UNIVERSITY OF TORONTO SCARBOROUGH
Department of Computer & Mathematical Sciences

STAD70H3 Statistics & Finance II

April 2019 Final Examination

Duration: 3 hours
Instructor: Sotirios Damouras
Aids allowed: Open book/notes, scientific calculator

Last Name: Solutions
First Name: ____________
Student #: ____________

Instructions:

- Read the questions carefully and answer only what is being asked.
- Answer all questions directly on the examination paper; use the last pages if you need more space, and provide clear pointers to your work.
- Show your intermediate work, and write clearly and legibly.

Question:	1	2	3	4	5	6	Total
Points:	15	10	25	20	20	20	110
Score:							

1. [15 points] Assume the return on an asset follows a logistic distribution with CDF

$$F_X(x) = \frac{1}{1+\exp\left(-\frac{x-\mu}{\lambda}\right)}, \quad \forall x \in \mathbb{R},\ \lambda > 0,$$

where μ and λ are the location and scale parameters, respectively. Find a closed-form expression for $\mathrm{VaR}(\alpha)$ and $\mathrm{CVaR}(\alpha)$.
(Hint: $\int \log(x)dx = x\log(x) - x + c$)

For continuous distributions, the quantile function is given by the inverse CDF:

We want $P(R \leq -\mathrm{VaR}(\alpha)) = \alpha \Rightarrow$

$\Rightarrow F(-\mathrm{VaR}(\alpha)) = \alpha \Rightarrow \mathrm{VaR}(\alpha) = -F^{-1}(\alpha)$

$$\begin{cases} F(x) = \dfrac{1}{1+e^{-\frac{x-\mu}{\lambda}}} = \alpha \Rightarrow e^{-\frac{x-\mu}{\lambda}} = \dfrac{1}{\alpha} - 1 = \dfrac{1-\alpha}{\alpha} \Rightarrow \\ \Rightarrow \dfrac{x-\mu}{\lambda} = -\log\left(\dfrac{1-\alpha}{\alpha}\right) \Rightarrow x = \mu - \lambda\log\left(\dfrac{1-\alpha}{\alpha}\right) \end{cases}$$

$\Rightarrow \mathrm{VaR}(\alpha) = -\mu - \lambda\log\left(\frac{\alpha}{1-\alpha}\right)$

$= \log(u) - \log(1-u)$

$$\mathrm{CVaR}(\alpha) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}(a) = \frac{1}{\alpha}\cdot\int_0^\alpha -\mu - \lambda\log\left(\frac{u}{1-u}\right)du$$

$$= -\mu - \frac{\lambda}{\alpha}\left\{\int_0^\alpha \log(u)\,du - \int_0^\alpha \log(1-u)\,du\right\} =$$

$$= -\mu - \frac{\lambda}{\alpha}\left\{(\alpha\log(\alpha) - \cancel{\alpha}) - 0 - [(\alpha-1)\log(1-\alpha) - \cancel{\alpha}] - 0\right)$$

$$= -\mu - \lambda\cdot\left(\log(\alpha) - \frac{\alpha-1}{\alpha}\log(1-\alpha)\right)$$

(contd...)

2. [10 points] Assume the returns on all assets in a market follow multivariate Normal distribution with mean $\boldsymbol{\mu}$ and variance-covariance matrix Σ. Furthermore, assume investors pick portfolios with minimum VaR(α) for some α. Show that the only portfolios that investors would consider are the ones lying on the *efficient frontier* of mean-variance analysis.
(Hint: show that minimizing VaR for a *Normal* distribution is equivalent to minimizing the portfolio variance for a given mean return level.)

We know that the portfolio returns follow

$R_{port} \sim N(\mu_{port}, \sigma^2_{port})$, where: $\begin{cases} \mu_{port} = \underline{w}^T \mu \\ \sigma^2_{port} = \underline{w}^T \Sigma \underline{w} \\ \text{for weights } \underline{w} \end{cases}$

$\Rightarrow$ $VaR(\alpha) = -\{\mu_{port} + \sigma_{port}\Phi^{-1}(\alpha)\}$

$\Rightarrow$ For given $(\mu_{port}, \alpha < \frac{1}{2})$, the minimum $VaR(\alpha)$ is given when $-\sigma_{port}\Phi^{-1}(\alpha)$ is minimized when

$\Rightarrow$ (if $\alpha < \frac{1}{2} \Rightarrow \Phi^{-1}(\alpha) < 0 \Rightarrow$) σ_{port} is minimized!

