richardwu.ca

MATH 247 Course Notes

Calculus 3 (Advanced)

Spiro Karigiannis • Winter 2018 • University of Waterloo

Last Revision: January 20, 2018

Table of Contents

Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

1 January 3, 2018

1.1 Euclidean space $\mathbb{R}^n$

Most postulates and theorems apply to any n-dimensional real vector space with a positive-definite inner product.

$$\mathbb{R}^n = \{x = (x_1, x_2, \dots, x_n); x_j \in \mathbb{R}, j = 1, \dots, n\}$$

Some properties of vectors in $\mathbb{R}^n$ where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, and $t \in \mathbb{R}$:

$$\begin{aligned}
&x + y = (x_1 + y_1, \dots, x_n + y_n) \\
&tx = (tx_1, \dots, tx_n) \\
&x + y = y + x \\
&(x + y) + z = x + (y + z) \\
&s(tx) = (st)x \\
&t\vec{0} = \vec{0} \\
&\vec{0}x = \vec{0} \\
&(t + s)x = tx + sx \\
&t(x + y) = tx + ty
\end{aligned}$$

1.2 Euclidean inner product

An important additional structure on $\mathbb{R}^n$ is the natural **Euclidean inner product** (aka the *dot product*).

$$\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

which can be written as $x \cdot y \in \mathbb{R}$.
Dot products are **billinear**, **symmetric**, and **positive-definite**. **Bilinear forms** satisfy

$$\begin{aligned}
(x + y) \cdot z &= x \cdot z + y \cdot z \\
x \cdot (y + z) &= x \cdot y + x \cdot z \\
(tx) \cdot y = x \cdot (ty) &= t(x \cdot y)
\end{aligned}$$

symmetric denotes

$$x \cdot y = y \cdot x$$

and **positive-definiteness** means $x \cdot x \geq 0$ with equality $\iff x = \vec{0}$.

Definition 1.1. The dot product is defined for $y = (y_1, \ldots, y_n)$ and $y = (y_1, \ldots, y_n)$

$$x \cdot y := \sum_{k=1}^{n} x_k y_k$$

Definition 1.2. The norm $\|x\|$ of $x \in \mathbb{R}^n$ (induced by some inner product $\langle x, x \rangle = x \cdot x$) is defined as

$$\begin{aligned} \|x\|^2 &= x \cdot x \\ \|x\|^2 &= \sqrt{x \cdot x} \end{aligned}$$

1.3 Triangle inequality

Proposition 1.1. Triangle inequality states

$$\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^n$$

To prove the above, we need the **Cauchy-Schwarz Inequality**.

Theorem 1.1. The Cauchy-Schwarz inequality states that

$$|x \cdot y| \leq \|x\|\|y\|$$

with equality iff $x = ty$ or $y = tx$ for some $t \in \mathbb{R}$.

Proof. For the equality case, WLOG if $x = ty$

$$\begin{aligned} x \cdot y = ty \cdot y &= t\|y\|^2 \\ &= |t|\|y\|^2 \\ &= \|x\|\|y\| \end{aligned}$$

Let $t \in \mathbb{R}$. Note for all t

$$\begin{aligned} 0 \leq \|x - ty\|^2 &= (x - ty) \cdot (x - ty) \\ &= x \cdot x - ty \cdot x - tx \cdot y + t^2 y \cdot y \\ &= \|x\|^2 + t^2\|y\|^2 - 2t(x \cdot y) \end{aligned}$$

Thus we have

$$at^2 + bt + c \geq 0 \quad \forall t \in \mathbb{R}$$

where $a = \|y\|^2$, $b = -2x \cdot y$ and $c = \|x\|^2$. Note there can exist at most one root (positive parabola where all values are non-negative). For $at^2 + bt + c = 0$ to have at most one real root (such that t exists), we need $b^2 - 4ac \leq 0$ (from the quadratic formula).

$$\begin{aligned} 4(x \cdot y)^2 &\leq 4\|x\|^2\|y\|^2 \\ |x \cdot y| &\leq \|x\|\|y\| \end{aligned}$$

If we have equality $\exists \quad t_0$ such that $at_0^2 + bt_0 + c = 0$ or $\|x - t_0 y\|^2 = 0$ so $x = t_0 y$. □

Corollary 1.1. The triangle inequality

$$\begin{aligned}\|x+y\|^2 &= (x+y)\cdot(x+y)\\ &= \|x\|^2 + 2x\cdot y + \|y\|^2\\ &\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\|+\|y\|)^2\end{aligned}$$

where the last line follows from the Cauchy-Schwarz inequality.

Definition 1.3. The **distance** between two points $x, y \in \mathbb{R}^n$ is defined to be

$$d(x,y) = \|x-y\|$$

which satisfies the properties

$$\begin{aligned}&d(x,y) = d(y,x)\\ &d(x,x) = 0\\ &d(x,y) \ge 0 \quad \text{with equality iff} \quad x = y\end{aligned}$$

so we can restate the triangle inequality as $d(x,y) \le d(x,z) + d(z,x) \quad \forall x,y,z \in \mathbb{R}^n$.

1.4 Norms

There exists different "natural" norms on $\mathbb{R}^n$

Definition 1.4. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is a map

$$\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^{\ge 0}$$

such that

1. $\|x\| = 0 \iff x = \vec{0}$
2. $\|tx\| = |t|\|x\|$
3. $\|x+y\| \le \|x\| + \|y\|$

All inner products determine a norm but not all norms are from inner products. We saw that the dot product determines a norm called the Euclidean norm.

l^1 norm $\|x\|_1 = \sum_{k=1}^n |x_k|$

l^p norm $\|x\|_p = (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}}$

sup norm (aka l^∞ norm) $\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$

One can see that l^∞ norm is a "limit" of l^p norms as $p \to \infty$.
Note the l^2 norm is the Euclidean norm.

Why are norms important? **A norm determines a distance**. For example

$$d(x,y) = \|x-y\|$$

(all norms determine a distance but not all distances are from norms).
Distance is important to define a **limit** which is crucial for differentiability/integrability.

1.5 Angle between two vectors

A corollary to C-S for $x, y \neq \vec{0}$

$$-1 \leq \frac{x \cdot y}{\|x\|\|y\|} \leq 1$$

Define the angle $\theta \in [0, \pi]$ between x and y to be

$$\cos\theta = \frac{x \cdot y}{\|x\|\|y\|}$$

so we have another definition of the dot product

$$x \cdot y = \|x\|\|y\| \cos\theta$$

We say x, y are **orthogonal** if $\theta = \frac{\pi}{2} \iff x \cdot y = 0$.
Why is this the correct definition?

$$\begin{aligned} \|y - x\|^2 &= (y - x) \cdot (y - x) \\ &= \|x\|^2 + \|y\|^2 - 2x \cdot y \\ &= \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos\theta \end{aligned}$$

This aligns with the Law of Cosines $c^2 = a^2 + b^2 - 2ab\cos\theta$.

2 January 5, 2018

2.1 Linear maps

Definition 2.1. A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if T takes linear combinations to linear combinations i.e.

$$T(\sum_{k=1}^{N} t_k v_k) = \sum_{k=1}^{N} T(v_k) \quad t_i \in \mathbb{R} \quad v_j \in \mathbb{R}^n$$

We will see linear maps are closely related to **differentiability**.
Some facts about linear maps: let $e_1, \ldots, e_n$ be the standard basis.

$$x \in \mathbb{R}^n = (x_1, \ldots, x_n) = \sum_{k=1}^{n} x_k e_k$$

Let $f_1, \ldots, f_m$ be the standard basis of $\mathbb{R}^m$ where $f_j = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^m$.

$$y \in \mathbb{R}^m = (y_1, \ldots, y_n) = \sum_{k=1}^{m} y_k f_k$$

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear and let

$$\begin{aligned} y = \sum_{e=1}^{m} y_l f_l = T(x) &= T(\sum_{k=1}^{n} x_k e_k) \\ &= \sum_{k=1}^{n} x_k T(e_k) \\ &= \sum_{k=1}^{n} x_k (\sum_{l=1}^{m} A_{lk} f_l) \\ &= \sum_{k=1}^{n} (\sum_{l=1}^{m} A_{lk} x_k) f_l \end{aligned}$$

By uniqueness of the expansion of a vector in terms of a basis (f_js) we conclude that

$$y_l = \sum_{k=1}^{n} A_{lk} x_k \quad l = 1, \ldots, m$$

or in matrix form

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \ldots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

