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Handbook of Mathematical Functions

With

Formulas, Graphs, and Mathematical Tables

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The text relating to physical constants and conversion factors (page 6) has been modified to take into account the newly adopted Système International d'Unites (SI).

ERRATA NOTICE

The original printing of this Handbook (June 1964) contained errors that have been corrected in the reprinted editions. These corrections are marked with an asterisk (*) for identification. The errors occurred on the following pages: 2–3, 6–8, 10, 15, 19–20, 25, 76, 85, 91, 102, 187, 189–197, 218, 223, 225, 233, 250, 255, 260–263, 268, 271–273, 292, 302, 328, 332, 333–337, 362, 365, 415, 423, 438–440, 443, 445, 447, 449, 451, 484, 498, 505–506, 509–510, 543, 556, 558, 562, 571, 595, 599, 600, 722–723, 739, 742, 744, 746, 752, 756, 760–765, 774, 777–785, 790, 797, 801, 822–823, 832, 835, 844, 886–889, 897, 914, 915, 920, 930–931, 936, 940–941, 944–950, 953, 960, 963, 989–990, 1010, 1026.

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Contents

16. Jacobian Elliptic Functions and Theta Functions

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Contents

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16. Jacobian Elliptic Functions and Theta Functions

Mathematical Properties

Jacobian Elliptic Functions

16.1. Introduction

A doubly periodic meromorphic function is called an *elliptic function*.

Let m, m_1 be numbers such that

$$m+m_1=1.$$

We call m the *parameter*, m_1 the *complementary parameter*.

In what follows we shall assume that the parameter m is a real number. Without loss of generality we can then suppose that $0 \leq m \leq 1$ (see **16.10, 16.11**).

We define *quarter-periods* K and iK' by

16.1.1

$$K(m)=K=\int_0^{\pi/2}\frac{d\theta}{(1-m\sin^2\theta)^{1/2}},$$

$$iK'(m)=iK'=i\int_0^{\pi/2}\frac{d\theta}{(1-m_1\sin^2\theta)^{1/2}}$$

so that K and K' are real numbers. K is called the real, iK' the imaginary quarter-period.

We note that

16.1.2 $\qquad K(m)=K'(m_1)=K'(1-m).$

We also note that if any *one* of the numbers m, m_1, $K(m)$, $K'(m)$, $K'(m)/K(m)$ is given, all the rest are determined. Thus K and K' can not both be chosen arbitrarily.

In the Argand diagram denote the points 0, K, $K+iK'$, iK' by s, c, d, n respectively. These points are at the vertices of a rectangle. The translations of this rectangle by λK, $\mu iK'$, where λ, μ are given all integral values positive or negative, will lead to the lattice

.s	.c	.s	.c
.n	.d	.n	.d
.s	.c	.s	.c
.n	.d	.n	.d

the pattern being repeated indefinitely on all sides.

Let p, q be any two of the letters s, c, d, n. Then p, q determine in the lattice a minimum rectangle whose sides are of length K and K' and whose vertices s, c, d, n are in counterclockwise order.

Definition

The Jacobian elliptic function pq u is defined by the following three properties.

(i) pq u has a simple zero at p and a simple pole at q.

(ii) The step from p to q is a half-period of pq u. Those of the numbers K, iK', $K+iK'$ which differ from this step are only quarter-periods.

(iii) The coefficient of the leading term in the expansion of pq u in ascending powers of u about $u=0$ is unity. With regard to (iii) the leading term is u, $1/u$, 1 according as $u=0$ is a zero, a pole, or an ordinary point.

Thus the functions with a pole or zero at the origin (i.e., the functions in which one letter is s) are odd, and the others are even.

Should we wish to call explicit attention to the value of the parameter, we write pq $(u|m)$ instead of pq u.

The Jacobian elliptic functions can also be defined with respect to certain integrals. Thus if

16.1.3 $\qquad u=\int_0^{\varphi}\frac{d\theta}{(1-m\sin^2\theta)^{1/2}},$

the angle φ is called the *amplitude*

16.1.4 $\quad \varphi=\text{am}\, u$

and we define

16.1.5

$$\text{sn}\, u=\sin\varphi,\ \text{cn}\, u=\cos\varphi,$$

$$\text{dn}\, u=(1-m\sin^2\varphi)^{1/2}=\Delta(\varphi).$$

Similarly all the functions pq u can be expressed in terms of φ. This second set of definitions, although seemingly different, is mathematically equivalent to the definition previously given in terms of a lattice. For further explanation of notations, including the interpretation, of such expressions as sn $(\varphi\backslash\alpha)$, cn $(u|m)$, dn (u, k), see **17.2.**

16.2. Classification of the Twelve Jacobian Elliptic Functions

According to Poles and Half-Periods

	Pole iK'	Pole $K+iK'$	Pole K	Pole 0	
Half period iK'	sn u	cd u	dc u	ns u	Periods $2iK'$, $4K+4iK'$, $4K$
Half period $K+iK'$	cn u	sd u	nc u	ds u	Periods $4iK'$, $2K+2iK'$, $4K$
Half period K	dn u	nd u	sc u	cs u	Periods $4iK'$, $4K+4iK'$, $2K$

The three functions in a vertical column are *copolar*.

The four functions in a horizontal line are *coperiodic*. Of the periods quoted in the last line of each row only two are independent.

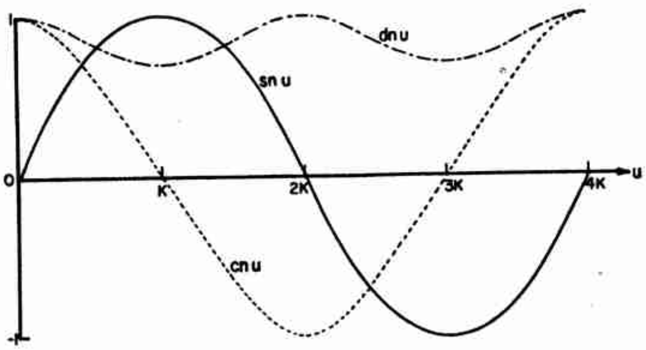

FIGURE 16.1. *Jacobian elliptic functions*

sn u, cn u, dn u

$$m=\frac{1}{2}$$

The curve for cn $(u|\frac{1}{2})$ is the boundary between those which have an inflexion and those which have not.

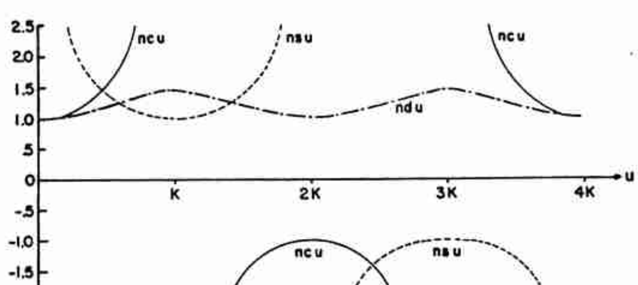

FIGURE 16.2. *Jacobian elliptic functions*

ns u, nc u, nd u

$$m=\frac{1}{2}$$

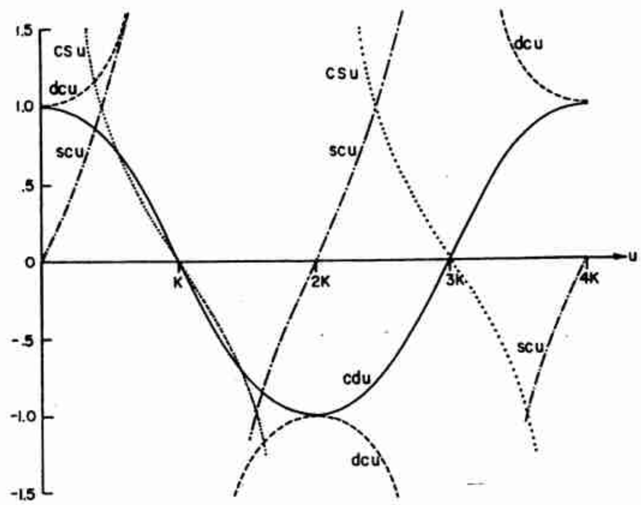

FIGURE 16.3. *Jacobian elliptic functions*

sc u, cs u, cd u, dc u

$$m=\frac{1}{2}$$

16.3. Relation of the Jacobian Functions to the Copolar Trio sn u, cn u, dn u

16.3.1 $\quad \operatorname{cd} u=\dfrac{\operatorname{cn} u}{\operatorname{dn} u} \qquad \operatorname{dc} u=\dfrac{\operatorname{dn} u}{\operatorname{cn} u} \qquad \operatorname{ns} u=\dfrac{1}{\operatorname{sn} u}$

16.3.2 $\quad \operatorname{sd} u=\dfrac{\operatorname{sn} u}{\operatorname{dn} u} \qquad \operatorname{nc} u=\dfrac{1}{\operatorname{cn} u} \qquad \operatorname{ds} u=\dfrac{\operatorname{dn} u}{\operatorname{sn} u}$

16.3.3 $\quad \operatorname{nd} u=\dfrac{1}{\operatorname{dn} u} \qquad \operatorname{sc} u=\dfrac{\operatorname{sn} u}{\operatorname{cn} u} \qquad \operatorname{cs} u=\dfrac{\operatorname{cn} u}{\operatorname{sn} u}$

And generally if p, q, r are any three of the letters s, c, d, n,

16.3.4 $$\operatorname{pq} u=\frac{\operatorname{pr} u}{\operatorname{qr} u}$$

provided that when two letters are the same, e.g., pp u, the corresponding function is put equal to unity.

16.4. Calculation of the Jacobian Functions by Use of the Arithmetic-Geometric Mean (A.G.M.)

For the A.G.M. scale see **17.6**.

To calculate sn $(u|m)$, cn $(u|m)$, and dn $(u|m)$ form the A.G.M. scale starting with

16.4.1 $$a_0=1,\ b_0=\sqrt{m_1},\ c_0=\sqrt{m},$$

terminating at the step N when c_N is negligible to the accuracy required. Find φ_N in degrees where

16.4.2 $$\varphi_N=2^N a_N u\,\frac{180°}{\pi}$$

and then compute successively φ_{N-1}, φ_{N-2}, . . ., φ_1, φ_0 from the recurrence relation

16.4.3 $$\sin\,(2\varphi_{n-1}-\varphi_n)=\frac{c_n}{a_n}\sin\varphi_n.$$

Then

16.4.4

$$\mathrm{sn}\,(u|m)=\sin\varphi_0,\quad \mathrm{cn}\,(u|m)=\cos\varphi_0$$

$$\mathrm{dn}\,(u|m)=\frac{\cos\varphi_0}{\cos\,(\varphi_1-\varphi_0)}.$$

From these all the other functions can be determined.

16.5. Special Arguments

	u	sn u	cn u	dn u
16.5.1	0	0	1	1
16.5.2	$\frac{1}{2}K$	$\frac{1}{(1+m_1^{1/2})^{1/2}}$	$\frac{m_1^{1/4}}{(1+m_1^{1/2})^{1/2}}$	$m_1^{1/4}$
16.5.3	K	1	0	$m_1^{1/2}$
16.5.4	$\frac{1}{2}(iK')$	$im^{-1/4}$	$\frac{(1+m^{1/2})^{1/2}}{m^{1/4}}$	$(1+m^{1/2})^{1/2}$
* **16.5.5**	$\frac{1}{2}(K+iK')$	$2^{-1/2}m^{-1/4}[(1+m^{1/2})^{1/2}+i(1-m^{1/2})^{1/2}]$	$\left(\frac{m_1}{4m}\right)^{1/4}(1-i)$	$\left(\frac{m_1}{4}\right)^{1/4}[(1+m_1^{1/2})^{1/2}-i(1-m_1^{1/2})^{1/2}]$
16.5.6	$K+\frac{1}{2}(iK')$	$m^{-1/4}$	$-i\left(\frac{1-m^{1/2}}{m^{1/2}}\right)^{1/2}$	$(1-m^{1/2})^{1/2}$
16.5.7	iK'	∞	∞	∞
16.5.8	$\frac{1}{2}K+iK'$	$(1-m_1^{1/2})^{-1/2}$	$-i\left(\frac{m_1^{1/2}}{1-m_1^{1/2}}\right)^{1/2}$	$-im_1^{1/4}$
16.5.9	$K+iK'$	$m^{-1/2}$	$-i(m_1/m)^{1/2}$	0

16.6. Jacobian Functions when $m=0$ or 1

		$m=0$	$m=1$
16.6.1	sn $(u\|m)$	$\sin u$	$\tanh u$
16.6.2	cn $(u\|m)$	$\cos u$	sech u
16.6.3	dn $(u\|m)$	1	sech u
16.6.4	cd $(u\|m)$	$\cos u$	1
16.6.5	sd $(u\|m)$	$\sin u$	$\sinh u$
16.6.6	nd $(u\|m)$	1	$\cosh u$
16.6.7	dc $(u\|m)$	$\sec u$	1
16.6.8	nc $(u\|m)$	$\sec u$	$\cosh u$
16.6.9	sc $(u\|m)$	$\tan u$	$\sinh u$
16.6.10	ns $(u\|m)$	$\csc u$	$\coth u$
16.6.11	ds $(u\|m)$	$\csc u$	csch u
16.6.12	cs $(u\|m)$	$\cot u$	csch u
16.6.13	am $(u\|m)$	u	gd u

*See page II

16.7. Principal Terms

When the elliptic function pq u is expanded in ascending powers of $(u-K_r)$, where K_r is one of $0, K, iK', K+iK'$, the first term of the expansion is called the principal term and has one of the forms A, $B\times(u-K_r)$, $C\div(u-K_r)$ according as K_r is an ordinary point, a zero, or a pole of pq u. The following list gives these forms, where $\times$ means that the factor $(u-K_r)$ has to be supplied and $\div$ means that the divisor $(u-K_r)$ has to be supplied.

	$K_r=$	0	K	iK'	$K+iK'$
16.7.1	sn u	$1\times$	1	$m^{-1/2}\div$	$m^{-1/2}$
16.7.2	cn u	1	$-m_1^{1/2}\times$	$-im^{-1/2}\div$	$-i\left(\frac{m_1}{m}\right)^{1/2}$
16.7.3	dn u	1	$m_1^{1/2}$	$-i\div$	$im_1^{1/2}\times$
16.7.4	cd u	1	$-1\times$	$m^{-1/2}$	$-m^{-1/2}\div$
16.7.5	sd u	$1\times$	$m_1^{-1/2}$	$im^{-1/2}$	$-i\frac{1}{(mm_1)^{1/2}}\div$
16.7.6	nd u	1	$m_1^{-1/2}$	$i\times$	$-im_1^{-1/2}\div$
16.7.7	dc u	1	$-1\div$	$m^{1/2}$	$-m^{1/2}\times$
16.7.8	nc u	1	$-m_1^{-1/2}\div$	$im^{1/2}\times$	$i\left(\frac{m}{m_1}\right)^{1/2}$
16.7.9	sc u	$1\times$	$-m_1^{-1/2}\div$	i	$im_1^{-1/2}$
16.7.10	ns u	$1\div$	1	$m^{1/2}\times$	$m^{1/2}$
16.7.11	ds u	$1\div$	$m_1^{1/2}$	$-im^{1/2}$	$i(mm_1)^{1/2}\times$
16.7.12	cs u	$1\div$	$-m_1^{1/2}\times$	$-i$	$-im_1^{1/2}$

16.8. Change of Argument

		u	$-u$	$u+K$	$u-K$	$K-u$	$u+2K$	$u-2K$	$2K-u$	$u+iK'$	$u+2iK'$	$u+K+iK'$	$u+2K+2iK'$
16.8.1	sn	sn u	$-$sn u	cd u	$-$cd u	cd u	$-$sn u	$-$sn u	sn u	$m^{-1/2}$ns u	sn u	$m^{-1/2}$dc u	$-$sn u
16.8.2	cn	cn u	cn u	$-m_1^{1/2}$sd u	$m_1^{1/2}$sd u	$m_1^{1/2}$sd u	$-$cn u	$-$cn u	$-$cn u	$-im^{-1/2}$ds u	$-$cn u	$-im_1^{1/2}m^{-1/2}$nc u	cn u
16.8.3	dn	dn u	dn u	$m_1^{1/2}$nd u	$m_1^{1/2}$nd u	$m_1^{1/2}$nd u	dn u	dn u	dn u	$-i$cs u	$-$dn u	$im_1^{1/2}$sc u	$-$dn u
16.8.4	cd	cd u	cd u	$-$sn u	sn u	sn u	$-$cd u	$-$cd u	$-$cd u	$m^{-1/2}$dc u	cd u	$-m^{-1/2}$ns u	$-$cd u
16.8.5	sd	sd u	$-$sd u	$m_1^{-1/2}$cn u	$-m_1^{-1/2}$cn u	$m_1^{-1/2}$cn u	$-$sd u	$-$sd u	sd u	$im^{-1/2}$nc u	$-$sd u	$-im_1^{-1/2}m^{-1/2}$ds u	sd u
16.8.6	nd	nd u	nd u	$m_1^{-1/2}$dn u	$m_1^{-1/2}$dn u	$m_1^{-1/2}$dn u	nd u	nd u	nd u	$i$sc u	$-$nd u	$-im_1^{-1/2}$cs u	$-$nd u
16.8.7	dc	dc u	dc u	$-$ns u	ns u	ns u	$-$dc u	$-$dc u	$-$dc u	$m^{1/2}$cd u	dc u	$-m^{1/2}$sn u	$-$dc u
16.8.8	nc	nc u	nc u	$-m_1^{-1/2}$ds u	$m_1^{-1/2}$ds u	$m_1^{-1/2}$ds u	$-$nc u	$-$nc u	$-$nc u	$im^{1/2}$sd u	$-$nc u	$im_1^{-1/2}m^{1/2}$cn u	nc u
16.8.9	sc	sc u	$-$sc u	$-m_1^{-1/2}$cs u	$-m_1^{-1/2}$cs u	$m_1^{-1/2}$cs u	sc u	sc u	$-$sc u	ind u	$-$sc u	$im_1^{-1/2}$dn u	$-$sc u
16.8.10	ns	ns u	$-$ns u	dc u	$-$dc u	dc u	$-$ns u	$-$ns u	ns u	$m^{1/2}$sn u	ns u	$m^{1/2}$cd u	$-$ns u
16.8.11	ds	ds u	$-$ds u	$m_1^{1/2}$nc u	$-m_1^{1/2}$nc u	$m_1^{1/2}$nc u	$-$ds u	$-$ds u	ds u	$-im^{1/2}$cn u	$-$ds u	$im_1^{1/2}m^{1/2}$sd u	ds u
16.8.12	cs	cs u	$-$cs u	$-m_1^{1/2}$sc u	$-m_1^{1/2}$sc u	$m_1^{1/2}$sc u	cs u	cs u	$-$cs u	$-i$dn u	$-$cs u	$-im_1^{1/2}$nd u	$-$cs u

16.9. Relations Between the Squares of the Functions

16.9.1 $-\mathrm{dn}^2u+m_1=-m\ \mathrm{cn}^2u=m\ \mathrm{sn}^2u-m$

16.9.2 $-m_1\mathrm{nd}^2u+m_1=-mm_1\mathrm{sd}^2u=m\ \mathrm{cd}^2u-m$

16.9.3 $m_1\mathrm{sc}^2u+m_1=m_1\mathrm{nc}^2u=\mathrm{dc}^2u-m$

16.9.4 $\mathrm{cs}^2u+m_1=\mathrm{ds}^2u=\mathrm{ns}^2u-m$

In using the above results remember that $m+m_1=1$.

