AE 352, Fall 2024

Problem Set 2: Vibration Theory

1. (25 points) The motion of the wingtip of your glider can be modeled as a damped, driven harmonic oscillator of the form

$$m\ddot{z} = -kz - c\dot{z} + d\sin(\omega_f t). \tag{1}$$

The mass of this system is $m = 4\,\mathrm{kg}$, k=10 N/m and you know that c is about one tenth of the system's critical damping coefficient. The forcing amplitude $d = 0.5\,\mathrm{m}$.

   (a) (5 pts) What is the natural frequency of the system without damping ($c = 0$, $d = 0$)?

   (b) (5 pts) What is the frequency of your system with damping, i.e. what is the frequency of the homogeneous solution ($c \neq 0$, $d = 0$)?

   (c) (10 pts) What frequency ω_f would drive your system (1) to a maximum resonant response?

   (d) (5 pts) What is the amplitude ($A = a \cdot d$, i.e. gain x forcing amplitude) of the solution of the above system when driven at the resonance frequency?

2. (25 points) A system of two coupled harmonic oscillators is shown in Figure 1. We will assume gravity and friction are negligible.

   (a) (10 pts) Find the equations of motion for both masses m_1 and m_2!

   (b) (10 pts) What would the eigensystem of the coupling matrix look like?

   (c) (5 pts) Find the eigenfrequencies of this system!

3. (25 points) Carbon monoxide (CO) is diatomic molecule consisting of a carbon atom with a mass of roughly 12 amu and an oxygen atom with a mass of 16 amu. In a simple spring-mass model (Figure 2) the constant of restitution for this setup is $k = 1860\,\mathrm{N/m}$. One atomic mass unit (amu) $= 1.66 \times 10^{-27}$ kg. Damping is negligible.

   (a) (10 pts) What are the equations of motion for this system?

   (b) (5 pts) What is/are the eigenfrequency/frequencies of this system?

   (c) (5 pts) What would the eigenmode(s), $q_i(t)$, look like?

   (d) (5 pts) What wavelength would you tune your laser to in order to excite the molecules eigenfrequency into resonance? The following equation should help,

$$\lambda\,\nu = c, \tag{2}$$

   where λ is the wavelength in meters, $\nu = \frac{\omega}{2\pi}$ is the frequency of the laser in [Hz] and c is the speed of light.

4. (25 pts) The system of three coupled harmonic oscillators presented in Figure 3 has masses and constants of restitution $\{m_i\} = \{1, 2, 3\}$ metric tons and $\{k_i\} = \{3000, 2000, 1000\}\,\mathrm{N/m}$, $i \in \{1, 2, 3\}$, respectively. What are the eigenmodes / mode shapes of this system, if the first mass is displaced by $x_1 = 1\,\mathrm{m}$ and then released? All other initial conditions are equal to zero. Analytic and numeric results are acceptable!

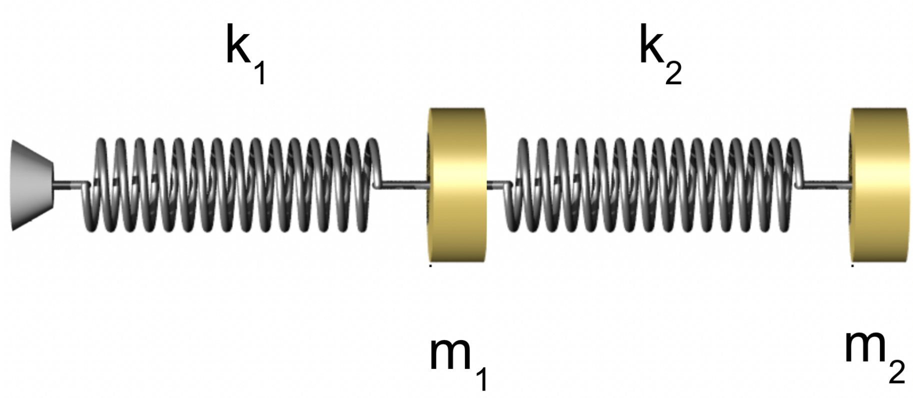

Figure 1: Two coupled springs with masses m_1 and m_2. The left spring connects to m_1 and is attached to a wall. The right spring connects masses m_1 and m_2. The mass m_2 is not connected to a wall. Both m_1 and m_2 can only move along the axis that connects the springs.

