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STAT 433/833 Course Notes

Stochastic Processes

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. These notes are my interpretation and transcription of the content covered in lectures. The instructor has not verified or confirmed the accuracy of these notes, and any discrepancies, misunderstandings, typos, etc. as these notes relate to course's content is not the responsibility of the instructor. If you spot any errors or would like to contribute, please contact me directly.

1 September 6, 2018

1.1 Example 1.2 solution

Use the definition of the Markov property to show that

$$\begin{aligned}&P(X_{n+1}=x_{n+1} \mid X_n=x_n, X_{n-1}=x_{n-1},\ldots,X_{n-k+1}=x_{n-k+1},X_{n-k-1}=x_{n-k-1},\ldots,X_0=x_0)\\ &\qquad = P(X_{n+1}=x_{n+1}\mid X_n=x_n), \quad k=1,2,\ldots,n\end{aligned}$$

(i.e. we are missing one past observation).

Solution. Applying the definition of conditional probability, our expression is equivalent to

$$\frac{P(X_{n+1}=x_{n+1},X_n=x_n,X_{n-1}=x_{n-1},\ldots,X_{n-k+1}=x_{n-k+1},X_{n-k-1}=x_{n-k-1},\ldots,X_0=x_0)}{P(X_n=x_n,X_{n-1}=x_{n-1},\ldots,X_{n-k+1}=x_{n-k+1},X_{n-k-1}=x_{n-k-1},\ldots,X_0=x_0)}=\frac{N}{D}$$

By the law of total probability

$$\begin{aligned}N &= \sum_{x_{n-k}\in S} P(X_{n+1}=x_{n+1},\ldots,X_{n-k}=x_{n-k},\ldots,X_0=x_0)\\ &= \sum_{x_{n-k}\in S} P(X_{n+1}=x_{n+1}\mid X_n=x_n,\ldots,X_{n-k}=x_{n-k},\ldots,X_0=x_0)\times P(X_n=x_n,\ldots,X_{n-k}=x_{n-k},\ldots,X_0=x_0)\end{aligned}$$

By the Markov property

$$\begin{aligned}&= P(X_{n+1}=x_{n+1}\mid X_n=x_n)\sum_{x_{n-k}\in S} P(X_n=x_n,\ldots,X_{n-k}=x_{n-k},\ldots,X_0=x_0)\\ &= P(X_{n+1}=x_{n+1}\mid X_n=x_n)P(X_n=x_n,\ldots,X_{n-k}\in S,\ldots,X_0=x_0)\end{aligned}$$

Since $X_{n-k}\in S$ is an event with probability 1

$$\begin{aligned}&= P(X_{n+1}=x_{n+1}\mid X_n=x_n)P(X_n=x_n,\ldots,X_{n-k+1}=x_{n-k+1},X_{n-k-1}=x_{n-k-1},\ldots,X_0=x_0)\\ &= P(X_{n+1}=x_{n+1}\mid X_n=x_n)\cdot D\end{aligned}$$

The result follow.

2 September 11, 2018

2.1 Section 1.2: Transitivity of communication relation

Prove that if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$ (and thus the communication relation "$\leftrightarrow$" is an equivalence relation).

Proof. $\exists n, m \in \mathbb{N}$ such that $P^{(n)}_{i,j} > 0$ and $P^{(m)}_{j,k} > 0$.
Note that

$$P^{(n+m)}_{i,k} = \sum_{l \in S} P^{(n)}_{i,l} P^{(m)}_{l,k} \geq P^{(n)}_{i,j} P^{(m)}_{j,k} > 0$$

Similarly we can show $k \to i$, thus $i \leftrightarrow k$. □

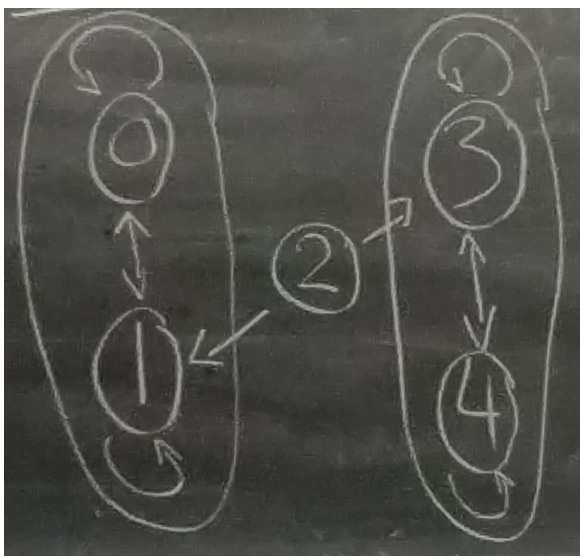
2.2 Example 1.3 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 0.1 & 0.9 \end{bmatrix} \end{array}$$

Use a state transition diagram to determine the equivalence classes.

Solution. We draw the following state transition diagram and note that there are three communication classes: $\{0,1\}, \{2\}, \{3,4\}$.



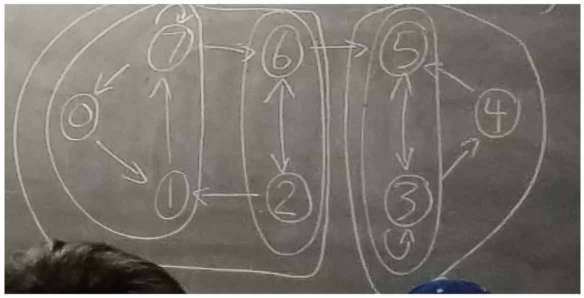
2.3 Example 1.4 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0.3 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

Use a state transition diagram to determine the equivalence classes.

Solution. We draw the following state transition diagram and note that there are two communication classes: $\{0, 1, 2, 6, 7\}, \{3, 4, 5\}$.



2.4 Example 1.5 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0.3 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

Use sample paths to prove that all states within the communication classes found in Example 1.4 communicate.

