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AE 410/CSE 461 – Computational Aerodynamics
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Final Project

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The objective of this project is to write a finite-volume solver for the inviscid flow in a shock-driven wind tunnel, commonly used for experiments at universities and national laboratories. This flow is governed by the Euler equations, which are given in their 1D conservative form as

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \tag{1}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) = 0 \tag{2}$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x}[(\rho E + p)u] = 0 \tag{3}$$

where

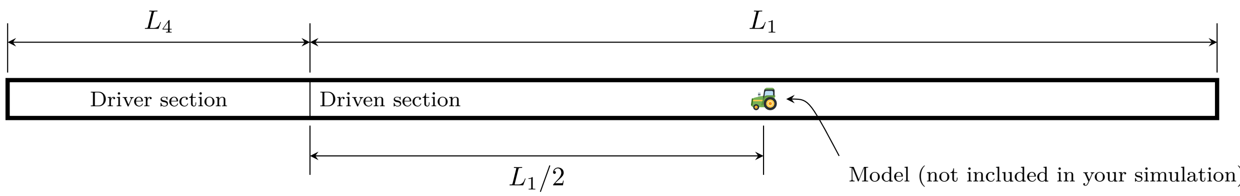
- ρ is the fluid density,
- u is the fluid velocity,
- p is the thermodynamic pressure,
- E is the total specific energy.

The pressure is related to the other quantities by the equation of state

$$p = (\gamma - 1)\left(\rho E - \frac{1}{2}\rho u^2\right) \tag{4}$$

for a perfect gas with ratio of specific heats γ.

The setup we consider in this project is a shock tube as drawn below. The shock tube contains a driver section, of length L_4, and a driven section of length L_1. In the driver section, air is compressed to a pressure p_4 and a temperature T_4. In the driven section, air is evacuated to a pressure $p_1 \ll p_4$ and a temperature T_1. The two sections have the same ratio of specific heats $\gamma = 1.4$, the same specific gas constant $R = 287$ J.kg^{-1}.K^{-1}, and are separated by a metal diaphragm that we assume to "magically" disappear at time $t = 0$. Located at a distance $L_1/2$ from the diaphragm, a small aerodynamic model is placed for experiments (but not included in your simulation!).



Problem 1 · In this problem, you will develop a finite-volume solver using Godunov's method to solve the 1D Euler equations. This method solves an exact Riemann problem at each face of your computational grid in order to reconstruct the time-averaged flux at the face. You are free to use any code provided on Canvas, including the code provided to solve the exact Riemann problem for the Euler equation.

Q1 → Write a finite-volume solver that uses Godunov's method to solve the 1D Euler equations for the shock-tube setup described above. Your code should accept the following physical parameter inputs:

- The lengths of the driver and driven sections, L_1 and L_4.
- The ratio of specific heats and the specific gas constant, γ and R.
- The initial pressure, temperature, and velocity in both sections, $\{p_1, T_1, u_1\}$ and $\{p_4, T_4, u_4\}$.

Note: In a real shock tube, $u_1 = u_4 = 0$. However, we want our solver to be able to consider non-zero initial velocities so as to validate it with the test-cases provided in the book of Toro [1].

Your code should also accept the following numerical parameter inputs:

- The number of grid points, N_x.
- The final simulation time, t_{final}.
- The timestep of the solver, Δt.

The output of your code should be:

- The discrete density, velocity, and pressure values inside the shock-tube for $t \in [0, t_{\text{final}}]$.

Your solver should be able to consider the following boundary conditions on either side of the domain:

- Solid-wall BC: Set velocity to zero, and the gradients of density and total energy density to zero.
- Far-field BC: Set the gradients of density, momentum density, and total energy density to zero.

It is recommended to enforce these conditions by introducing ghost cells on either side of the domain, and set the values in those ghost cells according to the boundary condition.

Problem 2 · In this problem, you will validate your finite-volume solver with known solutions to the 1D Euler equations.

