MXB261 Group Project

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Contents

1 Introduction

In this report the system dynamics of a simple parasite model were explored. The main aim was to investigate the affect that altering values of the models parameters would have on the underlying system dynamics. The system in explained in more depth in section 2.2, but in general it models the dynamics of a two population system where one population is a parasite, and the other populating may be considered as a food population. The model includes parameters for the birth and death rates of the parasites, as well as the rates of food growth, decay, and consumption of the food source.

To explore the relations between the values of these parameters and the system dynamics a variety of approaches were used. The first approach was performing first a one dimensional then series of two dimensional parameter sweeps, with the steady state solution to the system being solved via the use of MATLAB's `ODE45`.In this method the relations between the parameters relating food growth and food decay, and then food decay and food consumption rates were explored.

The second approach involved again solving the system with `ODE45` but this time performing a method of latin hypercube sampling to explore the relationship between the parameter values and the system dynamics this time in 3D parameter space. This method explored the relation's between the same parameters as the in the first method but this time the relations between all 3 at once was observed.

In the final section an agent based simulation examines the evolution of the population of parasites and prey in a $200x200$ grid simulation space with random movement of the parasites and interaction with the food prey. In addition to the original model parameters, the simulation examines two food growth methods, random food placement and localised food placement.

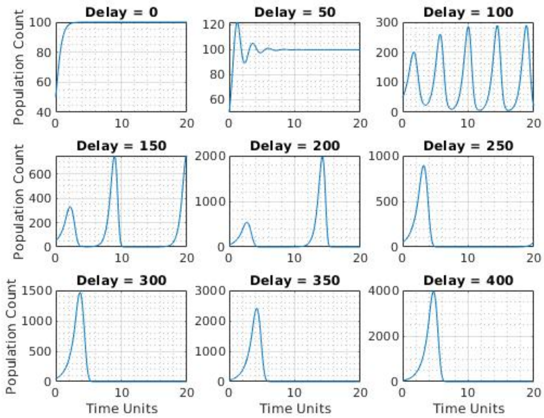
2 Task 1 Delay Dynamics and 1 and 2D Parameter Sweeps

2.1 Part a)

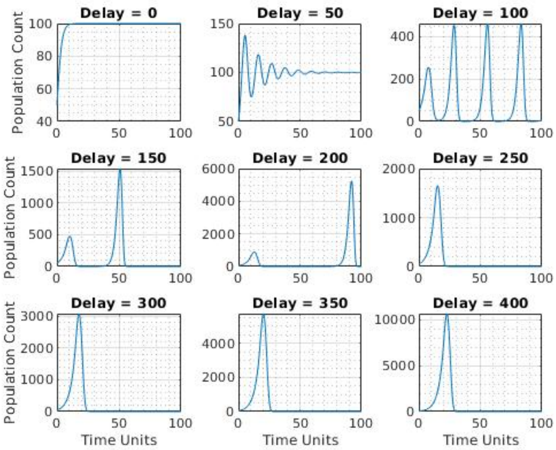
For part **a** of Task 1 a simple one dimensional logistic equation for parasite growth was modelled. This system was modelled iteratively with the $k+1$ value of the population being given by the equation,

$$X_{k+1} = X_k(1 + hr - hrX_{k-s}/K)$$

The model involves a carrying capacity (K), a growth rate (r) and a delayed response term (τ) which in our models is given as $\tau = sh$, where s is our delay, and h is our timestep. For our model we used a constant value of $K = 100$ and had an initial parasite population ($X0$) of 50. To account for the delay terms when $k < s$ it was necessary to define *Ghost Points* which would be used in place of a point that in time that would otherwise be before our initial start time.



(a) Delay $r = 2, h = 0.01$



(b) Delay $r = \frac{1}{2}, h = 0.05$

Figure 1: Oscillations Periods

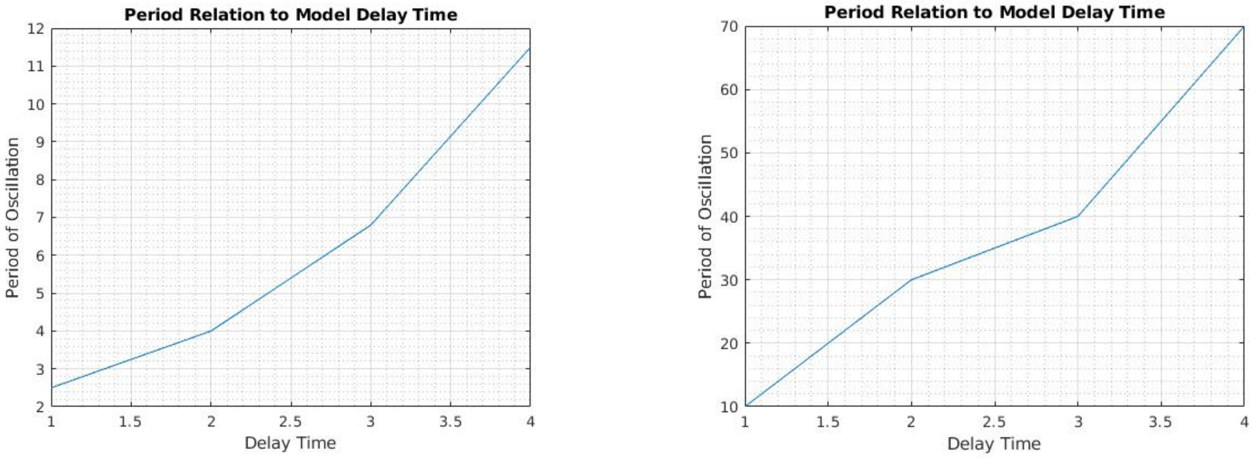
These *Ghost Points* we decided to be the population value at $X0$. The model was ran for 2000 time steps for a variety of delay values [50,100,150,200,250,300,350,400]. This was done first with a growth rate r of 2, and a step size h of 10.01. This was then repeated with a grow rate r of $\frac{1}{2}$ and a step size h of 0.05. The exact implementation of this can be seen in the MATLAB file `Part_1_Script.m`

