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STAT 333 Course Notes

Applied Probability

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

1 January 4, 2018

1.1 Example 1.1 solution

What is the probability that we roll a number less than 4 given that we know it's odd?

Solution. Let $A = \{1, 2, 3\}$ (less than 4) and $B = \{1, 3, 5\}$ (odd). We want to find $P(A \mid B)$. Note that $A \cap B = \{1, 3\}$ and there are six elements in the sample space S thus

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{2}{3}$$

1.2 Example 1.2 solution

Show that $BIN(n, p) \sim POI(\lambda)$ when $\lambda = np$ for n large and p small.

Solution. Let $\lambda = np$. Note that $p = \frac{\lambda}{n} \quad n > 0$. From the pmf for $X \sim BIN(n, p)$

$$\begin{aligned} p(x) &= \binom{n}{x} p^x (1-p)^{n-x} \\ &= \frac{n(n-1)...(n-x+1)}{x!} (\frac{\lambda}{n})^x (1 - \frac{\lambda}{n})^{n-x} \\ &= \frac{n(n-1)...(n-x+1)}{n^x} \cdot \frac{\lambda^x}{x!} \frac{(1-\frac{\lambda}{n})^n}{(1-\frac{\lambda}{n})^x} \end{aligned}$$

Recall $\lim_{n\to\infty}(1 - \frac{\lambda}{n})^n = e^{-\lambda}$ so

$$\lim_{n\to\infty} p(x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}$$

2 January 9, 2018

2.1 Example 1.3 solution

Find the mgf of $BIN(n, p)$ and use that to find $E[X]$ and $Var(X)$.

Solution. Recall the binomial series is

$$(a + b)^m = \sum_{x=0}^{m} \binom{m}{x} a^x b^{m-x} \quad a, b \in \mathbb{R}, m \in \mathbb{N}$$

Let $x \sim BIN(n, p)$ and so

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \ldots, n$$

Taking the mgf $E[e^{tX}]$

$$\begin{aligned}\Phi_X(t) = E[e^{tX}] &= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}\end{aligned}$$

from the binomial series we have

$$\Phi_x(t) = (pe^t + 1 - p)^n \quad t \in \mathbb{R}$$

We can take the first and second derivatives for the first and second moment

$$\begin{aligned}\Phi'_X(t) &= n(pe^t + 1 - p)^{n-1} pe^t \\ \Phi''_X(t) &= np[(pe^t + 1 - p)^{n-1} e^t + e^t (n-1)(pe^t + 1 - p)^{n-2} pe^t]\end{aligned}$$

So $E[X] = \Phi_X(t) \mid_{t=0} = np$.
For the variance, we need the second moment

$$\begin{aligned}E[X^2] &= \Phi_X(t) \mid_{t=0} \\ &= np[1 + (n-1)p] \\ &= np + (np)^2 - np^2\end{aligned}$$

So

$$\begin{aligned}Var(X) &= E[X^2] - E[X]^2 \\ &= np + (np)^2 - np^2 - (np)^2 \\ &= np(1-p)\end{aligned}$$

2.2 Example 1.4 solution

Show that $Cov(X, Y) = 0 \not\Rightarrow$ independence.

Solution. We show this using a counter example

		y		
	$p(x, y)$	0	1	$p_X(x)$
	0	0.2	0	0.2
x	1	0	0.6	0.6
	2	0.2	0	0.2
	$p_Y(y)$	0.4	0.6	1

Note that

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

where

$$E[XY] = \sum_{x=0}^{2}\sum_{y=0}^{1} xyp(x,y) = (1)(1)(0.6) = 0.6$$
$$E[X] = \sum_{x=0}^{2} xp_X(x) = (1)(0.6) + (2)(0.2) = 0.6 + 0.4 = 1$$
$$E[Y] = \sum_{y=0}^{1} yp_Y(y) = (1)(0.6) = 0.6$$

So $Cov(X,Y) = 0.6 - (1)(0.6) = 0$. However, $p(2,0) = 0.2 \neq p_X(2)p_Y(0) = (0.2)(0.4) = 0.08$, thus X and Y are not independent (they are dependent).

2.3 Example 1.5 solution

Given $X_1, \ldots, X_n$ are independent r.v's where $\Phi_X(t)$ is the mgf of X_i, show that $T = \sum_{i=1}^n X_i$ has mgf $\Phi_T(t) = \prod_{i=1}^n \Phi_{X_i}(t)$.

Solution. We take the definition of the mgf of T

$$\begin{aligned}
\Phi_T(t) &= E[e^{tT}] \\
&= E[e^{t(X_1+\ldots+X_n)}] \\
&= E[e^{tX_1} \cdot \ldots \cdot e^{tX_n}] \\
&= E[e^{tX_1}] \cdot \ldots \cdot E[e^{tX_n}] && \text{independence} \\
&= \prod_{i=1}^{n} \Phi_{X_i}(t)
\end{aligned}$$

2.4 Exercise 1.3

If $X_i \sim POI(\lambda_i)$ show that $T = \sum X_i \sim POI(\sum \lambda_i)$.

Solution. Recall that $POI(\lambda_i) \sim BIN(n_i, p)$ where $\lambda_i = n_i p$ and

$$\Phi_{X_i}(t) = (pe^t + 1 - p)^{n_i} \quad \forall t \in \mathbb{R}$$

where $X_i \sim BIN(n_i, p) \quad i = 1, \ldots, m$.
Therefore

$$\begin{aligned}
\Phi_T(t) &= \prod_{i=1}^{m} (pe^t + 1 - p)^{n_i} \\
&= (pe^t + 1 - p)^{n_1} \cdot \ldots \cdot (pe^t + 1 - p)^{n_m} \\
&= (pe^t + 1 - p)^{\sum n_i} \quad t \in \mathbb{R}
\end{aligned}$$

By the mgf uniqueness property, we have

$$T = \sum_{i=1}^{m} X_i \sim BIN(\sum_{i=1}^{m} n_i, p)$$

3 January 11, 2018

3.1 Theorem 2.1 (conditional variance)

Theorem 3.1.

$$Var(X_1 \mid X_2 = x_2) = E[X_1^2 \mid X_2 = x_2] - E[X_1 \mid X_2 = x_2]^2$$

Proof.

$$\begin{aligned}
Var(X_1 \mid X_2 = x_2) &= E[(X_1 - E[X_1 \mid X_2 = x_2])^2 \mid X_2 = x_2] \\
&= E[(X_1^2 - 2E[X_1 \mid X_2 = x_2]X_1 + E[X_1 \mid X_2 = x_2]^2) \mid X_2 = x_2] \\
&= E[X_1^2 \mid X_2 = x_2] - 2E[X_1 \mid X_2 = x_2]E[X_1 \mid X_2 = x_2] + E[X_1 \mid X_2 = x_2]^2 \\
&= E[X_1^2 \mid X_2 = x_2] - E[X_1 \mid X_2 = x_2]^2
\end{aligned}$$

□

3.2 Example 2.1

Suppose that X and Y are discrete random variables having join pmf of the form

$$p(x,y) = \begin{cases} 1/5 & \text{, if x = 1 and y = 0,} \\ 2/15 & \text{, if x = 0 and y = 1,} \\ 1/15 & \text{, if x = 1 and y = 2,} \\ 1/5 & \text{, if x = 2 and y = 0,} \\ 2/5 & \text{, if x = 1 and y = 1,} \\ 0 & \text{, otherwise.} \end{cases}$$

Find the conditional probability of $X \mid (Y = 1)$. Also calculate $E[X \mid Y = 1]$ and $Var(X \mid Y = 1)$.

Solution. Note: for problems of this nature, construct a table.

		y			
	$p(x,y)$	0	1	2	$p_X(x)$
	0	0	2/15	0	2/15
x	1	1/5	2/5	1/15	2/3
	2	1/5	0	0	1/5
	$p_Y(y)$	2/5	8/15	1/15	1

Then we have

$$\begin{aligned}
p(0 \mid 1) = P(X = 0 \mid Y = 1) = \frac{2/15}{8/15} = \frac{1}{4} \\
p(1 \mid 1) = P(X = 1 \mid Y = 1) = \frac{2/5}{8/15} = \frac{3}{4} \\
p(2 \mid 1) = P(X = 2 \mid Y = 1) = \frac{0}{8/15} = 0
\end{aligned}$$

The conditional pmf of $X \mid (Y = 1)$ can be represented as follows

x	0	1
$p(x \mid 1)$	1/4	3/4

We observe $X \mid (Y=1) \sim Bern(3/4)$. We can take the known $E[X]=p$ and $Var(X)p(1-p)$ for $X \sim Bern(p)$, thus

$$\begin{aligned}
&E[X \mid (Y=1)] = 3/4 \\
&Var(X \mid (Y=1)) = 3/4(1-3/4) = 3/16
\end{aligned}$$

3.3 Example 2.2

For $i=1,2$ suppose that $X_i \sim BIN(n_i, p)$ where X_1, X_2 are independent (but not identically distributed). Find conditional distribution of X_1 given $X_1 + X_2 = n$.

Solution. We want to find conditional pmf of $X \mid (X_1 + X_2 = n)$. Let this conditional pmf be denoted by

$$\begin{aligned}
p(x_1 \mid n) &= P(X_1 = x_1 \mid X_1 + X_2 = n) \\
&= \frac{P(X_1 = x_1, X_1 + X_2 = n)}{P(X_1 + X_2 = n)}
\end{aligned}$$

Recall: $X_1 + X_2 \sim BIN(n_1 + n_2, p)$ so

$$P(X_1 + X_2 = n) = \binom{n_1 + n_2}{n} p^n (1-p)^{n_1+n_2-n}$$

Next, consider

$$\begin{aligned}
P(X_1 = x_1, X_1 + X_2 = n) &= P(X_1 = x_1, x_1 + X_2 = n) \\
&= P(X_1 = x_1, X_2 = n - x_1) \\
&= P(X_1 = x_1)P(X_2 = n - x_1) && \text{independence} \\
&= \binom{n_1}{x_1} p^{x_1}(1-p)^{n_1-x_1} \cdot \binom{n_2}{n-x_1} p^{n-x_1}(1-p)^{n_2-(n-x_1)}
\end{aligned}$$

provided that $0 \le x_1 \le n_1$ **and**

$$\begin{aligned}
&0 \le n - x_1 \le n_2 \\
&-n_2 \le x_1 - n \le 0 \\
&n - n_2 \le x_1 \le n
\end{aligned}$$

(from the binomial coefficients). Therefore our domain for x_1 is

$$x_1 = \max\{0, n-n_2\}, \ldots, \min\{n_1, n\}$$

Thus we have

$$\begin{aligned}
p(x_1 \mid n) &= \frac{P(X_1 = x, X_1 + x_2 = n)}{P(X_1 + X_2 = n)} \\
&= \frac{\binom{n_1}{x_1} p^{x_1}(1-p)^{n_1 - x_1} \cdot \binom{n_2}{n-x_1} p^{n-x_1}(1-p)^{n_2-(n-x_1)}}{\binom{n_1+n_2}{n} p^n (1-p)^{n_1+n_2-n}} \\
&= \frac{\binom{n_1}{x_1}\binom{n_2}{n-x_1}}{\binom{n_1+n_2}{n}}
\end{aligned}$$

for $x_1 = \max\{0, n-n_2\}, \ldots, \min\{n_1, n\}$.
Recall: A $HG(N, r, n)$ (hypergeometric) distribution has pmf

$$\frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{x}} \quad x = \max\{0, n-N+r\}, \ldots, \min\{n, r\}$$

So this is precisely $HG(n_1 + n_2, x_1, n)$.
If you think about it: we are choosing x_1 successes from n_1 trials from the first set X_1 and choosing the remaining $n - x_1$ successes from n_2 trials from X_2.

4 Tutorial 1

4.1 Exercise 1: MGF of Erlang

Find the mgf of $X \sim Erlang(\lambda)$ and use it to find $E[X], Var(X)$.
Note that the Erlang's pdf is for $n \in \mathbb{Z}^+$ and $\lambda > 0$

$$f(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} \quad x > 0$$

Solution.

$$\begin{aligned}
\Phi_X(t) = E[e^{tX}] &= \int_0^\infty e^{tx} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} dx \\
&= \int_0^\infty \frac{\lambda^n x^{n-1} e^{-(\lambda - t)x}}{(n-1)!} dx
\end{aligned}$$

Note that the term in the integral is similar to the pdf of Erlang but for $\lambda = \lambda - t$. So we try to fix it so the integral is this pdf of Erlang

$$\begin{aligned}
\Phi_X(t) &= \int_0^\infty \frac{\lambda^n x^{n-1} e^{-(\lambda-t)x}}{(n-1)!} dx \\
&= (\frac{\lambda}{\lambda - t})^n \int_0^\infty \frac{(\lambda - t)^n x^{n-1} e^{-(\lambda-t)x}}{(n-1)!} dx \\
&= (\frac{\lambda}{\lambda - t})^n & t < \lambda
\end{aligned}$$

since the integral over the positive real line of the pdf of an $Erlang(n, \lambda - t)$ is 1 and $t < \lambda$ must hold so the rate parameter $\lambda - t$ is positive.

Differentiating,

$$\begin{aligned}
\Phi_X^{(1)}(t) &= \frac{d}{dt}(\frac{\lambda}{(\lambda-t)^n}) \\
&= \frac{n\lambda^n}{(\lambda-t)^{n+1}} \\
\Phi_X^{(2)}(t) &= \frac{d}{dt}(\frac{n\lambda^n}{(\lambda-t)^{n+1}}) \\
&= \frac{n(n+1)\lambda^n}{(\lambda-t)^{n+2}}
\end{aligned}$$

Thus we have

$$\begin{aligned}
E[X] = \Phi_X^{(1)}(0) &= \frac{n\lambda^n}{(\lambda-t)^{n+1}}\bigg|_{t=0} = \frac{n}{\lambda} \\
E[X^2] = \Phi_X^{(2)}(0) &= \frac{n(n+1)\lambda^n}{(\lambda-t)^{n+2}}\bigg|_{t=0} = \frac{n(n+1)}{\lambda^2} \\
Var(X) = E[X^2] - E[X]^2 &= \frac{n(n+1)}{\lambda^2} - \frac{n}{\lambda} = \frac{n}{\lambda^2}
\end{aligned}$$

Remark 4.1. To solve any of these mgfs, it is useful to see if one can reduce the integral into a pdf of a known distribution (possibly itself).

