0$. Since $B_Y(f(x_0), \epsilon)$ is open, we have that $f^{-1}(B_Y(f(x_0), \epsilon))$ is open.

But $x_0 \in V$, so there exists $\delta > 0$ such that $B(x_0, \delta) \subset V$ (we have a neighborhood around x_0) from 2).

Since $x_n \to x_0$, we can find an $N \in \mathbb{N}$ such that if $n \geq N$ then $x_n \in B(x_0, \delta)$, therefore $f(x_n) \in B_Y(f(x_0), \epsilon)$ from above for any $\epsilon > 0$.

$3 \Rightarrow 1$ Same as the proof for continuity at a point.

□

Remark 17.1. Note that if $f : X \to Y$ and $B \subset Y$, then $f(^{-1}(B))^c = f^{-1}(B^c)$.
hence $f : (X, d_X) \to (Y, d_Y)$ is continuous iff $f^{-1}(F)$ is closed for each closed subset F of Y.

Question 17.1. If $f : (X, d_X) \to (Y, d_X)$ is continuous and if $U \subset X$ is open is $f(U)$ open? No, not in general.

Example 17.1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1$ for all x. Clearly $f(\mathbb{R})$ is not open.

17.2 Homeomorphism

Definition 17.2 (Homeomorphism)**.** A function $\phi : (X, d_X) \to (Y, d_Y)$ is called a **homeomorphism** if ϕ is 1-1 and onto and if both ϕ and ϕ^{-1} are continuous.

Remark 17.2. $\phi(W)$ is open in Y iff W is open in X. $\phi(F)$ is closed in Y iff F is closed in X (we have pullbacks in both directions for continuous functions).

Definition 17.3 (Equivalence in metric spaces)**.** We say that $(X, d_X), (Y, d_Y)$ are **equivalent** if there exists $\phi : X \to Y$ that is 1-1 and onto and $c_1, c_2 > 0$ such that

$$c_1 d_X(x_1, x_2) \leq d_Y(\phi(x_1), \phi(x_2)) \leq c_2 d_X(x_1, x_2)$$

Claim. ϕ is a homeomorphism.
We can clearly find continuous functions ϕ and ϕ^{-1} (since we have inequalities, we can let $\delta = \frac{\epsilon}{c_2}$ for ϕ and $\delta = \frac{\epsilon}{c_1}$ for ϕ^{-1}).

Example 17.2. Let (X, d) be any set with the discrete metric. Let $f : (X, d) \to (Y, d_Y)$. Since (X, d) is discrete, if $W \subset Y$ is open, $f^{-1}(W)$ is open (since any subset of X is open in d).

Question 17.2. Suppose that $f : (\mathbb{R}, |\cdot|) \to (Y, d)$, d is the discrete metric. When is f continuous?
Note: let $y_0 \in Y$. Then $\{y_0\}$ is open and closed, therefore $f^{-1}(\{y_0\})$ is open and closed if f is continuous. The only sets in $\mathbb{R}$ that are both open and closed under $|\cdot|$ is $\varnothing$ and $\mathbb{R}$: thus f must be the constant function.
If $\mathbb{R}$ instead had the discrete metric, we can have arbitrary continuous f.

Definition 17.4 (Continuity on a set)**.** Let $A \subset (X, d)$. Let $f : X \to (Y, d_Y)$. We say that f is continuous on A iff $f_{|A}$ is continuous on (A, d_A), where $f_{|A}$ is the restriction of f to A and (A, d_A) is the induced metric, iff whenever $\{x_n\} \subset A$ and $x_n \to x_0 \in A$ (limit must be in A, so for an open interval we don't care about the endpoints), we have $f(x_n) \to f(x_0)$.

18 October 24, 2018

18.1 Completeness of metric spaces: Cauchy sequences

Question 18.1. Is there an instrinsic way to tell if a sequence $\{x_n\} \subset (X, d)$ converges?

Observation 18.1. g Assume $x_n \to x_0$. Let $\epsilon > 0$. Then we can find $N_0 \in \mathbb{N}$ such that if $n \geq N_0$ then $d(x_n, x_0) < \frac{\epsilon}{2}$.
Therefore if $m, n \geq N_0$ then

$$\begin{aligned} d(x_n, x_m) &\overset{\triangle}{\leq} d(x_n, x_0) + d(x_0, x_m) \\ &< \epsilon \end{aligned}$$

g

Definition 18.1 (Cauchy sequence)**.** We say that $\{x_n\} \subset (X, d)$ is **Cauchy** if every $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that if $n, m \geq N_0$ then $d(x_n, x_m) < \epsilon$.

Theorem 18.1. Every convergent sequence is Cauchy.

Question 18.2. Is every Cauchy sequence convergent? Not in general on generic (X, d) and even not in general on $\mathbb{R}$.

Example 18.1. Let $X = (0, 1)$ with the usual metric. Let $x_n = \frac{1}{n}$ thus $\{x_n\}$ is Cauchy in (X, d).
It does not converge (since the limit point 0 is outside of X).

Definition 18.2 (Completeness)**.** We say (X, d) is **complete** if and only if each Cauchy sequence $\{x_n\}$ in X converges (in X).

18.2 Properties of Cauchy sequences

Observation 18.2. g GIven a (general) sequence $\{x_n\} \subset (X, d)$ it is possible that $\{x_n\}$ diverges but that $\{x_n\}$ has a **subsequence** $\{x_{n_k}\}$ which converges.

g

Theorem 18.2. Let $\{x_n\} \subset (X, d)$ be Cuachy. Assume $x_{n_k} \to x_0$ for some subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Then $x_n \to x_0$.

Proof. Let $\epsilon > 0$. We can find N_0 such that if $n, m \geq N_0$ then $d(x_n, x_m) < \frac{\epsilon}{2}$.

Let $n \geq N_0$. Consider $d(x_n, x_0)$. We can find k_0 large enough so that $n_{k_0} \geq N_0$ and $d(x_{n_{k_0}} < \frac{\epsilon}{2}$. Hence if $n \geq N_0$ then

$$\begin{aligned} d(x_n, x_0) &\leq d(x_n, x_{n_{k_0}}) + d(x_{n_{k_0}}, x_0) \\ &< \epsilon \end{aligned}$$

□

Definition 18.3 (Boundedness)**.** Let $A \subset (X, d)$. We say that A is beounded if there exists $M > 0$ and $x_0 \in X$ such that $A \subset B[x_0, M]$.

Proposition 18.1. If $\{x_n\} \subset (X, d)$ is Cauchy then $\{x_n\}$ is bounded.

Proof. Let $\epsilon = 1$. There exists $N_0 \in \mathbb{N}$ such that if $n, m \geq N_0$ then $d(x_n, x_m) < \epsilon$. Choosing some arbitrary x_{N_0} if $n \geq N_0$, then $d(x_n, x_{N_0}) < 1$. Let $M = \max\{d(x_1, x_{N_0}), d(x_2, x_{N_0}), \dots, d_(x_{N_0-1}, x_{N_0}), 1\}$.
It is clear to see that $X \subset B[x_{N_0}, M]$. □

Theorem 18.3. $\mathbb{R}$ is complete.

Proof. We require the following theorem:

Theorem 18.4 (Bolzano-Weierstrass)**.** Every bounded sequence $\{x_n\} \subset \mathbb{R}$ has a convergent subsequence.

Proof. We can either use the Nested Interval Theorem or show that every sequence in $\mathbb{R}$ has a monotone sequence then apply the Monotone Convergence Theorem. □

Remark 18.1. The following are (logically) equivalent:

1. Bolzano-Weierstrass
2. Upper Bound Property
3. Monotone Convergence Theorem
4. Nested Interval Theorem

If $\{x_n\} \subset \mathbb{R}$ is Cauchy then $\{x_n\}$ is bounded.
By the BW theorem $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ with $x_{n_k} \to x_0$. Since $\{x_n\}$ is Cauchy then $x_n \to x_0$ hence $\mathbb{R}$ is complete. □

Remark 18.2. Consider $(\mathbb{R}^n, \|\cdot\|_p)$, $1 \leq p \leq \infty$.
Let $\{\vec{x}_k\} = \{(x_{k,1}, \dots, x_{k,n})\}$ be Cauchy in $(\mathbb{R}^n, \|\cdot\|_p)$. Since $|x_{k,j} - x_{m,j}| \leq \|\vec{x}_k - \vec{x}_m\|_p$ then $\{x_{k,j}\}_{k=1}^{\infty}$ is Cauchy for each $j = 1, \dots, n$.
Hence $x_{k_j} \to x_{0,j}$ for each $j = 1, \dots, n$ therefore $\vec{x}_k \to \vec{x}_0 = (x_{0,1}, \dots, x_{0,n})$ so $(\mathbb{R}^n, \|\cdot\|_p)$ is complete.

Example 18.2. Let (X, d) be discrete (i.e. d is the discrete metric).
If $\{x_n\}$ is Cauchy then $\exists N_0$ such that if $n, m \geq N_0$ then $x_n = x_m$. Therefore $\{x_n\}$ converges, thus discrete spaces are complete.

