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STAT 433/833 Course Notes

Stochastic Processes

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Table of Contents

Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. These notes are my interpretation and transcription of the content covered in lectures. The instructor has not verified or confirmed the accuracy of these notes, and any discrepancies, misunderstandings, typos, etc. as these notes relate to course's content is not the responsibility of the instructor. If you spot any errors or would like to contribute, please contact me directly.

1 September 6, 2018

1.1 Example 1.2 solution

Use the definition of the Markov property to show that

$$\begin{aligned}&P(X_{n+1}=x_{n+1}\mid X_n=x_n,X_{n-1}=x_{n-1},\dots,X_{n-k+1}=x_{n-k+1},X_{n-k-1}=x_{n-k-1},\dots,X_0=x_0)\\&\qquad=P(X_{n+1}=x_{n+1}\mid X_n=x_n),\quad k=1,2,\dots,n\end{aligned}$$

(i.e. we are missing one past observation).

Solution. Applying the definition of conditional probability, our expression is equivalent to

$$\frac{P(X_{n+1}=x_{n+1},X_n=x_n,X_{n-1}=x_{n-1},\dots,X_{n-k+1}=x_{n-k+1},X_{n-k-1}=x_{n-k-1},\dots,X_0=x_0)}{P(X_n=x_n,X_{n-1}=x_{n-1},\dots,X_{n-k+1}=x_{n-k+1},X_{n-k-1}=x_{n-k-1},\dots,X_0=x_0)}=\frac{N}{D}$$

By the law of total probability

$$\begin{aligned}N&=\sum_{x_{n-k}\in S}P(X_{n+1}=x_{n+1},\dots,X_{n-k}=x_{n-k},\dots,X_0=x_0)\\&=\sum_{x_{n-k}\in S}P(X_{n+1}=x_{n+1}\mid X_n=x_n,\dots,X_{n-k}=x_{n-k},\dots,X_0=x_0)\times P(X_n=x_n,\dots,X_{n-k}=x_{n-k},\dots,X_0=x_0)\end{aligned}$$

By the Markov property

$$\begin{aligned}&=P(X_{n+1}=x_{n+1}\mid X_n=x_n)\sum_{x_{n-k}\in S}P(X_n=x_n,\dots,X_{n-k}=x_{n-k},\dots,X_0=x_0)\\&=P(X_{n+1}=x_{n+1}\mid X_n=x_n)P(X_n=x_n,\dots,X_{n-k}\in S,\dots,X_0=x_0)\end{aligned}$$

Since $X_{n-k}\in S$ is an event with probability 1

$$\begin{aligned}&=P(X_{n+1}=x_{n+1}\mid X_n=x_n)P(X_n=x_n,\dots,X_{n-k+1}=x_{n-k+1},X_{n-k-1}=x_{n-k-1},\dots,X_0=x_0)\\&=P(X_{n+1}=x_{n+1}\mid X_n=x_n)\cdot D\end{aligned}$$

The result follow.

2 September 11, 2018

2.1 Section 1.2: Transitivity of communication relation

Prove that if $i\leftrightarrow j$ and $j\leftrightarrow k$, then $i\leftrightarrow k$ (and thus the communication relation "$\leftrightarrow$" is an equivalence relation).

Proof. $\exists n, m \in \mathbb{N}$ such that $P_{i,j}^{(n)} > 0$ and $P_{j,k}^{(m)} > 0$.
Note that

$$P_{i,k}^{(n+m)} = \sum_{l \in S} P_{i,l}^{(n)} P_{l,k}^{(m)} \geq P_{i,j}^{(n)} P_{j,k}^{(m)} > 0$$

Similarly we can show $k \to i$, thus $i \leftrightarrow k$. $\square$

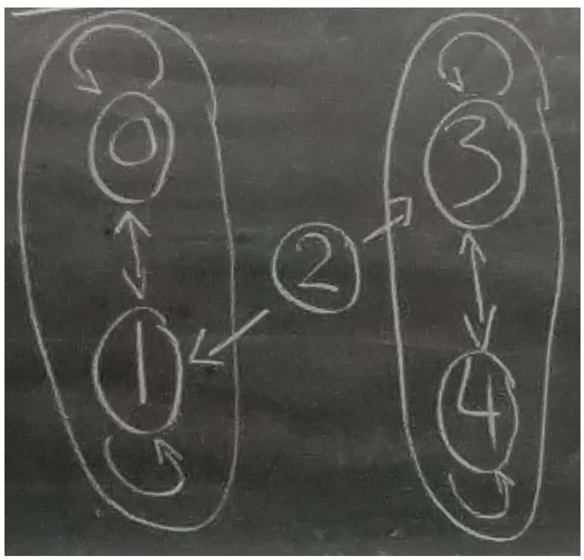
2.2 Example 1.3 solution

Given the DTMC with TPM

$$P = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 0.1 & 0.9 \end{bmatrix} \end{array}$$

Use a state transition diagram to determine the equivalence classes.