4.2 Exercise 2: MGF of Uniform

Find the mgf of the uniform distribution on $(0,1)$ and find $E[X]$ and $Var(X)$.

Solution. Let $X \sim U(0,1)$ so that $f(x) = 1 \quad 0 \le x \le 1$. We have

$$\begin{aligned}
\Phi_X(t) = E[e^{tX}] &= \int_0^1 e^{tx}(1)dx \\
&= \frac{1}{t}e^{tx}\bigg|_{x=0}^{x=1} \\
&= t^{-1}(e^t - 1) \quad t \neq 0
\end{aligned}$$

Differentiating

$$\begin{aligned}
\Phi_X^{(1)}(t) &= \frac{d}{dt}(t^{-1}(e^t-1)) \\
&= t^{-1}e^t - t^{-2}(e^t-1) \\
&= \frac{te^t - e^t + 1}{t^2} \\
\Phi_X^{(2)}(t) &= \frac{d}{dt}(\frac{te^t - e^t + 1}{t^2}) \\
&= \frac{t^2(te^t + e^t - e^t) - 2t(te^t - e^t + 1)}{t^4} \\
&= \frac{t^2e^t - 2te^t + 2e^t - 2}{t^3}
\end{aligned}$$

We may calculate the first two moments by applying **L'Hopital's rule** to calculate the limits

$$\begin{aligned}
E[X] = \Phi_X^{(1)}(t)\Big|_{t=0} &= \lim_{t\to\infty} \frac{te^t - e^t + 1}{t^2} \\
&= \lim_{t\to\infty} \frac{te^t + e^t - e^t}{2t} \\
&= \lim_{t\to\infty} \frac{e^t}{2} = \frac{1}{2}
\end{aligned}$$

Similarly

$$\begin{aligned}
E[X^2] = \Phi_X^{(2)}(t)\Big|_{t=0} &= \lim_{t\to\infty} \frac{t^2e^t - 2te^t + 2e^t - 2}{t^3} \\
&= \lim_{t\to\infty} \frac{t^2e^t + 2te^t - 2te^t - 2e^t + 2e^t}{3t^2} \\
&= \lim_{t\to\infty} \frac{e^t}{3} = \frac{1}{3}
\end{aligned}$$

So we have

$$Var(X) = E[X^2] - E[X]^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

4.3 Exercise 3: Moments from PGF

Suppose X is a discrete r.v. on $\mathbb{N}$ with pmf $p(x)$. Show how to find the first two moments of X from its pgf.

Solution. By definition, the pgf of X is $\Psi_X(z) = E[z^X] = \sum_{x=0}^{\infty} z^x p(x)$.
If we let $z = 1$, then the sum equals 1. However, if we take its derivative with respect to z just once

$$\Psi_X^{(1)}(z) = \frac{d}{dz}\sum_{x=0}^{\infty} z^x p(x) = \sum_{x=1}^{\infty} xz^{x-1}p(x)$$

Letting $z = 1$ we can find the first moment

$$\begin{aligned}
\Psi_X^{(1)}(1) &= \lim_{z\to 1}\sum_{x=1}^{\infty} xz^{x-1}p(x) \\
&= \sum_{x=1}^{\infty} xp(x) \\
&= \sum_{x=0}^{\infty} xp(x) && \text{when } x = 0 \text{ the term is 0 anyways} \\
&= E[X]
\end{aligned}$$

For the second moment, we consider the second derivative

$$\begin{aligned}
\Psi_X^{(1)}(z) &= \frac{d^2}{dz^2}\sum_{x=0}^{\infty} z^x p(x) \\
&= \sum_{x=2}^{\infty} x(x-1)z^{x-2}p(x)
\end{aligned}$$

Letting $z = 1$

$$\begin{aligned}
\Psi_X^{(2)}(1) &= \lim_{z\to 1}\sum_{x=2}^{\infty} x(x-1)z^{x-2}p(x) \\
&= \sum_{x=2}^{\infty} x(x-1)p(x) \\
&= \sum_{x=0}^{\infty} x(x-1)p(x) \\
&= E[X(X-1)] \\
&= E[X^2] - E[X]
\end{aligned}$$

So we have $E[X^2] = \Psi_X^{(2)}(1) + \Psi_X^{(1)}(1)$. To find the variance

$$Var(X) = \Psi_X^{(2)}(1) + \Psi_X^{(1)}(1) - (\Psi_X^{(1)}(1))^2$$

4.4 Exercise 4: PGF of Poisson

Suppose $X \sim POI(\lambda)$. Find the pgf of X and use it to find $E[X]$ and $Var(X)$. The pmf of $POI(\lambda)$ for $\lambda > 0$

$$f(x) = e^{-\lambda}\frac{\lambda^x}{x!} \quad x = 0, 1, 2, \ldots$$

Solution.

$$\begin{aligned}
\Psi_X(z) = E[z^X] &= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(z\lambda)^x}{x!} \\
&= e^{-\lambda}\cdot e^{z\lambda} \\
&= e^{\lambda(z-1)}
\end{aligned}$$

where the second equality holds since the summation is the Taylor expansion of $e^{z\lambda}$.
Differentiating

$$\begin{aligned}
\Psi_X^{(1)}(z) &= \frac{d}{dz}e^{\lambda(z-1)} \\
&= \lambda e^{\lambda(z-1)} \\
\Psi_X^{(2)}(z) &= \frac{d}{dz}\lambda e^{\lambda(z-1)} \\
&= \lambda^2 e^{\lambda(z-1)}
\end{aligned}$$

The moments are thus

$$\begin{aligned}
E[X] &= \Phi_X^{(1)}(1) = \lambda e^{\lambda(1-1)} = \lambda \\
E[X(X-1)] &= \Phi_X^{(2)}(1) = \lambda^2 e^{\lambda(1-1)} = \lambda^2 \\
E[X^2] &= E[X(X-1)] + E[X] = \lambda^2 + \lambda \\
Var(X) &= E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda
\end{aligned}$$

5 January 16, 2018

5.1 Example 2.3 solution

Let $X_1, \dots, X_m$ be independent r.v.'s where $X_i \sim POI(\lambda_i)$. Define $Y = \sum_{i=1}^m X_i$. Find the conditional distribution $X_j \mid (Y = n)$.

Solution. We set out to find

$$\begin{aligned}
p(x_j \mid n) = p(X_j = x_j \mid Y = n) &= \frac{P(X_j = x_j, Y = n)}{P(Y = n)} \\
&= \frac{P(X_j = x_j, \sum_{i=1}^m X_i = n)}{P(Y = n)} \\
&= \frac{P(X_j = x_j, X_j + \sum_{i=1, i\neq j}^m X_i = n)}{P(Y = n)} \\
&= \frac{P(X_j = x_j, \sum_{i=1, i\neq j}^m X_i = n - x_j)}{P(Y = n)} \\
&= \frac{P(X_j = x_j) P(\sum_{i=1, i\neq j}^m X_i = n - x_j)}{P(Y = n)} && \text{independence of } X_i
\end{aligned}$$

Remember that if $X_i \sim POI(\lambda_i)$, then

$$Y = \sum_{i=1}^m X_i \sim POI(\sum_{i=1}^m \lambda_i)$$

which can be derived from mgfs (Exercise 1.3). Therefore

$$\sum_{i=1, i\neq j}^m X_i \sim POI(\sum_{i=1, i\neq j}^m \lambda_i)$$

Expanding out $p(x_j \mid n)$ with the pdfs

$$p(x_j \mid n) = \frac{\frac{e^{-\lambda_j} \lambda_j^{x_j}}{x_j!} \cdot \frac{e^{-\sum_{i=1, i\neq j} \lambda_i} (\sum_{i=1, i\neq j} \lambda_i)^{n-x_j}}{(n-x_j)!}}{\frac{e^{-\sum_{i=1}^m \lambda_i} \cdot (\sum_{i=1}^m \lambda_i)^n}{n!}}$$

where $x_j \geq 0$ and $n - x_j \geq 0 \Rightarrow 0 \leq x_j \leq n$ (from the factorials).

Cancelling out the e^{λ} terms and let $\lambda_Y = \sum_{i=1}^{m} \lambda_i$

$$\begin{aligned} p(x_j \mid n) &= \frac{n!}{(n-x_j)!x_j!} \frac{\lambda_j^{x_j}}{\lambda_Y^{x_j}} \frac{(\lambda_Y - \lambda_j)^{n-x_j}}{\lambda_Y^{n-x_j}} \\ &= \binom{n}{x_j} (\frac{\lambda_j}{\lambda_Y})^{x_j} (1 - \frac{\lambda_j}{\lambda_Y})^{n-x_j} \end{aligned}$$

This is the binomial distribution, so we have

$$X_j \mid Y = n \sim BIN(n, \frac{\lambda_i}{\lambda_Y})$$

5.2 Example 2.4 solution

Suppose $X \sim POI(\lambda)$ and $Y \mid (X = x) \sim BIN(x, p)$. Find the conditional distribution $X \mid Y = y$.
(Note: range of y depends on x (that is $y \leq x$). Graphically, we have integral points on and below the $y = x$ line starting from 0 for both x and y).

Solution. We wish to find the conditional pmf given by $X \mid Y = y$ or

$$p(x \mid y) = P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Note that also

$$\begin{aligned} P(Y = y \mid X = x) &= \frac{P(Y = y, X = x)}{P(X = x)} \\ \Rightarrow P(X = x, Y = y) &= P(X = x)P(Y = y \mid X = x) \\ &= \frac{e^{-\lambda}\lambda^x}{x!} \cdot \binom{x}{y} p^y (1-p)^{x-y} \end{aligned}$$

for $x = 0, 1, 2, \ldots$ **and** $y = 0, 1, 2, \ldots, x$ (range of y depends on x).
To find the marginal marginal pmf of Y, we use

$$p_Y(y) = \sum_x p(x, y)$$

To find the support for x, note that from the graphical region, we realize that $x = 0, 1, 2, \ldots$ **and** $y = 0, 1, 2, \ldots, x$ is equivalent to $y = 0, 1, 2, \ldots$ **and** $x = y, y+1, y+2, \ldots$.

So

$$\begin{aligned}
p_Y(y) &= \sum_{x=y}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} \frac{x!}{(x-y)!y!} p^y(1-p)^{x-y} \\
&= \frac{\lambda^y e^{-\lambda} p^y}{y!} \sum_{x=y}^{\infty} \frac{\lambda^{x-y}(1-p)^{x-y}}{(x-y)!} \\
&= \frac{e^{-\lambda}(\lambda p)^y}{y!} \sum_{x=y}^{\infty} \frac{[\lambda(1-p)]^{x-y}}{(x-y)!} \\
&= \frac{e^{-\lambda}(\lambda p)^y}{y!} e^{\lambda(1-p)} \\
&= \frac{e^{-\lambda p}(\lambda p)^y}{y!} \qquad\qquad y = 0, 1, 2, \ldots
\end{aligned}$$

Note that $p_Y(y) \sim POI(\lambda p)$.
Thus

$$\begin{aligned}
p(x \mid y) &= \frac{P(X=x, Y=y)}{P(Y=y)} \\
&= \frac{\frac{e^{-\lambda}\lambda^x}{x!} \cdot \frac{x!}{(x-y)!y!} p^y(1-p)^{x-y}}{\frac{e^{-\lambda p}(\lambda p)^y}{y!}} \\
&= \frac{e^{-\lambda+\lambda p}[\lambda(1-p)]^{x-y}}{(x-y)!} \\
&= \frac{e^{-\lambda(1-p)}[\lambda(1-p)]^{x-y}}{(x-y)!} \qquad\qquad x = y, y+1, y+2, \ldots
\end{aligned}$$

This resembles the POIson distribution with $\lambda = \lambda(1-p)$ but with a slightly modified domain.
So we see that

$$W \mid (Y = y) \sim W + y$$

where $W \sim POI(\lambda(1-p))$. This is the **shifted Poisson pmf** y units to the right (note that W and y are random variables).
We can easily find the conditional expectations and variance e.g.

$$E[X \mid Y = y] = E[W + y] = E[W] + y$$

5.3 Example 2.5 solution

Suppose the joint pdf of X and Y is

$$f(x, y) = \begin{cases} \frac{12}{5}x(2-x-y) & , 0 < x < 1, 0 < y < 1, \\ 0 & , \text{ elsewhere} \end{cases}$$

Determine the conditional distribution of X given $Y = y$ where $0 < y < 1$. Also calculate the mean of $X \mid (Y = y)$. (Note: the graphical region is a unit square box where the bottom left corner is at $0, 0$: the inside of the box is the support).

