Problem Set 1: Kinematics & Newtonian Dynamics

Richie Ma
University of Illinois, Urbana-Champaign, Illinois, 61820

Question 1

(20 points) An airplane seems to be in trouble. It is spiralling from an altitude of 10km toward the ground at a constant rate of 100 m/s. As seen from below, the plane performs a circular motion with radius of 1km and a period of $6\pi \approx 18.3$ seconds.
(a) (10 points) What are the jet's position, velocity, acceleration and jerk? Derive the corresponding vector quantities!
(b) (5 points) Plot position, velocity and acceleration as function of time t!
(c) (5 points) How much acceleration will the passengers and pilots of this plane have to contend with? Interpret your findings!

(a)

- The plane performs a circular motion with a constant radius vector of $r = 1km = 1000m$.
- The period is given as $T = 6\pi$, which allows us to calculate the angular velocity ω as $\omega = \frac{2\pi}{T} = \frac{2\pi}{6\pi} = \frac{1}{3}$.
- The position vector of a circle is $\vec{r}(t) = r\cos(\omega t)\hat{\imath} + r\sin(\omega t)\hat{\jmath}$.
- The plane is falling from a height of 10km or 10,000 m at a constant rate of 100 m/s, which can be represented as: $\vec{r} = 10000 - 100t$.

Combining the above elements, we can write a position vector of:
$\vec{r}(t) = 1000\cos(\frac{1}{3}t)\hat{\imath} + 1000\sin(\frac{1}{3}t)\hat{\jmath} + (10000 - 100t)\hat{k}$,
which in turn lets us find velocity and acceleration and jerk by taking derivatives of $\hat{r}$ with respect to time:
$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{-1000}{3}\sin(\frac{1}{3}t)\hat{\imath} + \frac{1000}{3}\cos(\frac{1}{3}t)\hat{\jmath} - 100\hat{k}$
$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{-1000}{9}\cos(\frac{1}{3}t)\hat{\imath} - \frac{1000}{9}\sin(\frac{1}{3}t)\hat{\jmath} + 0\hat{k}$
$\vec{j}(t) = \frac{d\vec{a}}{dt} = \frac{1000}{27}\sin(\frac{1}{3}t)\hat{\imath} - \frac{1000}{27}\cos(\frac{1}{3}t)\hat{\jmath} + 0\hat{k}$

(b)

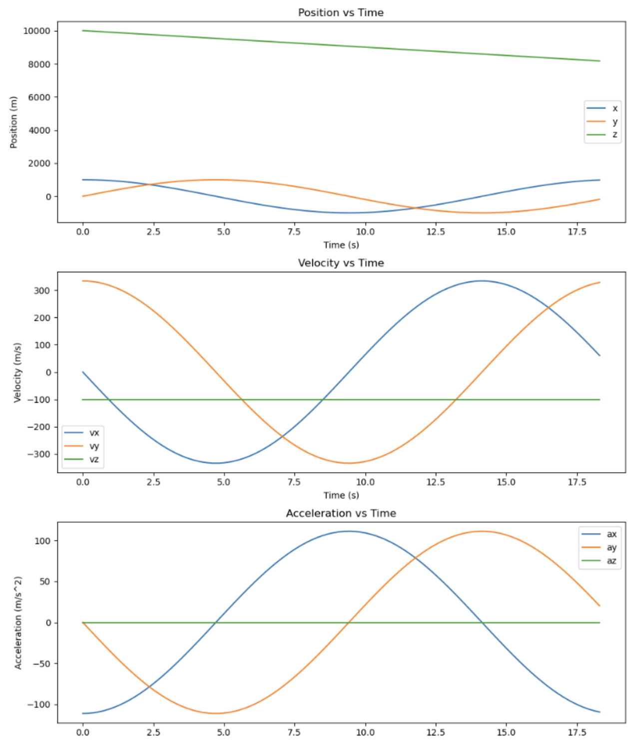


Fig. 1 Position, Velocity, and Acceleration

(c)