$\Rightarrow$ for given μ_{port}, $\min\{\sigma_{port}\}$, which is equivalent to mean-variance analysis

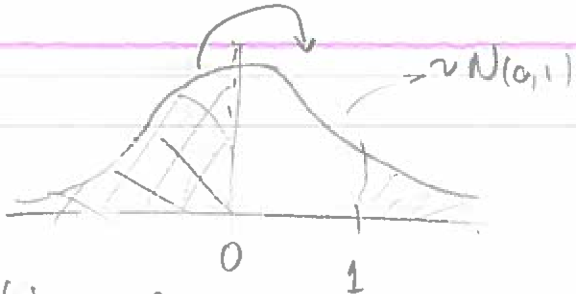
3. Consider a standard Brownian motion $\{W_t\}$ with $W_0 = 0$.

(a) [15 points] Let $M_1 = \max\{W_t : t \in [0,1]\}$ be the maximum of the process by time 1. Find the conditional expectation of M_1 given $(M_1 > 1)$, i.e $\mathbb{E}[M_1|M_1 > 1]$, in terms of the standard Normal CDF $\Phi(z)$.

(b) [10 points] Condition on the event $(W_1 = 1)$, and find the conditional (bivariate) distribution of $\begin{bmatrix} W_s \\ W_t \end{bmatrix} |(W_1 = 1)$, where $0 < s < t < 1$.

(Hint: for a multivariate Normal $\begin{bmatrix} \boldsymbol{X}_1 \\ \hline \boldsymbol{X}_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \hline \boldsymbol{\mu}_2 \end{bmatrix}, \left[\begin{array}{c|c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \hline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{array}\right]\right)$, we have $\boldsymbol{X}_1|(\boldsymbol{X}_2 = \boldsymbol{x}_2) \sim N\left(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2),\ \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right)$)

(a) We know that $M_t \sim |W_t| \Rightarrow M_1 \sim |W_1|$, where $W_1 \sim N(0,1)$. So, essentially, we want to find the conditional expectation of the tail of a "flipped" std Normal (this is similar to the CVaR(α)/ES calculation for Normal)



We have: $\mathbb{E}[M_1|M_1>1] = \int_1^\infty x \cdot \underbrace{f_{M_1|M_1>1}}_{\text{conditional PDF}}(x)\,dx =$

$$= \int_1^\infty x \cdot \frac{\overbrace{f_{M_1}(x)}^{\text{twice } N(0,1)}}{\mathbb{P}(M_1>1)}\,dx =$$

$$= \int_1^\infty x \cdot \frac{\cancel{2}\cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}{\cancel{2}\cdot \Phi(-1)}\,dx = \frac{1}{\Phi(-1)} \cdot \int_1^\infty \frac{1}{\sqrt{2\pi}}\left(-e^{-x^2/2}\right)'\,dx =$$

$$= \frac{1}{\Phi(-1)} \cdot \left[\underbrace{\frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}}_{=\varphi(x)\ \sim \text{ std Normal PDF}}\right]_\infty^1 = \frac{\varphi(1)}{\Phi(-1)}$$

(contd...)

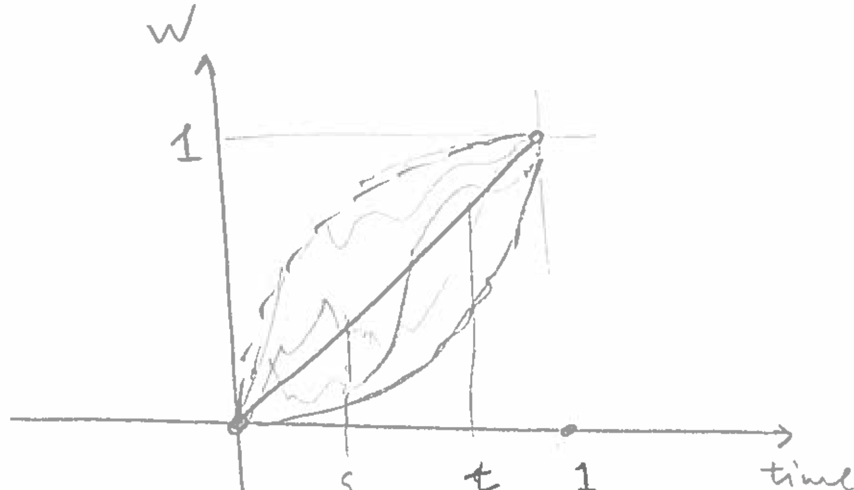
(b) $\begin{bmatrix} W_s \\ W_t \\ W_1 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} s & s & s \\ s & t & t \\ s & t & 1 \end{bmatrix} \right) \Rightarrow$

