UNIVERSITY OF TORONTO SCARBOROUGH Department of Computer & Mathematical Sciences

STAD70H3 Statistics & Finance II April 2020 Final Examination

Duration: 3 hours
Instructor: Sotirios Damouras
Aids allowed: Open book/notes, scientific calculator
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Last Name:
First Name:
Student #·

Instructions:

- Read the questions carefully and answer only what is being asked.
- Answer all questions directly on the examination paper; use the last pages if you need more space, and provide clear pointers to your work.
- Show your intermediate work, and write clearly and legibly.

Question:	1	2	3	4	5	Total
Points:	20	20	15	20	30	105
Score:						

1. (Modeling returns/Geometric Brownian Motion) Assume the prices of two assets (S_t, P_t) follow *correlated* geometric Brownian motions:

$$\begin{cases} d \log(S_t) = \mu_S dt + \sigma_S dW_t \\ d \log(P_t) = \mu_P dt + \sigma_P dV_t \end{cases}$$

where $S_0 = P_0 = 1$, and W_t, V_t are correlated standard Brownian motions with correlation $|\rho| < 1$.

- (a) [3 points] Express the (marginal) distribution of S_t as a log-Normal distribution, i.e. $S_t = S_0 \times \exp{\{\mathcal{N}(\cdot, \cdot)\}}$, and state the parameter values of the Normal distribution in the exponent.
- (b) [3 points] Show that S_t^a where a > 0 also follows log-Normal distribution, and find its parameter values as before.
- (c) [7 points] Show that the *conditional* distribution of $S_t|(P_t = e)$ is also log-Normal, and find its parameter values as before.
- (d) [7 points] Find the probability $\mathbb{P}(S_t^2 > P_t)$ in terms of the model parameters and the standard Normal CDF $\Phi(\cdot)$.

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- 2. (Betting Strategies) Consider a sequence of independent gambles, where the probability of winning each one is $1/2 . Moreover, for each gamble you either win or lose the amount you bet. Assume you start with initial wealth <math>V_0$ and at each step you bet a fixed fraction f of your wealth.
 - (a) [10 points] Show that if you win half and lose half of the first 2n bets (i.e. win any of the n bets), then the resulting wealth V_{2n} will always be less than your initial wealth V_0 , for any f > 0.
 - (b) [10 points] Show that if your goal is to maximize the expected square-root of the return (i.e. $\mathbb{E}[\sqrt{V_n/V_0}]$), then the optimal fraction for this strategy is $f^* = \frac{(p/q)^2 1}{(p/q)^2 + 1}$, where q = 1 p.
 - (*Hint*: the probability generating function of a Binomial RV $X \sim \text{Binomial}(n, p)$ is given by $g_X(z) = \mathbb{E}[z^X] = [q + pz]^n$.)

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- 3. (Risk Measures) Assume the loss of a portfolio is $L=X-\mu$, where X follows Exponential(1) distribution, i.e. $f_X(x)=e^{-x}, \forall x>0$.
 - (a) [6 points] Find a closed-form expression for the value at risk (VaR) at level α .
 - (b) [9 points] Find a closed-form expression for the conditional VaR at level α .

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Answer all questions below in an .R/.RMarkdown file and submit it Quercus.

4. • (Factor Models)

Use the following R code to download daily prices of 10 ETFs, from Jan 1, 2018 to Dec 31, 2019.

- (a) [5 points] Calculate the *log*-returns of the ETFs, and plot the price and return series for the first ETF (DVEM).
- (b) [5 points] Use factanal() to fit a 2-factor model to the correlation matrix of the returns. Report the factor loadings and idiosyncratic variances of you model.
- (c) [10 points] Simulate 250 daily *log*-returns using a multivariate Normal distribution with parameters given by the sample means and variances of the ETFs, and correlation matrix given by the previous factor model. Calculate and plot the cumulative *net*-returns of an equally weighted portfolio over the 10 ETFs.
- 5. R (Monte Carlo Simulation) Consider a European chooser option where the holder gets to decide at time T_1 whether the option becomes a call or a put with fixed strike K and maturity $T_2 > T_1$. In other words, the holder "chooses" at time T_1 the form of the option payoff: $(S_{T_2} K)_+$ for a call, or $(K S_{T_2})_+$ for a put. Note that because of put-call parity, i.e. $C(S_{T_1}, T_2 T_1, K) P(S_{T_1}, T_2 T_1, K) = S_{T_1} Ke^{-r(T_2 T_1)}$, the holder's optimal decision at time T_1 is straightforward:
 - If $S_{T_1} Ke^{-r(T_2 T_1)} > 0$, they choose the call (b/c it is more valuable, i.e. $C(S_{T_1}, T_2 T_1, K) > P(S_{T_1}, T_2 T_1, K)$)
 - If $S_{T_1} Ke^{-r(T_2 T_1)} < 0$, they choose the put

Let the current price of the underlying asset be $S_0 = 100$, the strike price be K = 100, the choosing and expiration times be $T_1 = 1$ and $T_2 = 2$, and the risk-free rate be r = 5%, and assume the standard geometric Brownian motion (GBM) asset price dynamics:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

with volatility $\sigma = 20\%$.

- (a) [5 points] Use fExoticOptions::SimpleChooserOption() to find the exact price of the option.
- (b) [10 points] Perform a simulation with n = 10,000 paths for pricing the chooser option. For each path, generate two prices: S_{T_1} at time T_1 for determining the form of the payoff, and S_{T_2} at time T_2 for determining the value of the payoff. Report the estimated price and its standard deviation.
- (c) [5 points] Repeat the simulation experiment, but this time use the Black-Scholes formula to value the chosen option at time T_1 . In other words, you don't have to simulate S_{T_2} ; just simulate S_{T_1} and calculate the exact value of the chosen call or put option at time T_1 using Black-Scholes. Report the estimated price and its standard deviation (should be smaller).

(*Hint*: use fOptions::GBSOption() to get the Black-Scholes price.)

(d) [10 points] Repeat part (b), using the following dynamics for the asset:

$$dS_t = rS_t dt + \sigma \log(S_t) dW_t$$

Note that the price process no longer follows GBM, so use path discretization with m = 20 steps per path to approximate S_{T_1} and S_{T_2} . Report the estimated price and its standard deviation.

(*Note*: the option choice rule at T_1 does not change with asset price dynamics, since put-call parity holds for any model; but your answer will be different from part (a) which only holds for GBM.)

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