UNIVERSITY OF TORONTO SCARBOROUGH
Department of Computer & Mathematical Sciences

STAD70H3 Statistics & Finance II

April 2020 Final Examination

Duration: 3 hours
Instructor: Sotirios Damouras
Aids allowed: Open book/notes, scientific calculator

Last Name: ______________________________
First Name: ______________________________
Student #: ______________________________

Instructions:

- Read the questions carefully and answer only what is being asked.
- Answer all questions directly on the examination paper; use the last pages if you need more space, and provide clear pointers to your work.
- Show your intermediate work, and write clearly and legibly.

Question:	1	2	3	4	5	Total
Points:	20	20	15	20	30	105
Score:						

1. (Modeling returns/Geometric Brownian Motion) Assume the prices of two assets (S_t, P_t) follow *correlated* geometric Brownian motions:

$$\begin{cases} d\log(S_t) = \mu_S dt + \sigma_S dW_t \\ d\log(P_t) = \mu_P dt + \sigma_P dV_t \end{cases}$$

where $S_0 = P_0 = 1$, and W_t, V_t are correlated standard Brownian motions with correlation $|\rho| < 1$.

(a) [3 points] Express the (marginal) distribution of S_t as a log-Normal distribution, i.e. $S_t = S_0 \times \exp\{\mathcal{N}(\cdot, \cdot)\}$, and state the parameter values of the Normal distribution in the exponent.

(b) [3 points] Show that S_t^a where $a > 0$ also follows log-Normal distribution, and find its parameter values as before.

(c) [7 points] Show that the *conditional* distribution of $S_t|(P_t = e)$ is also log-Normal, and find its parameter values as before.

(d) [7 points] Find the probability $\mathbb{P}(S_t^2 > P_t)$ in terms of the model parameters and the standard Normal CDF $\Phi(\cdot)$.

Solution:

(a) From GBM, we have

$$S_t = S_0 \times \exp\{\mu_S t + \sigma_S W_t\} = \exp\{\mathcal{N}(\mu_S t, \sigma_S^2 t)\}$$

(b) From the previous part, we have

$$S_t^a = S_0 \times \exp\{a\mu_S t + a\sigma_S W_t\} = \exp\{\mathcal{N}(a\mu_S t, a^2\sigma_S^2 t)\}$$

(c) We have

$$\begin{aligned} P_t = P_0 \times \exp\{\mu_p t + \sigma_p V_t\} = e \Leftrightarrow \mu_P t + \sigma_p V_t = 1 \Leftrightarrow V_t = \frac{1 - \mu_P t}{\sigma_P} &:= c \\ S_t|(P_t = e) = S_t|(V_t = c) = S_0 \times \exp\{\mu_S t + \sigma_S W_t|(V_t = c)\} \\ \begin{bmatrix} W_t \\ V_t \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, t\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right) \Rightarrow W_t|(V_t = c) \sim \mathcal{N}(\rho c, t(1 - \rho^2)) \\ \Rightarrow S_t|(P_t = e) = \exp\{\mu_S t + \sigma_S \mathcal{N}(\rho c, t(1 - \rho^2))\} \\ = \exp\{\mathcal{N}(\mu_S t + \sigma_S \rho c, \sigma_S^2 t(1 - \rho^2))\} \\ = \exp\{\mathcal{N}(\mu_S t + \rho\frac{\sigma_S}{\sigma_P}(1 - \mu_P t), \sigma_S^2 t(1 - \rho^2))\} \end{aligned}$$

(d)

$$\mathbb{P}(S_t^2 > P_t) = \mathbb{P}(S_t^2/P_t > 1), \text{ where}$$

$$\frac{S_t^2}{P_t} = S_0/P_0 \exp\{2\mu_S t + 2\sigma_S W_t - \mu_P t - \sigma_P V_t\} = \exp\{(2\mu_S - \mu_P)t + 2\sigma_S W_t - \sigma_P V_t\}$$

$$2\sigma_S W_t - \sigma_P V_t \sim \mathcal{N}(0, (4\sigma_S^2 + \sigma_P^2 - 4\rho\sigma_S\sigma_P)t)$$

$$\Rightarrow \mathbb{P}(S_t^2/P_t > 1) = \mathbb{P}(\mathcal{N}(\overbrace{(2\mu_S - \mu_P)t}^{=\mu}, \overbrace{(4\sigma_S^2 + \sigma_P^2 - 4\rho\sigma_S\sigma_P)t}^{=\sigma^2}) > 0)$$

$$= \mathbb{P}(\mu + \sigma Z > 0) = \mathbb{P}(Z > -\mu/\sigma) = 1 - \Phi(-\mu/\sigma)$$

2. (Betting Strategies) Consider a sequence of independent gambles, where the probability of winning each one is $1/2 < p < 1$. Moreover, for each gamble you either win or lose the amount you bet. Assume you start with initial wealth V_0 and at each step you bet a fixed fraction f of your wealth.