Figure 1 shows the solutions to our one-dimensional logistic equation for parasite growth.

Plot **a** from figure 1 shows our solutions with a growth rate of 2, and a timestep of 0.01. From the figure it can be seen that with no delay term, the population quickly approaches the carrying capacity of 100 then stabilises there. When the delay terms are added the solution now behaves differently appearing to oscillate around the carrying capacity. The period and amplitude of the oscillations appear to be related the the amount of delay that has been introduced. Plot **b** in figure 1 shows a similar dynamic however the amplitude of the oscillations have been increased and the period between them appears to be longer aswell.

Figure 2 shows the relationships between the period lengths and the delay times for both our trials with $r = 2, h = 0.01$ and $r = \frac{1}{2}, h = 0.05$. From this is can be seen that the amount of delay, and the period of the oscillations are positively related, with plot **a** showing what looks like a positive power-law relationship (possibly exponential) between the delay terms and the period, with plot **b** showing a similar relation. However with so few data points its hard to get an exact relation.

From figure 1 is can be seen that when the delay terms get larger the population of the parasites rapidly shoots up well over the carrying capacity, before then rapidly reducing. If the delay



(a) Delay $r = 2, h = 0.01$ (b) Delay $r = \frac{1}{2}, h = 0.05$

Figure 2: Oscillations Periods

term is great enough the parasite population approaches zero, however due to the nature of the model they never reach zero. This is an expected relationship as similar delay/response relations are common in ecological and predator prey modelling.

2.2 Part b)

For part be a system of both parasites and food were looked at. The population is governed by the following differential equations,

$$\frac{dX_1}{dt} = k1X_1X_2 - k2X_1$$

$$\frac{dX_2}{dt} = k3 - k4X_2 - k5X_1$$

Where X_1 is our parasite population, X_2 is the food population, $k1$ is the parasite birth rate, $k2$ is the parasite death rate, $k3$ is the rate of food growth, $k4$ is the rate of food decay, and $k5$ is the rate that the parasites eat the food.

This system was initially solved using `ODE45` with parameter values of,

- $k1 = 1$
- $k2 = 2$
- $k4 = 4$
- $k5 = 3$

- $X_1(0) = 1$
- $X_2(0) = 0$

This was solved over a time range of $0 \rightarrow 20$ temporal units, with the parameter $k3$ being swept over a range of values linearly spaced between $0 \rightarrow 50$. From these values there were two system dynamics that we looked for, the first was the solution where after the 20 units the population of parasites (X_1) had dropped below our tolerance level, which was set as 10^{-2}.

The second system dynamic that was looked for were when the food population (X_2) had approached 2 after our 20 temporal units. Parameter values, and parameter pairing which resulted in one of these two system dynamics being met were considered a useful parameter/pair and were stored. Any scenarios where either of the populations ended with negative values but were otherwise considered a success were discarded. The exact implementation of this can be seen in the MATLAB file `Part_1_Script.m`

Figure 4 shows the $k3$ values that resulted in our two different system dynamics. The two different dynamics were split into colours, blue representing the parasites $(X_1) \rightarrow$ Tol, and red for the food $(X_2) \rightarrow 2 \pm$ Tol. There was a no overlap observed between the two system dynamic solutions as can be seen by the gap between the red and blue points in the figure.

It was found that $k3$ values between $0.25 \rightarrow 7.25$ resulted in the parasites $\rightarrow$ Tol, while $k3$ values between $8.5 \rightarrow 50$ resulted in the food population $\rightarrow 2 \pm$ Tol.

2.3 Part c)

The population equations for this part were identical as in part **b**, with the exception of the parameter $k4$. For this section both $k3$ and $k4$ were swept over a range of linearly spaced values between $0 \rightarrow 50$. This was done via a nested for loop, meaning that for each of the n $k3$ values, the system was ran for each of the n $k4$ values, resulting in n^2 trials of the system. This was the general method used for the 2D parameter sweep in this part, as well as in part **d**.

