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STAT 433/833 Course Notes

Stochastic Processes

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Last Revision: September 11, 2018

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. These notes are my interpretation and transcription of the content covered in lectures. The instructor has not verified or confirmed the accuracy of these notes, and any discrepancies, misunderstandings, typos, etc. as these notes relate to course's content is not the responsibility of the instructor. If you spot any errors or would like to contribute, please contact me directly.

1 September 6, 2018

1.1 Example 1.2 solution

Use the definition of the Markov property to show that

$$\begin{aligned}
&P(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_{n-k+1} = x_{n-k+1}, X_{n-k-1} = x_{n-k-1}, \dots, X_0 = x_0) \\
&\qquad = P(X_{n+1} = x_{n+1} \mid X_n = x_n), \quad k = 1, 2, \dots, n
\end{aligned}$$

(i.e. we are missing one past observation).

Solution. Applying the definition of conditional probability, our expression is equivalent to

$$\frac{P(X_{n+1} = x_{n+1}, X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_{n-k+1} = x_{n-k+1}, X_{n-k-1} = x_{n-k-1}, \dots, X_0 = x_0)}{P(X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_{n-k+1} = x_{n-k+1}, X_{n-k-1} = x_{n-k-1}, \dots, X_0 = x_0)} = \frac{N}{D}$$

By the law of total probability

$$\begin{aligned}
N &= \sum_{x_{n-k} \in S} P(X_{n+1} = x_{n+1}, \dots, X_{n-k} = x_{n-k}, \dots, X_0 = x_0) \\
&= \sum_{x_{n-k} \in S} P(X_{n+1} = x_{n+1} \mid X_n = x_n, \dots, X_{n-k} = x_{n-k}, \dots, X_0 = x_0) \times P(X_n = x_n, \dots, X_{n-k} = x_{n-k}, \dots, X_0 = x_0)
\end{aligned}$$

By the Markov property

$$\begin{aligned}
&= P(X_{n+1} = x_{n+1} \mid X_n = x_n) \sum_{x_{n-k} \in S} P(X_n = x_n, \dots, X_{n-k} = x_{n-k}, \dots, X_0 = x_0) \\
&= P(X_{n+1} = x_{n+1} \mid X_n = x_n) P(X_n = x_n, \dots, X_{n-k} \in S, \dots, X_0 = x_0)
\end{aligned}$$

Since $X_{n-k} \in S$ is an event with probability 1

$$\begin{aligned}
&= P(X_{n+1} = x_{n+1} \mid X_n = x_n) P(X_n = x_n, \dots, X_{n-k+1} = x_{n-k+1}, X_{n-k-1} = x_{n-k-1}, \dots, X_0 = x_0) \\
&= P(X_{n+1} = x_{n+1} \mid X_n = x_n) \cdot D
\end{aligned}$$

The result follow.

2 September 11, 2018

2.1 Section 1.2: Transitivity of communication relation

Prove that if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$ (and thus the communication relation "$\leftrightarrow$" is an equivalence relation).

Proof. $\exists n, m \in \mathbb{N}$ such that $P_{i,j}^{(n)} > 0$ and $P_{j,k}^{(m)} > 0$.
Note that

$$P_{i,k}^{(n+m)} = \sum_{l \in S} P_{i,l}^{(n)} P_{l,k}^{(m)} \geq P_{i,j}^{(n)} P_{j,k}^{(m)} > 0$$

Similarly we can show $k \to i$, thus $i \leftrightarrow k$. $\square$

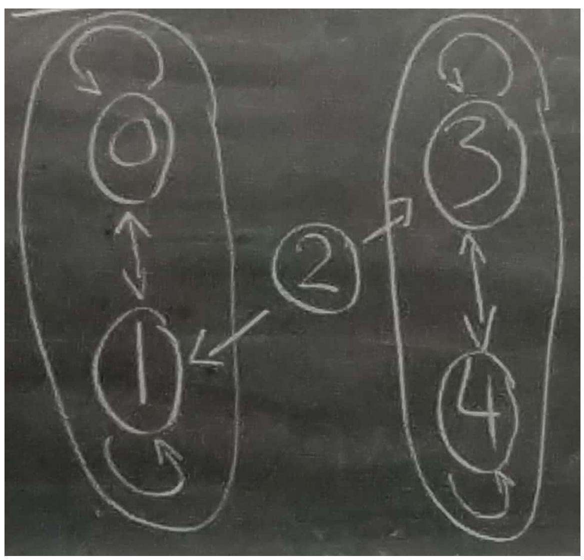
2.2 Example 1.3 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 0.1 & 0.9 \end{bmatrix} \end{array}$$

Use a state transition diagram to determine the equivalence classes.

