AE 352, Spring 2024

Problem Set 1: Kinematics & Newtonian Dynamics

1. (20 points) An airplane seems to be in trouble. It is spiralling from an altitude of 10km toward the ground at a constant rate of 100 m/s. As seen from below, the plane performs a circular motion with radius of 1km and a period of $6\pi \approx 18.3$ seconds.

   (a) (10 points) What are the jet's position, velocity, acceleration and jerk? Derive the corresponding vector quantities!

   (b) (5 points) Plot position, velocity and acceleration as function of time t!

   (c) (5 points) How much acceleration will the passengers and pilots of this plane have to contend with? Interpret your findings!

2. (20 points) Have you ever wondered why operators do not let anyone near the rear end of a jet engine when boarding a plane? A Boeing 777-300ER jet engine produces roughly 115,000 lbf of thrust.

   (a) (10 points) What acceleration would a person with a mass of 80 kg and a body-cross section of $7.5\,\mathrm{ft}^2$ if they were standing right behind the jet engine? For the sake of simplicity assume the entire thrust produced by the jet engine impacts the person.

   (b) (10 points) How far would the person be flung back due to the above acceleration? Assume the center of mass of the person is 1 m above ground level and the thrust vector only has horizontal components. Furthermore assume that the person is standing still before being swept off their feet by the jet engine, and that the person is not tumbling and that they will stop once the center of mass hits the ground.

3. (15 points) You've seen an amazing video and want to try skydiving. After jumping out of an airplane from an altitude of 3 km above Urbana-Campaign, unfortunately, your parachute does not deploy. In order to assess your chances of surviving the impact you decide to do a little math. Luckily you remember that the drag force can be described as

$$D = \frac{1}{2}\rho v^2 C_D A \tag{1}$$

   where ρ is the density of air, v is your velocity, C_D is the drag coefficient, and A is the cross section area. Assuming you have a mass of 75 kg, a cross section of $0.18\,\mathrm{m}^2$, a drag coefficient of 0.7, and the average air density over UIUC is $1.2\,\mathrm{kg/m}^3$.

   (a) (5 points) Draw a correct Free Body Diagram!

   (b) (5 points) What is your terminal velocity v_T, i.e. the speed at which gravity and air friction compensate each other?

   (c) (5 points) What acceleration would your body experience, when you hit the ground at terminal velocity and decelerate to zero velocity in $1/100^{th}$ of a second?

4. (20 points) After pulling the rip cord repeatedly, your parachute finally deploys at an altitude of 1km above ground. This increases your drag coefficient to 1.75. The canopy of your chute has an effective area of $20\,\mathrm{m}^2$. You are saved. Or are you?

   (a) (5 points) What is your new terminal velocity V_T?

   (b) (10 points) Assuming before you deployed the parachute your $v_T = 100\,\mathrm{m/s}$, do you reach the new terminal velocity V_T before you hit the ground? Determine how long it takes to decelerate to your new terminal velocity!

(c) (5 points) What acceleration would your body experience, when you hit the ground at terminal velocity V_T and decelerate to zero velocity in $1/6^{th}$ of a second?

5. (25 points) You are experiencing strong cross winds (100 km/h) during the last 500 m of your decent and you drift due East. The plane you jumped off of had a ground speed of 200 km/h due North. The effective area of you and your parachute with respect to horizontal wind is $5\,\mathrm{m}^2$.

(a) (15 points) Determine and plot your descent trajectory $\boldsymbol{q}(t)$!

(b) (10 points) How far off your original landing point will you have drifted?

Good luck!

Figure 1: Skydiver. Image credit: Wikipedia

Grading rubrics:			
Mastery	Grade Level	% of Points	Description
Problem Attempted	F	0-20	An honest attempt at tackling the problem, e.g. creating system diagrams, working through the problem with an incorrect or unclear approach.
Major Errors	F/D/C	20-60	Problem was approached correctly, but major mistakes occurred along the way. Implausible and/or unclear answers. Errors in plots.
Minor Errors	C/B/A	60-99	Problem was approached correctly, no major mistakes, but e.g. round off errors, or a typo, units along plot-axes are missing, etc.
Correct Answer	A+	100	Correct approach, no mistakes along the way. Clear answers are provided in complete sentences. Units are correct. Plots are legible and correctly labeled.