Solution. Using our theory, we wish to find the conditional pdf of $X \mid (Y = y)$ given by

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)}$$

For $0 < y < 1$

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\
&= \int_0^1 \frac{12}{5}x(2-x-y)dx \\
&= \frac{12}{5}\int_0^1 (2x - x^2 - xy)dx \\
&= \frac{12}{5}(x^2 - \frac{x^3}{3} - \frac{x^2 y}{2})\Big|_0^1 \\
&= \frac{12}{5}(1 - \frac{1}{3} - \frac{y}{2}) \\
&= \frac{2}{5}(4 - 3y)
\end{aligned}$$

So we have

$$\begin{aligned}
f_{X|Y}(x \mid y) &= \frac{\frac{12}{5}x(2-x-y)}{\frac{2}{5}(4-3y)} \\
&= \frac{6x(2-x-y)}{4-3y}
\end{aligned}$$

Thus we have

$$\begin{aligned}
E[X \mid Y] &= \int_0^1 x \cdot f_{X|Y}(x \mid y)dx \\
&= \frac{5-4y}{2(4-3y)}
\end{aligned}$$

6 January 18, 2018

6.1 Example 2.6 solution

Suppose the joint pdf of X and Y is

$$f(x,y) = \begin{cases} 5e^{-3x-y} & ,0 < 2x < y < \infty, \\ 0 & ,\text{ otherwise} \end{cases}$$

Find the conditional distribution of $Y \mid (X = x)$ where $0 < x < \infty$.
Note the region of support is a "flag" (upright triangle with downawrd point) where the slanted part is the line $y = 2x$.

Solution. We wish to find

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)}$$

For $0 < x < \infty$

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_{2x}^{\infty} 5e^{-3x-y}dy \\
&= 5e^{-3x}\int_{2x}^{\infty} 5e^{-y}dy \\
&= 5e^{-3x}(-e^{-y})\Big|_{2x}^{\infty} \\
&= 5e^{-3x}e^{-2x}) \\
&= 5e^{-5x}
\end{aligned}$$

so we have $f_X(x) \sim Exp(5)$.

Remark 6.1. The bounds on the integral are in terms of y: it is dependent on x in our $f(x,y)$ definition.

Now

$$\begin{aligned}
f_{Y|X}(y \mid x) &= \frac{5e^{-3x-y}}{5e^{-5x}} \\
&= e^{-y+2x} \quad y > 2x
\end{aligned}$$

Note: recognize the conditional pdf of $Y \mid (X = x)$ as that of a shifted exponential distribution ($2x$ units to the right). Specifically, we have

$$Y \mid (X = x) \sim W + 2x$$

where $W \sim Exp(1)$. Thus $E[Y \mid (X = x)] = E(W) + 2x$ and $Var[Y \mid (X = x)] = Var(W)$.

6.2 Example 2.7 solution

Suppose $X \sim U(0,1)$ and $Y \mid (X = x) \sim Bern(x)$. Find the conditional distribution $X \mid (Y = y)$.
Note: X is continuous and $Y \mid (X = x)$ is discrete.

Solution. We wish to find

$$f_{X|Y}(x \mid y) = \frac{p(y \mid x)f_X(x)}{p_Y(y)}$$

From the given information, we have $f_X(x) = 1$ for $0 < x < 1$ Furthermore $p(y \mid x) = Bern(x) = x^y(1-x)^{1-y}$ for $y = 0, 1$.
For $y = 0, 1$ note that (from $\int f(x \mid y)dx = 1$)

$$\begin{aligned}
p_Y(y) &= \int_{-\infty}^{\infty} p(y \mid x)f_X(x)dx \\
p_Y(y) &= \int_0^1 x^y(1-x)^{1-y}dx
\end{aligned}$$

To compute this integral, let's check $p_Y(0)$ and $p_Y(1)$

$$\begin{aligned} p_Y(0) &= \int_0^1 x^0(1-x)^{1-0}dx \\ &= \int_0^1 1 - xdx \\ &= x - \frac{x^2}{2}\Big|_0^1 \\ &= \frac{1}{2} \end{aligned}$$

Similarly, take $y = 1$ where $p_Y(1) = \frac{1}{2}$.
In other words, we have that $p_Y(y) = \frac{1}{2}$ $y = 0, 1$ so

$$Y \sim Bern\left(\frac{1}{2}\right)$$

So

$$\begin{aligned} f(x \mid y) &= \frac{p(y \mid x) f_X(x)}{p_Y(y)} \\ &= \frac{x^y(1-x)^{1-y} \cdot 1}{\frac{1}{2}} \\ &= 2x^y(1-x)^{1-y} \quad 0 < x < 1 \end{aligned}$$

6.3 Theorem 2.2 (law of total expectation)

Theorem 6.1. For random variables X and Y, $E[X] = E[E[X \mid Y]]$.

Proof. WLOG assume X, Y are jointly continuous random variables. We note

$$\begin{aligned} E[E[X \mid Y]] &= \int_{-\infty}^{\infty} E[X \mid Y = y] f_Y(y) dy \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x f_{X|Y}(x \mid y) dx\right] f_Y(y) dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \frac{f(x,y)}{f_Y(y)} \cdot f_Y(y) dx dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y) dx dy \\ &= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f(x,y) dy\right] dx \\ &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= E[X] \end{aligned}$$

□

6.4 Example 2.8 solution

Suppose $X \sim GEO(p)$ with pmf $p_X(x) = (1-p)^{x-1}p$ where $x = 1, 2, 3, \ldots$. Calculate $E[X]$ and $Var(X)$ using the law of total expectation.

Solution. Recall $E[X] = \frac{1}{p}$ and $Var(X) = \frac{1-p}{p^2}$ where X models the number of (independent) trials necessary to obtain the first success.
Remember: we could manually solve $E[X] = \sum_{x=1}^{\infty}(1-p)^{x-1}p$ and similarly $Var(X) = E[X^2] - E[X]$, or take the derivatives of the mgf $\Phi_X(t) = E[e^{tX}]$. This is tedious in general.

7 Tutorial 2

7.1 Sum of geometric distributions

Let X_i for $i = 1, 2, 3$ be independent geometric random variables having the same parameter p. Determine the value

$$P(X_j = x_j \mid \sum_{i=1}^{3} X_i = n)$$

Solution. Note that, by construction, the sum of k independent $GEO(p)$ random variables is distributed as $NB(k, p)$. Recall that

$$\begin{aligned}
&X_i \sim GEO(p) \Rightarrow P_{X_i}(x) = (1-p)^{x-1}p && x = 1, 2, 3, \ldots \\
&Y \sim NB(k, p) \Rightarrow P_Y(y) = \binom{y-1}{k-1} p^k (1-p)^{y-k} && y = k, k+1, k+2, \ldots
\end{aligned}$$

Breaking apart the summation

$$\begin{aligned}
P(X_j = x_j \mid \sum_{i=1}^{3} X_i = n) &= P(X_j = x_j \mid X_j + \sum_{i=1, i\neq j}^{3} X_i = n) \\
&= \frac{P(X_j = x_j, X_j + \sum_{i=1,i\neq j}^{3} X_i = n)}{P(\sum_{i=1}^{3} X_i = n)} \\
&= \frac{P(X_j = x_j, \sum_{i=1,i\neq j}^{3} X_i = n - x_j)}{P(\sum_{i=1}^{3} X_i = n)} \\
&= \frac{P(X_j = x_j) \cdot P(\sum_{i=1,i\neq j}^{3} X_i = n - x_j)}{P(\sum_{i=1}^{3} X_i = n)} && X_i\text{'s are independent} \\
&= \frac{(1-p)^{x_j-1}p \cdot \binom{n-x_j-1}{1} p^2 (1-p)^{n-x_j-2}}{\binom{n-1}{2} p^3 (1-p)^{n-3}} && \text{provided that } x_j \geq 1 \text{ and } n - x_j \geq 2 \\
&= \frac{(1-p)^{x_j-1}p \cdot \binom{n-x_j-1}{1} p^2 (1-p)^{n-x_j-2}}{\binom{n-1}{2} p^3 (1-p)^{n-3}} \\
&= \frac{(n-x_j-1)!}{1!(n-x_j-2)!} \cdot \frac{2!(n-3)!}{(n-1)!} \\
&= \frac{2(n-x_j-1)}{(n-1)(n-2)} \quad x_j = 1, 2, \ldots, n-2
\end{aligned}$$

Note this is a pmf so we can check

$$\begin{aligned}\sum_{x_1}^{n-2} \frac{2(n-x_1)}{(n-1)(n-2)} &= \sum_{x_1}^{n-2} \frac{2(n-1)}{(n-1)(n-2)} - \sum_{x_1}^{n-2} \frac{2x}{(n-1)(n-2)} \\ &= \frac{2(n-1)(n-2)}{(n-1)(n-2)} - \frac{2}{(n-1)(n-2)} \sum_{x=1}^{n-2} x \\ &= 2 - \frac{2}{(n-1)(n-2)} \cdot \frac{(n-2)(n-1)}{2} \\ &= 2-1 \\ &= 1\end{aligned}$$

which satisfies the cdf axiom.

7.2 Conditional card drawing

Given $N \in \mathbb{Z}^+$ cards labelled $1, 2, \dots, N$, let X represent the number that is picked. Suppose a second card Y is picked from $1, 2, \dots, X$.
Assuming $N = 10$, calculate the expected value of X given $Y = 8$.

Solution. Clearly we have that $P_X(x) = \frac{1}{N}$ where $x = 1, 2, \dots, N$ and $P_{Y|X}(y \mid x) = \frac{1}{x}$ for $y = 1, 2, \dots, x$.
To find the conditional distribution of $X \mid (Y = y)$ we must identify the joint distribution of X, Y. It immediately follows that

$$p(x,y) = P(X = x, Y = y) = P_{Y|X}(y \mid x) P_X(x) = \frac{1}{xN}$$

for $x = 1, 2, \dots, N$ and $y = 1, 2, \dots, x$. or equivalently the range can be re-expressed as

$$y = 1, 2, \dots, N \text{ and } x = y, y+1, \dots, N$$

Remark 7.1. Whenever we want to find the marginal pmf/pdf for a given rv Y, we generally need to re-map the support such that the support of Y is independent of the other rv X.

Note that

$$\begin{aligned}P_Y(y) = \sum_{x=y}^{N} p(x,y) &= \sum_{x=y}^{N} \frac{1}{xN} \\ &= \frac{1}{N} \sum_{x=y}^{N} \frac{1}{x} \quad y = 1, 2, \dots, N\end{aligned}$$

Letting $N = 10$, we can calculate

$$\begin{aligned}
E[X \mid Y = 8] &= \sum_{x=8}^{10} x P_{X|Y}(x \mid 8) \\
&= \sum_{x=8}^{10} x \frac{P(x,8)}{P_Y(8)} \\
&= \sum_{x=8}^{10} x \frac{\frac{1}{10x}}{\frac{1}{10}\sum_{z=8}^{10}\frac{1}{z}} \\
&= \sum_{x=8}^{10} x (\sum_{z=8}^{10} \frac{1}{z})^{-1} \\
&= 3(\frac{1}{8} + \frac{1}{9} + \frac{1}{10})^{-1} \\
&= 3(\frac{242}{720})^{-1} \\
&= \frac{1080}{121} \approx 8.9256
\end{aligned}$$

7.3 Conditional points from interval

Let us choose a random point from interal $(0, 1)$ denoted as rv X_1. We then choose a random point X_2 on the interval $(0, x_1)$ hwere x_1 is the realized value of X_1.

1. Make assumptions about the marginal pdf $f_1(x_1)$ and conditional pdf $f_{2|1}(x_2 \mid x_1)$.
2. Find the conditional mean $E[X_1 \mid X_2 = x_2]$.
3. Compute $P(X_1 + X_2 \geq 1)$.

Solution.

1. It makes sense that $X_1 \sim U(0, 1)$ and $X_2 \mid (X_1 = x_1) \sim U(0, x_1)$ so that $f_1(x_1) = 1$, $0 < x_1 < 1$ and $f_{2|1}(x_2 \mid x_1) = \frac{1}{x_1}$ for $0 < x_2 < x_1 < 1$.
2. Note that $f_{1|2}(x_1 \mid x_2) = \frac{f(x_1,x_2)}{f_2(x_2)}$ and so we need to identify the joint distribution of x_1 and x_2 as well as the marginal distribution of X_2. We have

$$\begin{aligned}
f(x_1, x_2) &= f_{2|1}(x_2 \mid x_1) \cdot f_1(x_1) \\
&= \frac{1}{x_1} & 0 < x_2 < x_1 < 1 \quad 0 < x_1 < 1
\end{aligned}$$

or equivalently, the region of support can be re-expressed as

$$\begin{aligned}
0 < x_2 < 1 \\
x_2 < x_1 < 1
\end{aligned}$$

so the marginal pdf of $f_2(x_2)$ is

$$\begin{aligned}
f_2(x_2) &= \int_{x_1=x_2}^{1} p(x_1,x_2)dx_1 \\
&= \int_{x_1=x_2}^{1} \frac{1}{x_1}dx_1 \\
&= \ln(x_1)\Big|_{x_1=x_2}^{x_1=1} \\
&= -\ln(x_2) \quad 0<x_2<1
\end{aligned}$$

so the conditional pdf is

$$\begin{aligned}
f_{1|2}(x_1 \mid x_2) &= \frac{f(x_1,x_2)}{f_2(x_2)} \\
&= \frac{1}{-x_1\ln(x_2)} \quad 0<x_2<x_1<1
\end{aligned}$$

Taking the expectation

$$\begin{aligned}
E[X_1 \mid X_2=x_2] &= \int_{x_1=x_2}^{1} x_1 p_{1|2}(x_1,x_2)dx_1 \\
&= \int_{x_1=x_2}^{1} x_1 \cdot \frac{1}{-x_1\ln(x_2)}dx_1 \\
&= \int_{x_1=x_2}^{1} \frac{1}{-\ln(x_2)}dx_1 \\
&= \frac{1-x_2}{-\ln(x_2)} \quad 0<x_2<1
\end{aligned}$$

Exercise: solve for $lim_{x_2\to 1}E[X_1 \mid X_2=x_2]$ (use LHR).

