Principle of Communications

Sampling and Analog-Digital Conversion

Dr. Xiang Cheng

Wireless Communications and Signal
Processing Research Center
Peking University

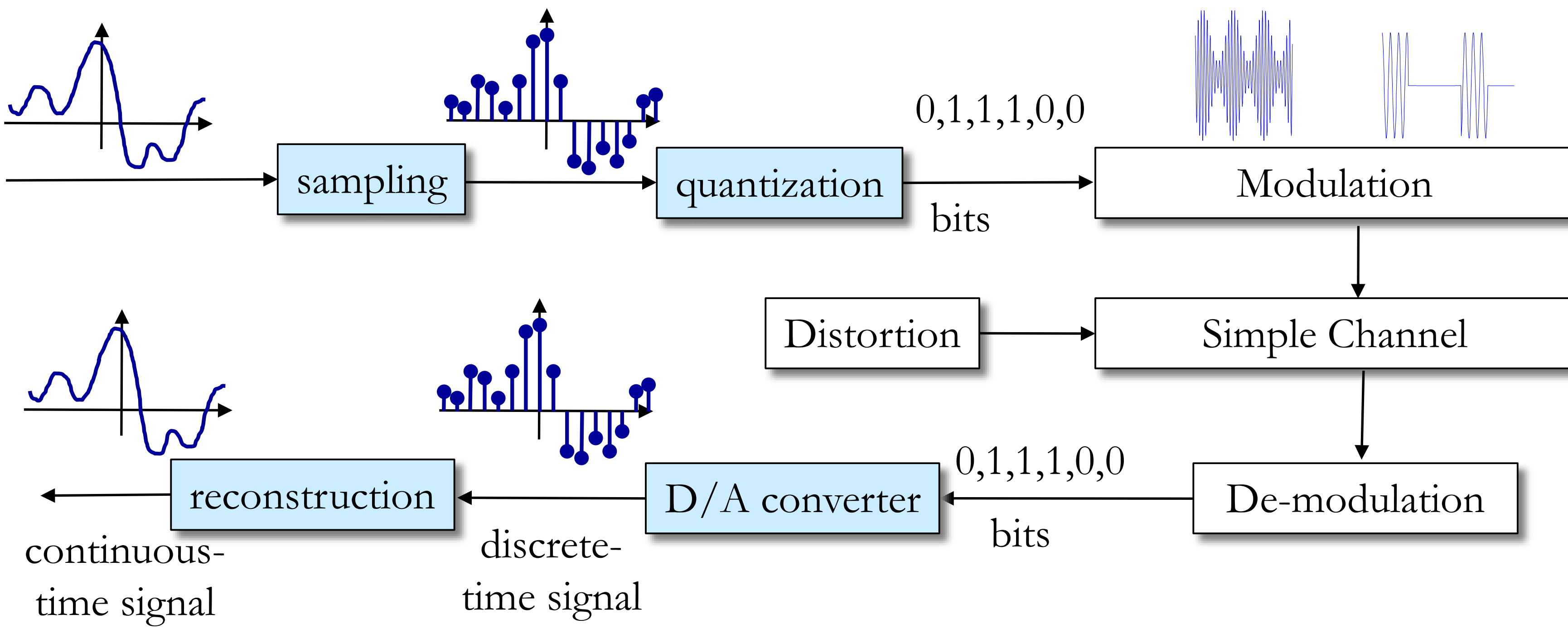
Outline

- Sampling Theorem
- Pulse Code Modulation (PCM)
- Delta Modulation & Differential Pulse Code Modulation
- Vocoders and Video Compression

