AE 352, Fall 2024

Problem Set 4: Everything all at once.

1. (20 points) The CEO of your aircraft manufacturing company decides that, instead of redesigning an aircraft from scratch, it is cheaper to simply exchange engines. Since the new engines have to be mounted slightly differently under the wings, the stability derivative M_q associated with the pitch moment of the aircraft changes from -0.2 to 0.1. The other stability derivatives remain unchanged, namely $X_u = -0.1$, $X_w = -0.05$, $Z_u = -0.1$, $Z_w = -0.2$, $Z_q = -250.1$, $M_u = -0.1$, $M_w = -0.1$ in corresponding units, so that the longitudinal stability of the aircraft is given by

$$\begin{pmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} X_u & X_w & 0 \\ Z_u & Z_w & (Z_q - U_0) \\ M_u & M_w & M_q \end{pmatrix} \begin{pmatrix} u \\ w \\ q \end{pmatrix}. \tag{1}$$

Here, u and w are tiny perturbations in the airspeed along the flight path and vertical to the flight path, respectively, and q is the rate of change of the pitch angle. U_0 is the nominal airspeed of 900 km/h.

   (a) (10 points) Find stable and - if present - unstable equilibrium points and manifolds in this system!

   (b) (5 points) Is simply exchanging engines a good idea? Compare the stability of your aircraft before and after the proposed change!

   (c) (5 points) Write an email to your CEO explaining your analysis professionally, but in layman's terms. Better be convincing.

2. (20 points) The Competitive Lotka-Volterra model can describe the dynamics of two industries competing for the same, limited resource. Consider, for instance, the number of aerospace engineers working in the space sector (x) vs the number of aerospace engineers working in aero (y), the limited resource being Aerospace Engineering graduates. The equations describing the yearly growth rate of the respective fields read something like (x and y are measured in thousands):

$$\begin{aligned} \dot{x} &= x\,(3 - x - 2y), \\ \dot{y} &= y\,(2 - x - y). \end{aligned}$$

   (a) (5 points) Which field is growing faster according to this model? For instance, if left untouched by the other field, which field would grow faster?

   (b) (5 points) Find the coordinates of all equilibrium points!

   (c) (10 points) Determine the stability of each equilibrium point!

   (d) (bonus, 10 points) What can you say about the global behavior of this system away from the equilibrium points?

3. (20 points) You are one of the lucky astronauts aboard the first crewed mission to Mars. Your spacecraft creates artificial gravity through spin in order to mitigate the debilitating effect 0G has on the physical health on your crew. The spacecraft consists of a thin spire with an engine and a massive habitat in the shape of a torus (doughnut) that rotates about the spire, in the mathematically positive sense, see Figure 1. The spacecraft has completed its last trajectory correction burn and is now about half way on a ballistic (coasting) trajectory to Mars.

(a) (5 points) Assuming the torus has a diameter of 500m, how fast would it have to rotate to provide 1g $= 9.81\,\mathrm{m/s^2}$ of artificial gravity for astronauts inside? Give the answer in radians per second!

(b) (5 points) Engineers have designed the spacecraft so that the attitude thrusters are on the outside of the spinning habitat. Assuming the z-axis of your body-fixed frame is pointing along the spire (axis of rotation of your habitat), at which position and in which direction would you have to fire your thrusters in order to yaw your spacecraft to the right?

(c) (5 points) You are in the habitat when you feel a sudden shock and notice that the habitat starts to spin up at a constant rate - about 2 seconds per minute. How long do you have before any of the accelerations you are experiencing in the torus will reach 7g and you start to pass out? Give the answer in seconds!

(d) (5 points) Which of the forces (fictitious or not) in your system of reference do you need to be most concerned about with respect to passing out? Centrifugal, Centripetal, Constant acceleration of the frame, Coriolis, the Sun's gravity, or Euler acceleration? Justify your choice!

4. (20 points) You are on patrol in an F-16 when your wingman starts banking hard right and accelerates to Mach 1.5 speeing away from you at a 90 deg angle. Your current airspeed is 900 km/h.

