University of Toronto Scarborough

Department of Computer & Mathematical Sciences

STAD70H3 Statistics and Finance II
Winter 2023 Midterm Examination

Last Name: ______________________________
First Name: ______________________________
Student #: ______________________________

Instructor: Sotirios Damouras
Duration: 120 minutes
Examination aids allowed: Open notes/books.

Question:	1	2	3	4	5	6	Total
Points:	20	20	10	10	30	10	100
Score:							

THEORY

1. (Portfolio Theory & Factor Models)
Consider 5 assets with net returns $(R_1, \ldots, R_5)$ following a Normal distribution with:

$$\mathbb{E}[R_i] = 0.05, \quad \mathbb{V}[R_i] = 0.04, \quad \text{Corr}[R_i, R_j] = \rho \in (-1, 1), \quad \forall\, i = 1, \ldots, 5,\ j \neq i.$$

   (a) [7 points] Calculate the *variance* of the minimum-variance portfolio for $\rho = 0.2$ and $\rho = -0.2$. Which of the two values of the correlation coefficient ρ would you prefer as an investor? (Feel free to use `R` as your calculator.)

   (b) [8 points] Now fix $\rho = -0.2$. Can the distribution of the assets be represented by a statistical factor model *with only one factor*? Justify your answer: if yes, provide such a model; if no, show why.

   (c) [5 points] Can *every* d-dimensional Normal distribution be represented by a statistical factor model with $\#d$ factors? Justify you answer.

Solution:

(a) By the symmetry of the assets (all share the same 1st and 2nd order moments), the minimum-variance weights are equal, i.e., $\boldsymbol{w}_{mv} = \frac{1}{5}\mathbf{1}$. The variance of the minimum-variance portfolio is given by

$$\begin{aligned}
\sigma_{mv} &= \boldsymbol{w}_{mv}^\top \Sigma_R \boldsymbol{w}_{mv} \\
&= \left(\frac{1}{5}\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}\right)\left(0.04\begin{bmatrix}1 & \rho & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho & \rho \\ \rho & \rho & 1 & \rho & \rho \\ \rho & \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & \rho & 1\end{bmatrix}\right)\left(\frac{1}{5}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right) \\
&= \frac{0.04}{25}(5 + 20\rho) = \frac{0.04}{25} + \begin{cases}5 + 4 = 9, & \rho = 0.2 \\ 5 - 4 = 1, & \rho = -0.2\end{cases} \\
&= \begin{cases}0.0144, & \rho = 0.2 \\ 0.0016, & \rho = -0.2\end{cases}
\end{aligned}$$

The negative correlation results in smaller variance, so it is preferable.
(Unfortunately, in reality assets are typically positively correlated.)

(b) NO, you cannot represent the variance-covariance matrix with this negative correlation.
The statistical factor model with 1 factor is:

$$\boldsymbol{R} = \boldsymbol{\mu} + \boldsymbol{\beta}^\top F + \boldsymbol{\epsilon} \Leftrightarrow \begin{bmatrix}R_1\\R_2\\R_3\\R_4\\R_5\end{bmatrix} = \begin{bmatrix}\mu_1\\\mu_2\\\mu_3\\\mu_4\\\mu_5\end{bmatrix} + \begin{bmatrix}\beta_1\\\beta_2\\\beta_3\\\beta_4\\\beta_5\end{bmatrix} F + \begin{bmatrix}\epsilon_1\\\epsilon_2\\\epsilon_3\\\epsilon_4\\\beta_5\end{bmatrix}$$

Because of the asset symmetry, the μ's, and error variances (σ_ϵ) are all equal, and the *beta*'s must have the same magnitude ($|\beta_i| = \beta$). We have, $\mu_1 = 0.05$, and the variance-covariance matrix satisfies:

$$\boldsymbol{\Sigma}_R = \boldsymbol{\beta}^\top \boldsymbol{\beta} + \Sigma_\epsilon$$

$$0.04 \begin{bmatrix} 1 & -0.2 & -0.2 & -0.2 & -0.2 \\ -0.2 & 1 & -0.2 & -0.2 & -0.2 \\ -0.2 & -0.2 & 1 & -0.2 & -0.2 \\ -0.2 & -0.2 & -0.2 & 1 & -0.2 \\ -0.2 & -0.2 & -0.2 & -0.2 & 1 \end{bmatrix} = \begin{bmatrix} \beta_1^2 & \beta_1\beta_2 & \beta_1\beta_3 & \beta_1\beta_4 & \beta_1\beta_5 \\ \beta_2\beta_1 & \beta_2^2 & \beta_2\beta_3 & \beta_2\beta_4 & \beta_2\beta_5 \\ \beta_3\beta_1 & \beta_3\beta_2 & \beta_3^2 & \beta_3\beta_4 & \beta_3\beta_5 \\ \beta_4\beta_1 & \beta_4\beta_2 & \beta_4\beta_3 & \beta_4^2 & \beta_4\beta_5 \\ \beta_5\beta_1 & \beta_5\beta_2 & \beta_5\beta_3 & \beta_5\beta_4 & \beta_5^2 \end{bmatrix} + $$

$$+ \sigma_\epsilon \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solving these equations, we get:

$$\begin{aligned} \sigma_\epsilon &= 0.8 \\ |\beta_i| &= \sqrt{0.2} \\ \beta_1\beta_2 &= -0.2 \Rightarrow \beta_2 = -\beta_1 \\ \beta_1\beta_3 &= -0.2 \Rightarrow \beta_3 = -\beta_1 \\ \beta_2\beta_3 &= -0.2 \Rightarrow \beta_2 = -\beta_3 \quad \underline{\text{impossible!}} \\ &\vdots \end{aligned}$$