We've shown that any linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is necessarily **matrix multiplication**

$$y = T(x) = A \cdot x$$

for some unique $m \times n$ matrix A (with respect to some bases in $\mathbb{R}^n$ and $\mathbb{R}^m$).
The rule of matrix multiplication is automatic from the composition of linear maps. Let

$$\begin{aligned} &T : \mathbb{R}^n \to \mathbb{R}^m \\ &S : \mathbb{R}^m \to \mathbb{R}^p \\ &y = T(x) = A \cdot x \quad m \times n \\ &z = S(y) = B \cdot y \quad p \times m \end{aligned}$$

Therefore $S \circ T : \mathbb{R}^n \to \mathbb{R}^p$ is linear.

$$\begin{aligned} (S \circ T)(\sum t_k v_k) &= S(T(\sum_k t_k v_k)) \\ &= S(\sum_k x_k T(v_k)) \\ &= \sum_k x_k S(T(v_k)) \\ &= \sum_k t_k (S \circ T)(v_k) \end{aligned}$$

So we have

$$\begin{aligned} z_l = \sum_{j=1}^{m} B_{lj} y_j &= \sum_{j=1}^{m} B_{lj} (\sum_{i=1}^{n} A_{ji} x_i) \\ &= \sum_{i=1}^{n} (\sum_{j=1}^{m} B_{lj} A_{ji}) x_i \\ &= \sum_{i=1}^{n} C_{li} x_i \end{aligned}$$

where

$$z = (S \circ T)(x) = C \cdot x \quad p \times n$$

Recall the space $L(\mathbb{R}^n, \mathbb{R}^m)$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ is itself a finite dimensional real vector space of dimension nm (isomorphic to $\mathbb{R}^{nm}$).

$$T \in L(\mathbb{R}^n, \mathbb{R}^m) \iff A \in M_{m \times n}(\mathbb{R})$$

where $M_{m\times n}(\mathbb{R})$ is the space of real $m \times n$ matrices. There is a unique 1-1 correspondence between T and A (as shown before).

2.2 Operator norm

Note one can define norm on matrices. The natural Euclidean norm for matrix A can be defined as

$$\|A\|_2 = \sqrt{\sum_{i=1,\dots,m; j=1,\dots,n} (A_{ij})^2}$$

Definition 2.2. The **operator norm** is defined for a $T : \mathbb{R}^n \to \mathbb{R}^m$ linear map as

$$\|T\|_{op} = \inf\{C > 0, \|T(x)\| \le C\|x\| \quad \forall x \in \mathbb{R}^n\}$$

We need to show this norm is

1. Well-defined
2. $\|\cdot\|_{op}$ is a norm

1. Show well-defined

$$T(x) = A \cdot x \quad A \quad m \times n$$

$$\begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} A_1 \cdot x \\ \vdots \\ A_m \cdot x \end{bmatrix} = T(x)$$

So the norm is

$$\begin{aligned} \|T(x)\|^2 &= (A_1 \cdot x)^2 + \dots + (A_m \cdot x)^2 \\ &\le \|A_1\|^2 \|x\|^2 + \dots + \|A_m\|^2 \|x\|^2 && \text{C-S} \\ &= (\|A_1\|^2 + \dots + \|A_m\|^2) \|x\|^2 \end{aligned}$$

Case 1 Assume $\|A_1\|^2 + \ldots + \|A_m\|^2 = 0$.

$$\begin{aligned} \|A_1\|^2 + \ldots + \|A_m\|^2 = 0 &\iff A = 0_{m\times n} \\ &\iff T = 0 \in L(\mathbb{R}^n, \mathbb{R}^m) \end{aligned}$$

Then $T(x) = 0 \quad \forall x$ so $\|T(x)\| \le C\|x\|$ holds $\forall C > 0$, thus the infimum of positive real numbers (0) implies $\|T\|_{op} = 0$.

Case 2 Assume $\|A_1\|^2 + \ldots + \|A_m\|^2 > 0$.

$\{C > 0, \|T(x)\| \le C\|x\| \quad \forall x \in \mathbb{R}^n\}$ is non-empty because $\sqrt{\|A_1\|^2 + \ldots + \|A_m\|^2}$ is in there. By the completeness of $\mathbb{R}$, $\|T\|_{op}$ exists and is ≥ 0.

2. We've shown $\|T\|_{op}$ exists and is ≥ 0 for all $T \in L(\mathbb{R}^n, \mathbb{R}^m)$. It remains to shown $\|T\|_{op}$ is a norm:

   (a) $\|T\|_{op} = 0$ only for the zero map

   (b) $\|\lambda T\|_{op} = |\lambda|\|T\|_{op} \quad \forall \lambda \in \mathbb{R}$

   (c) $\|T + S\|_{op} \le \|T\|_{op} + \|S\|_{op}$

   To see this, we note that since

$$\|T\|_{op} = \inf\{C > 0, \|T(x)\| \le C\|x\| \quad \forall x \in \mathbb{R}^n\}$$

   $\exists$ a **decreasing sequence** $c_k \ge 0$ such that $\|T(x)\| \le c_k\|x\| \quad \forall x \in \mathbb{R}^n$ and $\lim_{k\to\infty} c_k = \|T\|_{op}$.

   Take limit as $k \to \infty$ of the predicate in $\|T\|_{op}$.

$$\begin{aligned} \|T(x)\| &\le (\lim_{k\to\infty} c_k)\|x\| \\ \|T(x)\| &\le \|T\|_{op}\|x\| \end{aligned}$$

   So we have

$$\begin{aligned} \|T\|_{op} = 0 &\Rightarrow \|T(x)\| \le 0 \quad \forall x \\ &\Rightarrow T(x) = 0 \quad \forall x \\ &\Rightarrow T = 0 \in L(\mathbb{R}^n, \mathbb{R}^m) \end{aligned}$$

   which proves (a).

$$\|\lambda T\|_{op} = |\lambda|\|T\|_{op}$$

   follows from

$$\begin{aligned} \|(\lambda T)(x)\| &= \|\lambda(T(x))\| \\ &= |\lambda|\|T(x)\| \quad \forall x \end{aligned}$$

   If $\lambda = 0$, $\lambda T = 0 \Rightarrow \|\lambda T\|_{op} = 0 = |\lambda|\|T\|_{op}$.

If $\lambda \neq 0$

$$\begin{aligned}
\|\lambda T\|_{op} &= \inf\{C > 0, \|(\lambda T)(x)\| \leq C\|x\|\} \\
&= \inf\{C > 0, |\lambda|\|T(x)\| \leq C\|x\|\} \\
&= \inf\{C > 0, \|T(x)\| \leq \frac{C}{|\lambda|}\|x\|\} \\
&= |\lambda| \inf\{\tilde{C} > 0, \|T(x)\| \leq \tilde{C}\|x\|\} && \tilde{C} = \frac{C}{\lambda} \\
&= |\lambda|\|T\|_{op}
\end{aligned}$$

which proves (b). (c) is similar.

3 January 8, 2018

3.1 Topology of $\mathbb{R}^n$

Topology is the study of **closeness** in a space.

3.2 Open and closed balls

Definition 3.1. Let $x \in \mathbb{R}^n$ and $r > 0$. The **open ball** at radius r centred at x is denoted

$$B_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\| < r\}$$

It consists of all points in $\mathbb{R}^n$ whose distance from x is *strictly less than* r.

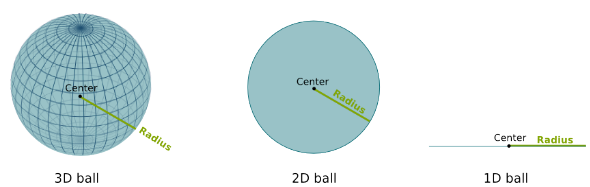

Figure 3.1: Open balls in R, R^2, and R^3.

In R, $B_r(x) = (x - r, x + r)$. In R^3, $B_r(x)$ is the *interior* of a sphere of radius r centred at x.

Definition 3.2. Let $x \in \mathbb{R}^n$, $r > 0$. The **closed ball** of radius $r > 0$ centred at x is denoted

$$\overline{B_r(x)} = \{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$$

Remark 3.1. The notation will be explained in the following class/section. Note that

$$\overline{B_r(x)} = B_r(x) \cup \{\text{points exactly at distance } r\}$$

For $n = 1$, $\overline{B_r(x)} = [x - r, x + r]$.

3.3 Open sets

Definition 3.3. A subset $U \subseteq \mathbb{R}^n$ is called an **open set** (or open) iff $\forall x \in U$, $\exists r > 0$ (r depends on x) such that $B_r(x) \subseteq U$.