$$\Rightarrow \begin{bmatrix} W_s \\ W_t \end{bmatrix} \Big| (W_1 = 1) \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ t \end{bmatrix} 1^{-1} \cdot (1-0), \begin{bmatrix} s & s \\ s & t \end{bmatrix} - \begin{bmatrix} s \\ t \end{bmatrix} 1^{-1} [s \;\; t] \right)$$

$$\Rightarrow \sim N\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{bmatrix} s-s^2 & s-st \\ s-st & t-t^2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} s(1-s) & s(1-t) \\ s\cdot(1-t) & t\cdot(1-t) \end{bmatrix}$$

0e



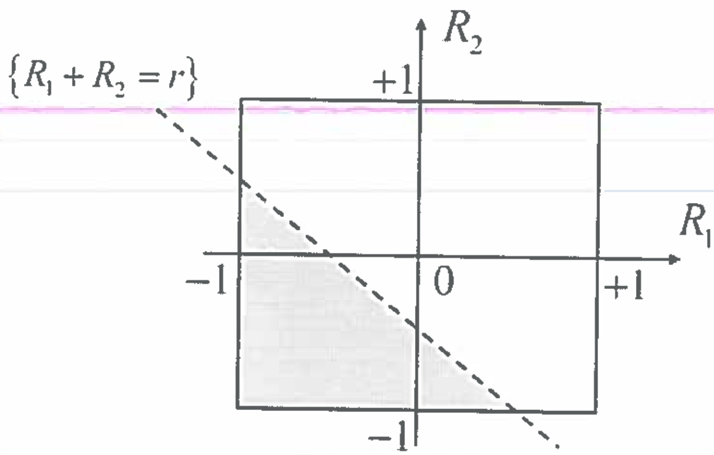
(contd...)

4. Consider two assets whose revenues are independently and uniformly distributed as $R_1, R_2 \sim^{iid} Unif(-1,1)$.

(a) [10 points] Find a closed-form expression for the individual asset value at risk, i.e. find $\mathrm{VaR}_\alpha(R_1)$ as a function of α.

(b) [10 points] Show that VaR is *not* subadditive in this case, i.e. show that $\mathrm{VaR}_\alpha(R_1+R_2) \not\leq \mathrm{VaR}_\alpha(R_1) + \mathrm{VaR}_\alpha(R_2)$ for some α.
(Hint: try $\alpha = 7/8$)

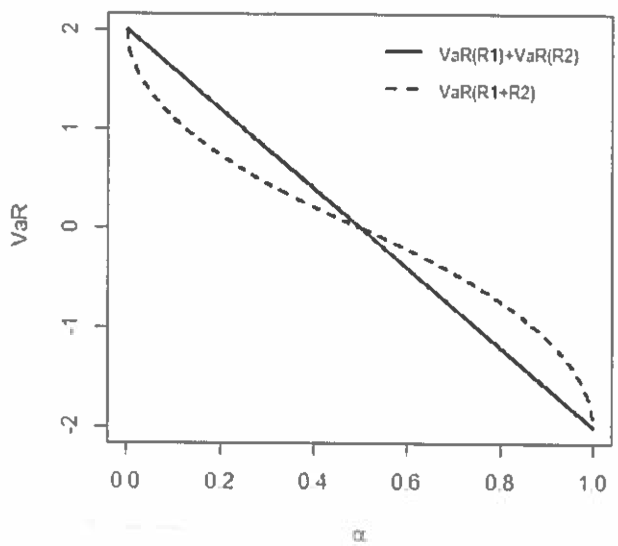
The CDF of R_i is $F_{R_i}(r) = \mathbb{P}(R_i \leq r) = \frac{r+1}{2}, r \in [-1,1], \forall i = 1,2$. Thus, $\mathrm{VaR}_\alpha(R_i) =$
$= -F_{R_i}^{-1}(\alpha) = -(2\alpha - 1) = 1 - 2\alpha, \forall i = 1,2$, and $\mathrm{VaR}_\alpha(R_1) + \mathrm{VaR}_\alpha(R_2) = 2(1-2\alpha)$.
The CDF of $R_1 + R_2$ is $F_{R_1+R_2}(r) = \mathbb{P}(R_1 + R_2 \leq r)$, $r \in [-2,2]$, where the probability is represented by the shaded region in the plot below:



It's straightforward to show that $\mathbb{P}(R_1 + R_2 \leq r) = \begin{cases} \frac{(2+r)^2}{8}, & r \leq 0 \\ 1 - \frac{(2-r)^2}{8}, & r > 0 \end{cases} \Rightarrow$

$$\Rightarrow \mathrm{VaR}_\alpha(R_1 + R_2) = \begin{cases} -2\left(\sqrt{2\alpha} - 1\right), & \alpha \leq \frac{1}{2} \\ -2\left(1 - \sqrt{2(1-\alpha)}\right), & \alpha > \frac{1}{2} \end{cases}.$$

As the following plot shows, the VaR is not subadditive in this case, since $\forall \alpha > \frac{1}{2}$ we have $\mathrm{VaR}_\alpha(R_1 + R_2) \not\leq \mathrm{VaR}_\alpha(R_1) + \mathrm{VaR}_\alpha(R_2)$.
(e.g. for $\alpha = \frac{7}{8}$ we have $\Rightarrow \mathrm{VaR}_\alpha(R_1 + R_2) = -2\left(1 - \sqrt{2(1-\frac{7}{8})}\right) = -2\left(1 - \sqrt{\frac{1}{4}}\right) = -1$, but
$\mathrm{VaR}_\alpha(R_1) + \mathrm{VaR}_\alpha(R_2) = 2(1 - 2\frac{7}{8}) = 2(1 - \frac{7}{4}) = 2(-\frac{3}{4}) = -\frac{3}{2} \leq \mathrm{VaR}_\alpha(R_1 + R_2)$)


VaR(R1)+VaR(R2)
VaR(R1+R2)
VaR
2
1
0
-1
-2
0.0
0.2
0.4
0.6
0.8
1.0
α

5. Consider two sets of assets whose return vectors $\boldsymbol{R}_1, \boldsymbol{R}_2$ are *independent* and have variance-covariance matrices $\mathbb{V}[\boldsymbol{R}_1] = \Sigma_1$, $\mathbb{V}[\boldsymbol{R}_2] = \Sigma_2$. Let the minimum-variance weights for a portfolio consisting only of assets in $\boldsymbol{R}_1$ be $\boldsymbol{w}_1$, and let the achieved minimum variance be σ_1^2. Similarly for the assets in $\boldsymbol{R}_2$, let the minimum-variance portfolio weights be $\boldsymbol{w}_2$ and the minimum variance be σ_2^2. Now turn attention to the minimum variance portfolio for *both* sets of assets $\left[\begin{array}{c}\boldsymbol{R}_1\\ \hline \boldsymbol{R}_2\end{array}\right]$.

(a) [10 points] Show that the combined minimum-variance portfolio weights are given by $\boldsymbol{w} = \frac{1}{\sigma_1^2+\sigma_2^2}\left[\begin{array}{c}\sigma_2^2\boldsymbol{w}_1\\ \hline \sigma_1^2\boldsymbol{w}_2\end{array}\right]$.

(b) [10 points] Show that the attained minimum variance is given by $\sigma^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}$.

a. For the individual minimum variance portfolios we have

$$\left.\begin{array}{l} \mathbf{w}_i = \dfrac{\Sigma_i^{-1}\mathbf{1}}{\mathbf{1}^\mathsf{T}\Sigma_i^{-1}\mathbf{1}} \\ \sigma_i^2 = \mathbb{V}\left[\mathbf{w}_i^\mathsf{T}\mathbf{R}_i\right] = \mathbf{w}_i^\mathsf{T}\underbrace{\mathbb{V}[\mathbf{R}_i]}_{=\Sigma_i}\mathbf{w}_i = \dfrac{\mathbf{1}^\mathsf{T}\Sigma_i^{-1}}{\mathbf{1}^\mathsf{T}\Sigma_i^{-1}\mathbf{1}}\cancel{\Sigma_i}\dfrac{\cancel{\Sigma_i^{-1}}\mathbf{1}}{\mathbf{1}^\mathsf{T}\Sigma_i^{-1}\mathbf{1}} = \dfrac{\mathbf{1}^\mathsf{T}\Sigma_i^{-1}\mathbf{1}}{\left(\mathbf{1}^\mathsf{T}\Sigma_i^{-1}\mathbf{1}\right)^2} = \dfrac{1}{\mathbf{1}^\mathsf{T}\Sigma_i^{-1}\mathbf{1}} \end{array}\right\} \Rightarrow$$