For this section, successful parameter pairings relating to the two system dynamics were stored and are displayed in figure 4

The left hand plot in figure 4 shows the parameter pairing that result in the dynamic of the parasites $(X_1) \rightarrow$ Tol. A plot of $1.9y = x$ is shown to help guide the eye to the relation between the parameter pairings that cause the system dynamic. From this it can be seen that when $k3 \leq 1.9k4$ we get the dynamics of the parasites tending to zero. The $1.9y = x$ relation isn't exact and wsa just fitted experimentally.

The plot on the RHS of figure 4 shows the parameter pairing that result in the food population

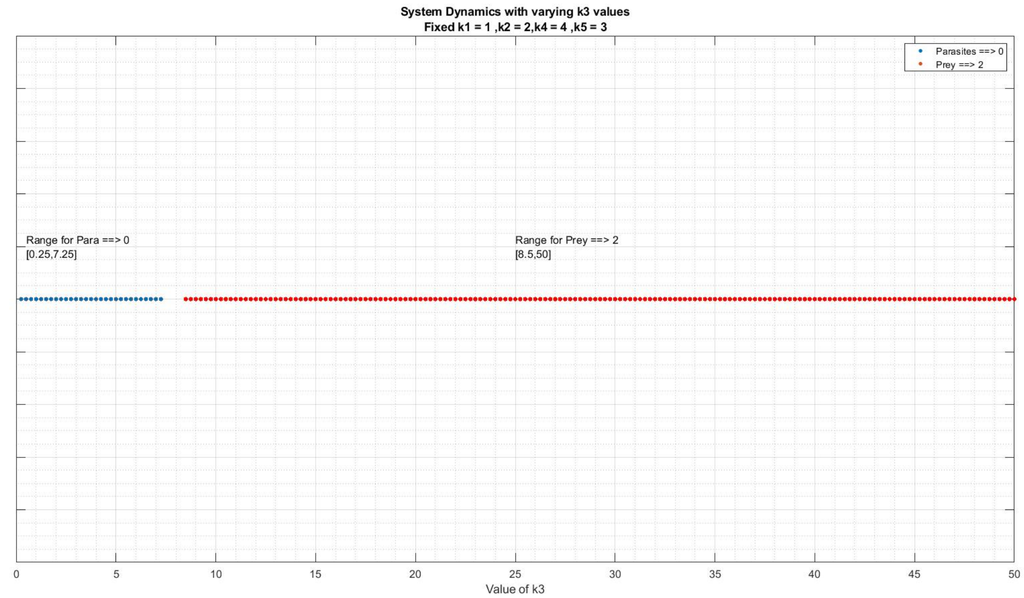


Figure 3: Parameters for System Dynamics

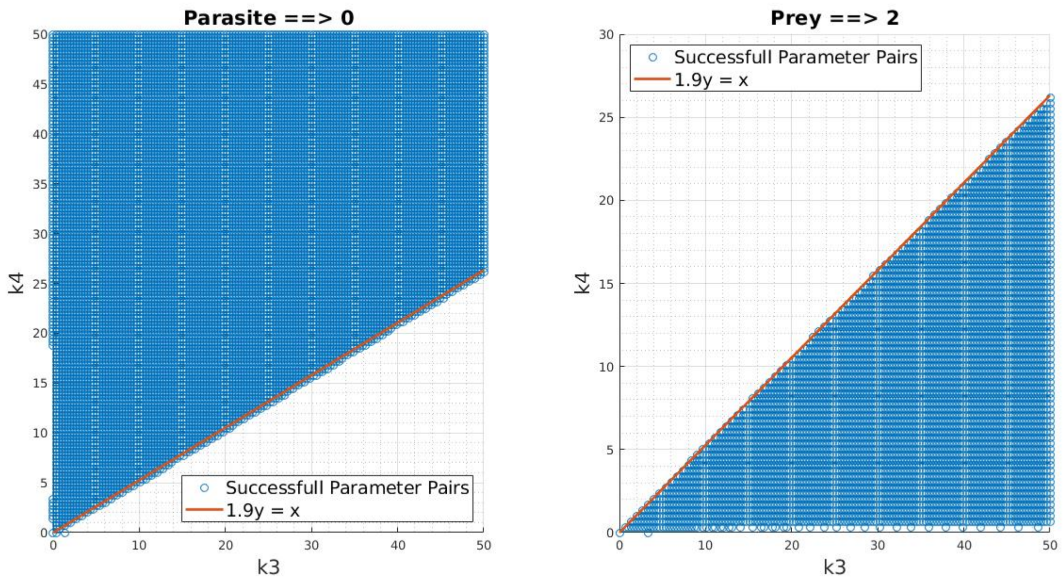


Figure 4: Two dimensional parameter sweep for parameters $k3$ and $k4$.