Solutions

Problem 1

(a) Given:

$$\boldsymbol{q}(t) = 1000(\cos(t/3), \sin(t/3), 10 - t/10)^T \, \text{m}.$$

Velocity, acceleration, and jerk:

$$\boldsymbol{v}(t) = \dot{\boldsymbol{q}}(t) = \frac{d}{dt}\boldsymbol{q}(t) = 1000/3(-\sin(t/3), \cos(t/3), -3/10)^T \, \text{m/s},$$

$$\boldsymbol{a}(t) = \ddot{\boldsymbol{q}}(t) = \frac{d}{dt}\boldsymbol{v}(t) = 1000/9(-\cos(t/3), -\sin(t/3), 0)^T \, \text{m/s}^2,$$

$$\boldsymbol{j}(t) = \dddot{\boldsymbol{q}}(t) = \frac{d}{dt}\boldsymbol{a}(t) = 1000/27(\sin(t/3), -\cos(t/3), 0)^T \, \text{m/s}^3.$$

Interpretation: The jet is spiralling downward from 10,000 m at a rate of 100 m/s. The downward motion is not accelerated, hence, some lift must still counteract gravity. However, the spiral is tight, with accelerations of more than 11G for the pilot away from the center of the spiral (centrifugal acceleration). Pilots typically can withstand onset rates of up to 40G, but sustained accelerations beyond 16G can be lethal. This scenario here would not end well for pilots or passengers. Note the phase shift between position and the time derivatives for $t = 0$s.

Graphs: Clear, visual depiction of alternating sinusoidal waves in x and y directions with labeled axes. 3D graphs should have x, y, z axes labeled and indicate function graphed varies with time, see Figure 2. 2D graphs should have both axes labeled, and either separated variables over graphs for clarity or short enough time interval on x axis to clearly read.

Problem 2

(a) What acceleration would a person experience? Assuming the person does not disintegrate

$$\frac{d\boldsymbol{q}}{dt} = \frac{dm}{dt}\dot{\boldsymbol{q}} + m\frac{d\dot{\boldsymbol{q}}}{dt} \stackrel{!}{=} \boldsymbol{F}, \quad \text{with} \, \dot{m} = 0 \, \text{we have} \, m\boldsymbol{a} \stackrel{!}{=} \boldsymbol{F} \tag{2}$$

Since we are on the ground, we have gravity, the normal force of the ground and the jet engine acting on the body to take into account. From a free body diagram we see that the person would experience the force of the jet engine in positive x direction and the force of gravity acting in -z direction, and the normal force opposite to the latter. If $\boldsymbol{a}$ is the acceleration of the center of mass of the person we have

$$m\boldsymbol{a} = \boldsymbol{F}_{jet} + \boldsymbol{F}_g + \boldsymbol{F}_N = \left((511543.0, 0)^T + (0, -784.8)^T + (0, 784.8)^T\right) \text{N}. \tag{3}$$

The acceleration would then be

$$\boldsymbol{a} = \boldsymbol{F}_{jet} + \boldsymbol{F}_g + \boldsymbol{F}_N \approx (6394.29, 0)^T \text{m/s}^2, \tag{4}$$

which corresponds to roughly 651G.

(b) How far would the person be flung back? Determine the time it takes for the center of mass to fall to the ground assuming the person is standing still before being swept off their feet by the jet engine.

$$z(t) = -\frac{1}{2}gt^2 + \dot{z}_0 + z_0, \text{where} \, \dot{z}_0 = 0 \, \text{m/s}, \; z_0 = 1 \, \text{m} \tag{5}$$

For $g = 9.81 \, \text{m/s}^2$, $t_{fall} \approx 0.45$ s. Now insert that time into the equation for the horizontal motion

$$x(t) = \frac{a_{jet}}{2}t^2 + \dot{x}_0 + x_0, \text{where} \, \dot{x}_0 = 0 \, \text{m/s}, \; x_0 = 0 \, \text{m}. \tag{6}$$

