UNIVERSITY OF TORONTO SCARBOROUGH
Department of Computer & Mathematical Sciences

STAD70H3 Statistics & Finance II

April 2021 Final Examination

Duration: 2 hours
Aids allowed: Open book/notes, scientific calculator
Instructor: Sotirios Damouras

Last Name: ______________________
First Name: ______________________
Student #: ______________________

Instructions:

- Read the questions carefully and answer only what is being asked.
- Show your intermediate work, and write clearly and legibly.

Question:	1	2	3	4	Total
Points:	25	25	25	25	100
Score:					

1. (Risk Management & Betting Strategies)
Assume you start with an initial wealth of V_0 and you can invest any amount on a sequence of investment opportunities with *net returns* $\{R_i\}_{i\geq 1}$ which follow independent and identically distributed Uniform$(-1, 2)$ distributions. For example, if you invest \$1 in such an opportunity, your resulting wealth will be $(1 + R) \in (0, 3)$, i.e. anywhere between losing to tripling your wealth.

   (a) [2 points] Assume you bet your entire initial capital of $V_0 = \$1$ on the first opportunity. Calculate the Value-at-Risk (VaR) of the investment at the $\alpha = 5\%$ level.

   (b) [3 points] Calculate the *Conditional* VaR (CVaR) at the $\alpha = 5\%$ level for the investment in part (a).

   (c) [5 points] Calculate the *Entropic* VaR (EVaR) at the $\alpha = 5\%$ level for the investment in part (a). This is defined as $\text{EVaR}_\alpha = \inf_{z>0}\{z^{-1}\ln(M_L(z)/\alpha)\}$, where $M_L(z)$ is the moment generating function of the *loss* (L). You are given that the infimum occurs at $z \approx 18.122$.

   (d) [5 points] Assume your strategy is to invest a *fixed fraction* (f) of your current wealth in each step. Show that your final wealth after n investments is

$$V_n = V_0 \prod_{i=1}^{n}(1 + fR_i)$$

   (e) [2 points] Assume you invest your *entire wealth* ($f = 1$) at each step. Find the probability of going bankrupt ($\mathbb{P}(V_n = 0)$) after n steps?

   (f) [3 points] Assume you invest your *entire wealth* ($f = 1$) at each step. Find the expected value of your wealth ($\mathbb{E}[V_n]$) after n steps.

   (g) [5 points] For general $f \in (0, 1)$, show that the expected log-wealth after n investments is

$$\mathbb{E}[\ln(V_n)] = \ln(V_0) + \frac{n}{3}\left[\left(2 + \frac{1}{f}\right)\ln(1 + 2f) - \left(\frac{1}{f} - 1\right)\ln(1 - f) - 1\right]$$

   (Hint: Use the fact that $\int \ln(1 + fx)dx = (x + 1/f)\ln(1 + fx) - x + c.$)

Solution:

(a) The loss distribution is $L = -R \sim \text{Uniform}(-2, 1)$, whose 5% quantile is

$$\text{VaR}_{.05} = 1 - .05 \times 3 = 0.85$$

(b) For the Uniform, the (conditional) expected value is the the midpoint of the range:

$$\text{CVaR}_{.05} = \frac{1 + 0.85}{2} = 9.25$$

(c) The MGF of the uniform distribution for the loss (L) is

$$M_L(z) = \mathbb{E}[e^{zL}] = \int_{-2}^{1} e^{z\ell}/3d\ell$$
$$= \frac{1}{3z}\left[e^{z\ell}\right]_{\ell=-2}^{1} = \frac{e^{z} - e^{-2z}}{3z}$$

The EVaR_α is

$$\begin{aligned}
\mathrm{EVaR}_\alpha &= \ln\left(\frac{e^z - e^{-2z}}{3z(0.05)}\right)/z\Big|_{z=18.122} \\
&= \ln\left(\frac{e^{18.122} - e^{-2(18.122)}}{0.15(18.122)}\right)/18.122 = 0.9448181
\end{aligned}$$

Note that: $\mathrm{VaR}_{.05} \leq \mathrm{CVaR}_{.05} \leq \mathrm{EVaR}_{.05}$, as expected.

(d) Assuming you invest a fraction f, then your wealth after the first step will be:

$$V_1 = (1-f)V_0 + fV_0(1+R_1) = V_0 - fV_0 + fV_0 + fV_0R_1 = V_0(1+fR_1)$$

Similarly for the next steps, we get

$$\begin{aligned}
V_n &= V_{n-1}(1+fR_n) \\
&= V_{n-2}(1+fR_{n-1})(1+fR_n) \\
&= \vdots \\
&= V_0 \prod_{i=1}^{n}(1+fR_i)
\end{aligned}$$

(e) If $f = 1$, you will go bankrupt if and only if any of the returns R_i is exactly equal -1. Since the return distribution is continuous, the probability of that happening is 0.