Obviously, there is NO way to satisfy this system.
Intuitively, if R_1 negatively correlated to R_2, and R_2 negatively correlated to R_3 in a single factor model, then you need R_1 to be positively correlated to R_3. This is not a problem for positively correlated variables in a single factor model (e.g., CAPM).

(c) YES. Consider the eigen-decomposition of the variance-covariance matrix of returns:

$$\begin{aligned} \boldsymbol{\Sigma}_R &= \boldsymbol{P}^\top \boldsymbol{\Lambda} \boldsymbol{R} = \boldsymbol{P}^\top \boldsymbol{\Lambda}^{1/2} \boldsymbol{\Lambda}^{1/2} \boldsymbol{P} \\ &= \overbrace{\left(\boldsymbol{P}\boldsymbol{\Lambda}^{1/2}\right)^\top}^{=\boldsymbol{\beta}^\top} \overbrace{\left(\boldsymbol{P}\boldsymbol{\Lambda}^{1/2}\right)}^{=\boldsymbol{\beta}} \end{aligned}$$

where we used the fact that the eigen-matrix is diagonal with positive elements (hence we can take its square root). We can represent *any* multivariate Normal return distribution $\boldsymbol{R} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}_R)$ as

$$\boldsymbol{R} = \boldsymbol{\mu} + \boldsymbol{\beta}^\top \boldsymbol{F} + \cancelto{\mathbf{0}}{\boldsymbol{\epsilon}} = \boldsymbol{\mu} + \left(\boldsymbol{P}\boldsymbol{\Lambda}^{1/2}\right)^\top \boldsymbol{F}$$

where $\boldsymbol{F} \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{I})$ is a standard multivariate Normal of the same dimension (d) as $\boldsymbol{R}$.

2. (Risk Management)
Consider two independent Uniform(0,1) loss RVs: $L_1, L_2 \sim^{iid} \text{Uniform}(0,1)$.

   (a) [5 points] Find the Value-at-Risk (VaR) of L_1 at confidence level $1-\alpha$, expressed as a function of α.

   (b) [5 points] Find the VaR of the *sum* $L_1 + L_2$ of the two losses at confidence level $1-\alpha$, expressed as a function of α.
   (Hint: the PDF of the sum of two Uniform(0,1) RV's is *triangular*.)

   (c) [5 points] Show that VaR is *not sub-additive* in this case for $\alpha > 1/2$.

   (d) [5 points] Find a close-form expression for the Expected Shortfall (CVaR) of L_1 at confidence level $1-\alpha$, expressed as a function of α.

Solution:

(a) For a uniform Normal, the $1-\alpha$ quantile is $\text{VaR}_\alpha = 1-\alpha$.

(b) The distribution of the sum of two uniforms is:

$$f(u) = \begin{cases} u, & 0 < u < 1 \\ 2-u, & 1 \le u < 2 \end{cases}$$

The VaR is given by the distribution's quantiles.
For $\alpha \le 1/2$, the top α quantile is given by setting the area of the right tail (which is a square triangle) equal to α:

$$\begin{aligned} \alpha &= (2-\text{VaR}_\alpha) \times (2-\text{VaR}_\alpha)/2 \\ \Rightarrow (2-\text{VaR}_\alpha)^2 &= 2\alpha \\ \Rightarrow \text{VaR}_\alpha &= 2-\sqrt{2\alpha} \end{aligned}$$

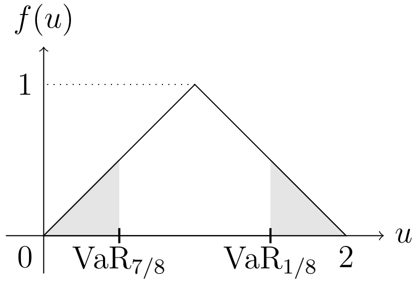
For $\alpha > 1/2$, the top α quantile is given by setting the area of the left tail (which is a square triangle) equal to $1-\alpha$:

$$\begin{aligned} 1-\alpha &= \text{VaR}_\alpha \times \text{VaR}_\alpha /2 \\ \Rightarrow \text{VaR}_\alpha &= \sqrt{2-2\alpha} \end{aligned}$$

Overall, we have

$$\text{VaR}_\alpha(L_1+L_2) = \begin{cases} 2-\sqrt{2\alpha}, & \alpha \le 1/2 \\ \sqrt{2-2\alpha}, & \alpha > 1/2 \end{cases}$$

The plot below shows the distribution and two sample values of VaR for $\alpha < 1/2$ and $\alpha > 1/2$:



(c) For VaR to be sub-additive, we need:

$$VaR_\alpha(L_1 + L_2) \leq VaR_\alpha(L_1) + VaR_\alpha(L_2)$$

But for $\alpha > 1/2$, we have:

$$\begin{aligned} VaR_\alpha(L_1 + L_2) &= \sqrt{2 - 2\alpha} \quad \text{(where } 0 < 2 - 2\alpha < 1\text{)} \\ &> 2 - 2\alpha = (1 - \alpha) + (1 - \alpha) \\ &> VaR_\alpha(L_1) + VaR_\alpha(L_2) \end{aligned}$$

Which contradicts sub-additivity.