18.3 Homeomorphism and completeness

Observation 18.3. Observe that if $X = \{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\} \subset \mathbb{R}$ with the induced metric, each $\{\frac{1}{n}\}$ is open (we can find a small enough ball so that it doesn't contain other points), therefore X does not converge.
Given $\{1, 2, \ldots, n, \ldots\} = \mathbb{N}$ and the discrete metric, define $\phi : \mathbb{N} \to \{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$ by $\phi(n) = \frac{1}{n}$.
Note that ϕ is a homeomorphism between $\mathbb{N}$ and $\{\frac{1}{n}\}$.
Note that $(\mathbb{N}, \text{discrete})$ is complete, but $X = \{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$ is not complete since its sequence $\{\frac{1}{n}\}$ is Cauchy but not convergent.
Therefore homeomorphism does not guarantee completeness equivalence.

19 October 26, 2018

19.1 Completeness of l_p

Theorem 19.1 (Completeness of l_p)**.** l_p is complete for every $1 \leq p \leq \infty$.

Proof. $p = \infty$ Let $\{\vec{x}_k\} \subset l_\infty$ (sequence of sequences) be Cauchy in $\|\cdot\|_\infty$ where $\vec{x}_k = \{x_{k,1}, x_{k,2}, ldots, x_{k,j}, \ldots\}$.

Note for each $j \in \mathbb{N}$ we have $|x_{k,j} - x_{m,j}| \leq \|\vec{x}_k - \vec{x}_m\|_\infty$ hence $\{x_{k,j}\}_{k=1}^\infty \subset \mathbb{R}$ is also Cauchy for each $j \in \mathbb{N}$. Let $x_{0,j} = \lim_{k\to\infty} x_{k,j}$ (by completness of $\mathbb{R}$).

Claim. We claim $\vec{x}_k \to \vec{x}_0 = \{x_{0,1}, x_{0,2}, \ldots\}$.

Let $\epsilon > 0$. Then there exists $N_0 \in N$ such that if $k, m \geq N_0$ then $\|\vec{x}_k - \vec{x}_m\| < \frac{\epsilon}{2}$. Let $k \geq N_0$ then $|x_{k,j} - x_{m,j}| < \frac{\epsilon}{2}$ for all $m \geq N_0$.

Hence $|x_{k,j} - x_{0,j}| = \lim_{m\to\infty} |x_{k,j} - x_{m,j}| \leq \frac{\epsilon}{2} < \epsilon$ for all $j \in \mathbb{N}$.

Therefore $\{x_{k,j} - x_{0,j}\}_{k=1}^\infty \in l_\infty$ and so $\{x_{0,j}\} \in l_\infty$.

From $|x_{k,j} - x_{0,j}| < \epsilon$ we get that $\|\vec{x}_k - \vec{x}_0\|_\infty < \epsilon$ so $\vec{x}_k \to \vec{x}_0$.

$1 \leq p < \infty$ Let $\{\vec{x}_k\} \subset l_p$ be Cauchy. Again $|x_{k,j} - x_{m,j}| \leq \|\vec{x}_k - \vec{x}_m\|_p$ so $\{x_{k,j}\}_{k=1}^\infty \subset \mathbb{R}$ is Cauchy for each j.

Let $x_{0,j} = \lim_{k\to\infty} x_{k,j}$ (by completeness of $\mathbb{R}$). Let $\epsilon > 0$. we can find N_0 such that if $k, m \geq N_0$ then $\|\vec{x}_k - \vec{x}_m\|_p < \frac{\epsilon}{2}$.

Let $j \in \mathbb{N}$. If $k, m \geq N_0$ then

$$\Big(\sum_{i=1}^{j} |x_{k,i} - x_{m,i}|^p\Big)^{\frac{1}{p}} \leq \|\vec{x}_k - \vec{x}_m\|_p < \frac{\epsilon}{2}$$

Let $k \geq N_0$, then

$$\Big(\sum_{i=1}^{j} |x_{k,i} - x_{0,i}|^p\Big)^{\frac{1}{p}} = \lim_{m\to\infty} \Big(\sum_{i=1}^{j} |x_{k,i} - x_{m,i}|^p\Big)^{\frac{1}{p}} \leq \frac{\epsilon}{2} < \epsilon$$

Since j was arbitrary, then

$$\Big(\sum_{i=1}^{\infty} |x_{k,i} - x_{0,i}|^p\Big)^{\frac{1}{p}} < \epsilon$$

So $\{x_{k,i} - x_{0,i}\}_{i=1}^\infty$ is in l_p thus $x_0 = \{x_{0,1}, x_{0,2}, \ldots\} \in l_p$ and for $k \geq N_0$ we have $\|\vec{x}_k - \vec{x}_0\|_p < \epsilon$ so $\vec{x}_k \to \vec{x}_0$. $\square$

19.2 Uniform and pointwise convergence

Definition 19.1 (Pointwise convergence)**.** A sequence $f_n : (X, d_X) \to (Y, d_Y)$ is said to converge **pointwise** to $f_0 : (X, d_X) \to (Y, d_Y)$ if for each $x_0 \in X$, the sequence $f_n(x_0) \to f_0(x_0)$ in Y.

Definition 19.2 (Uniform convergence)**.** We say $f_n \to f_0$ **uniformly** if for each $\epsilon > 0$ there exist $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, then $d(f_n(x), f_0(x)) < \epsilon$ for all $x \in X$.

Remark 19.1. Uniform convergence implies pointwise convergence **but the converse may not be necessarily true**.

Example 19.1. Let $X = [0, 1]$ and $Y = \mathbb{R}$ where $f_n(x) = x^n$. Note that

$$f_n(x) \to f_0(x) = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1 \end{cases}$$

which is pointwise convergence but not uniformly convergent.

19.3 Uniform convergence carries continuity (*********)

Theorem 19.2. Assume that $f_n : (X, d_X) \to (Y, d_Y)$ converges uniformly to $f_0 : X \to Y$. If each f_n is continuous at x_0, then f_0 is continuous at x_0.

Proof. Assume $\epsilon > 0$. Then there is some $N_0 \in \mathbb{N}$ such that if $n \geq N_0$ then $d_Y(f_n(x), f_0(x)) < \frac{\epsilon}{3}$ for all $x \in X$. Let $n_0 \geq N_0$. Since f_{n_0} is continuous at x_0 there exists a $\delta > 0$ such that if $x \in B(x_0, \delta)$ then $d_Y(f_{n_0}(x), f_{n_0}(x_0)) < \frac{\epsilon}{3}$. Let $d_X(x, x_0) < \delta$, then

$$\begin{aligned} d_Y(f_0(x), f_0(x_0)) &\leq d_Y(f_0(x), f_{n_0}(x)) + d_Y(f_{n_0}(x), f_{n_0}(x_0)) + d_Y(f_{n_0}(x_0), f_0(x_0)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

so f_0 is continuous at x_0. ☐

20 October 31, 2018

20.1 Banach space

Definition 20.1 (Banach space)**.** (X, d) is complete if every Cauchy sequence converges. A normed linear space V is called a **Banach space** if $(V, \|\cdot\|)$ is complete with respect to d_V.

Theorem 20.1. assume that $f_n : (X, d_X) \to (Y, d_Y)$ converges uniformly to f_0. If each f_n is continuous at x_0, then f_0 is continuous at x_0.

Corollary 20.1. Assume that $f_n : (X, d_X) \to (Y, d_Y)$ is continuous. If $f_n \to f_0$ uniformly on X, then $f_0 : X \to Y$ is continuous.

20.2 Completeness theorem for $C_b(X)$ (*****)

Theorem 20.2 (Completeness theorem for $C_b(X)$)**.** $(C_b(X), \|\cdot\|_\infty)$ is complete.

Proof. Let $\{f_n\} \subset C_b(x)$ be Cauchy. Given $\epsilon > 0$ we can find N_0 such that if $n, m \geq N_0$ then $\|f_n - f_m\|_\infty < \epsilon$.
Let $x \in X$. Then

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \epsilon$$

so $\{f_n(x)\}$ is Cauchy at x which implies convergence at x.
For each $x \in X$ let $f_0(x) = \lim_{n\to\infty} f_n(x)$.
Given $\epsilon > 0$, choose N_0 so that if $n, m \geq N_0$ then $\|f_n - f_m\|_\infty < \frac{\epsilon}{2}$. Then if $x \in X$

$$|f_n(x) - f_0(x)| \leq \lim_{m\to\infty} \|f_n - f_m\|_\infty \leq \frac{\epsilon}{2} < \epsilon$$

Hence $f_n \to f_0$ uniformly so f_0 is continuous.
Note since $\{f_n\}$ is bounded (Cauchy sequence) there exists $M > 0$ such that $\|f_n\|_\infty \leq M$ for all $n \in \mathbb{N}$. Let $x \in X$.
We can find n_0 such that $|f_0(x) - f_{n_0}(x)| < 1$ so

$$|f_0(x)| \leq |f_0(x) - f_{n_0}(x)| + |f_{n_0}(x)| < 1 + M$$

so $f_0 \in C_b(X)$. □

Remark 20.1. Given any X, if (X, d) is X with the discrete metric then $C_b(X), \|\cdot\|_\infty) = (l_\infty, \|\cdot\|_\infty)$.

