AE352 HW3 richiem2

1. (20 points) Determine the (approximate)

   (a) (10p) Eigenfrequencies and

   (b) (10p) Eigenmodes

   of the Double Pendulum! Make use of small angle approximations!

from lecture — equation of motion

$$(m_1+m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}\cos(\theta_2-\theta_1) = m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2-\theta_1) - (m_1+m_2)g\sin(\theta_1)$$

$$l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1\cos(\theta_2-\theta_1) = -l_1\dot{\theta}_1\sin(\theta_2-\theta_1) - g\sin(\theta_2)$$

small angle approximation — $\cos(\theta_2-\theta_1)\approx 1$, $\sin(\theta_2-\theta_1)\approx 0$

EOM becomes:

$$(m_1+m_2)l_1\ddot{\theta}_1 + m_2 l_2\ddot{\theta}_2 + (m_1+m_2)g\theta_1 = 0$$

$$m_2 l_2\ddot{\theta}_2 + m_2 l_1\ddot{\theta}_1 + m_2 g\theta_2 = 0$$

$$M = \begin{bmatrix} (m_1+m_2)l_1 & m_2 l_2 \\ m_2 l_1 & m_2 l_2 \end{bmatrix} \qquad K = \begin{bmatrix} (m_1+m_2)g & 0 \\ 0 & m_2 g \end{bmatrix}$$

$$\det(K - w^2 M) = 0$$

$$\det\begin{vmatrix} (m_1+m_2)(g-w^2 l) & -w^2 m_2 l_2 \\ -w^2 m_2 l_1 & m_2(g-w^2 l_2) \end{vmatrix}$$

$$= (m_1+m_2)g^2 - w^2(m_1+m_2)(l_1+l_2)g + w^4 m_1 l_1 l_2 = 0$$

assuming $m_1=m_2=1$ $\quad l_1=l_2=1$ $\quad g=9.81$

$$w^2 = \frac{g}{l}\left[1+\frac{m_2}{m_1} \pm \sqrt{\left(1+\frac{m_2}{m_1}\right)\frac{m_2}{m_1}}\right] = g\left[1+1\pm\sqrt{(1+1)1}\right] = g(2\pm\sqrt{2})$$

$$w = \sqrt{g(2\pm\sqrt{2})} \xrightarrow{e} \begin{aligned} w_1 &= \sqrt{9.81(2+\sqrt{2})} = 5.787 \\ w_2 &= \sqrt{9.81(2-\sqrt{2})} = 2.397 \end{aligned}$$

For eigenmodes: Ansatz $q(t) = ce^{iwt}$

$$q_1 = C_1 e^{i5.787t}$$
$$q_2 = C_2 e^{i2.397t}$$

but if l & g can change

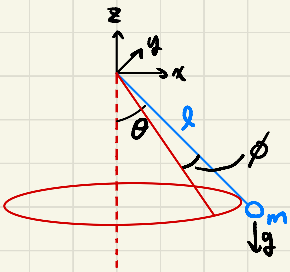
$$q_1 = C_1 e^{i\sqrt{2-\sqrt{2}}\sqrt{\frac{g}{l}}}$$
$$q_2 = C_2 e^{i\sqrt{2-\sqrt{2}}\sqrt{\frac{g}{l}}}$$

2. (20 points) Consider the example of the sky crane and the rover as discussed in the lecture. If we relax the constraint that the rover can only swing in one plane, and assume at the same time that the control of the sky crane is extremely efficient in keeping the sky crane at the origin, we have a spherical pendulum with a fixed anchor point. The spherical pendulum has constant length l, just like its 2D counterpart, but the rover can now swing freely in 3D.

   (a) (5p) Draw a system diagram with origin, axes, and the relevant variables, etc.

   (b) (5p) Find suitable generalized coordinates!

   (c) (10p) Find the equations of motion using Lagrangian Mechanics!