(f) For $f = 1$

$$\begin{aligned}
\mathbb{E}[V_n] &= V_0\,\mathbb{E}\left[\prod_{i=1}^{n}(1+R_i)\right] \\
&= \prod_{i=1}^{n}\mathbb{E}\left[1+R_i\right] \quad \text{(by independence)} \\
&= \left(\mathbb{E}\left[1+R_i\right]\right)^n \qquad\qquad = (1.5)^n
\end{aligned}$$

(g)

$$
\begin{aligned}
\mathbb{E}[\ln(V_n)] &= \mathbb{E}\left[\ln\left(V_0 \prod_{i=1}^{n}(1+fR_i)\right)\right] \\
&= \mathbb{E}\left[\ln(V_0) + \sum i = 1^n \ln(1+fR_i)\right] \\
&= \ln(V_0) + n\,\mathbb{E}[\ln(1+fR)] \\
&= \ln(V_0) + n\int_{-1}^{2} \ln(1+fx)/3dx \\
&= \ln(V_0) + \frac{n}{3}\left[\left(x+\frac{1}{f}\right)\ln(1+fx) - x\right]_{x=-1}^{2} \\
&= \ln(V_0) + \frac{n}{3}\left[\left(2+\frac{1}{f}\right)\ln(1+f2) - 2 - \left(-1+\frac{1}{f}\right)\ln(1-f) + 1\right] \\
&= \ln(V_0) + \frac{n}{3}\left[\left(2+\frac{1}{f}\right)\ln(1+2f) - \left(\frac{1}{f}-1\right)\ln(1-f) - 1\right]
\end{aligned}
$$

2. (Brownian Motion & Simulation)
Consider the standard Brownian motion $\{W_t\}_{t\geq 0}$ and its running maximum $M_t = \max_{0\leq s\leq t}\{W_s\}$. For this question you will prove that the *conditional* distribution of the maximum given the final point of the Brownian motion (i.e. $M_1|W_1$) follows a Rayleigh distribution, something we used in class.

(a) [7 points] Show that the joint PDF of M_1, W_1 is

$$f_{W_1,M_1}(x,y) = \frac{2(2y-x)}{\sqrt{2\pi}} \exp\left\{-\frac{(2y-x)^2}{2}\right\}, \ \forall y > 0, x < y$$

(Hint: Differentiate the joint CDF $F_{W_1,M_1}(x,y)$ w.r.t. x and y, using the fact that $\mathbb{P}(W_1 \leq x, M_1 \leq y) = \mathbb{P}(W_1 \leq x) - \mathbb{P}(W_1 \leq x, M_1 > y)$, where the latter probability is given by the reflection principle.)

(b) [7 points] Show that the conditional PDF of $M_1|(W_1 = x)$ is given by

$$f_{M_1|W_1}(y|x) = 2(2y-x)e^{-2y(y-x)}, \ \forall y > 0, x < y$$

(Hint: Use the fact that $f_{M_1|W_1}(y|x) = f_{W_1,M_1}(x,y)/f_{W_1}(x)$.)

(c) [4 points] Show that the conditional CDF of $M_1|(W_1 = x)$ is given by

$$F_{M_1|W_1}(y|x) = 1 - e^{-2y(y-x)}, \ \forall x \in \mathbb{R}, y > \min\{0, x\}$$

(d) [7 points] Let $L_t = \min_{0\leq s\leq t}\{W_t\}$ be the running *minimum* of the standard Brownian motion. Find a formula that simulates the *conditional minimum* $L_1|(W_1 = x)$ of the standard Brownian motion given $W_1 = x \in \mathbb{R}$, based on a single Uniform(0,1) random number U.
(Hint: use the inverse conditional CDF of $M_1|(W_1 = x)$ together with the symmetry of standard Brownian motion.)