Since there is seen to be no acceleration in the $\hat{k}$ direction, the acceleration felt by the pilots and passengers can be represented as $|a| = \sqrt{(\frac{-1000}{9}\cos(\frac{1}{3}t))^2 + (\frac{-1000}{9}\sin(\frac{1}{3}t))^2}$. Evaluating the acceleration at time $t = 0$, we find that the acceleration felt by the passengers is $|a| = \sqrt{(\frac{-1000}{9})^2} = \frac{1000}{9} \approx 111.1\ m/s^2$ which seems to be an high acceleration. Converting this number to G's felt by dividing it by 9.807, we get a total G's felt by the pilots and passengers of 11.33. Most people probably pass out at around 4 to 5 G's, so unless the pilots were ex-military fighter pilots, the whole plane is asleep.

Question 2

(20 points) Have you ever wondered why operators do not let anyone near the rear end of a jet engine when boarding a plane? A Boeing 777-300ER jet engine produces roughly 115,000 lbf of thrust.

(a) (10 points) What acceleration would a person with a mass of 80 kg and a body-cross section of 7.5 ft2 if they were standing right behind the jet engine? For the sake of simplicity assume the entire thrust produced by the jet engine impacts the person.

(b) (10 points) How far would the person be flung back due to the above acceleration? Assume the center of mass of the person is 1 m above ground level and the thrust vector only has horizontal components. Furthermore assume that the person is standing still before being swept off their feet by the jet engine, and that the person is not tumbling and that they will stop once the center of mass hits the ground.

(a)

Given the thrust force is 115,000 lbf, or converting to Newtons 511520 N, using the equation $F = ma$, to evaluate the acceleration, we get $511545.48 = 80 \times a$. Which gives us an acceleration of 6394 m/s^2.

(b)

Assuming the center of mass of the person is 1 meter above ground, and that they stop once the center of mass hits the ground, and knowing that the engine only produces horizontal thrust components, the person will free fall for 1 meter. The time it takes for the person to free fall from a height of 1 meter can be calculated using the equation $h = \frac{1}{2}gt^2$. because, they are at ground level, we substitute the appropriate values into the equation to get $1 = \frac{1}{2}(9.81)t^2 = 4.905t^2$. $t = \sqrt{\frac{1}{4.905}} = 0.452\ s$ Using the time free-falling, we can use the equation $s = v_0 t + \frac{1}{2}at^2$ to calculate the distance the person is swept off their feet. Plugging in appropriate values with initial velocity $v_0 = 0$, we get $s = 0 + \frac{1}{2}(6394)(0.452)^2 = (3197)(0.204) = 653.16\ m$. Which suggests that the person will be flung back a total of 653 meters before hitting the ground again.

Question 3

(15 points) You've seen an amazing video and want to try skydiving. After jumping out of an airplane from an altitude of 3 km above Urbana-Campaign, unfortunately, your parachute does not deploy. In order to assess your chances of surviving the impact you decide to do a little math. Luckily you remember that the drag force can be described as

$$D = \frac{1}{2}\rho v^2 C_D A$$

where ρ is the density of air, v is your velocity, C_D is the drag coefficient, and A is the cross section area. Assuming you have a mass of 75 kg, a cross section of 0.18 m^2, a drag coefficient of 0.7, and the average air density over UIUC is $1.2 kg/m^3$.
(a) (5 points) Draw a correct Free Body Diagram!
(b) (5 points) What is your terminal velocity v_T , i.e. the speed at which gravity and air friction compensate each other?
(c) (5 points) What acceleration would your body experience, when you hit the ground at terminal velocity and decelerate to zero velocity in 1/100th of a second?