   (a) (10 points) How quickly can you follow your wingman, i.e. how long does it take you (in seconds) to complete a 90deg turn to your right and accelerate to Mach 1.5 without exceeding 9G? You can safely assume the plane can withstand far higher accelerations than your body.

   (b) (10 points) What is the maximum bank angle you can adopt (see Figure 2) given the 9G constraint and the fact that you do not want to lose altitude during the turn?

5. (20 points) Your company has lost control of a communications satellite. You are asked to recover the satellite from its current "tumbling" state, i.e. $\boldsymbol{\omega})_{body} = (0.1, 0.2, 0.1)$ rad/s. You know the satellite principal moments of inertia which are $I_{xx,yy,zz} = (600, 500, 500)$ kg m^2. Luckily the satellite has three recently de-saturated reaction wheels oriented along the body axes, each of them capable of producing a constant torque of 0.1 Nm. Unfortunately, you only have enough battery power left to run one of them for 10 minutes...

   (a) (10 points) Which one of the reaction wheels should you run to reduce the tumbling of your satellite the most, i.e. minimize the satellite's rotational Kinetic Energy?

   (b) (10 points) Your colleagues are desperately trying to understand where the solar panels are pointing. If you can give them the Euler / Bryan-Tait angles as a function of time over 10 minutes or so they could probably figure it out. Hint: You may assume all angles are initially non-zero when you start your calculations.

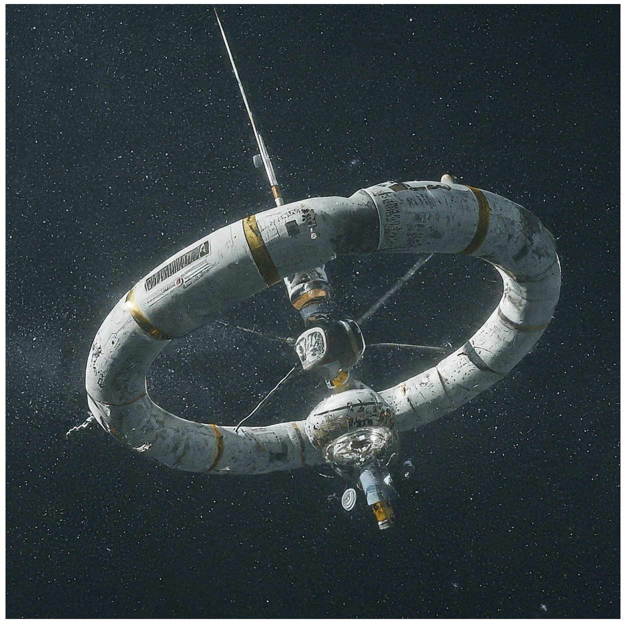

Figure 1: Artist's impression of a spaceship featuring a toroidal habitat.

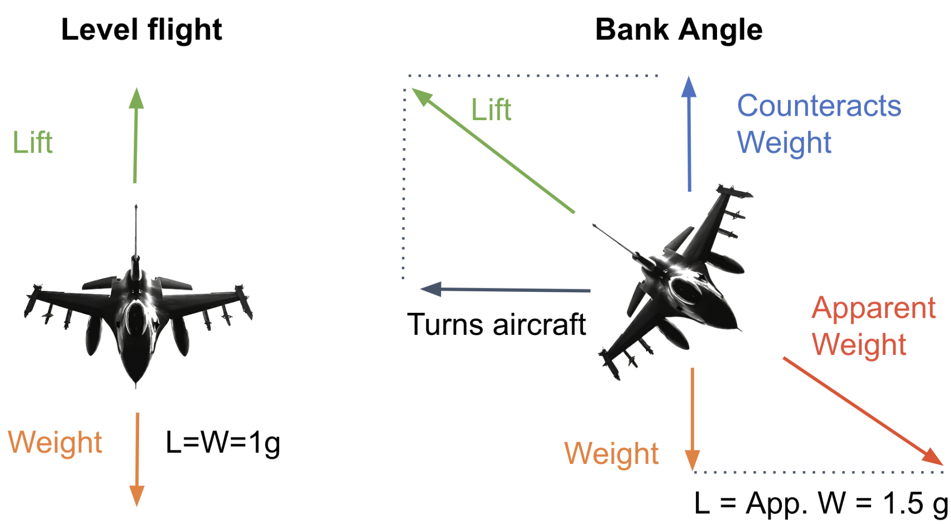

Figure 2: Banking aircraft.