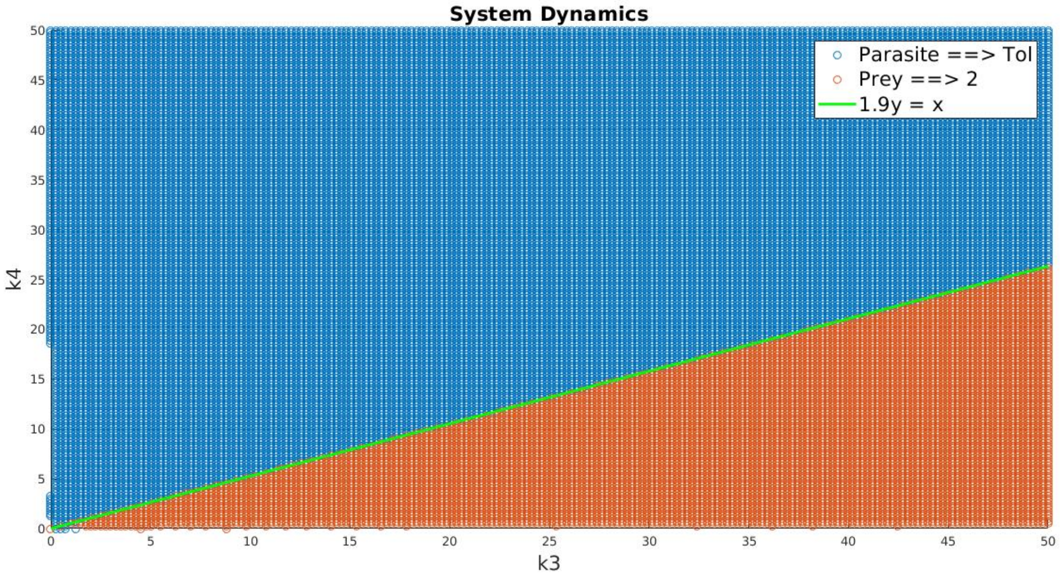


Figure 5: Two dimensional parameter sweep for parameters $k3$ and $k4$ shown same space.

$(X_2) \rightarrow 2 \pm$ Tol. A plot of $1.9y = x$ is also shown again as an aid to guide the eye to the area in the 2D parameter space which results in the specific system dynamics. From this it can be seen that whenever $k3 \ \geq 1.9k4$ we get the dynamics of the food approaching 2. In terms of the physical context of this question, it means that whenever the rate of food growth if double that of food decay, we will have these dynamics.

Figure 5 shows the two different system dynamics shown on the same plot. It can be seen that the two dynamics span nearly all of the parameter space in the $k3$ $k4$ plot. If the value of n was scaled up, it would be expected that all of the parameter space wold be filled.

2.4 Part d)

In part **d** a two dimensional parameter sweep was again completed, this time sweeping from $0 \rightarrow 50$ for parameters $k4$ and $k5$. The parameter $k3$ was fixed to a constant value of 10. The results of this parameter sweep are shown in figure 6 with the system dynamics of the parasites tending to 0 in green, and the dynamics of the food tending to 2 in blue. A plot of $x = 5.2$ is also shown on the graph which was experimentally derived to split the $k4 - k5$ between the two system dynamics. It should be noted that there is some slight overlap, as well as interestingly some green points along the $k5$ axis indicating that with $k5 = 0$ is is possible to have the parasites $(X_1) \rightarrow$ Tol even with $k4$ values less than this ≈ 5.2, although this could be due to some errors..

In section **b** we have $k4 = 4$, $k5 = 3$ and a sweep of $k3$. In this section $k3$ is fixed at 10. Therefore we can see that in section **b** when $k3 = 10$ we have the dynamics of the food tending

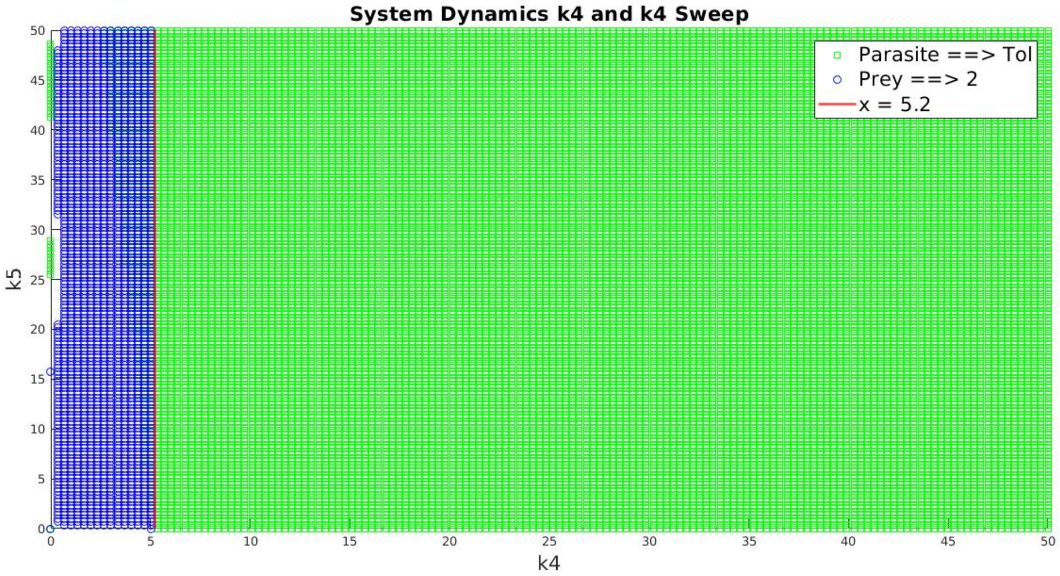


Figure 6: Two dimensional parameter sweep for parameters $k3$ and $k4$.

to 2. In figure 6 we can see that at $k4 = 4, k5 = 3$ we have the system dynamics of the food tending to 2, which is in agreement with the previous section. This is to be expected since they both rely on the same exact system of differential equations, therefore with the same parameters, we expect the same results.