3. The probability that $X_1+X_2 \geq 1$ may be calculated by taking the double integral over the region R of their support where $X_1+X_2\geq 1$ holds. This region may be found as follows:

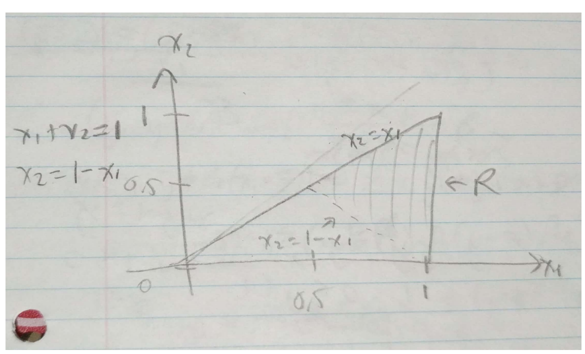


Figure 7.1: The region R is the support where $X_1+X_2\geq 1$.

The region R is equivalent to the bounds $\frac{1}{2}<x_1<1$ and $1-x_1<x_2<x_1$.

Integrating $f(x_1, x_2)$ over R we obtain

$$\begin{aligned}
P(X_1 + X_2 \geq 1) &= \int_R \int f(x_1, x_2) dx_2 dx_1 \\
&= \int_{\frac{1}{2}}^{1} \int_{1-x_1}^{x_1} \frac{1}{x_1} dx_2 dx_1 \\
&= \int_{\frac{1}{2}}^{1} \frac{x_2}{x_1} \Big|_{x_2=1-x_1}^{x_2=x_1} dx_1 \\
&= \int_{\frac{1}{2}}^{1} (2 - \frac{1}{x_1}) dx_1 \\
&= (2x_1 - \ln(x_1)) \Big|_{x_1=\frac{1}{2}}^{x_1=1} \\
&= 1 + \ln(\frac{1}{2}) \\
&= 1 - \ln(2) \\
&\approx 0.3068528
\end{aligned}$$

8 January 23, 2018

8.1 Example 2.8 solution

Suppose $X \sim GEO(p)$ with pmf $p_X(x) = (1-p)^{x-1}p$ for $x = 1, 2, 3 \ldots$. Calculate $E[X], Var(X)$ using the law of total expectation.

Solution. Recall X is modelling the number of trials needed to obtain the **1st success**. We want to calculate $E[X]$ and $Var(X)$ using the total law of expectation.
Define

$$Y = \begin{cases} 0 & \text{if the 1st trial is a failure} \\ 1 & \text{if the 1st trial is a success} \end{cases}$$

Note that $Y \sim Bern(p)$ so that $P_Y(0) = P(Y = 0) = 1 - p$ and similarly $P_Y(1) = P(Y = 1) = p$.
Thus by the law of total expectation

$$\begin{aligned}
E[X] &= E[E[X \mid Y]] \\
&= \sum_{y=0}^{1} E[X \mid Y = y] p_Y(y) \\
&= (1-p)E[X \mid Y = 0] + pE[X \mid Y = 1]
\end{aligned}$$

Note that

$$X \mid (Y = 1) = 1$$

with probability 1 (one success is equivalent to $X = 1$ for $GEO(p)$), and

$$X \mid (Y = 0) \sim 1 + X$$

(the first one failed, we expect to take X more trials; same initial problem - recurse. See course notes for formal proof).

Thus we have

$$\begin{aligned}
E[X] &= (1-p)E[1+X] + p(1) \\
&= (1-p)(1+E[X]) + p \\
&= 1 + (1-p)E[X] \\
\Rightarrow & E[X](1-(1-p)) = 1 \\
\Rightarrow & E[X] = \frac{1}{p}
\end{aligned}$$

as expected.
For $Var(X)$, notice that

$$\begin{aligned}
E[X^2] &= E[E[X^2 \mid Y]] \\
&= \sum_{y=0}^{1} E[X^2 \mid Y=y]p_Y(y) \\
&= (1-p)E[X^2 \mid Y=0] + pE[X^2 \mid Y=1] \\
&= (1-p)E[(1+X)^2] + p(1)^2 && \text{from above} \\
&= (1-p)E[1+2X+X^2] + p \\
&= (1-p)(1+2E[X]+E[X^2]) + p \\
&= 1 + 2(1-p)E[X] + (1-p)E[X^2] \\
\Rightarrow & E[X^2](1-(1-p)) = 1 + \frac{2(1-p)}{p} \\
\Rightarrow & E[X^2] = \frac{1}{p} + \frac{2(1-p)}{p^2}
\end{aligned}$$

So we have

$$\begin{aligned}
Var(X) &= E[X^2] - E[X]^2 \\
&= \frac{1}{p} + \frac{2(1-p)}{p^2} - \frac{1}{p^2} \\
&= \frac{p+2-2p-1}{p^2} \\
&= \frac{1-p}{p^2}
\end{aligned}$$

Remark 8.1. For law of total expectations, a large part of it is choosing the right random variable to condition on (i.e. $Y = Bern(p)$ in this example).

8.2 Theorem 2.3 (variance as expectation of conditionals)

Theorem 8.1. For random variables X and Y

$$Var(X) = E[Var(X \mid Y)] + Var(E[X \mid Y])$$

Proof. Recall that

$$Var(X \mid Y=y) = E[X^2 \mid Y=y] + E[X \mid Y=y]^2$$

so more generally we have

$$Var(X \mid Y) = E[X^2 \mid Y] + E[X \mid Y]^2$$

Taing the expectation of this

$$\begin{aligned} E[Var(X \mid Y)] &= E[E[X^2 \mid Y] - E[X \mid Y]^2] \\ &= E[E[X^2 \mid Y]] - E[E[X \mid Y]^2] \\ &= E[X^2] - E[E[X \mid Y]^2] && E[A] = E[E[A \mid B]] \text{ (law of total expectation)} \end{aligned}$$

Note that

$$Var(E[X \mid Y]) = Var(v(Y))$$

where $v(Y) = E[X \mid Y]$ is a function of Y (not X!).

$$\begin{aligned} Var(v(Y)) &= E[v(Y)^2] - E[v(Y)]^2 \\ &= E[E[X \mid Y]^2] - E[X]^2 && \text{law of total expectation} \end{aligned}$$

Therefore we have

$$\begin{aligned} E[Var(X \mid Y)] + Var(E[X \mid Y]) &= E[X^2] - E[E[X \mid Y]^2] + E[E[X \mid Y]^2] - E[X]^2 \\ &= E[X^2] - E[X]^2 \\ &= Var(X) \end{aligned}$$

as desired. □

8.3 Example 2.9 solution

Suppose $\{X_i\}_{i=1}^{\infty}$ is an iid sequence of random variables with common mean μ and variance σ^2. Let N be a discrete, non-negative integer-valued rv that is independent of each X_i.
Find the mean and variance of $T = \sum_{i=1}^{N} X_i$ (referred to as a **random sum**).

Solution. To find the mean:
We condition on N since the value of our T depends on how many X_i's there are which depends on N. By the law of total expectations

$$E[T] = E[E[T \mid N]]$$

Note that

$$\begin{aligned} E[T \mid N = n] &= E[\sum_{i=1}^{N} X_i \mid N = n] \\ &= E[\sum_{i=1}^{n} X_i \mid N = n] \\ &= \sum_{i=1}^{n} E[X_i \mid N = n] && \text{due to independence of } X_i \text{ and } N \\ &= \sum_{i=1}^{n} E[X_i] \\ &= n\mu \end{aligned}$$

So we have $E[T \mid N] = N\mu$.

Remark 8.2. We needed to first condition on a concrete $N = n$ in order to unwrap the summation, then revert back to the random variable N.

Thus we have

$$E[T] = E[E[T \mid N]] = E[N\mu] = \mu E[N]$$

which intuitively makes sense.

To find the variance:

We use our previous theorem on variance as expectation of conditionals

$$Var(T) = E[Var(T \mid N)] + Var(E[T \mid N])$$

We know from before that

$$Var(E[T \mid N]) = Var(N\mu) = \mu^2 Var(N)$$

We can break apart the variance as

$$\begin{aligned}
Var(T \mid N = n) &= Var(\sum_{i=1}^{N} X_i \mid N = n) \\
&= Var(\sum_{i=1}^{n} X_i \mid N = n) \\
&= Var(\sum_{i=1}^{n} X_i \\
&= \sum i = 1^n Var(X_i) && \text{independence of } X_i \\
&= \sigma^2 n
\end{aligned}$$

Therefore $Var(T \mid N) Var(T \mid N = n)\Big|_{n=N} = \sigma^2 N$.

So

$$E[Var(T \mid N)] = E[\sigma^2 N] = \sigma^2 E[N]$$

and thus

$$Var(T) = \sigma^2 E[N] + \mu^2 Var(N)$$

9 January 25, 2018

9.1 Example 2.10 solution ($P(X < Y)$)

Suppose X and Y are independent continuous random variables. Find an expression for $P(X < Y)$.

Solution. Define our event of interest as

$$A = \{X < Y\}$$

Thus we have

$$
\begin{aligned}
P(X<Y)=P(A) &= \int_{-\infty}^{\infty} P(A \mid Y=y) f_Y(y)dy && \text{law of total probability}\\
&= \int_{-\infty}^{\infty} P(X<Y \mid Y=y) f_Y(y)dy\\
&= \int_{-\infty}^{\infty} P(X<y \mid Y=y) f_Y(y)dy\\
&= \int_{-\infty}^{\infty} P(X<y) f_Y(y)dy && X<y \text{ only depends on X; } Y=y \text{ only depends on Y}\\
&= \int_{-\infty}^{\infty} P(X\le y) f_Y(y)dy && X \text{ is a continuous rv}\\
&= \int_{-\infty}^{\infty} F_X(y) f_Y(y)dy
\end{aligned}
$$

Suppose that X and Y have the same distribution. We expect $P(X<Y)=\frac{1}{2}$. Let's verify it with our expression

$$
\begin{aligned}
P(X<Y) &= \int_{-\infty}^{\infty} F_X(y) f_Y(y)dy\\
&= \int_{-\infty}^{\infty} F_Y(y) f_Y(y)dy && X \sim Y
\end{aligned}
$$

Let $u=F_Y(y)$, thus $\frac{du}{dy}=f_Y(y) \iff du=f_Y(y)dy$. So we have

$$
\begin{aligned}
P(X<Y) &= \int_0^1 udu && \text{domain for a CDF is } [0,1]\\
&= \left.\frac{u^2}{2}\right|_0^1\\
&= \frac{1}{2}
\end{aligned}
$$

9.2 Example 2.11 solution

Suppose $X \sim Exp(\lambda_1)$ and $Y \sim Exp(\lambda_2)$ are independent exponential rvs. Show that

$$P(X<Y)=\frac{\lambda_1}{\lambda_1+\lambda_2}$$

Solution. Since $Y \sim Exp(\lambda_2)$, then we have $f_Y(y)=\lambda_2 e^{-\lambda y}$ for $y>0$.
Since $X \sim Exp(\lambda_1)$, we have

$$
\begin{aligned}
F_X(x)=P(X\le x) &= \int_0 x\lambda_1 e^{-\lambda_1 x}dx\\
&= \left.-e^{-\lambda_1 x}\right|_0^x\\
&= 1-e^{-\lambda_1 x} \quad x\ge 0
\end{aligned}
$$

From the expression in Example 2.10, we have

$$\begin{aligned}
P(X < Y) &= \int_0^\infty F_X(y) f_Y(y) dy \\
&= \int_0^\infty (1 - e^{-\lambda_1 y})(\lambda_2 e^{-\lambda_2 y}) dy \\
&= \int_0^\infty \lambda_2 e^{-\lambda y} - \lambda_2 e^{-(\lambda_1+\lambda_2)y} dy \\
&= \int_0^\infty \lambda_2 e^{-\lambda y} + \frac{\lambda_2}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)y} \Big|_0^\infty \qquad = 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2}
\end{aligned}$$

9.3 Example 2.12 solution

Consider an experiment in which independent trials each having probability $p \in (0, 1)$ are performed until $k \in \mathbb{Z}^+$ consecutive successes are achieved. Determined the expected number of trails for k consecutive successes.