Solutions

1. Problem 1:

   (a) System (1) is linear in u, w, and q and therefore only has one equilibrium point at $(u, w, q)^* = (0, 0, 0)^T$. In order to assess the stability and find the associated manifolds we have to insert the numerical values into the coupling matrix. Let us start with the old aircraft, i.e. $M_q = -0.2$. The Eigenvalues and associated Eigenvectors are

   $$\lambda_{1,2,3} = (-0.32, -0.1, -0.08) \tag{2}$$

   and

   $$\begin{aligned} \boldsymbol{v}_1 &= (0.156949, 0.698343, 0.698343)^T, \\ \boldsymbol{v}_2 &= (0.707107, 0, 0.707107)^T, \\ \boldsymbol{v}_3 &= (0.843925, -0.379336, -0.379336)^T. \end{aligned} \tag{3}$$

   The general solution to this problem is of the form

   $$\boldsymbol{x} = A_1 e^{\lambda_1 t} \boldsymbol{v}_1 + A_2 e^{\lambda_2 t} \boldsymbol{v}_2 + A_3 e^{\lambda_3 t} \boldsymbol{v}_3, \tag{4}$$

   where $\boldsymbol{x}(t) = (u, w, q)^T(t)$ and $A_{1,2,3}$ are amplitudes that depend on the exact initial conditions of the ODE. We can see that the amplitudes and exponentials merely scale the eigenvectors without changing their direction. We can, thus, identify three manifolds as.

   $$\begin{aligned} \mathcal{W}_1 &= A_1 e^{\lambda_1 t} \boldsymbol{v}_1, \\ \mathcal{W}_2 &= A_2 e^{\lambda_2 t} \boldsymbol{v}_2, \\ \mathcal{W}_3 &= A_3 e^{\lambda_3 t} \boldsymbol{v}_3. \end{aligned} \tag{5}$$

   Since all eigenvalues $\lambda_{1,2,3} = (-0.32, -0.1, -0.08)$ are negative in this case, all manifolds are stable manifolds, i.e. $\mathcal{W}_i = \mathcal{W}_i^s$ for $i = 1, 2, 3$. This makes the sole equilibrium point a sink and the system stable.

   (b) Changing the engine, i.e. $M_q = -0.2 \to M_q = 0.1$ yields

   $$\lambda_{1,2,3} = (-0.265754, 0.125649, -0.0598953) \tag{6}$$

   and

   $$\begin{aligned} \boldsymbol{v}_1 &= (0.273367, 0.906232, 0.322512)^T, \\ \boldsymbol{v}_2 &= (0.0691781, -0.312199, 0.947495)^T, \\ \boldsymbol{v}_3 &= (0.77646, -0.622795, 0.0961041)^T, \end{aligned} \tag{7}$$

   which means the system has become unstable due to $\lambda_2 > 0$. The corresponding manifold $\mathcal{W}_2$ has turned into an unstable manifold $\mathcal{W}_2^u$. The system is now classified as a saddle with two stable dimensions and one unstable dimension along $\mathcal{W}_2^u$.

   (c) The email should be professional in tone, explain the procedure and the impact that changing the engine has on the aircraft stability.

2. Problem 2:

   (a) Which field grows (or shrinks) faster depends on the state of the system. If the system were decoupled, i.e.

   $$\begin{aligned} \dot{x} &= x\,(3 - x), \\ \dot{y} &= y\,(2 - y). \end{aligned}$$

   it is pretty clear that the space field would grow faster. But that is not the case for all states of the system. Figure 3, for instance, shows that.

(b) Equilibrium points are found by simultaneously solving the equations

$$\begin{aligned} x\,(3-x-2y) &= 0, \\ y\,(2-x-y) &= 0. \end{aligned}$$

They are located at $(x^*, y^*)_i = (0,0), (3,0), (0,2), (1,1)$.