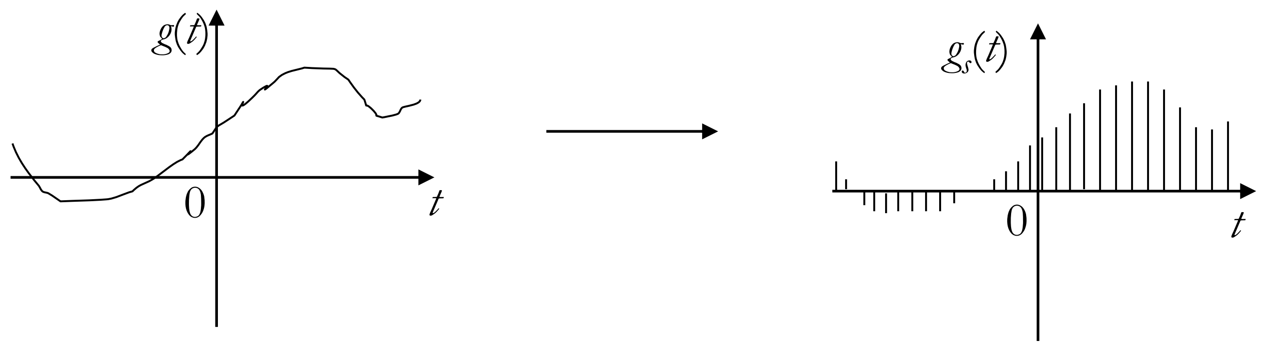
Roadmap

- Sampling Theorem
  - Sampling
  - Reconstruction
- Pulse Code Modulation (PCM)
- Delta Modulation & Differential Pulse Code Modulation
- Vocoders and Video Compression

Communication System Diagram



Sampling: Anything we should know?



Sampling & Reconstruction ABC

A

Sampling is the process of converting a continuous-time signal to a sequence of numbers.

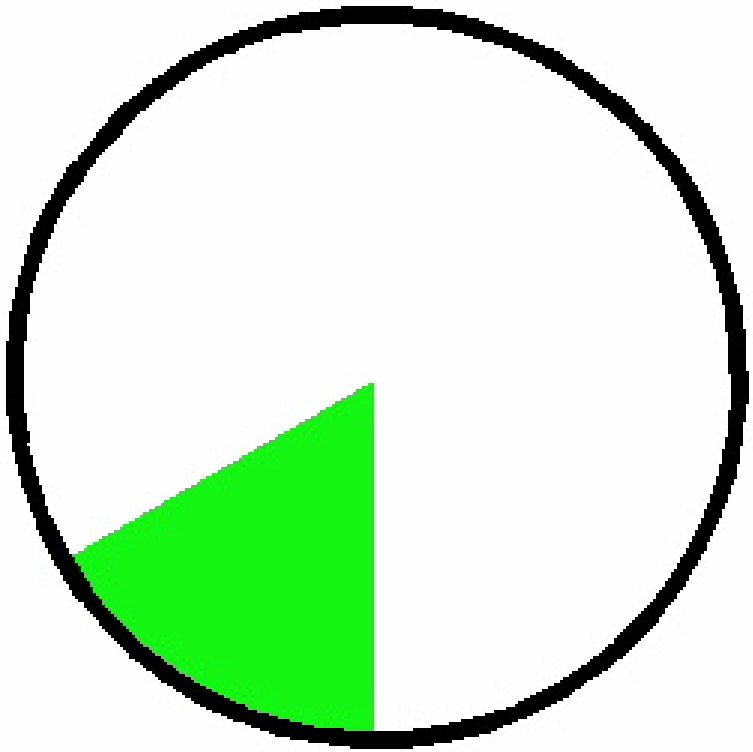
B

Reconstruction is the operation of filling the gaps in the sampled data.

C

Aliasing refers to the fundamental distortion introduced by sampling.