2.5 Part e)

Our results suggest a few different parameter relations that results in specific system dynamics. Firstly with $k1 = 1, k2 = 2, k4 = 4, k5 = 3$ is appears that the value of $k3$ which represents the rate of food growth, will determine what happens in this population system. It seems that whenever the rate of food growth is below ≈ 7.25 we will get a situation where the parasites will die out (tend to zero). If the rate of food growth is higher than ≈ 8.5 we will get the food population X_2 tending to 2. The physical interpretation of this could be that when the growth of food is high the parasites do not seem to be able to consume all of the food that is available, while when the food rate is low, there is not enough food for the parasites hence the will eventually die out. The next set of plots look at first the relation between food growth and food decay and its effect on system dynamics, and then the relation between food decay and food consumption rates by the parasites, and the relation on the system dynamics from that. It was found that unless the rate of food growth is approximately twice as much or more as the rate of food decay then the parasites will die out. If the rate of food growth is twice or more than that of food decay the food population (X_2) will tend to 2, noting this is with a fixed birth rate, death rate, and food consumption rates for X_1. The relation between $k4$ and $k5$, food consumption and food decay, appears to be relatively simple when we are working with a fixed $k3$ (rate of food growth). If the rate of food decay is greater than ≈ 5.2, then the parasites will die out regardless of the rate of food consumption (with some minimal exceptions for $k4 << 1$). This shows that the relation between food growth and food decay ($k3 - k4$) would possibly have a greater effect on the system dynamics then the rate of food consumption by the parasite..

3 Task 2 Latin Hypercube Sampling in 3D

The goal of task 2 was to build a population of successful parameter 3-tuples that reflect the system of equations exhibiting the characteristics that either the parasite population goes to zero, or the food population goes to two. The parameters, k1 and k2 which relate to the birth and death rate of the parasites were fixed, and so the successful 3-tuples related to k3, k4 and k5 – the food growth and decay rate and the consumption of food by the parasites, respectively. An additional condition was imposed that for a 3-parameter selection to be successful, neither population of parasites nor food could drop below zero in [0 T].

The method to find the successful parameter 3-tuples was to implement Latin Hypercube Sampling on the 3D space of parameters $k3$, $k4$ and $k5$ – each within the range [0 50]. Latin Hypercube Sampling is method to generate random samples of parameter values. A cube, $(n \times n)$ will be a Latin Hypercube if there is a sample taken from each row, column and depth. The rows, columns and depth are assigned to be a parameter (in this case, k3, k4 and k5). To find the value of a parameter at a sample point, first we define the vertices of the sub-cube in the sample region. Then a random value between the maximum and minimum vertex for each dimension is taken. These values for each dimension/parameter are stored.

To implement this in MATLAB, 3 arrays were created - corresponding to each dimension/ parameter. The values in the array were in the range [0 50], and linearly split into $n \times spacingFactor$ steps, referred to as the $sampleLength$. Using the `randperm` function, each array was shuffled to create random samples in each row, column and depth. For each element in the arrays, a random value was taken between the array vertices. These were stored in a $(samepleLength \times 3)$ parameter storage array. This process was looped over, such that the final parameter storage 3-tuple was $(no.iters \times samepleLength) \times 3$ in size.

The parameter array was then resulted into two system dynamics, either the population of parasites (X_1) going to below our tolerance level or the population of prey (food, X_2) approached 2. This was previously discussed in section 2.2 Part b).

There are distinct boundaries that are formed between regions in the $k3$, $k4$, 5 sample space, as demonstrated in Figure 7. First, we can note Figure 8(a), where a slice of the cube has been taken to show the relationship between $k3$ and $k4$. We can see that when when the rate of food decay is twice the rate of food growth ($k4 > 0.5k3$), the parasite population goes to 0 ($X_1 \Rightarrow 0$). The second boundary, shown in Figure 8(b) shows the relationship more clearly between $k3$ and $k5$. The plane $k5 = 1.25k3$ separates the region between where we will reach neither of our equilibrium solutions ($k5 > 1.25k3$) and where we will reach either of them (depending on the $k3$, $k4$ relations). This physically corresponds to when the rate the parasites eat the food is greater than 1.25 times the rate that the food grows, then an equilibrium solution is not reached.