   (a) [10 points] Show that if you win half and lose half of the first $2n$ bets (i.e. win any of the n bets), then the resulting wealth V_{2n} will always be less than your initial wealth V_0, for any $f > 0$.

   (b) [10 points] Show that if your goal is to maximize the *expected square-root* of the return (i.e. $\mathbb{E}[\sqrt{V_n/V_0}]$), then the optimal fraction for this strategy is $f^* = \frac{(p/q)^2-1}{(p/q)^2+1}$, where $q = 1-p$.
   (*Hint*: the probability generating function of a Binomial RV $X \sim \text{Binomial}(n,p)$ is given by $g_X(z) = \mathbb{E}[z^X] = [q + pz]^n$.)

Solution:

(a) Let the Binomial random variable $W_n \sim \text{Binomial}(n,p)$ denote the number of wins in n bets. At each bet we wager a proportion f of our wealth, so the total wealth after n gambles becomes

$$V_n = V_0(1+f)^{W_n}(1-f)^{n-W_n} = V_0\left(\frac{1+f}{1-f}\right)^{W_n}(1-f)^n$$

In this case, we have $W_{2n} = n$, therefore:

$$V_{2n} = V_0\left(\frac{1+f}{1-f}\right)^{W_{2n}}(1-f)^{2n} = V_0\left(\frac{1+f}{1-f}\right)^{n}(1-f)^{2n}$$

$$= V_0(1+f)^n(1-f)^n = V_0[(1+f)(1-f)]^n = V_0\,\overbrace{(1-f^2)^n}^{<1,\,\forall f>0}$$

which is always smaller than the initial wealth.

(b) The expected square root of the wealth after n bets is:

$$G(f) = \mathbb{E}\left[\sqrt{X_n/X_0}\right] = \mathbb{E}\left[\sqrt{\left(\frac{1+f}{1-f}\right)^{W_n}(1-f)^n}\right] = (1-f)^{n/2}\mathbb{E}\left[\left(\frac{1+f}{1-f}\right)^{W_n/2}\right]$$

$$= \left(\sqrt{1-f}\right)^n\mathbb{E}\left[\left(\sqrt{\frac{1+f}{1-f}}\right)^{W_n}\right] \overset{\text{(hint)}}{=} \left(\sqrt{1-f}\right)^n\left[q + p\sqrt{\frac{1+f}{1-f}}\right]^n$$

$$= \left[q\sqrt{1-f} + p\sqrt{1+f}\right]^n$$

We maximize this by taking the derivative w.r.t. f and setting it to 0:

$$\frac{dG(f)}{df} = 0 \Leftrightarrow \frac{d}{df}\left[q\sqrt{1-f} + p\sqrt{1+f}\right]^n = 0$$

$$\Rightarrow \underbrace{\cancel{n\left[q\sqrt{1-f} + p\sqrt{1+f}\right]^{n-1}}}_{>0}\left[-\frac{q}{2\sqrt{1-f}} + \frac{p}{2\sqrt{1+f}}\right] = 0$$

$$\Rightarrow (1+f)q^2 = (1-f)p^2 \Rightarrow f(p^2+q^2) = p^2 - q^2$$

$$\Rightarrow f^* = \frac{p^2 - q^2}{p^2 + q^2} = \frac{(p/q)^2 - 1}{(p/q)^2 + 1}$$

(contd...)

3. (Risk Measures) Assume the loss of a portfolio is $L = X - \mu$, where X follows Exponential(1) distribution, i.e. $f_X(x) = e^{-x}, \forall x > 0$.

   (a) [6 points] Find a closed-form expression for the value at risk (VaR) at level α.

   (b) [9 points] Find a closed-form expression for the *conditional* VaR at level α.

