# NATIONAL INSTITUTE OF TECHNOLOGY CALICUT
## Department of Mathematics
## MA 2002E: Mathematics III

Monsoon Semester 2024-25: Tutorial 4

## A Functions of complex variables, Limit, continuity

1. Show that $\lim_{x+iy\to 0}\left[xy/\left(x^2+y^2\right)\right]$ does not exist.

2. Show that $\lim_{x+iy\to 0}\left[x^2y/\left(x^4+y^2\right)\right]$ does not exist even though this function approaches the same limit along every straight line through the origin.

3. If
$$f(x,y)=\begin{cases} x\sin\left(\frac{1}{y}\right) & y\neq 0\\ 0 & y=0\end{cases}$$
show that $\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right]$ and $\lim_{x\to 0} f(x,y)$ exist and are equal, but that $\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right]$ does not exist.

4. Show that the various values approached by the difference quotient of $f(z)=\bar{z}$ as $\Delta z\to 0$ along the lines $y=mx$ all lie on a circle.

## B Complex differentiability, Cauchy-Riemann equations

5. Check whether the following functions analytic?

   (a) $f(z)=\operatorname{Im}\left(z^2\right)$
   (b) $f(x+iy)=e^{2x}(\cos y+i\sin y)$
   (c) $f(x+iy)=e^{-x}(\cos y-i\sin y)$
   (d) $f(z)=\operatorname{Arg} z$
   (e) $f(z)=\operatorname{Re} z+\operatorname{Im} z$
   (f) $f(z)=\ln|z|+i\operatorname{Arg} z$
   (g) $f(z)=i/z^8$
   (h) $f(z)=z^2+1/z^2$

6. Prove that the function $f(x+iy)=\sqrt{|x||y|}$ satisfies the Cauchy-Riemann equations at the origin, but $f$ is not holomorphic at origin.

7. Find the set points on which each of the following functions satisfies the Cauchy-Riemann equations. Also check whether the function is holomorphic and if so find its derivative.

   (a) $f(x+iy)=x$
   (b) $f(x+iy)=x^2+iy^2$
   (c) $f(x+iy)=x^3+i(y-1)^3$
   (d) $f(z)=\bar{z}$.
   (e) $f(z)=\bar{z}^2$.
   (f) $f(z)=|z|^2$.

8. Let $f$ be a holomorphic function in an open set $\Omega$. Prove that $f$ reduces to a constant if any of the following is true:

   (a) $\operatorname{Re}(f)$ is a constant
   (b) $\operatorname{Im}(f)$ is a constant
   (c) $|f|$ is a constant
   (d) $\operatorname{Arg}(f)$ is a constant
   (e) $f'=0$.
   (f) $\bar{f}$ is analytic.

9. If $f(z)$ is an analytic function, show that

   (a) $\left[\frac{\partial}{\partial x}|f(z)|\right]^2+\left[\frac{\partial}{\partial y}|f(z)|\right]^2=|f'(z)|^2$
   (b) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4\,|f'(z)|^2$

10. If $u$ and $v$ are respectively the real and imaginary parts of a holomorphic function $f$, prove that the level curves $u(x,y)=k$ and $v(x,y)=c$ are orthogonal to each other at all points $(x,y)$ where $f'(x+iy)\neq 0$.

## C Harmonic functions, Harmonic conjugate

11. Determine $a, b, c$ such that given the functions are harmonic and find a harmonic conjugate.

    (a) $u = e^{3x} \cos ay$ (b) $u = \sin x \cosh cy$ (c) $u = ax^3 + by^3$

12. Are the following functions harmonic? If yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$

    (a) $u = xy$ (c) $v = -y/\left(x^2 + y^2\right)$ (e) $u = x^3 - 3xy^2$
    (b) $v = xy$ (d) $v = \ln|z|$ (f) $u = e^{-x} \sin 2y$

13. Let $u$ be a real valued harmonic function in an open disc $\mathbb{D}$. Prove that any two harmonic conjugates of $u$ differ by a constant.

14. Let $u$ be a real valued harmonic function in an open disc $\mathbb{D}$. If $u^2$ is also harmonic, prove that $u$ is a constant.

15. Let $u$ be a real valued harmonic function in an open disc $\mathbb{D}$ and $v$ its harmonic conjugate. Prove that $-u$ is a harmonic conjugate of $v$.

16. Let $u$ be a real valued harmonic function in an open set $\Omega$ and $v$ its harmonic conjugate. Prove that $uv$ and $u^2 - v^2$ are also harmonic.

17. Let $u$ be a real-valued harmonic function in an open set $\Omega$. Prove that $\frac{\partial u}{\partial z}$ is holomorphic in $\Omega$.

18. Check whether the following functions $u$ are harmonic in some domain and if so find their harmonic conjugates. Also, find the holomorphic function $f(z)$ such that $u = \mathrm{Re}(f)$.

    (a) $u(x, y) = c$ (c) $u(x, y) = 2x(1 - y)$ (e) $u(x, y) = \sinh x \sin y$
    (b) $u(x, y) = y^3 - 3x^2 y$ (d) $u(x, y) = \frac{y}{x^2+y^2}$

19. If $f$ is holomorphic on a region $\Omega$, prove that $f$ is harmonic there. When is $|f|^2$ harmonic?

20. Let $f$ be a complex function on a region $\Omega$ and both $f$ and $f^2$ are harmonic in $\Omega$. Prove that either $f$ or $\bar{f}$ is harmonic in $\Omega$.

21. If $u$ and $v$ are harmonic in a region $\Omega \subset \mathbb{C}$, show that

$$\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

    is analytic in $\Omega$.

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