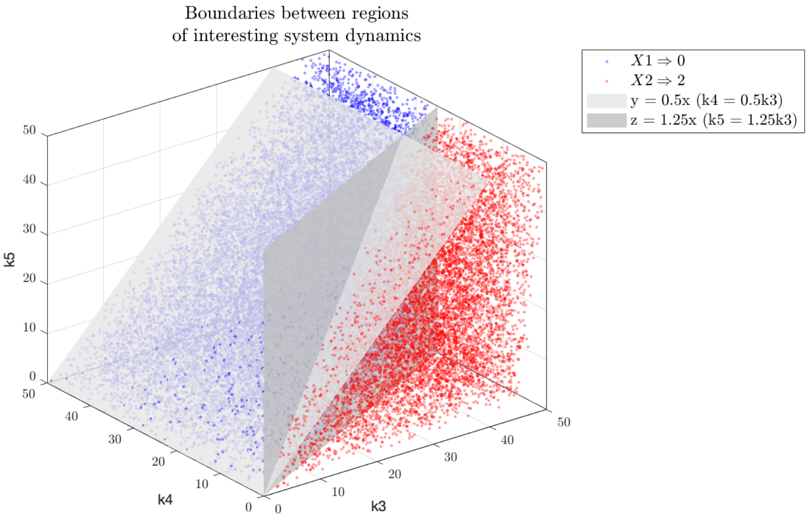
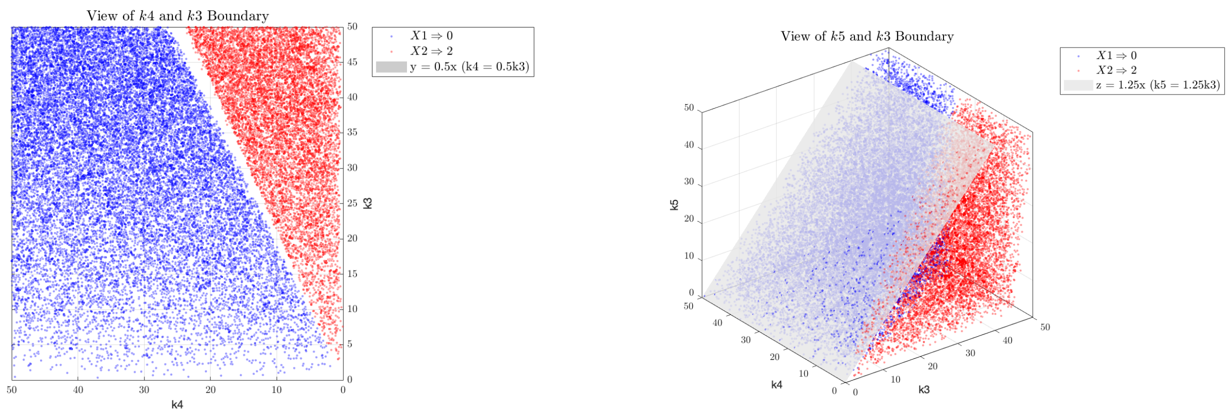


Figure 7: Caption



(a) Boundary k3 k4

(b) boundary other

Figure 8: boundaries

4 Task 4 Spatial agent-based implementation

Task 4 was to create an agent based simulation to examine the parasite system interactions at an individual level. The simulation was based in a $200x200$ grid simulation space, *map* with only one agent per cell. Each simulation had parameters for population density, parasite time-to-live , a food threshold, and new food growth per step. Two food placement types were modelled, random food placement and localised food placement.

Population density, pop_dens, was the initial percentage of parasite and food agents compared to the simulation space with values of 10%, 30% 30%, 40%. Parasite time-to-live, p_ttl, was the number of iterations till the parasite agent dies with values $p\_ttl \in [1, 15]$. The food threshold, $f\_th \in [0, 0.1]$, indicates the threshold for a uniform random sample, $u\ U(0, 1)$, generated each iteration for every living food agent, such that if $u < f\_th$ then the food agent dies. New food growth, new_food, is the number of new food agents "grown" in the simulation space after all previous interactions have occurred, with values of $100, 200, 300, 400$. Randomised food placement generated the new_food at randomised locations across the entire simulation grid-space. For localised food placement the simulation grid-space was divided into four quadrants, each iteration a quadrant was randomly selected and the new_food was added to that quadrant.

The simulation is initialized with parasite agents numbering $pop\_dens \times 200^2$, so for $pop\_dens = 0.3$, initial parasites population is 12000. The coordinates of each parasite within the simulation grid-space are uniformly random generated and stored in a vector, $para\_array$. Then food agents equal to the number of parasite agents are added to the grid-space, this is with either random or localised food placement, and their coordinates stored in a vector, $food\_array$. For randomised food placement the food agents coordinates are uniformly random generated, for localised food placement a quadrant is selected by a uniform random number and then within the selected quadrant food agents coordinates are uniformly random generated.

At each step the simulation loops through four functions; $para\_step$, $food\_update$, $food\_dies$, $food\_grows$, recording the food and parasite populations for the entire grid-space and each individual quadrant. The model simulates the individual movement of parasites by, each step, iterating over a random sequence for $para\_array$, selecting a random direction for each parasite to move and updating the parasite coordinates in $para\_array$ based on the destination cell's occupant. If the destination cell is empty the parasites moves there, if the cell is occupied by another parasite the current parasite's coordinates remain the same, if a food agent is there the current parasite moves there and a new parasite agent is placed in the original cell (a new parasite is born).