Example 20.1. Let $X = C[0,1]$ and $\|f\|_1 = \int_0^1 |f(x)|\, \mathrm{d}x$.
TODO revisit from picture

20.3 Characterization of completeness

Theorem 20.3 (Nested Interval)**.** If $\{[a_n, b_n]\}$ with $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$, then

$$\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \varnothing$$

This is actually a statement about completeness.

Question 20.1. How would the Nested Interval Theorem work in (X, d)?

Conjecture 20.1. If $\{F_n\}$ is a sequence of non-empty closed sets in (X, d) with $F_{n+1} \subseteq F_n$, then $\bigcap_{n=1}^{\infty} \neq \varnothing$.

Example 20.2. Let $X = \mathbb{R}$, $F_n = [n, \infty)$ where $F_{n+1} \subsetneq F_n$. Note that $\bigcap_{n=1}^{\infty} F_n = \varnothing$.

One might ask if this fails with a bounded set:

Example 20.3. Let $X = (0, 1]$, $F_n = (0, \frac{1}{n}]$ is closed in X. Note $F_{n+1} \subseteq F_n$ but $\bigcap_{n=1}^{\infty} F_n = \varnothing$.

Definition 20.2 (Diameter)**.** Given $A \subset (X, d)$ we define the **diameter** of A to be

$$\mathrm{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}$$

Proposition 20.1. Let $A \subset B \subset (X, d)$

1. $\mathrm{diam}(A) \leq \mathrm{diam}(B)$
2. $\mathrm{diam}(A) = \mathrm{diam}(\bar{A})$

Proof. 1. Proof is trivial.

2. If $\text{diam}(A) = \infty$, then $\text{diam}(\bar{A}) = \infty$ (since $\text{diam}(A) \leq \text{diam}(\bar{A})$).

   Assume $d = \text{diam}(A) < \infty$.

   Let $x_0, y_0 \in \bar{A}$. Then given $\epsilon > 0$ we can find $x_1, y_1 \in A$ with $d(x_0, x_1) < \frac{\epsilon}{2}$ and $d(y_0, y_1) < \frac{\epsilon}{2}$, hence

$$\begin{aligned} d(x_0, y_0) &\leq d(x_0, x_1) + d(x_1, y_1) + d(y_1, y_0) \\ &< \frac{\epsilon}{2} + d + \frac{\epsilon}{2} \\ &= d + \epsilon \end{aligned}$$

   So $\text{diam}(\bar{A}) < d + \epsilon$ for all $\epsilon > 0$ thus $\text{diam}(\bar{A}) \leq d$ but $d = \text{diam}(A) \leq \text{diam}(\bar{A})$ so $\text{diam}(\bar{A}) = d$.

□

21 November 2, 2018

21.1 Cantor's Intersection Principle

Theorem 21.1 (Cantor's Intersection Principle)**.** Let (X, d) be a metric space. Then TFAE:

1. (X, d) is complete.
2. If $\{F_n\}$ is a sequence of non-empty closed subsets such that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$ and if $\lim_{n\to\infty} \text{diam}(F_n) = 0$ then $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$.

Proof. $1 \Rightarrow 2$ For each $n \in \mathbb{N}$ pick $x_n \in F_n$. We claim that $\{x_n\}$ is Cauchy. Let $\epsilon > 0$. Then $\exists N_0 \in \mathbb{N}$ such that $\text{diam}(F_{N_0}) < \epsilon$. If $n, m \geq N_0$ then $x_n, x_m \in F_{N_0}$. Then $d(x_n, x_m) \leq \text{diam}(F_{N_0}) < \epsilon$ so $\{x_n\}$ is Cauchy.

Hence $x_n \to x_0 \in X$. Note $\{x_1, \ldots, x_n, \ldots\}$ converges to x_0 (i.e. $\lim_{n\to\infty} x_n = x_0$). Observe that $\{x_n, x_{n+1}, \ldots\} \subseteq F_n$. Hence $x_0 \in F_n$ for each $n \in \mathbb{N}$ therefore $x_0 \in \bigcap_{n=1}^{\infty} F_n$ (in fact, $\{x_0\} = \bigcap_{n=1}^{\infty} F_n$).

$2 \Rightarrow 1$ Let $\{x_n\} \subset X$ be Cauchy. Let $F_n = \overline{\{x_n, x_{n+1}, \ldots\}}$. Let F_n be closed and $F_{n+1} \subseteq F_n$.

Let $\epsilon > 0$. We can find N_0 such that if $n, m \geq N_0$ then $d(x_n, x_m) < \frac{\epsilon}{2}$. Hence $\text{diam}(\{x_{N_0}, x_{N_0+1}, \ldots\}) = \text{diam}(F_{N_0}) \leq \frac{\epsilon}{2}$ so $\text{diam}(F_n) \to 0$, hence from 2) $\bigcap_{n=1}^{\infty} F_n = \{x_0\}$ for some x_0.

Note: for any $k > 0$, $B(x_0, \frac{1}{k})$ will contain F_{i_k} for some i_k since $\text{diam}(F_n) \to 0$, therefore $B(x_0, \frac{1}{k})$ contains a tail of $\{x_n\}$ for each k.

Let $k = 1$. We can find $n_1 > 0$ such that $x_{n_1} \in B(x_0, \frac{1}{1})$.

Let $k = 2$. We can find $n_2 > n_1$ such that $x_{n_2} \in B(x_0, \frac{1}{2})$.

We can proceed inductively to construct $n_1 < n_2 < \ldots < n_k < \ldots$ such that $x_{n_k} \in B(x_0, \frac{1}{k})$. Hence $\{x_{n_k}\}$ converges to x_0. Since $\{x_n\}$ is Cauchy $\{x_n\}$ also converges to x_0.

□

21.2 Series and partial sums

Definition 21.1 (Series and partial sum)**.** Let $(X, \|\cdot\|)$ be a normed linear space. A **series** in X is a formal sum

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \ldots + x_n + \ldots$$

where $\{x_n\} \subset X$.

For each $k \in \mathbb{N}$, the kth **partial sum** of $\sum_{n=1}^{\infty} x_n$ is

$$S_k = \sum_{n=1}^{k} x_n = x_1 + \ldots + x_k$$

We say that $\sum_{n=1}^{\infty} x_n$ converges in $(X, \|\cdot\|)$ if $\{S_k\}_{k=1}^{\infty}$ converges. In this case we write $\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} S_k$. Otherwise $\sum_{n=1}^{\infty} x_n$ diverges.

22 November 5, 2018

22.1 Weierstrass M-Test

Theorem 22.1 (Weierstrass M-Test)**.** Let $(X, \|\cdot\|)$ be a normed linear space. Then TFAE:

1. $(X, \|\cdot\|)$ is complete i.e. $(X, \|\cdot\|)$ is a Banach space
2. If $\sum_{n=1}^{\infty} x_n$ is such that $\sum_{n=1}^{\infty}\|x_n\|$ converges then $\sum_{n=1}^{\infty} x_n$ converges (absolute convergence implies convergence).

Proof. $1 \Rightarrow 2$ Given $\sum_{n=1}^{\infty} x_n$, let $S_k = \sum_{n=1}^{k} x_n$, $T_k = \sum_{n=1}^{k}\|x_n\|$.

If $\sum_{n=1}^{\infty}\|x_n\|$ converges, then $\{T_k\}$ is Cauchy.

Hence given $\epsilon > 0$ we can find N_0 such that if $N_0 \leq m < k$ then

$$T_k - T_m = \sum_{n=1}^{k}\|x_n\| - \sum_{n=1}^{m}\|x_n\| = \sum_{n=k+1}^{m}\|x_n\| < \epsilon$$

So if $N_0 \leq m < k$ then

$$\|S_k - S_m\| = \|\sum_{n=1}^{k} x_n - \sum_{n=1}^{m} x_n\| = \|\sum_{n=k+1}^{m} x_n\| \leq \sum_{n=k+1}^{m}\|x_n\| < \epsilon$$

thus $\{S_k\}$ is Cauchy hene $\{S_k\}$ converges.

$2 \Rightarrow 1$ Assume that $\{x_n\}$ is Cauchy. We can find $n_1 < n_2 < \ldots < n_j < \ldots \subset \mathbb{N}$ such that $\|x_{n_j} - x_{n_{j+1}}\| < \frac{1}{2^j}$.

Note that

$$\sum_{j=1}^{\infty}\|x_{n_j} - x_{n_{j+1}}\| \leq \sum_{j=1}^{\infty}\frac{1}{2^j} < \infty$$

Hence $\sum_{j=1}^{\infty} x_{n_j} - x_{n_{j+1}}$ converges to some $x_0 \in X$.

In particular if

$$S_k = \sum_{j=1}^{k} x_{n_j} - x_{n_{j+1}} = x_{n_1} - x_{n_{k+1}} \to x_0$$

It follows that $x_{n_{k+1}} \overset{k\to\infty}{\to} x_{n_1} - x_0$ so $x_n \to x_{n_1} - x_0$, thus our arbitrary Cauchy sequence converges. $\square$

Example 22.1. Let

$$\phi(x) = \begin{cases} x \text{ if } x \in [0,1] \\ 2-x \text{ if } x \in [1,2] \end{cases}$$

Extend ϕ to $\mathbb{R}$ i.e. $\phi(x+2) = \phi(x)$ for all $x \in \mathbb{R}$.