Solution. class $\{3, 4, 5\}$ Note that $P_{3,4}P_{4,5}P_{5,3} > 0$ i.e. the sample path $3 \to 4 \to 5 \to 3$ has positive probability, thus states 3,4, and 5 communicate since for any pair of states $i, j \in \{3, 4, 5\}$, $\exists n_{i,j} \leq 3$ such that $P_{i,j}^{(n_{i,j})} > 0$.

class $\{0, 1, 2, 6, 7\}$ We have sample path $0 \to 1 \to 7 \to 6 \to 2 \to 1 \to 7 \to 0$ with positive probability.

By a similar argument as above the five states communicate.

2.5 Theorem 1.1 proof: periodicity is a class property

Theorem 2.1. If $i \leftrightarrow j$ then $d(i) = d(j)$ (equal periods).

Proof. Since $i \leftrightarrow j$, then $\exists n, m \in \mathbb{N}$ such that $P_{i,j}^{(n)} > 0$ and $P_{j,i}^{(m)} > 0$.
$\forall L \in \mathbb{Z}^+$ s.t. $P_{j,j}^{(L)} > 0$, we have

$$\begin{aligned} P_{i,i}^{(m+n+L)} &= \sum_{k \in S} P_{i,k}^{(n)} P_{k,i}^{(m+L)} \\ &= \sum_{k \in S} \sum_{l \in S} P_{i,k}^{(n)} P_{k,l}^{(L)} P_{l,i}^{(m)} \\ &\geq P_{i,j}^{(n)} P_{j,j}^{(L)} P_{j,i}^{(m)} \\ &> 0 \end{aligned}$$

Thus $d(i)$ divides $n+m+L$.
Note that $P_{i,i}^{(n+m)} = \sum_{k \in S} P_{i,k}^{(n)} P_{k,i}^{(m)} \geq P_{i,j}^{(n)} P_{j,i}^{(m)} > 0$, thus $d(i)$ divides $n+m$.
Therefore $d(i)$ divides $(n+m+L)-(n+m) = L$ $\forall L$ s.t. $P_{j,j}^{(L)} > 0$, thus $d(i)$ divides $gcd\{L \in \mathbb{Z}^+ \mid P_{j,j}^{(L)} > 0\} = d(j)$.
Similarly, $d(j)$ divides $d(i)$, thus $d(i) = d(j)$. □

3 September 13, 2018

3.1 Example 1.6 solution

Given the DTMC with TPM

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} \end{array}$$

Show that $d(i) = 1$ despite the fact that $P_{i,i}^{(1)} = P_{i,i} = 0$ for $i = 0, 1, 2$.

Solution. Consider state 0 where we have

$$\begin{aligned} P_{0,0}^{(1)} &= 0 \\ P_{0,0}^{(2)} &= \sum_{k \in S} P_{0,k} P_{k,0} \geq P_{0,1} P_{0,1} = \frac{1}{4} > 0 \\ P_{0,0}^{(3)} &= \sum_{k \in S} P_{0,k} P_{k,l} P_{l,0} \geq P_{0,1} P_{1,2} P_{2,0} = \frac{1}{8} > 0 \end{aligned}$$

Therefore $d(0) = \gcd\{2, 3, \ldots\} = 1$.

Since the sample path $0 \to 1 \to 2 \to 0$ has positive prob., all of the states communicate and the DTMC is irreducible, thus $d(2) = d(1) = d(0) = 1$ as well.

3.2 Theorem 1.2 proof: transience/recurrence are class properties

Theorem 3.1. Transience and recurrence are class properties i.e. if $i \leftrightarrow j$ and i is recurrent, then j is recurrent.

Proof. It clearly holds if $i = j$, so assume $i \neq j$. $i \leftrightarrow j$ so $\exists m, n \in \mathbb{Z}^+$ s.t. $P_{j,i}^{(m)} > 0$ and $P_{i,j}^{(n)} > 0$.
Note that

$$\begin{aligned} \sum_{n=1}^{\infty} P_{j,j}^{(n)} &\geq \sum_{l=m+n+1}^{\infty} P_{j,j}^{(l)} \\ &\geq \sum_{l=m+n+1}^{\infty} P_{j,i}^{(m)} P_{i,i}^{(l-m-n)} P_{i,j}^{(n)} \\ &= P_{j,i}^{(m)} P_{i,j}^{(n)} \sum_{l=m+n+1}^{\infty} P_{i,i}^{(l-m-n)} \\ &= P_{j,i}^{(m)} P_{i,j}^{(n)} \sum_{L=1}^{\infty} P_{i,i}^{(L)} \\ &= \infty \end{aligned}$$

since i is recurrent thus $\sum_{L=1}^{\infty} P_{i,i}^{(L)} = \infty$, thus state j is recurrent.
Transience is proven similarly. □

3.3 Theorem 1.3 proof: recurrent classes with states i, j imply $f_{i,j} = 1$

Theorem 3.2. If $i \leftrightarrow j$ and state i is recurrent, then

$$f_{i,j} = P(\text{DTMC ever makes future visit to state } j \mid X_0 = i) = 1$$

Proof. If $i = j$, then result follows by definition of recurrence.
Let $i \neq j$. Since $i \leftrightarrow j$, then $\exists n \in \mathbb{Z}^+$ s.t. $P_{j,i}^{(n)} > 0$.
State j is recurrent by theorem 1.2 so $f_{j,j} = 1$.
Assume that $f_{i,j} < 1$ for a contradiction.

Method 1 Note that

$$\begin{aligned}
f_{j,j} &= P(\text{DTMC ever makes future visit to j} \mid X_0 = j) \\
&= 1 - P(\text{never visits j} \mid X_0 = j) \\
&\leq 1 - P_{j,i}^{(n)}(1 - f_{i,j}) && P(\text{never visits j} \mid X_0 = j) \geq P_{j,i}^{(n)}(1 - f_{i,j}) \\
&< 1
\end{aligned}$$

which is a contradiction, so $f_{i,j} < 1$.

Method 2 Note that

$$\begin{aligned}
&\{X_n = i, \text{never visits j after i}\} \subseteq \{\text{never returns to state j}\} \\
\Rightarrow &P(X_n = i, \text{never visits j after i } \mid X_0 = j) \leq P(\text{never returns to j} \mid X_0 = j) \\
\Rightarrow &P_{j,i}^{(n)}(1 - f_{i,j}) \leq 1 - f_{j,j} \\
\Rightarrow &P_{j,i}^{(n)}(1 - f_{i,j}) \leq 0
\end{aligned}$$

which is a contradiction since $P_{j,i}^{(n)} > 0$ and $1 - f_{i,j} > 0$, so we must have $f_{i,j} = 1$.