Function $para\_step$ generates the new parasite coordinates placed in $para\_array$ and a new grid-space map with only parasites within it. Next $food\_update$ places the agents in $food\_array$ into the unoccupied new grid-space map, unplaced food agents are "eaten" and removed from $food\_array$. Then $food\_dies$ generates a uniformly random number for every food agent in $food\_array$, if this is below the parameter threshold f_th the agent "dies" and is removed from $food\_array$ and map. Finally $food\_grows$ generates new food agents equal to new_food, using either random or localised placement, adding them to $food\_array$ and map. The parasite and food populations are then recorded in vector pop_cnts for the entire map and $quad\_cnts1$, $quad\_cnts2$, $quad\_cnts3$, $quad\_cnts4$ for each quadrant.

4.1 Random food growth

Examining the results for the randomised food growth there are 3 notable sets of population dynamics that arise for different parameter sets. These will be referred to as parameter set A ($f\_th = 0.03$, $p\_ttl = 8$, $new\_food = 400$), parameter set B ($f\_th = 0.02$, $p\_ttl = 10$, $new\_food = 400$) and parameter set C ($f\_th = 0.01$, $p\_ttl = 15$, $new\_food = 300$). Figure 9 shows parameter sets A, B and C with increasing population density. Parameter set A shows a parasite population spike and crash early in the simulation. As the parasites time-to-live is $p\_ttl = 8$, food threshold is $f\_th = 0.03$, and new food growth is$new_food = 400$, the parasites death rate is too high and the food replacement rate too low for the parasite population to stabilise and they die out quickly. as the population density increases the only significant result is the higher parasite population spike and total food population, the population dynamics don't exhibit any change.

For parameter set B ($f\_th = 0.02$, $p\_ttl = 10$, $new\_food = 400$) the population is able to reach an equilibrium state with a slight oscillation around their averages. The food replacement rate is high enough and the parasite death rate low enough that the parasite population is able to recover after the initial population explosion and food consumption.

Parameter set C, ($f\_th = 0.01$, $p\_ttl = 15$, $new\_food = 300$), exhibits equilibrium with oscillatory behaviour. A delay between the parasite population growth and the food population growth occurs after the initial parasite population spike. It appears that the parasite death rate is at its lowest and the food replacement rate is slower than in parameter set B, resulting in parasite and food populations that are nearly equivalent. The oscillatory behaviour shows the parasites population recovering lagging behind the food growth. These 3 behaviours are exhibited by multiple parameters sets within the simulation, however they are the most prevalent population behaviours in the simulation.

For each parameter set (A, B and C) in Figure 9 as the population density increases the system dynamics remain consistent, with the only change exhibited being the overall population levels of both parasites and food.

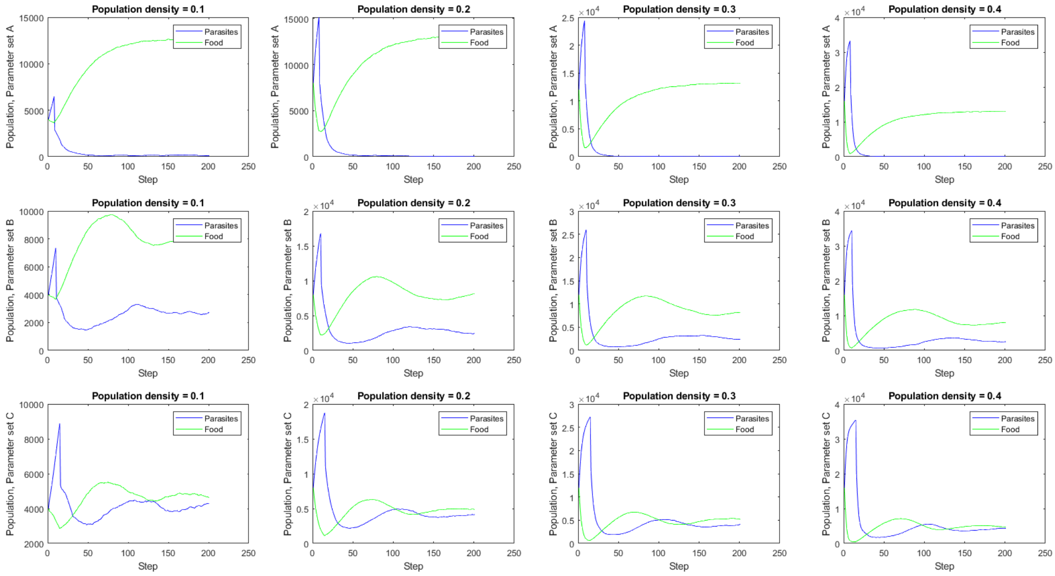
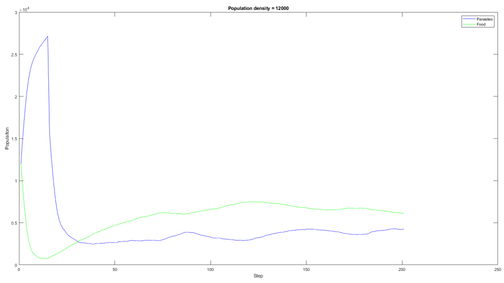