Solution. We draw the following state transition diagram and note that there are three communication classes: $\{0,1\}, \{2\}, \{3,4\}$.



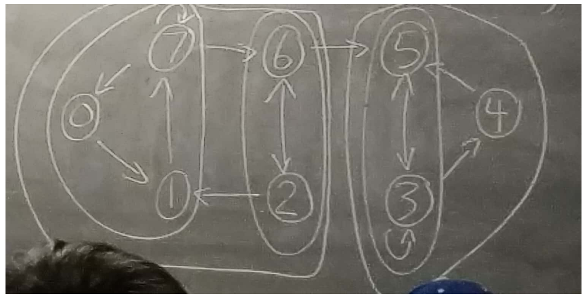
2.3 Example 1.4 solution

Given the DTMC with TPM

$$P = \begin{array}{cc} & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0.3 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

Use a state transition diagram to determine the equivalence classes.

Solution. We draw the following state transition diagram and note that there are two communication classes: $\{0, 1, 2, 6, 7\}, \{3, 4, 5\}$.



2.4 Example 1.5 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0.3 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

Use sample paths to prove that all states within the communication classes found in Example 1.4 communicate.

Solution. class $\{3, 4, 5\}$ Note that $P_{3,4}P_{4,5}P_{5,3} > 0$ i.e. the sample path $3 \to 4 \to 5 \to 3$ has positive probability, thus states 3,4, and 5 communicate since for any pair of states $i, j \in \{3, 4, 5\}$, $\exists n_{i,j} \leq 3$ such that $P_{i,j}^{(n_{i,j})} > 0$.

class $\{0, 1, 2, 6, 7\}$ We have sample path $0 \to 1 \to 7 \to 6 \to 2 \to 1 \to 7 \to 0$ with positive probability.

By a similar argument as above the five states communicate.

2.5 Theorem 1.1 proof: periodicity is a class property

Theorem 2.1. If $i \leftrightarrow j$ then $d(i) = d(j)$ (equal periods).

Proof. Since $i \leftrightarrow j$, then $\exists n, m \in \mathbb{N}$ such that $P_{i,j}^{(n)} > 0$ and $P_{j,i}^{(m)} > 0$.
$\forall L \in \mathbb{Z}^+$ s.t. $P_{j,j}^{(L)} > 0$, we have

$$\begin{aligned} P_{i,i}^{(m+n+L)} &= \sum_{k \in S} P_{i,k}^{(n)} P_{k,i}^{(m+L)} \\ &= \sum_{k \in S} \sum_{l \in S} P_{i,k}^{(n)} P_{k,l}^{(L)} P_{l,i}^{(m)} \\ &\geq P_{i,j}^{(n)} P_{j,j}^{(L)} P_{j,i}^{(m)} \\ &> 0 \end{aligned}$$

Thus $d(i)$ divides $n+m+L$.
Note that $P_{i,i}^{(n+m)} = \sum_{k \in S} P_{i,k}^{(n)} P_{k,i}^{(m)} \geq P_{i,j}^{(n)} P_{j,i}^{(m)} > 0$, thus $d(i)$ divides $n+m$.
Therefore $d(i)$ divides $(n+m+L)-(n+m) = L$ $\forall L$ s.t. $P_{j,j}^{(L)} > 0$, thus $d(i)$ divides $gcd\{L \in \mathbb{Z}^+ \mid P_{j,j}^{(L)} > 0\} = d(j)$.
Similarly, $d(j)$ divides $d(i)$, thus $d(i) = d(j)$. $\square$

3 September 13, 2018

3.1 Example 1.6 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} \end{array}$$

Show that $d(i) = 1$ despite the fact that $P_{i,i}^{(1)} = P_{i,i} = 0$ for $i = 0, 1, 2$.

Solution. Consider state 0 where we have

$$\begin{aligned} P_{0,0}^{(1)} &= 0 \\ P_{0,0}^{(2)} &= \sum_{k \in S} P_{0,k} P_{k,0} \geq P_{0,1} P_{0,1} = \frac{1}{4} > 0 \\ P_{0,0}^{(3)} &= \sum_{k \in S} P_{0,k} P_{k,l} P_{l,0} \geq P_{0,1} P_{1,2} P_{2,0} = \frac{1}{8} > 0 \end{aligned}$$

Therefore $d(0) = \gcd\{2, 3, \ldots\} = 1$.

Since the sample path $0 \to 1 \to 2 \to 0$ has positive prob., all of the states communicate and the DTMC is irreducible, thus $d(2) = d(1) = d(0) = 1$ as well.