□

3.4 Theorem 1.4 proof

Theorem 3.3. If state i is recurrent and state i does not communicate with state j, then $P_{i,j} = 0$.

Proof. Assume $i \neq j$. State i is recurrent so $f_{i,i} = 1$.
Assume that $P_{i,j} > 0$ for a contradiction so $i \to j$. Since i and j don't communicate and $i \to j$, then i is not accessible from j ($j \not\to i$).

Method 1 Note that

$$\begin{aligned}
f_{i,i} &= P(\text{DTMC ever makes future visit to i} \mid X_0 = i) \\
&= 1 - P(\text{never visits i} \mid X_0 = i) \\
&\leq 1 - P_{i,j} && P(\text{never visits i} \mid X_0 = i) \geq P_{i,j} \text{ since } j \not\to i \\
&< 1
\end{aligned}$$

which is a contradiction so $P_{i,j} = 0$.

Method 2 Note that

$$\begin{aligned}
&\{X_1 = j, \text{never returns to i after j}\} \subseteq \{\text{never returns to state i}\}\\
\Rightarrow &P(X_1 = j, \text{never visits j after i } \mid X_0 = i) \leq P(\text{never return to i} \mid X_0 = i)\\
\Rightarrow &P_{i,j} \leq 1 - f_{i,i}
\end{aligned}$$

where the last line follows since i is not accessible from j.

Since $f_{i,i} = 1$, we have $P_{i,j} \leq 0$ which is a contradiction, thus $P_{i,j} = 0$.

□

4 September 18, 2018

4.1 Example 1.7 solution

Consider the DTMC with one-step transition probabilities

$$\begin{aligned}
P_{1,j} = \frac{1}{2^j} \quad j = 2^n \quad n \in Z^+\\
P_{i,i-1}1 \quad i = 2, 3, 4, \ldots
\end{aligned}$$

Show that all states are null recurrent and check that a stationary distribution does not exist.

Solution. It is clear that evey state communicates an the DTMC is irreduiclb.e By Theorem 1.5, we only need to check one state for null reucrrence.
For state 1, note that

$$\begin{aligned}
f_{1,1} = \sum_{n=1}^{\infty} f_{1,1}^{(n)} &= \sum_{n=1}^{\infty} P_{1,n}\\
&= \sum_{m=1}^{\infty} P_{1,2^m}\\
&= \sum_{m=1}^{\infty} \frac{1}{2^m}\\
&= \frac{\frac{1}{2}}{1 - \frac{1}{2}}\\
&= 1
\end{aligned}$$

So state 1 is indeed recurrent. To show it is null recurrent, we look at its mean recurrent time m_1

$$
\begin{aligned}
m_1 = \sum_{n=1}^{\infty} n f_{1,1}^{(n)} \\
&= \sum_{n=1}^{\infty} n P_{1,n} \\
&= \sum_{m=1}^{\infty} 2^m P_{1,2^m} \\
&= \sum_{m=1}^{\infty} 2^m \frac{1}{2^m} \\
&= \sum_{m=1}^{\infty} 1 \\
&= \infty
\end{aligned}
$$

State 1 and hence the entire DTMC is null recurrent.
Does a stationary distribution exist? We observe $p = pP$ where $p = (p_1, p_2, \ldots)$ by vector-matrix multiplication

$$
\begin{aligned}
p_1 &= p_2 \\
p_2 &= \frac{1}{2}p_1 + p_3 \\
&\vdots \\
p_{2^m} &= \frac{1}{2^m}p_1 + p_{2^m+1} \qquad m \in \mathbb{Z}^+
\end{aligned}
$$

Also note that

$$
p_i = p_{i+1} \qquad i \neq 2^m \text{ for some } m \in \mathbb{Z}^+
$$

thus we have

$$
p_{2^m+1} = p_{2^m+2} = \ldots = p_{2^{m+1}-2} = p_{2^{m+1}-1} = p_{2^{m+1}}
$$

So our p vector is now

$$
\begin{aligned}
p &= (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, \ldots) \\
&= (p_1, p_2, p_4, p_4, p_8, p_8, p_8, p_8, p_{16}, \ldots)
\end{aligned}
$$

If we expand out our p_{2^m}

$$\begin{aligned}
p_{2^m} &= \frac{1}{2^m}p_1 + p_{2^m+1} \\
&= \frac{1}{2^m}p_1 + \frac{1}{2^{m+1}}p_1 + p_{2^m+2} \\
&= p_1 \sum_{l=m}^{\infty} \big(\frac{1}{2}\big)^l \\
&= p_1 \big(\frac{1}{2}\big)^m \sum_{n=0}^{\infty} \big(\frac{1}{2}\big)^n \\
&= p_1 \big(\frac{1}{2}\big)^{m-1}
\end{aligned}$$

We need $pe' = \sum_{n=1}^{\infty} p_n = 1$. Note that

$$\begin{aligned}
\sum_{n=1}^{\infty} p_n &= p_1 + \sum_{m=1}^{\infty} \sum_{l=2^{m-1}+1}^{2^m} p_l && \text{recall we have } 2^{m-1} \text{ of each } p_{2^m} \\
&= p_1 + \sum_{m=1}^{\infty} \sum_{l=2^{m-1}+1}^{2^m} p_{2^m} \\
&= p_1 + \sum_{m=1}^{\infty} 2^{m-1} \frac{1}{2^{m-1}} p_1 && 2^m - 2^{m-1} = 2^{m-1}(2-1) = 2^{m-1} \\
&= p_1 + \sum_{m=1}^{\infty} p_1 \\
&= \sum_{m=0}^{\infty} p_1
\end{aligned}$$

which is 0 if $p_1 = 0$ or ∞ if $p_1 > 0$. It can't hold that $pe' = 1$ while satisfying $p = pP$, thus a stationary distribution does not exist.

4.2 Example 1.8

Consider a DTMC with TPM

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Show that more than one stationary distribution exists.