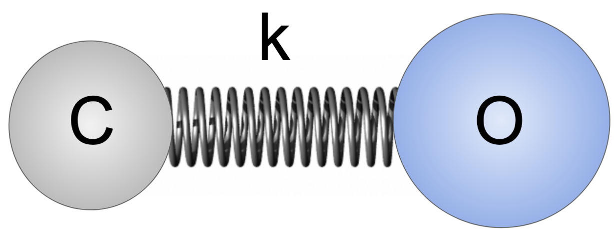

Figure 2: Schematic of a CO molecule.

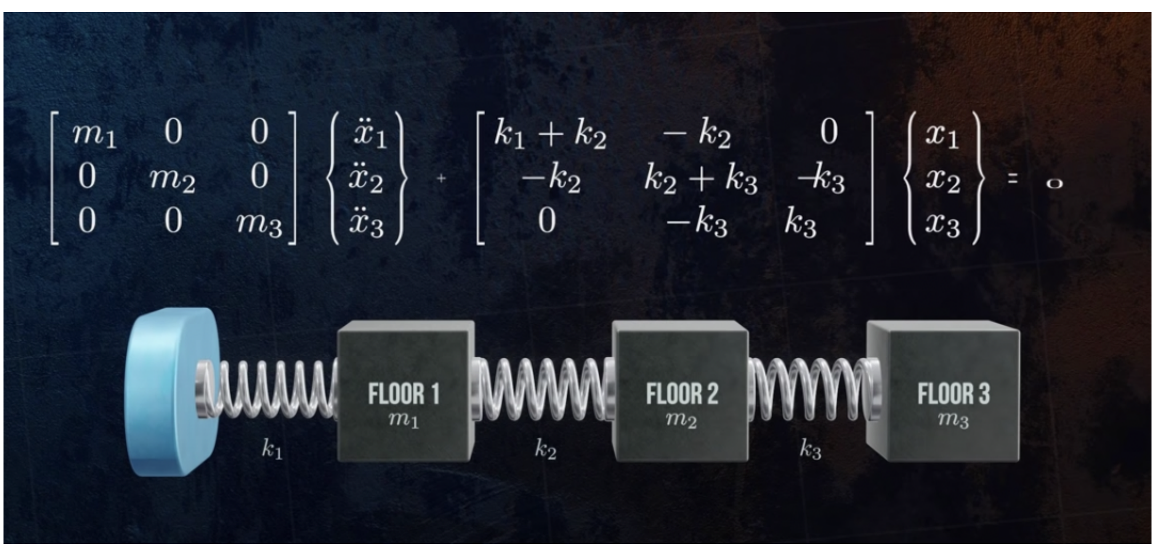

Figure 3: Three coupled oscillators after https://www.youtube.com/watch?v=vLaFAKnaRJU

Solutions

1. The 1D damped, driven harmonic oscillator from problem 1 has the following damping coefficient ($1/10\ c_{crit}$):

$$c = \frac{\sqrt{k\,m}}{5}, \tag{3}$$

(a) natural frequency,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5}{2}}\,\mathrm{Hz} \approx 1.58, \mathrm{Hz} \tag{4}$$

(b) the frequency the system attains without external forcing

$$\omega_0 = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \sqrt{\frac{99}{40}}\,\mathrm{Hz} \approx 1.57, \mathrm{Hz}. \tag{5}$$

In order to (c) find the maximum response we find the minimum of the denominator of amplitude

$$|a| = \left(m^2(k/m - \omega_f^2)^2 + c^2\omega_f^2\right)^{-1} \tag{6}$$

with respect to the forcing frequency ω_f.

$$\frac{da}{d\omega_f} == 0 \tag{7}$$

results in

$$\omega_f = \frac{\sqrt{2km - c^2}}{\sqrt{2}\,m} \approx 1.5653\,\mathrm{Hz}. \tag{8}$$

Finding the pole of the function by setting the denominator equal to zero and taking the real part of the resulting ω_r is also acceptable.

The (d) amplitude of the wingtip oscillation when driven at the maximum response frequency is $A = 0.251\,\mathrm{m}$.

2. The (a) equations of motion for this systems read

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{9}$$

The (b) eigensystem of the coupling matrix is

$$\lambda_{1,2} = -\frac{\mu \pm \sqrt{\Delta}}{2m_1m_2}, \tag{10}$$

where λ are eigenvalues and

$$\mu = k_1m_2 + k_2(m_1 + m_2), \quad \Delta = \mu^2 - 4k_1k_2m_1m_2. \tag{11}$$

and the eigenvectors $\boldsymbol{v}_{1,2}$ read

$$\boldsymbol{v}_{1,2} = \frac{\nu \pm \sqrt{\Lambda}}{2k_2m_1} \tag{12}$$

where

$$\nu = k_2(m_1 - m_2) - k_1m_2, \quad \Lambda = k_1^2m_2^2 + 2k_1k_2m_2(m_2 - m_1) + k_2^2(m_1 + m_2)^2. \tag{13}$$