$$\Rightarrow\ \mathbf{w}_i = \sigma_i^2 \times \Sigma_i^{-1}\mathbf{1},\ \forall i = 1,2$$

We also know by independence that the combined asset's variance-covariance matrix is $\Sigma = \mathbb{V}\left[\begin{array}{c}\mathbf{R}_1\\ \hline \mathbf{R}_2\end{array}\right] = \left[\begin{array}{c|c}\Sigma_1 & \mathbf{0}\\ \hline \mathbf{0} & \Sigma_2\end{array}\right]$. The combined minimum variance portfolio is thus given by

$$\mathbf{w} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\mathsf{T}\Sigma^{-1}\mathbf{1}} = \frac{\left[\begin{array}{c|c}\Sigma_1 & \mathbf{0}\\ \hline \mathbf{0} & \Sigma_2\end{array}\right]^{-1}\left[\begin{array}{c}\mathbf{1}\\ \hline \mathbf{1}\end{array}\right]}{\left[\begin{array}{c|c}\mathbf{1}^\mathsf{T} & \mathbf{1}^\mathsf{T}\end{array}\right]\left[\begin{array}{c|c}\Sigma_1 & \mathbf{0}\\ \hline \mathbf{0} & \Sigma_2\end{array}\right]^{-1}\left[\begin{array}{c}\mathbf{1}\\ \hline \mathbf{1}\end{array}\right]} = \frac{\left[\begin{array}{c|c}\Sigma_1^{-1} & \mathbf{0}\\ \hline \mathbf{0} & \Sigma_2^{-1}\end{array}\right]\left[\begin{array}{c}\mathbf{1}\\ \hline \mathbf{1}\end{array}\right]}{\left[\begin{array}{c|c}\mathbf{1}^\mathsf{T} & \mathbf{1}^\mathsf{T}\end{array}\right]\left[\begin{array}{c|c}\Sigma_1^{-1} & \mathbf{0}\\ \hline \mathbf{0} & \Sigma_2^{-1}\end{array}\right]^{-1}\left[\begin{array}{c}\mathbf{1}\\ \hline \mathbf{1}\end{array}\right]} =$$

$$= \frac{\left[\begin{array}{c}\Sigma_1^{-1}\mathbf{1}\\ \hline \Sigma_2^{-1}\mathbf{1}\end{array}\right]}{\underbrace{\mathbf{1}^\mathsf{T}\Sigma_1^{-1}\mathbf{1}}_{=1/\sigma_1^2}+\underbrace{\mathbf{1}^\mathsf{T}\Sigma_2^{-1}\mathbf{1}}_{=1/\sigma_2^2}} = \frac{1}{1/\sigma_1^2+1/\sigma_2^2}\left[\begin{array}{c}\Sigma_1^{-1}\mathbf{1}\\ \hline \Sigma_2^{-1}\mathbf{1}\end{array}\right] = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\left[\begin{array}{c}\Sigma_1^{-1}\mathbf{1}\\ \hline \Sigma_2^{-1}\mathbf{1}\end{array}\right] = \frac{1}{\sigma_1^2+\sigma_2^2}\left[\begin{array}{c}\sigma_2^2\mathbf{w}_1\\ \hline \sigma_1^2\mathbf{w}_2\end{array}\right]$$

b. The attained minimum variance is $\sigma^2 = \mathbb{V}\left[\mathbf{w}^{\mathsf{T}}\mathbf{R}\right] = \mathbf{w}^{\mathsf{T}}\mathbb{V}[\mathbf{R}]\mathbf{w} =$