Inserting the acceleration in x-direction from above we find $x(t = 0.45\text{s}) = 651.8\text{m}$. If we account for air drag our equation of motion reads as follows:

$$m\ddot{q}_x = ma_{jet} - \frac{1}{2}\,\rho\,C_D\,A\,\dot{q}_x^2 \tag{7}$$

The cross-section is given, $A \approx 0.7\,\text{m}^2$, and we can take $C_D = 0.7$ and $\rho = 1.2\,\text{kg/m}^3$ for the other problems. The solution is worked out in Problem 4, but we will state it here as well.

$$q_x(t) = \ln\left[\cosh\left(\sqrt{ab}\,t\right)\right]/b, \tag{8}$$

where $a = a_{jet}$ and $b = \frac{1}{2m}\rho\,C_D\,A$. Inserting numerical values results in $q_x(t = 0.45\text{s}) = 408.4\text{m}$. Of course, this assumes that the exhaust stream from the jet engine remains collimated over half a kilometer, which is unlikely.

Problem 3

(a) When jumping out of this airplane, there are two major forces acting on you: Your mass along with gravitational acceleration that is your weight, and the air resistance or Drag force opposing your weight. The force balance then becomes:

$$\begin{aligned} \sum_i \mathbf{F}_i &= 0, \\ -W + D &= 0, \\ W &= D, \end{aligned} \tag{9}$$

where W is your weight and D is the Drag. The Free Body Diagram could look like Figure 3.

(b) The Drag force can be described as:

$$D = \frac{1}{2}\rho v^2 C_D A, \tag{10}$$

where ρ is the density of air, v is your velocity, C_D is the drag coefficient, and A is the cross section area. So we get

$$\begin{aligned} W &= \frac{1}{2}\rho v^2 C_D A, \\ v_T &= \sqrt{\frac{2W}{\rho C_D A}}, \\ v_T &= \sqrt{\frac{2 \times 75 \times 9.81}{1.2 \times 0.7 \times 0.18}}\,\text{m/s}, \\ v_T &= 98.651\,\text{m/s}, \\ \boldsymbol{v}_T &= (0, 0, -98.651)^T\,\text{m/s}, \end{aligned} \tag{11}$$

where we used the coordinate system from the Free Body Diagram to generate the velocity vector.

(c) The deceleration when hitting the ground can be modeled as

$$\boldsymbol{a} \approx \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{\boldsymbol{v}_{end} - \boldsymbol{v}_T}{\Delta t} = \frac{\mathbf{0} - (0, 0, -98.651)^T}{0.01}\,\text{m/s}^2 \approx (0, 0, 9865.1)\,\text{m/s}^2. \tag{12}$$

This corresponds to approximately 1000G in the direction opposite to the velocity.

Problem 4

(a) The new terminal velocity $V_T = 5.92\text{m/s}$.

(b) The equations of motion derived via Newton's second law read

$$m\ddot{\boldsymbol{q}} = \boldsymbol{F}_{grav} + \boldsymbol{F}_{drag} \tag{13}$$

Rewriting the above equation in vector components we find that

$$m\ddot{q}_x = -\frac{1}{2}\,\rho\,\dot{q}_x^2 C_{D,x} A_x \tag{14}$$

$$m\ddot{q}_y = \frac{1}{2}\,\rho\,(w_y - \dot{q}_y)^2 C_{D,y} A_y \tag{15}$$

$$m\ddot{q}_z = -mg + \frac{1}{2}\,\rho\,\dot{q}_z^2 C_{D,z} A_z \tag{16}$$

where the x-axis is pointing due North, the y-axis is pointing due West and the z-axis skyward (zenith), and w is the wind speed in the respective direction. Note that in the presence of wind, drag is calculated with respect to the speed of the skydiver relative to the wind-speed. We cannot have the y-axis point due East since we want to retain a Right Hand System. Furthermore, we have $A_{x,y,z}$ and $C_{D,x,y,z}$ the exposed areas and drag coefficients in the directions of motion, respectively. Looking at the differential equations above, we see that they are of second order in q, but q does not appear. Hence, we can rewrite them as first order equations in $v = \dot{q}$ which means we can likely solve some of them through separation of variables. Using the proposed change in variables the system reads

$$\dot{v}_x = -av_x^2 \tag{17}$$

$$\dot{v}_y = b(w_y - v_y)^2 \tag{18}$$

$$\dot{v}_z = -g + cv_z^2 \tag{19}$$

where $a = \frac{1}{2m}\rho A_x C_{D,x}$, $b = \frac{1}{2m}\rho A_y C_{D,y}$, and $c = \frac{1}{2m}\rho A_z C_{D,z}$. These differential equations can be readily solved. The solutions to v_x and v_y are of the form