Q2 → Verify your code using the Tests 1 and 2 described in the book of Toro [1, p. 129, Section 4.3.3].

Test 1 is the Riemann problem with the initial conditions:

$$\left\{\begin{array}{lcll} \rho_L & = & 1 & \text{kg.m}^{-3} \\ u_L & = & 0 & \text{m.s}^{-1} \\ p_L & = & 1 & \text{Pa} \end{array}\right. , \quad \text{and} \quad \left\{\begin{array}{lcrl} \rho_R & = & 0.125 & \text{kg.m}^{-3} \\ u_R & = & 0 & \text{m.s}^{-1} \\ p_R & = & 0.1 & \text{Pa} \end{array}\right. \tag{5}$$

Test 2 is the Riemann problem with the initial conditions:

$$\left\{\begin{array}{lcrl} \rho_L & = & 1 & \text{kg.m}^{-3} \\ u_L & = & -2 & \text{m.s}^{-1} \\ p_L & = & 0.4 & \text{Pa} \end{array}\right. , \quad \text{and} \quad \left\{\begin{array}{lcrl} \rho_R & = & 1 & \text{kg.m}^{-3} \\ u_R & = & 2 & \text{m.s}^{-1} \\ p_R & = & 0.4 & \text{Pa} \end{array}\right. \tag{6}$$

For Test 1, consider the final time $t_{\text{final}} = 0.25$ s. For Test 2, consider the final time $t_{\text{final}} = 0.15$ s.

You can produce the exact solutions to these Riemann problems using the code provided on Canvas. The corresponding raw data is also provided on Canvas.

Note: You will need to think about how to set up L_1 and L_4 so that the solid-wall boundary conditions don't interfere with your verification, or use far-field boundary conditions.

Problem 3 · Let us now condider the following shock tube problem: $L_1 = 19$ m, $L_4 = 1$ m, $p_1 = 0.1$ bar, $p_4 = 100$ bar, $T_1 = T_4 = 293$ K. Both ends of the shock tube are solid walls.

Q3.1 → Run the simulation long enough for the primary shock to reflect off of both ends of the shock tube.

Q3.2 → Using your data, make a contour plot of the pressure, density, temperature, and velocity in the (x, t) plane, just like an (x, t) diagram. You should identify these features: primary shock wave, primary contact discontinuity, and primary expansion fan. Identify on each plot where:

1. the primary shock interacts with the right wall;
2. the primary shock interacts with contact discontinuity;
3. the primary expansion fan reflects off the left wall.

Note: In `Python`, such contour plots can be produced using `Matplotlib`'s `contourf` or `imshow` functions.

Q3.3 → Measure the primary shock Mach number using data from the (x, t) diagram and compare it to the theoretical value predicted by the so-called shock-tube equation,

$$\frac{p_4}{p_1} = \frac{2\gamma M_s^2 - (\gamma - 1)}{\gamma + 1} \left[1 - \frac{\gamma - 1}{\gamma + 1} \frac{c_1}{c_4} \left(M_s - \frac{1}{M_s}\right)\right]^{-\frac{2\gamma}{\gamma - 1}} , \tag{7}$$

where $M_s = S/c_1$ is the shock Mach number in the lab frame, c_1 is speed of sound in the driven section and c_4 is the speed of sound in the driver section.

Q3.4 → Evaluate M_s as a function of the timestep Δt and cell size $\Delta x = (L_4 + L_1)/N_x$. Does your numerical estimate of M_s approach the theoretical value as Δt and Δx decrease?

Q3.5 → Measure the "test time" the small aerodynamic model would experience. The test time is defined as the time just after the primary shock passes the model until just before the contact discontinuity impacts the model. During this test window, what are the flow conditions the model would experience?

Q3.6 → **`This question is optional. If successfully answered, it will grant you bonus points`**. In your exact Riemann solver code, replace the root-finding algorithm that is used to compute $p^\star$ by a new and different algorithm (for example: the secant method). Comment on the computational time that is now required to compute your solutions.