$$= \frac{1}{\sigma_1^2+\sigma_2^2}\left[\sigma_2^2\mathbf{w}_1^{\mathsf{T}} \;\vdots\; \sigma_1^2\mathbf{w}_2^{\mathsf{T}}\right]\left[\begin{array}{c|c} \Sigma_1 & 0 \\ \hline 0 & \Sigma_2 \end{array}\right]\left[\begin{array}{c} \sigma_2^2\mathbf{w}_1 \\ \hline \sigma_1^2\mathbf{w}_2 \end{array}\right]\frac{1}{\sigma_1^2+\sigma_2^2} =$$

$$= \left(\frac{1}{\sigma_1^2+\sigma_2^2}\right)^2\left(\sigma_2^2\mathbf{w}_1^{\mathsf{T}}\Sigma_1\mathbf{w}_1^2\sigma_2^2 + \sigma_1^4\mathbf{w}_2^{\mathsf{T}}\Sigma_2\mathbf{w}_2\sigma_1^4\right) =$$

$$= \frac{\sigma_2^4\sigma_2^2+\sigma_1^4\sigma_2^2}{\left(\sigma_1^2+\sigma_2^2\right)^2} = \frac{\sigma_1^2\sigma_2^2\cancel{\left(\sigma_1^2+\sigma_2^2\right)}}{\left(\sigma_1^2+\sigma_2^2\right)^{\cancel{2}}} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

6. Consider a standard Normal random variable $Z \sim N(0,1)$ and let $Y = e^Z$. Assume you want to estimate $\mathbb{E}[Y]$ using simulation, i.e. using $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}\sum_{i=1}^{n} e^{Z_i}$, where $Z_i \sim^{iid} N(0,1)$.

   (a) [7 points] Find the variance of your estimate $\mathbb{V}[\bar{Y}]$, as a function of n only. (Hint: the moment generating function of $Z \sim N(0,1)$ is $m_Z(t) = \mathbb{E}[e^{tZ}] = e^{t^2/2}$.)

   (b) [13 points] Assume you use $X = Z$ as a *control variable* for your simulation. Find the relative reduction in estimation accuracy, i.e. find the value of $\mathbb{V}[\bar{Y}_{ctrl}]/\mathbb{V}[\bar{Y}]$. (Hint: for $Z \sim N(0,1)$, we have $\mathbb{E}[Ze^Z] = \sqrt{e}$.)

(a) $\mathbb{V}[\bar{Y}] = \frac{1}{n}\cdot\mathbb{V}[Y] = \frac{1}{n}\mathbb{V}[e^Z] = \frac{1}{n}\left\{\mathbb{E}[(e^Z)^2] - (\mathbb{E}[e^Z])^2\right\}$

$= \frac{1}{n}\cdot\left\{\mathbb{E}[e^{2Z}] - (\mathbb{E}[e^Z])^2\right\} = \frac{1}{n}\cdot\left\{m_Z(2) - [m_Z(1)]^2\right\}$

$= \frac{1}{n}\left(e^{2^2/2} - (e^{1/2})^2\right) = \frac{1}{n}\cdot e^2 - e = \frac{1}{n}\cdot e\cdot(e-1$

(b) $\mathbb{V}[\bar{Y}_{ctrl}]/\mathbb{V}[Y] = 1 - \rho^2$, where $\rho = \frac{Cov(X,Y)}{\sqrt{\mathbb{V}(X)\cdot\mathbb{V}(Y)}}$

$Cov(X,Y) = \mathbb{E}[(X-\mathbb{E}(X))\cdot(Y-\mathbb{E}(Y))] =$

$= \mathbb{E}[(Z-\underbrace{\mathbb{E}[Z]}_{0})\cdot(e^Z - \mathbb{E}[e^Z])] =$

$= \mathbb{E}[Z\cdot(e^Z - m_Z(1))]$

$= \mathbb{E}[Ze^Z] - \underbrace{\mathbb{E}[Z]}_{0}\cdot m_Z(1) = \sqrt{e}$

$\mathbb{V}(X) = \mathbb{V}(Z) = 1$

Thus $\rho = \dfrac{Cov(X,Y)}{\sqrt{V(X)\,V[Y]}} = \dfrac{\sqrt{e}}{\sqrt{e(e-1)\cdot 1}} = \dfrac{1}{\sqrt{e-1}}$

$\Rightarrow 1-\rho^2 = 1-\dfrac{1}{e-1} = \dfrac{e-2}{e-1} = .418$

$= 41.8\%$

Extra Space (use if needed and clearly indicate which questions you are answering)