$q_1 = l \quad q_2 = \theta \quad q_3 = \phi$

$x = l\sin\theta\cos\phi \rightarrow x = q_1 \sin q_2 \cos q_3$

$y = l\sin\theta\sin\phi \rightarrow y = q_1 \sin q_2 \sin q_3$

$z = -l\cos\theta \rightarrow z = -q_1\cos q_2$

Kinetic energy

$$T = \frac{1}{2}m(\dot{q}_1^2 + q_1^2\dot{q}_2^2 + q_1^2\sin^2 q_2\dot{q}_3^2)$$

but $\dot{q}_1 = 0$ because q_1 constant

$$T = \frac{1}{2}mq_1^2(\dot{q}_2^2 + \sin^2 q_2 \dot{q}_3^2)$$

Potential energy

$$V = mgq_1(1-\cos q_2)$$

$$L = T - V = \frac{1}{2}m(q_1^2\dot{q}_2^2 + q_1^2\sin^2 q_2\dot{q}_3^2) - mgq_1(1-\cos q_2)$$

For $q_2 = \theta$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = 0$$

$$\frac{\partial}{\partial t}(ml^2\dot{q}_2) - mq_1^2\sin q_2\cos q_2\dot{q}_3^2 + mgq_1\sin q_2 = 0$$

$$\boxed{\ddot{\theta} = \sin\theta\cos\theta\dot{\phi}^2 - \frac{g}{l}\sin\theta}$$

For $q_3 = \phi$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_3}\right) - \frac{\partial L}{\partial q_3} = 0$$

$$\frac{\partial}{\partial t}(mq_1^2\sin^2 q_2\dot{q}_3) = 0$$

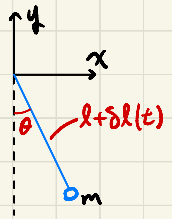
$$\boxed{\ddot{\phi} = \frac{-2\dot{\theta}\dot{\phi}\cos\theta}{\sin\theta}}$$

3. (20 points) Should the rope(s) connecting the rover to the sky crane be elastic? Use the anchored, 2D, mathematical pendulum as a model, but assume the length l is not constant. Instead the rope can change its length around an equilibrium $l \pm \delta l(t)$.

(a) (5p) Draw a system diagram!

(b) (5p) Find suitable generalized coordinates!

(c) (10p) Find the equations of motion using Lagrangian Mechanics!



$$q_1 = l + \delta l(t) \qquad q_2 = \theta$$

$$x = (l \pm \delta l)\sin\theta \longrightarrow x = q_1 \sin q_2 \longrightarrow \dot{x} = \dot{q}_1 \sin q_2 + q_1 \cos(q_2)\dot{q}_2$$

$$y = -(l \pm \delta l)\cos\theta \longrightarrow y = -q_1 \cos q_2 \longrightarrow \dot{y} = -\dot{q}_1 \cos q_2 + q_1 \sin(q_2)\dot{q}_2$$

Kinetic

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$T = \frac{1}{2}m\left((\dot{q}_1 \sin(q_2) + q_1\cos(q_2)\dot{q}_2)^2 + (-\dot{q}_1\cos(q_2) + q_2\sin(q_2)\dot{q}_2)^2\right)$$

↳ simplify: $T = \frac{1}{2}m(\dot{q}_1^2 + q_1^2\dot{q}_2^2)$

Potential

$$V = mgy = mg(-q_1\cos(q_2))$$

$$L = T - V = \frac{1}{2}m(\dot{q}_1^2 + q_1^2\dot{q}_2^2) + mgq_1\cos(q_2)$$

For $q_1 = l + \delta l(t)$

$$\frac{d}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0$$

$$\frac{\partial}{\partial t}(m\dot{q}_1) - (mq_1\dot{q}_2 - mg\cos q_2)) = 0$$

$$\boxed{(l + \ddot{\partial l}(t)) = (l + \partial l(t))\dot{\theta} - g\cos\theta}$$

For $q_2 = \theta$

$$\frac{d}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = 0$$

$$\frac{\partial}{\partial t}(mq_1^2\dot{q}_2) - (-mgq_1\sin(q_2)) = 0$$

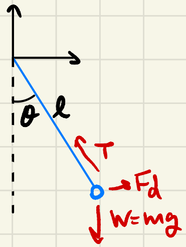
$$2m\dot{q}_1 q_2 + mq_1^2\ddot{q}_2 = mgq_1\sin q_2$$

$$\boxed{\ddot{\theta} = \frac{g(l + \delta l(t))\sin\theta - 2(l + \dot{\delta l}(t))\theta}{(l + \delta l(t))^2}}$$

4. (20 points) You are looking at an actual pendulum with a spherical bob. The bob has a diameter of 0.1m, a weight of 100 grams and a drag coefficient of 0.41. You may neglect the thickness of the rod. The air density is 1.2 kg/m^3. You may assume the pendulum swings in one plane.