(Informally: a subset U is open if for every $x \in U$, all points sufficiently close to x are *also* in U).

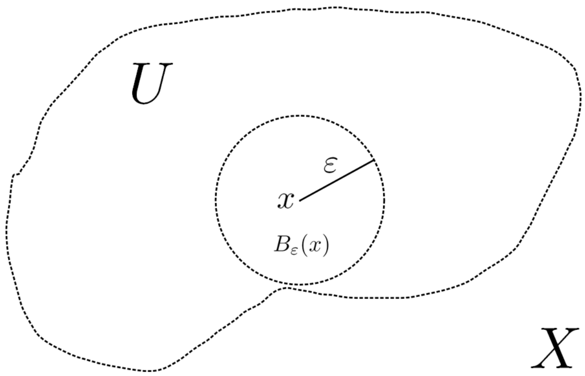

Figure 3.2: One can form an open ball for every point x in an open set U.

Example 3.1. Set that is not open

- $[0,1] \subseteq \mathbb{R}$. Note: $\nexists r > 0$ for $x = 1$ such that $B_r(x) \subseteq [0,1]$.

Sets that are open

- $\mathbb{R}^n$ since $x + \epsilon \in \mathbb{R}^n$ by definition.
- $\varnothing$ (vacuous: satisfied trivially $\varnothing$ has no points).

Proposition 3.1. An **open ball** is an **open set**.

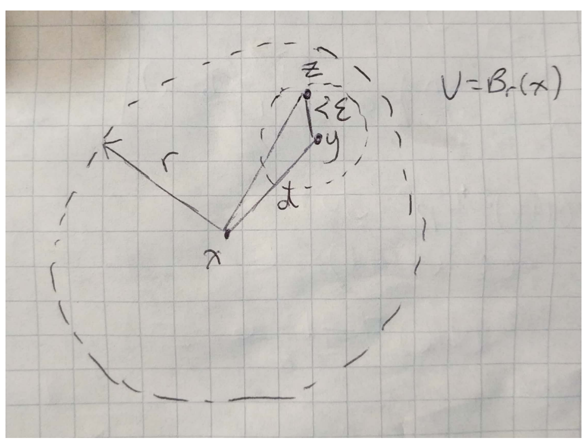

Figure 3.3: An open ball is an open set (see proof below).

Proof. Let $U = B_r(x)$ and $y \in U = B_r(x)$. We need to find some $\epsilon > 0$ such that $B_\epsilon(y) \subseteq U$.
Let $d = \|x - y\| < r$ since $y \in U = B_r(x)$.
Let $\epsilon = r - d > 0$.
Suppose $z \in B_\epsilon(y)$ thus $\|y - z\| < \epsilon$.
We thus have

$$\|z - x\| \overset{\triangle}{\leq} \|z - y\| + \|y - x\| < \epsilon + d = r$$

So $B_\epsilon(y) \subseteq U$ hence U is open. $\square$

We can construct more from open sets.

3.4 Properties of open sets

Lemma 3.1. 1. Let $U_\alpha \subseteq \mathbb{R}^n$ be open $\forall \alpha \in A$ (countably or uncountably many), then

$$\bigcup_{\alpha \in A} U_\alpha$$

is open.

2. Let $U_1, \ldots, U_k$ be open (**must be finite** number of sets). Then

$$\bigcap_{j=1}^{k} U_j$$

is open.

Informally, *arbitrary unions* of open sets are open. *Finite intersections* of open sets are open.

Proof.

1. We want to show $\bigcup_{\alpha \in A} U_\alpha$ is open.

   Let $x \in \bigcup_{\alpha \in A} U_\alpha$ so $\exists$ some $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$ (holds since union of sets).

   But U_{α_0} is open so $\exists r > 0$ such that $B_r(x) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in A} U_\alpha$.

2. Show $x \in \bigcap_{j=1}^{k} U_j$ so $x \in U_j$ for all $j = 1, \ldots, k$. Each U_j is open so $\forall j, \exists \epsilon_j > 0$ such that $B_{\epsilon_j}(x) \subseteq U_j$.

   Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_k\} > 0$. $\forall j$ we have $B_\epsilon(x) \subseteq B_{\epsilon_j}(x) \subseteq U_j$ hence $B_\epsilon(x) \subseteq \bigcap_{j=1}^{k} U_j$.

   Remark 3.2. Arbitrary (e.g. nonfinite) intersections of open sets need not be open (the min. of infinite numbers is not well defined. An infimum of positive numbers need not be > 0 i.e. it could be 0).

   Even intersection of countably infinite sets may not be open. Suppose $U_k = (0, 1 + \frac{1}{k}) \subseteq \mathbb{R} \quad \forall k \in \mathbb{N}$. Note that $\bigcap_{k=1}^{\infty} U_k = (0, 1]$ is not open.

$\square$

3.5 Closed sets

Definition 3.4. A subset $F \subseteq \mathbb{R}^n$ is called **closed** if $F^c = \mathbb{R} \setminus F$ is open (note: this definition is based on open's definition).

Proposition 3.2. A closed ball $\overline{B_r(x)} = \{y \in \mathbb{R}^n \mid \|y - x\| \leq r\}$ is a closed set.

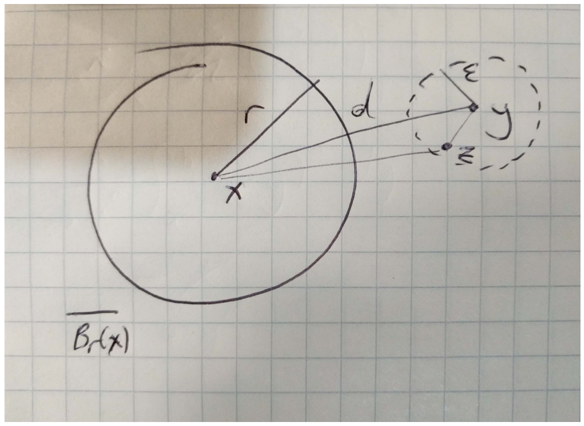

Figure 3.4: A closed ball is a closed set (see proof below).

Proof. Let $F = B_r(x)$ and

$$F^c = (\overline{B_r(x)})^c = \{y \in \mathbb{R}^n \mid \|y - x\| > r\}$$

Let $y \in \overline{B_r(x)}^c$: need to find $\epsilon > 0$ such that $B_\epsilon(y) \subseteq F^c$.
Let $d = \|x - y\| > r$ and let $\epsilon = d - r > 0$.
If $z \in B_\epsilon(y)$, then

$$\begin{aligned} \|x - y\| &\overset{\triangle}{\leq} \|x - z\| + \|z - y\| \\ d &\leq \|x - z\| + \|z - y\| \\ \|x - z\| &\geq d - \|z - y\| \\ &> d - \epsilon = r \end{aligned}$$

Hence $z \in F^c$ so $B_\epsilon(y) \subseteq F^c$, thus F^c is open and by definition F is closed. □

3.6 Properties of closed sets

Lemma 3.2. Note: this lemma is the inverse of the equivalent for open sets.

1. If $F_1, \ldots, F_k$ is closed, then $\bigcup_{j=1}^k F_j$ is closed.
2. If F_α is closed $\forall \alpha \in A$, then $\bigcap_{\alpha \in A} F_\alpha$ is closed.

Finite unions of closed sets are closed. Arbitrary intersections of closed sets are closed.

Proof. By De Morgan's laws

$$(\bigcup_{j=1}^k F_j)^c = \bigcap_{j=1}^k (F_j)^c$$
$$(\bigcap_{\alpha \in A} F_\alpha)^c = \bigcup_{\alpha \in A} (F_\alpha)^c$$

□

3.7 Neither open nor closed

A subset V of $\mathbb{R}^n$ need not be either open or closed. It can be open, closed, neither or both!

Example 3.2. Examples of non-exclusive open or closed sets are

- $(0, 1] \subseteq \mathbb{R}$ - neither
- $\mathbb{R}^n, \varnothing$ are *open and closed*

3.8 Interior

Sometimes a set is neither open or closed, but there are always **natural open (interior) and closed (closure) sets** which can be associated to any subset of $\mathbb{R}^n$.

Definition 3.5. Let $A \subseteq \mathbb{R}^n$ (could be $\varnothing$).

$$\begin{aligned} A^o &= \int(A) && \text{interior of } A \\ &= \bigcup_{\substack{V \subseteq A \\ V \text{open in } \mathbb{R}^n}} V && \text{union of \textbf{all} open subsets of } \mathbb{R}^n \text{ that are contained in } A \end{aligned}$$

Remark 3.3.