Solution:

(a) Let $\Phi()/\phi()$ be the standard normal CDF/PDF. The joint CDF of W_1, M_1 is given by:

$$\begin{aligned}
F_{W_1,M_1}(x,u) &= \mathbb{P}(W_1 \leq x, M_1 \leq y) \\
&= \mathbb{P}(W_1 \leq x) - \mathbb{P}(W_1 \leq x, M_1 > y) \\
&= \Phi(x) - \mathbb{P}(W_1 \geq 2y - x) \\
&= \Phi(x) - \Phi(-(2y-x)), \quad \forall y > 0, x < y
\end{aligned}$$

The joint PDF is

$$
\begin{aligned}
f_{W_1,M_1}(x,u) &= \frac{d^2}{dxdy} F_{W_1,M_1}(x,u) \\
&= \frac{d^2}{dxdy}[\Phi(W_1 \leq x) - \Phi(-(2y-x))] \\
&= \frac{d}{dx}[-\phi(-(2y-x))(-2)] \\
&= \frac{d}{dx}[2\phi(2y-x)] \\
&= \frac{d}{dx}\left[\frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{(2y-x)^2}{2}\right\}\right] \\
&= \frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{(2y-x)^2}{2}\right\} \frac{d}{dx}\left(-\frac{(2y-x)^2}{2}\right) \\
&= \frac{2(2y-x)}{\sqrt{2\pi}} \exp\left\{-\frac{(2y-x)^2}{2}\right\}, \quad \forall y > 0, x < y
\end{aligned}
$$

(b)

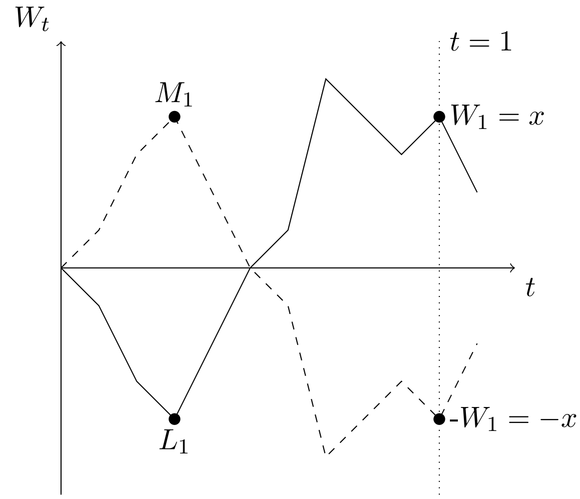
$$
\begin{aligned}
f_{M_1|W_1}(y|x) &= \frac{f_{W_1,M_1}(x,y)}{f_{W_1}(x)} \\
&= \frac{\frac{2(2y-x)}{\sqrt{2\pi}} \exp\left\{-\frac{(2y-x)^2}{2}\right\}}{\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}} \\
&= 2(2y-x) \exp\left\{-\frac{(2y-x)^2 - x^2}{2}\right\} \\
&= 2(2y-x) \exp\left\{-\frac{4y^2 - 4yx + x^2 - x^2}{2}\right\} \\
&= 2(2y-x) \exp\left\{-2y(y-x)\right\}, \quad \forall y > (0 \vee x)
\end{aligned}
$$

(c)

$$
\begin{aligned}
F_{M_1|W_1}(y|x) &= \int_{0\vee x}^{y} 2(2u-x)e^{-2u(u-x)}du \\
&= \int_{0\vee x}^{y} \left(-e^{-2u(u-x)}\right)' du \\
&= \left[-e^{-2u(u-x)}\right]_{u=(0\vee x)}^{y} \\
&= -e^{-2y(y-x)} + e^0, \qquad (\text{ because } (0\vee x)(x-(0\vee x)) = 0,\ \forall x \in \mathbb{R}) \\
&= 1 - e^{-2y(y-x)}, \quad y > (0 \vee u)
\end{aligned}
$$

(d) From the symmetry of standard Brownian motion, we have that the minimum conditional on $W_1 = x$ has the same distribution as *minus* the maximum conditional on

$W_1 = -x$, i.e. $L_1|(W_1 = x) \sim -M_1|(W_1 = -x)$. The following plot illustrates the point:



Moreover, we can simulate the conditional maximum M_1 given $W_1 = x$ using the inverse CDF method. Letting $M = M_1|(W_1 = x)$, we have

$$\begin{aligned}
F_{M_1|(W_1=x)}(M) \sim U &\Leftrightarrow 1 - e^{-2M(M-x)} = U \\
&\Rightarrow -2M(M-x) = \ln(1-U) \Rightarrow 2M^2 - 2xM + \ln(1-U) = 0 \\
\Rightarrow M = M_1|(W_1 = x) &= \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(2)\ln(1-U)}}{2(2)} \\
&= \frac{x + \sqrt{x^2 - 2\ln(1-U)}}{2} \quad \text{(rejecting the negative root)}
\end{aligned}$$

Combining the two, we get the minimum as :

$$\begin{aligned}
L_1|(W_1 = x) = -M_1|(W_1 = -x) &= -\frac{-x + \sqrt{x^2 - 2\ln(1-U)}}{2} \\
&= \frac{x - \sqrt{x^2 - 2\ln(1-U)}}{2}
\end{aligned}$$

Answer the following questions in an .R/.RMarkdown file.

