AE 352 - Spring 2024 - Quiz 1

Problems

1. (13 points) Carbon dioxide (CO_2) is molecule consisting of a carbon atom with a mass of roughly 12 amu and two oxygen atom with a mass of 16 amu. In a simple spring mass model the constant of restitution for this setup is $k \approx 1500\,\mathrm{N/m}$. One atomic mass unit (amu) $= 1.66 \times 10^{-27}$ kg. Damping is negligible.

   (a) (7 points) Find the equations of motion for this system!

   (b) (6 points) Find the 1D vibrational, angular eigenfrequencies/normal mode frequencies of this system!

   (c) (3 point Bonus) What wavelengths do those frequencies correspond to? As a reminder

   $$\lambda \nu = c, \tag{1}$$

   where λ is the wavelength in meters, $\nu = \frac{\omega}{2\pi}$ is the frequency of radiation Hz and $c = 299792458\,\mathrm{m/s}$ is the speed of light.

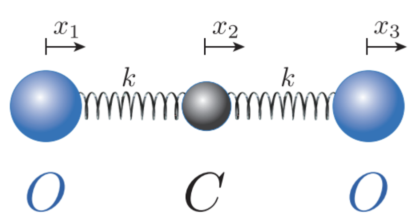

Figure 1: Schematic of a CO_2 molecule.

2. (12 points) A rocket with a mass of $m = 900\,\mathrm{kg}$ and a maximum thrust of $T = 140$ kN, streaks above the horizon and experiences a drag-force of $D = 0.13v^2$ Newtons, where v is the air speed of the rocket.

   (a) (7 points) What is the maximum air speed of the rocket?

   (b) (5 points) What angle of attack must the rocket have with respect to the local horizon in order to maintain its current altitude ($h = 1\,\mathrm{km}$) above the horizon?

Solutions

1. Let the mass of the oxygen atom be m_O and the mass of the carbon atom m_C.

   (a) the equations of motion in vector form read

   $$\begin{pmatrix} m_O & 0 & 0 \\ 0 & m_C & 0 \\ 0 & 0 & m_O \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \begin{pmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

   (b) The corresponding, collinear eigenfrequencies of the system are

   $$\omega_{1,2,3} = \sqrt{-\lambda_{1,2,3}} = \left(0, \sqrt{k/m_O}, \sqrt{k\frac{m_C + 2m_O}{m_C m_O}}\right)$$

   We see that ω_1 corresponds to no oscillation, because all the atoms remain at rest with respect to each other (no outer springs connected to a wall). The second frequency, ω_2 is a normal mode where the carbon atom remains at rest at the center while the oxygen atoms move in and out symmetrically, in opposing directions. The center of mass of the system remains at rest. Note that the carbon mass m_C does not appear in the eigenfrequency, since its mass is irrelevant to this vibration. The third frequency, ω_3 corresponds to a mode where the two oxygen atoms move in phase together left and right, while the carbon atom moves in the opposite direction by the distance required to keep the center of mass of the system at rest.

   (c) The wavelengths corresponding to the above eigenfrequencies are

   $$\lambda_{1,2,3} = \frac{2\pi c}{\omega_{1,2,3}} \approx (0, 7.93, 4.14)\,\mu\mathrm{m},$$

   which reside in the infrared spectrum.

2. (a) The maximum horizontal airspeed of the rocket is determined by the equilibrium of thrust and drag-force.

   $$T = D = \frac{1}{2}\rho \dot{q}_z^2 C_D A = 0.13 v^2,$$

   and hence

   $$v = \sqrt{T/0.13} = 1037.75\,\mathrm{m/s},$$

   which corresponds to Mach 3. This solution is correct. In order to understand the role of gravity for the maximum airspeed we can assume the rocket is headed straight towards the ground. Then equation (2a) has to include the Force of weight which is

   $$T + mg = D,$$

but since $mg = 9.81 \cdot 900 = 8829\,\mathrm{N}$ it is only roughly 6% of the Force exerted by the Thrust, the new maximum airspeed is

$$v = \sqrt{(|T| + m|g|)/0.13} = 1069.97\,\mathrm{m/s},$$

Anything in between $1036\,\mathrm{m/s}$ (see below) and $1070\,\mathrm{m/s}$ is permissible as answer for the max air speed.

(b) The rocket is traveling at max air speed, which means drag is already accounted for and needs to compensate for gravity in -z direction so that the angle of attack is

$$\alpha \approx \frac{9.81\,\mathrm{N/kg}}{140\,\mathrm{kN}/900\,\mathrm{kg}} = 0.063\,\mathrm{rad} \approx 3.6\,\mathrm{deg}.$$

This leads to a slightly reduced airspeed of $1036.72\,\mathrm{m/s}$.