1. A^o is open (arbitrary union of open sets) and $A^0 \subseteq A$
2. if V is any open subset of R^n that is contained in A, then $V \subseteq A^o$ (A^o is the largest open subset of R^n that is contained in A)
3. A is open iff $A^o = A$

   Proof. **Forwards:**

   A is open and $A \subseteq A$ thus A must be a V in the union, but since all $V \subseteq A$ then $A^o = A$.

   Backwards:

   $A^o = A$. Since A^o is open, A is open. □

3.9 Closure

Definition 3.6.

$$\begin{aligned} \overline{A} &= cl(A) && \text{closure of } A \\ &= \bigcap_{\substack{F \supseteq A \\ F \text{closed in } \mathbb{R}^n}} F && \text{intersection of \textbf{all} closed subsets of } \mathbb{R}^n \text{ that contains} A \end{aligned}$$

Remark 3.4.

1. $\overline{A}$ is closed (arbitrary intersection of closed sets) and $\overline{A} \supseteq A$
2. if F is any closed subset of R^n that contains A, then $F \supseteq \overline{A}$ ($\overline{A}$ is the smallest closed set of R^n containing A)
3. A is closed iff $\overline{A} = A$

4 January 10, 2018

4.1 Closure of open ball is closed ball

Proposition 4.1. The closure of the open ball $B_\epsilon(x)$ is the closed ball $\overline{B_\epsilon(x)}$ (hence the notation).

Proof. Remember

$$\overline{B_\epsilon(x)} = \{y \in \mathbb{R}^n \mid \|y - x\| \leq \epsilon\}$$

Let $A =$ is closure of $B_\epsilon(x)$.
Let $F = \{y \in \mathbb{R}^n \mid \|x - y\| \leq \epsilon\}$.
We want to show $A = F$.
We know F is closed and $F \supset B_\epsilon(x)$, so F contains $A =$ the closure of $B_\epsilon(x)$ (any closed set containing another set is in the intersection of the closure) or

$$F \supset A \supset B_\epsilon(x)$$

Suppose $F \neq A$, then $\exists y \in F$ with $y \notin A \Rightarrow y \notin B_\epsilon(x)$ so

$$\|x - y\| = \epsilon$$

(it's sandwiched between the closed ball ($\leq \epsilon$) and the open ball ($< \epsilon$), so it must hold with equality with ϵ).

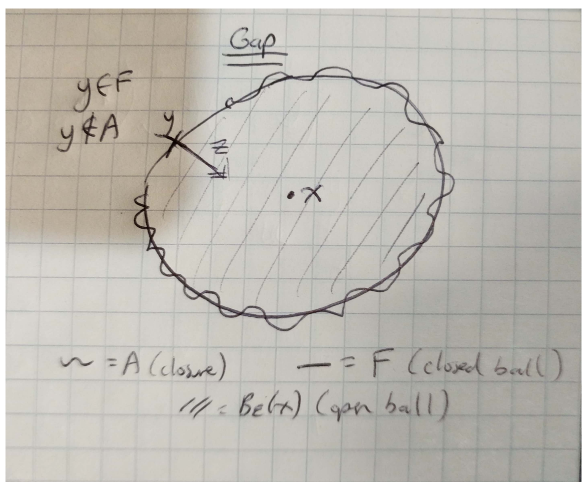

Figure 4.1: The closure of an open ball is the corresponding closed ball.

A is closed and $y \notin A$ so A^c is open and $y \in A^c$. So $\exists \delta > 0$ such that $B_\delta(y) \subseteq A^c$.
Let $t > 0$ with $t < \min\{\delta, \epsilon\}$.
Let

$$z = y + t\frac{(x - y)}{\|x - y\|}$$

(add t unit vectors from y to x). Note that

$$\|z - y\| = t < \delta$$

so $z \in B_\delta(y) \subseteq A^c$.

Also

$$\begin{aligned} x - z &= x - y - t\frac{(x-y)}{\|x-y\|} \\ &= (\|x-y\| - t)\frac{(x-y)}{\|x-y\|} \end{aligned}$$

where the left term is the norm of the vector and the right term is the unit vector.
Thus

$$\|x - z\| = |\|x-y\| - t| = |\epsilon - t| = \epsilon - t < \epsilon$$

So $z \in B_\epsilon(x) \subseteq A$, but we assumed $z \in A^c$ which is a contradiction.
So we must have $F = A$.

Remark 4.1. There is a much simpler proof of this using sequences and limit points.

□

4.2 Boundary

Definition 4.1. Let $A \subseteq \mathbb{R}^n$. We define the **boundary** of A denoted $\partial A = bd(A)$ to be

$$\partial A = bd(A) = \{x \in \mathbb{R}^n \mid B_\epsilon(x) \cap A \neq \varnothing, B_\epsilon(x) \cap A^c \neq \varnothing \quad \forall \epsilon > 0\}$$

That is, $x \in \partial A$ iff every open ball centred at x contains a point in A **and** a point in A^c.
Clearly

$$\begin{aligned} \partial B_\epsilon(x) &= \{y \in \mathbb{R}^n \mid \|y - x\| = \epsilon\} \\ &= \partial(\overline{B_\epsilon(x)}) \end{aligned}$$

4.3 Characterization of boundary

Proposition 4.2. Let $A \subseteq \mathbb{R}^n$: then

$$\begin{aligned} \partial A &= \overline{A} \setminus A^o \\ &= cl(A) \setminus int(A) \end{aligned}$$

Proof. The following two claims and proofs revolve around complements of sets and how if set A intersect a set B is the empty set, then A is a subset of B^c.

Claim 1

$$x \in \overline{A} \iff B_\epsilon(x) \cap A \neq \varnothing \quad \forall \epsilon > 0$$

Proof. **Forwards:**
Suppose $x \in \overline{A}$ but $\exists \epsilon_0 > 0$ $B_\epsilon(x) \cap A = \varnothing$.
So $B_\epsilon(x) \subseteq A^c \Rightarrow (B_\epsilon(x))^c \supset A$.
Since $(B_\epsilon(x))^c$ is closed, then $(B_\epsilon(x))^c \supset \overline{A}$ (by remark (2) after closure definition).
So $\overline{A} \cap B_\epsilon(x) = \varnothing$, but $x \in B_\epsilon(x) \Rightarrow x \notin \overline{A}$, which is a contradiction.
Backwards:

We prove the contrapositive

$$x \notin \overline{A} \Rightarrow B_\epsilon(x) \cap A = \varnothing \quad \forall \epsilon > 0$$

Assume $x \notin \overline{A} \Rightarrow x \in (\overline{A})^c$ which is open, so $\exists \epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \subseteq (\overline{A})^c$. Therefore $B_{\epsilon_0}(x) \cap \overline{A} = \varnothing$ (where $\overline{A} \supset A$), which proves our claim). □

Claim 2

$$x \notin A^o \iff B_\epsilon(x) \cap A^c \neq \varnothing \quad \forall \epsilon > 0$$

Proof. **Forwards:**

Suppose $x \notin A^o$. Assume (for contradiction) $\exists \epsilon_0 > 0$ such that

$$B_{\epsilon_0}(x) \cap A^c = \varnothing \Rightarrow B_{\epsilon_0}(x) \subseteq A$$

(nothing in A^c, thus all in A).

Ergo $x \in (A^o)^c$ and $B_{\epsilon_0}(x) \subseteq A^o$ (since $B_{\epsilon_0}(x)$ is a closed set contained in A - remark (2) after interior definition).

So $B_{\epsilon_0}(x) \cap (A^o)^c = \varnothing$ but $x \in B_{\epsilon_0}(x) \cap (A^o)^c$ which is a contradiction.

Backwards:

(Contrapositive): suppose $x \in A^o$. A^o is open so $\exists \epsilon > 0$ such that

$$B_{\epsilon_0}(x) \subseteq A^o \subseteq A$$

so $B_{\epsilon_0}(x) \cap A^c = \varnothing$. □

Putting the claims together:

$$\begin{aligned}
x \in \overline{A} &\iff B_\epsilon(x) \cap A \neq \varnothing \quad \forall \epsilon > 0 && (1)\\
x \in (A^o)^c &\iff B_\epsilon(x) \cap A \neq \varnothing \quad \forall \epsilon > 0 && (2)\\
x \in \partial A &\iff (1) + (2)\\
&\iff x \in \overline{A} \cap (A^o)^c = \overline{A} \setminus A^o
\end{aligned}$$

□

4.4 Sequences and limits

Definition 4.2. Let (x_k) be a sequence of points in $\mathbb{R}^n, k \in \mathbb{N}$. We say (x_k) **converges** to a point $x \in \mathbb{R}^n$ **iff** for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ (N depends on ϵ in general)

$$k \geq N \Rightarrow \|x_k - x\| < \epsilon$$

(i.e. for any $\epsilon > 0$, **all** the elements of sequence x_k after some $k = N$ are within ϵ of x).

