Bayesian modeling and clustering for spatio-temporal areal data: An application to Italian Fertility Rate

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Will Biden be revered in 50 years?

Bret Stephens

OPINION

A half-century from now, Joe Biden's presidency will be remembered, as most presidencies are, with a short summary sentence. It will read: "He defeated Donald Trump, and ..."

It won't be the infrastructure bill, the rate of inflation or the Inflation Reduction Act — which, so long as China, India, South Africa and other countries continue building huge coal-fired plants, probably won't lead to a major reduction in global greenhouse-gas emissions. It won't be Hunter's emails. Nor will it be whether he served one term or two.

He could be seen as the savior of the free world, or as the man who came up short.

What will matter in 2073 is whether he reversed the global tide of democratic retreat that began long before his presidency but reached new lows with the Taliban's victory in Afghanistan and Russia's invasion of Ukraine. If Biden can turn it, it will be a historic achievement. If not, much darker days will lie ahead.

He has a real chance.

On the positive side, there is last week's announcement of 31 M-1 Abrams tanks for Ukraine, unlocking German Leopard 2 tanks to be sent as well. The decision brings Ukraine a significant step closer to eventual NATO membership, to which it has more than earned the right.

Then there's the apparent end of attempts to revive the Iran nuclear deal and a visibly tougher posture by the administration toward Tehran's misogynistic tyrants, including, last week, the largest-ever joint military exercise with Israel.

And there is the president's repeated public statements that the United States will defend Taiwan in the event of a Chinese attack. Had Biden failed to say so, the island would be in even graver danger than it is now. Closer defense ties to Japan and Australia reinforce the point.

Each of these steps evoke the cautious but purposeful way in which Biden's political hero, Franklin Roosevelt, came to Britain's aid in 1941 with Lend-Lease while preparing America for the possibility of war. They come on top of Biden's other foreign-policy

STEPHENS, PAGE 10

The New York Times publishes opinion from a wide range of perspectives in hopes of promoting constructive debate about consequential questions.

The experimental Elderly and Children Together project in Piacenza, Italy, seeks to connect the very young with the elderly in a nation headed toward a "demographic disaster."

Italy: 'Destined to disappear'?

PIACENZA, ITALY

Fastest-shrinking nation in the West is a test case for the 'gray new world'

BY JASON HOROWITZ

On one side of a glass wall, three toddlers in a nursery school fastened playdough with plastic rolling pins. On the other, three old women in a nursing home tapped the pane to get their attention.

"Let's say hi to the nonni," the children's teacher said before leading them through a door that connected the two rooms.

The children stopped to play with the magnifying glass of a delighted 89-year-old woman who had been using it to read obituaries. Then the toddlers, all 2 years old, took an elevator upstairs, where nursing home residents waited to read them picture books in a small library.

"It's an extraordinary thing," said one of the residents, Giacomo Scaramuzza, 100. "People think we are from two different worlds, but it's not true. We are in the same world. And maybe I give them something, too. There is an exchange."

Piacenza's Elderly and Children Together, an experimental project, seeks to connect the vulnerable at both extremes of life. But it also puts the country's two existential challenges under one roof.

Italy's population is aging and shrinking at the fastest rate in the West, forcing the country to adapt to a booming population of elderly that puts it at the forefront of a global demographic trend that experts call the "silver tsunami." But it faces a demographic double whammy, with a drastically sinking birthrate — among the lowest in Europe. Prime Minister Giorgia Meloni has said Italy is "destined to disappear," unless it changes.

In January, Ms. Meloni's government approved a new "Pact for the Third Age," which she said would lay a foundation for health and social overhauls for Italy's exploding population of old people. "They represent the heart of society, and a patrimony of values, traditions and precious wisdom," Ms. Meloni said, adding that the law would prevent marginalization and the "parking" of elderly in institutions.

"To care for the old is to care for all of us," she said.

The overhaul essentially adopted, experts say nearly wholesale, a measure approved at the end of the previous administration of Prime Minister Mario Draghi. Critically, it followed Mr. Draghi's lead in wrapping the legislation into the European Union recovery fund program, which ensures that it will be enacted.

"This is the acknowledgment that long-term care is a welfare policy," said Cristiano Gori, who leads the Pact for a New Welfare on the Dependent, the umbrella organization that advocated the law.

The new law, he said, will fix a system that is "a mess," streamlining and simplifying government health care and social services, and getting local and national government into the growing field of long-term care. At the same time, it seeks to keep aging Italians in their own

ITALY, PAGE 8

Playing with clay at Elderly and Children Together. Italy's falling birthrate coupled with its booming population of elderly puts it at the forefront of a global demographic trend.

Neighbors help Russia sidestep sanctions

WASHINGTON

Though some goods are blocked, overall trade has recovered

BY ANA SWANSON

A strange thing happened with smartphones in Armenia last summer.

Shipments from other parts of the world into the tiny former Soviet republic began to balloon to more than 10 times the value of phone imports in previous months. At the same time, Armenia recorded an explosion in its exports of smartphones to a beleaguered ally: Russia.

The trend, which was repeated for washing machines, computer chips and other products in a handful of Asian countries last year, provides evidence of some of the new lifelines that are keeping the Russian economy afloat. Recent data show surges in trade for some of Russia's neighbors and allies, suggesting that countries like Turkey, China, Belarus, Kazakhstan and Kyrgyzstan are stepping in to provide Russia with many of the products that Western countries have tried to cut off as punishment for Moscow's invasion of Ukraine.

Those sanctions — which include restrictions on Russia's largest banks along with limits on the sale of technology that its military could use — are blocking access to a variety of products. Reports regularly filter out of Russia about consumers frustrated by high-priced or shoddy goods, whether milk, household appliances, computer software or medication, said Maria Snegovaya, a senior fellow for Russia and Eurasia at the Center for Strategic and International Studies, in an event at the think tank last month.

Even so, Russian trade appears to have largely bounced back to where it was before the invasion of Ukraine last February. Analysts estimate that Russia's imports may have already recovered to prewar levels, or will soon do so, depending on their models.