Solution. We have two equivalence classes: $\{0,2\}$ and $\{1\}$ and they are positive recurrent:

$$\begin{aligned}
&P(N_1 = 1 \mid X_0 = 1) = 1 \Rightarrow m_1 = 1 < \infty \\
&P(N_j = 2 \mid X_0 = j) = 2 \Rightarrow m_1 = 2 < \infty \qquad j = 0, 2
\end{aligned}$$

Consider $p = (\frac{1}{2}, 0, \frac{1}{2})$ and $q = (0, 1, 0)$.

For the former:

$$pP = (\frac{1}{2}, 0, \frac{1}{2}) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = (\frac{1}{2}, 0, \frac{1}{2}) = p$$

and

$$pe' = \frac{1}{2} + \frac{1}{2} = 1$$

Similarly for q, thus both p and q are both stationary.
In fact, any convex combination $\alpha p + (1-\alpha)q$, $\alpha \in [0,1]$ is a stationary distribution.
That is any $(\frac{\alpha}{2}, 1-\alpha, \frac{\alpha}{2})$ is stationary, thus there are infinitely many stationary distributions.

4.3 Theorem 1.7: Irreducible DTMC positive recurrent iff stationary distribution

Theorem 4.1. An irreducible DTMC is positive recurrent iff a stationary distribution exists.

Proof. Proof deferred. □

4.4 Uniqueness of stationary distributions

Theorem 4.2. Once we have theorem 1.7 we can prove uniqueness of stationary distributions i.e. the stationary distribution will not be unique if the DTMC has more than one positive recurrent equivalence class.

Proof. Consider a DTMC with two positive recurrent classes c_1, c_2. We can write the TPM as

$$P = \begin{array}{c c} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} c_1 \\ c_2 \end{array} & \left[\begin{array}{cc} P_1 & 0 \\ 0 & P_2 \end{array} \right] \end{array}$$

where P_1 and P_2 are irreducible TPMs when considered in isolation.
So if we had A DTMC $\{y_n, n \in \mathbb{N}\}$ with TPM P_1 then $\{y_n, n \in \mathbb{N}\}$ would be irreducible and positive recurrent i.e. $\exists p_1$ such that $p_1 P_1 = p_1$ and $p_1 e' = 1$.
Similarly $\exists p_2$ for P_2.
Consider

$$[\alpha p_1, (1-\alpha)p_2] = [(\alpha p_{1,1}, \ldots, \alpha p_{1,n}), ((1-\alpha)p_{2,1}, \ldots, (1-\alpha)p_{2,n})]$$

thus we have

$$\begin{aligned} pP &= [\alpha p_1, (1-\alpha)p_2] \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \\ &= [\alpha p_1 P_1, (1-\alpha)p_2 P_2] \\ &= [\alpha p_1, (1-\alpha)p_2] \\ &= p \end{aligned}$$

And note $pe' = \alpha p_1 e' + (1-\alpha)p_2 e' = \alpha + (1-\alpha) = 1$.
Thus we do not have a unique stationary distribution. □

5 September 20, 2018

5.1 Example 1.11 solution

Consider a DTMC with TPM

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{2}{3} & \frac{1}{12} \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Solve the limit TPM $\lim_{n\to\infty} P^{(n)}$. Does the limiting distribution of X_n depend on the initial distribution?

Solution. Clearly the equivalence classes are $\{0\}$ and $\{2\}$ (recurrent) and $\{1\}$ (transient).
$P_{0,0} = P_{2,2} = 1$ so states 0 and 2 are *absorbing states*.
We'd like to build up our matrix $\lim_{n\to\infty} P^{(n)}$.
Note $P_{0,0}^{(n)} = P_{2,2}^{(n)} = 1$ for all $n \in \mathbb{N}$ thus

$$\lim_{n\to\infty} P_{0,0}^{(n)} = \lim_{n\to\infty} P_{2,2}^{(n)} = 1$$

Thus

$$\lim_{n\to\infty} P_{0,1}^{(n)} = \lim_{n\to\infty} P_{0,2}^{(n)} = \lim_{n\to\infty} P_{2,0}^{(n)} = \lim_{n\to\infty} P_{2,1}^{(n)} = 0$$

Note that

$$\lim_{n\to\infty} P_{1,1}^{(n)} = \lim_{n\to\infty} (\frac{2}{3})^n = 0$$

Also

$$\begin{aligned}
P_{1,0}^{(n)} &= P(X_n = 0 \mid X_0 = 1) \\
&= \sum_{m=1}^{n} P(\text{DTMC first visits state 0 at time m} \mid X_0 = 1) \\
&= \sum_{m=1}^{n} P(X_n = 0, X_{n-1} = 0, \ldots, X_m = 0 \mid X_0 = 1) \\
&= \sum_{m=1}^{n} P(X_n = 0, X_{n-1} = 0, \ldots, X_m = 0, X_{m-1} = 1, \ldots, X_1 = 1 \mid X_0 = 1) \\
&= \sum_{m=1}^{n} P(X_n = 0 \mid X_m = 0) P(X_m = 0 \mid X_{m-1} = 1) P(X_{m-1} = 1 \mid X_{m-2} = 1) \ldots P(X_1 = 1 \mid X_0 = 1) \\
&= \sum_{m=1}^{n} 1 \cdot \frac{1}{4} \cdot (\frac{2}{3})^{m-1} \\
&= \frac{1}{4} \sum_{l=0}^{n-1} (\frac{2}{3})^l \\
&= \frac{1}{4} (\frac{1 - (\frac{2}{3})^n}{1 - \frac{2}{3}}) \\
&= \frac{3}{4} (1 - (\frac{2}{3})^n)
\end{aligned}$$

Similarly $P^{(n)}_{1,2} = \frac{1}{4}(1 - \left(\frac{2}{3}\right)^n)$.
Taking the limit of either, we get $\lim_{n\to\infty} P^{(n)}_{1,0} = \frac{3}{4}$ and $\lim_{n\to\infty} P^{(n)}_{1,2} = \frac{1}{4}$, thus we have

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

5.2 Theorem 1.8: Limiting probability of transient states

Theorem 5.1. For any state $i \in S$ ad transient state $j \in S$ of a DTMC, $\lim_{n\to\infty} P^{(n)}_{i,j} = 0$.

