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AE 410/CSE 461 – Computational Aerodynamics
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Take-home Midterm

Due date: March 22, 2024

The objective of this take-home midterm is to write a finite-difference solver for the inviscid flow in a 1D duct. This flow is governed by the 1D Euler equations, which are given in their conservative form as

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \tag{1}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) = 0 \tag{2}$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x}[(\rho E + p)u] = 0 \tag{3}$$

where

- ρ is the fluid density,
- u is the fluid velocity,
- p is the thermodynamic pressure,
- E is the total specific energy.

The pressure is related to the other quantities by the equation of state

$$p = (\gamma - 1)\left(\rho E - \frac{1}{2}\rho u^2\right) \tag{4}$$

for a perfect gas with ratio of specific heats γ.

The Euler equations (1)–(3) constitute a system of coupled nonlinear partial differential equations, which is typically very difficult to solve analytically. Instead, in this midterm, we will solve this system of equations numerically using finite differences and the method of lines on a periodic domain.

Problem 1 · The Euler equations (1)–(3) are written in terms of the conserved variables that are: the mass density ρ (mass per unit volume), the momentum density ρu (momentum per unit volume), and the total energy density ρE (total energy per unit volume). Let us assign these conserved variables to a single vector

$$\mathbf{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix} \tag{5}$$

Q1 → Using the equation of state (4), express the pressure as a function of the components of $\mathbf{q}$, i.e., as a function of q_0, q_1, and q_2.

Hint: For instance, the fluid velocity u can be expressed as

$$u = \frac{q_1}{q_0} \tag{6}$$

Problem 2 · The Euler equations (1)–(3) may be written in vector form as

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} = \mathbf{0} \tag{7}$$

where $\mathbf{f}(\mathbf{q})$ is the flux vector given as

$$\mathbf{f}(\mathbf{q}) = \begin{bmatrix} f_0(\mathbf{q}) \\ f_1(\mathbf{q}) \\ f_2(\mathbf{q}) \end{bmatrix} \tag{8}$$

Q2 → Write $\mathbf{f}(\mathbf{q})$ in terms of the components of $\mathbf{q}$.

Hint: For instance, it is pretty clear from Eq. (1) that

$$f_0(\mathbf{q}) = q_1 \tag{9}$$

Problem 3 · Now that we have expressed the Euler equations (1)–(3) in vector form, it is useful to express them in the quasi-linear form

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{M}\frac{\partial \mathbf{q}}{\partial x} = \mathbf{0} \tag{10}$$

where $\mathbf{M}$ is a 3×3 flux-Jacobian matrix defined as

$$\mathbf{M}(\mathbf{q}) = \frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial f_0(\mathbf{q})}{\partial q_0} & \frac{\partial f_0(\mathbf{q})}{\partial q_1} & \frac{\partial f_0(\mathbf{q})}{\partial q_2} \\ \frac{\partial f_1(\mathbf{q})}{\partial q_0} & \frac{\partial f_1(\mathbf{q})}{\partial q_1} & \frac{\partial f_1(\mathbf{q})}{\partial q_2} \\ \frac{\partial f_2(\mathbf{q})}{\partial q_0} & \frac{\partial f_2(\mathbf{q})}{\partial q_1} & \frac{\partial f_2(\mathbf{q})}{\partial q_2} \end{bmatrix} \tag{11}$$

Q3 → Find the expression of the matrix $\mathbf{M}$.

Remark: This matrix is **not** the same as the flux-Jacobian we have derived in class for the 1D Euler equations in non-conservative form. In class, we have considered the variables $(q_0, q_1, q_2) = (\rho, u, p)$; in this midterm, we consider the conserved variables $(q_0, q_1, q_2) = (\rho, \rho u, \rho E)$.

Problem 4 · The Jacobian matrix $\mathbf{M}$ admits a complete set of eigenvalues and eigenvectors $(\Omega_n, \mathbf{X}_n), n \in \{0, 1, 2\}$, such that

$$\mathbf{M}\mathbf{X}_n = \Omega_n \mathbf{X}_n \ , \quad \forall n \in \{0, 1, 2\} \tag{12}$$

which leads to the eigendecomposition

$$\mathbf{M} = \mathbf{X}\boldsymbol{\Omega}\mathbf{X}^{-1} \tag{13}$$

with

$$\boldsymbol{\Omega} = \operatorname{diag}(\Omega_0, \Omega_1, \Omega_2) \tag{14}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_0 & \mathbf{X}_1 & \mathbf{X}_2 \end{bmatrix} \tag{15}$$

Q4 → Find the eigenvalues and eigenvectors of $\mathbf{M}$ and order them such that $\Omega_0 \leq \Omega_1 \leq \Omega_2$.

