richardwu.ca

STAT 333 Course Notes

Applied Probability

Steve Drekic • Winter 2018 • University of Waterloo

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

1 January 4, 2018

1.1 Example 1.1 Solution

What is the probability that we roll a number less than 4 given that we know it's odd?

Solution. Let $A = \{1, 2, 3\}$ (less than 4) and $B = \{1, 3, 5\}$ (odd). We want to find $P(A \mid B)$. Note that $A \cap B = \{1, 3\}$ and there are six elements in the sample space S thus

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{2}{3}$$

1.2 Example 1.2 Solution

Show that $Bin(n, p) \sim Pois(\lambda)$ when $\lambda = np$ for n large and p small.

Solution. Let $\lambda = np$. Note that $p = \frac{\lambda}{n} \quad n > 0$. From the pmf for $X \sim Bin(n, p)$

$$\begin{aligned}
p(x) &= \binom{n}{x} p^x (1-p)^{n-x} \\
&= \frac{n(n-1)...(n-x+1)}{x!} (\frac{\lambda}{n})^x (1 - \frac{\lambda}{n})^{n-x} \\
&= \frac{n(n-1)...(n-x+1)}{n^x} \cdot \frac{\lambda^x}{x!} \frac{(1-\frac{\lambda}{n})^n}{(1-\frac{\lambda}{n})^x}
\end{aligned}$$

Recall $\lim_{n\to\infty}(1 - \frac{\lambda}{n})^n = e^{-\lambda}$ so

$$\lim_{n\to\infty} p(x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}$$

2 January 9, 2018

2.1 Example 1.3 Solution

Find the mgf of $Bin(n, p)$ and use that to find $E[X]$ and $Var(X)$.

Solution. Recall the binomial series is

$$(a+b)^m = \sum_{x=0}^{m} \binom{m}{x} a^x b^{m-x} \quad a, b \in \mathbb{R}, m \in \mathbb{N}$$

Let $x \sim Bin(n, p)$ and so

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \dots, n$$

Taking the mgf $E[e^{tX}]$

$$\begin{aligned}\Phi_X(t) = E[e^{tX}] &= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}\end{aligned}$$

from the binomial series we have

$$\Phi_x(t) = (pe^t + 1 - p)^n \quad t \in \mathbb{R}$$

We can take the first and second derivatives for the first and second moment

$$\begin{aligned}\Phi'_X(t) &= n(pe^t + 1 - p)^{n-1} pe^t \\ \Phi''_X(t) &= np[(pe^t + 1 - p)^{n-1} e^t + e^t (n-1)(pe^t + 1 - p)^{n-2} pe^t]\end{aligned}$$

So $E[X] = \Phi_X(t) \mid_{t=0} = np$.
For the variance, we need the second moment

$$\begin{aligned}E[X^2] &= \Phi_X(t) \mid_{t=0} \\ &= np[1 + (n-1)p] \\ &= np + (np)^2 - np^2\end{aligned}$$

So

$$\begin{aligned}Var(X) &= E[X^2] - E[X]^2 \\ &= np + (np)^2 - np^2 - (np)^2 \\ &= np(1-p)\end{aligned}$$

2.2 Example 1.4 Solution

Show that $Cov(X, Y) = 0 \not\Rightarrow$ independence.

Solution. We show this using a counter example

		y		
	$p(x,y)$	0	1	$p_X(x)$
	0	0.2	0	0.2
x	1	0	0.6	0.6
	2	0.2	0	0.2
	$p_Y(y)$	0.4	0.6	1

Note that

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

where

$$
\begin{aligned}
E[XY] &= \sum_{x=0}^{2}\sum_{y=0}^{1} xyp(x,y) = (1)(1)(0.6) = 0.6 \\
E[X] &= \sum_{x=0}^{2} xp_X(x) = (1)(0.6) + (2)(0.2) = 0.6 + 0.4 = 1 \\
E[Y] &= \sum_{y=0}^{1} yp_Y(y) = (1)(0.6) = 0.6
\end{aligned}
$$

So $Cov(X,Y) = 0.6 - (1)(0.6) = 0$. However, $p(2,0) = 0.2 \neq p_X(2)p_Y(0) = (0.2)(0.4) = 0.08$, thus X and Y are not independent (they are dependent).

2.3 Example 1.5 Solution

Given $X_1, \ldots, X_n$ are independent r.v's where $\Phi_X(t)$ is the mgf of X_i, show that $T = \sum_{i=1}^n X_i$ has mgf $\Phi_T(t) = \prod_{i=1}^n \Phi_{X_i}(t)$.

Solution. We take the definition of the mgf of T

$$
\begin{aligned}
\Phi_T(t) &= E[e^{tT}] \\
&= E[e^{t(X_1+\ldots+X_n)}] \\
&= E[e^{tX_1} \cdot \ldots \cdot e^{tX_n}] \\
&= E[e^{tX_1}] \cdot \ldots \cdot E[e^{tX_n}] && \text{independence} \\
&= \prod_{i=1}^n \Phi_{X_i}(t)
\end{aligned}
$$

2.4 Exercise 1.3

If $X_i \sim Pois(\lambda_i)$ show that $T = \sum X_i \sim Pois(\sum \lambda_i)$.

Solution. Recall that $Pois(\lambda_i) \sim Bin(n_i, p)$ where $\lambda_i = n_i p$ and

$$\Phi_{X_i}(t) = (pe^t + 1 - p)^{n_i} \quad \forall t \in \mathbb{R}$$

where $X_i \sim Bin(n_i, p) \quad i = 1, \ldots, m$.
Therefore

$$
\begin{aligned}
\Phi_T(t) &= \prod_{i=1}^m (pe^t + 1 - p)^{n_i} \\
&= (pe^t + 1 - p)^{n_1} \cdot \ldots \cdot (pe^t + 1 - p)^{n_m} \\
&= (pe^t + 1 - p)^{\sum n_i} \quad t \in \mathbb{R}
\end{aligned}
$$

By the mgf uniqueness property, we have

$$T = \sum_{i=1}^m X_i \sim Bin(\sum_{i=1}^m n_i, p)$$