Problem 4 · **`This problem is only required for those taking AE 410/CSE 461 for 4 credit hours`**. Repeat Problems 1–3 using an approximate Roe solver instead of an exact Riemann solver to reconstruct the face fluxes. As a reminder, the approximate Roe solver reconstructs time-averaged fluxes as

$$\mathbf{F}^{\mathrm{Roe}}_{n+1/2} = \frac{1}{2}(\mathbf{f}(\mathbf{q}_L) + \mathbf{f}(\mathbf{q}_R)) - \frac{1}{2} \sum_{k=0}^{3} \left|\hat{\Omega}_k\right| \alpha_k \hat{\mathbf{X}}_k , \tag{8}$$

with $\mathbf{q}_L = \mathbf{q}_n$ and $\mathbf{q}_R = \mathbf{q}_{n+1}$. In the previous expression, $\hat{\Omega}_k$ is the k^{th} eigenvalue of Roe's linearized flux Jacobian and $\hat{\mathbf{X}}_k$ the corresponding eigenvector. These eigenvalues and eigenvectors read as

$$\hat{\Omega}_0 = \hat{u} - \hat{c} \qquad \hat{\mathbf{X}}_0 = \begin{bmatrix} 1 & \hat{u} - \hat{c} & \hat{H} - \hat{u}\hat{c} \end{bmatrix}^\intercal \tag{9}$$

$$\hat{\Omega}_1 = \hat{u} \qquad \hat{\mathbf{X}}_1 = \begin{bmatrix} 1 & \hat{u} & \hat{u}^2/2 \end{bmatrix}^\intercal \tag{10}$$

$$\hat{\Omega}_2 = \hat{u} + \hat{c} \qquad \hat{\mathbf{X}}_2 = \begin{bmatrix} 1 & \hat{u} + \hat{c} & \hat{H} + \hat{u}\hat{c} \end{bmatrix}^\intercal \tag{11}$$

where

$$\hat{u} = \frac{\sqrt{\rho_L} u_L + \sqrt{\rho_R} u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{12}$$

$$\hat{H} = \frac{\sqrt{\rho_L} H_L + \sqrt{\rho_R} H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{13}$$

Finally, the coefficients α_k are obtained as

$$\boldsymbol{\alpha} = \begin{bmatrix} \frac{1}{2\hat{c}^2}\left(\Delta p - \hat{\rho}\hat{c}\Delta u\right) \\ \Delta\rho - \frac{1}{\hat{c}^2}\Delta p \\ \frac{1}{2\hat{c}^2}\left(\Delta p + \hat{\rho}\hat{c}\Delta u\right) \end{bmatrix} , \tag{14}$$

with

$$\Delta\rho = \rho_R - \rho_L \ , \tag{15}$$

$$\Delta u = u_R - u_L \ , \tag{16}$$

$$\Delta p = p_R - p_L \ , \tag{17}$$

$$\hat{\rho} = \sqrt{\rho_L \rho_R} \ . \tag{18}$$

Q4 $\rightarrow$ Using this approximate Roe solver, repeat the tests of Problem 2 and simulate the shock tube from Problem 3. In addition to making the contour plots requested in Q3.2, compare the exact Riemann solution to the Roe solution by plotting $\{\rho, u, p\}$ as functions of x every 0.01 second until $t_{\text{final}} = 0.1$ second after the diaphragm burst. Comment on what you see and on the computational time it takes the Roe method to work versus the Godunov method.

Note: Don't forget the entropy fix!

Submission guidelines · Instructions on how to prepare and submit your report are available on the course's Canvas page at https://canvas.illinois.edu/courses/43781/assignments/syllabus

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References

[1] E. Toto, Riemann Solvers and Numerical Methods for Fluid Dynamics. Springer Berlin (2009).