Solution. We draw the following state transition diagram and note that there are three communication classes: $\{0, 1\}, \{2\}, \{3, 4\}$.



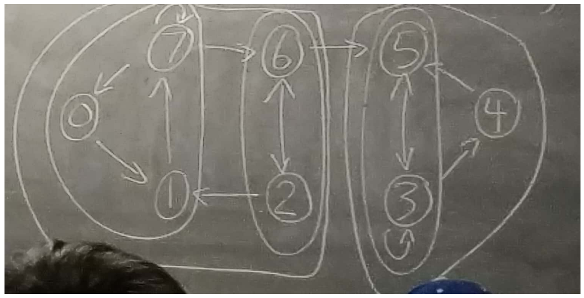
2.3 Example 1.4 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0.3 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

Use a state transition diagram to determine the equivalence classes.

Solution. We draw the following state transition diagram and note that there are two communication classes: $\{0, 1, 2, 6, 7\}, \{3, 4, 5\}$.



2.4 Example 1.5 solution

Given the DTMC with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0.3 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

Use sample paths to prove that all states within the communication classes found in Example 1.4 communicate.

Solution. class $\{3, 4, 5\}$ Note that $P_{3,4}P_{4,5}P_{5,3} > 0$ i.e. the sample path $3 \to 4 \to 5 \to 3$ has positive probability, thus states 3,4, and 5 communicate since for any pair of states $i, j \in \{3, 4, 5\}$, $\exists n_{i,j} \le 3$ such that $P_{i,j}^{(n_{i,j})} > 0$.

class $\{0, 1, 2, 6, 7\}$ We have sample path $0 \to 1 \to 7 \to 6 \to 2 \to 1 \to 7 \to 0$ with positive probability.

By a similar argument as above the five states communicate.

2.5 Theorem 1.1 proof: periodicity is a class property

Theorem 2.1. If $i \leftrightarrow j$ then $d(i) = d(j)$ (equal periods).

Proof. Since $i \leftrightarrow j$, then $\exists n, m \in \mathbb{N}$ such that $P_{i,j}^{(n)} > 0$ and $P_{j,i}^{(m)} > 0$.
$\forall L \in \mathbb{Z}^+$ s.t. $P_{j,j}^{(L)} > 0$, we have

$$\begin{aligned} P_{i,i}^{(m+n+L)} &= \sum_{k \in S} P_{i,k}^{(n)} P_{k,i}^{(m+L)} \\ &= \sum_{k \in S} \sum_{l \in S} P_{i,k}^{(n)} P_{k,l}^{(L)} P_{l,i}^{(m)} \\ &\ge P_{i,j}^{(n)} P_{j,j}^{(L)} P_{j,i}^{(m)} \\ &> 0 \end{aligned}$$

Thus $d(i)$ divides $n + m + L$.
Note that $P_{i,i}^{(n+m)} = \sum_{k \in S} P_{i,k}^{(n)} P_{k,i}^{(m)} \geq P_{i,j}^{(n)} P_{j,i}^{(m)} > 0$, thus $d(i)$ divides $n + m$.
Therefore $d(i)$ divides $(n+m+L) - (n+m) = L$ $\forall L$ s.t. $P_{j,j}^{(L)} > 0$, thus $d(i)$ divides $gcd\{L \in \mathbb{Z}^+ \mid P_{j,j}^{(L)} > 0\} = d(j)$.
Similarly, $d(j)$ divides $d(i)$, thus $d(i) = d(j)$. $\square$