(d) The conditional distribution that the uniform loss is greater than VaR_α is also Uniform in $(\text{VaR}_\alpha, 1) = (1-\alpha, 1)$. The mean of this is $\text{CVaR}_\alpha = (1-\alpha+1)/2 = 1 - \alpha/2, \ \forall \alpha \in (0, 1)$.

3. (Heavy Tails)
Consider two random variables X and Y. An alternative definition of heavy tailed distributions is those whose Moment Generating Function (MGF) is *infinite*. For the next two parts, assume that X has heavy tails, i.e., an infinite MGF.

   (a) [5 points] If X and Y are *independent*, show that the *sum* $S = X + Y$ will also have heavy tails (i.e., infinite MGF), irrespective of the behavior of Y. This shows that sums of independent heavy tail RVs are heavy tail RVs.

   (b) [5 points] Does the previous result necessarily hold for *dependent* RV's? Justify your answer (prove or disprove by counter example).

Solution:

(a) The MGF of independent RVs is given by the product of their MGFs:

$$M_{X+Y}(z) = \mathbb{E}[e^{z(X+Y)}] = \mathbb{E}[e^{zX}e^{zY}] \stackrel{\text{by indep.}}{=} \mathbb{E}[e^{zX}]\,\mathbb{E}[e^{zY}] = M_X(z)M_Y(z)$$

Since MGFs are positive, if $M_X(z)$ is infinite, then $M_X(z)M_Y(z)$ will also be infinite, irrespective of $M_Y(z)$. Hence, the sum of the RVs will also have heavy tails.

(b) No. If the RVs can be dependent, consider the extreme case of perfect negative dependence $Y = -X$. In this case, $S = X + Y = X - X = 0$, which is a constant RV with finite MGF.

PRACTICE

Answer the following questions in an .RMarkdown file, compile it to an .html file, and submit both to Quercus.

4. [10 points] (`R`; Risk Management)
Consider a Uniform(0,1) loss RV L again. Find the Entropic Value-at-Risk at level $\alpha = .05$; note that you will have to perform a univariate numerical optimization in `R`.
(Hint: the MGF of the Uniform(a,b) distribution is $M_U(z) = \mathbb{E}[e^{zU}] = \frac{e^{bz}-e^{az}}{z(b-a)}, \quad z > 0$.)

5. (`R`; Modeling Returns, Portfolio Theory)

   (a) [5 points] Download the daily adjusted closing prices from 2015-01-01 to 2020-12-31 of the S&P500 index (ticker `^GSPC`) and its top 10 companies w.r.t. market cap: `AAPL, MSFT, AMZN, NVDA, GOOGL, BRK-B, GOOG, TSLA, XOM, UNH`. Calculate the daily *net* returns of all assets.

   (b) [5 points] If you had invested $1,000 in Apple stock (`AAPL`) on 2016-10-31, how much would your investment be worth on 2018-11-08 (in $ terms)?

   (c) [5 points] Calculate the net returns of an equally weighted portfolio of `AAPL` and `XOM`. Then, fit a univariate t-distribution to the portfolio's returns using maximum likelihood and report the degrees of freedom (d.f.).
   (Hint: Use `MASS::fitdistr()`.)

   (d) [5 points] Fit a *2-dimensional* t-distribution to the returns of `AAPL` and `XOM` using profile likelihood. Based on this model, what would be the distribution of an equally weighted portfolio of the two stocks? (Specify its mean, variance, and d.f.) Are the d.f. close to those of the previous part?

   (e) [5 points] Consider all 10 stocks and plot the efficient frontier for portfolios with expected daily returns ranging from -0.05% to 0.1%.

   (f) [5 points] Consider ExxonMobil (`XOM`) alone, and calculate and report its Sharpe ratio, Treynor ratio, and Jensen's alpha; for the latter two, use the S&P500 index (`^GSPC`) as a market proxy, and a risk-free rate of 0.

6. (`R`; Simulation & Copulas)

   (a) [5 points] Specify a bivariate Clayton copula with parameter $\theta = 2$, and use it to generate n=1,000 random values; create a scatterplot of the simulated values.

   (b) [5 points] Transform the sample from the previous part so that its marginals have the following location-scale t-distributions: one dimension follows $t_{df=2}(\mu = 0.03, \sigma = 0.20)$ and the other follows $t_{df=4}(\mu = 0.05, \sigma = 0.30)$. Create a scatterplot of transformed values.

Extra Space (use if needed and clearly indicate which questions you are answering)

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