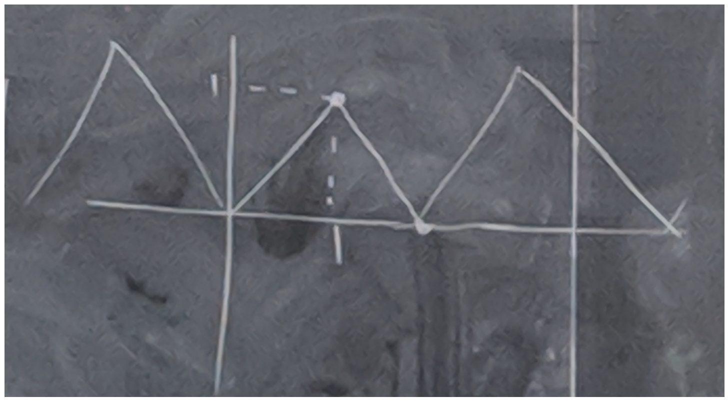

Figure 22.1: Our ϕ function extended to $\mathbb{R}$.

Define

$$f(x) = \sum_{n=1}^{\infty} \big(\frac{3}{4}\big)^n \phi(4^n x)$$

We will attempt to show that f is continuous but **nowhere differentiable**.
Note $\phi \in C_b(\mathbb{R})$, $\|\phi\|_\infty = 1$, hence

$$\sum_{n=1}^{\infty} \| \big(\frac{3}{4}\big)^n \phi(4^n x)\|_\infty = \sum_{n=1}^{\infty} \big(\frac{3}{4}\big)^n < \infty$$

Let

$$f(x) \xleftarrow{\text{in } \|\cdot\|_\infty, k\to\infty} \sum_{n=1}^{k} \big(\frac{3}{4}\big)^n \phi(4^n x) = S_k(x)$$

so $f(x) \in C_b(\mathbb{R})$ (continuous and bounded by Weierstrass M-test).
Let $x \in \mathbb{R}$. For each $m \in \mathbb{N}$ we find $k \in \mathbb{Z}$ such that $k \leq 4^m x < k+1$. Let $p_m = \frac{k}{4^m}$ and $q_m = \frac{k+1}{4^m}$. Note $p_m \to x$ and $q_m \to x$.
If $n > m$ then $|\phi(4^n p_m) - \phi(4^n q_m)| = 0$ since both $4^n p_m$ and $4^n q_m$ will be powers of 4 so ϕ will map them to the same values.
If $n = m$ then $|\phi(4^n p_m) - \phi(4^n q_m)| = 1$ (mapped to k and $k+1$ so ϕ will map to values with difference of 1).
If $n < m$ then

$$\begin{aligned} |\phi(4^n p_m) - \phi(4^n q_m)| &= |4^n p_m - 4^n q_m| && \text{on same line segment, difference on } \phi \text{ is difference of values} \\ &= |\frac{4^n k}{4^m} - \frac{4^n(k+1)}{4^m}| \\ &= 4^{n-m} \end{aligned}$$

Consider

$$\begin{aligned}
|f(p_m) - f(q_m)| &= |\sum_{n=1}^{\infty} (\frac{3}{4}))^n \big(\phi(4^n p_m) - \phi(4^n q_m)\big)| \\
&= |\sum_{n=1}^{m} (\frac{3}{4}))^n \big(\phi(4^n p_m) - \phi(4^n q_m)\big)| && n > m \Rightarrow 0 \\
&\overset{\triangle}{\geq} (\frac{3}{4})^m - \sum_{n=1}^{m-1} (\frac{3}{4})^n |\phi(4^n p_m) - \phi(4^n q_m)| \\
&= (\frac{3}{4})^m - \sum_{n=1}^{m-1} (\frac{3}{4})^n 4^{n-m} \\
&= (\frac{3}{4})^m - \frac{1}{4^m} \sum_{n=1}^{m-1} 3^n \\
&= (\frac{3}{4})^m - \frac{1}{4^m} (\frac{3^m - 1}{2}) \\
&= \frac{1}{4^m} (3^m - \frac{3^m - 1}{2}) \\
&= \frac{1}{4^m} (\frac{3^m + 1}{2}) \\
&> \frac{3^m}{2 \cdot 4^m}
\end{aligned}$$

Hence (approximating the derivative of $S_k(x) \overset{k\to\infty}{\to} f(x)$)

$$\frac{|f(p_m) - f(q_m)|}{|p_m - q_m|} > 4^m (\frac{3^m}{2 \cdot 4^m}) = \frac{3^m}{2}$$

Note that if $p_m = x$, then

$$\frac{|f(x) - f(q_m)|}{|x - q_m|} > \frac{3^m}{2}$$

If $p_m \neq x$, then Note that if $p_m = x$, then

$$\begin{aligned}
\frac{3^m}{2} &< \frac{|f(p_m) - f(q_m)|}{|p_m - q_m|} \\
&\leq \frac{|f(p_m) - f(x)|}{|p_m - q_m|} + \frac{|f(x) - f(q_m)|}{|p_m - q_m|} \\
&\leq \frac{|f(p_m) - f(x)|}{|p_m - x|} + \frac{|f(x) - f(q_m)|}{|x - q_m|}
\end{aligned}$$

So either $\frac{|f(p_m)-f(x)|}{|p_m-x|} \geq \frac{3^m}{4}$ or $\frac{|f(x)-f(q_m)|}{|x-q_m|} \geq \frac{3^m}{4}$. This gives us $\{t_m\}$ with $t_m \to x$ $t_m \neq x$ ($t_m = p_m$ or q_m) where

$$\frac{|f(x) - f(t_m)|}{|x - t_m|} > \frac{3^m}{4} \to \infty$$

so this means $f(x)$ continuous but nowhere differentiable.

23 November 7, 2018

23.1 Isometry

Definition 23.1. Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $\phi : X \to Y$ is said to be an isometry if $d_Y(\phi(x_1), \phi(x_2)) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$.
It is clear that isometries are 1-1. If ϕ is onto we say (X, d_X) and (Y, d_Y) are isometric metric spaces.

23.2 Completion of a metric space

Definition 23.2. A **completion of a metric space** (X, d_X) is a pair of $((X, d_X), \phi)$ whwere (Y, d_Y) is a complete metric space and $\phi : X \to Y$ is an *isometry* and $\overline{\phi(X)} = Y$.

Proposition 23.1. Let (X, d) be a complete metric space. Let $A \subset X$. Then (A, d_A) is complete if and only if A is closed.

Proof. Assume that A is complete. Let $\{x_n\} \subset A$ with $x_n \to x_0 \in X$. Then $\{x_n\}$ is Cauchy in (X, d) and hence Cauchy in (A, d_A). Therefore since (A, d_A) is complete $\{x_n\}$ converges in A so $x_0 \in A$ thus A is closed.
Assume that (X, d) is complete and that A is closed. let $\{x_n\} \subset A$ be Cauchy in (A, d_A) thus $\{x_n\}$ is Cauchy in (X, d). Hene $x_n \to x_0 \in X$ thus $\{x_n\}$ converges and so A is complete. □

23.3 Uniform continuity

Definition 23.3 (Uniform continuity)**.** We say that a function $f : (X, d_X) \to (Y, d_Y)$ is **uniformly continuous** if for every $\epsilon > 0$ there exists $\delta > 0$ for every $x_1, x_2 \in X$ such that $d_X(x_1, x_2) < \delta$ then $d_Y(f(x_1), f(x_2)) < \epsilon$.

Example 23.1. Given $(X, d), x_0 \in X$. Define $g_{x_0}(x) = d(x, x_0)$ for some fixed x_0.
So $|d(x, x_0) - d(y, x_0)| \leq d(x, y)$, i.e. $|g_{x_0}(x_1) - g_{x_0}(x_2)| \leq d(x_1, x_2)$ thus for $\epsilon > 0$, let $\delta = \epsilon$ then

$$d(x_1, x_2) < \delta = \epsilon \Rightarrow |g_{x_0}(x_1) - g_{x_0}(x_2)| < \epsilon$$

So g_{x_0} is uniformly continuous.

23.4 Completion theorem

Question 23.1. Does every (X, d) have a completion?

Theorem 23.1 (Completion theorem)**.** Every metric space (X, d) has a completion.

Proof. Pick $a \in X$. For each $u \in X$ define $\phi : X \to C_b(X)$ by

$$(\phi(u))(x) = f_u(x) = d(u, x) - d(x, a)$$

Clearly $\phi(u)$ is continuous.
Note that $|f_u(x)| = |d(u, x) - d(x, a)| \leq d(u, a)$ for all $x \in X$ so $\phi(u) \in C_b(X)$.
We need to show that ϕ is an isometry. Let $u, v \in X$

$$\begin{aligned} |f_u(x) - f_v(x)| &= |(d(u, x) - d(x, a)) - (d(v, x) - d(x, a))| \\ &= |d(u, x) - d(v, x)| \\ &\leq d(u, v) \end{aligned}$$

so $\|f_u - f_v\|_\infty \leq d(u, v)$.