Solution. Let N_k be the rv which counts the number of trials needed to obtain k consecutive successes.
Current goal: we want to find $E[N_k]$.
Note: when $n = 1$, then we have $N_1 \sim GEO(p)$, and so $E[N_1] = \frac{1}{p}$.
For arbitrary $k \geq 2$, we will try to find $E[N_k]$ using the law of total expectations, namely

$$E[N_k] = E[E[N_k \mid W]]$$

for some W rv we *choose carefully.*
Suppose we choose W where (we will later see why this won't work)

$$W = \begin{cases} 0 & \text{if first trial is a failure} \\ 1 & \text{if first trial is a success} \end{cases}$$

So we have

$$\begin{aligned}
E[N_k] &= \sum_w E[N_k \mid W = w] P(W = w) \\
&= P(W = 0) E[N_k \mid W = 0] + P(W = 1) E[N_k \mid W = 1] \\
&= (1 - p) E[N_k \mid W = 0] + p E[N_k \mid W = 1]
\end{aligned}$$

Note that

$$\begin{aligned}
&N_k \mid (W = 0) \sim 1 + N_k \\
&N_k \mid (W = 1) \sim ?
\end{aligned}$$

We can't simply have $N_k \mid (W = 1) \sim 1 + N_{k-1}$ since N_{k-1} does not guarantee that the $k - 1$ consecutive successes are followed immediately after our first $W = 1$.
Perhaps we need another W, $W = N_{k-1}$ so we attempt to find

$$E[N_k] = E[E[N_k \mid N_{k-1}]]$$

Consider

$$E[N_k \mid N_{k-1} = n]$$

conditional on $N_{k-1} = n$, defin

$$Y = \begin{cases} 0 & \text{if the } (n+1)\text{th trial is a failure} \\ 1 & \text{if the } (n+1)\text{th trial is a success} \end{cases}$$

Now we have

$$\begin{aligned} E[N_k \mid N_{k-1} = n] &= \sum_y E[N_k \mid N_{k-1} = n, Y = y] P(Y = y \mid N_{k-1} = n) \\ &= P(Y = 0 \mid N_{k-1} = n) E[N_k \mid N_{k-1} = n, Y = 0] \\ &+ P(Y = 1 \mid N_{k-1} = n) E[N_k \mid N_{k-1} = n, Y = 1] \\ &= (1-p) E[N_k \mid N_{k-1} = n, Y = 0] + pE[N_k \mid N_{k-1} = n, Y = 1] \quad Y \text{ is independent from } N_{k-1} \end{aligned}$$

Note that

$$\begin{aligned} N_k \mid (N_{k-1} = n \mid Y = 0) &\sim n + 1 + N_k \text{we need to start over again} \\ N_k \mid (N_{k-1} = n \mid Y = 1) &\sim n + 1 \text{ with probability 1} \end{aligned}$$

Therefore

$$\begin{aligned} E[N_k \mid N_{k-1} = n] = (1-p)(n + 1 + E[N_k]) + p(n+1) \\ = n + 1 + (1-p)E[N_k] \end{aligned}$$

which in terms of the rv N_{k-1}

$$E[N_k \mid N_{k=1}] = E[N_k \mid N_{k-1} = n]\Big|_{n=N_{k-1}} = N_{k-1} + 1 + (1-p)E[N_k]$$

Thus from the law of total expectations

$$\begin{aligned} E[N_k] &= E[E[N_k \mid N_{k-1}]] \\ &= E[N_{k-1} + 1 + (1-p)E[N_k]] \\ &= E[N_{k-1}] + 1 + (1-p)E[N_k] \\ \Rightarrow E[N_k] &= \frac{1}{p} + \frac{E[N_{k-1}]}{p} \end{aligned}$$

This is a recurrence relation for $k = 2, 3, 4, \ldots$. To solve, we check for some k values to gain some intuition

$$\begin{aligned} &k = 2 \Rightarrow E[N_2] = \frac{1}{p} + \frac{E[N_1]}{p} = \frac{1}{p} + \frac{1}{p^2} \\ &k = 3 \Rightarrow E[N_3] = \frac{1}{p} + \frac{E[N_2]}{p} = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} \\ &\vdots \\ &E[N_k] = \sum_{i=1}^{k} \frac{1}{p^i} \quad k = 1, 2, 3, \ldots \qquad \text{by induction} \end{aligned}$$

This is the finite geometric series for $r = \frac{1}{p}$, thus we have

$$E[N_k] = \frac{\frac{1}{p} - \frac{1}{p^{k+1}}}{1 - \frac{1}{p}}$$

10 Tutorial 3

10.1 Mixed conditional distribution

Suppose X is $Erlang(n, \lambda)$ with pdf

$$f_X(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} \quad x > 0$$

Suppose $Y \mid (X = x)$ is $POI(x)$ with pmf

$$p_{Y|X}(y \mid x) = \frac{e^{-x} x^y}{y!} \quad y = 0, 1, 2, \ldots$$

Find the condition distribution $X \mid (Y = y)$.

Solution. The marginal distribution of Y is characterized by its pmf

$$\begin{aligned}
p_Y(y) &= \int_{-\infty}^{\infty} p_{Y|X}(y \mid x) f_X(x) dx \\
&= \int_{-\infty}^{\infty} p_{Y|X}(y \mid x) f_X(x) dx \\
&= \int_0^{\infty} \frac{e^{-x} x^y}{y!} \cdot \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} dx \\
&= \frac{\lambda^n}{y!(n-1)!} \int_0^{\infty} x^{n+y-1} e^{-(\lambda+1)x} dx \\
&= \frac{\lambda^n (n+y-1)!}{(\lambda+1)^{n+y} y!(n-1)!} \int_0^{\infty} \frac{(\lambda+1)^{n+y} x^{n+y-1} e^{-(\lambda+1)x}}{(n+y-1)!} dx \\
&= \frac{\lambda^n (n+y-1)!}{(\lambda+1)^{n+y} y!(n-1)!} && \text{integral of pdf } Erlang(n+y, \lambda+1) \\
&= \binom{n+y-1}{n-1} \left(\frac{\lambda}{\lambda+1}\right)^n \left(\frac{1}{\lambda+1}\right)^y \quad y = 0, 1, 2, \ldots
\end{aligned}$$

Note that $p_Y(y)$ is the Negative Binomial distribution shifted to the left n units. In other words, it counts the number of "failures" before n successes, where the probability of success if $\lambda/(\lambda+1)$.
The distribution of $X \mid (Y = y)$ is thus

$$\begin{aligned}
f_{X|Y}(x \mid y) &= \frac{p_{Y|X}(y \mid x) f_X(x)}{p_Y(y)} \\
&= \frac{\frac{e^{-x} x^y}{y!} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}}{\frac{(n+y-1)!}{y!(n-1)!} \frac{\lambda^n}{(\lambda+1)^{n+y}}} \\
&= \frac{(\lambda+1)^{n+y} x^{n+y-1} e^{-(\lambda+1)x}}{(n+y-1)!} \quad x > 0
\end{aligned}$$

Note that $f_{X|Y}(x \mid y)$ is exactly the Erlang distribution $Erlang(n+y, \lambda+1)$.

10.2 Law of total expectations

1. Let $\{X_i\}_{i=1}^{\infty}$ an iid sequence of $EXP(\lambda)$ random variables and let $N \sim GEO(p)$ be independent of each X_i. Find $E[\prod_{i=1}^{N} X_i]$.
2. Let $\{X_i\}_{i=0}^{\infty}$ an iid sequence where $X_i \sim BIN(10, 1/2^i)$, $i = 0, 1, 2, \ldots$. Also let $N \sim POI(\lambda)$ be independent of each X_i. Find $E[X_N]$.

Solution. 1. We want to first find $E[\prod_{i=1}^{N} X_i \mid N = n]$ (conditioning on $N = n$)

$$\begin{aligned}
E[\prod_{i=1}^{N} X_i \mid N = n] &= E[\prod_{i=1}^{n} X_i \mid N = n] \\
&= E[\prod_{i=1}^{n} X_i] && \text{independence of } X_i's \text{ and } N \\
&= \prod_{i=1}^{n} E[X_i] && \text{independence of } X_i's \\
&= \prod_{i=1}^{n} \frac{1}{\lambda} \\
&= \frac{1}{\lambda^n}
\end{aligned}$$

Thus by the law of total expectations

$$\begin{aligned}
E[\prod_{i=1}^{N} X_i] &= E[E[\prod_{i=1}^{N} X_i \mid N = n]] \\
&= \sum_{n=1}^{\infty} \frac{1}{\lambda^n}(1-p)^{n-1}p \\
&= \frac{p}{\lambda^n} \sum_{n=1}^{\infty} (1-p)^{n-1} \\
&= \frac{p}{\lambda} \sum_{n=1}^{\infty} \left(\frac{1-p}{\lambda}\right)^{n-1} \\
&= \frac{p}{\lambda(1-\frac{1-p}{\lambda})} \sum_{n=1}^{\infty} \left(\frac{1-p}{\lambda}\right)^{n-1} \left(1 - \frac{1-p}{\lambda}\right) \\
&= \frac{p}{\lambda(1-\frac{1-p}{\lambda})} && \text{summation of pmf of } GEO(\frac{1-p}{\lambda}) \\
&= \frac{p}{\lambda - 1 + p}
\end{aligned}$$

provided that $\frac{1-p}{\lambda} < 1$ or $1 - p < \lambda$.

2. Condition on $N = n$ we have

$$E[X_N \mid N = n] = E[X_n \mid N = n] = E[X_n] = 10 \cdot \frac{1}{2^n} = \frac{10}{2^n}$$

From the law of total expectations

$$\begin{aligned}
E[X_N] &= E[E[X_N \mid N = n]] \\
&= \sum_{n=0}^{\infty} \frac{10}{2^n} \frac{e^{-\lambda}\lambda^n}{n!} \\
&= 10e^{-\lambda/2} \sum_{n=0}^{\infty} \frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^n}{n!} \\
&= 10e^{-\lambda/2} && \text{summation of pmf of } POI(\lambda/2)
\end{aligned}$$

10.3 Conditioning on wins and losses

A, B, C are evenly matched tennis players. Initially A and B play a set, and the winner plays C. The winner of each set continues playing the waiting player until one player wins two sets *in a row*. What is the probability that A is the overall winner?

Solution. Key idea: we condition on wins and losses each time until we can find some sort of recurrent relationship, eliminating trivial cases along the way.
Let A denote the event that A is the overall winner, and W_i, L_i denote that A wins or loses game i, respectively. Then we have

$$P(A) = P(W_1)P(A \mid W_1) + P(L_1)P(A \mid L_1) = \frac{1}{2}P(A \mid W_1) + \frac{1}{2}P(A \mid L_1)$$

We can then continue conditioning on subsequent games and their possible outcomes

$$\begin{aligned}
P(A \mid W_1) &= \frac{1}{2}P(A \mid W_1, W_2) + \frac{1}{2}P(A \mid W_1, L_2) \\
&= \frac{1}{2}(1) + \frac{1}{2}[P(A \mid W_1, L_2, C \text{ wins}) \\
&+ \frac{1}{2}P(A \mid W_1, L_2, C \text{ loses})] \\
&= \frac{1}{2} + \frac{1}{4}P(A \mid W_1, L_2, C \text{ loses}) && P(A \mid W_1, L_2, C \text{ wins}) = 0 \\
& && \text{since C wins twice in a row} \\
&= \frac{1}{2} + \frac{1}{4}[\frac{1}{2}P(A \mid W_1, L_2, C \text{ loses}, W_4) \\
&+ \frac{1}{2}P(A \mid W_1, L_2, C \text{ loses}, L_4)] \\
&= \frac{1}{2} + \frac{1}{8}P(A \mid W_1, L_2, C \text{ loses}, W_4) && P(A \mid W_1, L_2, C \text{ loses}, L_4) = 0 \\
& && \text{since B wins twice in a row} \\
&= \frac{1}{2} + \frac{1}{8}P(A \mid W_1) && \text{since the probability is the same as} \\
& && \text{A winning its second game after a win}
\end{aligned}$$

Solving this recurrence we get $P(A \mid W_1) = \frac{8}{14}$. Similarly

$$\begin{aligned}
P(A \mid L_1) &= \frac{1}{2}P(A \mid L_1, B \text{ wins}) + \frac{1}{2}P(A \mid L_1, B \text{ loses}) \\
&= \frac{1}{2}[\frac{1}{2}P(A \mid L_1, B \text{ loses}, W_3) + \frac{1}{2}P(A \mid L_1, B \text{ loses}, L_3)] && P(A \mid L_1, B \text{ wins}) = 0 \\
& && \text{since B wins twice in a row} \\
&= \frac{1}{4}P(A \mid L_1, B \text{ loses}, W_3) && P(A \mid L_1, B \text{ loses}, L_3) = 0 \\
& && \text{since C wins twice in a row} \\
&= \frac{1}{4}P(A \mid W_1)
\end{aligned}$$

So $P(A \mid L_1) = \frac{2}{14}$.
Plugging this into our initial equation we get $P(A) = \frac{5}{14}$.

11 February 1, 2018

11.1 Example 3.1 solution

A particle moves along the state $[0, 1, 2]$ according to a DTMC whose TPM is given by

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

where P_{ij} is the transition probability $P(X_n = j \mid X_{n-1} = i)$.
Let X_n denote the position of the particle after the nth move. Suppose the particle is likely to start in any of the three states.

1. Calculate $P(X_3 = 1 \mid X_0 = 0)$.
2. Calculate $P(X_4 = 2)$.
3. Calculate $P(X_6 = 0, X_4 = 2)$.

Solution. 1. We wish to determine $P^{(3)}_{0,1}$. To get this, we proceed to calculate $P^{(3)} = P^3$. So we have

$$P^3 = (P^2)P = \begin{bmatrix} 0.54 & 0.26 & 0.2 \\ 0.2 & 0.36 & 0.44 \\ 0.6 & 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{bmatrix} \tag{11.1}$$

$$= \begin{bmatrix} 0.478 & 0.264 & 0.258 \\ 0.36 & 0.256 & 0.384 \\ 0.57 & 0.18 & 0.25 \end{bmatrix} \tag{11.2}$$

So $P(X_3 = 1 \mid X_0 = 0) = P^{(3)}_{0,1} = 0.264$.

2. We wish to find $\alpha_{4,2} = P(X_4 = 2)$. So

$$\begin{aligned}
\alpha_4 &= (\alpha_{4,0}, \alpha_{4,1}, \alpha_{4,2}) \\
&= \alpha_0 P^{(4)} \\
&= (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) P^3 P \\
&= (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \begin{bmatrix} 0.478 & 0.264 & 0.258 \\ 0.36 & 0.256 & 0.384 \\ 0.57 & 0.18 & 0.25 \end{bmatrix} \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{bmatrix} \\
&= (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \begin{bmatrix} 0.4636 & 0.254 & 0.2824 \\ 0.444 & 0.2256 & 0.3304 \\ 0.524 & 0.222 & 0.254 \end{bmatrix} \\
&= (0.4772, 0.233867, 0.288933)
\end{aligned}$$

So we have $P(X_4 = 2) = 0.288933$.

