UNIVERSITY OF TORONTO SCARBOROUGH Department of Computer & Mathematical Sciences

STAD70H3 Statistics & Finance II April 2020 Final Examination

Duration: 3 hours
Instructor: Sotirios Damouras
Aids allowed: Open book/notes, scientific calculator
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Last Name:
First Name:
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Instructions:

- Read the questions carefully and answer only what is being asked.
- Answer all questions directly on the examination paper; use the last pages if you need more space, and provide clear pointers to your work.
- Show your intermediate work, and write clearly and legibly.

Question:	1	2	3	4	5	Total
Points:	20	20	15	20	30	105
Score:						

1. (Modeling returns/Geometric Brownian Motion) Assume the prices of two assets (S_t, P_t) follow *correlated* geometric Brownian motions:

$$\begin{cases} d \log(S_t) = \mu_S dt + \sigma_S dW_t \\ d \log(P_t) = \mu_P dt + \sigma_P dV_t \end{cases}$$

where $S_0 = P_0 = 1$, and W_t, V_t are correlated standard Brownian motions with correlation $|\rho| < 1$.

- (a) [3 points] Express the (marginal) distribution of S_t as a log-Normal distribution, i.e. $S_t = S_0 \times \exp{\{\mathcal{N}(\cdot, \cdot)\}}$, and state the parameter values of the Normal distribution in the exponent.
- (b) [3 points] Show that S_t^a where a > 0 also follows log-Normal distribution, and find its parameter values as before.
- (c) [7 points] Show that the *conditional* distribution of $S_t|(P_t = e)$ is also log-Normal, and find its parameter values as before.
- (d) [7 points] Find the probability $\mathbb{P}(S_t^2 > P_t)$ in terms of the model parameters and the standard Normal CDF $\Phi(\cdot)$.

Solution:

(a) From GBM, we have

$$S_t = S_0 \times \exp\{\mu_S t + \sigma_S W_t\} = \exp\{\mathcal{N}(\mu_S t, \sigma_S^2 t)\}\$$

(b) From the previous part, we have

$$S_t^a = S_0 \times \exp\{a\mu_S t + a\sigma_S W_t\} = \exp\{\mathcal{N}(a\mu_S t, a^2 \sigma_S^2 t)\}$$

(c) We have

$$P_{t} = P_{0} \times \exp\{\mu_{p}t + \sigma_{p}V_{t}\} = e \Leftrightarrow \mu_{p}t + \sigma_{p}V_{t} = 1 \Leftrightarrow V_{t} = \frac{1 - \mu_{p}t}{\sigma_{p}} := c$$

$$S_{t}|(P_{t} = e) = S_{t}|(V_{t} = c) = S_{0} \times \exp\{\mu_{S}t + \sigma_{S}W_{t}|(V_{t} = c)\}$$

$$\begin{bmatrix} W_{t} \\ V_{t} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, t \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right) \Rightarrow W_{t}|(V_{t} = c) \sim \mathcal{N}(\rho c, t(1 - \rho^{2}))$$

$$\Rightarrow S_{t}|(P_{t} = e) = \exp\{\mu_{S}t + \sigma_{S}\mathcal{N}(\rho c, t(1 - \rho^{2}))\}$$

$$= \exp\{\mathcal{N}(\mu_{S}t + \sigma_{S}\rho c, \sigma_{S}^{2}t(1 - \rho^{2}))\}$$

$$= \exp\{\mathcal{N}(\mu_{S}t + \rho\frac{\sigma_{S}}{\sigma_{p}}(1 - \mu_{p}t), \sigma_{S}^{2}t(1 - \rho^{2}))\}$$

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(d)
$$\mathbb{P}(S_t^2 > P_t) = \mathbb{P}(S_t^2/P_t > 1), \text{ where}$$

$$\frac{S_t^2}{P_t} = S_0/P_0 \exp\left\{2\mu_S t + 2\sigma_S W_t - \mu_P t - \sigma_P V_t\right\} = \exp\left\{(2\mu_S - \mu_P)t + 2\sigma_S W_t - \sigma_P V_t\right\}$$

$$2\sigma_S W_t - \sigma_P V_t \sim \mathcal{N}(0, (4\sigma_S^2 + \sigma_P^2 - 4\rho\sigma_S\sigma_P)t)$$

$$\Rightarrow \mathbb{P}(S_t^2/P_t > 1) = \mathbb{P}(\mathcal{N}((2\mu_S - \mu_P)t, (4\sigma_S^2 + \sigma_P^2 - 4\rho\sigma_S\sigma_P)t) > 0)$$

$$= \mathbb{P}(\mu + \sigma Z > 0) = \mathbb{P}(Z > -\mu/\sigma) = 1 - \Phi(-\mu/\sigma)$$

