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AE 410/CSE 461 – Computational Aerodynamics
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UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Homework #1

Due date: February 8, 2024

Differential matrix operators are key components of the computer programs developed in computational physics/computational aerodynamics. In this homework, you will develop your own code for constructing such an operator, and validate your implementation using test-functions with known derivatives.

Problem 1 · In the programming language of your choice, develop a piece of code that constructs the discrete operator $\mathbf{D}$, a $N \times N$ matrix corresponding to the finite-difference approximation

$$\left.\frac{\partial u}{\partial x}\right|_i \simeq \sum_{j=-L}^{R} a_j u_{i+j} \ , \quad i \in \{1, \ldots, N\}$$

on a one-dimensional domain discretized into N consecutive nodes located at $x_i = (i-1)\Delta x$ with $\Delta x = 1/N$. Moreover, this domain is assumed periodic, i.e., $u_i = u_{(i+kN)}, \forall k \in \mathbb{Z}$.

The scalars $a_j, j \in \{-L, \ldots, R\}$, are the coefficients of the finite-difference scheme that uses L left neighbors and R right neighbors to approximate the first spatial derivative at the location of any given discrete node. The ith row of the matrix $\mathbf{D}$ contains the coefficients for approximating the first spatial derivative at the location of the ith discrete node.

Your code/function should receive the following inputs:

- The positive integers L, R, and N (you may verify that $L + R + 1 \leq N$).
- The constant grid spacing Δx.
- The vector $\mathbf{a} = \begin{bmatrix} a_{-L} & a_{-L+1} & \cdots & a_R \end{bmatrix}^\intercal$ of length $L + R + 1$.

It should return:

- The matrix $\mathbf{D}$ of size $N \times N$.

Important remark: The discrete notations introduced in this problem correspond to a 1-based array indexing (i.e., all arrays start with the index 1). Depending on your choice of programming language, you may have to convert them into 0-based array indexing (i.e., all arrays start with the index 0). For reference, `Matlab` and `Fortan` use 1-based array indexing, whereas `Python` and `C/C++` use 0-based array indexing.

Problem 2 · In order to verify the code developed in the previous problem, we can test $\mathbf{D}$ against known data of the form

$$\mathbf{f} = \begin{bmatrix} f(x_1) & f(x_2) & \cdots & f(x_N) \end{bmatrix}^\intercal$$

where $f(x)$ is a function that is (at least) three times differentiable and whose derivatives can be calculated analytically. If $\mathbf{d}$ is the vector of the exact derivatives of $f(x)$ at the discrete points, i.e.,

$$\mathbf{d} = \begin{bmatrix} f'(x_1) & f'(x_2) & \cdots & f'(x_N) \end{bmatrix}^\intercal$$

then the vector containing the errors between the exact and numerically estimated derivatives of $f(x)$ is given as

$$\boldsymbol{\epsilon} = \mathbf{Df} - \mathbf{d}$$

The discrete ℓ_∞ and ℓ_2 norms of $\boldsymbol{\epsilon}$ are often used to study the accuracy of a numerical scheme. They are given as

$$\|\boldsymbol{\epsilon}\|_\infty = \max_i |\varepsilon_i|$$

$$\|\boldsymbol{\epsilon}\|_2 = \sqrt{\frac{1}{N}\sum_{i=1}^{N} \varepsilon_i^2}$$

Both $\|\boldsymbol{\epsilon}\|_\infty$ and $\|\boldsymbol{\epsilon}\|_2$ are scalar quantities, which depend on the type of finite-difference scheme used to construct $\mathbf{D}$, as well as on $\Delta x = 1/N$.

(a) Choose a function $f(x)$ that is well suited for the testing of your code. Justify this choice.

(b) Using a log-log scale, plot the ℓ_∞ and ℓ_2 errors norms corresponding to the forward difference scheme with coefficients

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{\Delta x}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(c) Using a log-log scale, plot the ℓ_∞ and ℓ_2 errors norms corresponding to the central difference scheme with coefficients

$$\mathbf{a} = \begin{bmatrix} a_{-1} \\ a_0 \\ a_1 \end{bmatrix} = \frac{1}{2\Delta x}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(d) Verify that the slope of these plotted error norms matches their expected value.

Problem 3 · **This problem is <u>required</u> for all students taking AE 410/CSE 461 <u>for four credit hours</u>. It is <u>not required</u> for students taking AE 410/CSE 461 <u>for three credit hours</u>.**

Using the Taylor series expansion of $f(x)$, calculate the analytically expected value of the discrete errors norms computed in Questions (b) and (c) of the previous problem.

Plot these analytically expected value alongside the corresponding previously computed errors norms.

<u>Hint:</u> You will need to use the midpoint rule

$$\int_0^1 g(x)\,\mathrm{d}x \simeq \sum_{i=1}^{N} g(x_i)\Delta x$$

Submission guidelines · Instructions on how to prepare and submit your report are available on the course's Canvas page at `https://canvas.illinois.edu/courses/43781/assignments/syllabus`