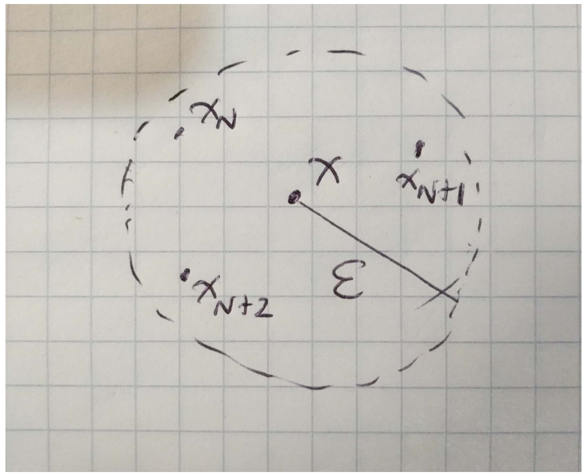

Figure 4.2: All points after $k = N$ for a converging sequence is within ϵ.

If (x_k) converges to x, we denote

$$\lim_{k \to \infty} x_k = x$$

where x is **the limit** of x_k.

4.5 Uniqueness of limits

Lemma 4.1. Suppose $\lim_{k \to \infty} x_k = x$ and $\lim_{k \to \infty} x_k = y$. Then $x = y$ (i.e. a sequence may not converge, but if it does the limit is unique).

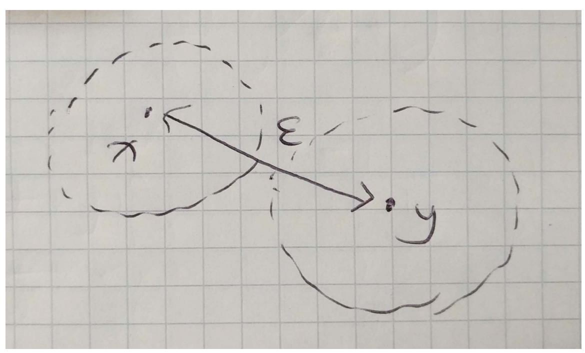

Figure 4.3: Sketch of proof with $x \neq y$ (see below).

Proof. Suppose $x \neq y$, so $\|x - y\| = \epsilon > 0$.
Since (x_k) converges to x, $\exists N_1 \in N$ such that $k \geq N_1$ and

$$\|x_k - x\| < \frac{\epsilon}{2}$$

Similarly for y $\exists k \geq N_2$.

Suppose $k \geq \max\{N_1, N_2\}$. Then

$$\begin{aligned} \|x-y\| &\overset{\triangle}{\leq} \|x-x_k\| + \|x_k - y\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

So $x = y$ by contradiction. □

4.6 Neighbourhood

Definition 4.3. Let $x \in \mathbb{R}^n$. A subset $U \in \mathbb{R}^n$ is called a **neighbourhood (n'h'd)** of x if $\exists \epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \subseteq U$.

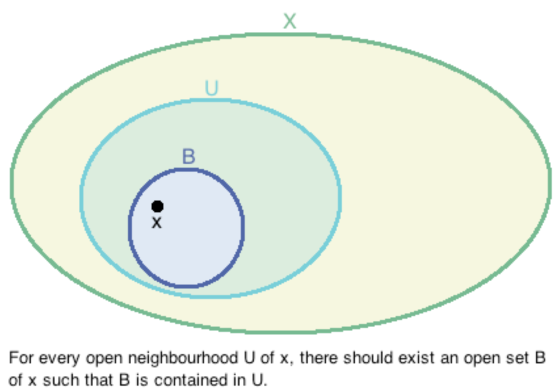

Figure 4.4: U is a neighbourhood of x since there exists an open set B of x contained in U.

(Equivalently, U is a n'h'd of $x \iff U$ contains an open set containing x.)

Definition 4.4. An open n'h'd of x is any open set containing x. (A set is an open n'h'd of x if it contains x **and** all points sufficiently close to x).

Lemma 4.2. Let (x_k) be a sequence in $\mathbb{R}^n$. Suppose $\lim_{k\to\infty} x_k$ exists and equal $x \in \mathbb{R}^n$. Then any n'h'd of x contains all x_k's for k sufficiently large, i.e. if U is a n'h'd of x, $\exists N \in \mathbb{N}$ (N depends on U) such that

$$k \geq N \Rightarrow x_k \in U$$

Proof. U is a n'h'd of x so $\exists \epsilon_0 > 0$ such that $B_\epsilon(x) \subseteq U$.
Since $\lim_{k\to\infty} x_k = x$, $\exists N \in N$ such that $k \geq N \Rightarrow \|x_k - x\| < \epsilon_0$ so $x_k \in B_\epsilon(x) \subseteq U \quad \forall k \geq \mathbb{N}$. □

5 January 12, 2018

5.1 Relations between convergent sequences and open/closed sets

Recall: $x \in \overline{A} \iff B_\epsilon(x) \cap A \neq \varnothing \quad \forall \epsilon > 0$.

Proposition 5.1. Suppose $x \in \overline{A}$. Take $\frac{1}{k} > 0$. From above: $exists x_k \in A$ such that $\|x_k - x\| < \frac{1}{k}$, then $\lim_{k\to\infty} x_k = x$.

Proof. Let $\epsilon > 0$ so $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$ (Archimedean Principle). $\forall k \geq N$, $\frac{1}{k} \leq \frac{1}{N} < \epsilon$ so $\|x_k - x\| < \frac{1}{k} < \epsilon$. □

To summarize, if $x \in \overline{A}$, then $\exists$ a sequence (x_k) such that $\lim_{k\to\infty} x_k = x$ **and** $x_k \in A \quad \forall \in \mathbb{N}$.

What about the converse?

Proposition 5.2. Suppose $x_k \in A \quad \forall k$ and $\lim_{k\to\infty} x_k = x$ **and** $x_k \in A \quad \forall k \in \mathbb{N}$. Then $x \in \overline{A}$.

Proof. If not, $x \in (\overline{A})^c$ so $\exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq (\overline{A})^c$. But $\exists N \in \mathbb{N}$ such that

$$k \geq N \Rightarrow x_k \in B_\epsilon(x)$$

and by hypothesis $x_k \in A \subseteq \overline{A}$.
So $k \geq N \Rightarrow x_k \in \overline{A}$ but we assumed $x_k \in (\overline{A})^c$ which is a contradiction. □

(i.e. whenever (x_k) is a convergent sequence of points all of whose elements are in A, then the limit is in $\overline{A}$).
Special case: If A is closed ($\overline{A} = A$) then if $(x_k) \to x$ and $x_k \in A \forall k$ then $x \in A$; this is **not** true for open sets A.

5.2 Bounded and Cauchy sequences

Definition 5.1. A sequence (x_k) in $\mathbb{R}^n$ is called **bounded** if $\exists M > 0$ such that

$$\|x_k\| \leq M \quad \forall k \in \mathbb{N}$$

(that is: all the x_k's lie in some closed ball $\overline{B_\epsilon(x)}$ centred at 0).

Definition 5.2. A sequence (x_k) is called **Cauchy** if for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$k, l \geq N \Rightarrow \|x_k - x_l\| < \epsilon$$

(eventually all points in the sequence are close to each other).

5.3 Convergent $\iff$ Cauchy

Proposition 5.3. Let (x_k) be a convergent sequence. Then (x_k) is Cauchy.

Proof. Let $x = \lim_{k\to\infty} x_k$. Let $\epsilon > 0$, then $\exists N$ such that

$$\|x_k - x\| < \frac{\epsilon}{2}$$

If $k, l \geq N$ then

$$\|x_k - x_l\| \overset{\triangle}{\leq} \|x_k - x\| + \|x - x_l\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Recall from MATH 147: In $\mathbb{R}$ every Cauchy sequence converges (equivalent to **completeness** of $\mathbb{R}$ or the real line). We show Cauchy converges in $\mathbb{R}^n$ in assignment 2 by showing that each j-th component $x^{(j)}$ converges then by the completeness of $\mathbb{R}$ this follows for $\mathbb{R}^n$.