Solution:

(a) The complementary CDF of the exponential is $\bar{F}_X(x) = e^{-x}, \ \forall x > 0$. Therefore:

$$\begin{aligned}
\mathbb{P}(L > \mathrm{VaR}) = \alpha \Rightarrow \mathbb{P}(X - \mu > \mathrm{VaR}) = \alpha \\
\Rightarrow \mathbb{P}(X > VaR + \mu) = \bar{F}_X(\mathrm{VaR} + \mu) = \alpha \\
\Rightarrow e^{-(\mathrm{VaR} + \mu)} = \alpha \Rightarrow -\mathrm{VaR} - \mu = \ln(\alpha) \\
\Rightarrow \mathrm{VaR} = -\ln(\alpha) - \mu
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{CVaR} &= \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}(u)du = \frac{1}{\alpha}\int_0^\alpha -\mu - \ln(u)du \\
&= -\mu - \frac{1}{\alpha}\int_0^\alpha [u\ln(u) - u]'du \\
&= -\mu - \frac{1}{\alpha}[u\ln(u) - u]_0^\alpha \\
&= -\mu - \frac{1}{\alpha}[\alpha\ln(\alpha) - \alpha - \underbrace{\lim_{u\to 0} u\ln(u)}_{=0}] \\
&= -\mu - \ln(\alpha) - 1
\end{aligned}$$

Answer all questions below in an .R/.RMarkdown file, and submit it on Quercus.

4. (Factor Models)
   Use the following R code to download daily prices of 10 ETFs, from Jan 1, 2018 to Dec 31, 2019.

```
library(zoo)
library(tseries)
tickers = c("DVEM", "EXT", "HYEM", "LTPZ", "SCHP",
            "EDV",  "SPMB", "TLT", "GRI", "GOVT")
S=list()
for(i in 1:length(tickers)){
  S[[i]] = get.hist.quote(tickers[i], start='2018-01-01',
             end='2019-12-31', quote='AdjClose', drop = TRUE)
}
```

   Consider the

   (a) [5 points] Calculate the *log*-returns of the ETFs, and plot the price and return series for the first ETF (DVEM).

   (b) [5 points] Use `factanal()` to fit a 2-factor model to the correlation matrix of the returns. Report the factor loadings and idiosyncratic variances of you model.

   (c) [10 points] Simulate 250 daily *log*-returns using a multivariate Normal distribution with parameters given by the sample means and variances of the ETFs, and correlation matrix given by the previous factor model. Calculate and plot the cumulative *net*-returns of an equally weighted portfolio over the 10 ETFs.

5. (Monte Carlo Simulation) Consider a European *chooser option* where the holder gets to decide at time T_1 whether the option becomes a call or a put with fixed strike K and maturity $T_2 > T_1$. In other words, the holder "chooses" at time T_1 the form of the option payoff: $(S_{T_2} - K)_+$ for a call, or $(K - S_{T_2})_+$ for a put. Note that because of put-call parity, i.e. $C(S_{T_1}, T_2 - T_1, K) - P(S_{T_1}, T_2 - T_1, K) = S_{T_1} - Ke^{-r(T_2 - T_1)}$, the holder's optimal decision at time T_1 is straightforward:

   - If $S_{T_1} - Ke^{-r(T_2-T_1)} > 0$, they choose the call
     (b/c it is more valuable, i.e. $C(S_{T_1}, T_2 - T_1, K) > P(S_{T_1}, T_2 - T_1, K)$)
   - If $S_{T_1} - Ke^{-r(T_2-T_1)} < 0$, they choose the put

   Let the current price of the underlying asset be $S_0 = 100$, the strike price be $K = 100$, the choosing and expiration times be $T_1 = 1$ and $T_2 = 2$, and the risk-free rate be $r = 5\%$, and assume the standard geometric Brownian motion (GBM) asset price dynamics:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

   with volatility $\sigma = 20\%$.

(a) [5 points] Use `fExoticOptions::SimpleChooserOption()` to find the exact price of the option.

(b) [10 points] Perform a simulation with $n = 10,000$ paths for pricing the chooser option. For each path, generate two prices: S_{T_1} at time T_1 for determining the form of the payoff, and S_{T_2} at time T_2 for determining the value of the payoff. Report the estimated price and its standard deviation.

(c) [5 points] Repeat the simulation experiment, but this time use the the Black-Scholes formula to value the chosen option at time T_1. In other words, you don't have to simulate S_{T_2}; just simulate S_{T_1} and calculate the exact value of the chosen call or put option at time T_1 using Black-Scholes. Report the estimated price and its standard deviation (should be smaller).
(*Hint*: use `fOptions::GBSOption()` to get the Black-Scholes price.)

(d) [10 points] Repeat part (b), using the following dynamics for the asset:

$$dS_t = rS_t dt + \sigma \log(S_t) dW_t$$

Note that the price process no longer follows GBM, so use path discretization with $m = 20$ steps per path to approximate S_{T_1} and S_{T_2}. Report the estimated price and its standard deviation.
(*Note*: the option choice rule at T_1 does not change with asset price dynamics, since put-call parity holds for any model.)

Extra Space (use if needed and clearly indicate which questions you are answering)

Extra Space (use if needed and clearly indicate which questions you are answering)

Extra Space (use if needed and clearly indicate which questions you are answering)