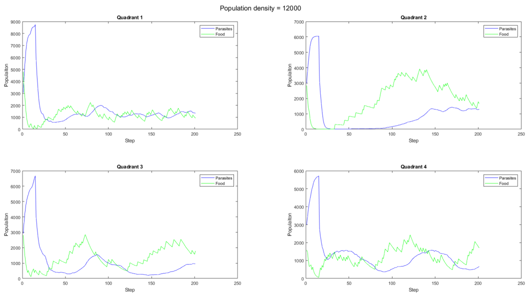
Figure 9: Population dynamics of randomised food placement

4.2 Localised food growth

The localised food placement strategy shows the same three dominant population dynamics as discussed in the random placement. While the initial localised quadrant placement of food would seemingly bias the simulation toward higher populations in that quadrant this is not the case. The initial parasite population spike occurs is balanced by the random selection of food growth in quadrants 1 to 4. Figure 10 compares the overall map populations to the individual quadrants for parameter set C. The oscillatory behaviour seen in Figure 9 is seen occurring individually in each of the quadrants in Figure 10 (b), 10 (a) shows an averaging effect across the entire simulation map, exhibiting more stable equilibrium behaviour. Additionally Figure 11 compares the random and localised food placement strategies, Figure 11 (b) shows how the dynamics in each quadrant can effect adjacent quadrants as parasites transition from on quadrant to another.

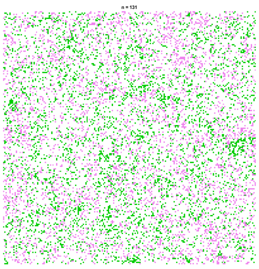


(a) Full map population

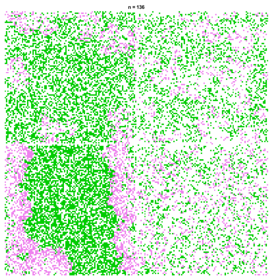


(b) Quadrant populations

Figure 10: Localised food placement



(a) Randomised food placement



(b) Localised food placement

Figure 11: Simulation map visualisation

5 Collated Results and Task Comparisons

The first set of results compared are the results from Task **1**, which involved the 1 and 2D parameter sweeps, and then the results from Task **2** which utilized latin hypercube sampling to explore the parameter space in 3 dimensions.

Figure 5 shows the relations between the parameters $k3$ and $k4$ and the resulting system dynamics. Figure 8(a) shows a 2D slice of the 3D plot created which shows the system dynamics observed in our $k3, k4, k5$ parameter space. The 2D slice displays the $k3 - k4$ side of our cube and is directly comparable to Figure 5. The same relation between the parameters of $k3$ and $k4$ in which that if the rate of food growth is more than $\approx$ double the rate of food decay, we get the food population $(X_2) \to 2 \pm$ Tol, and if the rate of food growth is less than $\approx$ double that of food decay we get the parasite population $(X_1) \to$ Tol. Figure 7 shows more information than figure 5 displaying that this relation holds true for all values [0,50] of $k5$.

Figure 6 shows the relation between $k4 - k5$ at a fixed $k3 = 10$ point. As stated above in section **2.4** it can be seen that for this $k3$ value whenever $k4 \approx> 5.2$ we get the dynamics of the parasites $\rightarrow$ Tol. Figure 7 from the Latin Hypercube sampling displays the same $k4 - k5$ relations but also how this relation varies with $k3$ ranging from [0,50]. From this is can be seen that the 'constant' relation of $k4 = 5.2$ does not hold as the values of $k3$ are changed. Instead is shows that the relation seen in figure 6 which was plotted as $k4 = 5.2$ in red, is simply the intercept of a plane which spans the $k3, k4, k5$ cube with the equation for the plane $k5 = 1.25k3$, which gives a more accurate representation of the parameter relations and subsequent system dynamic relationships.

Task 4, while being governed by a different model than Tasks **1** and **2** still demonstrates the strong influence on the population system dynamics that is caused by the relations between food growth, food depletion and parasite death rate. As discussed in section **4.1** the different parameter values result in dominant system dynamics.

6 Conclusion

In summary the methods were successful in finding how the parameter values affected the system dynamics of the parasite model. The parameters for food growth rate $k3$, food decay $k4$, and food consumption by the parasites $k5$ were looked at in task's **1** and **2**. It was found that when the food growth rate $k3$ is approximately double or more than that of the rate of food decay the population of the food (X_2) will approach 2, while if the rate of food growth is less than double the rate of food decay, the parasite population (X_1) will tend to 0. A relation between the parameters for food growth rate $k3$ and food consumption by the parasite $k5$ was also found. The relation showed that if the rate of food eaten by the parasites was roughly 1.25 times or more than the rate of food growth, no equilibrium solution was met, while if the rate of food consumption by that parasites was less than 1.25 times that of food growth, we would find one of the two system dynamics, and the dynamics were then dictated by the relation between parameters $k3$ and $k4$.