If pq u, rt u are any two of the twelve functions, one entry expresses tq^2u in terms of pq^2u and another expresses qt^2u in terms of rt^2u. Since $\mathrm{tq}^2u\cdot\mathrm{qt}^2u=1$, we can obtain from the table the bilinear relation between pq^2u and rt^2u. Thus for the functions cd u, sn u we have

16.9.5 $$\mathrm{nd}^2u=\frac{1-m\ \mathrm{cd}^2u}{m_1},\ \mathrm{dn}^2u=1-m\ \mathrm{sn}^2u$$

and therefore

16.9.6 $$(1-m\ \mathrm{cd}^2u)(1-m\ \mathrm{sn}^2u)=m_1.$$

16.10. Change of Parameter

Negative Parameter

If m is a positive number, let

16.10.1 $$\mu=\frac{m}{1+m},\ \mu_1=\frac{1}{1+m},\ v=\frac{u}{\mu_1^{\frac{1}{2}}}\qquad(0<\mu<1)$$

16.10.2 $\mathrm{sn}\ (u|-m)=\mu_1^{\frac{1}{2}}\mathrm{sd}\ (v|\mu)$

16.10.3 $\mathrm{cn}\ (u|-m)=\mathrm{cd}\ (v|\mu)$

16.10.4 $\mathrm{dn}\ (u|-m)=\mathrm{nd}\ (v|\mu)$.

16.11. Reciprocal Parameter (Jacobi's Real Transformation)

16.11.1 $m>0,\ \mu=m^{-1},\ v=um^{1/2}$

16.11.2 $\mathrm{sn}\ (u|m)=\mu^{1/2}\mathrm{sn}\ (v|\mu)$

16.11.3 $\mathrm{cn}\ (u|m)=\mathrm{dn}\ (v|\mu)$

16.11.4 $\mathrm{dn}\ (u|m)=\mathrm{cn}\ (v|\mu)$

Here if $m>1$ then $m^{-1}=\mu<1$.

Thus elliptic functions whose parameter is real can be made to depend on elliptic functions whose parameter lies between 0 and 1.

16.12. Descending Landen Transformation (Gauss' Transformation)

To decrease the parameter, let

16.12.1 $$\mu=\left(\frac{1-m_1^{1/2}}{1+m_1^{1/2}}\right)^2,\ v=\frac{u}{1+\mu^{1/2}},$$

then

16.12.2 $$\mathrm{sn}\ (u|m)=\frac{(1+\mu^{1/2})\ \mathrm{sn}\ (v|\mu)}{1+\mu^{1/2}\ \mathrm{sn}^2\ (v|\mu)}$$

16.12.3 $$\mathrm{cn}\ (u|m)=\frac{\mathrm{cn}\ (v|\mu)\ \mathrm{dn}\ (v|\mu)}{1+\mu^{1/2}\ \mathrm{sn}^2\ (v|\mu)}$$

16.12.4 $$\mathrm{dn}\ (u|m)=\frac{\mathrm{dn}^2(v|\mu)-(1-\mu^{1/2})}{(1+\mu^{1/2})-\mathrm{dn}^2(v|\mu)}.$$

Note that successive applications can be made conveniently to find sn $(u|m)$ in terms of sn $(v|\mu)$ and dn $(u|m)$ in terms of dn $(v|\mu)$, but that the calculation of cn $(u|m)$ requires all three functions.

16.13. Approximation in Terms of Circular Functions

When the parameter m is so small that we may neglect m^2 and higher powers, we have the approximations

16.13.1
$$\mathrm{sn}\ (u|m)\approx\sin u-\frac{1}{4}\ m(u-\sin u\cos u)\cos u$$

16.13.2
$$\mathrm{cn}\ (u|m)\approx\cos u+\frac{1}{4}\ m(u-\sin u\cos u)\sin u$$

16.13.3 $$\mathrm{dn}\ (u|m)\approx1-\frac{1}{2}\ m\sin^2 u$$

16.13.4 $$\mathrm{am}\ (u|m)\approx u-\frac{1}{4}\ m(u-\sin u\cos u).$$

One way of calculating the Jacobian functions is to use Landen's descending transformation to reduce the parameter sufficiently for the above formulae to become applicable. See also **16.14.**

16.14. Ascending Landen Transformation

To increase the parameter, let

16.14.1 $$\mu=\frac{4m^{1/2}}{(1+m^{1/2})^2},\ \mu_1=\left(\frac{1-m^{1/2}}{1+m^{1/2}}\right)^2,\ v=\frac{u}{1+\mu_1^{1/2}}$$

16.14.2 $$\mathrm{sn}\ (u|m)=(1+\mu_1^{1/2})\ \frac{\mathrm{sn}\ (v|\mu)\ \mathrm{cn}\ (v|\mu)}{\mathrm{dn}\ (v|\mu)}$$

16.14.3 $$\mathrm{cn}\ (u|m)=\frac{1+\mu_1^{1/2}}{\mu}\ \frac{\mathrm{dn}^2(v|\mu)-\mu_1^{1/2}}{\mathrm{dn}\ (v|\mu)}$$

16.14.4 $$\mathrm{dn}\ (u|m)=\frac{1-\mu_1^{1/2}}{\mu}\ \frac{\mathrm{dn}^2\ (v|\mu)+\mu_1^{1/2}}{\mathrm{dn}\ (v|\mu)}$$

Note that, when successive applications are to be made, it is simplest to calculate dn $(u|m)$ since this is expressed always in terms of the same function. The calculation of cn $(u|m)$ leads to that of dn $(v|\mu)$.

The calculation of sn $(u|m)$ necessitates the evaluation of all three functions.

16.15. Approximation in Terms of Hyperbolic Functions

When the parameter m is so close to unity that m_1^2 and higher powers of m_1 can be neglected we have the approximations

16.15.1

$$\operatorname{sn}(u|m)\approx\tanh u+\frac{1}{4}m_1(\sinh u\cosh u-u)\operatorname{sech}^2 u$$

16.15.2

$$\operatorname{cn}(u|m)\approx\operatorname{sech} u-\frac{1}{4}m_1(\sinh u\cosh u-u)\tanh u\operatorname{sech} u$$

16.15.3

$$\operatorname{dn}(u|m)\approx\operatorname{sech} u+\frac{1}{4}m_1(\sinh u\cosh u+u)\tanh u\operatorname{sech} u$$

16.15.4

$$\operatorname{am}(u|m)\approx\operatorname{gd} u+\frac{1}{4}m_1(\sinh u\cosh u-u)\operatorname{sech} u.$$

Another way of calculating the Jacobian functions is to use Landen's ascending transformation to increase the parameter sufficiently for the above formulae to become applicable. See also **16.13.**

16.16. Derivatives

	Function	Derivative	
16.16.1	sn u	cn u dn u	Pole n
16.16.2	cn u	$-$sn u dn u	
16.16.3	dn u	$-m$ sn u cn u	
16.16.4	cd u	$-m_1$ sd u nd u	Pole d
16.16.5	sd u	cd u nd u	
16.16.6	nd u	m sd u cd u	
16.16.7	dc u	m_1 sc u nc u	Pole c
16.16.8	nc u	sc u dc u	
16.16.9	sc u	dc u nc u	
16.16.10	ns u	$-$ds u cs u	Pole s
16.16.11	ds u	$-$cs u ns u	
16.16.12	cs u	$-$ns u ds u	

Note that the derivative is proportional to the product of the two copolar functions.

16.17. Addition Theorems

16.17.1 $$\operatorname{sn}(u+v)=\frac{\operatorname{sn}u\cdot\operatorname{cn}v\cdot\operatorname{dn}v+\operatorname{sn}v\cdot\operatorname{cn}u\cdot\operatorname{dn}u}{1-m\operatorname{sn}^2u\cdot\operatorname{sn}^2v}$$

16.17.2 $$\operatorname{cn}(u+v)=\frac{\operatorname{cn}u\cdot\operatorname{cn}v-\operatorname{sn}u\cdot\operatorname{dn}u\cdot\operatorname{sn}v\cdot\operatorname{dn}v}{1-m\operatorname{sn}^2u\operatorname{sn}^2v}$$

16.17.3 $$\operatorname{dn}(u+v)=\frac{\operatorname{dn}u\cdot\operatorname{dn}v-m\operatorname{sn}u\cdot\operatorname{cn}u\cdot\operatorname{sn}v\cdot\operatorname{cn}v}{1-m\operatorname{sn}^2u\cdot\operatorname{sn}^2v}$$

Addition theorems are derivable one from another and are expressible in a great variety of forms. Thus $\operatorname{ns}(u+v)$ comes from $1/\operatorname{sn}(u+v)$ in the form $(1-m\operatorname{sn}^2u\operatorname{sn}^2v)/(\operatorname{sn}u\operatorname{cn}v\operatorname{dn}v+\operatorname{sn}v\operatorname{cn}u\operatorname{dn}u)$ from **16.17.1.**

Alternatively $\operatorname{ns}(u+v)=m^{1/2}\operatorname{sn}\{(iK'-u)-v\}$ which again from **16.17.1** yields the form $(\operatorname{ns}u\operatorname{cs}v\operatorname{ds}u-\operatorname{ns}v\operatorname{cs}u\operatorname{ds}v)/(\operatorname{ns}^2u-\operatorname{ns}^2v)$.

The function $\operatorname{pq}(u+v)$ is a rational function of the four functions pq u, pq v, pq$'u$, pq$'v$.

16.18. Double Arguments

16.18.1 $$\operatorname{sn}2u=\frac{2\operatorname{sn}u\cdot\operatorname{cn}u\cdot\operatorname{dn}u}{1-m\operatorname{sn}^4u}=\frac{2\operatorname{sn}u\cdot\operatorname{cn}u\cdot\operatorname{dn}u}{\operatorname{cn}^2u+\operatorname{sn}^2u\cdot\operatorname{dn}^2u}$$

16.18.2 $$\operatorname{cn}2u=\frac{\operatorname{cn}^2u-\operatorname{sn}^2u\cdot\operatorname{dn}^2u}{1-m\operatorname{sn}^4u}=\frac{\operatorname{cn}^2u-\operatorname{sn}^2u\cdot\operatorname{dn}^2u}{\operatorname{cn}^2u+\operatorname{sn}^2u\cdot\operatorname{dn}^2u}$$

16.18.3 $$\operatorname{dn}2u=\frac{\operatorname{dn}^2u-m\operatorname{sn}^2u\cdot\operatorname{cn}^2u}{1-m\operatorname{sn}^4u}=\frac{\operatorname{dn}^2u+\operatorname{cn}^2u(\operatorname{dn}^2u-1)}{\operatorname{dn}^2u-\operatorname{cn}^2u(\operatorname{dn}^2u-1)}$$

16.18.4 $$\frac{1-\operatorname{cn}2u}{1+\operatorname{cn}2u}=\frac{\operatorname{sn}^2u\cdot\operatorname{dn}^2u}{\operatorname{cn}^2u}$$

16.18.5 $$\frac{1-\operatorname{dn}2u}{1+\operatorname{dn}2u}=\frac{m\operatorname{sn}^2u\cdot\operatorname{cn}^2u}{\operatorname{dn}^2u}$$

16.19. Half Arguments

16.19.1 $$\operatorname{sn}^2\tfrac{1}{2}u=\frac{1-\operatorname{cn}u}{1+\operatorname{dn}u}$$

16.19.2 $$\operatorname{cn}^2\tfrac{1}{2}u=\frac{\operatorname{dn}u+\operatorname{cn}u}{1+\operatorname{dn}u}$$

16.19.3 $$\operatorname{dn}^2\tfrac{1}{2}u=\frac{m_1+\operatorname{dn}u+m\operatorname{cn}u}{1+\operatorname{dn}u}$$

16.20. Jacobi's Imaginary Transformation

16.20.1 $$\operatorname{sn}(iu|m)=i\operatorname{sc}(u|m_1)$$

16.20.2 $$\operatorname{cn}(iu|m)=\operatorname{nc}(u|m_1)$$

16.20.3 $$\operatorname{dn}(iu|m)=\operatorname{dc}(u|m_1)$$

16.21. Complex Arguments

With the abbreviations

16.21.1

$$s=\mathrm{sn}(x|m),\ c=\mathrm{cn}(x|m),\ d=\mathrm{dn}(x|m),\ s_1=\mathrm{sn}(y|m_1),$$
$$c_1=\mathrm{cn}(y|m_1),\ d_1=\mathrm{dn}(y|m_1)$$

16.21.2 $$\mathrm{sn}(x+iy|m)=\frac{s\cdot d_1+ic\cdot d\cdot s_1\cdot c_1}{c_1^2+ms^2\cdot s_1^2}$$

16.21.3 $$\mathrm{cn}(x+iy|m)=\frac{c\cdot c_1-is\cdot d\cdot s_1\cdot d_1}{c_1^2+ms^2\cdot s_1^2}$$

16.21.4 $$\mathrm{dn}(x+iy|m)=\frac{d\cdot c_1\cdot d_1-ims\cdot c\cdot s_1}{c_1^2+ms^2\cdot s_1^2}$$

16.22. Leading Terms of the Series in Ascending Powers of u

16.22.1

$$\mathrm{sn}(u|m)=u-(1+m)\frac{u^3}{3!}+(1+14m+m^2)\frac{u^5}{5!}$$
$$-(1+135m+135m^2+m^3)\frac{u^7}{7!}+\ldots$$

16.22.2

$$\mathrm{cn}(u|m)=1-\frac{u^2}{2!}+(1+4m)\frac{u^4}{4!}$$
$$-(1+44m+16m^2)\frac{u^6}{6!}+\ldots$$

16.22.3

$$\mathrm{dn}(u|m)=1-m\frac{u^2}{2!}+m(4+m)\frac{u^4}{4!}$$
$$-m(16+44m+m^2)\frac{u^6}{6!}+\ldots$$

No formulae are known for the general coefficients in these series.