$$v_x(t) = \frac{v_{x,0}}{1 + av_{x,0}t} \tag{20}$$

$$v_y(t) = \frac{b\,w_y^2\,t - b\,v_{y,0}\,w_y\,t + v_{y,0}}{1 + b\,w_x\,t - b\,v_{0,y}\,t} \tag{21}$$

and the solution for the z-component reads

$$v_z(t) = -\sqrt{\frac{g}{c}}\tanh\left(\sqrt{cg}\,t - d\right), \quad d = \operatorname{arctanh}\sqrt{c/g}\,v_{0,z}. \tag{22}$$

The latter equation is the one we are interested in. Inserting the numerical values and solving for t we find that the deceleration to $V_T = -5.9\,\mathrm{m/s}$ takes about 2 seconds. Another way of getting an order of magnitude estimate is realizing that at a speed of $100\,\mathrm{m/s}$, gravity becomes negligible compare to the drag force produced by the newly deployed parachute. In fact, the drag force is about 300 times larger than gravity when the parachute opens. Only when the terminal velocity is reached are the two forces - by definition - comparable. Hence, one could also have used the approximate equation

$$\dot{v}_z \approx cv_z^2 \tag{23}$$

with solution

$$v_z(t) = \frac{v_{z,0}}{1 + cv_{z,0}t} \tag{24}$$

which can be readily solved for t to produce

$$t_{VT} = \frac{V_T - v_{z,0}}{c\,v_{z,0}\,V_T}. \tag{25}$$

This approximation yields $t_{VT} \approx 0.64\,\mathrm{s}$, which is on the order of a second, so the right order of magnitude. We should be safe.

(c) The new acceleration would be roughly 3.6G in the direction opposite to the velocity.

Problem 5

(a) Since we were able to express the velocities as function of time in equations (20-22), we can now integrate them with respect to time to get $\boldsymbol{q}(t)$.

$$q_x(t) = \ln[1 + av_{0,x}t]/a + q_x(0), \quad (26)$$

$$q_y(t) = w_y t - \frac{w_y}{b(v_{y,0} - w_y)} - \frac{\ln(1 + b(w_y - v_{y,0})t)}{b} + q_y(0), \quad (27)$$

$$q_z(t) = -\frac{\alpha}{\beta} \ln[\cosh(\beta\, t - \operatorname{arctanh}(v_{z,0}/\beta))] + q_z(0), \quad (28)$$

where $\alpha = \sqrt{g/c}$ and $\beta = \sqrt{gc}$. Using those results to calculate our paths, we find for the first part of the descent without parachute:

Path I:

x[m]	y[m]	z[m]	v_x[m/s]	v_y[m/s]	v_z[m/s]	t[s]
0	0	3000	55.56	0	0	0
918 ± 0.5	0	1000	22 ± 0.5	0	-98 ± 0.5	27.2 ± 0.5

The next segment starts with the deployment of the parachute:

Path II:

x[m]	y[m]	z[m]	v_x[m/s]	v_y[m/s]	v_z[m/s]	t[s]
918 ± 0.5	0	1000	22 ± 0.5	0	-98 ± 2	27.2 ± 0.5
988 ± 2	0	500	0.17 ± 0.1	0	-5.92 ± 0.1	110 ± 0.5

The cross wind sets in during the third segment. **Path III**:

x[m]	y[m]	z[m]	v_x[m/s]	v_y[m/s]	v_z[m/s]	t[s]
988 ± 2	0	500	0.17 ± 0.1	0	-5.92 ± 0.1	110 ± 0.5
998 ± 5	-2273 ± 5	0	0.09 ± 0.1	-27.6 ± 1	-5.92 ± 0.1	195 ± 2

The path is displayed in Figures 4 - 7.

(b) The difference in the precalculated path vs the actual path is due to the cross wind. Or descent analysis indicates that we are more than 2km off the original landing site, 2273 ± 5 meters to be precise.