(c) The corresponding eigenvalues are

i. $(x^*, y^*)_i = (0,0)$, $\lambda_{1,2} = (3,2)$, unstable

ii. $(x^*, y^*)_i = (3,0)$, $\lambda_{1,2} = (-3,-1)$, stable

iii. $(x^*, y^*)_i = (0,2)$, $\lambda_{1,2} = (-2,-1)$, stable

iv. $(x^*, y^*)_i = (1,1)$, $\lambda_{1,2} = (-1-\sqrt{2}, -1+\sqrt{2})$, unstable

(d) Figure 3 illustrates the global behavior of the system. If there are too few aerospace graduates to satisfy industry needs (lower left corner), the system encourages more students to graduate - all arrows flow to larger numbers in space (x) and aero (y). If the system is over-saturated, i.e. there are too many graduates to begin with, the number of all graduates decreases (upper right corner). The unstable equilibrium at (1,1) suggests that it is unlikely that an equal number of graduates are interested in space and aero and the program either tends to produce more aero or more space graduates. The stable equilibria indicate that once the industry decides to focus on space, equilibrium at (3,0) or aero equilibrium at (2,0), the other sector de-facto becomes extinct and is very difficult to revive (stable equilibria).

3. Problem 3:

(a) The distance to the center of the habitat ring is $d/2 = q = 250\,\mathrm{m}$. Since the habitat is spinning at a constant rate the Euler acceleration is zero. If astronauts do not run in the habitat the Coriolis effect is negligible. The artificial gravity should be working regardless of whether or not folks are moving inside the habitat, so we need it to be caused by the centrifugal acceleration experienced by the astronauts in the rotating habitat. Since we are coasting in deep space all external (inertial) forces are also equal to zero. Hence, $a_{rel} + a_{frame} = 0$ and

$$\boldsymbol{a}_{rel} = -\boldsymbol{a}_{frame} = -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{q}) \tag{8}$$

If we choose our coordinate system such that the spire is along the z-Axis, the habitat rotates around that axis. Let's then check the direction of this acceleration. If $\boldsymbol{\Omega} = (0,0,\omega)^T$ and $\boldsymbol{q} = (q,0,0)^T$ then $\boldsymbol{\Omega} \times \boldsymbol{q} = (0, \omega q, 0)$. And

$$a_{rel=} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{q}) = (\omega^2 q, 0, 0)^T, \tag{9}$$

The acceleration points outward away from the center of rotation, which is exactly what we want to simulate gravity. Folks have to stand with their heads toward the spire. As we only have one non-zero component in the acceleration, we can use that component to calculate the necessary spin rate

$$\omega = \sqrt{\frac{9.81\,\mathrm{m/s^2}}{250\,\mathrm{m}}} = 0.198 \text{ rad/s}. \tag{10}$$

This corresponds to a spin period of roughly $P_{1g} = 31.7$ seconds.

(b) If you are not sure what this problem was about check out this video. We assume each thruster is on the outside of the habitat, their trajectory is $\boldsymbol{q}(t) = q(\cos\omega t, \sin\omega t, 0)^T$. This means the angular momentum of the station is

$$\boldsymbol{L} = m(\boldsymbol{q} \times \dot{\boldsymbol{q}}), \tag{11}$$

where m is the mass of the station. We want e.g.

$$\dot{\boldsymbol{L}} = (0,1,0)\,\mathrm{Nm} \tag{12}$$

Hence we need to find the torque in the that would get us there.

$$\dot{\boldsymbol{L}} = m(\boldsymbol{q} \times \boldsymbol{a}), \tag{13}$$

where $\boldsymbol{a}$ is the acceleration caused by the thruster. Since we only care about the y coordinate of this vector equation we find

$$\dot{\boldsymbol{L}}_y = -a_z q \cos \omega t. \tag{14}$$

Hence the best result will be obtained when the thruster is fired in z direction to obtain thrust in -z direction when the thruster is on the positive x axis. The same effect would be accomplished if the thruster would fire in the opposite direction on the other side of the habitat, and if there is redundancy in that thrusters are separated by 180 degrees on the habitat we would want to fire both thrusters at the same time, on in z and one in -z direction. Of course, one could avoid this problem entirely, if there was a counterspinning section on the spacecraft that would cancel the habitat's angular momentum. In that case, thrusters can simply be placed on the spire and we would not have to worry about the transport theorem.

(c) We can use the same equation as in b) to estimate the spin rate that would lead to 7g acceleration

$$\omega_{7g} = \sqrt{\frac{7 \cdot 9.81\,\mathrm{m/s^2}}{250\,\mathrm{m}}} = 0.524\ \mathrm{rad/s}, \tag{15}$$

which is a spin period of roughly $P_{7g} = 12$ seconds. We can then find the time we have before passing out through

$$\frac{P_{1g} - P_{7g}}{\dot{P}} = 591.9\,\mathrm{s}, \tag{16}$$

so you have pretty much exactly 10 minutes.

(d) In order to judge which forces we have to be most concerned we should calculate the Euler force in the system. For that we need $\dot{\omega}$. We know the speed-up of the habitat rotation

$$\dot{\omega} = \frac{0.524 - 0.198}{591.9}\ \mathrm{rad/s^2} \approx 0.00055\,\mathrm{rad/s^2} \tag{17}$$

The magnitude of the Euler acceleration is thus

$$||\dot{\boldsymbol{\Omega}} \times \boldsymbol{q}|| = \dot{\omega} q = 0.14\,\mathrm{N/kg}. \tag{18}$$

In contrast the magnitude of the centrifugal acceleration at the end is

$$\omega^2 q \approx 68\,\mathrm{N/kg}, \quad a.k.a.\, 7\,\mathrm{g}. \tag{19}$$

In order for the Coriolis effect to matter the astronauts would have to run in the habitat. To find out how fast they would have to move in order to reach the same magnitude as the centrifugal acceleration we can equate

$$\omega^2 q = 2\omega v_{rel} \tag{20}$$

so that

$$v_{rel} = \omega q/2 \approx 65.5\,\mathrm{m/s}, \tag{21}$$

about 6 times faster than the fastest human sprinter on Earth. We are assuming we are in deep space, so Earth's gravity does not matter. Even if we were not, 1g on the surface of the Earth is still 7 times less than the 7g we face in the habitat.

4. Problem 4:

(a) This is another classical application of the "Transport Theorem". The acceleration experienced by the pilot during the turn can be written as

$$\boldsymbol{a}_r = \boldsymbol{a}_I - \omega \times \boldsymbol{v}, \tag{22}$$

where $\boldsymbol{a}_r$ and $\boldsymbol{a}_I$ are accelerations in the rotating and inertial frame, respectively, ω is the angular velocity of the turn and $\boldsymbol{v}$ the current airspeed. Here, we have made our life "easy" by assuming the plane maintains its velocity during the turn. With $\boldsymbol{a}_I = (0, 0, -g)$, $\boldsymbol{v}_I = (v_x, 0, 0)$, and $\omega = (0, 0, -\omega_z)$ (no change in altitude) and $\|\boldsymbol{a}_r\| \leq 9\text{G}$ we find

$$|\omega_z| = \sqrt{(9g)^2 - g^2}/v_x \approx 0.35 \text{ rad/s}. \tag{23}$$

At constant velocity, it takes $t_{turn} = \frac{\pi}{2}/\omega_z \approx 4.5\,\text{s}$ to turn 90 degrees to the right. After finishing the turn we need to speed up to Mach 1.5 again adhering to the 9G acceleration limit. The total acceleration reads

$$\boldsymbol{a}_{tot} = (\frac{\Delta v}{\Delta t}, 0, -g) \tag{24}$$

with the condition that

$$\|\boldsymbol{a}_{tot}\| \leq 9g \tag{25}$$

We can solve the above equation for Δt so that

$$\Delta t = \frac{\Delta v}{4\sqrt{5}g}. \tag{26}$$

Then, with $\Delta v = (1.5 \cdot 343 - 250)\,\text{m/s}$

$$t_{lin} = \Delta t \approx 3.01\,\text{s}. \tag{27}$$

The total maneuver time amounts to $t_{tot} = t_{turn} + t_{lin} \approx 7.5\,\text{s}$.

(b) According to Figure 2 and assuming a z axis pointing away form the ground, we can calculate the bank angle β from the y and the negative z components of $\boldsymbol{a}_r = (0, -v_x\omega_z, -g)$. We need an acceleration of $+g$ in the z component of the lift vector because lift in the z direction still has to compensate for gravity - we don't want to lose altitude. We also calculated in the previous example, that we can only turn at a rate of $\omega_z = 0.35\,\text{rad/s}$ in order not to exceed the 9G limit. Hence, the lift vector reads

$$\boldsymbol{l} = (0, -v_x\omega_z, g) \tag{28}$$

and the bank angle reads

$$\beta = \frac{\pi}{2} + \tan^{-1}\left(\frac{-v_x\omega_z}{g}\right) \approx 6\,\text{deg}. \tag{29}$$

5. Problem 5:

(a) The satellite is prolate with respect to its first axis and symmetric with respect to the other two. From the lecture we know that in this case the angular speed around the axis with the largest moment of inertia is constant when there are no external torques acting on the system. This can be seen from Figure 4, where $\omega_1(t) = const.$ The remaining angular velocities oscillate about the initial velocity with a phase shift of $\pi/2$ and a frequency of

$$\omega_n = (1 - I_1/I_2)\omega_1 = -0.02\,\text{rad/s}, \tag{30}$$

which translates into a period of $P \approx 314\,\text{s}$. We know that

$$\boldsymbol{L} = I\boldsymbol{\omega}. \tag{31}$$

Since I is diagonal in the body fixed frame it stands to reason that changing the spin around the axis with the highest moment of inertia will have the largest impact on the angular momentum. In our case that is ω_1 through N_1, respectively. Figures 5 and 6 confirm this notion as we see the effect of a corresponding torque on $\boldsymbol{\omega}$ as well as the (rotational) Kinetic Energy of the satellite.

(b) The corresponding Euler angles are best found by integrating the unit quaternion ODE and then transforming back to either Rotation, Precession, Nutation (3-1-3) or Tait-Bryan (3-2-1) angles. Example Results for Tait-Bryan angles are provided in Figures 7 without torque, and Figures 8 with torque. Please note that the range depends on the choice of the starting angles. If the frequencies in the angles roughly match those in the plots, the problem can be considered correctly solved.

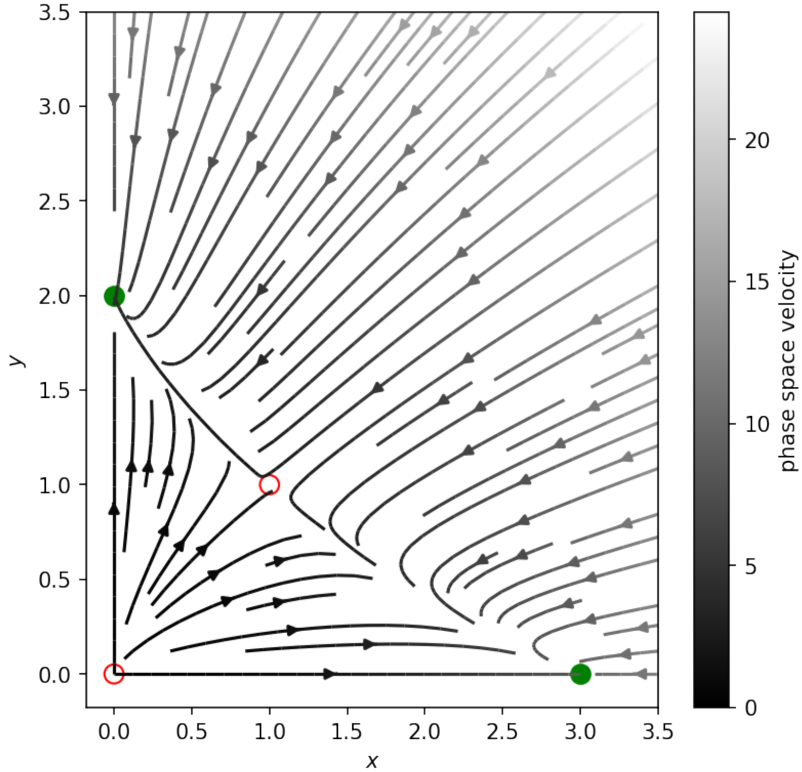

Figure 3: Phase diagram of the above Competitive Lotka Volterra system.

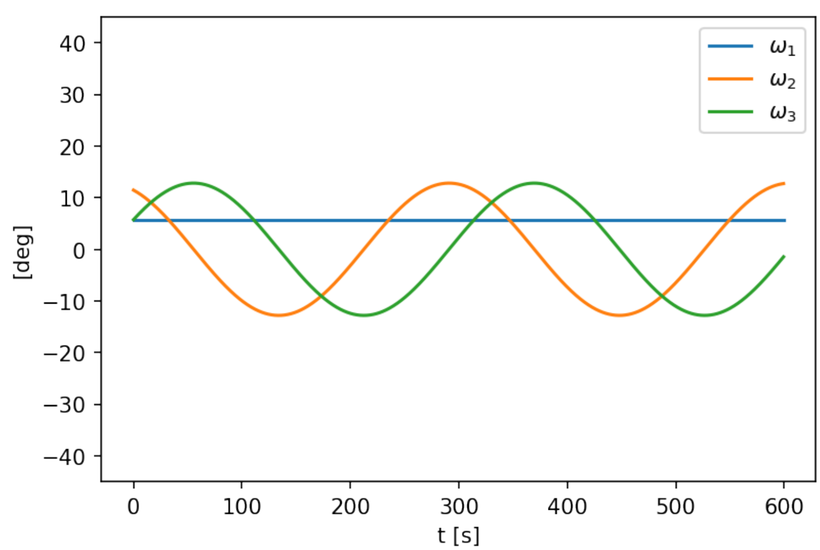

Figure 4: Angular velocities of the satellite without torques.

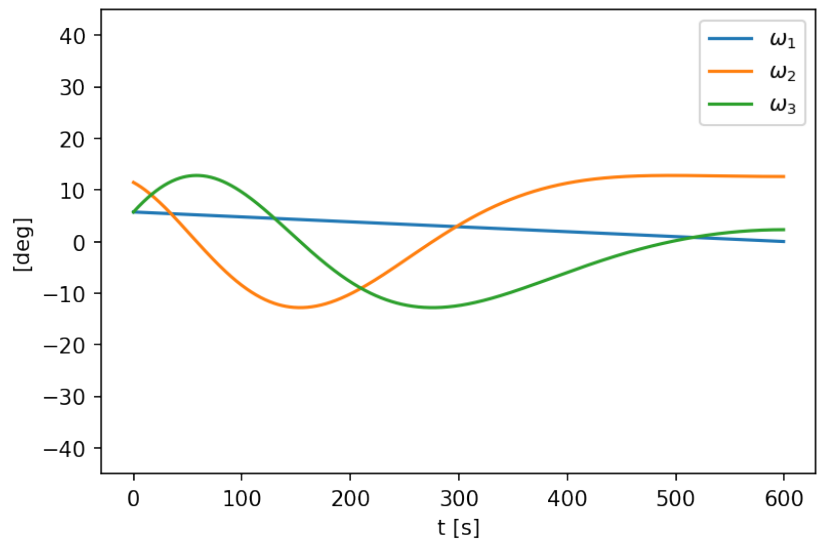

Figure 5: Angular velocities of the satellite torque along the axis of largest moment of inertia.

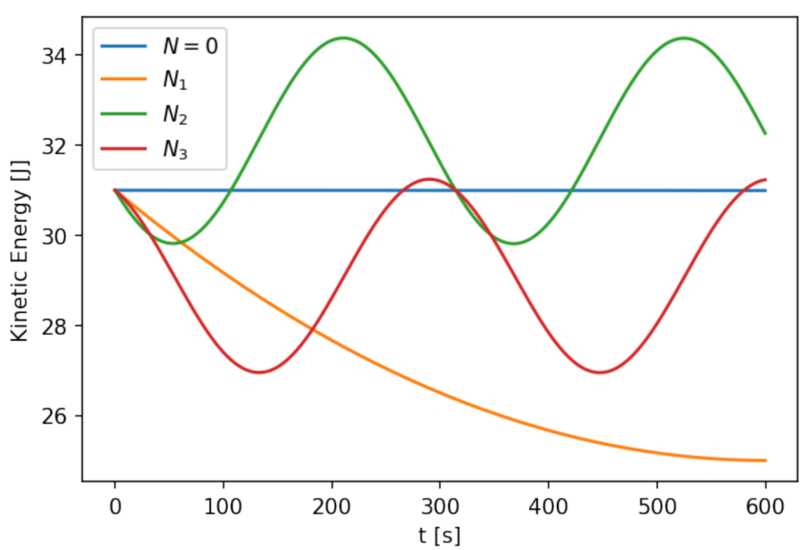

Figure 6: Satellite kinetic energy time evolution when 0.1 Nm of torque is applied to the different axes.

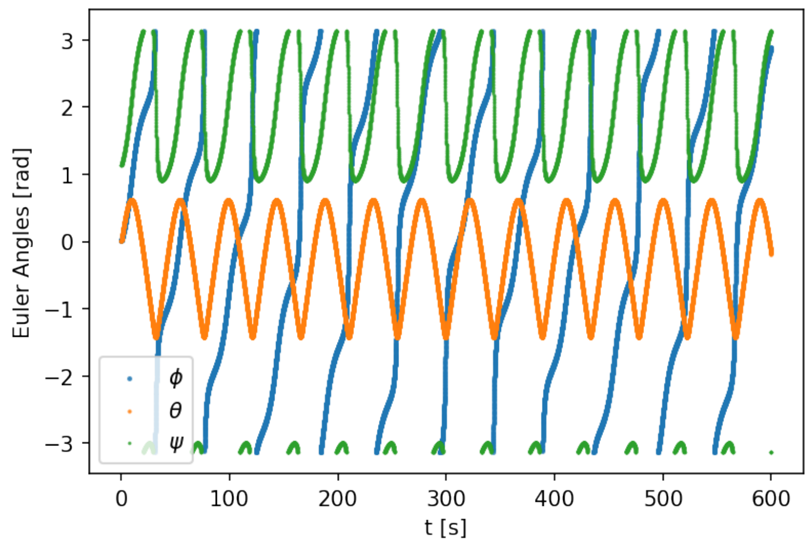

Figure 7: Tait-Bryan angles without torque.

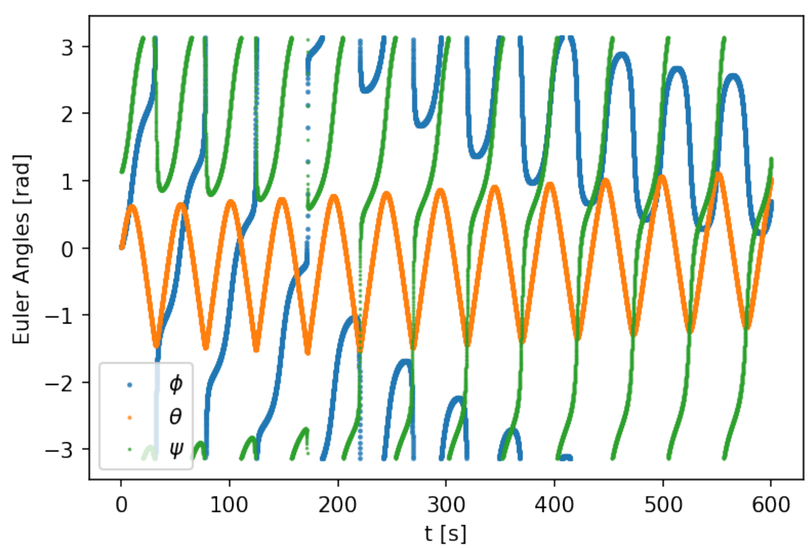

Figure 8: Tait-Bryan angles during torquing along the first body axis.