(a)

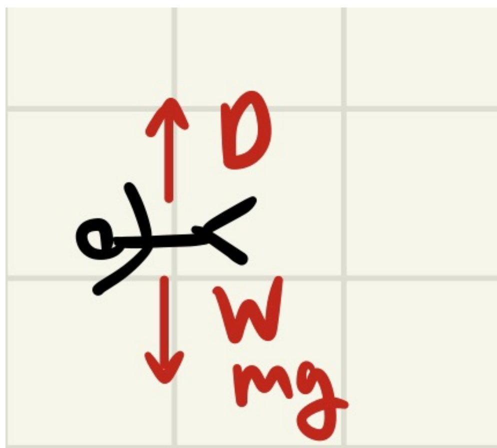


Fig. 2 Free Body Diagram of Skydiver

(b)

At terminal velocity, the net force on the falling object is zero. Using the free body diagram above, we can see the forces that oppose each other have to form the equation $mg = \frac{1}{2}\rho v_T^2 C_D A$ at terminal velocity. Plugging in values, we rearrange the equation to solve for terminal velocity $v_T = \sqrt{\frac{2mg}{\rho C_D A}} = \sqrt{\frac{1471.5}{0.1512}} = 98.65\ m/s$.

(c)

Acceleration can be written in the the form $a = \frac{\Delta v}{\Delta t}$, which gives us a deceleration of $a = \frac{0-98.65}{0.01} = -9865.16\ m/s^2$

Question 4

(20 points) After pulling the rip cord repeatedly, your parachute finally deploys at an altitude of 1km above ground. This increases your drag coefficient to 1.75. The canopy of your chute has an effective area of 20 m^2. You are saved. Or are you?

(a) (5 points) What is your new terminal velocity v_T ?

(b) (10 points) Assuming before you deployed the parachute your v_T = 100 m/s, do you reach the new terminal velocity v_T before you hit the ground? Determine how long it takes to decelerate to your new terminal velocity!

(c) (5 points) What acceleration would your body experience, when you hit the ground at terminal velocity VT and decelerate to zero velocity in 1/6th of a second?

(a)

Using the same equations as above, $v_T = \sqrt{\frac{2mg}{\rho C_D A}}$, but plugging in new values, we write $v_T = \sqrt{\frac{2mg}{\rho C_D A}} = \sqrt{\frac{2 \cdot 75 \cdot 9.81}{1.2 \cdot 1.75 \cdot 20}} = 5.92\ m/s$.

(b)

$a = \frac{dv}{dt} = A - Bv^2 = g - \frac{1}{2}\frac{\rho c_D A v^2}{m}$ integrating the equation on both sides, we get $\int_{t_1}^{t_2} dt = \int_{v_1}^{v_2} \frac{1}{g - \frac{1}{2}\frac{\rho c_D A v^2}{m}} dv$, which gives us a time of 2.88 seconds, which means we reach terminal velocity after 2.88 seconds.

$a = \frac{v_2 - v_1}{t} = \frac{5.92 - 100}{2.88} = -32.67$

The equation $v_2^2 = v_1^2 + 2ah$ gives us $5.92^2 = 100^2 - 2(32.67)h$, which results in a height of 152.47 meters. Because it takes only 152.47 meters to reach our new terminal velocity, and we have 1000 meters of room from the ground, we reach terminal velocity before hitting the ground.

(c)

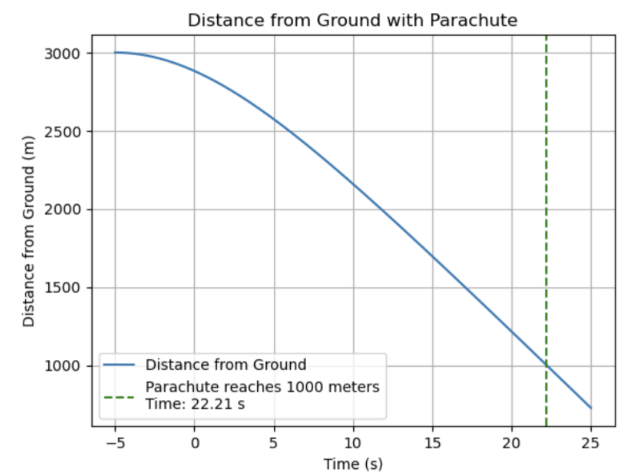
Similar to Question 3, acceleration can be written in the the form $a = \frac{\Delta v}{\Delta t}$, which gives us a deceleration of $a = \frac{0 - 5.92}{0.1667} = -35.51\ m/s^2$