In order to (c) find the eigenfrequencies we rewrite the system of differential equations in its eigenform

$$\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{14}$$

Inserting the Ansatz $q(t) = ce^{i\omega t}$ we find

$$\begin{aligned}\omega_1 &= \sqrt{-\lambda_1} = i\sqrt{\lambda_1},\\ \omega_2 &= \sqrt{-\lambda_2} = i\sqrt{\lambda_2}\end{aligned}$$

for the eigenfrequencies of the system. The eigenmodes would then read

$$\begin{aligned}q_1(t) &= c_1 e^{-t\sqrt{\lambda_1}},\\ q_2(t) &= c_2 e^{-t\sqrt{\lambda_2}}.\end{aligned}$$

Note that the eigenvalues $\lambda_{1,2}$ can be complex depending on Δ.

3. (a) The equations of motion for the CO molecule are

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -k & k \\ k & -k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{15}$$

(b) The corresponding eigensystem of the coupling matrix reads

$$\lambda_{1,2} = \left(-k\frac{m_1+m_2}{m_1 m_2}, 0\right) \qquad v_1 = \begin{pmatrix} -m_2/m_1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{16}$$

The normal mode ODEs read

$$\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} -k\frac{m_1+m_2}{m_1 m_2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{17}$$

The only non-trivial solution stems from the ansatz $q_1 = c_1 e^{i(\omega t + \phi_1)}$ that, if inserted into the normal mode ODEs gives the eigenfrequency

$$\omega_1 = \sqrt{k\frac{m_1+m_2}{m_1 m_2}} = 4.04 \times 10^{14}\,\text{Hz} \tag{18}$$

(c) The corresponding eigenmodes would then look like

$$\begin{aligned}q_1(t) &= c_1 e^{i(t\times 4.04\times 10^{14}\text{Hz}+\phi_1)} \qquad (19)\\ q_2(t) &= c_2 \qquad (20)\end{aligned}$$

(d) The resonance frequency for the laser is

$$\lambda = 2\pi c/\omega \approx 4.7\mu\text{m}, \tag{21}$$

which lies in the infrared spectrum.

4. The system in matrix form reads

$$M\ddot{\boldsymbol{x}} = -K\boldsymbol{x} \tag{22}$$

Since we have different masses to account for we rewrite the system and redefine the coupling matrix as follows

$$\ddot{\boldsymbol{x}} = A\boldsymbol{x}, \tag{23}$$

With $A = -M^{-1}K$ being the new coupling matrix. For our set of values the eigensystem for A reads

$$\lambda_{1,2,3} = (-5.5036, -1.1751, -0.154624)^T \tag{24}$$

The eigenvectors are columns of the S matrix. The latter reads

$$S = \begin{pmatrix} 0.96961 & 0.437229 & 0.191427 \\ -0.24415 & 0.836177 & 0.463767 \\ 0.0157406 & -0.331118 & 0.865029 \end{pmatrix} \tag{25}$$

The decoupled system of differential equations reads

$$\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{26}$$

Solving this decoupled system with the Ansatz $q(t) = ce^{i\omega t}$ line by line we find

$$\begin{aligned} q_1(t) &= c_1 e^{-it\sqrt{5.5036}}, \\ q_2(t) &= c_2 e^{-it\sqrt{1.1751}}, \\ q_3(t) &= c_3 e^{-it\sqrt{0.1546}}. \end{aligned}$$

In order to make use of the initial conditions we transfrom $\boldsymbol{x}(t=0)$ to $\boldsymbol{q}(t=0)$ via

$$\boldsymbol{q}(0) = S\boldsymbol{x}(0). \tag{27}$$

Since $e^{i\lambda 0} = 1$ we know $\boldsymbol{q}(0) = (c_1, c_2, c_3)^T$. Also,

$$\begin{aligned} S.\boldsymbol{x}(0)^T &= S.(1,0,0)^T && (28) \\ &= (0.96961, -0.24415, 0.0157406)^T && (29) \\ &= (c_1, c_2, c_3)^T. && (30) \end{aligned}$$

Hence, the eigenmodes for this system read

$$\begin{aligned} q_1(t) &= 0.96961\, e^{-it\sqrt{5.5036}}, \\ q_2(t) &= -0.24415\, e^{-it\sqrt{1.1751}}, \\ q_3(t) &= 0.0157406\, e^{-it\sqrt{0.1546}}. \end{aligned}$$

The solution in $\boldsymbol{x}$ coordinates can be acquired through

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = S^{-1} \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix}. \tag{31}$$