Proof. Note that

$$\begin{aligned}
\sum_{n=1}^{\infty} P^{(n)}_{i,j} &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} f^{(k)}_{i,j} P^{(n-k)}_{j,j} \\
&= \sum_{k=1}^{\infty} f^{(k)}_{i,j} \sum_{n=k}^{\infty} P^{(n-k)}_{j,j} \\
&= \sum_{k=1}^{\infty} f^{(k)}_{i,j} \sum_{l=0}^{\infty} P^{(l)}_{j,j} \\
&= f_{i,j}(1 + \sum_{l=1}^{\infty} P^{(l)}_{j,j}) \\
&\le 1 + \sum_{l=1}^{\infty} P^{(l)}_{j,j} && f_{i,j} \le 1 \\
&< \infty && \text{since } j \text{ is transient } \sum_{l=1}^{\infty} P^{(l)}_{j,j} < \infty
\end{aligned}$$

Therefore $\lim_{n\to\infty} P^{(n)}_{j,j} = 0$ by the nth term test for infinite series (i.e. otherwise the sum above will be infinite). □

6 September 25, 2018

6.1 Example 1.12 solution

Consider the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

which clearly has period $d = d(0) = d(1) = 2$. Confirm that the extended BLT for periodic DTMCs holds true for this DTMC.

Solution. Clearly $m_0 = m_1 = 2$. We will check the LHS and RHS of the extended BLT equation.

Note that

$$\begin{aligned}
\lim_{n\to\infty} P^{(2n)} &= \lim_{n\to 0} \Pi_{i=1}^{n} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \lim_{n\to 0} \Pi_{i=1}^{n} I_{2\times 2} \\
&= I_{2\times 2}
\end{aligned}$$

so $\lim_{n\to\infty} P_{j,j}^{(2n)} = 1 = \frac{d}{m} = 1$.

6.2 Theorem 1.10: null recurrence has limiting probability of 0

Theorem 6.1. If state i is null recurrent, then $\lim_{n\to\infty} P_{i,j}^{(n)} = 0$ for any state j.

Proof. **Case 1:** $i = j$ By the extended BLT $\lim_{n\to\infty} P_{i,i}^{(nd)} = \frac{d}{m_i} = 0$ since i is null recurrent so $m_i = \infty$.

Also $P_{i,i}^{(k)} = 0$ if k is not divisible by d, thus $\lim_{n\to\infty} P_{i,i}^{(n)} = 0$.

Case 2: $i \neq j$ $i \not\leftrightarrow j$ Since i is recurrent and it does not communicate with state j, then $P_{i,j}^{(n)} = 0 \; \forall n \in Z^+$ so the statement holds.

$i \leftrightarrow j$ Since i, j communicate, j is also null recurrent.

$$\begin{aligned}
\lim_{n\to\infty} P_{i,j}^{(n)} &= \lim_{n\to\infty} \sum_{k=1}^{n} f_{i,j}^{(k)} P_{j,j}^{(n-k)} \\
&= \lim_{n\to\infty} \sum_{k=1}^{\infty} f_{i,j}^{(k)} P_{j,j}^{(n-k)} && P_{j,j}^{(s)} = 0 \text{ if } s < 0 \\
&= \sum_{k=1}^{\infty} f_{i,j}^{(k)} (\lim_{n\to\infty} P_{j,j}^{(n-k)}) \\
&= 0 && \lim_{n\to\infty} P_{j,j}^{(n-k)} = 0 \text{ by case 1 since j is null recurrent}
\end{aligned}$$

Remark 6.1. Note that case 2(b) implies that $\lim_{n\to\infty} P_{i,j}^{(n)} = 0$ if j is null recurrent (regardless of i).

Remark 6.2. We can exchange order of limit and infinite summation by applying the dominated convergence theorem (DCT) $Y_n = P_{j,j}^{(n-k)}$ with probability $f_{i,j}^{(k)}$, $Y = 0$ and $Z = 1$.

□

6.3 Theorem 1.5: positive/null recurrence is a class property

Theorem 6.2. If $i \leftrightarrow j$ and state i is positive recurrent, then state j is also positive recurrent (null/positive recurrence is a class property).

Proof. Since $i \leftrightarrow j$, $d(i) = d(j) = d$. Since i is positive recurrent ($m_i < \infty$) by the extended LT $\lim_{n\to\infty} P_{i,i}^{(nd)} = \frac{d}{m_i} > 0$.

Since $i \leftrightarrow j$, $\exists a, b \in \mathbb{Z}^+$ such that $P_{j,i}^{(a)} > 0$ and $P_{i,j}^{(b)} > 0$, thus $P_{j,j}^{(a+b)} \geq P_{j,i}^{(a)} \cdot P_{i,j}^{(b)} > 0$, so d must divide $a + b$ i.e. $\exists k \in Z^+$ s.t. $a + b = kd$.

Let $l = n - k \to n = l + k$ thus

$$\begin{aligned}
\lim_{n\to\infty} P_{j,j}^{(nd)} &= \lim_{l\to\infty} P_{j,j}^{((l+k)d)} \\
&= \lim_{l\to\infty} P_{j,j}^{(a+ld+b)} \\
&\geq \lim_{\to\infty} P_{j,i}^{(a)} P_{i,i}^{(ld)} P_{i,j}^{(b)} \\
&= P_{j,i}^{(a)} (\lim_{\to\infty} P_{i,i}^{(ld)}) P_{i,j}^{(b)} \\
&> 0
\end{aligned}$$

Thus we have $\frac{d}{m_j} = \lim_{n\to\infty} P_{j,j}^{(nd)} > 0$ so $m_j < \infty$ thus j is positive recurrent. □

7 September 27, 2018

7.1 Theorem 1.6: Finite DTMCs have no null recurrent states

Theorem 7.1. In a finite-state DTMC, there can never be any null recurrent states.