In part, that could be because many nations have found Russia hard to quit. Recent research showed that fewer than 9 percent of companies based in the European Union and Group of 7 nations had divested one of their Russian subsidiaries. And maritime tracking firms have recorded a surge in activity by shipping fleets that may be helping Russia to export its energy, apparently bypassing Western restrictions on those sales.

While Western countries have not banned the shipment of consumer products like cellphones and washing machines to Russia, other sweeping penalties were expected to clamp down on its economy. They include a cap on the price that Russia can charge for its oil as well as restrictions on access to semi-

RUSSIA, PAGE 9

Fertility in Italy

During the last years, Italy has been experiencing a significant decrease in the birth rate. For the first time ever, in 2021 the births were below 400.000.

The birth rate has declined across the country but with a different spread across geographic areas and provinces.

It's important to understand which socio-economic factors influence the most this trend and to explain in which way the geographic location has an impact on this phenomenon.

Fertility in Italy - ISTAT Data

ISTAT has collected 10 years worth of data (from 2011 to 2020) about the **fertility rate** (the average number of children per woman of childbearing age, 15-49 years) in each province of Italy.

Alongside the fertility rate, the **average age of the mother at delivery** and the **average age of the father at the birth of the child** were computed.

We enriched this dataset with some socio-economic indices, such as **employment** and **inactivity** rate, in order to better explain the birth rate decrease phenomenon.

Fertility in Italy - ISTAT Data

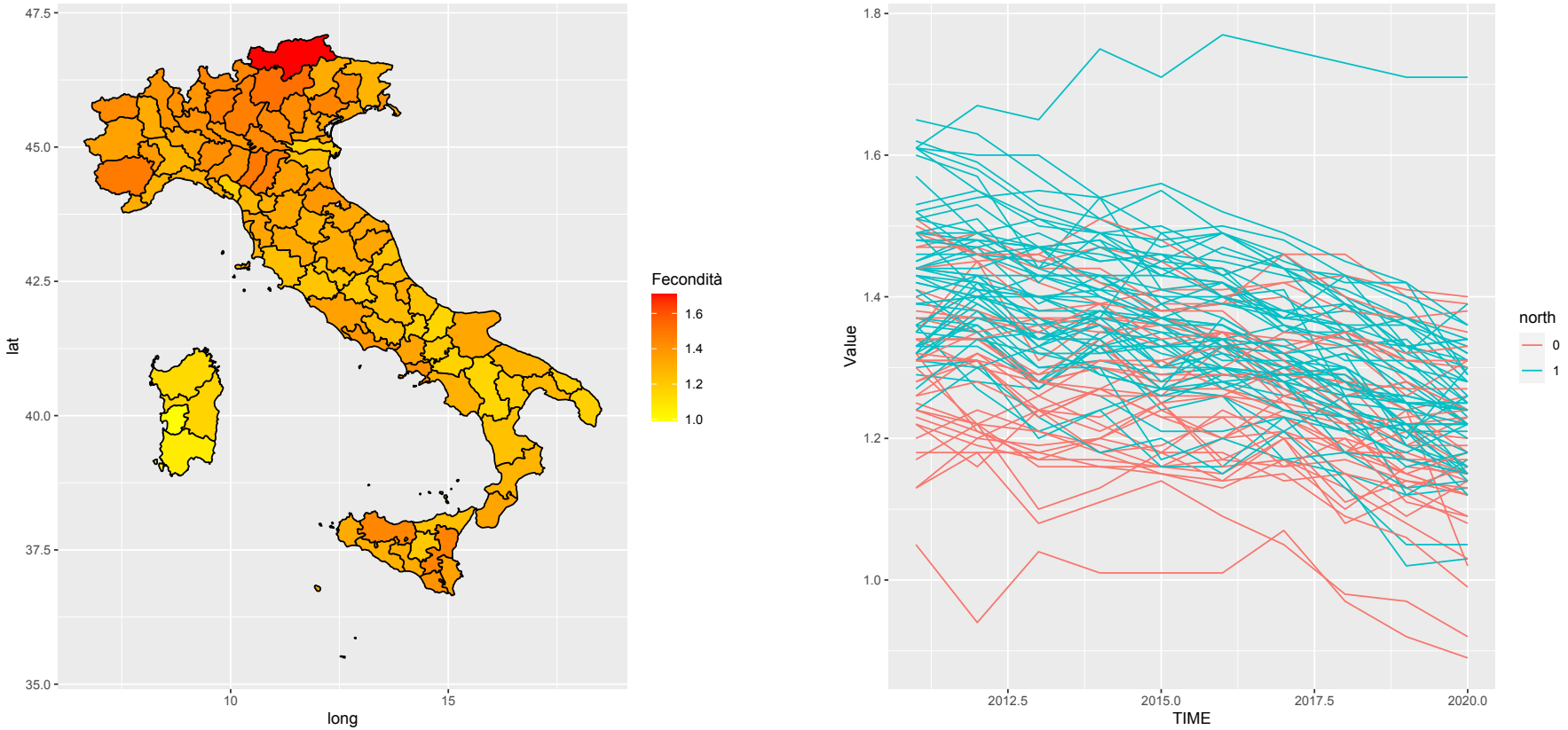

Figure: *Average fertility rate in each province (left) and time-series of fertility rate divided into north and south (right)*

Fertility in Italy - ISTAT Data

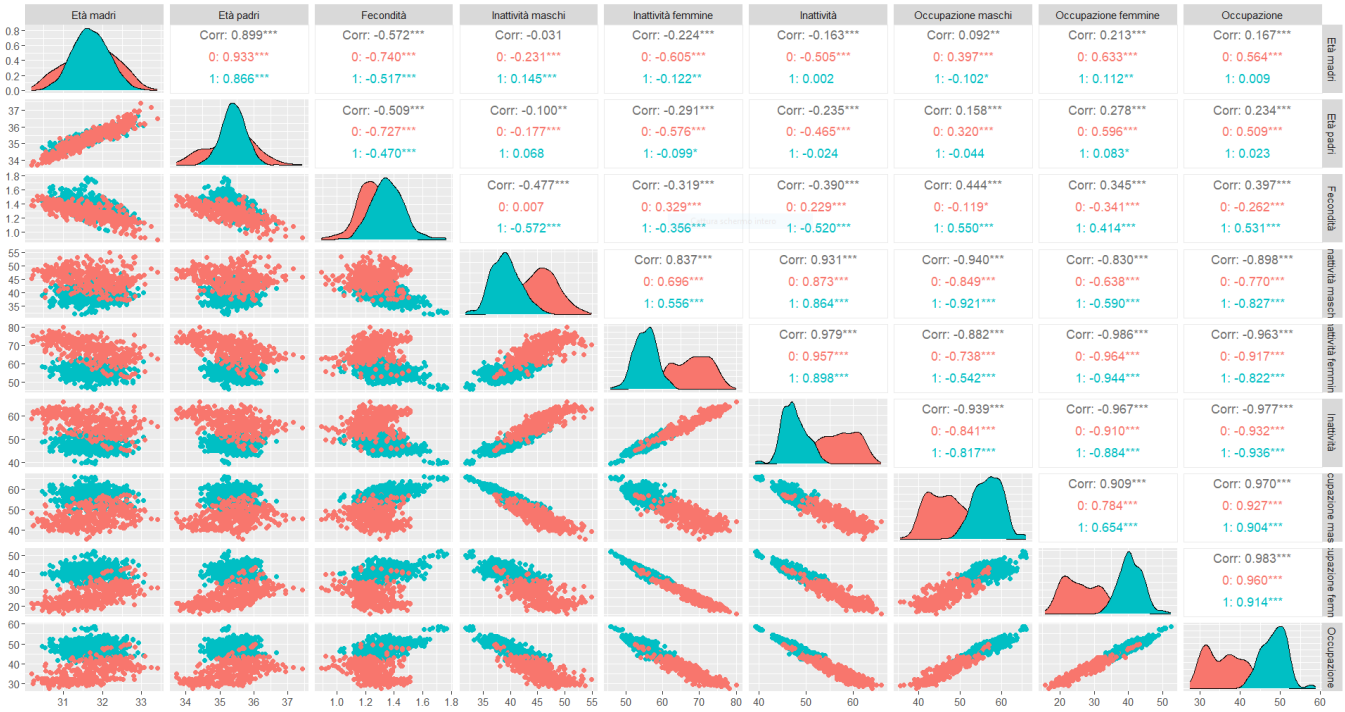

Figure: *GGpairs of the dataset divided between north and south*

Fertility rate is positively correlated with employment and negatively with inactivity rate. Fertility rate is negatively correlated with both the age of the mother and the age of the father. Employment and inactivity rate are strongly negatively correlated. north-south duality is reflected on the economic variables

Fertility in Italy - ISTAT Data

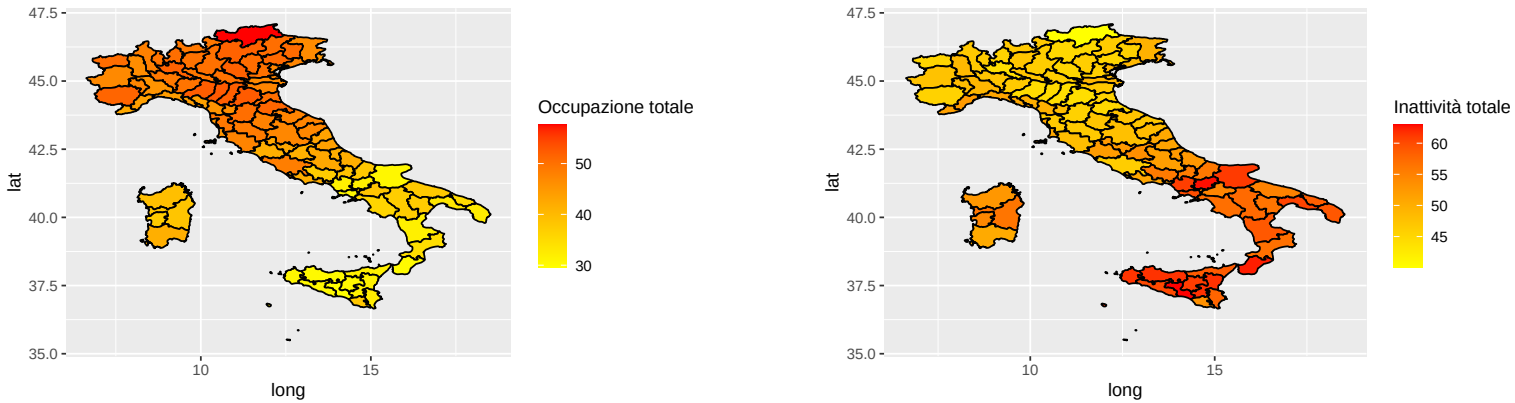

Figure: *Average employment rate in each province (left) and average inactivity rate in each province (right)*

	Moran's I	Geary's C
Fertility Rate	0.4378463	0.4286952
Age of the Mother	0.3901207	0.5001999
Female Inactivity	0.8496539	0.1227897
Male Inactivity	0.6619759	0.2628052
Female Employment	0.8588112	0.1089427
Male Employment	0.8010590	0.1487005

Full Model

Following paper (1) we consider the following model.
For all $i = 1, \dots, I$ and $t = 1, \dots, T$:

$$
\begin{aligned}
&Y_{it} | \mathbf{x}_{it}, \boldsymbol{\beta}^{\star}_{s_i}, w_{it}, \sigma^2, s_i \overset{\text{ind}}{\sim} N(\mathbf{x}_{it}^{\top} \boldsymbol{\beta}^{\star}_{s_i} + w_{it}, \sigma^2) \\
&\mathbf{w}_t | \mathbf{w}_{t-1}, \xi, \tau^2, \rho, W \sim N_I(\xi \mathbf{w}_{t-1}, \tau^2 Q(\rho, W)^{-1}) \\
&\mathbf{w}_1 | \tau^2, \rho, W \sim N_I(\mathbf{0}, \tau^2 Q(\rho, W)^{-1}) \\
&\xi \sim N(\mu_\xi, \sigma^2_\xi) \\
&\sigma^2 \sim \text{Inv-Gamma}(a_{\sigma^2}, b_{\sigma^2}) \\
&\tau^2 \sim \text{Inv-Gamma}(a_{\tau^2}, b_{\tau^2}) \\
&\mathbf{s} | \alpha \sim \text{PòlyaUrn}(\mathbf{s} | \alpha) \\
&\alpha \sim \text{Gamma}(a_\alpha, b_\alpha) \\
&\boldsymbol{\beta}^{\star}_1, \dots, \boldsymbol{\beta}^{\star}_{K_I} | \boldsymbol{\mu}_\beta, \Sigma_\beta, \overset{\text{iid}}{\sim} P_0, \\
&P_0(d\boldsymbol{\beta}^{\star}) = N_{p+1}(d\boldsymbol{\beta}^{\star} | \boldsymbol{\mu}_0, \Sigma_0)
\end{aligned}
\tag{1}
$$

where $Q(\rho, W) = \rho(\text{diag}(W\mathbf{1}) - W) + (1 - \rho)\mathbb{I}_I$

Equivalent Formulation

For all $i = 1, ..., I$ and $t = 1, ..., T$:

$$
\begin{aligned}
&\boldsymbol{Y}_i | X_i, \boldsymbol{w}_i, \sigma^2, P \overset{\text{ind}}{\sim} \int_{\mathbb{R}^{p+1}} N_T(X_i\boldsymbol{\beta} + \boldsymbol{w}_i, \sigma^2 I) P(d\boldsymbol{\beta}) \\
&P \sim DP(\alpha, P_0) \\
&\boldsymbol{w}_t | \boldsymbol{w}_{t-1}, \xi, \tau^2, \rho, W \sim N_I(\xi \boldsymbol{w}_{t-1}, \tau^2 Q(\rho, W)^{-1}) \\
&\boldsymbol{w}_1 | \tau^2, \rho, W \sim N_I(\boldsymbol{0}, \tau^2 Q(\rho, W)^{-1}) \\
&\xi \sim N(\mu_\xi, \sigma_\xi^2) \\
&\sigma^2 \sim \text{Inv-Gamma}(a_{\sigma^2}, b_{\sigma^2}) \\
&\tau^2 \sim \text{Inv-Gamma}(a_{\tau^2}, b_{\tau^2}) \\
&\alpha \sim \text{Gamma}(a_\alpha, b_\alpha) \\
&P_0(d\boldsymbol{\beta}) = N_{p+1}(d\boldsymbol{\beta} | \boldsymbol{\mu}_0, \Sigma_0)
\end{aligned}
\tag{2}
$$

where $\boldsymbol{Y}_i = (Y_{i1}, ..., Y_{iT})'$, $\boldsymbol{w}_i = (w_{i1}, ..., w_{iT})'$ and X_i is a matrix of T rows, which are the vector $\boldsymbol{x}'_{it}$ for $t = 1, .., T$.

Code Implementation

- Since the implementation of the MCMC algorithm for sampling from posterior distributions was performed in `STAN`, we used formulation 2 that did not include a Pólya Urn scheme as a prior for s.
- The Pólya Urn scheme is in fact a discrete distribution and therefore incompatible with the Hamiltonian Monte Carlo, that is the MCMC method used by `STAN` to sample from the posteriors.
- Using Stick-Breaking construction for the first line of 2 and truncating it to H equal to 10, a compromise value between the performance of the code and the goodness of the approximation, we obtained a finite mixture model

Code Optimization

- Fixing $\rho = 0.95$ implies fixing the covariance matrix of the random effects, that can be factorized with the Cholesky decomposition, obtaining L such that $L^\top L = Q(\rho, W)^{-1}$
- We can reparametrize the random effects in the following more efficient way

$$\begin{aligned} \boldsymbol{w}_1 &= L\boldsymbol{w}_1^{raw} \\ \boldsymbol{w}_t &= \xi \boldsymbol{w}_{t-1} + L\boldsymbol{w}_t^{raw} \\ \boldsymbol{w}_t^{raw} &\overset{\text{iid}}{\sim} N_I(\mathbf{0}, \tau^2 \mathbb{I}_{p+1}) \end{aligned}$$

- We used data structures that made it easy to use `STAN` *vectorization*, both in terms of matrix operations and vectorized probability functions
- We avoided using transpositions within the definition loops and adopted the most efficient solutions for accessing rows and columns of random effects

Simulation Study

To verify the correctness of the implementation and show the ability of the model to correctly estimate parameters and cluster areal locations we performed a study on simulated data.

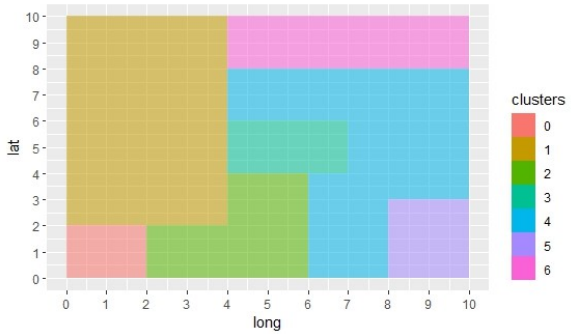

Figure: *Underlying cluster structure for simulated data.*

- Each areal location had three covariates and four regression coefficients
- ρ is set to 0.95
- $\tau^2 = \sigma^2 = 1$
- we used no prior for $\boldsymbol{\mu}_0$ and σ_0^2, set equal to zero and one respectively, as there was no reason to further facilitate the detection of clusters.

Simulation Study

- The density plots show that the true β coefficients are accurately recovered, while the trace plots imply good mixing and convergence.
- Estimating the clustering with the Binder loss and the VI we obtained the same results and we were able to find all the 7 clusters from which the data came, which still underlines the goodness of the used model.