In fact observe for $x = v$, we have $|f_u(v) - f_v(v)| = |d(u,v) - d(v,v)| = d(u,v)$ therefore $\|f_u - f_v\|_\infty = \|\phi(u) - \phi(v)\|_\infty = d(u,v)$ (since it's the supremum of all distances).
Let $Y = \phi(x) \subset C_b(X)$ our complete metric space that gives us our completion for X. $\square$

Question 23.2. If (X,d) has two completions, how are they related?

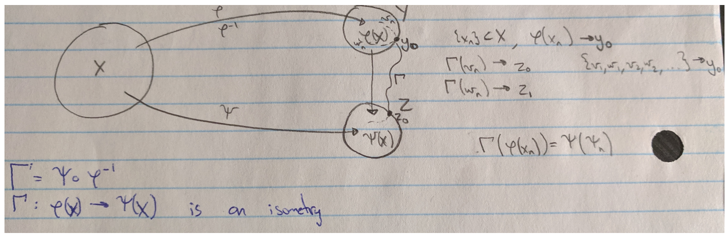

Figure 23.1: Suppose we have two completions with functions ϕ and ψ (both isometries).

Note that $\Gamma = \psi \circ \phi^{-1}$ where $\Gamma : \phi(x) \to \psi(x)$ is an isometry since both ϕ and ψ are isometries themselves).
So there exists a completion between completions.

24 November 9, 2018

24.1 Lipschitz and contractions

Definition 24.1 (Lipschitz)**.** A function $f : (X, d_X) \to (Y, d_Y)$ is said to be **Lipschitz** if there exists $\alpha \geq 0$ such that $d_Y(f(x_1), f(x_2)) \leq \alpha d_X(x_1, x_2)$ for all $x_1, x_2 \in X$.

Observation 24.1. Lipschitz implies uniformly continuous. For example if $f : [a,b] \to \mathbb{R}$ f is continuous on $[a,b]$.

Definition 24.2 (Contraction)**.** $f : X \to Y$ is a **contraction** if there exists $0 \leq k < 1$ with $d_Y(f(x_1), f(x_2)) \leq k d_X(x_1, x_2)$ $\forall x_1, x_2 \in X$.

24.2 Banach Contractive Mapping Theorem

Question 24.1. Does there exist $f \in C[0,1]$ such that $f(x) = e^x + \int_0^x \frac{\sin t}{2} f(t)\,dt$?

Definition 24.3 (Fixed points)**.** Given (X,d), $\Gamma : X \to X$, we say that x_0 is a fixed point of Γ if $\Gamma(x_0) = x_0$.

Note. Define $\Gamma : C[0,1] \to C[0,1]$ by $\Gamma(f)(x) = e^x \int_0^x \frac{\sin t}{2} f(t)\,dt$. Note f_0 is a solution to our previous question if f is a **fixed point** of Γ.

Theorem 24.1 (Banach Contractive Mapping Theorem)**.** Assume that (X,d) is complete. If $\Gamma : X \to X$ is contractive then there exists a **unique** $x_0 \in X$ such that $\Gamma(x_0) = x_0$ (fixed point).

Proof. Pick $x_1 \in X$. Let $x_2 = \Gamma(x_1), x_3 = \Gamma(x_2), \ldots, x_{n+1} = \Gamma(x_n)$.

Claim. $\{x_n\}$ is Cauchy.
Note $d(x_3, d_2) = d(\Gamma(x_2), \Gamma(x_1)) \leq k d(x_2, x_1)$, $d(x_4, d_3) = d(\Gamma(x_3), \Gamma(x_2)) \leq k d(x_3, x_2) = k^2 d(x_2, x_1)$. Therefore

$$d(x_{n+1}, x_n) = k^{n-1} d(x_1, x_2)$$

If $m > n$ then

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n) \\
&\leq k^{m-2} d(x_2, x_1) + k^{m-3} d(x_2, x_1) + \ldots + k^{n-1} d(x_2, x_1) \\
&\leq \sum_{j=n-1}^{\infty} k^{m-2} d(x_2, x_1) \\
&\leq \sum_{j=n-1}^{\infty} k^{j} d(x_2, x_1) \\
&\leq \frac{k^{n-1}}{1-k} d(x_2, x_1)
\end{aligned}$$

Since $k^{n-1} \overset{n\to\infty}{\to} 0$, then $\{x_n\}$ is Cauchy.

Since (X, d) is complete $x_n \to x_0 \in X$. Note that $x_{n+1} \to x_0$ so $\Gamma(x_n) \to x_0$. Since Γ is continuous we also have $\Gamma(x_n) \to \Gamma(x_0)$ so $\Gamma(x_0) = x_0$.
To show uniqueness, assume $\Gamma(y_0) = y_0$. Then

$$d(x_0, y_0) = d(\Gamma(x_0), \Gamma(y_0)) \leq k d(x_0, y_0)$$

since $0 < k < 1$ then $d(x_0, y_0) = 0$ so $x_0 = y_0$. $\square$

Example 24.1. Show that

$$f(x) = e^x \int_0^x \frac{\sin(t)}{2} f(t)\,\mathrm{d}t$$

has a unique solution in $C[0, 1]$.

Solution. Define $\Gamma : C[0, 1] \to C[0, 1]$ by

$$\Gamma(f)(x) = e^x + \int_0^x \frac{\sin(t)}{2} f(t)\,\mathrm{d}t$$

Let $f, g \in C[0, 1]$, then

$$\begin{aligned}
|\Gamma(f)(x) - \Gamma(g)(x)| &= \left| \left[e^x + \int_0^x \frac{\sin(t)}{2} f(t)\,\mathrm{d}t\right] - \left[e^x + \int_0^x \frac{\sin(t)}{2} g(t)\,\mathrm{d}t\right] \right| \\
&= \left| \int_0^x \frac{\sin(t)}{2} (f(t) - g(t))\,\mathrm{d}t \right| \\
&\leq \int_0^x \left| \frac{\sin(t)}{2} \right| |f(t) - g(t)|\,\mathrm{d}t \\
&\leq \int_0^1 \frac{1}{2} \|f - g\|_\infty\,\mathrm{d}t \\
&= \frac{1}{2} \|f - g\|_\infty
\end{aligned}$$

So $norm\Gamma(f) - \Gamma(g)_\infty \leq \frac{1}{2}\|f - g\|_\infty$ Thus Γ is contractive.
Fact: the unique fixed point is the solution.

Example 24.2. Show that

$$f(x) = e^x \int_0^x \frac{\sin(t)}{2} f(t)\,\mathrm{d}t$$

has a unique solution in $C[0,1]$.

Solution. Define $\Gamma : C[0,1] \to C[0,1]$ by

$$\Gamma(f)(x) = x + \int_0^x t^2 f(t)\,\mathrm{d}t$$

Let $f, g \in C[0,1]$, then

$$\begin{aligned} |\Gamma(f)(x) - \Gamma(g)(x)| &\leq \int_0^1 t^2 \|f-g\|_\infty \,\mathrm{d}t \\ &= \frac{1}{3}\|f-g\|_\infty \end{aligned}$$

By the BCMT, the above has a unique solution.
To find the solution, let $f_1(x) = x$. Then

$$\begin{aligned} f_2(x) = \Gamma(f_1)(x) &= x + \int_0^x t^2 f_1(t)\,\mathrm{d}t \\ &= x + \int_0^x t^3\,\mathrm{d}t \\ &= x + \frac{x^4}{4} \end{aligned}$$

We apply Γ again

$$\begin{aligned} f_3(x) = &= x + \int_0^x t^2(t + t^4/4)\,\mathrm{d}t \\ &= \frac{x}{1} + \frac{x^4}{1\cdot 4} + \frac{x^7}{1\cdot 4\cdot 7} \end{aligned}$$

Thus

$$f_n(x) = \frac{x}{1} + \frac{x^4}{1\cdot 4} + \frac{x^7}{1\cdot 4\cdot 7} + \ldots + \frac{x^{3n-2}}{1\cdot 4\cdot 7\cdot \ldots \cdot (3n-2)}$$

So $f_0 = \sum_{k=1}^{\infty} \frac{x^{3k-2}}{1\cdot 4\cdot \ldots \cdot (3k-2)}$.

Some applications of BCMT:

1. Newton's Method
2. Picard's Theorem: let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, where $(x,y) \in dom(f)$, be Lipschitz in y, that is

   $$|f(t,y_1) - f(t,y_2)| \leq \alpha|y_1 - y_2| \qquad \forall y_1, y_2 \in \mathbb{R}$$

   If $y_0 \in \mathbb{R}$ then there exists a unique $\phi \in C[a,b]$ such that

   $$\phi'(t) = f(t, \phi(t)) \qquad \forall t \in (a,b)$$

   with $\phi(0) = y_0$.