(a) (5p) Draw a system diagram!

(b) (15p) Find the equations of motion using... whatever method you like!



Drag force $F_d = \frac{1}{2}\rho v^2 C_d A$

Area $A = \frac{\pi d^2}{4} = \frac{\pi (0.1)^2}{4} = 0.00785$

Velocity can be related to angular velocity

$w = \dot{\theta} \rightarrow V = wr \qquad V = \dot{\theta} r$

w/o drag force

$ml\ddot{\theta} + mg\sin(\theta) = 0$

w/ drag force

$ml\ddot{\theta} + mg\sin\theta + \frac{1}{2}\rho(l\dot{\theta})^2 C_d A$

$(0.1)l\ddot{\theta} + (0.1)(9.8)\sin\theta + \frac{1}{2}(1.2)(l\dot{\theta})^2(0.41)(0.00785)$

$$\boxed{0.1l\ddot{\theta} + 0.98\sin\theta + 0.00193(l\dot{\theta})^2}$$

5. (20 points) A classical model for the Helium atom in its ground state consists of two electrons moving on a flat, infinitely thin super conducting hoop. They are subject to the Coulomb potential

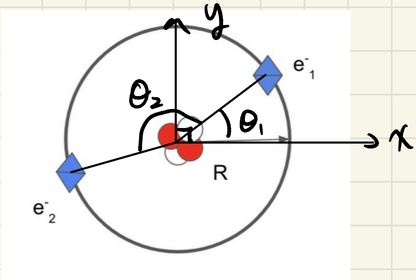
$$V = k\frac{q_1 + q_2}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|}. \quad (1)$$

Here, $q_1 = q_2$ are the electric charges of the electrons, k is the Coulomb constant and $\boldsymbol{r}_{1,2}$ the positions of the electrons on the hoop with constant radius R, see Figure 1.

(a) (10p) Find suitable generalized coordinates!

(b) (10p) Find the equations of motion using Lagrangian Mechanics!

(c) (bonus, 5 points) Describe the behavior of the resulting dynamical system! Do you think this is a good model for Helium?



$q_1 = \theta_1 \qquad q_2 = \theta_2$

$e_1: \ x = R\cos q_1 \qquad\qquad e_2: \ x = R\cos q_2$

$\quad\ \ y = R\sin q_1 \qquad\qquad\quad\ \ y = R\sin q_2$

Kinetic energy

$T_1 = \frac{1}{2} m A^2 \dot{q}_1^2$

$T_2 = \frac{1}{2} m A^2 \dot{q}_2^2$

$T = T_1 + T_2 = \frac{1}{2} m A (\dot{q}_1^2 + \dot{q}_2^2)$

Potential energy

$V = k\frac{q_1 + q_2}{|r_1 - r_2|}$ but $q_1 = q_2$

from law cosines

and $\|r_1 + r_2\| = R\sqrt{2 - 2\cos(q_1 - q_2)}$

$$V = \frac{2kq}{R\sqrt{2 - 2\cos(\theta_1 - \theta_2)}}$$

$L = T - V$

$$= \frac{1}{2} m R^2 \dot{q}_1^2 + \frac{1}{2} m R^2 \dot{q}_2^2 - \frac{2kq}{R\sqrt{2 - 2\cos(q_1 - q_2)}}$$

Euler for $\theta_1 = q_1$

$$\frac{d}{dt}(mR^2 \dot{q}_1) - \frac{\partial}{\partial q_1}\left(\frac{-2kq}{R\sqrt{2 - 2\cos(q_1 - q_2)}}\right) = 0$$

$$mR^2 \ddot{\theta}_1 = \frac{2kq\sin(\theta_1 - \theta_2)}{R(2 - 2\cos(\theta_1 - \theta_2))^{3/2}}$$

Euler for $\theta_2 = q_2$

$$\frac{d}{dt}(mR^2 \dot{q}_2) - \frac{\partial}{\partial q_2}\left(\frac{-2kq}{R\sqrt{2 - 2\cos(\theta_1 - \theta_2)}}\right) = 0$$

$$mR^2 \ddot{\theta}_2 = -\frac{2kq\sin(\theta_1 - \theta_2)}{R(2 - 2\cos(\theta_1 - \theta_2))^{3/2}}$$

c) the resulting system will be oscillatory due to coulomb repulsion and oscilate around a point

It is a good model as it captures repulsion, but not quantum effects