Aliasing Example 1: A Turning Wheel



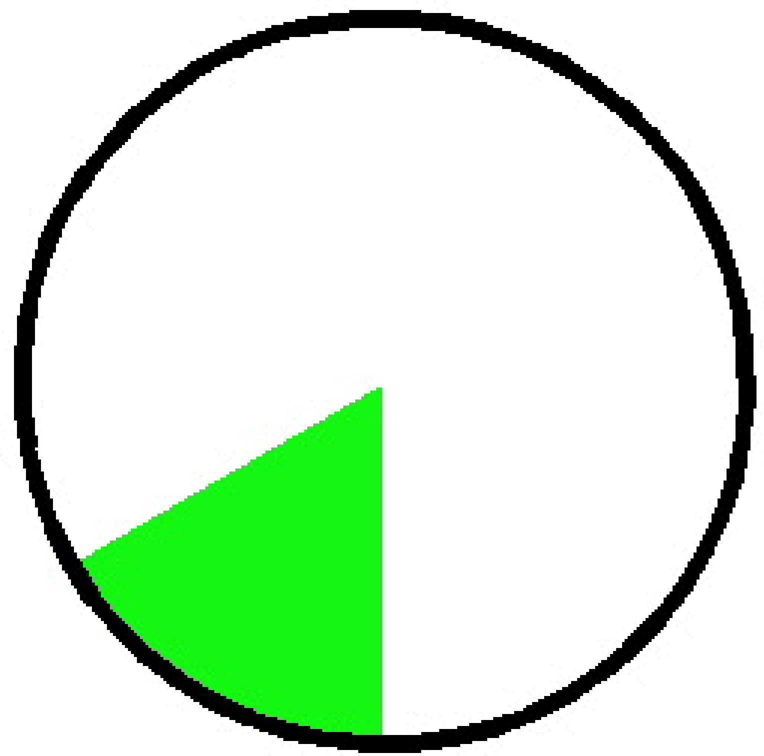
Rotating period: T

Sampling period: Ts=T/12

Rotating frequency: f=1/T

Sampling frequency: fs=12/T=12f

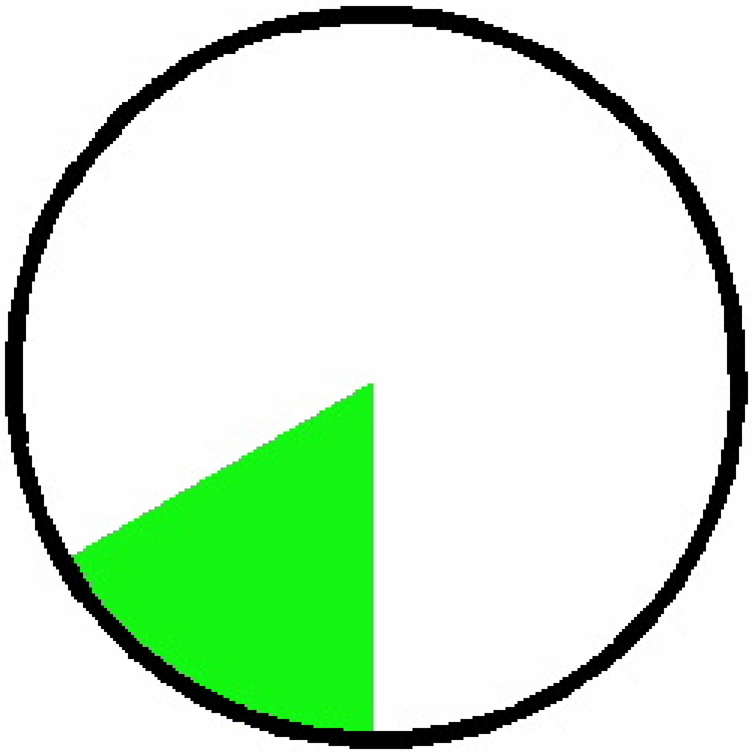
Aliasing Example 1: A Turning Wheel



Rotating period: T/2
Sampling period: Ts=T/12

Rotating frequency: f=2/T
Sampling frequency: fs=12/T=6f

Aliasing Example 1: A Turning Wheel