25 November 12, 2018

25.1 Motivation and definitions for categories

Example 25.1. Let

$$f(x) = \begin{cases} \frac{1}{m} & \text{if } x = \frac{n}{m}, n \in Z, m \neq 0, m \in \mathbb{N}, \gcd(m,n) = 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

f is continuous at each $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ (by density of irrationals), discontinuous otherwise (by density of rationals, find some $x \in \mathbb{Q}$ such that $|f(x) - f(x_0)| \geq \epsilon$).

Question 25.1. Does there exist a function f so that f is continuous on $\mathbb{Q}$ but not on $\mathbb{R} \setminus \mathbb{Q}$?

Definition 25.1. Let $f : X \to \mathbb{R}$. For each $n \in \mathbb{N}$ define

$$D_n(f) = \{x_0 \in X \mid \text{ for every } \delta > 0, \exists x, y \in B(x_0, \delta) \text{ s.t. } d(f(x), f(y)) \geq \frac{1}{n}\}$$

Facts:

1. $D_n(f)$ is closed for each $n \in \mathbb{N}_\infty$
2. f is continuous at x_0 iff $x_0 \notin \bigcap_{n=1}^\infty D_n(f)$, that is $D(f) = \{x_0 \in X \mid f \text{ is discont. at } x_0\}$ implies $D(f) = \bigcap_{n=1}^\infty D_n(f)$.

Definition 25.2 (F delta)**.** Let (X, d) be a metric space. We say that $A \subset X$ is F_δ if there exists $\{F_n\}_{n=1}^\infty$ closed sets with $A = \bigcup_{n=1}^\infty F_n$.

Definition 25.3 (G delta)**.** We say that $A \subset X$ is G_δ if $A = \bigcap_{n=1}^\infty U_n$ where U_n is open.

Remark 25.1. A if F_δ iff A^c is G_δ.

Definition 25.4 (Nowhere dense)**.** Recall we say that $A \subset X$ is **dense** if $\bar{A} = X$ or equivalently if $A \cap U \neq \varnothing$ for every non-empty open set U.
We say A is **nowhere dense** if $\text{int}(\bar{A}) = \varnothing$. This is equivalent to $(\bar{A})^c$ being dense (e.g. the Cantor set is nowhere dense in $\mathbb{R}$).

Remark 25.2. If A is dense then A^c is nowhere dense: suppose $A^c \cap U \neq \varnothing$ for U open. Then U is open but does not intersect A, which is a contradiction since dense sets intersect all open sets.

Definition 25.5 (1st category)**.** We say that A is a set of **1st category** if $A = \bigcup_{i=1}^\infty A_n$ where each A_n is nowhere dense.

Definition 25.6 (2nd category)**.** We say that A is of **2nd category** if A is not of 1st cateogry.

Definition 25.7 (Residual)**.** We say that A is **residual** in X if A^c is of 1st category.

Observation 25.1. If $F \subset (X, d)$ is closed then F is G_δ.
Since F is closed it must be the intersection of open sets i.e.

$$F = \bigcap_{n=1}^\infty (\bigcup_{x \in F} B(x, \frac{1}{n}))$$

Similarly open sets are F_δ.

25.2 Baire Category Theorem

Theorem 25.1 (Baire Category Theorem I)**.** Let (X, d) be complete. Let $\{U_n\}_{n=1}^{\infty}$ be a countable collection of dense open sets. Then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.

Proof. Assume $\{U_n\}_{n=1}^{\infty}$ are open and dense. Let $W \subset X$ be open and non-empty. Hence $W \cap U_1$ is open and non-empty. Hence there exists x_1 and $0 < r_n \leq 1$ such that

$$B(x_1, r_1) \subseteq B[x_1, r_1] \subseteq W \cap U_1$$

Similarly we can find x_2 and $0 \leq r_2 < \frac{1}{2}$ such that

$$B(x_2, r_2) \subseteq B[x_2, r_2] \subseteq B(x_1, r_1) \cap U_2$$

We can proceed recurisvley to construct $\{x_n\}$ and $\{r_n\}$ with $0 \leq r_n < \frac{1}{n}$ and

$$B(x_{n+1}, r_{n+1}) \subseteq B[x_{n+1}, r_{n+1}] \subseteq B(x_n, r_n) \cap U_{n+1}$$

Then $B[x_{n+1}, r_{n+1}] \subset B[x_n, r_n]$ and $\operatorname{diam}(B[x_n, r_n]) \leq 2r_n \to 0$ so by Cantor's Intersection Principle we have

$$x_0 = \bigcap_{n=1}^{\infty} B[x_n, r_n]$$

But $x_0 \in U_n$ for all n hence

$$x_0 \in W \cap U_1 \Rightarrow x_0 \in W \cap \Big(\bigcup_{n=1}^{\infty} U_n \Big)$$

□

Theorem 25.2 (Baire Category Theorem II)**.** If (X, d) is complete, then X is of 2nd category in itself.

Proof. Assume for a contradiction that $X = \bigcup_{n=1}^{\infty} A_n$ where A_n is nowhere dense (i.e. X is 1st category).
Then $X = \bigcup_{n=1}^{\infty} \bar{A}_n$. Let $U_n = \bar{A}_n^c$. Then U_n is open and dense. But $\big(\bigcup_{n=1}^{\infty} U_n\big)^c = \bigcup_{n=1}^{\infty} \bar{A}_n = X$ so $\bigcap_{n=1}^{\infty} U_n = \varnothing$, which is a contradiction by Baire Category Theorem I.
So X must be 2nd catgory. □

Example 25.2. $\mathbb{R}$ is of 2nd category and $\mathbb{R} \setminus \mathbb{Q}$ is of 2nd category (by Baire Category Theorem II since irrationals are complete), so $\mathbb{R} \setminus \mathbb{Q}$ is residual.

Theorem 25.3. $\mathbb{Q}$ is not a G_δ set.

Proof. Assume for a contradiction that $\mathbb{Q}$ is G_δ and $\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$ where U_n is open (it is dense since $U_n \supset \mathbb{Q}$). Let $F_n = U_n^c \Rightarrow F_n$ is closed and nowhere dense.
Let $\mathbb{Q} = \{r_1, r_2, \ldots\}$ and let $S_n = F_n \cup \{r_n\}$. Then S_n is closed and nowhere dense but $\mathbb{R} = \bigcup_{n=1}^{\infty} S_n$, which is a contradiction. □

Corollary 25.1. There is no $f : \mathbb{R} \to \mathbb{R}$ with $D(f) = \mathbb{R} \setminus \mathbb{Q}$.

Note. $\mathbb{Q} = F_\delta$ and $\mathbb{Q} = D(f)$ for some f.

Question 25.2. If $A \subset \mathbb{R}$ is F_δ is $A = D(f)$ for some f?

26 November 14, 2018

26.1 Uniform convergence at one point

Definition 26.1 (Uniform convergence at one point)**.** A sequence $\{f_n : (X, d_X) \to (Y, d_Y)\}$ of functions converges uniformly to $f_0 : X \to Y$. We say $\{f_n\}$, **converges uniformly at** $x_0 \in X$ if for each $\epsilon > 0$ there exists $\delta > 0$ and $N_0 \in \mathbb{N}$ such that if $d_X(x, x_0) < \delta$ and $m, n \geq N_0$ then $d_Y(f_n(x), f_m(x)) < \epsilon$.

Remark 26.1. This is like the Cauchy criterion for functions.

Theorem 26.1. Assume $f_n \to f_0$ pointwise on X and uniformly at $x_0 \in X$.
If each f_n is continuous at x_0 then f_0 is continuous at x_0.

Theorem 26.2. Let $\{f_n : (a, b) \to \mathbb{R}\}$ be a sequence of continuous functions converging pointwise to f_0. Then there exists $x_0 \in (a, b)$ such that $f_n \to f_0$ uniformly at x_0.

Proof. Assume $f_n \to f_0$ on (a, b).