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- 2. (Betting Strategies) Consider a sequence of independent gambles, where the probability of winning each one is $1/2 . Moreover, for each gamble you either win or lose the amount you bet. Assume you start with initial wealth <math>V_0$ and at each step you bet a fixed fraction f of your wealth.
 - (a) [10 points] Show that if you win half and lose half of the first 2n bets (i.e. win any of the n bets), then the resulting wealth V_{2n} will always be less than your initial wealth V_0 , for any f > 0.
 - (b) [10 points] Show that if your goal is to maximize the expected square-root of the return (i.e. $\mathbb{E}[\sqrt{V_n/V_0}]$), then the optimal fraction for this strategy is $f^* = \frac{(p/q)^2 1}{(p/q)^2 + 1}$, where q = 1 p.

(*Hint*: the probability generating function of a Binomial RV $X \sim \text{Binomial}(n, p)$ is given by $g_X(z) = \mathbb{E}[z^X] = [q + pz]^n$.)

Solution:

(a) Let the Binomial random variable $W_n \sim \text{Binomial}(n, p)$ denote the number of wins in n bets. At each bet we wager a proportion f of our wealth, so the total wealth after n gambles becomes

$$V_n = V_0 (1+f)^{W_n} (1-f)^{n-W_n} = V_0 \left(\frac{1+f}{1-f}\right)^{W_n} (1-f)^n$$

In this case, we have $W_{2n} = n$, therefore:

$$V_{2n} = V_0 \left(\frac{1+f}{1-f}\right)^{W_{2n}} (1-f)^{2n} = V_0 \left(\frac{1+f}{1-f}\right)^n (1-f)^{2n}$$
$$= V_0 (1+f)^n (1-f)^n = V_0 [(1+f)(1-f)]^n = V_0 \underbrace{(1-f)^{2n}}_{<1, \forall f>0} (1-f)^n$$

which is always smaller than the initial wealth.

(b) The expected square root of the wealth after n bets is:

$$G(f) = \mathbb{E}\left[\sqrt{X_n/X_0}\right] = \mathbb{E}\left[\sqrt{\left(\frac{1+f}{1-f}\right)^{W_n}(1-f)^n}\right] = (1-f)^{n/2}\mathbb{E}\left[\left(\frac{1+f}{1-f}\right)^{W_n/2}\right]$$
$$= \left(\sqrt{1-f}\right)^n\mathbb{E}\left[\left(\sqrt{\frac{1+f}{1-f}}\right)^{W_n}\right] \stackrel{\text{(hint)}}{=} \left(\sqrt{1-f}\right)^n\left[q+p\sqrt{\frac{1+f}{1-f}}\right]^n$$
$$= \left[q\sqrt{1-f} + p\sqrt{1+f}\right]^n$$

We maximize this by taking the derivative w.r.t. f and setting it to 0:

$$\frac{dG(f)}{df} = 0 \Leftrightarrow \frac{d}{df} \left[q\sqrt{1-f} + p\sqrt{1+f} \right]^n = 0$$

$$\Rightarrow \underbrace{n \left[q\sqrt{1-f} + p\sqrt{1+f} \right]^{n-1}}_{>0} \left[-\frac{q}{2\sqrt{1-f}} + \frac{p}{2\sqrt{1+f}} \right] = 0$$

$$\Rightarrow (1+f)q^2 = (1-f)p^2 \Rightarrow f(p^2+q^2) = p^2 - q^2$$

$$\Rightarrow f^* = \frac{p^2 - q^2}{p^2 + q^2} = \frac{(p/q)^2 - 1}{(p/q)^2 + 1}$$

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- 3. (Risk Measures) Assume the loss of a portfolio is $L = X \mu$, where X follows Exponential(1) distribution, i.e. $f_X(x) = e^-x, \forall x > 0$.
 - (a) [6 points] Find a closed-form expression for the value at risk (VaR) at level α .
 - (b) [9 points] Find a closed-form expression for the conditional VaR at level α .