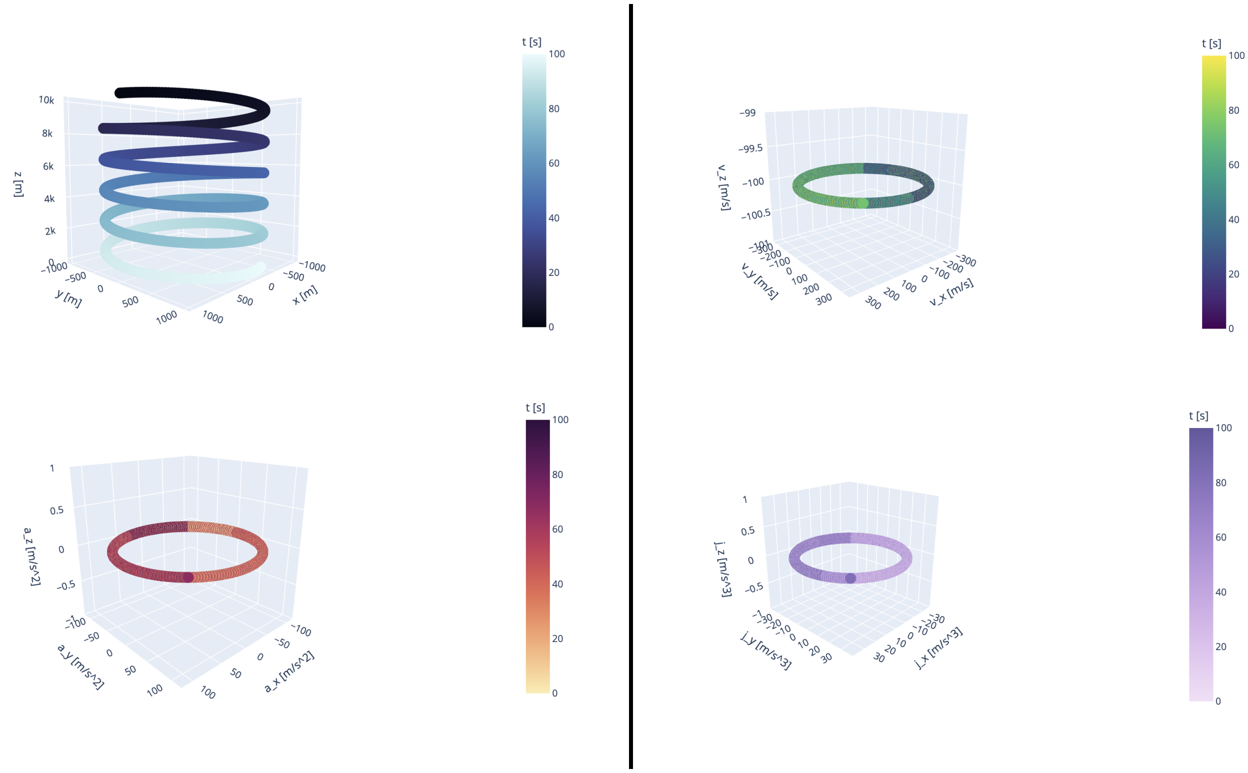

Figure 2: From top left to bottom right: position, velocity, acceleration and jerk as function of time for problem 1.

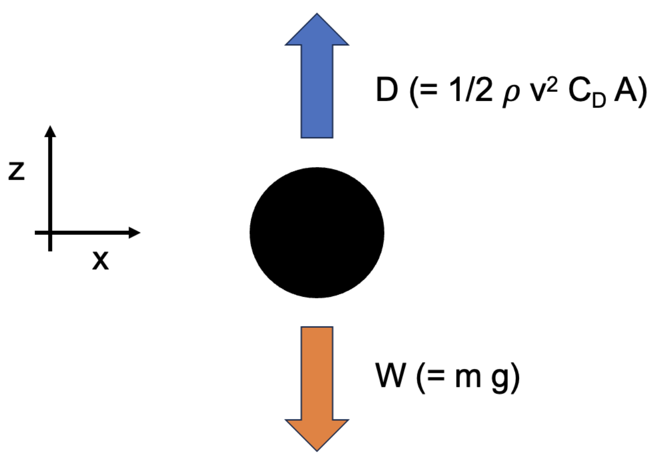

Figure 3: Free Body Diagram of a Skydiver without wind.