Claim. We claim there exists $[\alpha_1, \beta_1] \subset (a, b)$ and $N_1 \in \mathbb{N}$ such that if $x \in [\alpha_1, \alpha_1]$ and $n, m \geq N$ then $|f_n(x) - f_m(x)| \leq 1$.
Suppose our claim fails. Then there exists $t_1 \in (a, b]$ and $n_1, m_1 \in \mathbb{N}$ such that

$$|f_{n_1}(t_1) - f_{m_1}(t_1)| > 1$$

since $f_{n_1} - f_{m_1}$ is continuous, there exists an open interval $I_1 \subset \bar{I}_1 \subset (a, b)$ such that $|f_{n_1}(x) - f_{m_1}(x)| > 1$ for all $x \in I$.
Similarly we can find $t_2 \in I_1$, $n_2, m_2 \geq \max\{n_1, m_1\}$ such that

$$|f_{n_2}(t_2) - f_{m_2}(t_2)| > 1$$

This gives an open interval $I_2 \subset \bar{I}_2 \subset I_1 \subset \bar{I}_1 \subset (a, b)$.
We get a sequence $\{I_k\}$ of open intervals and $\{n_k\}, \{m_k\}$ such that $I_{k+1} \subset \bar{I}_{k+1} \subset I_k$, $n_{k+1}, m_{k+1} \geq \max\{n_k, m_k\}$ and if $x \in I_k$ then $|f_{n_k}(x) - f_{m_k}(x)| > 1$.
By the Nested Interval Theorem $\bigcap_{k=1}^{\infty} \bar{I}_k \neq \varnothing$.
If $x_* \in \bigcap_{k=1}^{\infty} \bar{I}_k$ then $|f_{n_k}(x_*) - f_{m_k}(x_*)| \geq 1$.
But since $\{f_n(x_*)\}$ is Cauchy (since it has a limit on $x_* \in (a, b)$) so this is impossible.
Hence our claim holds

In a similar manner we find a sequence $\{[\alpha_k, \beta_k]\}$ $(\alpha_k < \beta_k)$ such that

$$(\alpha_{k+1}, \beta_{k+1}) \subseteq [\alpha_{k+1}, \beta_{k+1}] \subset (\alpha_k, \beta_k) \subset (a, b)$$

and a sequence $N_1 < N_2 < \ldots < N_k < \ldots$ such that if $x \in [\alpha_k, \beta_k]$ and if $m, n \geq N_k$ then $|f_n(x) - f_m(x)| < \frac{1}{k}$.
Let $x_0 = \bigcap_{k=1}^{\infty} [\alpha_k, \beta_k]$. Given $\epsilon > 0$ choose k so that $\frac{1}{k} < \epsilon$. If $x \in [\alpha_{k+1}, \beta_{k+1}] \subset (\alpha_k, \beta_k) \subset [\alpha_k, \beta_k]$ $n, m \geq N_k$ then $|f_n(x) - f_m(x)| \leq \frac{1}{k} < \epsilon$.
If $\delta > 0$ is such that $(x_0 - \delta, x_0 + \delta) \subset (\alpha_k, \beta_k)$ (possible since (α_k, β_k) is an open ball), then if $x \in (x_0 - \delta, x_0 + \delta)$ we have $|f_n(x) - f_m(x)| < \epsilon$. □

Corollary 26.1. Assume that $\{f_n\} \subset C([a, b])$ (or $C_b(\mathbb{R})$) converges pointwise to f_0. Then f_0 is continuous on a residual set in $[a, b]$ (or in $\mathbb{R}$).

Proof. The previous theorem shows that continuity points are dense. We also know they are G_δ. □

Corollary 26.2. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable then f' is continuous on a dense G_δ.

Proof. Let $g_n(x) = \frac{f(x+h)-f(x)}{\frac{1}{n}}$. Note that $g_n \to f^{-1}$ pointwise on $\mathbb{R}$ so by our previous theorem f' is continuous on a dense G_δ. □

26.2 Compactness

Definition 26.2 (Cover and subcover)**.** Given (X,d), an **(open) cover** of X is a collection $\{U_\alpha\}_{\alpha\in I}$ of open sets with $X = \bigcup_{\alpha\in I} U_\alpha$.
A **subcover** is a subcollection $\{U_\alpha\}_{\alpha\in J\subseteq I}$ such that $X = \bigcup_{\alpha\in J} U_\alpha$.
If $A \subset X$, $\{U_\alpha\}_{\alpha\in I}$ covers A if $A \subset \bigcup_{\alpha\in I} U_\alpha$ or equivalent if $\{U_\alpha \cap A\}_{\alpha\in I}$ is a cover of (A, d_A).

Definition 26.3 (Compact)**.** We say that (X,d) is **compact** if and only if each cover $\{U_\alpha\}_{\alpha\in I}$ of X has a **finite** subcover.
We say that $A \subset X$ is compact if every cover $\{U_\alpha\}_{\alpha\in I}$ of A has a finite subcover ((A, d_A) is compact).

Example 26.1. We note that $[0,1] \subset \mathbb{R}$ is compact.
$(0,1) \subset \mathbb{R}$ is not compact (let our subcover be $U_n = (\frac{1}{n}, 1)$).

Theorem 26.3 (Heine-Borel)**.** $A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded (NB: this holds only for $\mathbb{R}^n$).

Proposition 26.1. let $A \subseteq (X,d)$ be compact, then A is closed and bounded.

Proof. Suppose A is not closed.
Let $x_0 \in \text{bdy}(A) \setminus A$. Then if $U_n = (B[x_0, \frac{1}{n}])^c$ then $A \subseteq \bigcup_{n=1}^\infty U_n$ and $\{U_n\}_{n=1}^\infty$ has no finite subcover.
Suppose A is not bounded. Let $x_0 \in X$. Let $U_n = B(x_0, n)$ so $A \subset \bigcup_{n=1}^\infty U_n = X$. But $\{U_n\}_{n=1}^\infty$ has no finite subcover. □

Example 26.2. Let $A = \{\{x_n\} \subset l_\infty \mid \|\{x_n\}\|_\infty \leq 1\}$. Note that A is closed and bounded.
Let $U_{\{x_n\}} = B(\{x_n\}, \frac{1}{2})$ so $A \subset \bigcup_{\{x_n\}\in A} U_{\{x_n\}}$.
Consider $S = \{\{x_n\} \mid x_n = 1 \text{ or } 0\}$. Note that $|S \cap B(\{x_n\}, \frac{1}{2}| \leq 1$ so $S \subset A$ but S has no finite subcover.
This is problem since closed and bounded sets are not compact in general metric spaces.
Therefore we only have compact $\Rightarrow$ closed and bounded and not necessarily the converse.

27 November 16, 2018

27.1 Closed sets are compact in a closed metric space

Proposition 27.1. If (X,d) is compact and A is closed, then A is compact.

Proof. Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of A. Then $\{U_\alpha\}_{\alpha\in I} \cup \{A^c\}$ is an open cover of X.
There exists $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ such that $\left(\bigcup_{k=1}^n U_{\alpha_k}\right) \cup A^c = X$ thus $A \subset \left(\bigcup_{k=1}^n U_{\alpha_k}\right)$. □

27.2 Sequentially compact and Bolzano-Weierstrass Property

Two variants of compactness include:

Definition 27.1 (Sequentially compact)**.** $A \subseteq (X, d)$ is **sequentially compact** if every sequence $\{x_n\} \subset A$ has a subsequence $\{x_{n_k}\}$ with $x_{n_k} \to x_0 \in A$.

Definition 27.2 (Bolzano-Weierstrass Property (BWP))**.** A has the **Bolzano-Weierstrass Property** if every infinite subset of A has a limit point in A.

Exercise 27.1. Show that if $A \subset \mathbb{R}^n$ then A is compact iff A is sequentially compact.

Theorem 27.1. Let (X, d) be a metric space. Then TFAE:

1. (X, d) is sequentially compact.
2. (X, d) has the Bolzano-Weierstrass Property.

Proof. $1 \Rightarrow 2$ Assume (X, d) is sequentially compact. Let $S \subset X$ be infinite. Then we can find a sequence $\{x_n\} \subset S$ with $x_n \neq x_m$ if $n \neq m$. By sequential compactness $x_n \to x_0 \in S$ hence x_0 is a limit point of S.

$2 \Rightarrow 1$ Let $\{x_n\} \subset X$. If $\{x_n\}$ is not infinite then there exists $\{x_{n_k}\}$ with $x_{n_{k_1}} = x_{n_{k_2}}$ for all k_1, k_2. Then $x_{n_k} \to x_{n_{k_0}}$ for any k_0.

If $\{x_n\}$ is infinite, then by BWP $\{x_n\}$ has a limit point x_0. Let $x_{n_1} \in B(x_0, 1)$. We can choose $n_2 > n_1$ with $x_{n_2} \in B(x_0, \frac{1}{2})$. Choose $x_{n_{k+1}} > x_{n_k}$ so that $x_{n_{k+1}} \in B(x_0, \frac{1}{k+1})$ so $\{x_{n_k}\}$ converges to x_0.

□

27.3 Finite Intersection Property

Definition 27.3 (Finite Intersection Property (FIP))**.** A collection $\{A_\alpha\}_{\alpha \in I}$ of subsets of X has the **Finite Intersection Property** if $\bigcap_{n=1}^{n} A_i \neq \varnothing$ for all finite subcollections $\{A_1, \ldots, A_n\}$.

Example 27.1. $F_n = [n, \infty)$. Note that $\{F_n\}_{n=1}^{\infty}$ has the FIP but $\bigcap_{n=1}^{\infty} F_n = \varnothing$.

Theorem 27.2. Let (X, d) be a metric space. Then TFAE

1. (X, d) is compact
2. If $\{F_\alpha\}_{\alpha \in I}$ is a non-empty collection of closed sets with the FIP, then $\bigcap_{\alpha \in I} \neq \varnothing$.

Proof. $1 \Rightarrow 2$ Assume that (X, d) is compact and that $\{F_\alpha\}_{\alpha \in I}$ is a non-empty collection of closed sets with $\bigcap_{\alpha \in I} F_\alpha = \varnothing$ (we show the contrapositive of 2).

Let $U_\alpha = F_\alpha^c$. Then $X = \bigcup_{\alpha \in I} U_\alpha$. Hence there exists $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ with $X = \bigcup_{i=1}^{n} U_{\alpha_i}$ thus $\bigcap_{i=1}^{n} F_{\alpha_i} = \varnothing$.