3. (Simulation & Betting Strategies)
Consider the same setup as in question 1., i.e. a sequence of investment opportunities with i.i.d. Uniform$(-1, 2)$ returns and an initial capital of $V_0 = 1$.
   (a) [5 points] Use the `optimize()` function to find the Kelly fraction f^* to invest, i.e. value of f which maximizes the expression in Q1.(g).
   (b) [10 points] Simulate $1,000$ wealth paths consisting of 30 investments steps each using the Kelly fraction from the previous part (or, if you didn't get it, just $f = .7$). On the same axes, plot the *mean* simulated wealth, and the *5% and 95% quantiles* of the simulated wealth at each step.
   (c) [10 points] In class, we mentioned that the Kelly criterion strategy reaches any wealth level faster than other strategies. You will test this claim with a simulation experiment. Fix a target wealth of $10,000$ and simulate $1,000$ paths *with as many steps necessary* to reach or exceed this level, under both the Kelly strategy (f^*) and the all-in strategy ($f = 1$). Report the *average* number of steps required to reach or exceed the target wealth under each strategy. Does it confirm the claim?

4. (Industry Factor Models)
We have seen how to fit macroeconomic factor models, by regressing returns ($\boldsymbol{R}$) on *observed* factors ($\boldsymbol{F}$) one asset at a time, to estimate their loadings ($\boldsymbol{\beta}$). There is a related class of models called *fundamental factor models*, which are also fit by regression, but taking a different approach. The idea is to look at asset characteristics (i.e. fundamentals) to determine the asset *loadings* ($\boldsymbol{\beta}$) based of their grouping with respect to common but unobserved factors. The simplest method is to group assets in non-overlapping *industry categories*, and assume all assets in a category are exposed to the same factor. In terms of the model, their loadings (β's) are either 1 for their industry's factor, or 0 for all other industry factors. The following model with 4 assets in 2 industries serves as an illustration:

$$\begin{bmatrix} R_{1,t} \\ R_{2,t} \\ R_{3,t} \\ R_{4,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F_{1,t} \\ F_{2,t} \end{bmatrix} + \begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \\ e_{4,t} \end{bmatrix} \Leftrightarrow \boldsymbol{R} = \boldsymbol{\beta}^\top \boldsymbol{F} + \boldsymbol{e}$$

In order to estimate the factor values ($F_{1/2,t}$), we use *cross-sectional* regression, i.e. we fix time t and look at returns as observations from the model

$$R_{i,t} = F_{1,t} I_{i,1} + F_{2,t} I_{i,2} + e_{i,t}, \; i = 1, \ldots, n$$

where $I_{i,1/2}$ are the indicator variables of asset i belonging to industry 1/2, serving as the explanatory variables, and the factors values $F_{1/2,t}$ are the regression coefficients [1]. The following code loads 250 daily *net* returns for $n = 121$ assets (matrix `R`) belonging in 2 industries (factor `sector`). The loop fits 250 regressions to estimate the factor values (matrix `f`) and asset idiosyncratic errors (matrix `e`).

[1] In this simple case, this is effectively an ANOVA model, where factor values are simple averages of each industry's returns.

```
load("STAD70_W21_Final.RData")

b = model.matrix( object = ~ sectors - 1 )
f = matrix( 0, nrow = nrow(R), ncol = nlevels(sectors) )
e = matrix(0, nrow = nrow(R), ncol = ncol(R) )

for( i in 1:n_days){
tmp = lm( R[i,] ~ b )
f[i,] = tmp$coefficients
e[i,] = tmp$residuals
}
```

(a) [5 points] Create a scatter-plot of the values of the two industry factors. Do the factors look independent? Explain your answer based on what you know about asset behaviour.

(b) [3 points] Use the *sample covariance* of the returns (`R`) to calculate the minimum-variance-portfolio weights.

(c) [10 points] Use the *industry factor model* to estimate the return covariance matrix (from $\boldsymbol{\beta}\boldsymbol{\Sigma}_F\boldsymbol{\beta}^\top + \boldsymbol{S}_e$), and use that to calculate the minimum-variance-portfolio weights.

(d) [3 points] Use the weights you found in the last two parts to create the returns of each portfolio, and calculate their Sharpe ratio assuming a risk-free rate of 0.

(e) [4 points] Plot the *cumulative gross returns* of the two portfolios on the same axes.