NATIONAL INSTITUTE OF TECHNOLOGY CALICUT
Department of Mathematics

MA 2002E: Mathematics III

Monsoon Semester 2024-25: Tutorial 3

A Adjoint of an operator, matrix representation

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (x + 2y, 3x - 4z, y)$. Find $T^*(x, y, z)$.
2. Let $T : \mathbb{C}^3 \to \mathrm{C}^3$ be defined by
$$T(x, y, z) = (ix + (2 + 3i)y, 3x + (3 - i)z, (2 - 5i)y + iz)$$
Find $T^*(x, y, z)$.
3. Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by
$$T(z_1, \ldots, z_n) = (0, z_1, \ldots, z_{n-1}).$$
Find a formula for $T^*(z_1, \ldots, z_n)$.
4. Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that λ is an eigenvalue of T if and only if $\bar{\lambda}$ is an eigenvalue of T^*.
5. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under T^*.
6. Suppose $T \in \mathcal{L}(V, W)$. Prove that
   (a) T is injective if and only if T^* is surjective;
   (b) T is surjective if and only if T^* is injective.
7. Prove that
$$\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$$
and dim range T^* = dim range T for every $T \in \mathcal{L}(V, W)$.
8. Show that $T^*T = 0$ implies $T = 0$.
9. Make $\mathcal{P}_2(\mathbb{R})$ into an inner-product space by defining
$$\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx.$$
Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ by $T(a_0 + a_1x + a_2x^2) = a_1x$.
   (a) Show that T is not self-adjoint.
   (b) The matrix of T with respect to the basis $(1, x, x^2)$ is
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
This matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

B Self-adjoint operators

10. Suppose $S, T \in \mathcal{L}(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.
11. Suppose V is a real inner product space. Show that the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$.
12. Suppose V is a complex inner product space with $V \neq \{0\}$. Show that the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.
13. Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that there is a subspace U of V such that $P = P_U$ if and only if P is self-adjoint.
14. Suppose that T is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of T. Prove that $T^2 - 5T + 6I = 0$.
15. Give an example of an operator $T \in \mathcal{L}(V)$ such that 2 and 3 are the only eigenvalues of T and $T^2 - 5T + 6I \neq 0$.
16. For any operator T, show that $T + T^*$ is self-adjoint and $T - T^*$ is skew-adjoint.
17. Suppose T is self-adjoint. Show that $T^2(v) = 0$ implies $T(v) = 0$. Use this to prove that $T^n(v) = 0$ also implies $T(v) = 0$ for $n > 0$.
18. Prove that there does not exist a self-adjoint operator $T \in \mathbb{R}^3$ such that $T(1, 2, 3) = (0, 0, 0)$ and $T(2, 5, 7) = (2, 5, 7)$.
19. Prove or give a counterexample: the product of any two self-adjoint operators on a finite-dimensional inner-product space is self-adjoint.
20. Suppose T_1 and T_2 are self-adjoint. Show that T_1T_2 is self-adjoint if and only if T_1 and T_2 commute, i.e. $T_1T_2 = T_2T_1$.
21. True or false (and give a proof of your answer): There exists $T \in \mathcal{L}(\mathbb{R}^3)$ such that T is not self-adjoint (with respect to the usual inner product) and such that there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of T.

C Normal operators

22. Suppose $\dim V \geq 2$. Show that the set of normal operators on V is not a subspace of $\mathcal{L}(V)$.
23. Suppose that T is a normal operator on V and that 3 and 4 are eigenvalues of T. Prove that there exists a vector $v \in V$ such that $||v|| = \sqrt{2}$ and $||Tv|| = 5$.
24. Give an example. of an operator $T \in \mathcal{L}(\mathbb{C}^4)$ such that T is normal but not self-adjoint.
25. Suppose T is a normal operator on V. Suppose also that $v, w \in V$ satisfy the equations $||v|| = ||w|| = 2$, $Tv = 3v$, $Tw = 4w$. Show that $||T(v + w)|| = 10$.
26. Prove that if $T \in \mathcal{L}(V)$ is normal, then range $T =$ range T^*
27. Prove that if $T \in \mathcal{L}(V)$ is normal, then
$$\text{null } T^k = \text{null } T \quad \text{and} \quad \text{range } T^k = \text{range } T$$
for every positive integer k.
28. Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $\|Te_j\| = \|T^*e_j\|$ for each j, then T is normal.
29. Suppose $T \in \mathcal{L}(\mathbb{C}^3)$ is normal and $T(1, 1, 1) = (2, 2, 2)$. Suppose $(z_1, z_2, z_3) \in \text{null } T$. Prove that $z_1 + z_2 + z_3 = 0$.