5.4 Convergence implies bounded

Lemma 5.1. Every convergent sequence is bounded.

Proof. Let $x = \lim_{k\to\infty} x_k$. Let $M_0 = \|x\| + \epsilon$ for $\epsilon > 0$. Then $\exists N$ such that

$$k \geq N \Rightarrow \|x_k - x\| < \epsilon$$

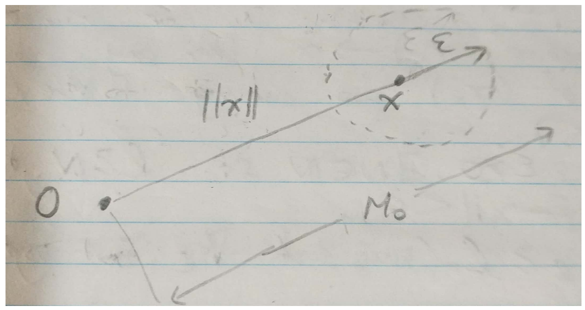

Figure 5.1: Convergent sequences can be bounded by the limit and ϵ and finite points in the sequence.

Note that

$$k \geq N \Rightarrow \|x_k\| \overset{\triangle}{\leq} \|x_k - x\| + \|x\| < \epsilon + \|x\| = M_0$$

Thus we let $M = \max\{\|x_1\|, \ldots, \|x_{N-1}\|, M_0\}$ then $\|x_k\| \leq M \quad \forall k \in \mathbb{N}$. □

Note: not every bounded sequence is Cauchy. Consider $x_k = (-1)^{k+1}$ is $\mathbb{R}$, which is bounded but not convergent. Can we find a weaker statement that's true i.e. given a bounded sequence, can we somehow obtain from it a convergent sequence?

5.5 Subsequences

Let (x_k) be a sequence in $\mathbb{R}^n$. Let $1 \leq k_1 < k_2 < \ldots < k_e < k_{e+1} < \ldots$ be a sequence of $1, 2, 3, 4, \ldots$. Then the corresponding sequence (y_l) (or (x_{k_l})) in $\mathbb{R}^n$ given by $y_l = x_{k_l}$ is called a **subsequence** of (x_k).

Example 5.1. The subsequence given by $k_l = 2l - 1$ (odd numbers) is

$$(x_{2l-1}) = x_1, x_3, x_5, \ldots$$

Proposition 5.4. Suppose $(x_k) \to x$. Then any subsequence (x_{k_l}) of (x_k) also converges to the same limit x.

Proof. Let $\epsilon > 0$. $\exists N \in \mathbb{N}$ such that $l \geq N$ then $\|x_l - x\| < \epsilon$, but $k_l \geq l$ (since each k_e must be strictly larger $> k_{e-1}$), so $\|x_k - x\| < \epsilon \quad \forall l \geq N$ hence $\lim_{k\to\infty} x_{k_l} = x$. □

Note: A sequence (x_k) that does not converge can have

1. Subsequences that don't converge (e.g. $k_l = l$ so $x_{k_l} = x_l$).
2. Distinct subsequences with different limits.

   For example, $x_k = (-1)^{k+1}$ which is $1, -1, 1, -1, \ldots$, we can have two subsequences

$$x_{2l-1} = (-1)^{2l} = 1, 1, 1, \ldots \qquad (x_{2l-1}) \to 1$$
$$x_{2l} = (-1)^{2l-1} = -1, -1, -1, \ldots \qquad (x_{2l}) \to -1$$

5.6 Bolzano-Weierstrass (B-W) Theorem

Theorem 5.1. In $\mathbb{R}^n$, every bounded sequence has a convergent subsequence.

Remark 5.1. This convergent subsequence is **not** unique. We'll see in the proof that we make many arbitrary choices.

Proof. By induction on n.
Case $n = 1$: Let (x_k) be a sequence in $\mathbb{R}$ that is **bounded**. So $\exists M > 0$ such that $|x_k| \leq M \quad \forall k \in \mathbb{N} \iff x_k \in [-M, M]$.

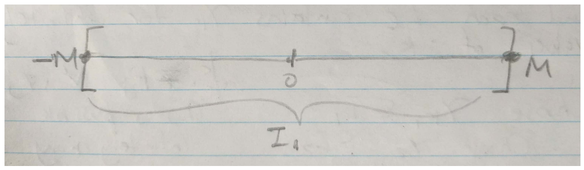

Figure 5.2: I_1 is the interval of our bounded sequence in $\mathbb{R}$.

Define

$$I_1 = [-M, M] = [-M, 0] \cup [0, M]$$

At least one (maybe both) of $[-M, 0]$ and $[0, M]$ contains x_k for infinite many values of k (the x_k's could initially be all in one side then infinitely many in the other, or the x_k's could jump back and forth so both would have infinitely many).
Let I_2 denote the one with infinitely many. That is $x_k \in I_2$ for infinitely many x_k's. Note that

$$I_2 \subseteq I_1$$
$$I_2 = [a, b] = [a, \frac{a+b}{2}] \cup [\frac{a+b}{2}, b]$$

Again, at least one of these halves contains infinitely many x_k's. Let I_3 be that one.
Keep subdividing in this way and choosing a half which contains x_k for infinitely many k's. We have

$$\text{length } I_1 = 2M$$
$$\text{length } I_2 = M$$
$$\text{length } I_3 = \frac{M}{2}$$
$$\vdots$$
$$\text{length } I_l = \frac{2M}{2^{l-1}}$$

moreover,

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_e \supseteq I_{e+1} \supseteq \ldots$$

and each I_l contains x_k for infinitely many values of k.
We can thus choose some $x_{k_1} \in I_1, x_{k_2} \in I_2, \ldots, x_{k_l} \in I_l \quad \forall l \in \mathbb{N}$ where $1 \leq k_1 < k_2 < \ldots < k_e < k_{e+1} < \ldots$. This is possible since there are infinitely many x_k's in each interval.
We claim:

1.
$$\bigcap_{l=1}^{\infty} I_l \neq \varnothing$$

and in fact contains **exactly one point** x.

Note that
$$I_l = [a_l, b_l] \quad \text{for some } a_l < b_l$$

and also
$$I_l \supset I_{l+1} \Rightarrow a_1 \leq a_l \leq a_{l+1} < b_{l+1} \leq b_l \leq b_1 \quad \forall l$$

(i.e. either endpoints move inwards for each successive interval).

So (a_l) is an increasing sequence bounded by b_1, therefore $\exists a$ such that $\lim_{l\to\infty} a_l = a$ and $a_l \leq a \leq b_1 \quad \forall l$.

Similarly (b_l) is a decreasing sequence bounded by a_1, so $\exists b$ such that $\lim_{l\to\infty} b_l = b$ and $a_1 \leq b \leq b_l \quad \forall l$.

We have $a_l < b_l \quad \forall l$. Taking the limit we have $a \leq b$ (limit can only be guaranteed with potential for equality).

$$a_1 \leq a_l \leq a_{l+1} \leq a \leq b \leq b_{l+1} \leq b_l \leq b_1$$

Note that

$$\begin{aligned} 0 \leq b - a \leq b_l - a_l &= length(I_l) \\ &= \frac{2M}{2^{l-1}} \to 0 \text{ as } l \to \infty \end{aligned}$$

hence $a = b$ (call this x).

By construction $x = a = b \in [a_l, b_l] = I_l \quad \forall l$ so

$$x \in \bigcap_{l=1}^{\infty} I_l$$

so there exists an element. Suppose $y \in \bigcap_{l=1}^{\infty} I_i$ then $x, y \in I_l \quad \forall l$ and

$$|x - y| \leq \frac{2M}{2^{l-1}} \quad \forall l \Rightarrow x = y (\text{as } l \to 0)$$

2.
$$\lim_{l\to\infty} x_{k_l} = x$$

Assume $x_{k_l} \in I_l$ and $x \in I_l \quad \forall l$ (from claim 1). So

$$|x_{k_l} - x| \leq \frac{2M}{2^{l-1}} \to 0 \text{ as } l \to \infty$$

thus $\lim_{l\to\infty} x_{k_l} = x$.

The above two claims prove the theorem for $n = 1$.
Now suppose the thoerem is true for n, we show it is true for $n + 1$.
Let (x_k) be a bounded sequence in $\mathbb{R}^{n+1}$, so $\exists M$ such that $\|x_k\| \leq M \quad \forall k$.
We write $x_k = (x_k^1, x_k^2, \ldots, x_k^{n+1})$ where x_k^j is the j-th component of vector $x_k \in \mathbb{R}^{n+1}$.