3. We wish to calculate $P(X_6 = 0, X_4 = 2)$, which is

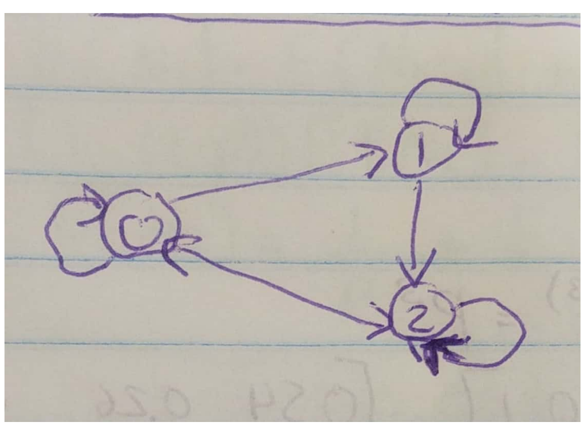
$$\begin{aligned}
P(X_6 = 0, X_4 = 2) &= P(X_4 = 2) P(X_6 = 0 \mid X_4 = 2) \\
&= (0.288433) P(X_2 = 0 \mid X_0 = 2) && \text{by stationary assumption} \\
&= (0.288433) P^{(2)}_{2,0} \\
&= (0.288433)(0.6) \\
&= 0.1733598
\end{aligned}$$

Continued: what are the equivalence classes of the DTMC?

Solution. Remember we have the TPM

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

To answer questions of this nature, it is useful to draw a **statement transition diagram**



We see that all states communicate with each other (there is some path from state i to j and vice versa). Thre is only one equivalence class, namely $\{0, 1, 2\}$. This is an **irreduicble DTMC**.

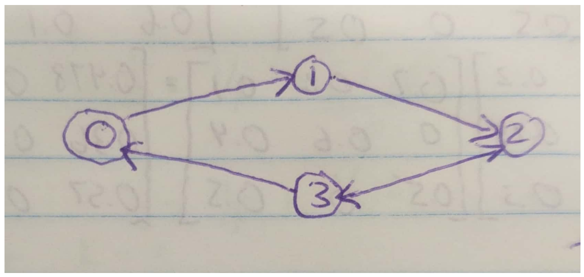
11.2 Example 3.2 solution

Consider a DTMC with TPM

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix}$$

What are the equivalence classes of this DTMC?

Solution. Using a state diagram we have



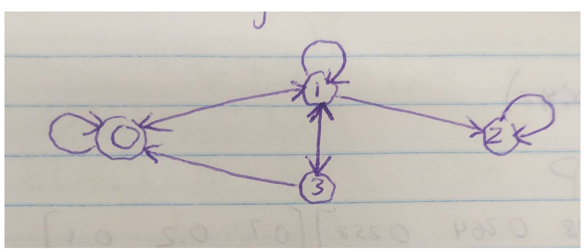
From the diagram, there is only one equivalence class $\{0, 1, 2, 3\}$. This DTMC is irreducible.

11.3 Example 3.3 solution

Consider a DTMC with TPM

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ 0 & 0 & 1 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 \end{bmatrix}$$

What are the equivalence classes of this DTMC?



Solution. From the state diagram there are two equivalence classes: $\{2\}$ and $\{0, 1, 3\}$. Thus this DTMC is not irreducible.

12 Tutorial 4

12.1 Law of total expectations with indicator variables

Suppose the number of people who get on the ground floor of an elevator follows $POI(\lambda)$. If there are m floors above the ground floor and if each person is equally likely to get off at each of the m floors, independent of where the others get off, calculate the expected number of stops the elevator will make before discharging all passengers.

Solution. Let X_i denote the whether or not someone gets off at floor i, that is

$$X_i = \begin{cases} 0 & \text{no one gets off at floor } i \\ 1 & \text{someone gets off at floor } i \end{cases}$$

It is easier to think of the case where no one gets off at a floor. That is for N people

$$P(X_i) = (1 - \frac{1}{m})^N$$

Since X_i is bernoulli, we have

$$E[X_i \mid N = n] = 1 - (1 - \frac{1}{m})^n$$

Let $X = X_1 + \ldots + X_m$ denote the total number of stops for the elevator. Thus we have

$$\begin{aligned} E[X] = E[E[X \mid N = n]] &= \sum_{n=0}^{\infty} E[\sum_{i=1}^{m} X_i \mid N = n] p_N(n) \\ &= \sum_{n=0}^{\infty} m(1 - (1 - \frac{1}{m})^n) \cdot \frac{e^{-\lambda}\lambda^n}{n!} \\ &= m[\sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} - \sum_{n=0}^{\infty} \frac{e^{-\lambda}((1 - \frac{1}{m})\lambda)^n}{n!}] \\ &= m[1 - e^{\frac{-\lambda}{m}} \sum_{n=0}^{\infty} \frac{e^{-(1-\frac{1}{m})\lambda}((1 - \frac{1}{m})\lambda)^n}{n!}] \\ &= m(1 - e^{\frac{-\lambda}{m}}) \end{aligned}$$

where the second last and third last equality follows from the summation of Poisson distributions.

13 February 6, 2018

13.1 Example 3.4 solution

Consider the DTMC with TPM

$$P = \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ 0 & 0 & 1 & 0 \\ \frac{3}{4} & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

which has equivalence classes $\{0, 3\}$, $\{1\}$, and $\{2\}$. Determine the period of each state.

Solution. Consider the state 0. There are many paths which we can go from state 0 to 0 in n steps, but one obvious one is simply going from 0 to 0 n times which has probability $(P_{0,0})^n$. Therefore

$$P_{0,0}^{(n)} \geq (P_{0,0})^n = (1/3)^n > 0 \quad \forall n \in \mathbb{Z}^+$$

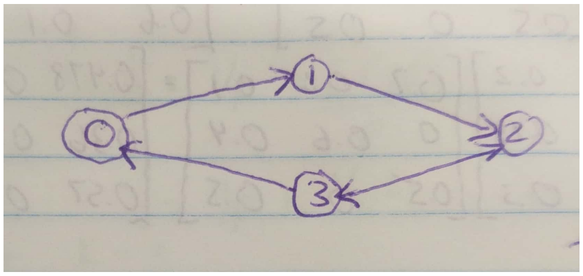
Thus $d(0) = \gcd\{n \in \mathbb{Z}^+ \mid P_{0,0}^{(n)} > 0\} = \gcd\{1, 2, 3, \ldots\} = 1$ (there is a way to get from 0 to 0 in any $n \in \mathbb{Z}^+$ steps, so we take the gcd of $\mathbb{Z}^+$ which is 1).
In fact, since every term on the main diagonal of P is positive, the same argument holds for every state. Thus $d(1) = d(2) = d(3) = 1$.

13.2 Example 3.2 (continued) solution

Recall for the DTMC with TPM

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix}$$

there is 1 equivalence class $\{0, 1, 2, 3\}$ with state diagram



(note the bi-direction between 2 and 3). Determine the period for each state.

Solution. We see that in one of the loops $0 \to 1 \to 2 \to 3 \to 0$ we have

$$P^{(n)}_{0,0} > 0 \text{ for } n = 4, 8, 12, 16, \ldots$$

Also we have (for the cycle $0 \to 1 \to 2 \to 3 \to 2 \to 3 \to 0$)

$$P^{(n)}_{0,0} > 0 \text{ for } n = 6, 10, 14, 18, \ldots$$

Thus $d(0) = \gcd\{4, 6, 8, 10, 12\} = 2$.
Following a similar line of logic, we find

$$\begin{aligned} d(1) &= \gcd\{4, 6, 8, 10, 12, \ldots\} = 2 \\ d(2) &= \gcd\{2, 4, 6, 8, 10, 12, \ldots\} = 2 \\ d(3) &= \gcd\{2, 4, 6, 8, 10, 12, \ldots\} = 2 \end{aligned}$$

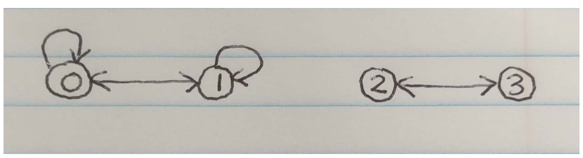
13.3 Example 3.5 solution

Consider the DTMC with TPM

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Find the equivalence class of this DTMC and determine the period of each state.

Solution. To determine the equivalence class we draw the state diagram



Clearly the equivalence classes are $\{0, 1\}$ and $\{2, 3\}$.
As in Example 3.4, the main diagonal terms for rows 0 and 1 are positive (i.e. $P_{0,0}, P_{1,1} > 0$) and so $d(0) = d(1) = 1$.
For states 2 and 3 the DTMC will continually alternate (with probability 1) between each other at every step (i.e. $2 \to 3 \to 2 \to 3 \to \ldots$). Thus it is clear that

$$d(2) = \gcd\{n \in \mathbb{Z}^+ \mid P_{2,2}^{(n)} > 0\} = \gcd\{2, 4, 6, 8, \ldots\} = 2$$
$$d(3) = \gcd\{n \in \mathbb{Z}^+ \mid P_{3,3}^{(n)} > 0\} = \gcd\{2, 4, 6, 8, \ldots\} = 2$$

13.4 Theorem 3.1 (equivalent states have equivalent periods)

Theorem 13.1. If $i \leftrightarrow j$ (they communicate), then $d(i) = d(j)$.

Proof. Assume $i \neq j$. Since $i \leftrightarrow j$, we know by definition that $P_{i,j}^{(n)} > 0$ for some $n \in \mathbb{Z}^+$ and $P_{j,i}^{(m)} > 0$ for some $m \in \mathbb{Z}^+$. Moreover since state i is accessible from state j and state j is accessible from state i, $\exists s \in \mathbb{Z}^+$ such that $P_{j,j}^{(s)} > 0$.
Clearly we have that

$$P_{i,i}^{(n+m)} \geq P_{i,j}^{(n)} \cdot P_{j,i}^{(m)} > 0$$

(paths that take n steps to i to j then m steps to j to i is one such possible path from i to i in $n + m$ steps. There could be more $n + m$ paths hence the $\geq$. This also follows from the Chapman-Kolmogorov equations.)
In addition,

$$P_{i,i}^{(n+s+m)} \geq P_{i,j}^{(n)} \cdot P_{j,j}^{(s)} \cdot P_{j,i}^{(m)} > 0$$

So we have paths with $n + m$ and $n + s + m$ steps, thus $d(i)$ divides both $n + m$ and $n + s + m$. Therefore it follows that $d(i)$ divides their differencely, namely $(n + s + m) - (n + m) = s$. Since this holds true for *any* s which satisfies $P_{j,j}^{(s)} > 0$, then it must be the case that $d(i)$ divides $d(j)$.
Using the same line of logic, it is straightforward to show $d(j)$ divides $d(i)$.
Putting these two arguments together, we deduce that $d(i) = d(j)$. $\square$

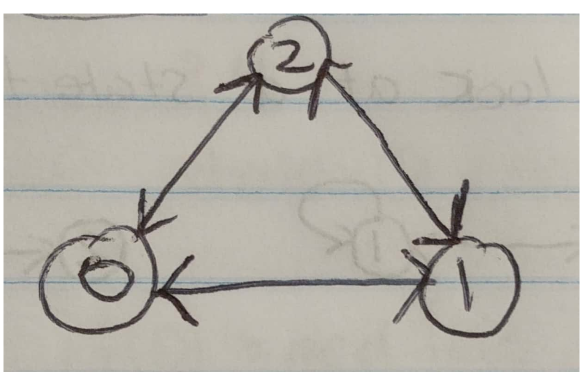
13.5 Example 3.6 solution

Consider the DTMC with TPM

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Find the equivalence classes and determine the period of the states.

Solution. The state transition diagram looks like



Clearly the DTMC is irreducible (i.e. there is just one class $\{0,1,2\}$).
Note that

$$\begin{aligned}
P^{(1)}_{0,0} &= 0 \\
P^{(2)}_{0,0} &\geq P_{0,1}P_{1,0} = \left(\frac{1}{2}\right)^2 = \frac{1}{4} > 0 \\
P^{(3)}_{0,0} &\geq P_{0,1}P_{1,2}P_{2,0} = \left(\frac{1}{2}\right)^3 = \frac{1}{8} > 0
\end{aligned}$$

Clearly $d(0) = \gcd\{2,3,\ldots\} = 1$.
By the above theorem, we have $d(0) = d(1) = d(2) = 1$.

14 February 8, 2018

14.1 Theorem 3.2 (communication and recurrent state i implies recurrent state j)

Theorem 14.1. If $i \leftrightarrow j$ (communicate) and state i is recurrent, then state j is recurrent.

Proof. Since $j \leftrightarrow j$, $\exists m,n \in \mathbb{N}$ such that

$$\begin{aligned}
P^{(m)}_{i,j} &> 0 \\
P^{(n)}_{j,i} &> 0
\end{aligned}$$

Also since state i is recurrent, then we have

$$\sum_{l=1}^{\infty} P^{(l)}_{i,i} = \infty$$

Suppose that $s \in \mathbb{Z}^+$. Note that

$$P^{(n+s+m)}_{j,j} \geq P^{(m)}_{j,i} \cdot P^{(s)}_{i,i} \cdot P^{(n)}_{i,j}$$

Now to show that j is recurrent, we show that the following series diverges

$$\begin{aligned}
\sum_{k=1}^{\infty} P^{(k)}_{j,j} &\geq \sum_{k=n+m+1}^{\infty} P^{(k)}_{j,j} \\
&= \sum_{s=1}^{\infty} P^{(n+s+m)}_{j,j} \\
&\geq \sum_{s=1}^{\infty} P^{(m)}_{j,i} \cdot P^{(s)}_{i,i} \cdot P^{(n)}_{i,j} \\
&= P^{(m)}_{j,i} \cdot P^{(n)}_{i,j} \sum_{s=1}^{\infty} P^{(s)}_{i,i} \\
&= \infty
\end{aligned}$$

since $P^{(m)}_{j,i}, P^{(n)}_{i,j} > 0$ and the series diverges by our premise. Therefore j is recurrent. $\square$

Remark 14.1. A by-product of the above theorem is that if $i \leftrightarrow j$ and state i is transient, then state j is transient.