Proof. Assume there exists a null recurrent state i. By theorem 1.10, $\lim_{n\to\infty} P_{i,j}^{(n)} = 0$ for all $j \in S$.
We have $1 = \sum_{j\in S} P_{i,j}^{(n)}$. Take the limiting of both sides

$$\begin{aligned}
1 &= \lim_{n\to\infty} \sum_{j\in S} P_{i,j}^{(n)} \\
&= \sum_{j\in S} \lim_{n\to\infty} P_{i,j}^{(n)} \\
&= \sum_{j\in S} 0 \\
&= 0
\end{aligned}$$

a contradiction. □

7.2 Theorem 1.7: Irreduicle DTMC positive recurrent iff stationary distribution

Theorem 7.2. An irreducible DTMC is positive recurrent iff a stationary distribution exists.

Proof. **Forwards** $\Rightarrow$ Assume the DTMC is positive recurrent. For some state i, define $\gamma = (\gamma_0, \gamma_1, \ldots)$ where $\gamma_j = \lim_{n\to\infty} \gamma_{i,j}^{(n)}$ and

$$\begin{aligned}
\gamma_{i,j}^{(n)} &= E[\frac{1}{n} \sum_{k=1}^{n} 1_{\{X_k = j\}} \mid X_0 = i] \\
&= \frac{1}{n} \sum_{k=1}^{n} P_{i,j}^{(k)}
\end{aligned}$$

Suppose that $S = \mathbb{N}$ (state space). For any $m = 1, 2, \ldots$ (so up to some state m) we have

$$\begin{aligned}
\sum_{j=0}^{m} \gamma_j &= \sum_{j=0}^{m} \lim_{n\to\infty} \gamma_{i,j}^{(n)} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \sum_{j=0}^{m} P_{i,j}^{(k)} \\
&\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}(1) && \text{row sum of n-step TPM up to state m, so } \leq 1 \\
&= \lim_{n\to\infty} 1 \\
&= 1
\end{aligned}$$

So $\sum_{j=0}^{\infty} \gamma_j \leq 1$ In fact, $\sum_{j=0}^{\infty} \gamma_j = 1$:

$$\begin{aligned}
\sum_{j=0}^{\infty} \gamma_j &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \sum_{j=0}^{\infty} P_{i,j}^{(k)} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}(1) \\
&= 1
\end{aligned}$$

We want to show that $\gamma = \gamma P$. Note that the RHS is

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} P^{(k)} P &= \frac{1}{n}\sum_{k=1}^{n} P^{(k)}(I + P - I) \\
&= \frac{1}{n}\sum_{k=1}^{n} P^{(k)} + \frac{1}{n}\sum_{k=1}^{n}(P^{(k+1)} - P^{(k)}) \\
&= \frac{1}{n}\sum_{k=1}^{n} P^{(k)} + \frac{1}{n}(P^{(n+1)} - P^{(1)}) && \text{telescoping}
\end{aligned}$$

Note that the ith row of $\frac{1}{n}\sum_{k=1}^{n} P^{(k)}$ is $(\gamma_{i,0}^{(n)}, \gamma_{i,1}^{(n)}, \ldots)$. Taking the limit of the ith row of the RHS

$$\begin{aligned}
\lim_{n\to\infty} \big[\frac{1}{n}\sum_{k=1}^{n} P^{(k)} + \frac{1}{n}(P^{(n+1)} - P^{(1)})\big]_{(i,\cdot)} &= \gamma + 0 \\
&= \gamma
\end{aligned}$$

where the second line follows since every element of a TPM is $\in [0, 1]$ where

$$\big|[P^{(n+1)} - P^{(1)}]_{(i,j)}\big| \leq 1 \Rightarrow \lim_{n\to\infty} \frac{1}{n}(P^{(n+1)} - P^{(1)})_{(i,\cdot)} = 0$$

Now taking the limit of the (i,l)th element of the LHS

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[P^{(k)}P\right]_{(i,l)} &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\sum_{j\in S}P_{i,j}^{(k)}P_{j,l} \\
&= \sum_{j\in S}\Big[\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}P_{i,j}^{(k)}\Big]P_{j,l} && \text{DCT} \\
&= \sum_{j\in S}\gamma_j P_{j,l} \\
&= [\gamma P]_l
\end{aligned}$$

where DCT is applied where $Y_n = P_{j,l}$ with probability $\frac{1}{n}\sum_{k=1}^{n}P_{i,j}^{(k)}$ i.e. $Y = P_{j,l}$ with probability γ_j, $Z = 1$.

So the jth row of the LHS converges to γP.

$\gamma = \gamma P$ and so γ satisfies the stationary condition and is a stationary distribution (actually: we let $\pi = \frac{\gamma}{\gamma e'}$ to make it a true distribution).

Backwards $\Leftarrow$ Assume there exists a stationary distribution. Assume the DTMC is null recurrent or transient.

In either case, $\lim_{n\to\infty}P_{i,j}^{(n)} = 0$ for all $j \in S$.

We have $\pi = \pi P^{(n)} \Rightarrow \pi_j = \sum_{i\in S}\pi_i P_{i,j}^{(n)}$ for all $j \in S$, for all $n \in \mathbb{N}$.

Taking the limit of both sides as $n \to \infty$

$$\begin{aligned}
\pi_j &= \lim_{n\to\infty}\sum_{j\in S}\pi_i P_{i,j}^{(n)} \\
&= \sum_{j\in S}\pi_i(\lim_{n\to\infty}P_{i,j}^{(n)}) && \text{DCT} \\
&= \sum_{j\in S}\pi_i 0 \\
&= 0 \qquad \forall j \in S
\end{aligned}$$

where the second line follows from applying DCT where $Y_n = P_{i,j}^{(n)}$ with probability π_j, where $Y = 0$ and $Z = 1$.

Thus $\pi = (0, 0, \ldots)$ which is not a distribution, thus we have a contradiction.

$\square$

8 October 2, 2018

8.1 Example 1.13 solution

Consider a DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} p & 0 & 1-p & 0 \\ 0 & r & 0 & 1-r \\ q & 0 & 1-q & 0 \\ 0 & s & 0 & 1-s \end{bmatrix} \end{array}$$

with $o < p, q, r, s < 1$ so that it has two positive recurrent communication classes $C_1 = \{0, 2\}$ and $C_2 = \{1, 3\}$. Rewrite P according to its canonical decomposition and solve for $\lim_{n\to\infty} P^n$.

