STAD70 W23 Name (f,l): _____
Quiz 1 Student #: _____

1. (6 points) Assume that an asset's limit order book at depth-5 looks as follows:

orders	5 best bids					5 best asks					
price	$\cdots$	5.50	5.55	5.56	5.58	5.60	5.65	5.66	5.68	5.69	5.71	$\cdots$
volume	$\cdots$	150	90	80	170	30	20	80	135	25	90	$\cdots$

Furthermore, assume that a *market* order arrives for *selling* 100 shares. What is the *average price* (per share) received by the seller of this market order?

Solution: The market order will be fulfilled by the best available bids until the required volume is achieved. This means that the seller will sell 30 shares at \$5.60 and 70 shares at \$5.58, which results in an average share price of:

$$\frac{5.6 \times 30 + 5.58 \times 70}{100} = 5.586$$

2. Let the random variable (RV) $X \sim \text{Exp}(1)$ follow exponential distribution with CDF $F_X(x) = 1 - e^{-x},\ x > 0$. Define the transformed RV $Y = 1/X$, known as an *inverse exponential*.

   (a) (6 points) Show that the CDF of Y is $F_Y(y) = e^{-1/y},\ y > 0$.
   (Hint: Express the CDF of Y in terms of the CDF of X.)

   (b) (8 points) Find the *tail index* α of the inverse exponential distribution.
   (Hint: Find the PDF of Y using its CDF in the previous part, and look at its asymptotic behavior as $y \to \infty$.)

Solution:

(a)

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(1/X \le y) \qquad (\text{where } y > 0) \\ &= P(X \ge 1/y) = 1 - P(X \le 1/y) = 1 - F_X(1/y) \\ &= 1 - (1 - e^{-1/y}) = e^{-1/y} \end{aligned}$$

(b)

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(e^{-1/y}\right) \\ &= e^{-1/y} \frac{d}{dy}(-1/y) = \frac{e^{-1/y}}{y^2} \end{aligned}$$

$$\Rightarrow f_Y(y) \sim \frac{1}{y^2},\ \text{as } y \to \infty \Rightarrow \text{ the tail index is } \alpha = 1$$