$2 \Rightarrow 1$ We show the contrapositive.

Suppose $\{U_\alpha\}_{\alpha \in I}$ is a cover of X with no finite subcover.

For any $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ we have $X \setminus \bigcup_{i=1}^{n} U_{\alpha_i} \neq \varnothing$. Then $\bigcap_{i=1}^{n} U_{\alpha_i}^c \neq \varnothing$. Hence $\{F_\alpha\}_{\alpha \in I}$ ($F_\alpha = U_\alpha^c$) is a collection of closed sets with the FIP but $\bigcap_{\alpha \in I} F_\alpha = \varnothing$.

□

Corollary 27.1. Let (X, d) is compact and if $\{F_n\}_{n=1}^{\infty}$ is a sequence of non-empty closed sets with $F_{n+1} \subset F_n$, then $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$.

Corollary 27.2. If (X, d) is compact then (X, d) is complete.

27.4 ϵ-net

Note. Compactness tells us that we can finitely approximate some metric space (X, d) for some $\epsilon > 0$.
Assume that (X, d) is compact. Let $\epsilon > 0$ where $X = \bigcup_{x \in X} B(x, \epsilon)$. Then there exists $x_1, \dots, x_n$ such that $X = \bigcup_{i=1}^{n} B(x_i, \epsilon)$.

Definition 27.4 (ϵ-net)**.** Given $A \subset (X, d)$ and an $\epsilon > 0$. An **ϵ-net** for A is a finite set $\{x_1, \dots, x_n\}$ such that $A \subset \bigcup_{i=1}^{n} B(x_i, \epsilon)$.

Definition 27.5 (Totally bounded)**.** We say A is **totally bounded** if there exists an ϵ-net for every $\epsilon > 0$.

Theorem 27.3. If (X, d) is compact, (X, d) is totally bounded.

Example 27.2. $S = \{\{x_n\} \in l_\infty \mid \|\{x_n\}\|_\infty \le 1\}$. S is bounded but it has no $\frac{1}{2}$-net (as demonstrated before).

Proposition 27.2. $A \subset (X, d)$ is totally bounded if and only if $\bar{A}$ is totally bounded.

28 November 19, 2018

28.1 Compactness implies Bolzano-Weierstrass Property

Theorem 28.1. If (X, d) is compact, then (X, d) has the BWP.

Proof. Assume $S \subset X$ is infinite. Then there exists $\{x_n\}$ with $x_n \neq x_m$ if $n \neq m$.
Let $F_n = \{x_n, x_{n+1}, \dots, \}$. Then $F_{n+1} \subseteq F_n$ so $\{F_n\}$ has the FIP. Therefore there exists $x_0 \in \bigcap_{n=1}^{\infty} F_n$. Hence $B(x_0, \epsilon) \cap F_n \neq \varnothing$ for all $n \in \mathbb{N}$. In fact $B(x_0, \epsilon) \cap \{x_n\}$ must be infinite so $x_0 \in \mathrm{Lim}(S)$. $\square$

28.2 Sequentially compact implies complete and totally bounded

Proposition 28.1. If (X, d) is sequentially compact, then (X, d) is complete and totally bounded.

Proposition 28.2. Let $\{x_n\}$ be Cauchy. Then sequential compactness implies $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with $x_{n_k} \to x_0$. Hence $x_n \to x_0$ so (X, d) is complete.
Assume (X, d) is not totally bounded. Then there exists an $\epsilon_0 > 0$ with no finite ϵ_0-net.
Pick $x_1 \in X$. $B(x_1, \epsilon_0) \neq X$ so there exists $x_2 \in X \setminus B(x_1, \epsilon_0)$. Since $X \neq B(x_1, \epsilon_0) \cup B(x_2, \epsilon_0)$ then there exists $x_3 \in X \setminus \big(B(x_1, \epsilon_0) \cup B(x_2, \epsilon_0)\big)$.
We get $\{x_n\} \subset X$ with $d(x_n, x_m) \ge \epsilon_0$ for all $n \neq m$, then $\{x_n\}$ has no convergent subsequence, which is a contradiction of sequential compactness, so (X, d) must be totally bounded.

28.3 Sequential compactness is preserved under continuous functions

Theorem 28.2. If (X, d_X) is sequentially compact and if $f : (X, d_X) \to (Y, d_Y)$ is continuous, then $f(X)$ is seuqnetially compact.

Proof. Let $\{y_n\} \subset f(X)$. Let $x_n \in X$ so that $y_n = f(x_n)$. Then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0$. Hence $y_{n_k} = f(x_{n_k}) \to f(x_0) = y_0$. $\square$

Corollary 28.1 (Extreme value theorem)**.** If (X, d) is sequentially compact and if $f : X \to \mathbb{R}$ is continuous then there exists $c, d \in X$ with $f(c) \le f(x) \le f(d)$ for all $x \in X$.

Proof. $f(X)$ is sequentially compact in $\mathbb{R}$ by our theorem above so $f(X)$ is closed and bounded. Therefore $\mathrm{glb}(f(X)) \in f(X)$ and $\mathrm{lub}(f(X)) \in f(X)$ as desired. $\square$

28.4 Lebesgue's theorem

Theorem 28.3 (Lebesgue's theorem)**.** Let (X, d) be sequentially compact. Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of X. Then there exists $\epsilon > 0$ such that if $0 < \delta < \epsilon$ and $x_0 \in X$ then $B(x_0, \delta) \subset U_{\alpha_0}$ for some α_0.

Proof. Assume $U_{\alpha_0} = X$. Then any $\epsilon > 0$ works. We may assume $U_\alpha \neq X$ for any $\alpha \in I$.
Define $\phi : X \to \mathbb{R}$ by $\phi(x) = \sup\{\delta > 0 \mid B(x, \delta) \subset U_{\alpha_0} \text{ for some } \alpha_0\}$.
Then $\phi(x) > 0$. We also have $\phi(x) < \infty$ since (X, d) is bounded and $U_\alpha \neq X$ for any α.
For any $x, y \in X$ we have $\phi(x) \leq \phi(y) + d(x, y)$ by the triangle inequality (exercise: verify).
Therefore $|\phi(x) - \phi(y)| \leq d(x, y)$ so ϕ is (uniformly) continuous. Since (X, d) is sequentially compact there exists some $\epsilon > 0$ such that $\phi(x) \geq \epsilon$ for all x (by EVT).
If $0 < \delta < \epsilon$, then $B(x, \delta) \subset U_{\alpha_0}$ for some α_0. $\square$

Theorem 28.4 (Lebesgue-Borel)**.** Let (X, d) is a metric space. Then TFAE:

1. (X, d) is compact
2. (X, d) has the BWP
3. (X, d) is sequentially compact

Proof. We've shown all implications except for $3 \Rightarrow 1$.
Let $\{U_\alpha\}_{\alpha\in I}$ be a cover of X. Let ϵ_0 be as in Lebesgue's theorem. Fix $0 < \delta < \epsilon_0$.
Since (X, d) is totally bounded (by sequential compactness) we can find $\{x_1, \ldots, x_n\}$ with $X = \bigcup_{i=1}^n B(x_i, \delta)$.
For each $i = 1, \ldots, n$, $B(x_i, \delta) \subset U_{\alpha_i}$ for some $\alpha_i \in I$ by Lebesgue's theorem, so $\bigcup_{i=1}^n U_{\alpha_i} = X$ a finite subcover. $\square$

Remark 28.1. The ϵ in Lebesgue's theorem is called a **Lebesgue number** for our cover $\{U_\alpha\}_{\alpha\in I}$.

28.5 Complete and totally bounded implies compactness

Theorem 28.5. let (X, d) be a metric space. TFAE:

1. (X, d) is compact
2. (X, d) is complete and totally bounded

Proof. We've shown $1 \implies 2$ so it remains to show $2 \implies 1$.
We need only show (X, d) is sequentially compact. Assume $\{x_n\} \subset (X, d)$. Since (X, d) is totally bounded then there exists $y_1 \in X$ such that if $S_1 = B(y_1, 1)$ then S_1 contains infinitely many terms in $\{x_n\}$. We can find y_2 such that if $S_2 = B(y_2, \frac{1}{2})$ then S_2 contains infinitely many terms in $\{x_n\} \cap S_1$.
We can construct $\{S_k\}$ with $S_k = B(y_k, \frac{1}{k})$ and S_{k+1} contains infinite many terms in $\{x_n\}$ which are in $S_1 \cap S_2 \cap \ldots \cap S_k$.
Note $\operatorname{diam}(S_k) \to 0$. We can pick $n_1 < n_2 < \ldots < n_k < \ldots$ so that $x_{n_{k+1}} \in S_1 \cap \ldots \cap S_k$.
Assume $k, m \geq N$ then $x_{n_k}, x_{n_m} \in S_n$ where $d(x_{n_k}, x_{n_m}) \leq \operatorname{diam}(S_N)$ so $\{x_n\}$ is Cauchy. Since (X, d) is complete then $x_{n_k} \to x_0$ so (X, d) is sequentially compact. $\square$