So

$$\|x_k\|^2 = |x_k^1|^2 + |x_k^2|^2 + \ldots + |x_k^n|^2 + |x_k^{n+1}|^2 \leq M^2 \tag{5.1}$$

Define a sequence (y_k) in $\mathbb{R}^n$ as the first n components of x_k

$$y_k = (x_k^1, \ldots, x_k^n)$$

therefore $\|y_k\| \leq M \quad \forall k$ by equation 5.1.
By the inductive hypothesis, $\exists$ a subsequence (y_{k_l}) of (y_k) that converges to some point $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$.
Consider the sequence $(x_{k_1}^{n+1})$ in $\mathbb{R}^1$ **(TODO(richardwu): why can't we just use (x_k^{n+1}) here instead?)**. By equation 5.1, $|x_{k_l}^{n+1}| \leq M \quad \forall l$, so $(x_{k_l}^{n+1})$ is a bounded sequence in $\mathbb{R}$. By B-W for $n = 1$, $\exists$ subsequence $(x_{k_{l_j}}^{n+1})$ that converges to some $x^{n+1} \in \mathbb{R}$.
Consider the subsequence $(y_{k_{l_j}})$ of (y_{k_l}), which also converges to $(x^1, \ldots, x^n) \in \mathbb{R}^n$.
So $x_{k_{l_j}}^a \to x^a$ as $j \to \infty$ for $a = 1, \ldots, n$ and $a = n + 1$.
Thus the sequence $x_{k_{l_j}} \to x$ as $j \to \infty$.

Remark 5.2. We used the IH/B-W on the first n components and then the $n+1$ component to find corresponding convergent subsequences. In order to "meld" them together, we needed to take the subsequence of either subsequence (to have a 2-level subsequence) to ensure it converges for the same k_{l_j}'s as the other 1-level subsequence.
TODO(richardwu): see the above TODO for why we don't just take k_l's instead of k_{l_j}'s.

□

6 Janaury 15, 2017

6.1 Connectedness

Definition 6.1. Let E be a non-empty subset of $\mathbb{R}^n$.
We say E is **disconnected** if there exists a **separation** for E. A separation of E is a pair U, V open sets in $\mathbb{R}^n$ such that

1. $E \cap U \neq \varnothing$
2. $E \cap V \neq \varnothing$
3. $E \cap U \cap V = \varnothing$
4. $E \subseteq U \cup V$

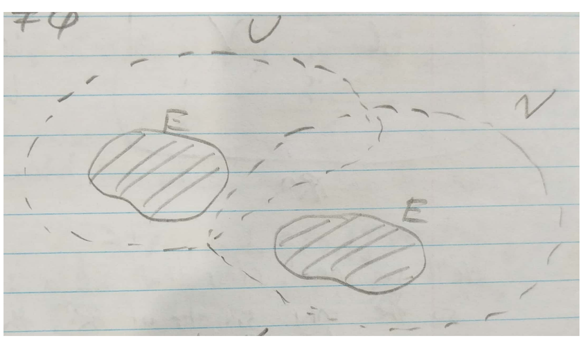

Figure 6.1: E is disconnected since there are open sets U, V that form a separation.

Note that $U \cap V$ need not be empty, but it must be disjoint from E.
(intuitively a set is disconnected if it is more than one piece).

Definition 6.2. E is **connected** if $\nexists$ any separation of E.

Remark 6.1. Connectedness and disconnectedness do not apply to $\varnothing$.

6.2 Is R^n connected?

(Yes it is).
Suppose $\exists$ a separation of $\mathbb{R}^n$ of open sets U, V such that

1. $$\varnothing \neq U \cap \mathbb{R}^n = U$$
$$\varnothing \neq V \cap \mathbb{R}^n = V$$

   which implies U, V both non-empty. Furthermore

2. $$U \cap V \cap \mathbb{R}^n = U \cap V = \varnothing$$

   which implies U, V are disjoint.

3. $$\mathbb{R}^n \subseteq U \cup V \subseteq \mathbb{R}^n$$

   so $\mathbb{R}^n = U \cup V$. Since $U \cap V = \varnothing$, then $U^c = V$ and $V^c = U$.

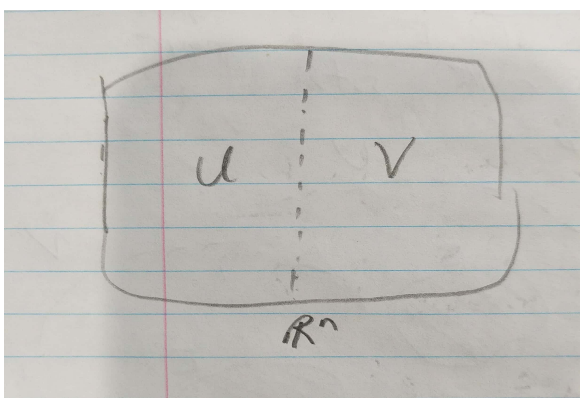

Figure 6.2: Sketch of what disconnected $\mathbb{R}^n$ would look like.

This would mean U, V are both **non-empty** subsets that are **both open and closed** and $U, V \neq \mathbb{R}^n$ (since they are non-empty disjoint).
In other words, if $\exists U$ such that $U \neq 0, U \neq \mathbb{R}^n$ and U is both open and closed, then $U, V = U^c$ gives a separation of $\mathbb{R}^n$.
We'll see (next class) that $\nexists$ a separation of $\mathbb{R}^n$ for any n, so the only subsets of $\mathbb{R}^n$ that are both open and closed are $\varnothing, \mathbb{R}^n$.

6.3 $[0, 1]$ is connected

This is an example of a connected subset in $\mathbb{R}$ and will be used next time to prove $\mathbb{R}^n$ is connected and *more*.

Theorem 6.1. Let $E = [0, 1] \subseteq \mathbb{R}$. Then E is connected.
(Aside: in fact: *any* interval $[a, b], [a, b), (a, b], (a, b)$ in $\mathbb{R}$ is connected and thesse are the **only** connected subsets in $\mathbb{R}$ i.e. connectedness $\Rightarrow$ interval).

Proof. By contradiction.
Suppose $[0, 1]$ is not connected. $\exists$ a separation U, V open subsets of $[0, 1]$ where

1. $U \cap [0, 1] \neq \varnothing$
2. $V \cap [0, 1] \neq \varnothing$
3. $U \cap V \cap [0, 1] = \varnothing$
4. $[0, 1] \subseteq U \cup V$

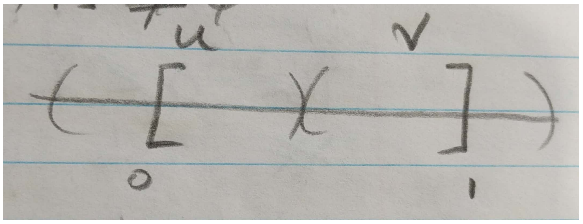

Figure 6.3: Sketch of U, V open sets as (potential) separation for $[0, 1]$.

WLOG $0 \in U$. Since U is open and $0 \in U$, $\exists \epsilon_0 > 0$ such that $B_{\epsilon_0}(0) = (-\epsilon_0, \epsilon_0) \subseteq U$.
WLOG, $\epsilon_0 < 1$ so $[0, \epsilon_0) \subseteq U \cap [0, 1]$.
Define t_0 as

$$\sup\{\epsilon \in (0, 1) \mid [0, \epsilon) \subseteq U \cap [0, 1]\}$$

note: the above is a non-empty subset of $\mathbb{R}$ since ϵ_0 is in the set. It's bounded above by 1, so the supremum or t_0 must exist.
We have $0 < \epsilon_0 \leq t_0 \leq 1$ so $t_0 \in (0, 1]$, thus $t_0 \in U$ or $t_0 \in V$.

Case 1: $t_0 \in U$ Since U is open (all open sets have some open ball around every point) $\exists \delta > 0$ such that

$$(t_0 - \delta, t_0 + \delta) \subseteq U \tag{6.1}$$

WLOG $\delta < t_0$ but $0 < t_0 - \delta < t_0$ so by definition of t_0 (as supremum), $\exists \hat{\epsilon} > 0$ with $t_0 - \delta < \hat{\epsilon} < t_0$ such that

$$[0, \hat{\epsilon}) \subseteq U \cap [0, 1] \tag{6.2}$$

Combining equation 6.1 and 6.2 (joining the two intervals together since we do not know if either separately are in U), we have

$$[0, t_0 + \delta) \subseteq U \cap [0, 1] \tag{6.3}$$

We have two subcases:

$t_0 < 1$ Then we can shrink $\delta > 0$ further to ensure $t_0 + \delta < 1$ ($\delta < 1 - t_0$).
Then $0 < t_0 + \delta < 1$ and $[0, t_0 + \delta) \subseteq U \cap [0,1]$ which contradicts t_0 as the supremum.