Question 5

(25 points) You are experiencing strong cross winds (100 km/h) during the last 500 m of your decent and you drift due East. The plane you jumped off of had a ground speed of 200 km/h due North. The effective area of you and your parachute with respect to horizontal wind is 5 m^2.
(a) (15 points) Determine and plot your descent trajectory q(t)!
(b) (10 points) How far off your original landing point will you have drifted?

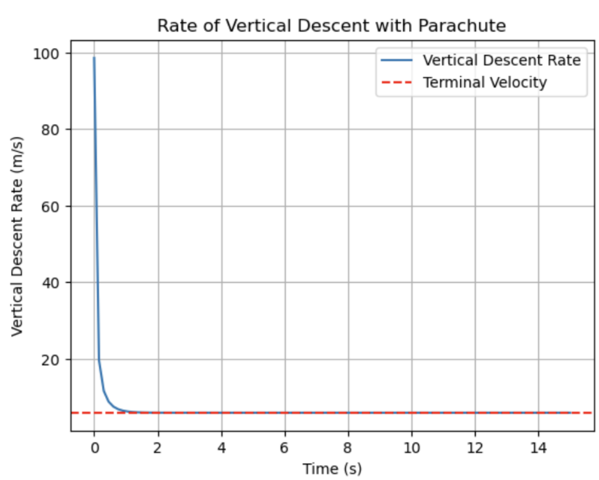
(a)

The skydiver initially jumps out of a plane at 3000 meters. They accelerate to a terminal velocity of 98.65 seconds and descends towards the ground before finally successfully deploying their parachute at 1000 meters above the ground. The first 2000 meters of their travel can be graphed as:



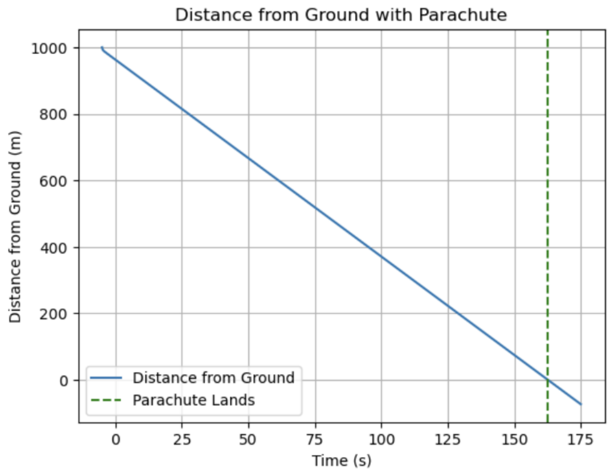
The time it takes for the skydiver to reach 1000 meters is also calculated and found to be 22.21 seconds.

The parachute has a stage of deceleration, which is modelled in question 4. The terminal velocity of the opened parachute is 5.92 m/s. The equation that is used to garner that information is then integrated upon to find the vertical descent rate, or the vertical trajectory of the parachute.



Notice that the initial velocity is somewhere close to 98.65 m/s, which is the free fall terminal velocity without a parachute. The terminal velocity with a parachute is also clearly labelled with a horizontal line. The image is zoomed in to show the details of the deceleration.