3.2 Theorem 1.2 proof: transience/recurrence are class properties

Theorem 3.1. Transience and recurrence are class properties i.e. if $i \leftrightarrow j$ and i is recurrent, then j is recurrent.

Proof. It clearly holds if $i = j$, so assume $i \neq j$. $i \leftrightarrow j$ so $\exists m, n \in \mathbb{Z}^+$ s.t. $P_{j,i}^{(m)} > 0$ and $P_{i,j}^{(n)} > 0$.
Note that

$$\begin{aligned} \sum_{n=1}^{\infty} P_{j,j}^{(n)} &\geq \sum_{l=m+n+1}^{\infty} P_{j,j}^{(l)} \\ &\geq \sum_{l=m+n+1}^{\infty} P_{j,i}^{(m)} P_{i,i}^{(l-m-n)} P_{i,j}^{(n)} \\ &= P_{j,i}^{(m)} P_{i,j}^{(n)} \sum_{l=m+n+1}^{\infty} P_{i,i}^{(l-m-n)} \\ &= P_{j,i}^{(m)} P_{i,j}^{(n)} \sum_{L=1}^{\infty} P_{i,i}^{(L)} \\ &= \infty \end{aligned}$$

since i is recurrent thus $\sum_{L=1}^{\infty} P_{i,i}^{(L)} = \infty$, thus state j is recurrent.
Transience is proven similarly. □

3.3 Theorem 1.3 proof: recurrent classes with states i, j imply $f_{i,j} = 1$

Theorem 3.2. If $i \leftrightarrow j$ and state i is recurrent, then

$$f_{i,j} = P(\text{DTMC ever makes future visit to state } j \mid X_0 = i) = 1$$

Proof. If $i = j$, then result follows by definition of recurrence.
Let $i \neq j$. Since $i \leftrightarrow j$, then $\exists n \in \mathbb{Z}^+$ s.t. $P_{j,i}^{(n)} > 0$.
State j is recurrent by theorem 1.2 so $f_{j,j} = 1$.
Assume that $f_{i,j} < 1$ for a contradiction.

Method 1 Note that

$$\begin{aligned} f_{j,j} &= P(\text{DTMC ever makes future visit to j} \mid X_0 = j) \\ &= 1 - P(\text{never visits j} \mid X_0 = j) \\ &\leq 1 - P_{j,i}^{(n)}(1 - f_{i,j}) && P(\text{never visits j} \mid X_0 = j) \geq P_{j,i}^{(n)}(1 - f_{i,j}) \\ &< 1 \end{aligned}$$

which is a contradiction, so $f_{i,j} < 1$.

Method 2 Note that

$$\begin{aligned} &\{X_n = i, \text{never visits j after i}\} \subseteq \{\text{never returns to state j}\} \\ \Rightarrow &P(X_n = i, \text{never visits j after i } \mid X_0 = j) \leq P(\text{never returns to j} \mid X_0 = j) \\ \Rightarrow &P_{j,i}^{(n)}(1 - f_{i,j}) \leq 1 - f_{j,j} \\ \Rightarrow &P_{j,i}^{(n)}(1 - f_{i,j}) \leq 0 \end{aligned}$$

which is a contradiction since $P_{j,i}^{(n)} > 0$ and $1 - f_{i,j} > 0$, so we must have $f_{i,j} = 1$.

□

3.4 Theorem 1.4 proof

Theorem 3.3. If state i is recurrent and state i does not communicate with state j, then $P_{i,j} = 0$.

Proof. Assume $i \neq j$. State i is recurrent so $f_{i,i} = 1$.
Assume that $P_{i,j} > 0$ for a contradiction so $i \to j$. Since i and j don't communicate and $i \to j$, then i is not accessible from j ($j \not\to i$).

Method 1 Note that

$$\begin{aligned} f_{i,i} &= P(\text{DTMC ever makes future visit to i} \mid X_0 = i) \\ &= 1 - P(\text{never visits i} \mid X_0 = i) \\ &\leq 1 - P_{i,j} && P(\text{never visits i} \mid X_0 = i) \geq P_{i,j} \text{ since } j \not\to i \\ &< 1 \end{aligned}$$

which is a contradiction so $P_{i,j} = 0$.

Method 2 Note that

$$\begin{aligned} &\{X_1 = j, \text{never returns to i after j}\} \subseteq \{\text{never returns to state i}\} \\ \Rightarrow &P(X_1 = j, \text{never visits j after i } \mid X_0 = i) \leq P(\text{never return to i} \mid X_0 = i) \\ \Rightarrow &P_{i,j} \leq 1 - f_{i,i} \end{aligned}$$

where the last line follows since i is not accessible from j.

Since $f_{i,i} = 1$, we have $P_{i,j} \leq 0$ which is a contradiction, thus $P_{i,j} = 0$.

□