14.2 Theorem 3.3 (communication and recurrent state i implies mutual recurrence among all states)

Theorem 14.2. If $i \leftrightarrow j$ and state i is recurrent, then

$$f_{i,j} = P(\text{DTMC ever makes a future visit to state } j \mid X_0 = i) = 1$$

Proof. Clearly the result is true if $i = j$. Therefore suppose that $i \neq j$. Since $i \leftrightarrow j$, the fact that state i is recurrent implies that state j is recurrent by the previous theorem and $f_{j.j} = 1$.
To prove $f_{i,j} = 1$, suppose that $f_{i,j} < 1$ and try to get a contradiction.
Since $i \leftarrow j$, $\exists n \in \mathbb{Z}^+$ such that $P^{(n)}_{j,i} > 0$ i.e. each time the DTMC visits state j, there is the possibility of being in state i n time units later with probability $P^{(n)}_{j,i} > 0$.
If we are assuming that $f_{i,j} < 1$, then this implies that the probability of returning to state j after visiting i in the future is not guaranteed (as $1 - f_{i,j} > 0$). Therefore

$$\begin{aligned} 1 - f_{j,j} = P(\text{DTMC never makes a future visit to state } j \mid X_0 = j) \\ &= P^{(n)}_{j,i} \cdot (1 - f_{i,j}) \\ &> 0 && \text{both } > 0 \end{aligned}$$

This implies that $1 - f_{j,j} > 0$ or $f_{j,j} < 1$, which is a contradiction. Therefore $f_{i,j} = 1$. □

14.3 Theorem 3.4 (finite-state DTMCs have at least one recurrent state)

Theorem 14.3. A finite-state DTMC has at least one recurrent state.

Proof. Equivalently, we want to show that not all states can be transient.
Suppose that $\{0, 1, 2, \ldots, N\}$ represents the states of the DTMC where $N < \infty$ (finite).
To prove that not all states can be transient, we suppose they are all transient and try to get a contradiction.
Now for each $i = 0, 1, \ldots, N$, if state i is assumed to be transient, we know that after a *finite* amount of time (denoted by T_i), state i will never be visited again. As a result, after a finite amount of time $T = \{T_0, T_1, \ldots, T_n\}$ has gone by *none of the states will be visited again.*
However, the DTMC **must be in some state** after time T but we have exhausted all states fro the DTMC to be in. This is a contradiction thus not all states can be transient in a finite state DTMC. □

15 February 13, 2018

15.1 Theorem 3.5 (recurrent i and not communicate with j implies $P_{i,j} = 0$)

Theorem 15.1. If state i is recurrent and state i does not communicate with state j, then $P_{i,j} = 0$.

Proof. Let us assume that $P_{i,j} > 0$ (one-step transition probability) and try to get a contradiction.
Then $P^{(n)}_{j,i} = 0$ $\forall n \in \mathbb{Z}^+$ otherwise states i, j communicate.
However, the DTMC, starting in state i, would have a positive probability of at least $P_{i,j}$ of never returning to state i. This contradicts the recurrence of state i, therefore we must have $P_{i,j} = 0$. □

15.2 Example 3.3 (continued) solution

Recall our earlier DTMC with TPM

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ 0 & 0 & 1 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 \end{bmatrix}$$

Determine whether each state is transient or recurrent.

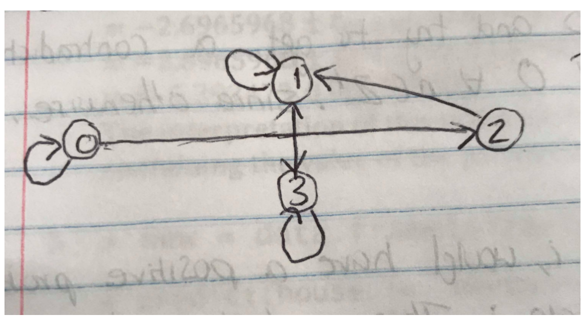
Solution. Note that $\sum_{n=1}^{\infty} P_{2,2}^{(n)} = \sum_{n=1}^{\infty} 1 = \infty$ which implies state 2 is recurrent.
Looking at the possible transitions that can take place among states 0, 1, and 3 we strongly suspect state 1 to be transient (since there is a positive probability of never returning to state 1 if a transition to state 2 occurs). To show this formally assume state 1 is recurrent and try to get a contradiction.
So if state 1 is recurrent, note that state 1 does not communicate with state 2. By theorem 3.5, we have that $P_{1,2} = 0$, but in fact $P_{1,2} = \frac{1}{8} > 0$. This is a contradiction, so state 1 must be transient and so $\{0, 1, 3\}$ is transient.

15.3 Example 3.7 solution

Consider a DTMC with TPM

$$P = \begin{bmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & \frac{3}{5} \end{bmatrix}$$

Determine whether each state is transient or recurrent.



Solution. The equivalence classes are $\{0\}, \{2\}, \{1, 3\}$.
Note that (since 0 is in its own equivalence classes)

$$\sum_{n=1}^{\infty} P_{0,0}^{(n)} = \sum_{n=1}^{\infty} (\frac{1}{4})^n = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3} < \infty$$

so state 0 is transient.
Furthermore (since state 2 is in its own equivalence class)

$$\sum_{n=1}^{\infty} P_{2,2}^{(n)} = \sum_{n=1}^{\infty} 0 = 0 < \infty$$

so state 2 is also transient.

On the other hand, concerning the class $\{1,3\}$ we observe

$$f_{1,1}^{(1)} = P_{1,1} = \frac{1}{3}$$

AND

$$f_{1,1}^{(n)} = \sum_{n=2}(2/3)(3/5)^{n-2}(2/5) \quad n \geq 2$$

Now

$$\begin{aligned} f_{1,1} = \sum_{n=1}^{\infty} f_{1,1}^{(n)} &= \frac{1}{3}\sum_{n=2}^{\infty}(\frac{2}{3})(\frac{3}{5})^{n-2}(\frac{2}{5}) \\ &= \frac{1}{3} + (\frac{2}{3})(\frac{2}{5})\sum_{n=2}(\frac{3}{5})^{n-2} \\ &= \frac{1}{3} + (\frac{2}{3})(\frac{2}{5}) \cdot \frac{1}{1-\frac{3}{5}} \\ &= 1 \end{aligned}$$

By definition since $f_{1,1} = 1$ state 1 is recurrent and so class $\{1,3\}$ is recurrent.
(Or we could have concluded from Theorem 3.4 that because $\{0\}$ and $\{2\}$ are both transient, $\{1,3\}$ must be recurrent).

15.4 Example 3.8 solution

Consider a DTMC $\{X_n, n \in \mathbb{N}\}$ whose state space is all integers i.e. $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Suppose the TPM for the DTMC satisfies

$$P_{i,i-1} = 1 - p \text{ and } P_{i,i+1} = p \quad \forall i \in \mathbb{Z} \text{ and } 0 < p < 1$$

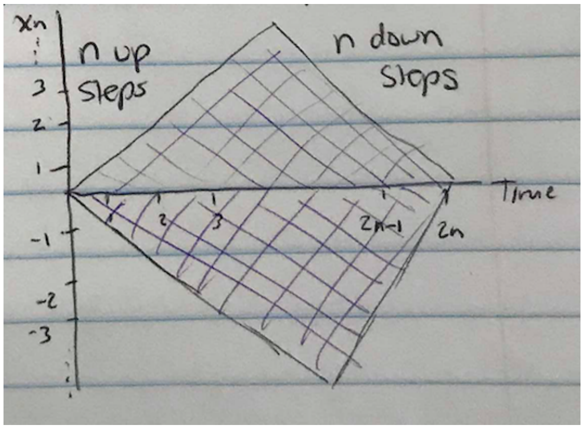
In other words from any state, we can either jump up or down one state. This is often referred to as the *Random Walk*. Characterize the behaviour of this DTMC in terms of its equivalence classes, periodicty, and transience/recurrence.

Solution. First of all since $0 < p < 1$ all states clearly communicate with each other. This implies that $\{X_n, n \in \mathbb{N}\}$ is an irreducible DTMC.
Hence we can determine its periodicity (and likewise its transcience/recurrence) by analyzing any state we wish.
Let us select state 0. Starting from state 0, note that state 0 cannot possibly visited in an odd number of transitions since we are guaranteed to have the number of up(down) jumps exceed the number of down(up) jumps (??? why).
Thus $P_{0,0}^{(1)} = P_{0,0}^{(3)} = P_{0,0}^{(5)} = \ldots = 0$ or equivalent

$$P_{0,0}^{(2n-1)} = 0 \quad \forall n \in \mathbb{Z}^+$$

However since it is clearly possible to return to state 0 in an even number of transitions, it immediately follows that $P_{0,0}^{(2n)} > 0 \ \forall n \in \mathbb{Z}^+$.
Hence $d(0) = gcd\{n \in \mathbb{Z}^+ \mid P_{0,0}^{(n)} > 0\} = gcd\{2, 4, 6, \ldots\} = 2$.
Finally to determine whether state 0 is transient or recurrent, let us consider

$$\sum_{n=1}^{\infty} P_{0,0}^{(n)} = \sum_{n=1}^{\infty} P_{0,0}^{(2n)} = \sum_{n=1}^{\infty} \binom{2n}{n} p^n (1-p)^n$$



That is we need to take n up steps with probability p and n down steps with probability $1-p$.
Note that $\binom{2n}{n}$ accounts for the number of ways to arrange this.
Recall: ratio test for series.
Suppose that $\sum_{n=1}^{\infty} a_n$ is a series of positive terms and

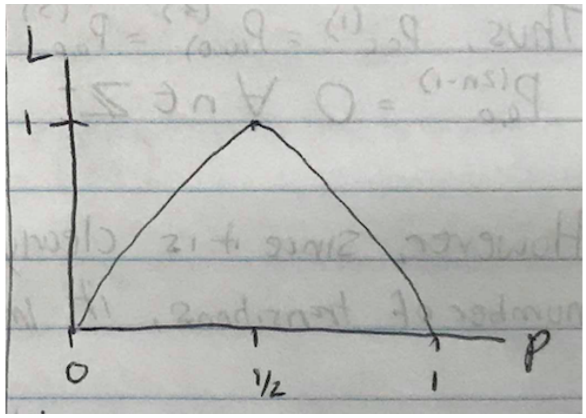
$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

1. If $L < 1$ then the series converges.
2. If $L > 1$ then the series diverges.
3. If $L = 1$ then the test is inconclusive.

In our case we obtain

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{\frac{(2(n+1))!}{(n+1)!(n+1)!} \cdot p^{n+1}(1-p)^{n+1}}{\frac{2n!}{n!n!} \cdot p^n(1-p)^n} \\
&= \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{n!n!}{2n!} \cdot p(1-p) \\
&= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \cdot p(1-p) \\
&= \lim_{n\to\infty} \frac{4n^2+6n+2}{n^2+2n+1} \cdot p(1-p) \\
&= 4p(1-p)
\end{aligned}$$

A plot of $L = 4p(1-p)$ reveals the following shape



Note that if $p \neq \frac{1}{2}$ then $L < 1$ which means state 0 is transient (and so is DTMC).
However, if $p = \frac{1}{2}$ then $L = 1$ and the ratio test is inconclusive. We need another approach to handle $p = \frac{1}{2}$ (this approach can handle both $p = \frac{1}{2}$ and $p \neq \frac{1}{2}$).
Recall that $f_{i,j} = P(\text{DTMC ever makes a future visit to state } j \mid X_0 = i)$.
Letting $q = 1 - p$ and conditioning on the state of the DTMC at time 1 we have

$$\begin{aligned}
f_{0,0} &= P(\text{DTMC ever makes a future visit to state } 0 \mid X_0 = 0) \\
&= P(X_1 = -1 \mid X_0 = 0) \cdot P(\text{DTMC ever makes a future visit to state } 0 \mid X_1 = -1, X_0 = 0) \\
&+ P(X_1 = 1 \mid X_0 = 0) \cdot P(\text{DTMC ever makes a future visit to state } 0 \mid X_1 = 1, X_0 = 0) \\
&\text{by Markov property (only most recent state matters) and stationarity (we can treat time 1 as time 0)} \\
&= q \cdot f_{-1,0} + p \cdot f_{1,0}
\end{aligned}$$

Using precisely the same logic it follows that

$$f_{1,0} = q \cdot 1 + p \cdot f_{2,0} \tag{15.1}$$

Consider $f_{2,0}$: in order to visit state 0 from state 2, we first have to visit state 1 again, which happens with probability $f_{2,1}$. This is **equivalent to** $f_{1,0}$, so $f_{2,1} = f_{1,0}$ (probability of ever making a visit to the state one step down).
Given you make a visit back to state 1, then we must make a visit back to state 0, which happens to with probability $f_{1,0}$.
Putting this together we have

$$f_{2,0} = f_{2,1} \cdot f_{1,0} = f_{1,0}^2$$

Thus from equation 15.1 we have

$$f_{1,0} = q + p \cdot f_{1,0}^2$$

which is a quadratic in the form

$$pf_{1,0}^2 - f_{1,0} + q = 0$$

Applying the quadratic formula we get

$$\begin{aligned} f_{1,0} &= \frac{1 \pm \sqrt{1-4pq}}{2p} \\ &= \frac{1 \pm \sqrt{(p+q)^2 - 4pq}}{2p} \\ &= \frac{1 \pm \sqrt{p^2 - 2pq + q^2}}{2p} \\ &= \frac{1 \pm |p-q|}{2p} \end{aligned}$$

There can only be **one unique solution** for $f_{1,0}$ which means that one of

$$\begin{aligned} r_1 &= \frac{1+|p-q|}{2p}; \text{ or} \\ r_2 &= \frac{1-|p-q|}{2p} \end{aligned}$$

must be inadmissible (intuitively: we see that if $p < 0.5$ then r_1 will be bigger than 1 which is not possible for probabilities so r_1 does not work for all p).
To determine which it is, suppose that $q > p$. Then $|p-q| = -(p-q)$ and the 2 roots become

$$\begin{aligned} r_1 &= \frac{1-(p-q)}{2p} = \frac{q+q}{2p} = \frac{q}{p} > 1 \\ r_2 &= \frac{1+(p-q)}{2p} = \frac{p+p}{2p} = 1 \end{aligned}$$

Thus we must have

$$f_{1,0} = \frac{1-|p-q|}{2p}$$

Note: using the exact same approach

$$f_{-1,0} = \frac{1-|p-q|}{2q}$$

With knowledge of $f_{1,0}$ and $f_{-1,0}$ we find that

$$\begin{aligned} f_{0,0} &= qf_{-1,0} + pf_{1,0} \\ &= q(\frac{1-|p-q|}{2q}) + p(\frac{1-|p-q|}{2p}) \\ &= 1 - |p-q| \end{aligned}$$

Suppose $p > q$ i.e. $1-q > q$ or $2q < 1$:

$$\begin{aligned} f_{0,0} &= 1-(p-q) \\ &= 1-p+q \\ &= 2q < 1 \end{aligned}$$

since $f_{0,0} < 1$ we have that state 0 is transient.
Similarly for $p < q$ we get $f_{0,0} = 2p < 1$ so state 0 is transient.