16.23. Series Expansions in Terms of the Nome $q=e^{-\pi K'/K}$ and the Argument $v=\pi u/(2K)$

16.23.1 $$\mathrm{sn}\ (u|m)=\frac{2\pi}{m^{1/2}K}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\sin\ (2n+1)v$$

16.23.2 $$\mathrm{cn}\ (u|m)=\frac{2\pi}{m^{1/2}K}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\cos\ (2n+1)v$$

16.23.3 $$\mathrm{dn}\ (u|m)=\frac{\pi}{2K}+\frac{2\pi}{K}\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\cos 2nv$$

16.23.4

$$\mathrm{cd}\ (u|m)=\frac{2\pi}{m^{1/2}K}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n+1/2}}{1-q^{2n+1}}\cos\ (2n+1)v$$

16.23.5

$$\mathrm{sd}\ (u|m)=\frac{2\pi}{(mm_1)^{1/2}K}\sum_{n=0}^{\infty}(-1)^n\frac{q^{n+1/2}}{1+q^{2n+1}}\sin\ (2n+1)v$$

16.23.6

$$\mathrm{nd}\ (u|m)=\frac{\pi}{2m_1^{1/2}K}+\frac{2\pi}{m_1^{1/2}K}\sum_{n=1}^{\infty}(-1)^n\frac{q^n}{1+q^{2n}}\cos 2nv$$

16.23.7

$$\mathrm{dc}\ (u|m)=\frac{\pi}{2K}\sec v$$
$$+\frac{2\pi}{K}\sum_{n=0}^{\infty}(-1)^n\frac{q^{2n+1}}{1-q^{2n+1}}\cos\ (2n+1)v$$

16.23.8

$$\mathrm{nc}\ (u|m)=\frac{\pi}{2m_1^{1/2}K}\sec v$$
$$-\frac{2\pi}{m_1^{1/2}K}\sum_{n=0}^{\infty}(-1)^n\frac{q^{2n+1}}{1+q^{2n+1}}\cos\ (2n+1)v$$

16.23.9

$$\mathrm{sc}\ (u|m)=\frac{\pi}{2m_1^{1/2}K}\tan v$$
$$+\frac{2\pi}{m_1^{1/2}K}\sum_{n=1}^{\infty}(-1)^n\frac{q^{2n}}{1+q^{2n}}\sin 2nv$$

16.23.10

$$\mathrm{ns}\ (u|m)=\frac{\pi}{2K}\csc v-\frac{2\pi}{K}\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1-q^{2n+1}}\sin\ (2n+1)v$$

16.23.11

$$\mathrm{ds}\ (u|m)=\frac{\pi}{2K}\csc v-\frac{2\pi}{K}\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1+q^{2n+1}}\sin\ (2n+1)v$$

16.23.12

$$\mathrm{cs}\ (u|m)=\frac{\pi}{2K}\cot v-\frac{2\pi}{K}\sum_{n=1}^{\infty}\frac{q^{2n}}{1+q^{2n}}\sin 2nv$$

16.24. Integrals of the Twelve Jacobian Elliptic Functions

16.24.1 $\int \mathrm{sn}\ u\ du=m^{-1/2}\ln\ (\mathrm{dn}\ u-m^{1/2}\mathrm{cn}\ u)$

16.24.2 $\int \mathrm{cn}\ u\ du=m^{-1/2}\arccos\ (\mathrm{dn}\ u)$

16.24.3 $\int \mathrm{dn}\ u\ du=\arcsin\ (\mathrm{sn}\ u)$

16.24.4 $\int \mathrm{cd}\ u\ du=m^{-1/2}\ln\ (\mathrm{nd}\ u+m^{1/2}\mathrm{sd}\ u)$

16.24.5 $\int \mathrm{sd}\ u\ du=(mm_1)^{-1/2}\arcsin\ (-m^{1/2}\mathrm{cd}\ u)$

16.24.6 $\int \mathrm{nd}\ u\ du=m_1^{-1/2}\arccos\ (\mathrm{cd}\ u)$

16.24.7 $\int \mathrm{dc}\ u\ du=\ln\ (\mathrm{nc}\ u+\mathrm{sc}\ u)$

16.24.8 $\int \mathrm{nc}\ u\ du=m_1^{-1/2}\ln\ (\mathrm{dc}\ u+m_1^{1/2}\mathrm{sc}\ u)$

16.24.9 $\int \mathrm{sc}\ u\ du=m_1^{-1/2}\ln\ (\mathrm{dc}\ u+m_1^{1/2}\mathrm{nc}\ u)$

16.24.10 $\int \mathrm{ns}\ u\ du=\ln\ (\mathrm{ds}\ u-\mathrm{cs}\ u)$

16.24.11 $\int \mathrm{ds}\ u\ du=\ln\ (\mathrm{ns}\ u-\mathrm{cs}\ u)$

16.24.12 $\int \mathrm{cs}\ u\ du=\ln\ (\mathrm{ns}\ u-\mathrm{ds}\ u)$

In numerical use of the above table certain restrictions must be put on u in order to keep the arguments of the logarithms positive and to avoid trouble with many-valued inverse circular functions.

16.25. Notation for the Integrals of the Squares of the Twelve Jacobian Elliptic Functions

16.25.1 $$\mathrm{Pq}\, u=\int_0^u \mathrm{pq}^2 t\, dt \text{ when } \mathrm{q}\neq \mathrm{s}$$

16.25.2 $$\mathrm{Ps}\, u=\int_0^u \left(\mathrm{pq}^2 t-\frac{1}{t^2}\right) dt-\frac{1}{u}$$

Examples

$$\mathrm{Cd}\, u=\int_0^u \mathrm{cd}^2 t\, dt, \mathrm{Ns}\ u=\int_0^u\left(\mathrm{ns}^2 t-\frac{1}{t^2}\right)dt-\frac{1}{u}$$

16.26. Integrals in Terms of the Elliptic Integral of the Second Kind (see 17.4)

16.26.1 $$m\mathrm{Sn}\ u=-E(u)+u$$

16.26.2 $$m\mathrm{Cn}\ u=E(u)-m_1 u$$ Pole n

16.26.3 $$\mathrm{Dn}\ u=E(u)$$

16.26.4 $$m\mathrm{Cd}\ u=-E(u)+u+m \operatorname{sn} u \operatorname{cd} u$$

16.26.5

$$mm_1\mathrm{Sd}\ u=E(u)-m_1 u-m \operatorname{sn} u \operatorname{cd} u$$ Pole d

16.26.6 $$m_1\mathrm{Nd}\ u=E(u)-m \operatorname{sn} u \operatorname{cd} u$$

16.26.7 $$\mathrm{Dc}\ u=-E(u)+u+\operatorname{sn} u \operatorname{dc} u$$

16.26.8

$$m_1\mathrm{Nc}\ u=-E(u)+m_1 u+\operatorname{sn} u \operatorname{dc} u$$ Pole c

16.26.9 $$m_1\mathrm{Sc}\ u=-E(u)+\operatorname{sn} u \operatorname{dc} u$$

16.26.10 $$\mathrm{Ns}\ u=-E(u)+u-\operatorname{cn} u \operatorname{ds} u$$

16.26.11

$$\mathrm{Ds}\ u=-E(u)+m_1 u-\operatorname{cn} u \operatorname{ds} u$$ Pole s

16.26.12 $$\mathrm{Cs}\ u=-E(u)-\operatorname{cn} u \operatorname{ds} u$$

All the above may be expressed in terms of Jacobi's zeta function (see **17.4.27**).

$$Z(u)=E(u)-\frac{E}{K}u, \text{ where } E=E(K)$$

16.27. Theta Functions; Expansions in Terms of the Nome q

16.27.1

$$\vartheta_1(z, q)=\vartheta_1(z)=2q^{1/4}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)} \sin (2n+1)z$$

16.27.2

$$\vartheta_2(z, q)=\vartheta_2(z)=2q^{1/4}\sum_{n=0}^{\infty} q^{n(n+1)} \cos (2n+1)z$$

16.27.3 $$\vartheta_3(z, q)=\vartheta_3(z)=1+2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz$$

16.27.4

$$\vartheta_4(z, q)=\vartheta_4(z)=1+2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz$$

Theta functions are important because every one of the Jacobian elliptic functions can be expressed as the ratio of two theta functions. See **16.36.**

The notation shows these functions as depending on the variable z and the nome q, $|q|<1$. In this case, here and elsewhere, the convergence is not dependent on the trigonometrical terms. In their relation to the Jacobian elliptic functions, we note that the nome q is given by

$$q=e^{-\pi K'/K},$$

where K and iK' are the quarter periods. Since $q=q(m)$ is determined when the parameter m is given, we can also regard the theta functions as dependent upon m and then we write

$$\vartheta_a(z, q)=\vartheta_a(z|m),\ a=1, 2, 3, 4$$

but when no ambiguity is to be feared, we write $\vartheta_a(z)$ simply.

The above notations are those given in Modern Analysis [16.6].

There is a bewildering variety of notations, for example the function $\vartheta_4(z)$ above is sometimes denoted by $\vartheta_0(z)$ or $\vartheta(z)$; see the table given in Modern Analysis [16.6]. Further the argument $u=2Kz/\pi$ is frequently used so that in consulting books caution should be exercised.

16.28. Relations Between the Squares of the Theta Functions

16.28.1 $$\vartheta_1^2(z)\vartheta_4^2(0)=\vartheta_3^2(z)\vartheta_2^2(0)-\vartheta_2^2(z)\vartheta_3^2(0)$$

16.28.2 $$\vartheta_2^2(z)\vartheta_4^2(0)=\vartheta_4^2(z)\vartheta_2^2(0)-\vartheta_1^2(z)\vartheta_3^2(0)$$

16.28.3 $$\vartheta_3^2(z)\vartheta_4^2(0)=\vartheta_4^2(z)\vartheta_3^2(0)-\vartheta_1^2(z)\vartheta_2^2(0)$$

16.28.4 $$\vartheta_4^2(z)\vartheta_4^2(0)=\vartheta_3^2(z)\vartheta_3^2(0)-\vartheta_2^2(z)\vartheta_2^2(0)$$

16.28.5 $$\vartheta_2^4(0)+\vartheta_4^4(0)=\vartheta_3^4(0)$$

Note also the important relation

16.28.6 $$\vartheta_1'(0)=\vartheta_2(0)\vartheta_3(0)\vartheta_4(0) \text{ or } \vartheta_1'=\vartheta_2\vartheta_3\vartheta_4$$

16.29. Logarithmic Derivatives of the Theta Functions

16.29.1 $$\frac{\vartheta_1'(u)}{\vartheta_1(u)}=\cot u+4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin 2nu$$

16.29.2

$$\frac{\vartheta_2'(u)}{\vartheta_2(u)}=-\tan u+4\sum_{n=1}^{\infty}(-1)^n\frac{q^{2n}}{1-q^{2n}}\sin 2nu$$

16.29.3 $$\frac{\vartheta_3'(u)}{\vartheta_3(u)}=4\sum_{n=1}^{\infty}(-1)^n\frac{q^n}{1-q^{2n}}\sin 2nu$$

16.29.4 $$\frac{\vartheta_4'(u)}{\vartheta_4(u)}=4\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin 2nu$$

16.30. Logarithms of Theta Functions of Sum and Difference

16.30.1

$$\ln\frac{\vartheta_1(\alpha+\beta)}{\vartheta_1(\alpha-\beta)}=\ln\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}+4\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{2n}}{1-q^{2n}}\sin 2n\alpha\sin 2n\beta$$

16.30.2

$$\ln\frac{\vartheta_2(\alpha+\beta)}{\vartheta_2(\alpha-\beta)}=\ln\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{q^{2n}}{1-q^{2n}}\sin 2n\alpha\sin 2n\beta$$

16.30.3

$$\ln\frac{\vartheta_3(\alpha+\beta)}{\vartheta_3(\alpha-\beta)}=4\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{q^n}{1-q^{2n}}\sin 2n\alpha\sin 2n\beta$$

16.30.4

$$\ln\frac{\vartheta_4(\alpha+\beta)}{\vartheta_4(\alpha-\beta)}=4\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1-q^{2n}}\sin 2n\alpha\sin 2n\beta$$

The corresponding expressions when $\beta=i\gamma$ are easily deduced by use of the formulae **4.3.55** and **4.3.56.**

16.31. Jacobi's Notation for Theta Functions

16.31.1 $\Theta(u|m)=\Theta(u)=\vartheta_4(v),\qquad v=\frac{\pi u}{2K}$

16.31.2 $\Theta_1(u|m)=\Theta_1(u)=\vartheta_3(v)=\Theta(u+K)$

16.31.3 $\mathrm{H}(u|m)=\mathrm{H}(u)=\vartheta_1(v)$

16.31.4 $\mathrm{H}_1(u|m)=\mathrm{H}_1(u)=\vartheta_2(v)=\mathrm{H}(u+K)$

16.32. Calculation of Jacobi's Theta Function $\Theta(u|m)$ by Use of the Arithmetic-Geometric Mean

Form the A.G.M. scale starting with

16.32.1 $a_0=1,\ b_0=\sqrt{m_1},\ c_0=\sqrt{m}$

terminating with the Nth step when c_N is negligible to the accuracy required. Find φ_N in degrees, where

16.32.2 $$\varphi_N=2^N a_N u\,\frac{180^\circ}{\pi}$$

and then compute successively $\varphi_{N-1}, \varphi_{N-2}, \ldots, \varphi_1, \varphi_0$ from the recurrence relation

16.32.3 $$\sin(2\varphi_{n-1}-\varphi_n)=\frac{c_n}{a_n}\sin\varphi_n.$$

Then

16.32.4

$$\ln\Theta(u|m)=\frac{1}{2}\ln\frac{2m_1^{1/2}K(m)}{\pi}+\frac{1}{2}\ln\frac{\cos(\varphi_1-\varphi_0)}{\cos\varphi_0}+\frac{1}{4}\ln\sec(2\varphi_0-\varphi_1)+\frac{1}{8}\ln\sec(2\varphi_1-\varphi_2)+\ldots+\frac{1}{2^{N+1}}\ln\sec(2\varphi_{N-1}-\varphi_N)$$

16.33. Addition of Quarter-Periods to Jacobi's Eta and Theta Functions

u	$-u$	$u+K$	$u+2K$	$u+iK'$	$u+2iK'$	$u+K+iK'$	$u+2K+2iK'$
16.33.1 $\mathrm{H}(u)$	$-\mathrm{H}(u)$	$\mathrm{H}_1(u)$	$-\mathrm{H}(u)$	$iM(u)\Theta(u)$	$-N(u)\mathrm{H}(u)$	$M(u)\Theta_1(u)$	$N(u)\mathrm{H}(u)$
16.33.2 $\mathrm{H}_1(u)$	$\mathrm{H}_1(u)$	$-\mathrm{H}(u)$	$-\mathrm{H}_1(u)$	$M(u)\Theta_1(u)$	$N(u)\mathrm{H}_1(u)$	$-iM(u)\Theta(u)$	$-N(u)\mathrm{H}_1(u)$
16.33.3 $\Theta_1(u)$	$\Theta_1(u)$	$\Theta(u)$	$\Theta_1(u)$	$M(u)\mathrm{H}_1(u)$	$N(u)\Theta_1(u)$	$iM(u)\mathrm{H}(u)$	$N(u)\Theta_1(u)$
16.33.4 $\Theta(u)$	$\Theta(u)$	$\Theta_1(u)$	$\Theta(u)$	$iM(u)\mathrm{H}(u)$	$-N(u)\Theta(u)$	$M(u)\mathrm{H}_1(u)$	$-N(u)\Theta(u)$

where

$$M(u)=\left[\exp\left(-\frac{\pi i u}{2K}\right)\right]q^{-\frac{1}{4}},$$

$$N(u)=\left[\exp\left(-\frac{\pi i u}{K}\right)\right]q^{-1}$$

$\mathrm{H}(u)$ and $\mathrm{H}_1(u)$ have the period $4K$. $\Theta(u)$ and $\Theta_1(u)$ have the period $2K$.

$2iK'$ is a quasi-period for all four functions, that is to say, increase of the argument by $2iK'$ multiplies the function by a factor.

16.34. Relation of Jacobi's Zeta Function to the Theta Functions

$$Z(u)=\frac{\partial}{\partial u}\ln\Theta(u)$$

16.34.1 $$Z(u)=\frac{\pi}{2K}\frac{\vartheta_1'\left(\frac{\pi u}{2K}\right)}{\vartheta_1\left(\frac{\pi u}{2K}\right)}-\frac{\text{cn } u \text{ dn } u}{\text{sn } u}$$

16.34.2 $$=\frac{\pi}{2K}\frac{\vartheta_2'\left(\frac{\pi u}{2K}\right)}{\vartheta_2\left(\frac{\pi u}{2K}\right)}+\frac{\text{dn } u \text{ sn } u}{\text{cn } u}$$

16.34.3 $$=\frac{\pi}{2K}\frac{\vartheta_3'\left(\frac{\pi u}{2K}\right)}{\vartheta_3\left(\frac{\pi u}{2K}\right)}-m\frac{\text{sn } u \text{ cn } u}{\text{dn } u}$$

16.34.4 $$=\frac{\pi}{2K}\frac{\vartheta_4'\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)}$$

16.35. Calculation of Jacobi's Zeta Function *Z(u|m)* by Use of the Arithmetic-Geometric Mean

Form the A.G.M. scale **17.6** starting with

16.35.1 $$a_0=1,\ b_0=\sqrt{m_1},\ c_0=\sqrt{m}$$

terminating at the Nth step when c_N is negligible to the accuracy required. Find φ_N in degrees where

16.35.2 $$\varphi_N=2^N a_N u\frac{180°}{\pi}$$

and then compute successively $\varphi_{N-1}, \varphi_{N-2}, \ldots, \varphi_1, \varphi_0$ from the recurrence relation

16.35.3 $$\sin(2\varphi_{n-1}-\varphi_n)=\frac{c_n}{a_n}\sin\varphi_n.$$

Then

16.35.4

$$Z(u|m)=c_1\sin\varphi_1+c_2\sin\varphi_2+\ldots+c_N\sin\varphi_N.$$

16.36. Neville's Notation for Theta Functions

These functions are defined in terms of Jacobi's theta functions of **16.31** by

16.36.1 $$\vartheta_s(u)=\frac{\text{H}(u)}{\text{H}'(0)},\ \vartheta_c(u)=\frac{\text{H}(u+K)}{\text{H}(K)}$$

16.36.2 $$\vartheta_d(u)=\frac{\Theta(u+K)}{\Theta(K)},\ \vartheta_n(u)=\frac{\Theta(u)}{\Theta(0)}.$$

If λ, μ are any integers positive, negative, or zero the points $u_0+2\lambda K+2\mu iK'$ are said to be *congruent to* u_0.

$\vartheta_s(u)$ has zeros at the points congruent to 0
$\vartheta_c(u)$ has zeros at the points congruent to K
$\vartheta_n(u)$ has zeros at the points congruent to iK'
$\vartheta_d(u)$ has zeros at the points congruent to $K+iK'$

Thus the suffix secures that the function $\vartheta_p(u)$ has zeros at the points marked p in the introductory diagram in **16.1.2**, and the constant by which Jacobi's function is divided secures that the leading coefficient of $\vartheta_p(u)$ at the origin is unity. Therefore the functions have the fundamentally important property that if p, q are any two of the letters s, c, n, d, the Jacobian elliptic function pq u is given by

16.36.3 $$\text{pq } u=\frac{\vartheta_p(u)}{\vartheta_q(u)}.$$

These functions also have the property

16.36.4 $$m_1^{-1/4}\vartheta_c(K-u)=\vartheta_s(u)$$

16.36.5 $$m_1^{-1/4}\vartheta_d(K-u)=\vartheta_n(u),$$

for complementary arguments u and $K-u$.

In terms of the theta functions defined in **16.27**, let $v=\pi u/(2K)$, then

16.36.6 $$\vartheta_s(u)=\frac{2K\vartheta_1(v)}{\vartheta_1'(0)},\ \vartheta_c(u)=\frac{\vartheta_2(v)}{\vartheta_2(0)}$$

16.36.7 $$\vartheta_d(u)=\frac{\vartheta_3(v)}{\vartheta_3(0)},\ \vartheta_n(u)=\frac{\vartheta_4(v)}{\vartheta_4(0)}.$$

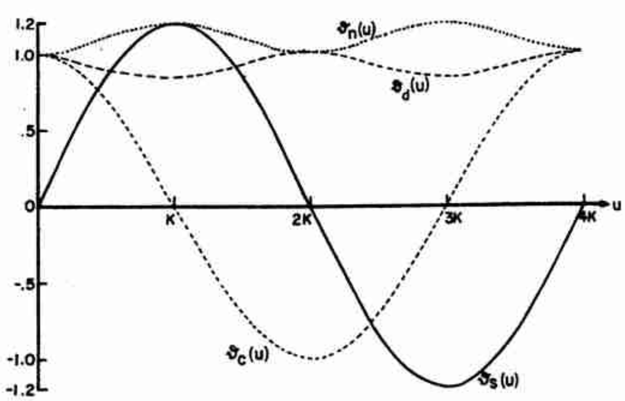

FIGURE 16.4. *Neville's theta functions* $\vartheta_s(u)$, $\vartheta_c(u)$, $\vartheta_d(u)$, $\vartheta_n(u)$
$m=\frac{1}{2}$

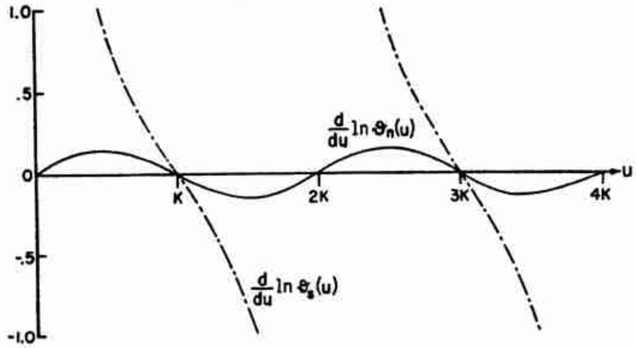

FIGURE 16.5. *Logarithmic derivatives of theta functions*

$$\frac{d}{du}\ln\vartheta_s(u),\ \frac{d}{du}\ln\vartheta_n(u)$$

$$m=\frac{1}{2}$$

16.37. Expression as Infinite Products

$$q=q(m),\ v=\pi u/(2K)$$

16.37.1

$$\vartheta_s(u)=\left(\frac{16q}{mm_1}\right)^{1/6}\sin v\prod_{n=1}^{\infty}(1-2q^{2n}\cos 2v+q^{4n})$$

16.37.2

$$\vartheta_c(u)=\left(\frac{16qm_1^{1/2}}{m}\right)^{1/6}\cos v\prod_{n=1}^{\infty}(1+2q^{2n}\cos 2v+q^{4n})$$

16.37.3

$$\vartheta_d(u)=\left(\frac{mm_1}{16q}\right)^{1/12}\prod_{n=1}^{\infty}(1+2q^{2n-1}\cos 2v+q^{4n-2})$$

16.37.4

$$\vartheta_n(u)=\left(\frac{m}{16qm_1^2}\right)^{1/12}\prod_{n=1}^{\infty}(1-2q^{2n-1}\cos 2v+q^{4n-2})$$

16.38. Expression as Infinite Series

Let $v=\pi u/(2K)$

16.38.1

$$\vartheta_s(u)=\left[\frac{2\pi q^{1/2}}{m^{1/2}m_1^{1/2}K}\right]^{1/2}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\sin(2n+1)v$$

16.38.2 $\vartheta_c(u)=\left[\frac{2\pi q^{1/2}}{m^{1/2}K}\right]^{1/2}\sum_{n=0}^{\infty}q^{n(n+1)}\cos(2n+1)v$

16.38.3 $\vartheta_d(u)=\left[\frac{\pi}{2K}\right]^{1/2}\{1+2\sum_{n=1}^{\infty}q^{n^2}\cos 2nv\}$

16.38.4

$$\vartheta_n(u)=\left[\frac{\pi}{2m_1^{1/2}K}\right]^{1/2}\{1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nv\}$$

16.38.5 $(2K/\pi)^{1/2}=1+2q+2q^4+2q^9+\ldots=\vartheta_3(0,q)$

16.38.6

$$(2K'/\pi)^{1/2}=1+2q_1+2q_1^4+2q_1^9+\ldots=\vartheta_3(0,q_1)$$

16.38.7

$$(2m^{1/2}K/\pi)^{1/2}=2q^{1/4}(1+q^2+q^6+q^{12}+q^{20}+\ldots)=\vartheta_2(0,q)$$

16.38.8

$$(2m_1^{1/2}K/\pi)^{1/2}=1-2q+2q^4-2q^9+\ldots=\vartheta_4(0,q).$$

Numerical Methods

16.39. Use and Extension of the Tables

Example 1. Calculate nc $(1.99650|.64)$ to 4S.

From **Table 17.1,** $1.99650=K+.001$. From the table of principal terms

$$\text{nc } u=-m_1^{-1/2}/(u-K)+\ldots$$

$$\text{nc }(K+.001|.64)=\frac{-(.36)^{-1/2}}{.001}+\ldots=-\frac{10000}{6}+\ldots=-1667+\ldots$$

and since the next term is of order .001 this value -1667 is correct to at least 4S.

Example 2. Use the descending Landen transformation to calculate dn $(.20|.19)$ to 6D.

Here $m=.19$, $m_1^{1/2}=.9$ and so from **16.12.1**

$$\mu=\left(\frac{1}{19}\right)^2,\ 1+\mu^{1/2}=\frac{20}{19},\ v=.19.$$

Also

$$\mu^2=\left(\frac{1}{19}\right)^4=10^{-6}\times 7.67$$

which is negligible.

From **16.12.4**

$$\text{dn}(.20|.19)=\frac{\text{dn}^2\left[.19\left|\left(\frac{1}{19}\right)^2\right.\right]-\left(1-\frac{1}{19}\right)}{\left(1+\frac{1}{19}\right)-\text{dn}^2\left[.19\left|\left(\frac{1}{19}\right)^2\right.\right]}.$$

Now from **16.13.3**

$$\text{dn}\left[.19\left|\left(\frac{1}{19}\right)^2\right.\right]=.999951$$

whence dn $(.20|.19)=.996253$.

Example 3. Use the ascending Landen transformation to calculate dn $(.20|.81)$ to 5D.

From **16.14.1**

$$\mu=\frac{4(.9)}{(1.9)^2}=\frac{360}{361},\ \mu_1=\left(\frac{1}{19}\right)^2$$

$$1+\mu_1^{1/2}=\frac{20}{19},\quad v=\frac{19}{20}\times .20=.19,$$

μ_1^2 is negligible to 4D. Thus

Example 7. Use the q-series to compute cs $(.53601\ 62|.09)$.

Here we use the series **16.23.12**, $K=1.60804\ 862$, $q=.00589\ 414$, $v=\frac{\pi u}{2K}=\frac{\pi}{6}$ radians or $30°$.

Since q^4 is negligible to 8D, we have to 7D

cs $(.53601\ 62|.09)$

$$=\frac{\pi}{2K}\cot 30°-\frac{2\pi}{K}\left\{\frac{q^2}{1+q^2}\sin 60°\right\}$$

$$=(.97683\ 3852)(1.73205\ 081)$$

$$-3.90733\ 541[(.00003\ 4740)(.86602\ 5404)]$$

$$=1.69180\ 83.$$

Example 8. Use theta functions to compute sn $(.61802|.5)$ to 5D.

Here $K(\frac{1}{2})=1.85407$

$$\epsilon°=\frac{.61802}{1.85407}\times 90°=30°$$

$$\sin^2\alpha=1/2,\ \alpha=45°.$$

Thus

$$\text{sn}\ (.61802|.5)=\frac{\vartheta_s(30°\backslash 45°)}{\vartheta_n(30°\backslash 45°)}$$

$$=\frac{.59128}{1.04729}=.56458$$

from **Table 16.1.**

Example 9. Use theta functions to compute sc $(.61802|.5)$ to 5D.

As in the preceding example

$$\epsilon°=30°,\ \alpha°=45°$$

so that

$$\text{sc}\ (.61802|.5)=\frac{\vartheta_s(30°\backslash 45°)}{\vartheta_c(30°\backslash 45°)}.$$

We use **Table 16.1** to give

$$\vartheta_s(30°\backslash 45°)=.59128$$

$$(\sec 45°)^{\frac{1}{2}}\vartheta_c(30°\backslash 45°)=1.02796.$$

Therefore

$$\text{sc}\ (.61802|.5)=\frac{.59128}{1.02796}(\sec 45°)^{\frac{1}{2}}$$

$$=.68402.$$

Example 10. Find sn $(.75342|.7)$ by inverse interpolation in **Table 17.5.**

This method is explained in chapter **17, Example 7.**

Example 11. Find u, given that cs $(u|.5)=.75$.

From **16.9.4** we have

$$\text{sn}^2\ u=\frac{1}{1+\text{cs}^2 u}.$$

Thus

$$\text{sn}^2\ (u|.5)=.64$$

and

$$\text{sn}\ (u|.5)=.8.$$

We have therefore replaced the problem by that of finding u given sn $(u|m)$, where m is known. If $\varphi=\text{am}\ u$

$\sin\varphi=\text{sn}\ u$ and so

$$\varphi=.9272952 \text{ radians or } 53.13010°.$$

From **Table 17.5,**

$$u=F(53.13010°\backslash 45°)=.99391.$$

Alternatively, starting with the above value of φ we can use the A.G.M. scale to calculate $F(\varphi\backslash\alpha)$ as explained in **17.6.** This method is to be preferred if more figures are required, or if α differs from a tabular value in **Table 17.5.**

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17. Elliptic Integrals

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Contents

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17. Elliptic Integrals

Mathematical Properties

17.1. Definition of Elliptic Integrals

If $R(x, y)$ is a rational function of x and y, where y^2 is equal to a cubic or quartic polynomial in x, the integral

17.1.1
$$\int R(x,y)dx$$

is called an *elliptic integral*.

The elliptic integral just defined can not, in general, be expressed in terms of elementary functions.

Exceptions to this are

(i) when $R(x, y)$ contains no odd powers of y.

(ii) when the polynomial y^2 has a repeated factor.

We therefore exclude these cases.

By substituting for y^2 and denoting by $p_s(x)$ a polynomial in x we get [2]

$$R(x,y)=\frac{p_1(x)+yp_2(x)}{p_3(x)+yp_4(x)}$$
$$=\frac{[p_1(x)+yp_2(x)][p_3(x)-yp_4(x)]y}{\{[p_3(x)]^2-y^2[p_4(x)]^2\}y}$$
$$=\frac{p_5(x)+yp_6(x)}{yp_7(x)}=R_1(x)+\frac{R_2(x)}{y}$$

where $R_1(x)$ and $R_2(x)$ are rational functions of x. Hence, by expressing $R_2(x)$ as the sum of a polynomial and partial fractions

$$\int R(x,y)dx=\int R_1(x)dx+\Sigma_s A_s\int x^s y^{-1}dx$$
$$+\Sigma_s B_s\int [(x-c)^s y]^{-1}dx$$

Reduction Formulae

Let

17.1.2

$$y^2=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4 \qquad (|a_0|+|a_1|\neq 0)$$
$$=b_0(x-c)^4+b_1(x-c)^3+b_2(x-c)^2+b_3(x-c)+b_4 \qquad (|b_0|+|b_1|\neq 0)$$

17.1.3 $$I_s=\int x^s y^{-1}dx,\ J_s=\int [y(x-c)^s]^{-1}dx$$

By integrating the derivatives of yx^s and $y(x-c)^{-s}$ we get the reduction formulae

17.1.4

$$(s+2)a_0I_{s+3}+\tfrac{1}{2}a_1(2s+3)I_{s+2}+a_2(s+1)I_{s+1}$$
$$+\tfrac{1}{2}a_3(2s+1)I_s+sa_4I_{s-1}=x^sy \quad (s=0, 1, 2, \ldots)$$

[2] See [17.7] **22.72.**

17.1.5

$$(2-s)b_0J_{s-3}+\tfrac{1}{2}b_1(3-2s)J_{s-2}+b_2(1-s)J_{s-1}$$
$$+\tfrac{1}{2}b_3(1-2s)J_s-sb_4J_{s+1}=y(x-c)^{-s}$$
$$(s=1, 2, 3, \ldots)$$

By means of these reduction formulae and certain transformations (see **Examples 1** and **2**) every elliptic integral can be brought to depend on the integral of a rational function and on three canonical forms for elliptic integrals.

17.2. Canonical Forms

Definitions

17.2.1

$m=\sin^2 \alpha$; m is the parameter, α is the modular angle

17.2.2 $$x=\sin\varphi=\mathrm{sn}\, u$$

17.2.3 $$\cos\varphi=\mathrm{cn}\, u$$

17.2.4

$(1-m\sin^2\varphi)^{\frac{1}{2}}=\mathrm{dn}\, u=\Delta(\varphi)$, the delta amplitude

17.2.5 $\varphi=\arcsin(\mathrm{sn}\, u)=\mathrm{am}\, u$, the amplitude

Elliptic Integral of the First Kind

17.2.6 $$F(\varphi\backslash\alpha)=F(\varphi|m)=\int_0^{\varphi}(1-\sin^2\alpha\sin^2\theta)^{-\frac{1}{2}}d\theta$$

17.2.7 $$=\int_0^x[(1-t^2)(1-mt^2)]^{-\frac{1}{2}}dt$$
$$=\int_0^u dw=u$$

Elliptic Integral of the Second Kind

17.2.8 $$E(\varphi\backslash\alpha)=E(u|m)=\int_0^x(1-t^2)^{-\frac{1}{2}}(1-mt^2)^{\frac{1}{2}}dt$$

17.2.9 $$=\int_0^{\varphi}(1-\sin^2\alpha\sin^2\theta)^{\frac{1}{2}}d\theta$$

17.2.10 $$=\int_0^u \mathrm{dn}^2\, w\, dw$$

17.2.11 $$=m_1u+m\int_0^u \mathrm{cn}^2\, w\, dw$$

17.2.12 $E(\varphi\backslash\alpha)=u-m\int_0^u \mathrm{sn}^2 w\,dw$

17.2.13 $$=\frac{\pi}{2K(m)}\frac{\vartheta_4'(\pi u/2K)}{\vartheta_4(\pi u/2K)}+\frac{E(m)u}{K(m)}$$

(For theta functions, see chapter **16.**)

Elliptic Integral of the Third Kind

17.2.14

$$\Pi(n;\varphi\backslash\alpha)=\int_0^\varphi (1-n\sin^2\theta)^{-1}[1-\sin^2\alpha\sin^2\theta]^{-1/2}d\theta$$

If $x=\mathrm{sn}\,(u|m)$,

17.2.15

$$\Pi(n;u|m)=\int_0^x (1-nt^2)^{-1}[(1-t^2)(1-mt^2)]^{-1/2}dt$$

17.2.16 $$=\int_0^u (1-n\,\mathrm{sn}^2\,(w|m))^{-1}dw$$

The Amplitude φ

17.2.17 $\varphi=\mathrm{am}\,u=\arcsin\,(\mathrm{sn}\,u)=\arcsin x$

can be calculated from **Tables 17.5** and **4.14.**

The Parameter m

Dependence on the *parameter* m is denoted by a vertical stroke preceding the parameter, e.g., $F(\varphi|m)$.

Together with the parameter we define the *complementary parameter* m_1 by

17.2.18 $$m+m_1=1$$

When the parameter is real, it can always be arranged, see **17.4,** that $0\leq m\leq 1$.

The Modular Angle α

Dependence on the modular angle α, defined in terms of the parameter by **17.2.1,** is denoted by a backward stroke $\backslash$ preceding the modular angle, thus $E(\varphi\backslash\alpha)$. The *complementary modular* angle is $\pi/2-\alpha$ or $90°-\alpha$ according to the unit and thus $m_1=\sin^2(90°-\alpha)=\cos^2\alpha$.

The Modulus k

In terms of Jacobian elliptic functions (chapter **16**), the modulus k and the complementary modulus are defined by

17.2.19 $k=\mathrm{ns}\,(K+iK'),\ k'=\mathrm{dn}\,K.$

They are related to the parameter by $k^2=m$, $k'^2=m_1$.

Dependence on the modulus is denoted by a comma preceding it, thus $\Pi(n;u,k)$.

In computation the modulus is of minimal importance, since it is the parameter and its complement which arise naturally. The parameter and the modular angle will be employed in this chapter to the exclusion of the modulus.

The Characteristic n

The elliptic integral of the third kind depends on three variables namely (i) the parameter, (ii) the amplitude, (iii) the characteristic n. When real, the characteristic may be any number in the interval $(-\infty,\infty)$. The properties of the integral depend upon the location of the characteristic in this interval, see **17.7.**

17.3. Complete Elliptic Integrals of the First and Second Kinds

Referred to the canonical forms of **17.2,** the elliptic integrals are said to be *complete* when the amplitude is $\frac{1}{2}\pi$ and so $x=1$. These complete integrals are designated as follows

17.3.1

$$[K(m)]=K=\int_0^1[(1-t^2)(1-mt^2)]^{-1/2}dt$$
$$=\int_0^{\pi/2}(1-m\sin^2\theta)^{-1/2}d\theta$$

17.3.2 $$K=F(\tfrac{1}{2}\pi|m)=F(\tfrac{1}{2}\pi\backslash\alpha)$$

17.3.3

$$E[K(m)]=E=\int_0^1(1-t^2)^{-1/2}(1-mt^2)^{1/2}dt$$
$$=\int_0^{\pi/2}(1-m\sin^2\theta)^{1/2}d\theta$$

17.3.4 $$E=E[K(m)]=E(m)=E(\tfrac{1}{2}\pi\backslash\alpha)$$

We also define

17.3.5

$$K'=K(m_1)=K(1-m)=\int_0^{\pi/2}(1-m_1\sin^2\theta)^{-1/2}d\theta$$

17.3.6 $$K'=F(\tfrac{1}{2}\pi|m_1)=F(\tfrac{1}{2}\pi\backslash\tfrac{1}{2}\pi-\alpha)$$

17.3.7

$$E'=E(m_1)=E(1-m)=\int_0^{\pi/2}(1-m_1\sin^2\theta)^{1/2}d\theta$$

17.3.8 $E'=E[K(m_1)]=E(m_1)=E(\tfrac{1}{2}\pi\backslash\tfrac{1}{2}\pi-\alpha)$

K and iK' are the "real" and "imaginary" *quarter-periods* of the corresponding Jacobian elliptic functions (see chapter **16**).

Relation to the Hypergeometric Function
(see chapter 15)

17.3.9 $$K=\tfrac{1}{2}\pi F(\tfrac{1}{2},\tfrac{1}{2};1;m)$$

17.3.10 $$E=\tfrac{1}{2}\pi F(-\tfrac{1}{2},\tfrac{1}{2};1;m)$$

Infinite Series

17.3.11

$$K(m)=\frac{1}{2}\pi\left[1+\left(\frac{1}{2}\right)^2 m+\left(\frac{1\cdot3}{2\cdot4}\right)^2 m^2+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 m^3+\ldots\right]\quad(|m|<1)$$

17.3.12

$$E(m)=\frac{1}{2}\pi\left[1-\left(\frac{1}{2}\right)^2\frac{m}{1}-\left(\frac{1\cdot3}{2\cdot4}\right)^2\frac{m^2}{3}-\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\frac{m^3}{5}-\ldots\right]\quad(|m|<1)$$

Legendre's Relation

17.3.13 $$EK'+E'K-KK'=\tfrac{1}{2}\pi$$

Auxiliary Function

17.3.14 $$L(m)=\frac{K'(m)}{\pi}\ln\frac{16}{m_1}-K(m)$$

17.3.15 $$m=1-16\exp\left[-\pi(K(m)+L(m))/K'(m)\right]$$

17.3.16 $$m=16\exp\left[-\pi(K'(m)+L(m_1))/K(m)\right]$$

The function $L(m)$ is tabulated in **Table 17.4.**

***q*-Series**

The Nome q and the Complementary Nome q_1

17.3.17 $$q=q(m)=\exp\left[-\pi K'/K\right]$$

17.3.18 $$q_1=q(m_1)=\exp\left[-\pi K/K'\right]$$

17.3.19 $$\ln\frac{1}{q}\ln\frac{1}{q_1}=\pi^2$$

17.3.20

$$\log_{10}\frac{1}{q}\log_{10}\frac{1}{q_1}=(\pi\log_{10}e)^2=1.86152\ 28349\text{ to 10D}$$

17.3.21

$$q=\exp\left[-\pi K'/K\right]=\frac{m}{16}+8\left(\frac{m}{16}\right)^2+84\left(\frac{m}{16}\right)^3+992\left(\frac{m}{16}\right)^4+\ldots\quad(|m|<1)$$

17.3.22 $$K=\frac{1}{2}\pi+2\pi\sum_{s=1}^{\infty}\frac{q^s}{1+q^{2s}}$$

17.3.23

$$\frac{E}{K}=\frac{1}{3}(1+m_1)+(\pi/K)^2\left[1/12-2\sum_{s=1}^{\infty}q^{2s}(1-q^{2s})^{-2}\right]$$

17.3.24 $$\operatorname{am}u=v+\sum_{s=1}^{\infty}\frac{2q^s\sin 2sv}{s(1+q^{2s})}\text{ where }v=\pi u/(2K)$$

Limiting Values

17.3.25 $$\lim_{m\to0}K'(E-K)=0$$

17.3.26 $$\lim_{m\to1}\left[K-\tfrac{1}{2}\ln(16/m_1)\right]=0$$

17.3.27 $$\lim_{m\to0}m^{-1}(K-E)=\lim_{m\to0}m^{-1}(E-m_1K)=\pi/4$$

17.3.28 $$\lim_{m\to0}q/m=\lim_{m_1\to1}q_1/m_1=1/16$$

Alternative Evaluations of K and E (see also **17.5**)

17.3.29

$$K(m)=2[1+m_1^{1/2}]^{-1}K([(1-m_1^{1/2})/(1+m_1^{1/2})]^2)^*$$

17.3.30

$$E(m)=(1+m_1^{1/2})E([(1-m_1^{1/2})/(1+m_1^{1/2})]^2)-2m_1^{1/2}(1+m_1^{1/2})^{-1}K([(1-m_1^{1/2})/(1+m_1^{1/2})]^2)$$

17.3.31 $$K(\alpha)=2F(\arctan(\sec^{1/2}\alpha)\backslash\alpha)$$

17.3.32 $$E(\alpha)=2E(\arctan(\sec^{1/2}\alpha)\backslash\alpha)-1+\cos\alpha$$

Polynomial Approximations[3] $(0\le m<1)$

17.3.33

$$K(m)=[a_0+a_1m_1+a_2m_1^2]+[b_0+b_1m_1+b_2m_1^2]\ln(1/m_1)+\epsilon(m)$$

$$|\epsilon(m)|\le3\times10^{-5}$$

$a_0=1.38629\ 44$	$b_0=.5$
$a_1=\ .11197\ 23$	$b_1=.12134\ 78$
$a_2=\ .07252\ 96$	$b_2=.02887\ 29$

17.3.34

$$K(m)=[a_0+a_1m_1+\ldots+a_4m_1^4]+[b_0+b_1m_1+\ldots+b_4m_1^4]\ln(1/m_1)+\epsilon(m)$$

$$|\epsilon(m)|\le2\times10^{-8}$$

$a_0=1.38629\ 436112$	$b_0=.5$
$a_1=\ .09666\ 344259$	$b_1=.12498\ 593597$
$a_2=\ .03590\ 092383$	$b_2=.06880\ 248576$
$a_3=\ .03742\ 563713$	$b_3=.03328\ 355346$
$a_4=\ .01451\ 196212$	$b_4=.00441\ 787012$

[3] The approximations **17.3.33–17.3.36** are from C. Hastings, Jr., Approximations for Digital Computers, Princeton Univ. Press, Princeton, N. J. (with permission).

*See page II.

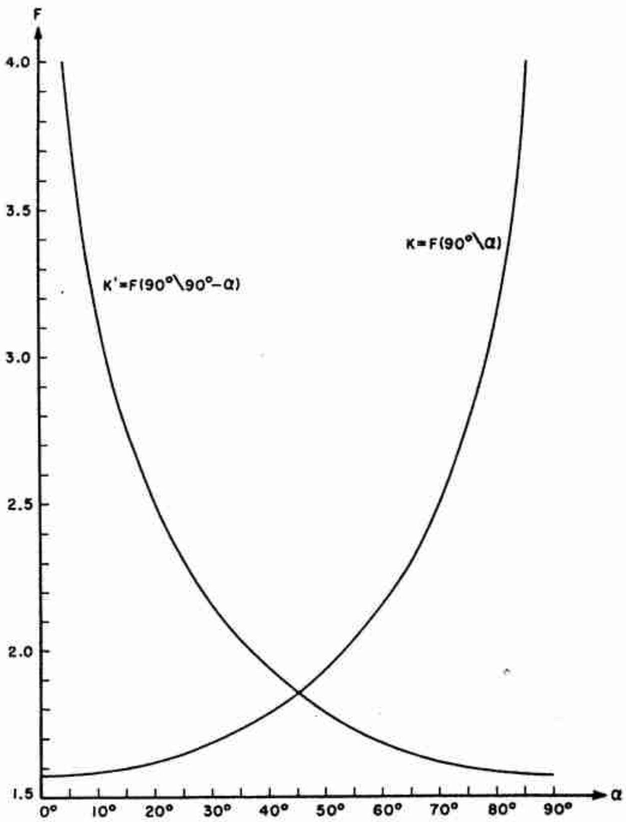

FIGURE 17.1. *Complete elliptic integral of the first kind.*

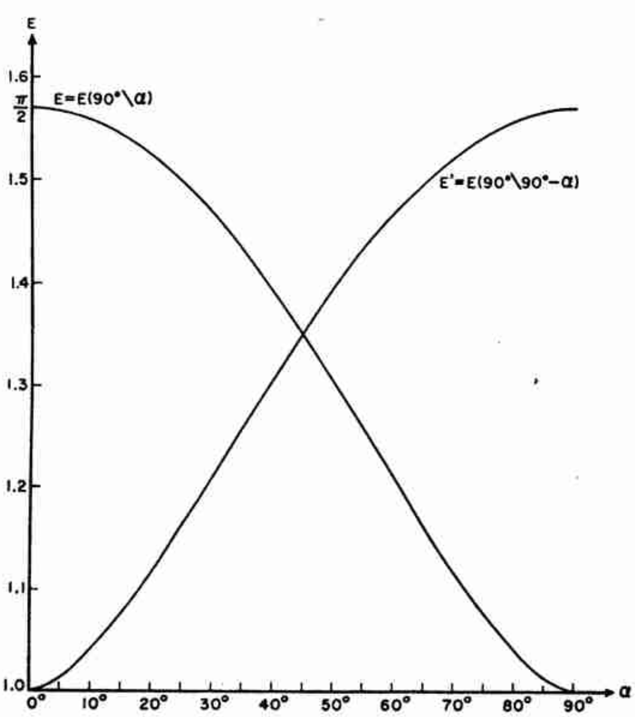

FIGURE 17.2. *Complete elliptic integral of the second kind.*

17.3.35

$$E(m)=[1+a_1m_1+a_2m_1^2]+[b_1m_1+b_2m_1^2]\ln(1/m_1)+\epsilon(m)$$

$$|\epsilon(m)|<4\times10^{-5}$$

$a_1=.46301\ 51$	$b_1=.24527\ 27$
$a_2=.10778\ 12$	$b_2=.04124\ 96$

17.3.36

$$E(m)=[1+a_1m_1+\ldots+a_4m_1^4]+[b_1m_1+\ldots+b_4m_1^4]\ln(1/m_1)+\epsilon(m)$$

$$|\epsilon(m)|<2\times10^{-8}$$

$a_1=.44325\ 141463$	$b_1=.24998\ 368310$
$a_2=.06260\ 601220$	$b_2=.09200\ 180037$
$a_3=.04757\ 383546$	$b_3=.04069\ 697526$
$a_4=.01736\ 506451$	$b_4=.00526\ 449639$

17.4. Incomplete Elliptic Integrals of the First and Second Kinds

Extension of the Tables

Negative Amplitude

17.4.1 $$F(-\varphi|m)=-F(\varphi|m)$$

17.4.2 $$E(-\varphi|m)=-E(\varphi|m)$$

Amplitude of Any Magnitude

17.4.3 $$F(s\pi\pm\varphi|m)=2sK\pm F(\varphi|m)$$

17.4.4 $$E(u+2K)=E(u)+2E$$

17.4.5 $$E(u+2iK')=E(u)+2i(K'-E')$$

17.4.6

$$E(u+2mK+2niK')=E(u)+2mE+2ni(K'-E')$$

17.4.7 $$E(K-u)=E-E(u)+m\,\mathrm{sn}\,u\,\mathrm{cd}\,u$$

Imaginary Amplitude

If $\tan\theta=\sinh\varphi$

17.4.8 $$F(i\varphi\backslash\alpha)=iF(\theta\backslash\tfrac{1}{2}\pi-\alpha)$$

17.4.9

$$E(i\varphi\backslash\alpha)=-iE(\theta\backslash\tfrac{1}{2}\pi-\alpha)+iF(\theta\backslash\tfrac{1}{2}\pi-\alpha)+i\tan\theta(1-\cos^2\alpha\sin^2\theta)^{\frac{1}{2}}$$

Jacobi's Imaginary Transformation

17.4.10

$$E(iu|m)=i[u+\mathrm{dn}(u|m_1)\mathrm{sc}(u|m_1)-E(u|m_1)]$$

Complex Amplitude

17.4.11 $$F(\varphi+i\psi|m)=F(\lambda|m)+iF(\mu|m_1)$$

where $\cot^2\lambda$ is the positive root of the equation $x^2-[\cot^2\varphi+m\sinh^2\psi\csc^2\varphi-m_1]x-m_1\cot^2\varphi=0$ and $m\tan^2\mu=\tan^2\varphi\cot^2\lambda-1$.

17.4.12

$$E(\varphi+i\psi\backslash\alpha)=E(\lambda\backslash\alpha)-iE(\mu\backslash 90^\circ-\alpha)+iF(\mu\backslash 90^\circ-\alpha)+\frac{b_1+ib_2}{b_3}$$

where

$$b_1=\sin^2\alpha\sin\lambda\cos\lambda\sin^2\mu(1-\sin^2\alpha\sin^2\lambda)^{1/2}$$

$$b_2=(1-\sin^2\alpha\sin^2\lambda)(1-\cos^2\alpha\sin^2\mu)^{1/2}\sin\mu\cos\mu$$

$$b_3=\cos^2\mu+\sin^2\alpha\sin^2\lambda\sin^2\mu$$

Amplitude Near to $\pi/2$ (see also 17.5)

If $\cos\alpha\tan\varphi\tan\psi=1$

17.4.13 $F(\varphi\backslash\alpha)+F(\psi\backslash\alpha)=F(\pi/2\backslash\alpha)=K$

17.4.14

$$E(\varphi\backslash\alpha)+E(\psi\backslash\alpha)=E(\pi/2\backslash\alpha)+\sin^2\alpha\ \sin\varphi\ \sin\psi$$

Values when φ is near to $\pi/2$ and m is near to unity can be calculated by these formulae.

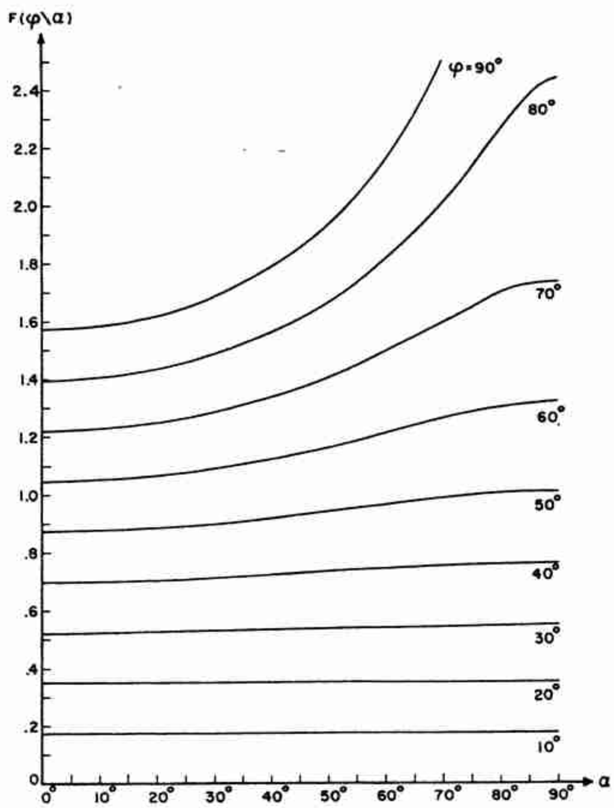

FIGURE 17.3. *Incomplete elliptic integral of the first kind.*

$F(\varphi\backslash\alpha)$, φ constant

Parameter Greater Than Unity

17.4.15 $F(\varphi|m)=m^{-1/2}F(\theta|m^{-1})$, $\sin\theta=m^{1/2}\sin\varphi$

17.4.16 $E(u|m)=m^{1/2}E(um^{1/2}|m^{-1})-(m-1)u$

by which a parameter greater than unity can be replaced by a parameter less than unity.

Negative Parameter

17.4.17

$$F(\varphi|-m)=(1+m)^{-1/2}K(m(1+m)^{-1})-(1+m)^{-1/2}F\left(\frac{\pi}{2}-\varphi\,|\,m(1+m)^{-1}\right)$$

17.4.18

$$E(u|-m)=(1+m)^{1/2}\{E(u(1+m)^{1/2}|m(m+1)^{-1})-m(1+m)^{-1/2}\mathrm{sn}(u(1+m)^{1/2}|m(1+m)^{-1})\,\mathrm{cd}(u(1+m)^{1/2}|m(1+m)^{-1})\}$$

whereby computations can be made for negative parameters, and therefore for pure imaginary modulus.

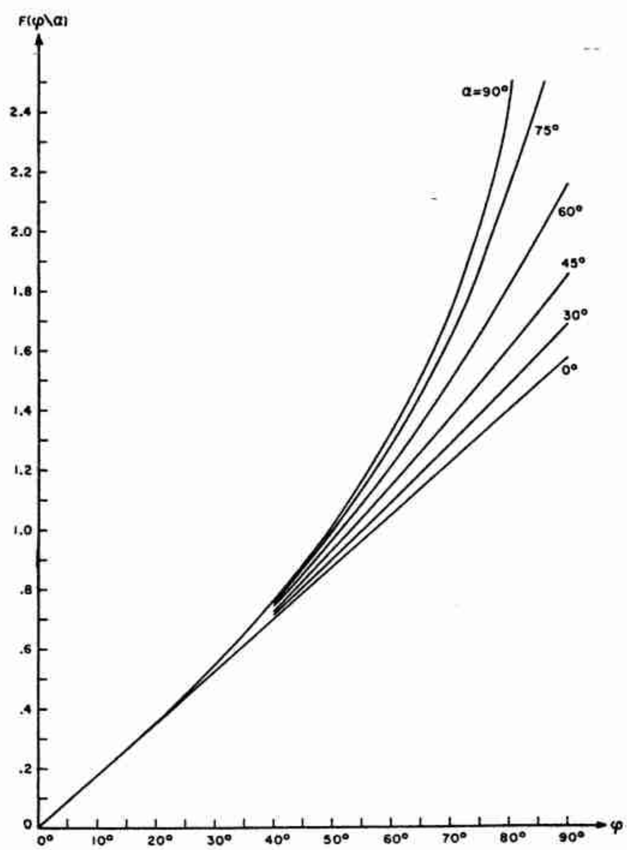

FIGURE 17.4. *Incomplete elliptic integral of the first kind.*

$F(\varphi\backslash\alpha)$, α constant

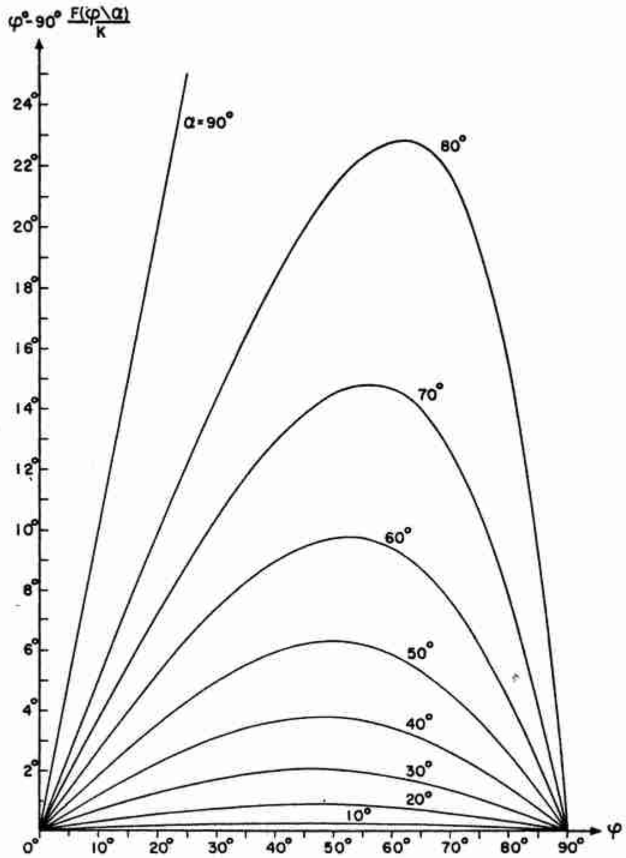

FIGURE 17.5. $\varphi-90° \dfrac{F(\varphi\backslash\alpha)}{K}$, *α constant.*

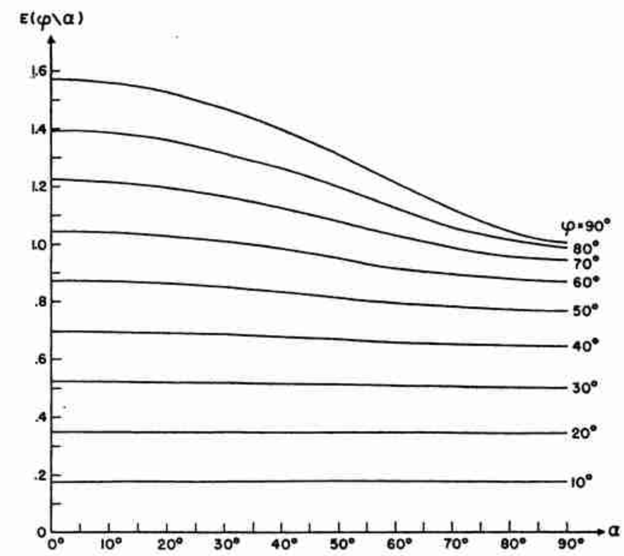

FIGURE 17.6. *Incomplete elliptic integral of the second kind.*

$E(\varphi\backslash\alpha)$, φ constant

Special Cases

17.4.19 $$F(\varphi\backslash 0)=\varphi$$

17.4.20 $$F(i\varphi\backslash 0)=i\varphi$$

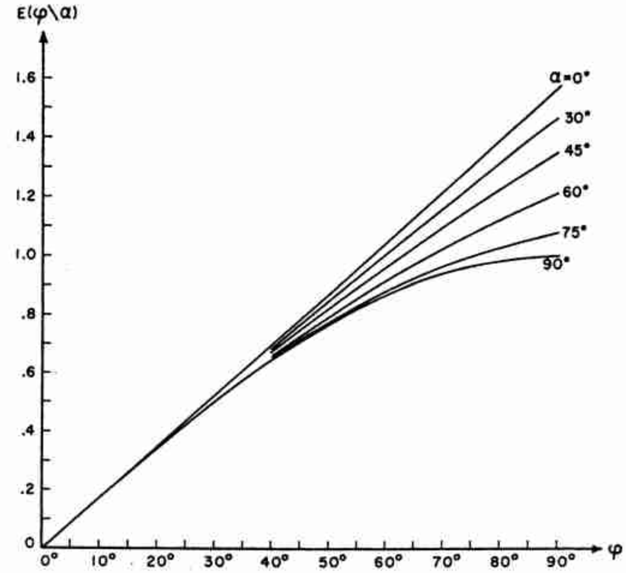

FIGURE 17.7. *Incomplete elliptic integral of the second kind.*

$E(\varphi\backslash\alpha)$, α constant

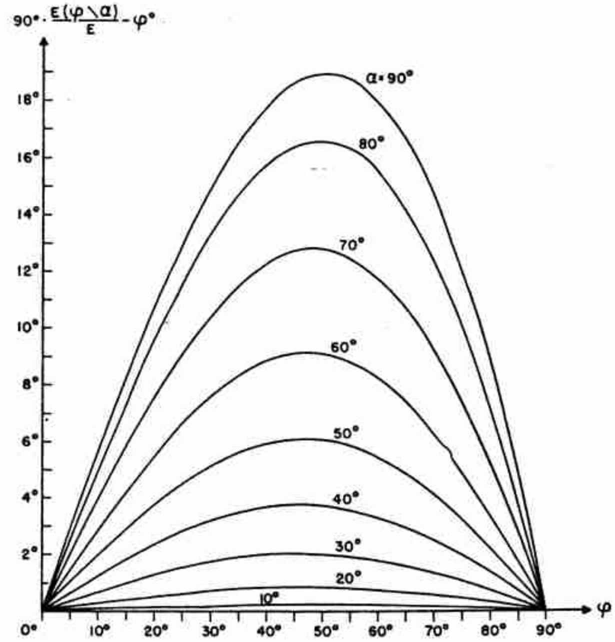

FIGURE 17.8. $90° \dfrac{E(\varphi\backslash\alpha)}{E}-\varphi$, *α constant.*

17.4.21

$$F(\varphi\backslash 90°)=\ln\,(\sec\varphi+\tan\varphi)=\ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$$

17.4.22 $$F(i\varphi\backslash 90°)=i\arctan\,(\sinh\varphi)$$

17.4.23 $$E(\varphi\backslash 0)=\varphi$$

17.4.24 $$E(i\varphi\backslash 0)=i\varphi$$

17.4.25 $$E(\varphi\backslash 90°)=\sin\varphi$$

17.4.26 $$E(i\varphi\backslash 90°)=i\sinh\varphi$$

Jacobi's Zeta Function

17.4.27 $Z(\varphi\backslash\alpha)=E(\varphi\backslash\alpha)-E(\alpha)F(\varphi\backslash\alpha)/K(\alpha)$

17.4.28 $Z(u|m)=Z(u)=E(u)-uE(m)/K(m)$

17.4.29 $Z(-u)=-Z(u)$

17.4.30 $Z(u+2K)=Z(u)$

17.4.31 $Z(K-u)=-Z(K+u)$

17.4.32 $Z(u)=Z(u-K)-m\operatorname{sn}(u-K)\operatorname{cd}(u-K)$

Special Values

17.4.33 $Z(u|0)=0$

17.4.34 $Z(u|1)=\tanh u$

Addition Theorem

17.4.35

$$Z(u+v)=Z(u)+Z(v)-m\operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u+v)$$

Jacobi's Imaginary Transformation

17.4.36

$$iZ(iu|m)=Z(u|m_1)+\frac{\pi u}{2KK'}-\operatorname{dn}(u|m_1)\operatorname{sc}(u|m_1)$$

Relation to Jacobi's Theta Function

17.4.37 $Z(u)=\Theta'(u)/\Theta(u)=\frac{d}{du}\ln\Theta(u)$

q-Series

17.4.38 $Z(u)=\frac{2\pi}{K}\sum_{s=1}^{\infty} q^s(1-q^{2s})^{-1}\sin(\pi su/K)$

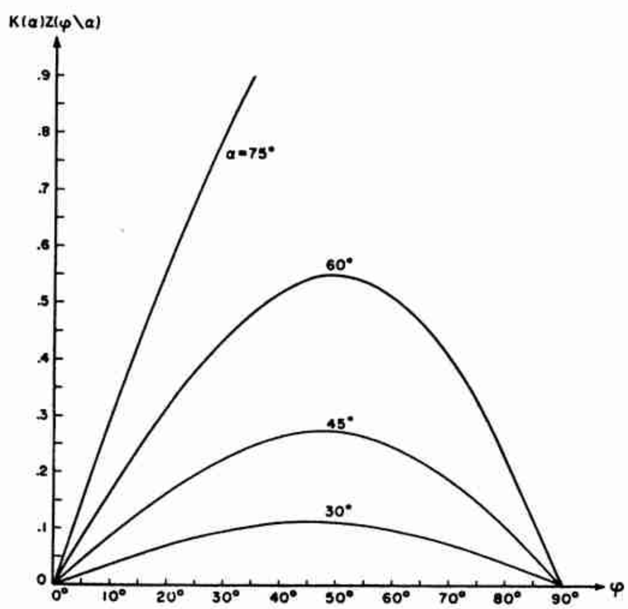

FIGURE 17.9. *Jacobian zeta function* $K(\alpha)Z(\varphi\backslash\alpha)$.

*See page II.

Heuman's Lambda Function

17.4.39

$$\Lambda_0(\varphi\backslash\alpha)=\frac{F(\varphi\backslash 90^\circ-\alpha)}{K'(\alpha)}+\frac{2}{\pi}K(\alpha)Z(\varphi\backslash 90^\circ-\alpha)$$

17.4.40

$$=\frac{2}{\pi}\{K(\alpha)E(\varphi\backslash 90^\circ-\alpha)-[K(\alpha)-E(\alpha)]F(\varphi\backslash 90^\circ-\alpha)\}$$

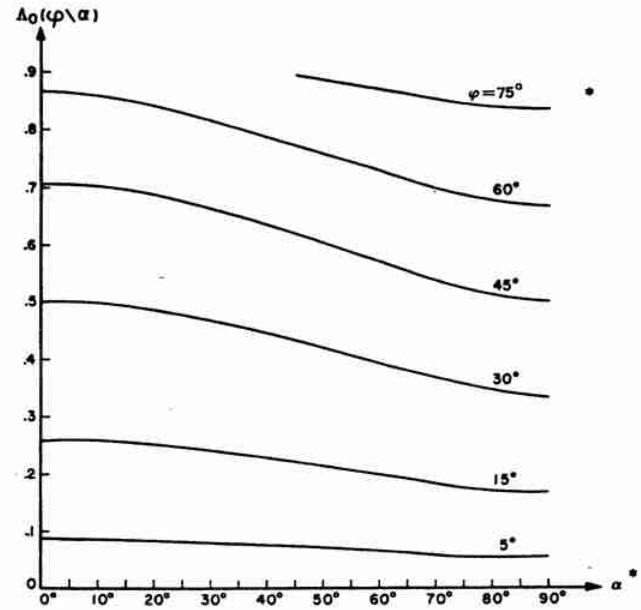

FIGURE 17.10. *Heuman's lambda function* $\Lambda_0(\varphi\backslash\alpha)$.

Numerical Evaluation of Incomplete Integrals of the First and Second Kinds

For the numerical evaluation of an elliptic integral the quartic (or cubic[4]) under the radical should first be expressed in terms of t^2, see **Examples 1** and **2.** In the resulting quartic there are only six possible sign patterns or combinations of the factors namely

$$(t^2+a^2)(t^2+b^2),\ (a^2-t^2)(t^2-b^2),$$
$$(a^2-t^2)(b^2-t^2),\ (t^2-a^2)(t^2-b^2),\ (t^2+a^2)(t^2-b^2),$$
$$(t^2+a^2)(b^2-t^2).$$

The list which follows is then exhaustive for integrals which reduce to $F(\varphi\backslash\alpha)$ or $E(\varphi\backslash\alpha)$.

The value of the elliptic integral of the first kind is also expressed as an *inverse* Jacobian elliptic function. Here, for example, the notation $u=\operatorname{sn}^{-1}x$ means that $x=\operatorname{sn} u$.

The column headed "t substitution" gives the Jacobian elliptic function substitution which is appropriate to reduce every elliptic integral which contains the given quartic.

[4] For an alternate treatment of cubics see **17.4.61** and **17.4.70.**

	$F(\varphi\backslash\alpha)$	Equivalent Inverse Jacobian Elliptic Function	φ	t Substitution	$E(\varphi\backslash\alpha)$
$\cos\alpha=b/a$ $a>b$ $m=(a^2-b^2)/a^2$	**17.4.41** $a\int_0^x \frac{dt}{[(t^2+a^2)(t^2+b^2)]^{1/2}}$	$\mathrm{sc}^{-1}\left(\frac{x}{b}\middle\vert\frac{a^2-b^2}{a^2}\right)$	$\tan\varphi=\frac{x}{b}$	$t=b\ \mathrm{sc}\ v$	$\frac{b^2}{a}\int_0^x\left(\frac{t^2+a^2}{t^2+b^2}\right)\frac{dt}{[(t^2+a^2)(t^2+b^2)]^{1/2}}$
	17.4.42 $a\int_x^\infty \frac{dt}{[(t^2+a^2)(t^2+b^2)]^{1/2}}$	$\mathrm{cs}^{-1}\left(\frac{x}{a}\middle\vert\frac{a^2-b^2}{a^2}\right)$	$\tan\varphi=\frac{a}{x}$	$t=a\ \mathrm{cs}\ v$	$a\int_x^\infty\left(\frac{t^2+b^2}{t^2+a^2}\right)\frac{dt}{[(t^2+a^2)(t^2+b^2)]^{1/2}}$
	17.4.43 $a\int_b^x \frac{dt}{[(a^2-t^2)(t^2-b^2)]^{1/2}}$	$\mathrm{nd}^{-1}\left(\frac{x}{b}\middle\vert\frac{a^2-b^2}{a^2}\right)$	$\sin^2\varphi=\frac{a^2(x^2-b^2)}{x^2(a^2-b^2)}$	$t=b\ \mathrm{nd}\ v$	$ab^2\int_b^x\frac{1}{t^2}\frac{dt}{[(a^2-t^2)(t^2-b^2)]^{1/2}}$
	17.4.44 $a\int_x^a \frac{dt}{[(a^2-t^2)(t^2-b^2)]^{1/2}}$	$\mathrm{dn}^{-1}\left(\frac{x}{a}\middle\vert\frac{a^2-b^2}{a^2}\right)$	$\sin^2\varphi=\frac{a^2-x^2}{a^2-b^2}$	$t=a\ \mathrm{dn}\ v$	$\frac{1}{a}\int_x^a\frac{t^2dt}{[(a^2-t^2)(t^2-b^2)]^{1/2}}$
$\sin\alpha=b/a$ $a>b$ $m=b^2/a^2$	**17.4.45** $a\int_0^x \frac{dt}{[(a^2-t^2)(b^2-t^2)]^{1/2}}$	$\mathrm{sn}^{-1}\left(\frac{x}{b}\middle\vert\frac{b^2}{a^2}\right)$	$\sin\varphi=\frac{x}{b}$	$t=b\ \mathrm{sn}\ v$	$\frac{1}{a}\int_0^x\frac{(a^2-t^2)dt}{[(a^2-t^2)(b^2-t^2)]^{1/2}}$
	17.4.46 $a\int_x^b \frac{dt}{[(a^2-t^2)(b^2-t^2)]^{1/2}}$	$\mathrm{cd}^{-1}\left(\frac{x}{b}\middle\vert\frac{b^2}{a^2}\right)$	$\sin^2\varphi=\frac{a^2(b^2-x^2)}{b^2(a^2-x^2)}$	$t=b\ \mathrm{cd}\ v$	$a(a^2-b^2)\int_x^b\left(\frac{1}{a^2-t^2}\right)\frac{dt}{[(a^2-t^2)(b^2-t^2)]^{1/2}}$
	17.4.47 $a\int_a^x \frac{dt}{[(t^2-a^2)(t^2-b^2)]^{1/2}}$	$\mathrm{dc}^{-1}\left(\frac{x}{a}\middle\vert\frac{b^2}{a^2}\right)$	$\sin^2\varphi=\frac{x^2-a^2}{x^2-b^2}$	$t=a\ \mathrm{dc}\ v$	$\frac{a^2-b^2}{a}\int_a^x\left(\frac{t^2}{t^2-b^2}\right)\frac{dt}{[(t^2-a^2)(t^2-b^2)]^{1/2}}$
	17.4.48 $a\int_x^\infty \frac{dt}{[(t^2-a^2)(t^2-b^2)]^{1/2}}$	$\mathrm{ns}^{-1}\left(\frac{x}{a}\middle\vert\frac{b^2}{a^2}\right)$	$\sin\varphi=\frac{a}{x}$	$t=a\ \mathrm{ns}\ v$	$a\int_x^\infty\left(\frac{t^2-b^2}{t^2}\right)\frac{dt}{[(t^2-a^2)(t^2-b^2)]^{1/2}}$
$\cot\alpha=\frac{b}{a}$ $m=a^2/(a^2+b^2)$	**17.4.49** $(a^2+b^2)^{1/2}\int_b^x \frac{dt}{[(t^2+a^2)(t^2-b^2)]^{1/2}}$	$\mathrm{nc}^{-1}\left(\frac{x}{b}\middle\vert\frac{a^2}{a^2+b^2}\right)$	$\cos\varphi=\frac{b}{x}$	$t=b\ \mathrm{nc}\ v$	$\frac{b^2}{(a^2+b^2)^{1/2}}\int_b^x\frac{t^2+a^2}{t^2}\frac{dt}{[(t^2+a^2)(t^2-b^2)]^{1/2}}$
	17.4.50 $(a^2+b^2)^{1/2}\int_x^\infty \frac{dt}{[(t^2+a^2)(t^2-b^2)]^{1/2}}$	$\mathrm{ds}^{-1}\left(\frac{x}{(a^2+b^2)^{1/2}}\middle\vert\frac{a^2}{a^2+b^2}\right)$	$\sin^2\varphi=\frac{a^2+b^2}{a^2+x^2}$	$t=(a^2+b^2)^{1/2}\,\mathrm{ds}\,v$	$(a^2+b^2)^{1/2}\int_x^\infty\frac{t^2}{(t^2+a^2)}\frac{dt}{[(t^2+a^2)(t^2-b^2)]^{1/2}}$
$\tan\alpha=\frac{b}{a}$ $m=b^2/(a^2+b^2)$	**17.4.51** $(a^2+b^2)^{1/2}\int_0^x \frac{dt}{[(t^2+a^2)(b^2-t^2)]^{1/2}}$	$\mathrm{sd}^{-1}\left(\frac{x(a^2+b^2)^{1/2}}{ab}\middle\vert\frac{b^2}{a^2+b^2}\right)$	$\sin^2\varphi=\frac{x^2(a^2+b^2)}{b^2(a^2+x^2)}$	$t=\frac{ab}{(a^2+b^2)^{1/2}}\,\mathrm{sd}\,v$	$a^2(a^2+b^2)^{1/2}\int_0^x\frac{1}{(t^2+a^2)}\frac{dt}{[(t^2+a^2)(b^2-t^2)]^{1/2}}$
	17.4.52 $(a^2+b^2)^{1/2}\int_x^b \frac{dt}{[(t^2+a^2)(b^2-t^2)]^{1/2}}$	$\mathrm{cn}^{-1}\left(\frac{x}{b}\middle\vert\frac{b^2}{a^2+b^2}\right)$	$\cos\varphi=\frac{x}{b}$	$t=b\ \mathrm{cn}\ v$	$\frac{1}{(a^2+b^2)^{1/2}}\int_x^b\frac{(t^2+a^2)dt}{[(t^2+a^2)(b^2-t^2)]^{1/2}}$

Some Important Special Cases

$\frac{1}{2}F(\varphi\backslash\alpha)$	$\cos\varphi$	α	$\frac{1}{3^{1/4}}F(\varphi\backslash\alpha)$	$\cos\varphi$	α
17.4.53 $\int_x^\infty \frac{dt}{(1+t^4)^{\frac{1}{2}}}$	$\frac{x^2-1}{x^2+1}$	45°	**17.4.57** $\int_x^\infty \frac{dt}{(t^3-1)^{\frac{1}{2}}}$	$\frac{x-1-\sqrt{3}}{x-1+\sqrt{3}}$	15°
17.4.54 $\int_0^x \frac{dt}{(1+t^4)^{\frac{1}{2}}}$	$\frac{1-x^2}{1+x^2}$	45°	**17.4.58** $\int_1^x \frac{dt}{(t^3-1)^{\frac{1}{2}}}$	$\frac{\sqrt{3}+1-x}{\sqrt{3}-1+x}$	15°
17.4.55 * $\frac{1}{2^{\frac{1}{2}}}\int_1^x \frac{dt}{(t^4-1)^{\frac{1}{2}}}$	$\frac{1}{x}$	45°	**17.4.59** $\int_x^1 \frac{dt}{(1-t^3)^{\frac{1}{2}}}$	$\frac{\sqrt{3}-1+x}{\sqrt{3}+1-x}$	75°
17.4.56 * $\frac{1}{2^{\frac{1}{2}}}\int_x^1 \frac{dt}{(1-t^4)^{\frac{1}{2}}}$	x	45°	**17.4.60** $\int_{-\infty}^x \frac{dt}{(1-t^3)^{\frac{1}{2}}}$	$\frac{1-\sqrt{3}-x}{1+\sqrt{3}-x}$	75°

Reduction of $\int dt/\sqrt{P}$ where $P=P(t)$ is a cubic polynomial with three real factors $P=(t-\beta_1)(t-\beta_2)(t-\beta_3)$ where $\beta_1>\beta_2>\beta_3$. Write

17.4.61

$$\lambda=\frac{1}{2}(\beta_1-\beta_3)^{1/2},\ m=\sin^2\alpha=\frac{\beta_2-\beta_3}{\beta_1-\beta_3},$$
$$m_1=\cos^2\alpha=\frac{\beta_1-\beta_2}{\beta_1-\beta_3}$$

17.4.62 $\lambda\int_{\beta_3}^x \frac{dt}{\sqrt{P}}$	$F(\varphi\backslash\alpha)$	$\sin^2\varphi=\frac{x-\beta_3}{\beta_2-\beta_3}$
17.4.63 $\lambda\int_x^{\beta_2} \frac{dt}{\sqrt{P}}$	$F(\varphi\backslash\alpha)$	$\cos^2\varphi=\frac{(\beta_1-\beta_2)(x-\beta_3)}{(\beta_2-\beta_3)(\beta_1-x)}$
17.4.64 $\lambda\int_{\beta_1}^x \frac{dt}{\sqrt{P}}$	$F(\varphi\backslash\alpha)$	$\sin^2\varphi=\frac{x-\beta_1}{x-\beta_2}$
17.4.65 $\lambda\int_x^\infty \frac{dt}{\sqrt{P}}$	$F(\varphi\backslash\alpha)$	$\cos^2\varphi=\frac{x-\beta_1}{x-\beta_3}$
17.4.66 $\lambda\int_{-\infty}^x \frac{dt}{\sqrt{-P}}$	$F(\varphi\backslash(90°-\alpha°))$	$\sin^2\varphi=\frac{\beta_1-\beta_3}{\beta_1-x}$
17.4.67 $\lambda\int_x^{\beta_3} \frac{dt}{\sqrt{-P}}$	$F(\varphi\backslash(90°-\alpha°))$	$\cos^2\varphi=\frac{\beta_2-\beta_3}{\beta_2-x}$
17.4.68 $\lambda\int_{\beta_2}^x \frac{dt}{\sqrt{-P}}$	$F(\varphi\backslash(90°-\alpha°))$	$\sin^2\varphi=\frac{(\beta_1-\beta_3)(x-\beta_2)}{(\beta_1-\beta_2)(x-\beta_3)}$
17.4.69 $\lambda\int_x^{\beta_1} \frac{dt}{\sqrt{-P}}$	$F(\varphi\backslash(90°-\alpha°))$	$\cos^2\varphi=\frac{x-\beta_2}{\beta_1-\beta_2}$

*See page II.

Reduction of $\int dt/\sqrt{P}$ when $P=P(t)=t^3+a_1t^2+a_2t+a_3$ is a cubic polynomial with only one real root $t=\beta$. We form the first and second derivatives $P'(t)$, $P''(t)$ with respect to t and then write

17.4.70 $\lambda^2=[P'(\beta)]^{1/2},\ m=\sin^2\alpha=\frac{1}{2}-\frac{1}{8}\frac{P''(\beta)}{[P'(\beta)]^{1/2}}$

17.4.71 $\lambda\int_\beta^x \frac{dt}{\sqrt{P}}$	$F(\varphi\backslash\alpha)$	$\cos\varphi=\frac{\lambda^2-(x-\beta)}{\lambda^2+(x-\beta)}$
17.4.72 $\lambda\int_x^\infty \frac{dt}{\sqrt{P}}$	$F(\varphi\backslash\alpha)$	$\cos\varphi=\frac{(x-\beta)-\lambda^2}{(x-\beta)+\lambda^2}$
17.4.73 $\lambda\int_{-\infty}^x \frac{dt}{\sqrt{(-P)}}$	$F(\varphi\backslash(90°-\alpha°))$	$\cos\varphi=\frac{(\beta-x)-\lambda^2}{(\beta-x)+\lambda^2}$
17.4.74 $\lambda\int_x^\beta \frac{dt}{\sqrt{(-P)}}$	$F(\varphi\backslash(90°-\alpha°))$	$\cos\varphi=\frac{\lambda^2-(\beta-x)}{\lambda^2+(\beta-x)}$

17.5. Landen's Transformation

Descending Landen Transformation [5]

Let α_n, α_{n+1} be two modular angles such that

17.5.1 $(1+\sin\alpha_{n+1})(1+\cos\alpha_n)=2 \qquad (\alpha_{n+1}<\alpha_n)$

and let φ_n, φ_{n+1} be two corresponding amplitudes such that

17.5.2 $\tan(\varphi_{n+1}-\varphi_n)=\cos\alpha_n\tan\varphi_n \quad (\varphi_{n+1}>\varphi_n)$

[5] The emphasis here is on the modular angle since this is an argument of the Tables. All formulae concerning Landen's transformation may also be expressed in terms of the modulus $k=m^{\frac{1}{2}}=\sin\alpha$ and its complement $k'=m_1^{\frac{1}{2}}=\cos\alpha$.

Thus the step from n to $n+1$ decreases the modular angle but increases the amplitude. By iterating the process we can descend from a given modular angle to one whose magnitude is negligible, when **17.4.19** becomes applicable.

With $\alpha_0=\alpha$ we have

17.5.3

$$F(\varphi\backslash\alpha)=(1+\cos\,\alpha)^{-1}F(\varphi_1\backslash\alpha_1)$$

$$=\tfrac{1}{2}(1+\sin\,\alpha_1)F(\varphi_1\backslash\alpha_1)$$

17.5.4 $$F(\varphi\backslash\alpha)=2^{-n}\prod_{s=1}^{n}(1+\sin\alpha_s)F(\varphi_n\backslash\alpha_n)$$

17.5.5 $$F(\varphi\backslash\alpha)=\Phi\prod_{s=1}^{\infty}(1+\sin\,\alpha_s)$$

17.5.6 $$\Phi=\lim_{n\to\infty}\frac{1}{2^n}F(\varphi_n\backslash\alpha_n)=\lim_{n\to\infty}\frac{\varphi_n}{2^n}$$

17.5.7 $$K=F(\tfrac{1}{2}\pi\backslash\alpha)=\tfrac{1}{2}\pi\prod_{s=1}^{\infty}(1+\sin\alpha_s)$$

17.5.8 $$F(\varphi\backslash\alpha)=2\pi^{-1}K\Phi$$

17.5.9

$$E(\varphi\backslash\alpha)=F(\varphi\backslash\alpha)\left[1-\frac{1}{2}\sin^2\alpha\left(1+\frac{1}{2}\sin\alpha_1\right.\right.$$

$$\left.\left.+\frac{1}{2^2}\sin\alpha_1\sin\alpha_2+\ldots\right)\right]+\sin\alpha\left[\frac{1}{2}(\sin\alpha_1)^{1/2}\sin\varphi_1\right.$$

$$\left.+\frac{1}{2^2}(\sin\alpha_1\sin\alpha_2)^{1/2}\sin\varphi_2+\ldots\right]$$

17.5.10

$$E=K\left[1-\frac{1}{2}\sin^2\alpha\left(1+\frac{1}{2}\sin\alpha_1+\frac{1}{2^2}\sin\alpha_1\sin\alpha_2\right.\right.$$

$$\left.\left.+\frac{1}{2^3}\sin\alpha_1\sin\alpha_2\sin\alpha_3+\ldots\right)\right]$$

Ascending Landen Transformation

Let α_n, α_{n+1} be two modular angles such that

17.5.11 $$(1+\sin\,\alpha_n)(1+\cos\,\alpha_{n+1})=2\qquad(\alpha_{n+1}>\alpha_n)$$

and let φ_n, φ_{n+1} be two corresponding amplitudes such that

17.5.12 $$\sin\,(2\varphi_{n+1}-\varphi_n)=\sin\,\alpha_n\sin\,\varphi_n\qquad(\varphi_{n+1}<\varphi_n)$$

Thus the step from n to $n+1$ increases the modular angle but decreases the amplitude. By iterating the process we can ascend from a given modular angle to one whose difference from a right angle is so small that **17.4.21** becomes applicable.

With $\alpha_0=\alpha$ we have

17.5.13 $$F(\varphi\backslash\alpha)=2(1+\sin\,\alpha)^{-1}F(\varphi_1\backslash\alpha_1)$$

17.5.14 $$F(\varphi\backslash\alpha)=2^n\prod_{s=0}^{n-1}(1+\sin\,\alpha_s)^{-1}F(\varphi_n\backslash\alpha_n)$$

17.5.15 $$F(\varphi\backslash\alpha)=\prod_{s=1}^{n}(1+\cos\,\alpha_s)F(\varphi_n\backslash\alpha_n)$$

17.5.16 $$F(\varphi\backslash\alpha)=[\csc\,\alpha\prod_{s=1}^{\infty}\sin\,\alpha_s]^{\frac{1}{2}}\ln\,\tan\,(\tfrac{1}{4}\pi+\tfrac{1}{2}\Phi)$$

17.5.17 $$\Phi=\lim_{n\to\infty}\varphi_n$$

Neighborhood of a Right Angle (see also 17.4.13)

When both φ and α are near to a right angle, interpolation in the table $F(\varphi\backslash\alpha)$ is difficult. Either Landen's transformation can then be used with advantage to increase the modular angle and decrease the amplitude or vice-versa.

17.6. The Process of the Arithmetic-Geometric Mean

Starting with a given number triple (a_0, b_0, c_0) we proceed to determine number triples (a_1, b_1, c_1), (a_2, b_2, c_2), . . ., (a_N, b_N, c_N) according to the following scheme of arithmetic and geometric means

17.6.1

$$\begin{array}{lll}
a_0 & b_0 & \\
a_1=\tfrac{1}{2}(a_0+b_0) & b_1=(a_0b_0)^{\frac{1}{2}} & \\
a_2=\tfrac{1}{2}(a_1+b_1) & b_2=(a_1b_1)^{\frac{1}{2}} & \\
\vdots & \vdots & \\
a_N=\tfrac{1}{2}(a_{N-1}+b_{N-1}) & b_N=(a_{N-1}b_{N-1})^{\frac{1}{2}} & \\
 & & c_0\\
 & & c_1=\tfrac{1}{2}(a_0-b_0)\\
 & & c_2=\tfrac{1}{2}(a_1-b_1)\\
 & & \vdots\\
 & & c_N=\tfrac{1}{2}(a_{N-1}-b_{N-1}).
\end{array}$$

We stop at the Nth step when $a_N=b_N$, i.e., when $c_N=0$ to the degree of accuracy to which the numbers are required.

To determine the complete elliptic integrals $K(\alpha)$, $E(\alpha)$ we start with

17.6.2 $$a_0=1,\ b_0=\cos\,\alpha,\ c_0=\sin\,\alpha$$

whence

17.6.3 $$K(\alpha)=\frac{\pi}{2a_N}$$

17.6.4 $$\frac{K(\alpha)-E(\alpha)}{K(\alpha)}=\frac{1}{2}[c_0^2+2c_1^2+2^2c_2^2+\ldots+2^Nc_N^2]$$

To determine $K'(\alpha)$, $E'(\alpha)$ we start with

17.6.5 $$a_0'=1,\ b_0'=\sin\alpha,\ c_0'=\cos\alpha$$

whence

17.6.6 $$K'(\alpha)=\frac{\pi}{2a_N'}$$

17.6.7

$$\frac{K'(\alpha)-E'(\alpha)}{K'(\alpha)}=\frac{1}{2}[c_0'^2+2c_1'^2+2^2c_2'^2+\ldots+2^Nc_N'^2]$$

To calculate $F(\varphi\backslash\alpha)$, $E(\varphi\backslash\alpha)$ start from **17.5.2** which corresponds to the descending Landen transformation and determine $\varphi_1, \varphi_2, \ldots, \varphi_N$ successively from the relation

17.6.8 $$\tan(\varphi_{n+1}-\varphi_n)=(b_n/a_n)\tan\varphi_n,\ \varphi_0=\varphi$$

Then to the prescribed accuracy

17.6.9 $$F(\varphi\backslash\alpha)=\varphi_N/(2^Na_N)$$ *

17.6.10

$$Z(\varphi\backslash\alpha)=E(\varphi\backslash\alpha)-(E/K)F(\varphi\backslash\alpha)$$

* $$=c_1\sin\varphi_1+c_2\sin\varphi_2+\ldots+c_N\sin\varphi_N$$

17.7. Elliptic Integrals of the Third Kind

17.7.1

$$\Pi(n;\varphi\backslash\alpha)=\int_0^\varphi(1-n\sin^2\theta)^{-1}(1-\sin^2\alpha\sin^2\theta)^{-\frac{1}{2}}d\theta$$

17.7.2 $$\Pi(n;\tfrac{1}{2}\pi\backslash\alpha)=\Pi(n\backslash\alpha)$$

Case (i) Hyperbolic Case $0<n<\sin^2\alpha$

$$\epsilon=\arcsin(n/\sin^2\alpha)^{\frac{1}{2}},\qquad 0\le\epsilon\le\tfrac{1}{2}\pi$$

$$\beta=\tfrac{1}{2}\pi F(\epsilon\backslash\alpha)/K(\alpha)$$

$$q=q(\alpha)$$

$$v=\tfrac{1}{2}\pi F(\varphi\backslash\alpha)/K(\alpha),$$

$$\delta_1=[n(1-n)^{-1}(\sin^2\alpha-n)^{-1}]^{\frac{1}{2}}$$

17.7.3

$$\Pi(n;\varphi\backslash\alpha)=\delta_1[-\tfrac{1}{2}\ln[\vartheta_4(v+\beta)/\vartheta_4(v-\beta)]$$
$$+v\vartheta_1'(\beta)/\vartheta_1(\beta)]$$

*See page II.

17.7.4

$$\tfrac{1}{2}\ln\frac{\vartheta_4(v+\beta)}{\vartheta_4(v-\beta)}=2\sum_{s=1}^\infty s^{-1}q^s(1-q^{2s})^{-1}\sin 2sv\sin 2s\beta$$

17.7.5

$$\frac{\vartheta_1'(\beta)}{\vartheta_1(\beta)}=\cot\beta+4\sum_{s=1}^\infty q^{2s}(1-2q^{2s}\cos 2\beta+q^{4s})^{-1}\sin 2\beta$$

In the above we can also use Neville's theta functions **16.36.**

17.7.6 $$\Pi(n\backslash\alpha)=K(\alpha)+\delta_1K(\alpha)Z(\epsilon\backslash\alpha)$$

Case (ii) Hyperbolic Case $n>1$

The case $n>1$ can be reduced to the case $0<N<\sin^2\alpha$ by writing

17.7.7 $$N=n^{-1}\sin^2\alpha,\ p_1=[(n-1)(1-n^{-1}\sin^2\alpha)]^{\frac{1}{2}}$$

17.7.8

$$\Pi(n;\varphi\backslash\alpha)=-\Pi(N;\varphi\backslash\alpha)+F(\varphi\backslash\alpha)$$
$$+\frac{1}{2p_1}\ln[(\Delta(\varphi)+p_1\tan\varphi)(\Delta(\varphi)-p_1\tan\varphi)^{-1}]$$

where $\Delta(\varphi)$ is the delta amplitude, **17.2.4.**

17.7.9 $$\Pi(n\backslash\alpha)=K(\alpha)-\Pi(N\backslash\alpha)$$

Case (iii) Circular Case $\sin^2\alpha<n<1$

$$\epsilon=\arcsin[(1-n)/\cos^2\alpha]^{\frac{1}{2}}\qquad 0\le\epsilon\le\tfrac{1}{2}\pi$$

$$\beta=\tfrac{1}{2}\pi F(\epsilon\backslash 90^\circ-\alpha)/K(\alpha)$$

$$q=q(\alpha)$$

17.7.10

$$v=\tfrac{1}{2}\pi F(\varphi\backslash\alpha)/K(\alpha),\ \delta_2=[n(1-n)^{-1}(n-\sin^2\alpha)^{-1}]^{\frac{1}{2}}$$

17.7.11 $$\Pi(n;\varphi\backslash\alpha)=\delta_2(\lambda-4\mu v)$$

17.7.12

$$\lambda=\arctan(\tanh\beta\tan v)$$
$$+2\sum_{s=1}^\infty(-1)^{s-1}s^{-1}q^{2s}(1-q^{2s})^{-1}\sin 2sv\sinh 2s\beta$$

17.7.13

$$\mu=\left[\sum_{s=1}^\infty sq^{s^2}\sinh 2s\beta\right]\left[1+2\sum_{s=1}^\infty q^{s^2}\cosh 2s\beta\right]^{-1}$$

17.7.14 $$\Pi(n\backslash\alpha)=K(\alpha)+\tfrac{1}{2}\pi\delta_2[1-\Lambda_0(\epsilon\backslash\alpha)]$$

where Λ_0 is Heuman's Lambda function, **17.4.39.**

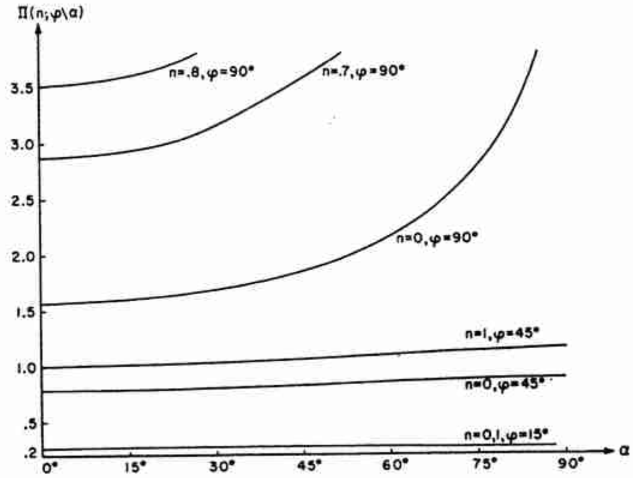

FIGURE 17.11. *Elliptic integral of the third kind* $\Pi(n;\, \varphi\backslash\alpha)$.

Case (iv) Circular Case $n<0$

The case $n<0$ can be reduced to the case $\sin^2\alpha<N<1$ by writing

17.7.15

$$N=(\sin^2\alpha-n)(1-n)^{-1}$$

$$p_2=[-n(1-n)^{-1}(\sin^2\alpha-n)]^{\frac{1}{2}}$$

17.7.16

$$[(1-n)(1-n^{-1}\sin^2\alpha)]^{\frac{1}{2}}\Pi(n;\,\varphi\backslash\alpha)$$
$$=[(1-N)(1-N^{-1}\sin^2\alpha)]^{\frac{1}{2}}\Pi(N;\,\varphi\backslash\alpha)$$
$$+p_2^{-1}\sin^2\alpha F(\varphi\backslash\alpha)+\arctan\,[\tfrac{1}{2}p_2\sin 2\varphi/\Delta(\varphi)]$$

17.7.17

$$\Pi(n\backslash\alpha)=(-n\cos^2\alpha)(1-n)^{-1}(\sin^2\alpha-n)^{-1}\Pi(N\backslash\alpha)$$
$$+\sin^2\alpha(\sin^2\alpha-n)^{-1}K(\alpha)$$

Special Cases

17.7.18 $\qquad n=0$

$$\Pi(0;\,\varphi\backslash\alpha)=F(\varphi\backslash\alpha)$$

17.7.19 $\qquad n=0,\ \alpha=0$

$$\Pi(0;\,\varphi\backslash 0)=\varphi$$

17.7.20 $\qquad \alpha=0$

$$\Pi(n;\,\varphi\backslash 0)=(1-n)^{-\frac{1}{2}}\arctan\,[(1-n)^{\frac{1}{2}}\tan\varphi], \quad n<1$$ *

$$=(n-1)^{-\frac{1}{2}}\operatorname{arctanh}\,[(n-1)^{\frac{1}{2}}\tan\varphi], \quad n>1$$

$$=\tan\varphi \qquad n=1$$

17.7.21 $\qquad \alpha=\pi/2$

$$\Pi(n;\,\varphi\backslash\pi/2)=(1-n)^{-1}[\ln\,(\tan\varphi+\sec\varphi)$$
$$-\tfrac{1}{2}n^{\frac{1}{2}}\ln\,(1+n^{\frac{1}{2}}\sin\varphi)(1-n^{\frac{1}{2}}\sin\varphi)^{-1}] \qquad n\neq 1$$

17.7.22 $\qquad n=\pm\sin\alpha$

$$(1\mp\sin\alpha)\{2\Pi(\pm\sin\alpha;\,\varphi\backslash\alpha)-F(\varphi\backslash\alpha)\}$$
$$=\arctan\,[(1\mp\sin\alpha)\tan\varphi/\Delta(\varphi)]$$

17.7.23 $\qquad n=1\pm\cos\alpha$

$$2\cos\alpha\Pi(1\pm\cos\alpha;\,\varphi\backslash\alpha)=\pm\tfrac{1}{2}\ln\,[(1+\tan\varphi$$
$$\cdot\Delta(\varphi))(1-\tan\varphi\cdot\Delta(\varphi))^{-1}]+\tfrac{1}{2}\ln\,[(\Delta(\varphi)$$
$$+\cos\alpha\cdot\tan\varphi)(\Delta(\varphi)-\cos\alpha\tan\varphi)^{-1}]$$
$$\mp(1\mp\cos\alpha)F(\varphi\backslash\alpha)$$

17.7.24 $\qquad n=\sin^2\alpha$

$$\Pi(\sin^2\alpha;\,\varphi\backslash\alpha)=\sec^2\alpha E(\varphi\backslash\alpha)-(\tan^2\alpha\sin 2\varphi)/(2\Delta(\varphi))$$

17.7.25 $\qquad n=1$

$$\Pi\,(1;\,\varphi\backslash\alpha)=F(\varphi\backslash\alpha)-\sec^2\alpha E(\varphi\backslash\alpha)+\sec^2\alpha\tan\varphi\Delta(\varphi)$$

Numerical Methods

17.8. Use and Extension of the Tables

Example 1. Reduce to canonical form $\int y^{-1}dx$, where

$$y^2=-3x^4+34x^3-119x^2+172x-90$$

By inspection or by solving an equation of the fourth degree we find that

$$y^2=Q_1Q_2 \text{ where } Q_1=3x^2-10x+9,\ Q_2=-x^2+8x-10$$

First Method

$Q_1-\lambda Q_2=(3+\lambda)x^2-(10+8\lambda)x+9+10\lambda$ is a perfect square if the discriminant

$$(10+8\lambda)^2-4(3+\lambda)(9+10\lambda)=0;\ \text{i.e., if } \lambda=-\frac{2}{3} \text{ or } \frac{1}{2}$$

and then

$$Q_1+\frac{2}{3}Q_2=\frac{7}{3}(x-1)^2,\ Q_1-\frac{1}{2}Q_2=\frac{7}{2}(x-2)^2$$

Solving for Q_1 and Q_2 we get

$$Q_1=(x-1)^2+2(x-2)^2,\ Q_2=2(x-1)^2-3(x-2)^2$$

The substitution $t=(x-1)/(x-2)$ then gives

$$\int y^{-1}dx=\pm\int[(t^2+2)(2t^2-3)]^{-\frac{1}{2}}dt$$

*See page II.

If the quartic $y^2=0$ has four real roots in x (or in the case of a cubic all three roots are real), we must so combine the factors that no root of $Q_1=0$ lies between the roots of $Q_2=0$ and no root of $Q_2=0$ lies between the roots of $Q_1=0$. Provided this condition is observed the method just described will always lead to real values of λ. These values may, however, be irrational.

Second Method

Write

$$t^2=\frac{Q_1}{Q_2}=\frac{3x^2-10x+9}{-x^2+8x-10}$$

and let the discriminant of $Q_2t^2-Q_1$ be

$$4T^2=(8t^2+10)^2-4(t^2+3)(10t^2+9)$$
$$=4(3t^2+2)(2t^2-1)$$

Then

$$\int y^{-1}dx=\pm\int T^{-1}dt=\pm\int[(3t^2+2)(2t^2-1)]^{-\frac{1}{2}}dt$$

This method will succeed if, as here, T^2 as a function of t^2 has real factors. If the coefficients of the given quartic are rational numbers, the factors of T^2 will likewise be rational.

Third Method

Write

$$w=\frac{Q_1}{Q_2}=\frac{3x^2-10x+9}{-x^2+8x-10}$$

and let the discriminant of Q_2w-Q_1 be

$$4W=4(3w+2)(2w-1)=4(Aw^2+Bw+C)$$

Then if

$$z^2=W/w \text{ and } Z^2=(B-z^2)^2-4AC=(z^2-1)^2+48$$

$$\int y^{-1}dx=\pm\int Z^{-1}dz$$

However, in this case the factors of Z are complex and the method fails.

Of the second and third methods one will always succeed where the other fails, and if the coefficients of the given quartic are rational numbers, the factors of T^2 or Z^2, as the case may be, will be rational.

Example 2. Reduce to canonical form $\int y^{-1}dx$ where $y^2=x(x-1)(x-2)$.

We use the third method of **Example 1** taking $Q_1=(x-1)$, $Q_2=x(x-2)$ and writing

$$w=\frac{Q_1}{Q_2}=\frac{x-1}{x^2-2x}$$

The discriminant of $Q_2w-Q_1=x^2w-(2w+1)x+1$ is

$$4W=(2w+1)^2-4w=4w^2+1$$

so that

$$W=Aw^2+Bw+C \text{ where } A=1,\ B=0,\ C=\frac{1}{4}$$

and if we write $z^2=W/w$ and

$$Z^2=(B-z^2)^2-4AC=(z^2)^2-1=(z^2-1)(z^2+1),$$

$$\int y^{-1}dx=\pm\int[(z^2-1)(z^2+1)]^{-1/2}\,dz$$

The first method of **Example 1** fails with the above values of Q_1 and Q_2 since the root of $Q_1=0$ lies between the roots of $Q_2=0$, and we get imaginary values of λ. The method succeeds, however, if we take $Q_1=x$, $Q_2=(x-1)(x-2)$, for then the roots of $Q_1=0$ do not lie between those of $Q_2=0$.

Example 3. Find $K(80/81)$.

First Method

Use **17.3.29** with $m=80/81$, $m_1=1/81$, $m_1^{1/2}=1/9$. Since $[(1-m_1^{1/2})(1+m_1^{1/2})^{-1}]^2=.64$, $K(80/81)=1.8\ K(.64)=3.59154\ 500$ to 8D, taking $K(.64)$ from **Table 17.1.**

Second Method

Table 17.4 giving $L(m)$ is useful for computing $K(m)$ when m is near unity or $K'(m)$ when m is near zero.

$$K(80/81)=\frac{1}{\pi}K'(80/81)\ln(16\times 81)-L(80/81).$$

By interpolation in **Tables 17.1** and **17.4,** since $80/81=.98765\ 43210$,

$$K'(80/81)=1.57567\ 8423$$
$$L(80/81)=.00311\ 16543$$
$$K(80/81)=\pi^{-1}(1.57567\ 8423)(7.16703\ 7877)$$
$$-.00311\ 16543$$
$$=3.59154\ 5000 \text{ to 9D.}$$

Third Method

The polynomial approximation **17.3.34** gives to 8D

$$K(80/81)=3.59154\ 501$$

Fourth Method, Arithmetic-Geometric Mean

Here $\sin^2\alpha=80/81$ and we start with

$$a_0=1,\ b_0=\frac{1}{9},\ c_0=\sqrt{80/81}=.99380\ 79900$$

giving

$$\epsilon=\arcsin\,[(1-n)/\cos^2\alpha]^{\frac{1}{2}}=45°$$
$$\beta=\tfrac{1}{2}\pi F(45°\backslash 60°)/K(30°)=.79317\ 74$$
$$v=\tfrac{1}{2}\pi F(45°\backslash 30°)/K(30°)=.74951\ 51$$
$$\delta_2=(40/9)^{\frac{1}{2}}$$
$$q=.01797\ 24$$

and so from **17.7.11**

$$\Pi\ (\tfrac{5}{8};\,45°\backslash 30°)=(40/9)^{1/2}(\lambda-4\mu v)$$
$$=2.10818\ 51\{.55248\ 32-4(.03854\ 26)(.74951\ 51)\}=.921129.$$

Table 17.9 gives .92113 with 4 point Lagrangian interpolation.

Example 18. Evaluate the complete elliptic integral

$$\Pi\ (\tfrac{5}{8}\backslash 30°)\ \text{to 5D.}$$

From **17.7.14** we have

$$\Pi\ (\tfrac{5}{8}\backslash 30°)=K(30°)+\frac{\pi}{2}\sqrt{\frac{40}{9}}\,[1-\Lambda_0(\epsilon\backslash 30°)]$$

where $\epsilon=\arcsin\,[(1-n)/\cos^2\alpha]^{1/2}=45°$. Thus using **Table 17.8**

$$\Pi\ (\tfrac{5}{8}\backslash 30°)=2.80099.$$

Table 17.9 gives 2.80126 by 6 point Lagrangian interpolation. The discrepancy results from interpolation with respect to n for $\varphi=90°$ in **Table 17.9.**

Example 19. Evaluate

$$\Pi\ (\tfrac{5}{4};\,45°\backslash 30°)=\int_0^{\pi/4}(1-\tfrac{5}{4}\sin^2\theta)^{-1}(1-\tfrac{1}{4}\sin^2\theta)^{-1/2}\,d\theta$$

to 5D.

Here $n=\frac{5}{4}$, $\varphi=45°$, $\alpha=30°$ and since the characteristic is greater than unity we use **17.7.7**

$$N=n^{-1}\sin^2\alpha=.2,\ p_1=(1/5)^{\frac{1}{2}}$$
$$\Pi\ (\tfrac{5}{4};\,45°\backslash 30°)=-\Pi(\,2;\,45°\backslash 30°)+F(45°\backslash 30°)+(\tfrac{1}{2}\sqrt{5})\ \ln\frac{(7/8)^{\frac{1}{2}}+(1/5)^{\frac{1}{2}}}{(7/8)^{\frac{1}{2}}-(1/5)^{\frac{1}{2}}}$$
$$=-.83612+.80437+\tfrac{1}{2}\sqrt{5}\ \ln\frac{\sqrt{35}+\sqrt{8}}{\sqrt{35}-\sqrt{8}}$$
$$=1.13214.$$

Numerical quadrature gives the same result.

Example 20. Evaluate

$$\Pi\ (-\tfrac{1}{4};\,45°\backslash 30°)=\int_0^{\pi/4}(1+\tfrac{1}{4}\sin^2\theta)^{-1}(1-\tfrac{1}{4}\sin^2\theta)^{-\frac{1}{2}}d\theta$$

to 5D.

Here the characteristic is negative and we therefore use **17.7.15** with $n=-\frac{1}{4}$, $\sin^2\alpha=\frac{1}{4}$

$$N=(1-n)^{-1}(\sin^2\alpha-n)=.4,\ \ p_2=\sqrt{.1}$$

and therefore

$$(5/2)^{\frac{1}{2}}\ \Pi\ (-\tfrac{1}{4};\,45°\backslash 30°)=(9/40)^{\frac{1}{2}}\Pi\ (\tfrac{2}{5};\,45°\backslash 30°)+\frac{1}{2}(5/2)^{\frac{1}{2}}F(45°\backslash 30°)+\arctan\,(35)^{-\frac{1}{2}}$$

Using **Tables 4.14, 17.5,** and **17.9** we get

$$\Pi\ (-\tfrac{1}{4};\,45°\backslash 30°)=.76987$$

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