UNIVERSITY OF TORONTO SCARBOROUGH
Department of Computer & Mathematical Sciences

STAD70H3 Statistics & Finance II

April 2019 Final Examination

Duration: 3 hours
Instructor: Sotirios Damouras
Aids allowed: Open book/notes, scientific calculator

Last Name: ______________________________
First Name: ______________________________
Student #: ______________________________

Instructions:

- Read the questions carefully and answer only what is being asked.
- Answer all questions directly on the examination paper; use the last pages if you need more space, and provide clear pointers to your work.
- Show your intermediate work, and write clearly and legibly.

Question:	1	2	3	4	5	6	Total
Points:	15	10	25	20	20	20	110
Score:							

1. [15 points] Assume the return on an asset follows a logistic distribution with CDF

$$F_X(x) = \frac{1}{1 + \exp\left(-\frac{x-\mu}{\lambda}\right)}, \quad \forall x \in \mathbb{R},\ \lambda > 0,$$

where μ and λ are the location and scale parameters, respectively. Find a closed-form expression for $\mathrm{VaR}(\alpha)$ and $\mathrm{CVaR}(\alpha)$.
(Hint: $\int \log(x)dx = x\log(x) - x + c$)

2. [10 points] Assume the returns on all assets in a market follow multivariate Normal distribution with mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$. Furthermore, assume investors pick portfolios with minimum VaR(α) for some α. Show that the only portfolios that investors would consider are the ones lying on the *efficient frontier* of mean-variance analysis.
(Hint: show that minimizing VaR for a *Normal* distribution is equivalent to minimizing the portfolio variance for a given mean return level.)

3. Consider a standard Brownian motion $\{W_t\}$ with $W_0 = 0$.

   (a) [15 points] Let $M_1 = \max\{W_t : t \in [0, 1]\}$ be the maximum of the process by time 1. Find the conditional expectation of M_1 given $(M_1 > 1)$, i.e $\mathbb{E}[M_1 | M_1 > 1]$, in terms of the standard Normal CDF $\Phi(z)$.

   (b) [10 points] Condition on the event $(W_1 = 1)$, and find the conditional (bivariate) distribution of $\begin{bmatrix} W_s \\ W_t \end{bmatrix} |(W_1 = 1)$, where $0 < s < t < 1$.

   (Hint: for a multivariate Normal $\left[\begin{array}{c} \boldsymbol{X}_1 \\ \hline \boldsymbol{X}_2 \end{array}\right] \sim N\left(\left[\begin{array}{c} \boldsymbol{\mu}_1 \\ \hline \boldsymbol{\mu}_2 \end{array}\right], \left[\begin{array}{c|c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \hline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{array}\right]\right)$, we have $\boldsymbol{X}_1 | (\boldsymbol{X}_2 = \boldsymbol{x}_2) \sim N\left(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2),\ \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right)$)

4. Consider two assets whose revenues are independently and uniformly distributed as $R_1, R_2 \sim^{iid} Unif(-1, 1)$.

   (a) [10 points] Find a closed-form expression for the individual asset value at risk, i.e. find $\mathrm{VaR}_\alpha(R_1)$ as a function of α.

   (b) [10 points] Show that VaR is *not* subadditive in this case, i.e. show that $\mathrm{VaR}_\alpha(R_1 + R_2) \nleq \mathrm{VaR}_\alpha(R_1) + \mathrm{VaR}_\alpha(R_2)$ for some α.
   (Hint: try $\alpha = 7/8$)

5. Consider two sets of assets whose return vectors $\boldsymbol{R}_1, \boldsymbol{R}_2$ are *independent* and have variance-covariance matrices $\mathbb{V}[\boldsymbol{R}_1] = \boldsymbol{\Sigma}_1$, $\mathbb{V}[\boldsymbol{R}_2] = \boldsymbol{\Sigma}_2$. Let the minimum-variance weights for a portfolio consisting only of assets in $\boldsymbol{R}_1$ be $\boldsymbol{w}_1$, and let the achieved minimum variance be σ_1^2. Similarly for the assets in $\boldsymbol{R}_2$, let the minimum-variance portfolio weights be $\boldsymbol{w}_2$ and the minimum variance be σ_2^2. Now turn attention to the minimum variance portfolio for *both* sets of assets $\left[\begin{array}{c}\boldsymbol{R}_1 \\ \hline \boldsymbol{R}_2\end{array}\right]$.

   (a) [10 points] Show that the combined minimum-variance portfolio weights are given by $\boldsymbol{w} = \frac{1}{\sigma_1^2+\sigma_2^2}\left[\begin{array}{c}\sigma_2^2\boldsymbol{w}_1 \\ \hline \sigma_1^2\boldsymbol{w}_2\end{array}\right]$.

   (b) [10 points] Show that the attained minimum variance is given by $\sigma^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}$.

6. Consider a standard Normal random variable $Z \sim N(0,1)$ and let $Y = e^Z$. Assume you want to estimate $\mathbb{E}[Y]$ using simulation, i.e. using $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i = \frac{1}{n}\sum_{i=1}^n e^{Z_i}$, where $Z_i \sim^{iid} N(0,1)$.

   (a) [7 points] Find the variance of your estimate $\mathbb{V}[\bar{Y}]$, as a function of n only.
   (Hint: the moment generating function of $Z \sim N(0,1)$ is $m_Z(t) = \mathbb{E}[e^{tZ}] = e^{t^2/2}$.)

   (b) [13 points] Assume you use $X = Z$ as a *control variable* for your simulation. Find the relative reduction in estimation accuracy, i.e. find the value of $\mathbb{V}[\bar{Y}_{ctrl}]/\,\mathbb{V}[\bar{Y}]$.
   (Hint: for $Z \sim N(0,1)$, we have $\mathbb{E}[Ze^Z] = \sqrt{e}$.)

Extra Space (use if needed and clearly indicate which questions you are answering)

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