Solution. We can rearrange the TPM as

$$P^* = \begin{array}{c} \\ 0 \\ 2 \\ 1 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 2 & 1 & 3 \end{array} \\ \begin{bmatrix} p & 1-p & 0 & 0 \\ q & 1-q & 0 & 0 \\ 0 & 0 & r & 1-r \\ 0 & 0 & s & 1-s \end{bmatrix} \end{array} = \begin{array}{c} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \end{array} \Rightarrow (P^*)^n = \begin{array}{c} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} P_1^n & 0 \\ 0 & P_2^n \end{bmatrix} \end{array}$$

We can find the limiting probabilities within each class in isolation.

C_1 The conditions of the BLT are satisfied so $\exists \tilde{\pi}_1$ such that $\tilde{\pi}_1 P_1 = \tilde{\pi}_1$ and $\tilde{\pi}_1 e' = 1$ ($\tilde{\pi}_1 = (\pi_0, \pi_2)$). We have

$$\begin{aligned} \pi_0 &= p\pi_0 + q\pi_2 \\ \Rightarrow \pi_2 &= \frac{1-p}{q}\pi_0 \end{aligned}$$

and also

$$\begin{aligned} 1 &= \pi_0 + \pi_2 \\ \Rightarrow \pi_0 &= \frac{q}{q+1-p}, \pi_2 = \frac{1-p}{q+1-p} \end{aligned}$$

C_2 Similarly (as above) $\pi_1 = \frac{s}{s+1-r}, \pi_3 = \frac{1-r}{s+1-r}$.

Returning to our original TPM

$$\lim_{n\to\infty} P^n = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \pi_0 & 0 & \pi_2 & 0 \\ 0 & \pi_1 & 0 & \pi_3 \\ \pi_0 & 0 & \pi_2 & 0 \\ 0 & \pi_1 & 0 & \pi_3 \end{bmatrix} \end{array} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{q}{q+1-p} & 0 & \frac{1-p}{q+1-p} & 0 \\ 0 & \frac{s}{s+1-r} & 0 & \frac{1-r}{s+1-r} \\ \frac{q}{q+1-p} & 0 & \frac{1-p}{q+1-p} & 0 \\ 0 & \frac{s}{s+1-r} & 0 & \frac{1-r}{s+1-r} \end{bmatrix} \end{array}$$

8.2 Random walk transience/recurrence

Theorem 8.1. The simple random walk is transient if $p \neq q$, and null recurrent if $p = q = \frac{1}{2}$.

Proof. **Case 1:** $p \neq q$ Without loss of generality, $p > q$. By the strong law of large numbers

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = E[X_1] = p - q > 0$$

thus we have

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty}\sum_{i=1}^{n} X_i \\
&= \lim_{n\to\infty} n\Big(\frac{1}{n}\sum_{i=1}^{n} X_i\Big) \\
&= \infty\cdot(p-q) \\
&= \infty && p-q>0
\end{aligned}$$

There is a last visit to state 0 (since we may go off to infinity) thus 0 is transient hence the DTMC is transient (class property).

Case 2: $p=q=\frac{1}{2}$ We want to show $\sum_{n=1}^{\infty} P_{0,0}^{(n)}=\infty$ to show that state 0 is null recurrent.

We know $P_{0,0}^{(2n+1)}=0$, $n\in\mathbb{N}$ and

$$P_{0,0}^{(2n)} = P(\text{n steps to the right and n steps to the left})$$

This is in fact $BIN(2n,\frac{1}{2})$ thus

$$P_{0,0}^{(2n)} = \binom{2n}{n}\Big(\frac{1}{2}\Big)^{2n} = \frac{(2n)!}{n!n!}\Big(\frac{1}{4}\Big)^{n}$$

By Stirling's Formula for large n: $n! = \sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}$, thus

$$\begin{aligned}
\frac{(2n)!}{n!n!} &= \frac{\sqrt{2\pi}e^{-2n}(2n)^{2n+\frac{1}{2}}}{(\sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}})^2} \\
&= \frac{2^{2n+\frac{1}{2}}}{\sqrt{2\pi}\sqrt{n}} \\
&= \frac{4^n}{\sqrt{n\pi}}
\end{aligned}$$

Thus for large n, $P_{0,0}^{(2n)} = \frac{1}{\sqrt{n\pi}}$, therefore

$$\begin{aligned}
\sum_{n=1}^{\infty} P_{0,0}^{(n)} &= \sum_{m=1}^{\infty} P_{0,0}^{(2m)} \\
&\sim \sum_{m=1}^{\infty}\frac{1}{\sqrt{n\pi}} \\
&\geq \frac{1}{\sqrt{\pi}}\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}} \\
&= \infty
\end{aligned}$$

Thus state 0 is recurrent.

We consider $\pi = \pi P$ and $\pi e' = 1$ for this DTMC i.e.

$$\begin{aligned}
&\pi_i = P_{i-1,i}\pi_{i-1} + P_{i+1,i}\pi_{i+1} \\
\Rightarrow &\pi_i = \frac{1}{2}\pi_{i-1} + \frac{1}{2}\pi_{i+1} \\
\Rightarrow &\pi_{i+1} - \pi_i = \pi_i - \pi_{i-1} && \forall i \in \mathbb{Z} \\
\Rightarrow &\pi_i = \pi_0 + id && d = \pi_1 - \pi_0
\end{aligned}$$

Since $\pi_i \in [0,1]$ for all i we must have $d = 0$, so $\pi_i = \pi_0$, but $\pi e' = \sum_{i=-\infty}^{\infty} \pi_i = \sum_{i=-\infty}^{\infty} \pi_0$ which is 0 if $\pi_0 = 0$ or ∞ if $\pi_0 > 0$.

Therefore there is no stationary distribution and hence state 0 is not positive recurrent.

□

9 October 4, 2018

9.1 Section 1.7.2: conditions for positive recurrent G/M/1 queue

Question 9.1. What are the conditions for a G/M/1 queue and its associated DTMC where $b_0 > 0$ (can transition to the right) and $b_0 + b_1 < 1$ (can transition to the left) to be positive recurrent?
Note under the above two conditions the DTMC is irreducible and aperiodic.