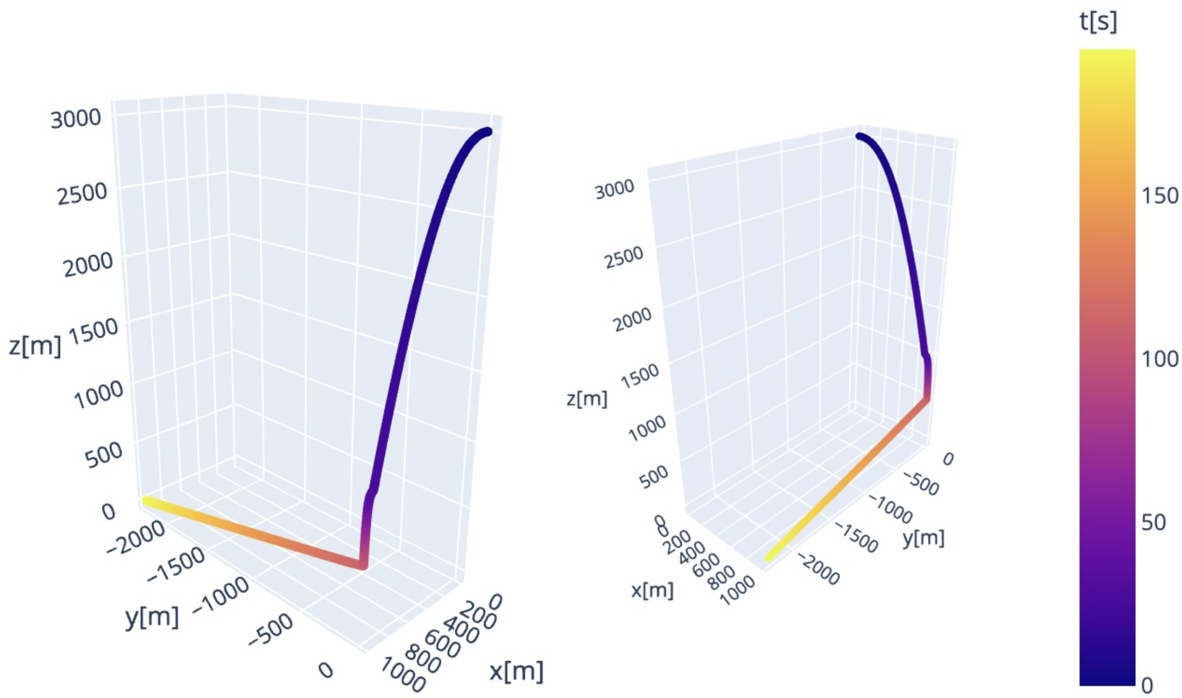

Figure 4: Path of skydiver. Directions: x (due North), y (due West), z (zenith), color code is time.

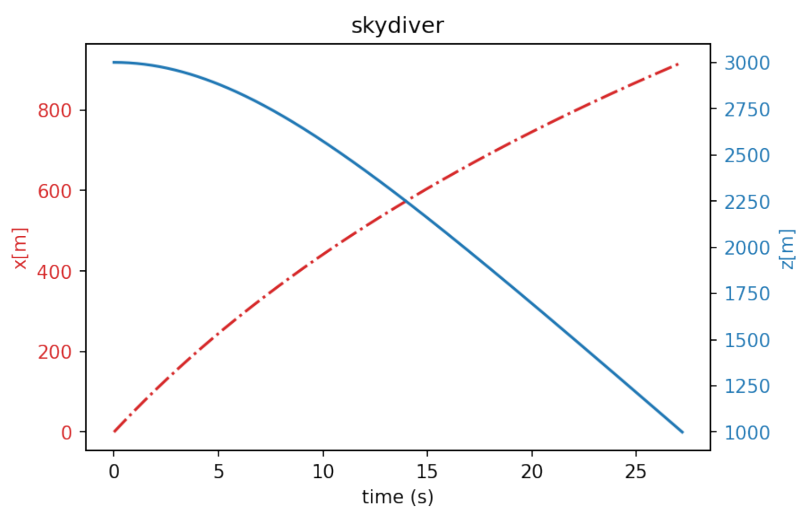

Figure 5: Path I of skydiver. Directions: x (due North), y (due West), z (zenith).