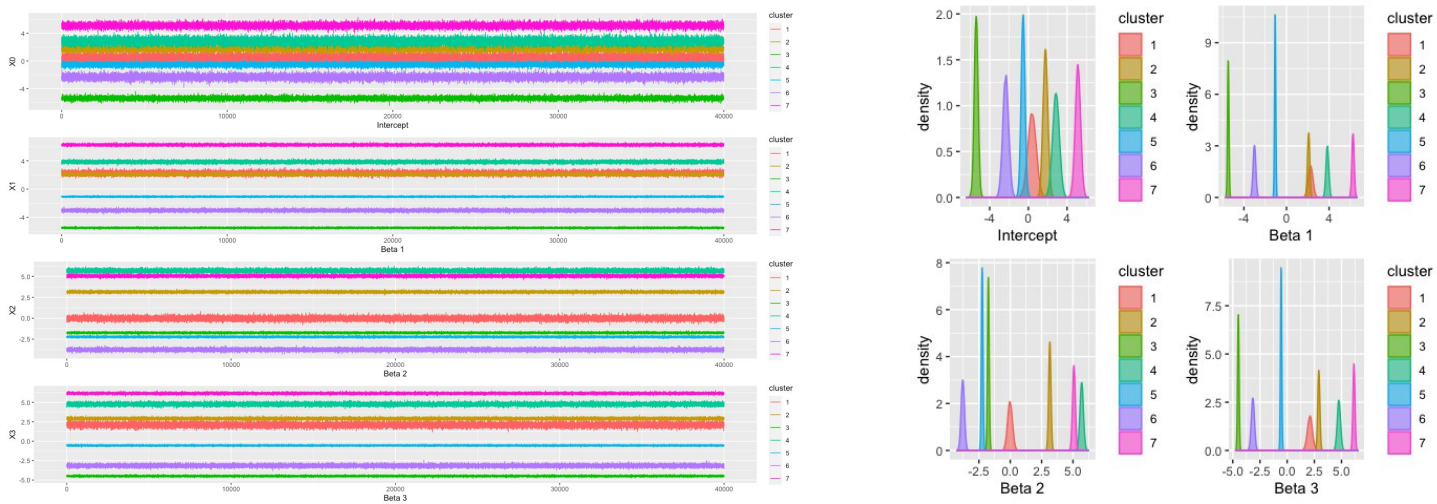

Figure: *Posterior inference of the cluster-specific regression coefficients* β .

Prior Elicitation

To help the model in recognizing clusters, we put priors on the hyperparameters of the base measure of the Dirichlet Process:

$$
\begin{aligned}
P_0(d\boldsymbol{\beta}^\star) &= N_{p+1}(d\boldsymbol{\beta}^\star | \boldsymbol{\mu}_0, \Sigma_0) \\
\boldsymbol{\mu}_0 &\sim N_{p+1}(\boldsymbol{\eta}_0, \tau_0^2 \mathbb{I}_{p+1}) \\
\Sigma_0 &= \sigma_0^2 \mathbb{I}_{p+1} \\
\sigma_0^2 &\sim \text{Inv-Gamma}(a_{\sigma_0^2}, b_{\sigma_0^2})
\end{aligned}
$$

Following (1), we fixed the hyperparameters of α to $a_\alpha = 3$ and $b_\alpha = 2$, in order to obtain a prior expected number of clusters of 6.75 with prior variance of number of clusters equal to 7.32.

