Variance, Standard Deviation of Discrete and Continuous Random Variables

Lesson 5

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DISCRETE

Expected Value

Definition: Suppose X is a discrete random variable that takes values $x_1, x_2, ..., x_n$ with probabilities $p(x_1), p(x_2), \dots , p(x_n)$.

The expected value of X is denoted E(X) and defined by:

$$E(X) = p(x_1) \,.\, x_1 + p(x_2) \,.\, x_2 + \dots + p(x_n) \,.\, x_n$$

Properties of Expected Value

1. If X and Y are random variables on a sample space Ω then:

$$E(X + Y) = E(X) + E(Y)$$

2. If a and b are constants then:

$$E(a X + b) = a E(X) + b$$

Example 1: Roll two dice and let X be the sum. Find E(X).

Properties of Expected Value

Example 2: (For infinite random variables the "mean" (expected value) does not always exist.) Suppose X has an infinite number of values according to the following table.

	x_1	x_2	x_3	...	x_k	
values x:	2	2^2	2^3	...	2^k	...
pmf p(x):	$1/2$	$1/2^2$	$1/2^3$	...	$1/2^k$	...

Expected values of functions

If X is a discrete random variable taking values $x_1, x_2, \ldots$ and h is a function the $h(X)$ is a new random variable. Its expected value is

$$E(h(X)) = \sum(h(x_j) \,.\, p(x_j))$$

Example 3: Let X be the value of a roll of one die and let $Y = X^2$. Find $E(Y)$.

Example 4: Roll two dice and let X be the sum. Suppose the payoff function is given by $Y = X^2 - 6X + 1$.

Variance

Definition: If X is a discrete random variable with mean $E(X) = \mu$, then the variance of X is defined by:

$$\mathrm{Var}(X) = E((X - \mu)^2)$$

And is denoted by σ^2.

“ Is a measure of how much the probability mass is spread out around this center.”

Standard Deviation

Definition: If X is a discrete random variable, then the standard deviation of X is defined by:

$$\sigma = (\mathrm{Var}(X))^{1/2}$$

Example 5: Compute the mean (expected), variance and standard deviation of the random variable X with the following table of values and probabilities.

value x	1	3	5
pmf $p(x)$	¼	¼	½

Example 6: For each random variable X, Y , Z, and W plot the pmf and compute the mean (expected value) and variance.

(i)

value x	1	2	3	4	5
pmf $p(x)$	1/5	1/5	1/5	1/5	1/5

(ii)

value y	1	2	3	4	5
pmf $p(y)$	1/10	2/10	4/10	2/10	1/10

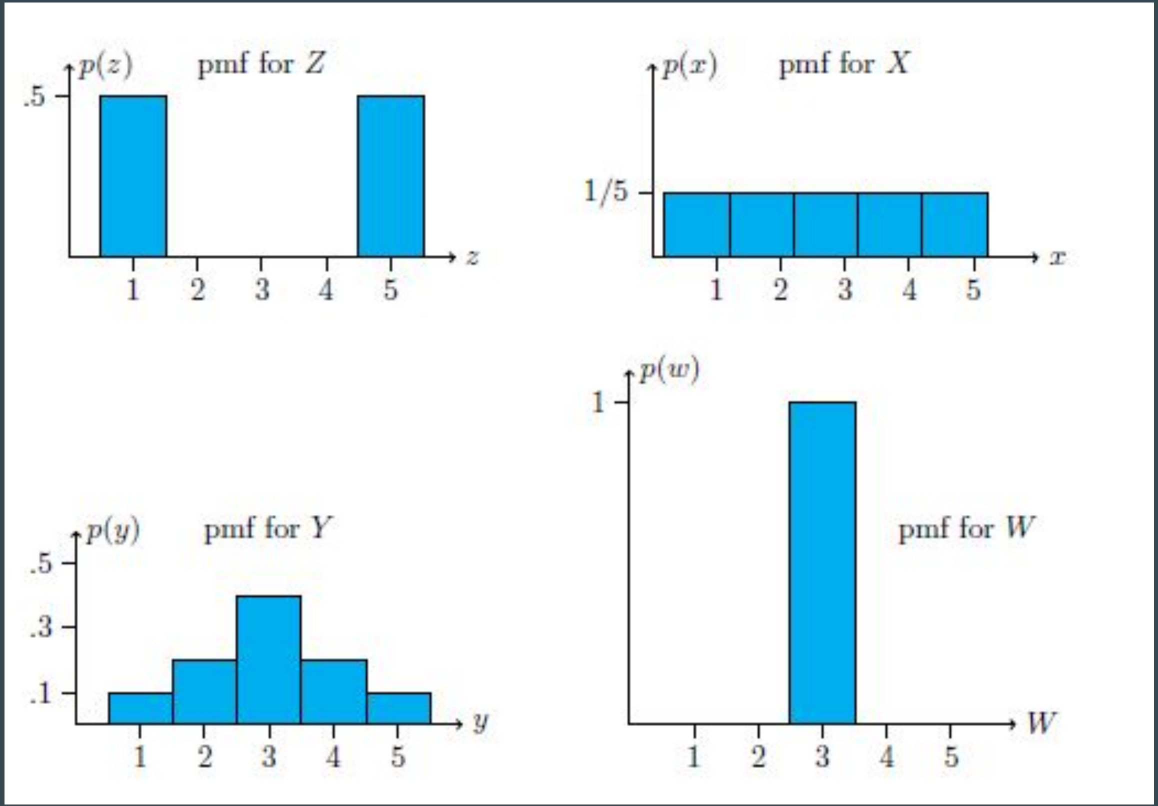
(iii)

value z	1	2	3	4	5
pmf $p(z)$	5/10	0	0	0	5/10

(iv)

value w	1	2	3	4	5
pmf $p(w)$	0	0	1	0	0

- Var(X) = 2
- Var(Y) = 1.2
- Var(Z) = 4
- Var(W) = 0



Properties of Variance

- If X and Y are independent then $Var(X + Y) = Var(X) + Var(Y)$
- For constans a and b, then: $Var(a \,.\, X + b) = a^2\, Var(X)$
- $Var(X) = E(X^2) - E(X)^2$