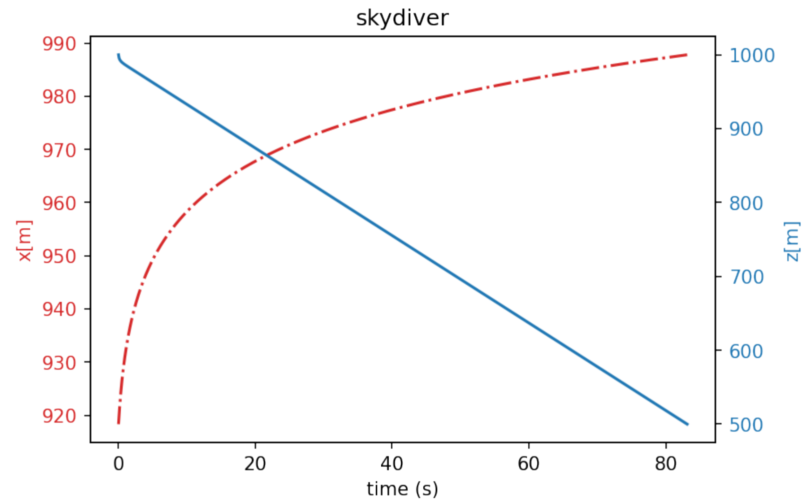

Figure 6: Path II of skydiver. Directions: x (due North), y (due West), z (zenith).

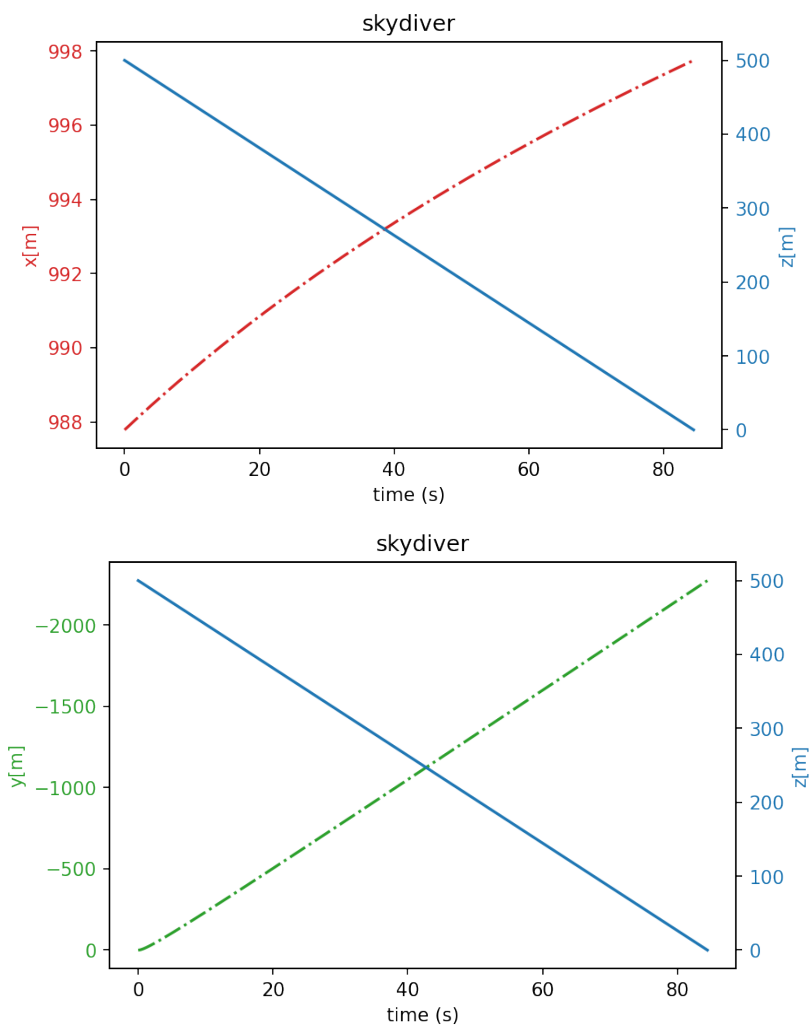

Figure 7: Path III of skydiver. Directions: x (due North), y (due West), z (zenith).