AE 352 - Spring 2024 - Quiz 2

Problem

1. (25 points) A metallic bead of mass m is sliding on a ring with radius R. The ring is stationary, as shown in Figure 1, but the bead can move along the ring. The bead is also subject to gravity and friction, where

$$\boldsymbol{F}_{fric} = -k\,\boldsymbol{v}(t), \tag{1}$$

is the force of friction experienced by the bead. Here, $\boldsymbol{v}$ is the instantaneous velocity vector of the bead and k is a constant of friction with dimension [kg/s].

   (a) (5 points) Find suitable generalized coordinates!

   (b) (5 points) Find the Lagrangian!

   (c) (15 points) Find the equations of motion for the bead!

   (d) (5 point Bonus) Does this system have equilibrium points? If so where are they?

Figure 1: Bead on a non-rotating ring with friction.

Solution

(a) Generalized Coordinates:

$$q_1 = R, \quad q_2 = \theta \tag{2}$$

Expressing our coordinates in therms of Generalized Coordinates

$$x = q_1 \sin(q_2) \tag{3}$$

$$y = -q_1 \cos(q_2) \tag{4}$$

$$\dot{q}_1 = \dot{R} = 0 \tag{5}$$

Calculating derivatives for the kinetic energy.

$$\dot{x} = \dot{q_1} \sin(q_2) + q_1 \cos(q_2)\dot{q_2} = q_1\dot{q}_2 \cos(q_2) \tag{6}$$

$$\dot{y} = -\dot{q_1} \cos(q_2) + q_1 \sin(q_2)\dot{q_2} = q_1\dot{q}_2 \sin(q_2) \tag{7}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}mq_1^2\dot{q}_2^2(\cos^2(q_2) + \sin^2(q_2)) = \frac{1}{2}mq_1^2\dot{q}_2^2 \tag{8}$$

The potential energy accounts for gravity.

$$V = mgy = -m\,g\,q_1 \cos(q_2) \tag{9}$$

(b) The Lagrangian then reads

$$L = T - V = \frac{1}{2}m(q_1\dot{q}_2)^2 + m\,g\,q_1\cos(q_2) \tag{10}$$

And finally, D'Alembert's equations of motion can be derived via

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j, \quad j = 1, 2. \tag{11}$$

Here, Q_j are the Generalized forces, i.e.

$$Q_j = \sum_i \boldsymbol{F}_i \frac{\partial \boldsymbol{r}_i}{\partial q_j} \tag{12}$$

$$\boldsymbol{F} = -k\,(\dot{x}, \dot{y})^T \tag{13}$$

Q_j are the sum of the components in the q_j direction for all external forces that have not been taken into account by the scalar potential.

$$\begin{aligned}
Q_1 &= -k\left(\dot{x}\frac{\partial x}{\partial q_1} + \dot{y}\frac{\partial y}{\partial q_1}\right) = -k(\dot{x}\sin q_2 - \dot{y}\cos q_2) && (14)\\
&= -k(q_1\dot{q}_2\cos q_2 \sin q_2 - q_1\dot{q}_2 \sin q_2 \cos q_2) = 0 && (15)\\
Q_2 &= -k\left(\dot{x}\frac{\partial x}{\partial q_2} + \dot{y}\frac{\partial y}{\partial q_2}\right) = -k\,q_1^2\dot{q}_2 && (16)
\end{aligned}$$

Since q_1 is an ignorable coordinate we only need to calculate the equations of motion for q_2

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = \frac{d}{dt}(mq_1^2\dot{q}_2) = mq_1^2\ddot{q}_2 \tag{17}$$

$$\left(\frac{\partial L}{\partial q_2}\right) = mgq_1 \sin q_2 \tag{18}$$

(c) The equations of motion then read

$$ml^2\ddot{\theta} = -mgl\sin\theta - kl^2\dot{\theta} \tag{19}$$

or

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{k}{m}\dot{\theta}. \tag{20}$$

(d) To answer the bonus question we split this EoM into first order ODEs

$$\frac{d}{dt}\theta = \dot{\theta} \tag{21}$$

$$\frac{d}{dt}\dot{\theta} = -\frac{g}{l}\sin\theta - \frac{k}{m}\dot{\theta} \tag{22}$$

This system has equilibria at $\frac{g}{l}\sin\theta = \frac{k}{m}\dot{\theta}$, but since we also require $\frac{d}{dt}\theta = 0$ the equilibrium points are the same as those of the pendulum, namely $\dot{\theta} = 0$, $\theta = \pm n\pi$.