STAD70 W23 Name (f,l): ______________________

Quiz 3 Student #: ______________________

1. Assume that the loss L of an investment follows continuous Uniform(-60,40) distribution.
   (a) (4 points) Find the Value-at-Risk (VaR) at confidence level $1-\alpha = 95\%$.
   (b) (4 points) Find the *Conditional*-VaR/Expected Shortfall at the same level.

Solution:

(a) This is the 95% quantile of the Uniform$(-60, 40)$ loss distribution, which is

$$\text{VaR}_{\alpha=.05} = -60 + 100 * (0.95) = 35$$

(b) The conditional distribution of the loss above the previous VaR will also be Uniform in $(35, 40)$, therefore the CVaR/ES is the midpoint of that interval, i.e., $\text{CVaR}_{\alpha=.05} = 37.5$ level.

2. Consider the general *statistical factor model* $\underbrace{\boldsymbol{R}_t}_{n\times 1} = \underbrace{\boldsymbol{\mu}}_{n\times 1} + \underbrace{\boldsymbol{\beta}^\top}_{n\times m} \underbrace{\boldsymbol{F}_t}_{m\times 1} + \underbrace{\boldsymbol{\epsilon}_t}_{n\times 1}$, with n variables and $m < n$ factors.
   (a) (6 points) List *all* the conditions on the 1st and 2nd order moments (i.e., means, variance, and covariances) of the random vectors $\boldsymbol{F}_t$ and $\boldsymbol{\epsilon}_t$, so that the model is *identifiable* (i.e., there is a unique representation for given moments of $\boldsymbol{R}_t$).
   (b) (6 points) Specify a statistical factor model with the *smallest possible number of factors* that can represent $\boldsymbol{R}_t$ with moments $\mathbb{E}[\boldsymbol{R}_t] = \mathbf{0}$ and $\mathbb{V}[\boldsymbol{R}_t] = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$.

Solution:

(a) For i.i.d. data over time, we need

$$\begin{aligned} \mathbb{E}[\boldsymbol{F}_t] = \mathbb{E}[\boldsymbol{\epsilon}_t] &= \mathbf{0} \\ \mathbb{V}[\boldsymbol{F}_t] &= \boldsymbol{I} \\ \mathbb{V}[\boldsymbol{\epsilon}_t] &= \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n^2 \end{bmatrix} \\ \text{Cov}[\boldsymbol{F}_t, \boldsymbol{\epsilon}_t] &= 0 \end{aligned}$$

(b) We just need 1 factor as follows:

$$
\begin{aligned}
\boldsymbol{\mu} &= \mathbf{0}_{4\times 1} \\
\boldsymbol{\beta}^\top &= \mathbf{1}_{4\times 1} \\
\mathbb{V}[\boldsymbol{\epsilon}_t] &= \boldsymbol{I}_{4\times 4} \\
\Rightarrow \mathbb{V}[\boldsymbol{R}_t] &= \boldsymbol{\beta}^\top \boldsymbol{\beta} + \mathbb{V}[\boldsymbol{\epsilon}_t] \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}
\end{aligned}
$$