STAD70 W23 Name (f,l): ____________________

Quiz 5 Student #: ____________________

1. (10 points) Consider a *digital call option* that pays \$1 if the asset price at expiration (S_T) exceeds the strike price (K). Assume the asset price follows the usual Black-Scholes dynamics $dS_t = rS_t dt + \sigma S_t dW_t$, where $r > 0$ is the risk-free interest rate, $\sigma > 0$ is the volatility, and $\{W_t\}$ is standard Brownian Motion.
   Let $Z_i \sim^{iid} \mathrm{N}(0,1)$, and write the digital call price estimate $\bar{X}_n$ as a function of the normal variates $\{Z_i\}_{i=1}^n$ and the parameters S_0, K, T, r, σ.

Solution: From GBM, the asset price at expiration is given by

$$S_i(T) = S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z_i\right\}$$

and the payoff by $\mathbb{I}(S(T) \geq K) = \begin{cases} 1, & S(T) \geq K \\ 0, & \text{otherwise} \end{cases}$. The MC estimator is given by the average discounted payoff

$$\begin{aligned} \bar{X}_n &= \frac{e^{-rT}}{n}\sum_{i=1}^n \mathbb{I}(S_i(T) \geq K) \\ &= \frac{e^{-rT}}{n}\sum_{i=1}^n \mathbb{I}\left(S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z_i\right\} \geq K\right) \end{aligned}$$

2. (10 points) Assume you are using Monte Carlo simulation with *path discretization* for estimating the price of various path-dependent options under the Black-Scholes model (i.e., the price follows Geometric Brownian Motion). For each of the following option types, determine whether the estimate will under-estimate, over-estimate, or be unbiased.

   i Down-&-Out Call

   ii Down-&-Out Put

   iii Up-&-Out Call

   iv Up-&-In Call

   v Asian Call with payoff $\left(\frac{1}{m}\sum_{j=1}^m S_{t_j} - K\right)_+$ depending on the average price at simulated times $t_1, \ldots, t_m$.

Solution:

i Down-&-Out Call: Over-estimate

ii Down-&-Out Put: Over-estimate

iii Up-&-Out Call: Over-estimate

iv Up-&-In Call: Under-estimate

v Asian Call: unbiased

All knock-out (Down/Up, Call/Put) barrier option prices will be overestimated, because the discretized minima/maxima will not be as extreme as the true ones. For similar reasons, the knock-in option prices will be underestimated. The Asian option prices will be unbiased, because the simulated values come from their exact distribution.