# AVL Tree

The height of an AVL tree, T, storing n items is O(log n)

# Height of an AVL Tree

- Let n(h) be the minimum number of internal nodes of an AVL tree with height h
- As base cases for a recursive definition, let n(1) = 1, because an AVL tree of height 1 must have at least one internal node
- n(2) = 2, because an AVL tree of height 2 must have at least two internal nodes

## Find the minimum number of nodes for height, h:

- The root of a tree with height, h will have as its childrens subtrees as AVL trees
  - What are the possible values for the height of its children subtrees?

- With the minimum number of nodes for height, h, the root of such a tree will have as its childrens subtrees as AVL trees
  - with the minimum number of nodes for height h - 1 and
  - an AVL tree with the minimum number of nodes for height h - 2
- Why?
- Draw the structure of such an AVL tree.

- In general for h>=3, $n(h) = 1 + n(h-1) + n(h-2)$
- The n(h) values are strictly increasing as h increases, in a way that corresponds to the Fibonacci sequence
- Note that, $n(h-1) >= n(h-2)$, we approximate here by applying this in the above equation, i.e
- $n(h) >= n(h-2) + n(h-2) + 1$

$$> 2\, n(h-2)$$

$$> 4\, n(h-4)$$

$$> 8\, n(h-6)$$

$$\ldots > 2^i\, n(h - 2i)$$

- When $i = h/2 - 1$ (even integer), $n(h) > 2^{(h/2 - 1)}\, n(2)$
- $n(h) > 2^{h/2}$, when we substitute $n(2) = 2$

AVL tree of height h has at least $2^{h/2}$ internal nodes (Min), Max=$2^h$

Taking Log on both sides,

$h/2 < \log n(h)$

$h < 2 \log n(h)$

$h = O(\log n)$

n - Number of nodes in an AVL tree

n(h) - Minimum number of internal nodes in an AVL tree of height h

Note that, $n > n(h)$

$h = O(\log n)$

Thus, the height of an AVL tree is, $O(\log n)$

# Sharper bound on the height of an AVL tree

- ⋆ Let $m_h$ be the minimum number of nodes in an AVL tree of height h
- ⋆ We have $m_h = 1 + m_{h-1} + m_{h-2}$ (with $m_0 = 1$ and $m_1 = 2$)
- ⋆ An approximate function is the fibonacci function $f_i = f_{i-1} + f_{i-2}$
- ⋆ $m_h = \phi^{h+3}/\sqrt{5} - 1$ where $\phi$ is $(1 + \sqrt{5})/2$
- ⋆ $\log_\phi(m_h) \approx (h+3) - \log_\phi \sqrt{5}$
- ⋆ $\log_2(m_h)/\log_2(\phi) \approx (h+3) - 1.6723$
- ⋆ $\log_2(m_h)/\log_2(\phi) \approx (h+3) - 1.6723$
- ⋆ $\log_2(m_h)/\ 0.694 \approx (h+3) - 1.6723$
- ⋆ $1.44 \log_2(m_h) \approx h + 1.3277$
- ⋆ $h \approx 1.44 \log_2(m_h) - 1.3277$

# Structure of an AVL tree

**Consider an AVL tree on n nodes.**

**Consider a leaf node which is closest to the root node.**

**Suppose this leaf node is at level k.**

**We will show that height of the tree is at most 2k-1.**
**(considering the ht of the leaf node as 1)**

When k = 2, ht = 3

k = 3, ht = 5

k = 4, ht = 7

# Another Property of an AVL tree

Since the closest leaf is at level k, all nodes at levels k - 2 have 2 children.

Prove this by contradiction.

Exercise

- In an AVL tree of height h, the leaf closest to the root is at level at least (h+1) / 2.
- On the first (h-1)/2 levels the AVL tree is a complete binary tree.
- After (h-1)/2 levels the AVL tree may start thinning out.
- Number of nodes in an AVL tree
  - $2^{(h/2 - 1)} <= n <= 2^h$

# Smallest AVL tree of height 9: