AE 352, Fall 2024

Problem Set 3: Lagrangian Mechanics

1. (20 points) Determine the (approximate)

   (a) (10p) Eigenfrequencies and

   (b) (10p) Eigenmodes

   of the Double Pendulum! Make use of small angle approximations!

2. (20 points) Consider the example of the sky crane and the rover as discussed in the lecture. If we relax the constraint that the rover can only swing in one plane, and assume at the same time that the control of the sky crane is extremely efficient in keeping the sky crane at the origin, we have a spherical pendulum with a fixed anchor point. The spherical pendulum has constant length l, just like its 2D counterpart, but the rover can now swing freely in 3D.

   (a) (5p) Draw a system diagram with origin, axes, and the relevant variables, etc.

   (b) (5p) Find suitable generalized coordinates!

   (c) (10p) Find the equations of motion using Lagrangian Mechanics!

3. (20 points) Should the rope(s) connecting the rover to the sky crane be elastic? Use the anchored, 2D, mathematical pendulum as a model, but assume the length l is not constant. Instead the rope can change its length around an equilibrium $l \pm \delta l(t)$.

   (a) (5p) Draw a system diagram!

   (b) (5p) Find suitable generalized coordinates!

   (c) (10p) Find the equations of motion using Lagrangian Mechanics!

4. (20 points) You are looking at an actual pendulum with a spherical bob. The bob has a diameter of 0.1m, a weight of 100 grams and a drag coefficient of 0.41. You may neglect the thickness of the rod. The air density is 1.2 kg/m^3. You may assume the pendulum swings in one plane.

   (a) (5p) Draw a system diagram!

   (b) (15p) Find the equations of motion using... whatever method you like!

5. (20 points) A classical model for the Helium atom in its ground state consists of two electrons moving on a flat, infinitely thin super conducting hoop. They are subject to the Coulomb potential

$$V = k\frac{q_1 + q_2}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|}. \tag{1}$$

   Here, $q_1 = q_2$ are the electric charges of the electrons, k is the Coulomb constant and $\boldsymbol{r}_{1,2}$ the positions of the electrons on the hoop with constant radius R, see Figure 1.

   (a) (10p) Find suitable generalized coordinates!

   (b) (10p) Find the equations of motion using Lagrangian Mechanics!

   (c) (bonus, 5 points) Describe the behavior of the resulting dynamical system! Do you think this is a good model for Helium?

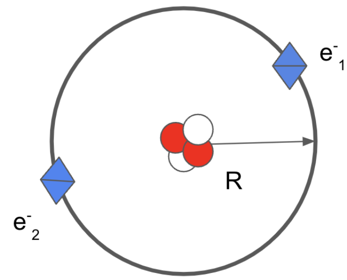

Figure 1: A classical model for the Helium atom.

Solutions

1. We start with the "cleaned-up" set of equations from the lecture where we have assumed $\theta_1 = \theta_2$.

$$\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} = -\frac{g}{l_1 l_2(1-\mu)}\begin{pmatrix} l_2 & -l_2\mu \\ -l_1 & l_1\end{pmatrix}\begin{pmatrix}\sin(\theta_1)\\ \sin(\theta_2).\end{pmatrix} \tag{2}$$

Really, if we make a small angle approximation, i.e. $\theta_1, \theta_2 \ll 1$ we end up with the same solution except for the sines on the right hand side,

$$\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} = -\frac{g}{l_1 l_2(1-\mu)}\begin{pmatrix} l_2 & -l_2\mu \\ -l_1 & l_1\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2.\end{pmatrix} \tag{3}$$

Here

$$\frac{m_2}{m_1+m_2} = \mu. \tag{4}$$

The eigenvalues of this system read

$$\lambda_{1,2} = \frac{g}{2l_1 l_2(\mu-1)}\left(l_1 + l_2 \pm \sqrt{(l_1-l_2)^2 + 4l_1 l_2\mu}\right) \tag{5}$$

with eigenfrequencies

$$\omega_{1,2} = \sqrt{-\lambda_{1,2}}. \tag{6}$$

The eigenmodes (b) then read

$$q(t)_{1,2} = A_{1,2}\, e^{i\omega_{1,2}t}. \tag{7}$$

The above solution is sufficient to receive full points. For $l_1 = l_2 = 1\,\mathrm{m}$ and $m_1 = m_2 = 1\,\mathrm{kg}$, the eigenfrequencies (a) (angular) are 1.94 Hz and 5.06 Hz. The eigenvectors in this case are

$$v_1 = \begin{pmatrix} 0.525731 \\ 0.850651 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -0.850651 \\ 0.525731 \end{pmatrix}. \tag{8}$$

2. Generalized coordinates: the position of the mass is expressed along Cartesian axes. Here, following the conventions shown in the diagram,

$$x = l\sin\theta\cos\phi \tag{9}$$
$$y = l\sin\theta\sin\phi \tag{10}$$
$$z = -l\cos\theta. \tag{11}$$

Next, time derivatives of these coordinates are taken, to obtain velocities along the axes

$$\dot{x} = l\cos\theta\cos\phi\,\dot{\theta} - l\sin\theta\sin\phi\,\dot{\phi} \tag{12}$$
$$\dot{y} = l\cos\theta\sin\phi\,\dot{\theta} + l\sin\theta\cos\phi\,\dot{\phi} \tag{13}$$
$$\dot{z} = l\sin\theta\,\dot{\theta}. \tag{14}$$

With

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \tag{15}$$

$$= l^2 \left(\dot{\theta}^2 + \sin^2 \theta \, \dot{\phi}^2 \right) \tag{16}$$

we find the Kinetic and Potential Energy

$$T = \tfrac{1}{2} m v^2 \tag{17}$$

$$= \tfrac{1}{2} m l^2 \left(\dot{\theta}^2 + \sin^2 \theta \, \dot{\phi}^2 \right) \tag{18}$$

$$V = mgz = -mg \, l \cos \theta \tag{19}$$

The Lagrangian is then

$$L = \frac{1}{2} m l^2 \left(\dot{\theta}^2 + \sin^2 \theta \; \dot{\phi}^2 \right) + mgl \cos \theta. \tag{20}$$

The Euler–Lagrange equations involving the polar angle θ and azimuth angle ϕ read

$$\frac{d}{dt} \frac{\partial}{\partial \dot{\theta}} L - \frac{\partial}{\partial \theta} L = 0, \tag{21}$$

$$\frac{d}{dt} \frac{\partial}{\partial \dot{\phi}} L - \frac{\partial}{\partial \phi} L = 0. \tag{22}$$

That gives

$$\frac{d}{dt} \left(m l^2 \dot{\theta} \right) - m l^2 \sin \theta \cdot \cos \theta \, \dot{\phi}^2 + mgl \sin \theta = 0, \tag{23}$$

$$\frac{d}{dt} \left(m l^2 \sin^2 \theta \cdot \dot{\phi} \right) = 0. \tag{24}$$

The last equation shows that angular momentum around the vertical axis, $|\mathbf{L}_z| = l \sin \theta \times ml \sin \theta \, \dot{\phi}$ is conserved. The final Equations of Motion read

$$\ddot{\theta} = \sin \theta \cos \theta \dot{\phi}^2 - \frac{g}{l} \sin \theta, \tag{25}$$

$$\ddot{\phi} \sin \theta = -2 \, \dot{\theta} \, \dot{\phi} \, \cos \theta. \tag{26}$$

When $\dot{\phi} = 0$ the first of the above equations reduces to the differential equation for the motion of a simple gravity pendulum.

The azimuth ϕ , being absent from the Lagrangian, is a cyclic coordinate, which implies that its conjugate momentum is a constant of motion.

3. This is the "Elastic Pendulum" problem in disguise. In order to construct the Lagrangian we need generalized coordinates. Polar coordinates are still our best bet even-though the length is no longer constant. We can use one trick, though, namely applying Hooke's law is the potential energy of the elastic rope itself

$$V_k = \frac{1}{2} k x^2 = \frac{1}{2} k (\delta l)^2 \tag{27}$$

where k is the spring constant. The potential energy from gravity is determined by the relative height of the mass, i.e.,

$$V_g = -gm(l + \delta l) \cos \theta \tag{28}$$

where g is the gravitational acceleration at the surface of Mars, and we assume $l = l_0 = const$ so that $\delta l(t)$ reflects the harmonic motion of the rope around l.

The kinetic energy is simply

$$T = \frac{1}{2} m v^2 \tag{29}$$

where v is the velocity of the mass, in our case

$$T = \frac{m}{2}(\dot{\delta l}^2 + (l+\delta l)^2\dot{\theta}^2) \tag{30}$$

The Lagrangian is then $L = T - V_k - V_g$, i.e.

$$L[\delta l, \dot{\delta l}, \theta, \dot{\theta}] = \frac{1}{2}m(\dot{\delta l}^2 + (l+\delta l)^2\dot{\theta}^2) - \frac{1}{2}k\delta l^2 + gm(l+\delta l)\cos\theta \tag{31}$$

With two degrees of freedom, for δl and θ, the equations of motion can be found using two Euler-Lagrange equations:

$$\frac{\partial L}{\partial \delta l} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\delta l}} = 0 \tag{32}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \tag{33}$$

This leads to the following equations of motion

$$\ddot{\delta l} = (l+\delta l)\dot{\theta}^2 - \frac{k}{m}\delta l + g\cos\theta \tag{34}$$

$$\ddot{\theta} = -\frac{g}{l+\delta l}\sin\theta - \frac{2\dot{\delta l}}{l+\delta l}\dot{\theta} \tag{35}$$

The elastic pendulum is now described with two coupled ordinary differential equations. If $\dot{\theta} \ll 1$, i.e. the centrifugal forces are small, and $\theta \approx 0$, i.e. the pendulum is hanging straight down, the first equation becomes

$$\ddot{\delta l} = -\frac{k}{m}\delta l + g, \tag{36}$$

which is a harmonic oscillator subject to gravity. Similarly, if $\dot{\delta l}$ is small, we recover the equation of motion of the mathematical pendulum for θ.

4. The Pendulum with Air Resistance.

   (a) Newtonian Mechanics: From the lecture we know that the equation of motion for the unperturbed pendulum is

   $$ml^2\ddot{\theta} = -mgl\sin\theta. \tag{37}$$

   We divide by l to make the units on each side become Newton

   $$ml\ddot{\theta} = -mg\sin\theta. \tag{38}$$

   Air resistance is proportional to the velocity squared namely

   $$D = \frac{1}{2}\rho v^2 C_D A. \tag{39}$$

   Since $v = l\dot{\theta}$ in our case,

   $$D = \frac{1}{2}\rho(l\dot{\theta})^2 C_D A. \tag{40}$$

   Adding the latter to the force balance gives

   $$ml\ddot{\theta} = -mg\sin\theta - \frac{1}{2}\rho C_D A l^2\dot{\theta}^2. \tag{41}$$

   Inserting the given values

   $$\ddot{\theta} = -9.81/l\,\sin\theta - 0.019\,l\,\dot{\theta}^2. \tag{42}$$

(b) D'Alembert: From the Lecture we know the constraints on the system as well as a suitable set of generalized coordinates. Note, that, since gravity is a conservative force, we do not have to include it as a Generalized force. Instead we can use the potential energy to account for gravity in the Lagrangian and only include air resistance as Generalize force. Generalized Coordinates

$$x = l \sin(\theta) \tag{43}$$

$$y = -l \cos(\theta) \tag{44}$$

$$q_1 = l, \quad q_2 = \theta \tag{45}$$

Expressing our coordinates in therms of Generalized Coordinates

$$x = q_1 \sin(q_2) \tag{46}$$

$$y = -q_1 \cos(q_2) \tag{47}$$

$$\dot{q}_1 = \dot{l} = 0 \tag{48}$$

Calculating derivatives for the kinetic energy.

$$\dot{x} = \dot{q_1} \sin(q_2) + q_1 \cos(q_2)\dot{q_2} = q_1 \dot{q_2} \cos(q_2) \tag{49}$$

$$\dot{y} = -\dot{q_1} \cos(q_2) + q_1 \sin(q_2)\dot{q_2} = q_1 \dot{q_2} \sin(q_2) \tag{50}$$

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m q_1^2 \dot{q}_2^2 (\cos^2(q_2) + \sin^2(q_2)) = \frac{1}{2} m q_1^2 \dot{q}_2^2 \tag{51}$$

The potential energy accounts for gravity.

$$V = mgy = -m\, g\, q_1 \cos(q_2) \tag{52}$$

The Lagrangian then reads

$$L = T - V = \frac{1}{2} m (q_1 \dot{q}_2)^2 + m\, g\, q_1 \cos(q_2) \tag{53}$$

And finally, D'Alembert's equations of motion can be derived via

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j, \quad j = 1, 2. \tag{54}$$

Here, Q_j are the Generalized forces, i.e.

$$Q_j = \sum_i \boldsymbol{F}_i \frac{\partial \boldsymbol{r}_i}{\partial q_j} \tag{55}$$

Q_j are the sum of the components in the q_j direction for all external forces that have not been taken into account by the scalar potential.
In matrix form the same equation reads

$$\boldsymbol{Q} = \boldsymbol{J} \cdot \boldsymbol{F}. \tag{56}$$

Where

$$\boldsymbol{J} = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial y}{\partial q_1} \\ \frac{\partial x}{\partial q_2} & \frac{\partial y}{\partial q_2} \end{pmatrix} \tag{57}$$

Air resistance acts against the direction of motion, i.e. against the direction of the velocity vector.

$$\boldsymbol{F} = -\frac{D}{v} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}. \tag{58}$$

$$
\begin{aligned}
Q_1 &= -\frac{D}{v}\left(\dot{x}\frac{\partial x}{\partial q_1} + \dot{y}\frac{\partial y}{\partial q_1}\right) = -\frac{D}{v}(\dot{x}\sin\theta - \dot{y}\cos\theta) = 0 && (59)\\
Q_2 &= -\frac{D}{v}\left(\dot{x}\frac{\partial x}{\partial q_2} + \dot{y}\frac{\partial y}{\partial q_2}\right) = -\frac{1}{2}\rho C_D A l^3 \dot{\theta}^2. && (60)
\end{aligned}
$$

The equations of motion then read

$$
ml\ddot{\theta} + mg\sin\theta = -\frac{1}{2}\rho C_D A l^2 \dot{\theta}^2. \tag{61}
$$

5. The Classical Helium Atom.

   (a) The system diagram must contain the positions of the electrons with masses $m_{1,2}$ as function of R as well as the polar angles $\theta_{1,2}$. Alternatively, diagrams containing the position vectors $\boldsymbol{r}_{1,2}$ are permitted.