When $p = q$, then we have $f_{0,0} = 1$ so state 0 is recurrent under $p = q = \frac{1}{2}$.

16 February 15, 2018

16.1 Example 3.9 solution

Consider the DTMC with TPM

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Determine if $\lim_{n\to\infty} P^{(n)}$ exists (i.e. the limiting behavior of the DTMC).

Solution. There are 2 equivalence classes, namely $\{0, 2\}$ and $\{1\}$.
Each class is recurrent with periods 2 and 1, respectively.
For $n \in \mathbb{Z}^+$, note that

$$P^{(2n)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which follows by taking $P^{(2)}$ and multiplying by itself arbitrarily many times. Also

$$P^{(2n-1)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which follows by taking $P^{(1)}$ and multiply it arbitrarily many times by $P^{(2)}$ (identity).
As such, $\lim_{n\to\infty} P^{(n)}$ does not exist since $P^{(n)}$ alternates between those two matrices.
However, while $\lim_{n\to\infty} P^{(n)}_{0,0}$ and $\lim_{n\to\infty} P^{(n)}_{0,2}$ (and many other) do not exist, note that some limits do exist such as

$$\lim_{n\to\infty} P^{(n)}_{0,1} = 0$$

and

$$\lim_{n\to\infty} P^{(n)}_{1,1} = 1$$

16.2 Example 3.10 solution

Consider the DTMC with TPM

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Determine if $\lim_{n\to\infty} P^{(n)}$ exists.

Solution. There is only one equivalence class and so the DTMC is irreducible. Also it is straightforward to show that the DTMC is aperiodic and recurrent.
It can be shown that

$$\lim_{n\to\infty} P^{(n)} = \begin{bmatrix} \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \end{bmatrix}$$

(we can find this by using some software and repeatedly applying matrix exponentiation; we will later demonstrate a way to solve this for certain DTMCs (see Example 3.10 (continued) solution).
Note that this matrix has **identical rows**. This implies that $P_{i,j}^{(n)}$ converges to a value as $n \to \infty$ which is the same for all initial states i. In other words, there is a limiting probability the DTMC will be in state j as $n \to \infty$ and this probability is indepdendent of the initial states.

17 February 27, 2018

17.1 Example 3.11 solution

Consider the DTMC with TPM

$$P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 \end{bmatrix}$$

Examine $\lim_{n\to 0} P^{(n)}$ and explain why the limiting probability of being in a state depends on the initial state of this DTMC.

Solution. We have three equivalence classes $\{0\}, \{1\}, \{2\}$. Since $P_{i,i} > 0 \; \forall i = 0, 1, 2$ each state is aperiodic.

$$\sum_{n=1}^{\infty} P_{0,0}^{(n)} = \sum_{n=1}^{\infty} P_{2,2}^{(n)} = \sum_{n=1}^{\infty} = \infty$$

which implies state 0 and 2 are recurrent.
We can also show that

$$\sum_{n=1}^{\infty} P_{1,1}^{(n)} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = 1 < \infty$$

which means state 1 is transient.
In fact, states 0 and 2 are examples of **absorbing states** (i.e. $P_{0,0} = P_{2,2} = 1$).
If one begins in state 1, note that with probability $\frac{1}{2}$ one can end up in one of the two recurrent classes in the next transition i.e. the DTMC will eventually be absorbed into either state 0 or 2 so state 1 is *transient*.
It can be shown that

$$\lim_{n\to 0} P^{(n)} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

Intuitively this makes sense: eventually state 1 will go into 0 or 2. The relative probability of going into state 0 and going into state 2 is equivalent to our limiting probability entries.

Remark 17.1. Unlike the previous example, it *does matter* from which state one begins in this DTMC. Furthermore the second column of the above contains all zeros.

17.2 Theorem 3.6 solution

Theorem 17.1. For any state i and transient state j of a DTMC, $\lim_{n\to\infty} P_{i,j}^{(n)} = 0$.

Proof. Recall that we defined the "first visit probability in n steps" as

$$f_{i,j}^{(n)} = P(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i)$$

Furthermore we defined

$$f_{i,j} = P(\text{DTMC ever makes a future visit to state } j \mid X_0 = i) = \sum_{n=1}^{\infty} f_{i,j}^{(n)}$$

We can then use the fact that

$$P_{i,j}^{(n)} = \sum_{k=1}^{n} f_{i,j}^{(k)} P_{j,j}^{(n-k)}$$

So we have

$$\begin{aligned}
\sum_{n=1}^{\infty} P_{i,j}^{(n)} &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} f_{i,j}^{(k)} P_{j,j}^{(n-k)} \\
&= \sum_{k=1}^{\infty} f_{i,j}^{(k)} \sum_{n=k}^{\infty} P_{j,j}^{(n-k)} \\
&= \sum_{k=1}^{\infty} f_{i,j}^{(k)} \sum_{l=0}^{\infty} P_{j,j}^{(l)} && l = n-k \\
&= (\sum_{k=1}^{\infty} f_{i,j}^{(k)})(1 + \sum_{l=1}^{\infty} P_{j,j}^{(l)}) && P_{i,i}^{(0)} = 1 \quad \forall i \\
&= f_{i,j}(1 + \sum_{l=1}^{\infty} P_{j,j}^{(l)}) \\
&\le 1 + \sum_{l=1}^{\infty} P_{j,j}^{(l)} && f_{i,j} \le 1 \text{ since it's a probability} \\
&< \infty \text{ since } j is transient so the summation is finite
\end{aligned}$$

Recall the **nth term test**: If $\lim_{n\to 0} a_n \neq 0$ or if the limit does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.
We can show by contradiction that if the series converges then the limit is 0.
By the **nth term test** for infinite series

$$\lim_{n\to 0} P_{i,j}^{(n)} = 0$$

□

17.3 Example 3.10 (continued) solution

Recall we were given the DTMC with TPM

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Find the limiting probability of this DTMC.

Solution. Clearly DTMC is irreducible, aperiodic and positive recurrent. The BLT conditions are met and $\pi = (\pi_0, \pi_1, \pi_2)$ exists and satisfies

$$\begin{aligned}
&\pi = \pi P \\
&\pi e' = 1 && e' = (1, 1, \dots, 1)^T
\end{aligned}$$

Thus we have

$$(\pi_0, \pi_1, \pi_2) = (\pi_0, \pi_1, \pi_2) \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Expanding the matrix multiplication we get

$$\begin{aligned} \pi_0 &= \frac{1}{2}\pi_0 + \frac{1}{2}\pi_1 \\ \pi_1 &= \frac{1}{2}\pi_0 + \frac{1}{4}\pi_1 + \frac{1}{3}\pi_2 \\ \pi_2 &= \frac{1}{4}\pi_1 + \frac{2}{3}\pi_2 \\ 1 &= \pi_0 + \pi_1 + \pi_2 \end{aligned}$$

(we can disregard the $\pi_1 = \ldots$ equation since it requires solving for the most terms and we have 4 equations for 3 unknowns). Solving the system we get

$$\begin{aligned} \pi_0 = \pi_1 &= \frac{4}{11} \\ \pi_2 &= \frac{3}{11} \end{aligned}$$

Recall we had

$$\lim_{n\to\infty} P^{(n)} = \begin{bmatrix} \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \end{bmatrix} = \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \\ \pi_0 & \pi_1 & \pi_2 \\ \pi_0 & \pi_1 & \pi_2 \end{bmatrix}$$

We observe that each row of $P^{(n)}$ converges to π as $n \to \infty$.

17.4 Example 3.12 solution (Gambler's Ruin)

Consider a gambler who, at each play of a game, has probability $p \in (0, 1)$ of winning one unit and probability $q = 1 - p$ of losing one unit. Assume that successive plays of the game are independent. If the gambler initially begins with i units, what is the probability that the gambler's fortune will reach N units ($N < \infty$) before reaching 0 units? This problem is often referred to as the Gambler's Ruin Problem, with state 0 representing bankruptcy and state N representing the jackpot.

Solution. For $n \in \mathbb{N}$, define X_n as the gambler's fortune after the nth play of the game. Therefore $\{X_n, n \in \mathbb{N}\}$ is a DTMC with TPM

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ q & 0 & p & 0 & \ldots & 0 & 0 & 0 \\ 0 & q & 0 & p & \ldots & 0 & 0 & 0 \\ \vdots \\ 0 & 0 & 0 & 0 & \ldots & q & 0 & p \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix}$$

Where the TPM has entries P_{ij} where $i, j \in [0, N]$ (for each $i \in [1, N-1]$, we have $P_{i(i-1)} = q$ and $P_{i(i+1)} = p$. $P_{00} = P_{NN} = 1$ and all other $P_{mn} = 0$).
Note: States 0 and N are treated as **absorbing states**, therefore they are both recurrent.
States $\{1, 2, \ldots, N-1\}$ belong to the same equivalence class and it is straightforward to see it is a transient class.

Goal: Determine $P(i)$ for $i = 0, 1, \ldots, N$ which represents the probability that starting with i units, the gambler's fortune will eventually reach N units.
It follows that (for the limiting TPM)

$$\lim_{n\to\infty} P^{(n)} = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1-P(1) & 0 & 0 & 0 & \ldots & 0 & 0 & P(1) \\ 1-P(2) & 0 & 0 & 0 & \ldots & 0 & 0 & P(2) \\ \vdots & & & & & & & \\ 1-P(N-1) & 0 & 0 & 0 & \ldots & 0 & 0 & P(N-1) \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix}$$

where the $(0,0)$ entry is 1 (0 state is recurrent), $(0,N)$ is 0 (can never reach state N from 0) and similarly for (N,N) and $(N,0)$.
Entries $(i,N) = P(i)$ by definition where $P(0) = 0$ and $P(N) = 1$. Since the sum of the entries for a given row must be 0, we get for entry $(i,0)$ as $1 - P(i)$.
Conditioning on the outcome of the very first game $i = 1, 2 \ldots, N-1$, we have

$$P(i) = p \cdot P(i+1) + q \cdot P(i-1)$$

(if we go up by 1 with probability p, then the probability of winning afterwards is $P(i+1)$ i.e. $P(i+1) = P(i \mid \text{win})$ and $p = P(\text{win})$. Similarly for q and $P(i-1)$).
Note that $p + q = 1$, so we have

$$\begin{aligned} P(i) &= pP(i) + qP(i) \\ \Rightarrow p(P(i+1) - P(i)) &= q(P(i) - P(i-1)) \\ P(i+1) - P(i) &= \frac{q}{p}(P(i) - P(i-1)) \end{aligned}$$

To determine if an explicit solution exists, consider several values of i

$$\begin{aligned} &i = 1 \Rightarrow P(2) - P(1) = \frac{q}{p}(P(1) - P(0)) = \frac{q}{p}P(1) \\ &i = 2 \Rightarrow P(3) - P(2) = \frac{q}{p}(P(2) - P(1)) = \left(\frac{q}{p}\right)^2 P(1) \\ &i = 3 \Rightarrow P(4) - P(3) = \frac{q}{p}(P(3) - P(2)) = \left(\frac{q}{p}\right)^3 P(1) \\ &\vdots \\ &i = k \Rightarrow P(k+1) - P(k) = \frac{q}{p}(P(k) - P(k-1)) = \left(\frac{q}{p}\right)^k P(1) \end{aligned}$$

Note: the above k equations are linear in terms of the unknowns $P(1), P(2), \ldots, P(k+1)$. Summing these k

equations yields

$$
\begin{aligned}
P(k+1) - P(1) &= \sum_{i=1}^{k} \left(\frac{q}{p}\right)^i P(1) \\
\Rightarrow P(k+1) &= \sum_{i=0}^{k} \left(\frac{q}{p}\right)^i P(1) \quad k = 0, 1, \ldots, N-1 \\
\Rightarrow P(k) &= \sum_{i=0}^{k-1} \left(\frac{q}{p}\right)^i P(1) \quad k = 1, 2, \ldots, N
\end{aligned}
$$

Applying the formula for a finite geometric series

$$
P(k) = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^k}{1-\frac{q}{p}} P(1) & p \neq \frac{1}{2} \\ kP(1) & p = \frac{1}{2} \end{cases}
$$

Letting $k = N$, we obtain for $p \neq \frac{1}{2}$

$$
1 = P(N) = \frac{1-\left(\frac{q}{p}\right)^N}{1-\frac{q}{p}} P(1) \Rightarrow P(1) = \frac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^N}
$$

Similarly for $p = \frac{1}{2}$, we have $P(1) = \frac{1}{N}$.
So for $k = 0, 1, \ldots, N$ we have

$$
P(k) = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N} & p \neq \frac{1}{2} \\ \frac{k}{N} & p = \frac{1}{2} \end{cases}
$$

Think about what happens when $N \to \infty$ for each case.