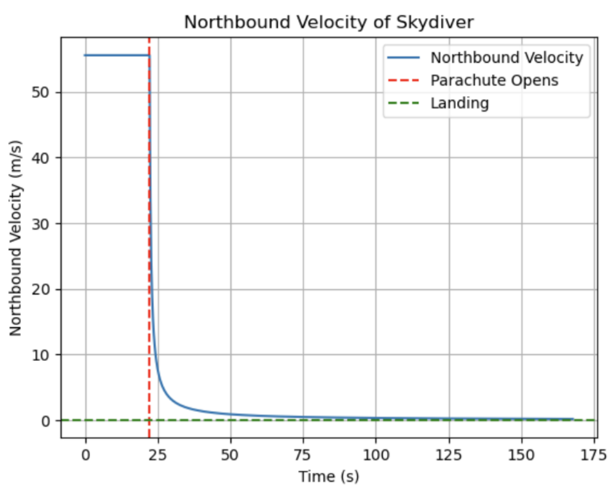
In order to figure out the length of time it takes for the parachute to reach the land, we have to again take the integral of the velocity/rate of vertical descent.



The graph shows that the skydiver reaches the ground at mostly a stable rate of descent, which corresponds to the terminal velocity displayed above, with a slight concave up bend at the beginning where the parachute is decelerating the person. The time it takes for the skydiver to reach the ground from 1000 meters is 167.7 seconds.

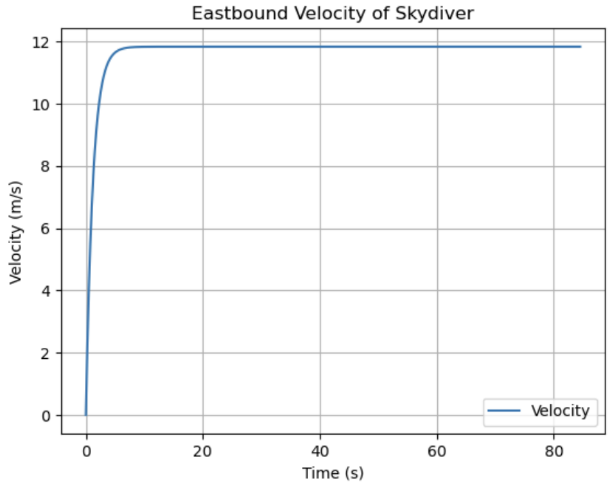
The total time it take for the person to reach the ground from the plane is calculated to be 22.21 + 167.7 = 189.91 seconds. The time it takes for the person to reach the ground from 500 meters is also found to be 84.5 seconds.

The skydiver first jumps out of the plane moving Northbound at 200km/h, which converts to 55.56 m/s. Without opening the parachute, the horizontal cross sectional area is assumed to be too small to make a difference, which means there is no deceleration or acceleration from the original 55.56 m/s. After opening the parachute at 1000 meters from the ground, the cross sectional area changes and the drag force also changes.



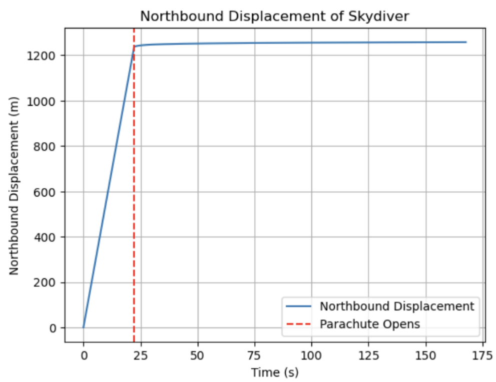
The same is said for the Eastbound velocity, the skydiver experiences no wind for the first 2500 meters, before finally experiencing the 100km/h winds at the last 500 meters. Because there is not initial eastbound movement, and we know the time it takes for the skydiver to travel from 500 meters to the floor, we can plot the velocity eastbound due to acceleration from the 100km/h wind.

As seen in the graph, the skydiver starts to accelerate before reaching a terminal velocity eastbound until they land after 84.5 seconds.