Example 7: Suppose X and Y are independent and $Var(X) = 3$ and $Var(Y\ ) = 5$. Find:

(i) $Var(X + Y)$

(ii) $Var(3X + 4)$

(iii) $Var(X + X)^*$

(iv) $Var(X + 3Y)$

Summary

Expected Value:		**Variance:**
Synonyms:	mean, average	
Notation:	$E(X)$, μ	$\text{Var}(X)$, σ^2
Definition:	$E(X) = \sum_j p(x_j) x_j$	$E((X-\mu)^2) = \sum_j p(x_j)(x_j - \mu)^2$
Scale and shift:	$E(aX+b) = aE(X) + b$	$\text{Var}(aX+b) = a^2\text{Var}(X)$
Linearity:	(for any X, Y) $E(X+Y) = E(X) + E(Y)$	(for X, Y independent) $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$
Functions of X:	$E(h(X)) = \sum p(x_j)\, h(x_j)$	
Alternative formula:		$\text{Var}(X) = E(X^2) - E(X)^2 = E(X^2) - \mu^2$

CONTINUOUS

Expected Value

Definition: Let X be a continuous random variable with range [a, b] and probability density function f(x). The expected value of X is defined by:

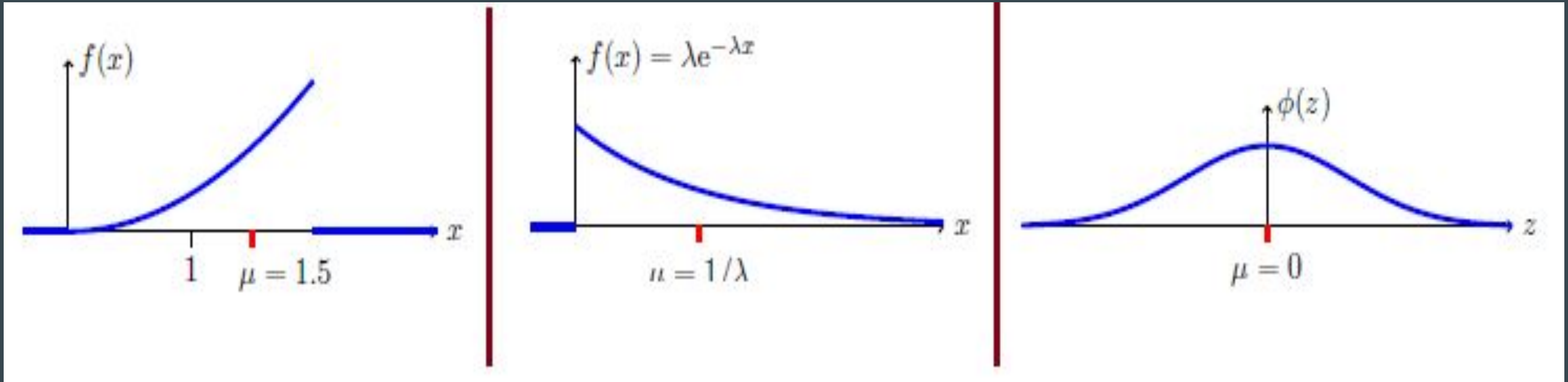
$$E(X) = \int_a^b x\, f(x)\, dx$$

Example 8: Let X have range [0, 2] and density $f(x) = 3x^2/8$. Find E(X).

Example 9: The range of X is [0, ∞) and its pdf is $f(x) = \lambda e^{-\lambda x}$

Example 10: The range of Z is (-∞, ∞) and its pdf is $f(x) = e^{-Z^2/2}/\sqrt{2\pi}$

Expected Value



Example 8

Example 9

Example 10

Properties of Expected Value

The properties of $E(X)$ for continuous random variables are the same as for discrete ones:

1. If X and Y are random variables on a sample space Ω then:

$$E(X + Y) = E(X) + E(Y)$$

2. If a and b are constants then:

$$E(a \, . \, X + b) = a \, . \, E(X) + b$$

Expectation of Functions of X

This works exactly the same as the discrete case. If $h(x)$ is a function then $Y = h(X)$ is a random variable and

$$E(Y) = E(\ h(X)\ ) = \int h(x)\ .\ f(x)\ dx$$

Example 11: Find $E(X^2)$ if the range of X is $[0, \infty)$ and its pdf is $f(x) = \lambda e^{-\lambda x}$

Variance

Definition: Let X be a continuous random variable with mean. The variance of X is:

$$\mathrm{Var}(X) = E((X - \mu)^2)$$

Properties:

- If X and Y are independent then: $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$
- For constants *a* and *b* : $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$
- $\mathrm{Var}(X) = E(X^2) - E(X)^2$

Example 12: Find Var(X). X has range [0, 1] and density f(x) = 1.

Dessert

We roll two standard 6-sided dice. You win \$1000 if the sum is 2 and lose \$100 otherwise. How much do you expect to win on average per trial?

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Statistics
Density
population
Probability
experiment
standard
intersection
box-plot
observation
distribution
Venn-Diagram
probability-tree
Standard
randomness
histogram
bargraph
event
z-score
Median
union
Mode
Normal
independence
Mean
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Sample