   (b) System constraints:

$$
\begin{aligned}
z_1 &= 0, && (62)\\
z_2 &= 0, && (63)\\
x_1^2 + y_1^2 &= R && (64)\\
x_2^2 + y_2^2 &= R && (65)\\
& && (66)
\end{aligned}
$$

   Hence, polar coordinates are an excellent choice, i.e.

$$
\begin{aligned}
\boldsymbol{r}_1 &= (x_1, y_1, z_1) \to (R, \theta_1, 0) && (67)\\
\boldsymbol{r}_2 &= (x_2, y_2, z_2) \to (R, \theta_2, 0), && (68)\\
& && (69)
\end{aligned}
$$

   with $\dot{R} = 0$, since that encodes all our constraints. We thus have R, θ_1 and θ_2 as generalized coordinates, but R is an ignorable coordinate, so really we only need θ_1 and θ_2.

   (c) First we need to find the Lagrangian in generalized coordinates. To this end we construct the total kinetic and potential energies in Cartesian coordinates.

$$
\begin{aligned}
T &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) && (70)\\
V &= k\frac{q_1 + q_2}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|} && (71)
\end{aligned}
$$

   We will use the following transformation to find our generalized (polar) coordinates

$$
\begin{aligned}
\boldsymbol{r}_1 &= (R\cos\theta_1, R\sin\theta_1, 0) && (72)\\
\boldsymbol{r}_2 &= (R\cos\theta_2, R\sin\theta_2, 0) && (73)\\
& && (74)
\end{aligned}
$$

   In polar coordinates the energies read

$$
\begin{aligned}
T &= \frac{1}{2}R^2(m_1\dot{\theta}_1^2 + m_2\dot{\theta}_2^2) && (75)\\
V &= \frac{k(q_1 + q_2)}{R\sqrt{2(1 - \cos(\theta_1 - \theta_2))}}. && (76)
\end{aligned}
$$

   The Lagrangian is then

$$
\begin{aligned}
L &= T - V && (77)\\
&= \frac{1}{2}R^2(m_1\dot{\theta}_1^2 + m_2\dot{\theta}_2^2) - \frac{k(q_1 + q_2)}{R\sqrt{2(1 - \cos(\theta_1 - \theta_2))}}. && (78)
\end{aligned}
$$

Using the Euler Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = 0, \tag{79}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = 0, \tag{80}$$

$$\tag{81}$$

we can generate the equations of motion for $\theta_{1,2}(t)$.

$$\ddot{\theta}_1 = \frac{k(q_1+q_2)}{\sqrt{8}\,m_1}\frac{\sin(\Delta\theta)}{R^3(1-\cos(\Delta\theta))^{3/2}} \tag{82}$$

$$\ddot{\theta}_2 = -\frac{k(q_1+q_2)}{\sqrt{8}\,m_2}\frac{\sin(\Delta\theta)}{R^3(1-\cos(\Delta\theta))^{3/2}} \tag{83}$$

$$\tag{84}$$

with $\Delta\theta = \theta_1 - \theta_2$. Subtracting the above equations we get the equation of motion for $\Delta\theta$,

$$\frac{d^2\Delta\theta}{dt^2} = \frac{\sqrt{2}kq}{R^3 m}\frac{\sin(\Delta\theta)}{(1-\cos(\Delta\theta))^{3/2}}. \tag{85}$$

This tells us that the equations of motion only depend on the difference between the angles and, hence, only on the relative position of the electrons on their orbit. Furthermore, we can see that for $\Delta\theta = 0$ the cosine term approaches one and the equations become singular. This is the classical equivalent of the Pauli exclusion principle, that states that two electrons with the same quantum numbers cannot occupy the same coordinate in phase space.

(d) The system appears to consist of two coupled pendula with an amplitude that itself depends on how close the electrons are, e.g. at the initial time. This model contains a few desirable properties, that describe nature accurately, such as the fact that electrons on the same "orbit" cannot have the exact same initial conditions (Pauli exclusion). What this model does not describe correctly, however, is that the motion actually occurs in a 3D shell. Also, there is no quantization of angular momenta, etc, and the statistical nature implied by the Schrödinger equation is not modeled either. Finally, Heisenberg's uncertainty principle is violated here, since we can tell exactly where the electrons are and what their velocity is at any moment.