(b)

Integrating the northbound velocity, we get:

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Total Distance Traveled: 1257.84 meters
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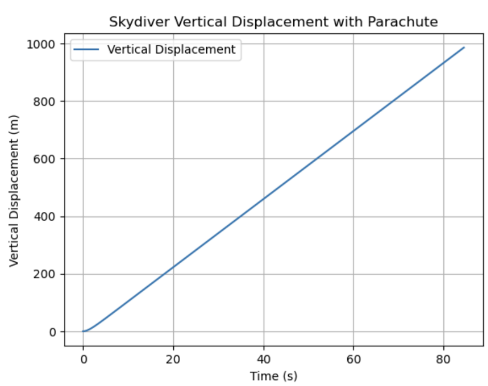
Which also tells use the total displacement Northbound as 1257.84 meters.

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Total Eastbound Displacement: 2347.22 meters
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The Eastbound shows similar results, with total displacement Eastbound of 2347.22 meters.

Example code for rate of vertical descent with parachute:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint

# Constants
mass = 75  # kg
cross_sectional_area = 20  # m^2
drag_coefficient = 1.75
air_density = 1.2
initial_velocity = 98.65  # m/s
initial_height = 1000  # meters
terminal_velocity = 5.92  # m/s

# Define the differential equation
def model(v, t):
    # Calculate drag force
    drag_force = 0.5 * drag_coefficient * cross_sectional_area * air_density * v**2

    # Calculate net force
    net_force = mass * 9.8 - drag_force

    # Calculate acceleration
    acceleration = net_force / mass

    return acceleration

# Time points
t = np.linspace(0, 50, 100)  # Adjust the time range accordingly

# Solve the differential equation
v = odeint(model, initial_velocity, t)

# Plot the rate of descent
plt.plot(t, v[:, 0], label='Vertical Descent Rate')
plt.axhline(y=terminal_velocity, color='r', linestyle='--', label='Terminal Velocity')
plt.xlabel('Time (s)')
plt.ylabel('Vertical Descent Rate (m/s)')
plt.title('Rate of Vertical Descent with Parachute')
plt.legend()
plt.grid(True)
plt.show()
```

Example code for Northbound velocity with parachute:

```
# Constants
mass = 75  # kg
initial_plane_speed = 200 / 3.6  # Initial speed of the plane converted to m/s
parachute_opening_time = 22.21  # seconds
parachute_opening_height = 1000  # meters
parachute_effective_area = 5  # m^2
air_density = 1.2
initial_height = 3000  # meters
terminal_velocity = 98.65  # m/s

# Time points
t1 = np.linspace(0, parachute_opening_time, 1000)  # Time before parachute opening
t2 = np.linspace(parachute_opening_time, 167.7, 1000)  # Time after parachute opening
t = np.concatenate((t1, t2))

# Define the differential equation for northbound velocity before parachute opening
def model1(v, t):
    # Initial acceleration due to the northbound motion of the plane
    plane_acceleration = 0

    # Calculate drag force without cross-sectional area initially
    drag_force = 0

    # Calculate net force considering the northbound motion
    net_force = -drag_force  # Negative sign for northbound direction
    acceleration = net_force / mass

    return acceleration + plane_acceleration

# Define the differential equation for northbound velocity after parachute opening
def model2(v, t):
    # Calculate drag force with effective parachute area
    drag_force = 0.5 * air_density * parachute_effective_area * v**2

    # Calculate net force considering the northbound motion
    net_force = -drag_force  # Negative sign for northbound direction
    acceleration = net_force / mass

    return acceleration

# Solve the differential equations
v1 = odeint(model1, initial_plane_speed, t1)
v2 = odeint(model2, v1[-1, 0], t2)  # Use the final velocity of the first part as initial velocity for the second part
v = np.concatenate((v1, v2))

# Plot the northbound velocity
plt.plot(t, v, label='Northbound Velocity')
plt.axvline(x=parachute_opening_time, color='r', linestyle='--', label='Parachute Opens')
plt.axhline(y=0, color='g', linestyle='--', label='Landing')
plt.xlabel('Time (s)')
plt.ylabel('Northbound Velocity (m/s)')
plt.title('Northbound Velocity of Skydiver')
plt.legend()
plt.grid(True)
plt.show()
```