Posterior Inference

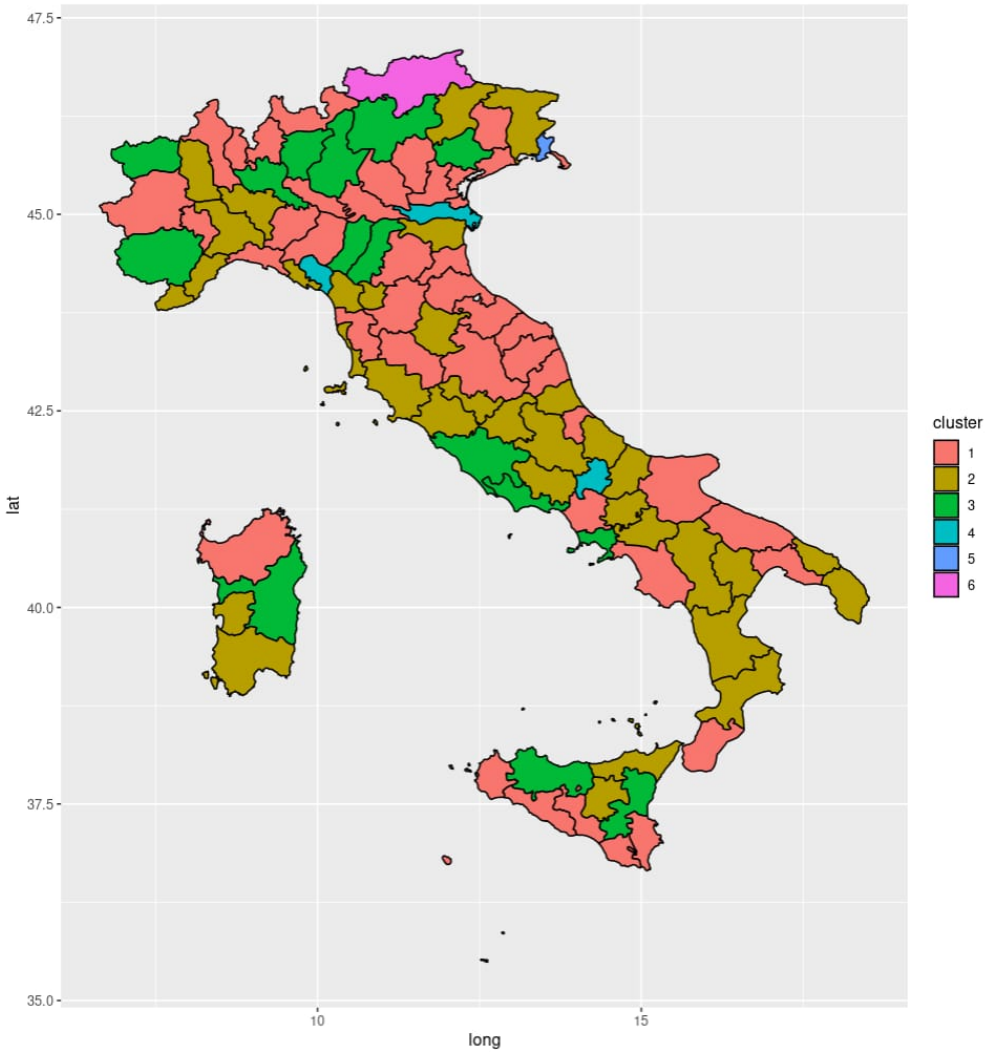

Figure: *Partition of the Italian provinces obtained by minimizing the posterior expectation of the VI loss function*