Hint: To keep the expressions relatively compact, it is useful to introduce the speed of sound c given by

$$c = \sqrt{\frac{\gamma p}{\rho}} \tag{16}$$

Problem 5 · We will use a flux-splitting strategy to solve the system of equations (10). As a result, we need to construct finite-difference operators that are biased in the positive and negative directions. We choose to use the finite-difference operators that approximate a given function $f(x)$ as

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_n^+ = \sum_{j=-3}^{1} a_j^+ f_{n+j} \tag{17}$$

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_n^- = \sum_{j=-1}^{3} a_j^- f_{n+j} \tag{18}$$

In other terms, $\mathrm{d}\cdot/\mathrm{d}x|^+$ will be upwind if information propagates from left to right, and $\mathrm{d}\cdot/\mathrm{d}x|^-$ will be upwind is information propagates from right to left.

Q5.1 → Find the coefficients $\{a_j^+\}_{j=-3}^{1}$ and $\{a_j^-\}_{j=-1}^{3}$ that lead to the finite-difference approximations (17) and (18).

Q5.2 → Derive the leading truncation error term for each scheme. What is the order of accuracy of these two finite-difference schemes?

Problem 6 · We want to know whether the schemes constructed in Problem 5 are dispersive and/or dissipative.

Q6.1 → Using Fourier error analysis, derive the expression of the modified wavenumber $\kappa^\star$ associated with each scheme.

Q6.2 → Plot the real and imaginary parts of $\kappa^\star \Delta x$ as a function of $\kappa \Delta x$ between 0 and π, and explain if one should expect the schemes to be dispersive and/or dissipative.

Problem 7 · We want to solve the system of equations (10) in a stable manner by splitting the flux-Jacobian matrix $\mathbf{M}$ as

$$\mathbf{M} = \underbrace{\mathbf{X}\boldsymbol{\Omega}^+\mathbf{X}^{-1}}_{\mathbf{M}^+} + \underbrace{\mathbf{X}\boldsymbol{\Omega}^-\mathbf{X}^{-1}}_{\mathbf{M}^-} \tag{19}$$

with

$$\boldsymbol{\Omega}^+ = \frac{1}{2}\left(\boldsymbol{\Omega} + |\boldsymbol{\Omega}|\right) \tag{20}$$

$$\boldsymbol{\Omega}^- = \boldsymbol{\Omega} - \boldsymbol{\Omega}^+ \tag{21}$$

This leads to the system of equations

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{M}^+ \frac{\partial \mathbf{q}}{\partial x} + \mathbf{M}^- \frac{\partial \mathbf{q}}{\partial x} = \mathbf{0} \tag{22}$$

Following the method of lines and making use of our knowledge of the sign of $\boldsymbol{\Omega}^+$ and of $\boldsymbol{\Omega}^-$, we approximate the spatial derivative in the term $\mathbf{M}^+\partial\mathbf{q}/\partial x$ with the biased scheme given in Eq. (17), and the spatial derivative in the term $\mathbf{M}^-\partial\mathbf{q}/\partial x$ with the biased scheme given in Eq. (18), so as to transform Eq. (22) into a system of ordinary differential equations (i.e., the semi-discrete 1D Euler problem).

Q7 → Develop a code for solving the semi-discrete 1D Euler problem on the periodic domain $[0, L]$ discretized into N_x grid points.

Hints: You may reuse parts of the code that was developed in HW3 for the semi-discrete 1D linear advection problem. Remember that in this midterm, we consider the three conserved variables $(\rho, \rho u, \rho E)$, therefore there will be three discrete variables at each grid point (i.e., $3N_x$ discrete variables in total). You may order the vector of discrete variables in whichever way you see fit; this should not affect your end results.

Problem 8 · Solve the semi-discrete 1D Euler periodic problem with:

- $L = 10$ m, $N_x = 150$
- $\gamma = 1.4$
- $\rho(x, t = 0) = 1.225$ kg/m^3
- $u(x, t = 0) = 100$ m/s
- $p(x, t = 0) = p_\infty \left(1 + \frac{1}{10} \exp\left(-10(x - L/2)^2\right)\right)$, with $p_\infty = 101325$ Pa.

Q8 → Plot the spatial variations of (ρ, u, p) at 4 millisecond intervals between $t = 0$ and $t = 0.02$ s, and explain what you observe.

Problem 9 · **This problem is required for students taking AE 410/CSE 461 for four credit hours. It is not required for students taking AE 410/CSE 461 for three credit hours.**

It is customary in CFD development to verify ones code before using it, so as to guarantee the correctness of the simulation results. There exist several ways to do so, including:

Option 1: Develop an exact solution to the governing equations using the method of characteristics, and compare your numerical solution against this analytical solution.

Option 2: Research and implement the method of manufactured solutions.

Option 3: Compare your numerical solution against the numerical solution of another, already established code (this is called cross-code verification).

Q9 → Verify the code you have developed for Problems 7/8 using one of the listed approaches.

Submission guidelines · Instructions on how to prepare and submit your report are available on the course's Canvas page at https://canvas.illinois.edu/courses/43781/assignments/syllabus