D Spectral theorem

30. Suppose $F = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Prove that T is normal if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and
$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_n, T),$$
where $\lambda_1, \ldots, \lambda_n$ denote the distinct eigenvalues of T.
31. Give an example of an operator T on a complex vector space such that $T^9 = T^8$ but $T^2 \neq T$.

32. Suppose V is a complex inner product space. Prove that every normal operator on V has a square root. (An operator $S \in \mathcal{L}(V)$ is called a square root of $T \in \mathcal{L}(V)$ if $S^2 = T$.)

33. Prove that a normal operator on a complex inner product space is self-adjoint if and only if all its eigenvalues are real.

34. Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

35. Suppose $F = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Prove that T is self-adjoint if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_n, T),$$

where $\lambda_1, \ldots, \lambda_n$ denote the distinct eigenvalues of T.

36. Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbb{F}$, and $\epsilon > 0$. Suppose there exists $v \in V$ such that $\|v\| = 1$ and

$$\|Tv - \lambda v\| < \epsilon.$$

Prove that T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.

37. Suppose V is a finite-dimensional real vector space and $T \in \mathcal{L}(V)$. Prove that T has a basis consisting of eigenvectors of T if and only if there is an inner product on V that makes T into a self-adjoint operator.

E Singular Value Decomposition (SVD)

38. Suppose $T \in \mathcal{L}(V)$. Prove that if both T and $-T$ are positive operators, then $T = 0$.

39. Suppose $T \in \mathcal{L}\left(\mathbf{F}^4\right)$ is the operator whose matrix (with respect to the standard basis) is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

Show that T is an invertible positive operator.

40. Suppose n is a positive integer and $T \in \mathcal{L}(\mathbf{F}^n)$ is the operator whose matrix (with respect to the standard basis) consists of all 1's. Show that T is a positive operator.

41. Prove that the sum of two positive operators on V is a positive operator.

42. Suppose $S \in \mathcal{L}(V)$ is an invertible positive operator and $T \in \mathcal{L}(V)$ is a positive operator. Prove that $S+T$ is invertible.

43. Let $T(x, y) = (-4y, x)$. Find the singular values of T.

44. Suppose $T \in \mathcal{L}(V, W)$. Show that $T = 0$ if and only if all singular values of T are 0 .

45. Find the singular values of the differentiation operator $D \in \mathcal{L}(\mathcal{P}_2(\mathbf{R})\ )$ defined by $Dp = p'$, where the inner product on $\mathcal{P}_2(\mathbf{R})$ is

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\, dx.$$

46. Find $A^{\mathrm{T}}A$ and AA^{T} and the singular vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{u}_1, \boldsymbol{u}_2$ for A :

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{has rank } r = 2. \text{ The eigenvalues are } 0, 0, 0$$

Check the equations $Av_1 = \sigma_1 u_1$ and $Av_2 = \sigma_2 u_2$ and $A = \sigma_1 u_1 v_1^{\mathrm{T}} + \sigma_2 u_2 v_2^{\mathrm{T}}$. If you remove row 3 of A (all zeros), show that σ_1 and σ_2 don't change.

47. Find the singular values and also the eigenvalues of B :

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 8 \\ \frac{1}{1000} & 0 & 0 \end{bmatrix} \quad \text{has rank } r = 3 \text{ and determinant } \frac{8}{1000}$$

Compared to A in the previous exercise, eigenvalues have changed much more than singular values.

48. Find the eigenvalues and the singular values of this 2 by 2 matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \quad \text{with} \quad A^{\mathrm{T}}A = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \quad \text{and} \quad AA^{\mathrm{T}} = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$$

49. Find a closest rank-1 approximation to these matrices (in Frobenius norm):

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

50. Find a closest rank-1 approximation to $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

*****