$t_0 = 1$ This implies $U \cap [0,1] = [0,1]$ by equation 6.3 but then $V \cap [0,1] = \varnothing$ (since $U \cap V \cap [0,1] = \varnothing$), which is a contradiction since V must be non-empty.

Case 2: $t_0 \in V$ Since V is open $\exists \zeta > 0$ such that

$$(t_0 - \zeta, t_0 + \zeta) \subseteq V \tag{6.4}$$

WLOG $\zeta < t_0$ but $0 < t_0 - \zeta < t_0$ so by definition of t_0 (as supremum) $\exists \tilde{\epsilon} > 0$ with $t_0 - \zeta < \tilde{\epsilon} \leq t_0$ such that

$$[0, \tilde{\epsilon}) \subseteq U \cap [0,1]$$

(it's U since that was the set t_0 was defined with).

Pick $s \in (t_0 - \zeta, \tilde{\epsilon})$. Then $s \in U \cap [0,1]$ by equation 6.1 but also $s \in V \cap [0,1]$ by equation 6.4, which is a contradiction.

By the contradiction of the two cases above, $[0,1]$ is connected. □

7 January 17, 2017

7.1 Convex sets

Definition 7.1. A **non-empty** subset E of $\mathbb{R}^n$ is called **convex** if whenever $x, y \in E$ then

$$tx + (1-t)y \in E \quad \forall t \in [0,1]$$

i.e. the line segment between any 2 points in E is contained in E.

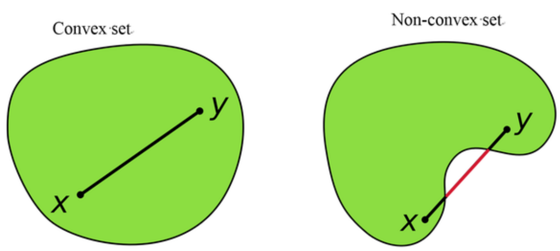

Figure 7.1: Convex and non-convex sets in $\mathbb{R}^2$.

7.2 Convex ⇒ connected

Corollary 7.1. Any convex subset E of $\mathbb{R}^n$ is connected. This implies two corollaries:

Corollary 7.2. $\mathbb{R}^n$ is connected $\forall n$ since $\mathbb{R}^n$ is trivially connected.

Corollary 7.3. The only subsets of $\mathbb{R}^n$ that are both open and closed are $\varnothing, \mathbb{R}^n$ (see the remark about $\mathbb{R}^n$ connectedness from above).

Proof. Let E be convex and suppose E is *not* connected. $\exists$ open subsets U, V such that

1. $U \cap E \neq \varnothing$

2. $V \cap E \neq \varnothing$
3. $U \cap V \cap E = \varnothing$
4. $E \subseteq U \cup V$

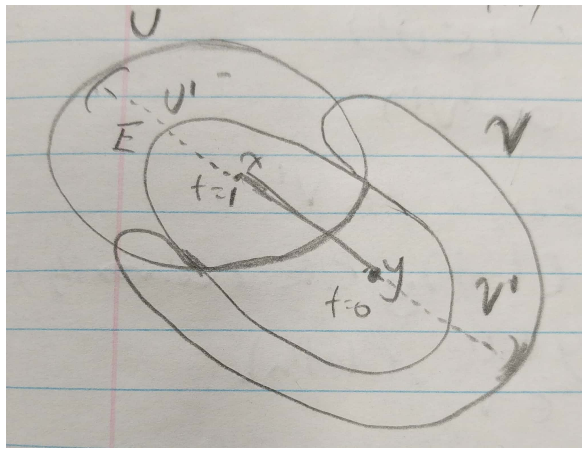

Figure 7.2: Suppose convex E is not connected and there exists a separation U, V.

Let $x \in U \cap E$ and $y \in V \cap E$ (therefore $x \neq y$ since $U \cap V \cap E = \varnothing$). Since E is convex,

$$tx + (1-t)y \in E \quad \forall t \in [0,1]$$

Define U', V' subsets of $\mathbb{R}^n$ by

$$U' = \{t \in \mathbb{R} : tx + (1-t)y \in U\} \qquad V' = \{t \in \mathbb{R} : tx + (1-t)y \in V\}$$

(note: U', V' is not restricted to elements $[0,1]$: t could extend arbitrarily into E^c).
Claim: U', V' are open subsets of $\mathbb{R}$. Let $t_0 \in U'$ so $x_0 = t_0 + (1-t_0)y \in U$. Since U is open in $\mathbb{R}^n$ $\exists \epsilon_0 > 0$ such that $B_{\epsilon_0}(x_0) \in U$. We pick $t \in \mathbb{R}$ such that

$$|t - t_0| < \frac{\epsilon_0}{\|x\| + \|y\|}$$

then

$$\begin{aligned} B_{\epsilon_0}(x_0) \Rightarrow \|(tx + (1-t)y) - x_0\| &= \|tx + (1-t)y - t_0x - (1-t_0)y\| \\ &= \|(t-t_0)x + (t_0 - t)y\| \\ &\overset{\triangle}{\leq} |t - t_0|(\|x\| + \|y\|) \\ &< \epsilon_0 \end{aligned}$$

But $B_{\epsilon_0}(x_0) \subseteq U$ so if $|t - t_0| < \frac{\epsilon_0}{\|x\|+\|y\|}$ then $t \in U'$ (we want our choice of t to imply $t \in U'$).
So $\frac{B_{\epsilon_0}(t_0)}{\|x\|+\|y\|} \subseteq U'$ so U' is open.
Similarly, V' is open.
Thus here are the properties of U', V'. They are both open in $\mathbb{R}$ and

1. $U' \cap [0,1] \neq \varnothing$ (since $1 \in U'$)

2. $V' \cap [0, 1] \neq \varnothing$ (since $0 \in V'$)

3. $U' \cap V' \cap [0, 1] = \varnothing$ (since $0 \in V'$)

   Given some $t \in [0, 1]$ (since $tx \in (1 - t)y \in E$ from convexity), note that either $t \in U'$ from $tx + (1 - t)y \in U$ or $t \in V'$ from $tx + (1 - t)y \in V$ (we know from before that $U \cap V \cap E = \varnothing$ thus this must hold for the subsets U', V').

4. $[0, 1] \subseteq U' \cup V'$

   If $t \in [0, 1]$, then $z = tx + (1 - t)y \in E$ so $z \in U \cup V$ from before, so $z \in U$ or $z \in V$, thus by their definitions $t \in U'$ or $t \in V'$.

Then U', V' is a separation for $[0, 1]$, which is a contradiction. Thus E is connected. □

Remark 7.1. In general, to prove a set E is connected it is generally easier to assume it is *not* connected and there exists a separation, then derive a contradiction.

7.3 Open cover and compactedness

Definition 7.2. Let E be a subset of $\mathbb{R}^n$. An **open cover** of E is a collection of open subsets U_α $\alpha \in A$ of $\mathbb{R}^n$ such that

$$E \subseteq \bigcup_{\alpha \in A} U_\alpha$$

(finite or infinite union of open subsets).

Definition 7.3. The subset E is called **compact** *iff* every open cover of E admits a **finite subcover**.
That i: if $\bigcup U_\alpha$ $\alpha \in A$ is an open cover of E, then $\exists$ a finite subset A_0 of A such that

$$E \subseteq \bigcup_{\alpha \in A_0} U_\alpha$$

Informally, whenever a compact E is covered by a collection of open sets, it is actually covered by just finitely many of those open sets.

Remark 7.2. This definition is not very useful for checking if a subset is compact (because you would have to check every open cover of E).

Definition 7.4. A subset $E \subseteq \mathbb{R}^n$ is called **bounded** if $\exists M > 0$ such that $E \subseteq B_M(0)$. That is $\|x\| < M$ $\forall x \in E$.

7.4 Heine-Borel theorem

Theorem 7.1. Let E be a subset of $\mathbb{R}^n$. E is **compact** *iff* E is both **closed and bounded**.
The following proof uses the *density of rationals.*