Solution:

(a) The complementary CDF of the exponential is $\bar{F}_X(x) = e^{-x}$, for all x > 0. Therefore:

$$\mathbb{P}(L > VaR) = \alpha \Rightarrow \mathbb{P}(X - \mu > VaR) = \alpha$$

$$\Rightarrow \mathbb{P}(X > VaR + \mu) = bar F_X(VaR + \mu) = \alpha$$

$$\Rightarrow e^{-VaR + \mu} = \alpha \Rightarrow -VaR + \mu = \ln(\alpha)$$

$$\Rightarrow VaR = \mu - \ln(\alpha)$$

(b)

$$\begin{aligned} CVaR &= \frac{1}{\alpha} \int_0^\alpha VaR(u)du = \frac{1}{\alpha} \int_0^\alpha \mu - \ln(u)du \\ &= \mu - \frac{1}{\alpha} \int_0^\alpha [u \ln(u) - u]'du = \mu - \frac{1}{\alpha} [u \ln(u) - u]_0^\alpha \\ &= \mu - \frac{1}{\alpha} [\alpha \ln(\alpha) - \alpha - \varprojlim_{u \to 0} u \ln(u)] \\ &= \mu - \ln(\alpha) - 1 \end{aligned}$$

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Answer all questions below in an .R/.RMarkdown file, and submit it on Quercus.

4. • (Factor Models)

Use the following R code to download daily prices of 10 ETFs, from Jan 1, 2018 to Dec 31, 2019.

Consider the

- (a) [5 points] Calculate the *log*-returns of the ETFs, and plot the price and return series for the first ETF (DVEM).
- (b) [5 points] Use factanal() to fit a 2-factor model to the correlation matrix of the returns. Report the factor loadings and idiosyncratic variances of you model.
- (c) [10 points] Simulate 250 daily *log*-returns using a multivariate Normal distribution with parameters given by the sample means and variances of the ETFs, and correlation matrix given by the previous factor model. Calculate and plot the cumulative *net*-returns of an equally weighted portfolio over the 10 ETFs.
- 5. R (Monte Carlo Simulation) Consider a European chooser option where the holder gets to decide at time T_1 whether the option becomes a call or a put with fixed strike K and maturity $T_2 > T_1$. In other words, the holder "chooses" at time T_1 the form of the option payoff: $(S_{T_2} K)_+$ for a call, or $(K S_{T_2})_+$ for a put. Note that because of put-call parity, i.e. $C(S_{T_1}, T_2 T_1, K) P(S_{T_1}, T_2 T_1, K) = S_{T_1} Ke^{-r(T_2 T_1)}$, the holder's optimal decision at time T_1 is straightforward:
 - If $S_{T_1} Ke^{-r(T_2 T_1)} > 0$, they choose the call (b/c it is more valuable, i.e. $C(S_{T_1}, T_2 T_1, K) > P(S_{T_1}, T_2 T_1, K)$)
 - If $S_{T_1} Ke^{-r(T_2 T_1)} < 0$, they choose the put

Let the current price of the underlying asset be $S_0 = 100$, the strike price be K = 100, the choosing and expiration times be $T_1 = 1$ and $T_2 = 2$, and the risk-free rate be r = 5%, and assume the standard geometric Brownian motion (GBM) asset price dynamics:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

with volatility $\sigma = 20\%$.

- (a) [5 points] Use fExoticOptions::SimpleChooserOption() to find the exact price of the option.
- (b) [10 points] Perform a simulation with n = 10,000 paths for pricing the chooser option. For each path, generate two prices: S_{T_1} at time T_1 for determining the form of the payoff, and S_{T_2} at time T_2 for determining the value of the payoff. Report the estimated price and its standard deviation.
- (c) [5 points] Repeat the simulation experiment, but this time use the Black-Scholes formula to value the chosen option at time T_1 . In other words, you don't have to simulate S_{T_2} ; just simulate S_{T_1} and calculate the exact value of the chosen call or put option at time T_1 using Black-Scholes. Report the estimated price and its standard deviation (should be smaller).

(*Hint*: use fOptions::GBSOption() to get the Black-Scholes price.)

(d) [10 points] Repeat part (b), using the following dynamics for the asset:

$$dS_t = rS_t dt + \sigma \log(S_t) dW_t$$

Note that the price process no longer follows GBM, so use path discretization with m = 20 steps per path to approximate S_{T_1} and S_{T_2} . Report the estimated price and its standard deviation.

(*Note*: the option choice rule at T_1 does not change with asset price dynamics, since put-call parity holds for any model.)

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