Claim. The DTMC with the above two conditions is positive recurrent iff $E[B] = \sum_{k=1}^{\infty} kb_k > 1$. That is, the expected number of potential service completions during a single interarrival time is greater than 1.
The stationary distribution $p = (p_0, p_1, \ldots)$ when it exists, satisfies $p_k = r_0^k(1 - r_0)$ for $k \in \mathbb{N}$ where $r_0 \in (0,1)$ is the solution to $r_0 = \Phi_B(r_0)$ and $\Phi_B(z) = E[z^B] = \sum_{k=0}^{\infty} z^k b_k$ is the probability generating function of B.

Proof. Recall from theorem 1.7 that a DTMC is positive recurrent iff a stationary distribution exists.
We will confirm that stationary distribution exists to prove the claim. Note that we want $p = pP$, thus

$$\begin{aligned}
p_0 &= p_0(1 - b_0) + p_1(1 - b_0 - b_1) + \ldots \\
&= \sum_{i=0}^{\infty} p_i\big(1 - \sum_{j=0}^{i} b_j\big)
\end{aligned}$$

Also

$$\begin{aligned}
p_1 &= p_0 b_0 + p_1 b_1 + p_2 b_2 + \ldots \\
p_2 &= p_1 b_0 + p_2 b_1 + p_3 b_2 + \ldots \\
&\vdots \\
p_k &= \sum_{i=0}^{\infty} p_{k-1+i} b_i
\end{aligned}$$

We also want $pe' = 1$ i.e. $\sum_{i=0}^{\infty} p_i = 1$.
Assume $p_k = r_0^k(1 - r_0)$, $k \in \mathbb{N}$ (geometric distribution), where $r_0 \in (0,1)$.
We want to check under what conditions this equation for p_k satisfies our three equations for p_0, p_k and $pe' = 1$.

Note that

$$1 = \sum_{i=0}^{\infty} p_i = \sum_{i=0}^{\infty} r_0^k(1-r_0) = \frac{1-r_0}{1-r_0} = 1$$

For our p_k we have

$$\begin{aligned}
p_k &= \sum_{i=0}^{\infty} p_{k-1+i} b_i \\
\Rightarrow r_0^k(1-r_0) &= \sum_{i=0}^{\infty} r_0^{k-1+i}(1-r_0) b_i \\
\Rightarrow r_0 &= \sum_{i=0}^{\infty} r_0^i b_i \\
\Rightarrow r_0 &= \Phi_B(r_0)
\end{aligned}$$

since $\Phi_B(z) = E[z^B]$ is the pgf of B.
When does $r_0 = \Phi_B(r_0)$ have a solution for $r_0 \in (0,1)$? If r_0 is a solution to $z = \Phi_B(z)$ then it is the intersection of the lines $y = z$ and $y = \Phi_B(z)$.
Properties of $\Phi_B(z) = \sum_{i=0}^{\infty} z^i b_i$:

1. $\Phi_B(z)$ is continuous
2. $$\Phi_B(0) = \sum_{i=0}^{\infty} z^i b_i \mid_{z=0} = b_0 > 0$$
3. $$\Phi_B(1) = \sum_{i=0}^{\infty} z^i b_i \mid_{z=1} = \sum_{i=0}^{\infty} b_i = 1$$
4. $$\Phi_B'(z) = \frac{d}{dz}\Big(\sum_{i=0}^{\infty} z^i b_i\Big) = \sum_{i=0}^{\infty} i z^{i-1} b_i > 0 \qquad \forall z \in (0,1)$$
   Note that
   $$\Phi_B'(1) = \sum_{i=0}^{\infty} i 1^{i-1} b_i = E[B]$$
5. $$\Phi_B''(z) = \sum_{i=0}^{\infty} i(i-1) z^{i-2} b_i > 0 \qquad \forall z \in (0,1)$$
   therefore $y = \Phi_B(z)$ is convex.

We have two cases from $E[B]$:

Case 1 $E[B] > 1$ We are guaranteed an intersection at $z \in (0,1)$ since the slope $y = \Phi_B(z)$ is greater than that of $y = z$ at $z = 1$ (from $\Phi_B'(1) = E[B]$).

Case 2 $E[B] \leq 1$ There is not intersection before $z = 1$.

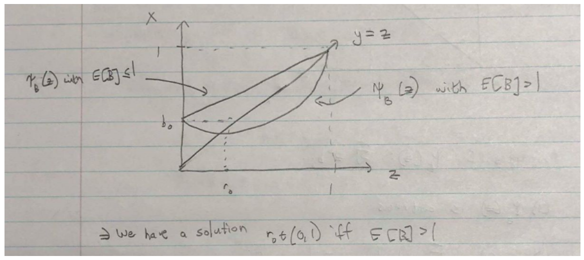


Figure 9.1: Diagram of $y = z$ and $y = \Phi_B(z)$ when $E[B] \leq 1$ and $E[B] > 1$. Note that $E[B]$ is exactly the derivative $\Phi'_B(z)$ at $z = 1$.

Thus we have a solution for $r_0 \in (0, 1)$ iff $E[B] > 1$.
We verifying that our equation for p_0 is satisfied

$$
\begin{aligned}
& p_0 = \sum_{i=0}^{\infty} p_i (1 - \sum_{j=0}^{i} b_j) \\
\iff & r_0^0 (1 - r_0) = \sum_{i=0}^{\infty} p_i - \sum_{i=0}^{\infty} \sum_{j=0}^{i} p_i b_j \\
\iff & 1 - r_0 = 1 - \sum_{j=0}^{\infty} b_j \sum_{i=j}^{\infty} r_0^i (1 - r_0) && pe' = 1 \\
\iff & 1 - r_0 = 1 - \sum_{j=0}^{\infty} b_j r_0^j && \sum_{i=j}^{\infty} r_0^i (1 - r_0) = r_0^j \text{ (geometric series)} \\
\iff & 1 - r_0 = 1 - \Phi_B(r_0) \\
\iff & 1 - r_0 = 1 - r_0
\end{aligned}
$$

So when $E[B] > 1$ $p_k = r_0^k (1 - r_0)$ for $k \in \mathbb{N}$ is a stationary distribution hence the DTMC is positive recurrent. $\square$