Posterior Inference

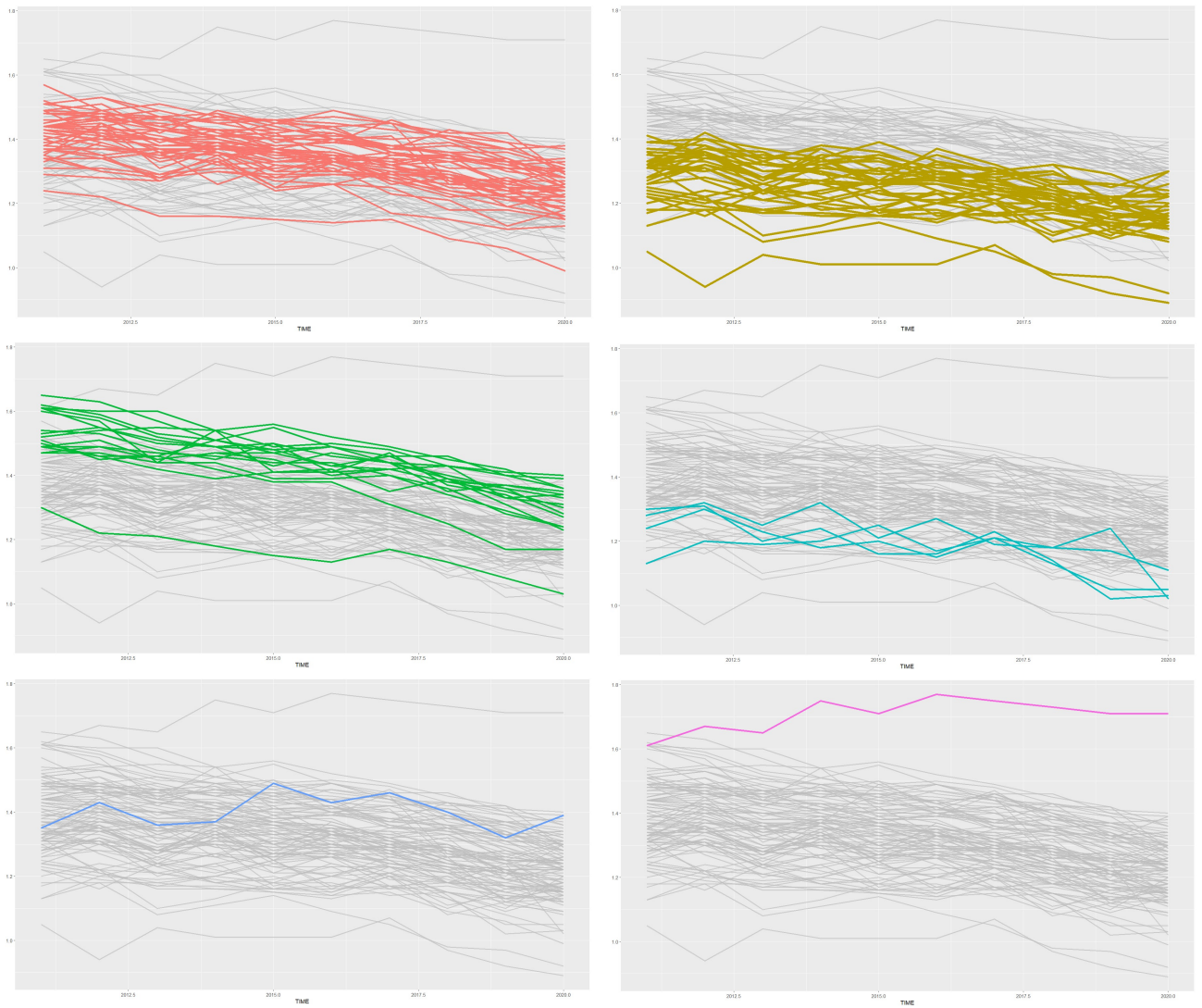

Figure: *Fertility rate time-series divided for each cluster*

Posterior Inference

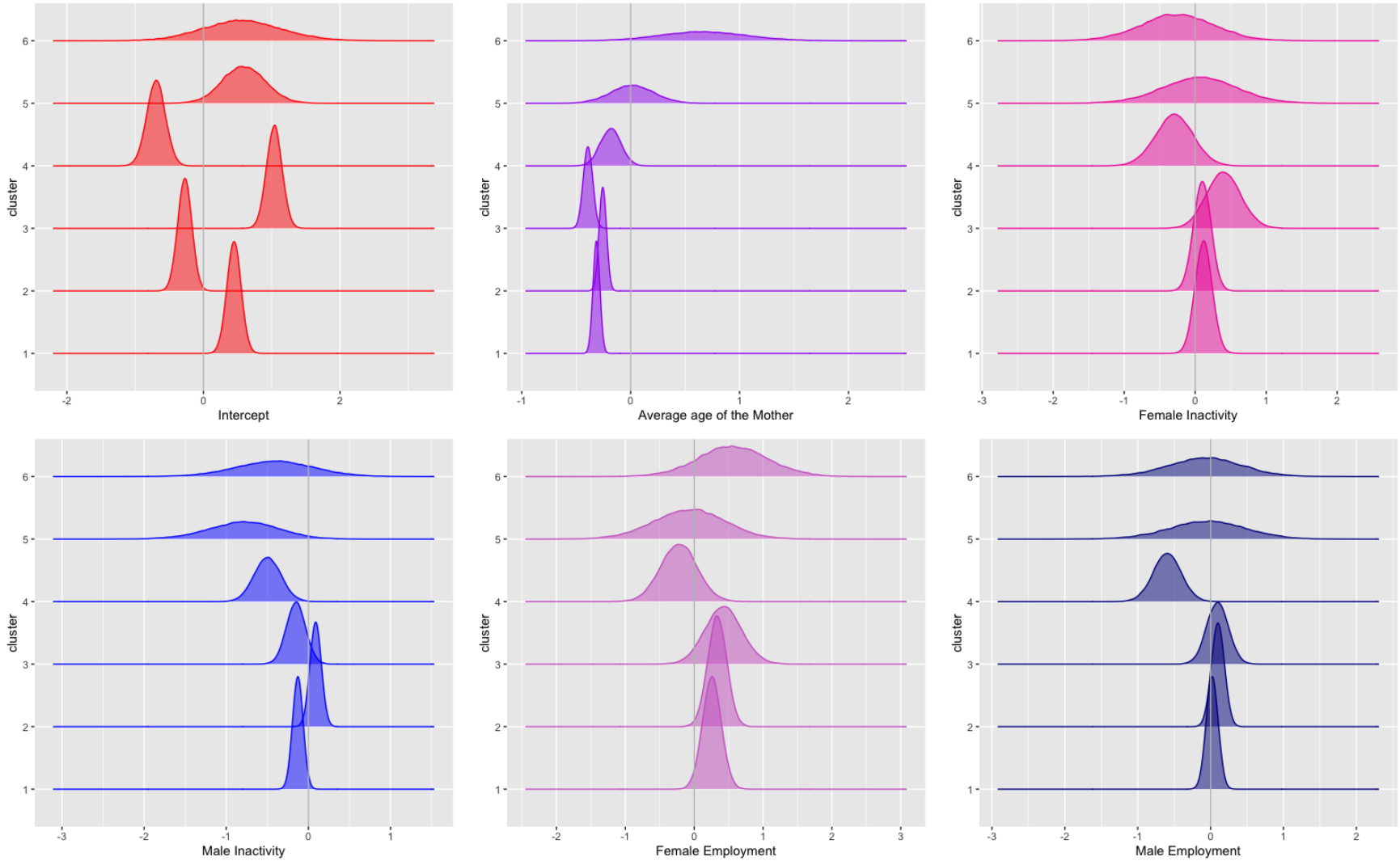

Figure: *Posterior kernel density estimates of the regression coefficients*

Posterior Inference

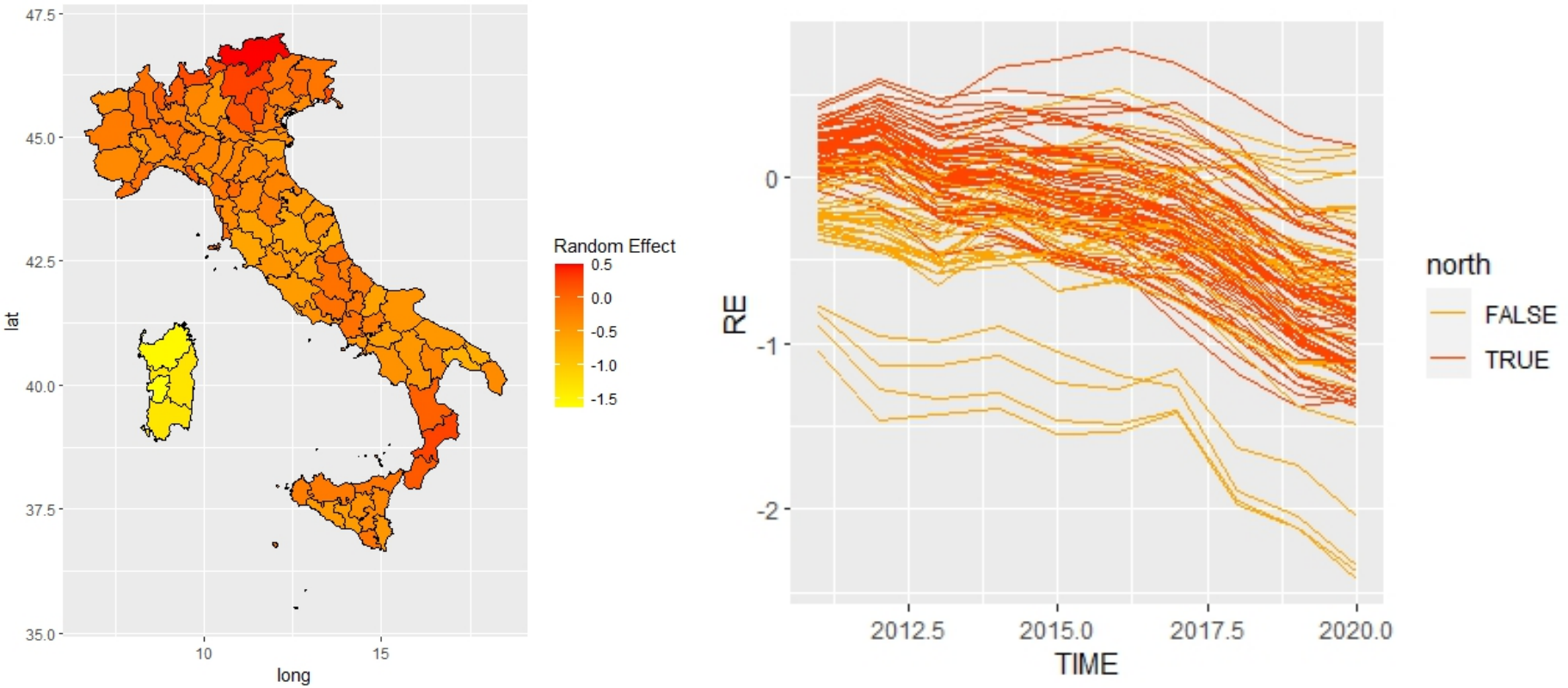

Figure: *Estimated spatio-temporal random effects*

MCMC Diagnostics

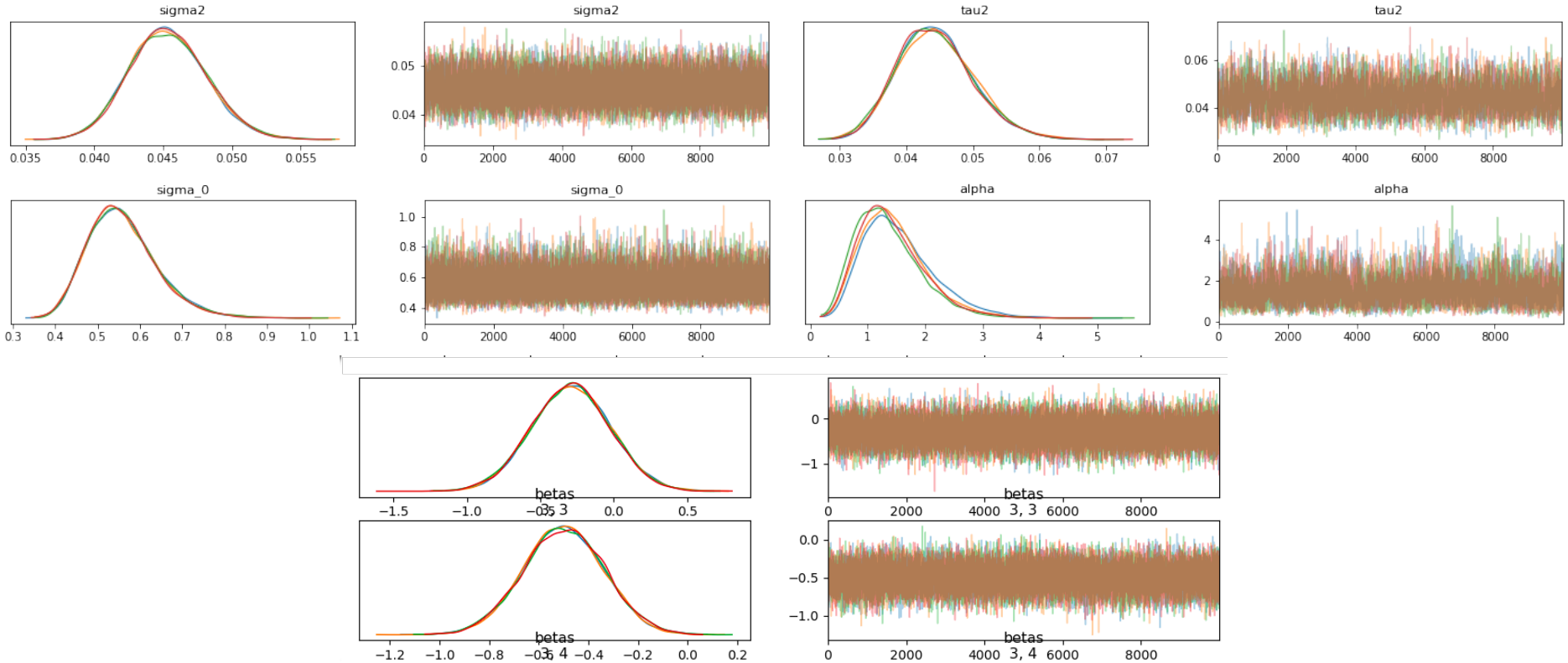

Figure: *Traceplots for some variables in the state of the MCMC*

Competitor Model

`ST.CARar()` model in R-package `CARBayesST`:

$$
\begin{aligned}
&Y_{it}|\mathbf{x}_{it}, \boldsymbol{\beta}, w_{it}, \sigma^2 \overset{\text{ind}}{\sim} N(\mathbf{x}_{it}^\top \boldsymbol{\beta} + w_{it}, \sigma^2) \\
&\mathbf{w}_t|\mathbf{w}_{t-1}, \xi, \tau^2, \rho, W \sim N_I(\xi \mathbf{w}_{t-1}, \tau^2 Q(\rho, W)^{-1}) \\
&\mathbf{w}_1|\tau^2, \rho, W \sim N_I(\mathbf{0}, \tau^2 Q(\rho, W)^{-1}) \\
&\boldsymbol{\beta} \sim N_{p+1}(\mu_\beta, \Sigma_\beta) \\
&\xi \sim U([0,1]) \\
&\rho \sim U([0,1]) \\
&\sigma^2 \sim \text{Inv-Gamma}(a_{\sigma^2}, b_{\sigma^2}) \\
&\tau^2 \sim \text{Inv-Gamma}(a_{\tau^2}, b_{\tau^2})
\end{aligned}
\tag{3}
$$

	WAIC	LPML
`ST.CARar()`	-29	-191
Model (1)	4	-62

References

[1] Alexander Mozdzen, Andrea Cremaschi, Annalisa Cadonna, Alessandra Guglielmi, and Gregor Kastner. Bayesian modeling and clustering for spatio-temporal areal data: An application to italian unemployment. Spatial Statistics, 2022

[2] Stan Development Team. Stan user's guide, 2011–2022

[3] Duncan Lee, Alastair Rushworth, Gary Napier, and William Pettersson. Carbayesst version 3.3: Spatio- temporal areal unit modelling in r with conditional autoregressive priors

[4] Rushworth A, Lee D, and Mitchell R. A spatio-temporal model for estimating the long- term effects of air pollution on respiratory hospital admissions in greater london. Spatial and Spatio-temporal Epidemiology, 2014

[5] Redazione OpenPolis. Il calo delle nascite dopo l'emergenza covid, May 2022