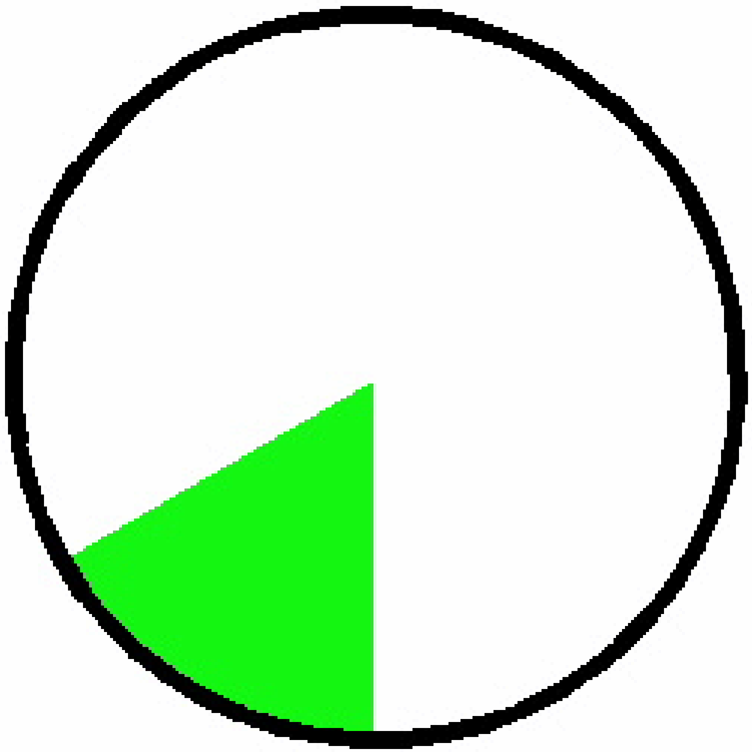
Rotating period: T/3

Sampling period: Ts=T/12

Rotating frequency: f=3/T

Sampling frequency: fs=12/T=4f

Aliasing Example 1: A Turning Wheel



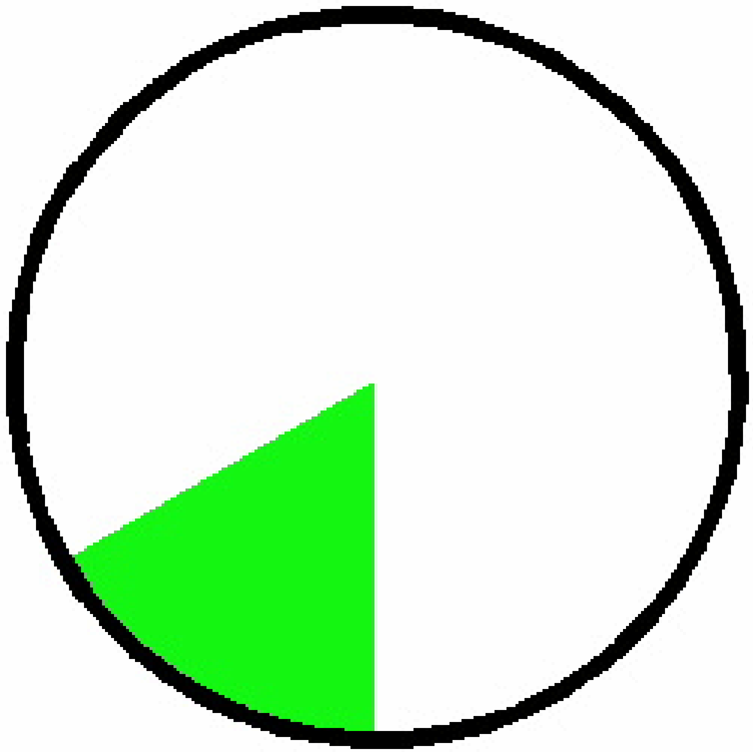
Rotating period: T/6

Sampling period: Ts=T/12

Rotating frequency: f=6/T

Sampling frequency: fs=12/T=2f

Aliasing Example 1: A Turning Wheel



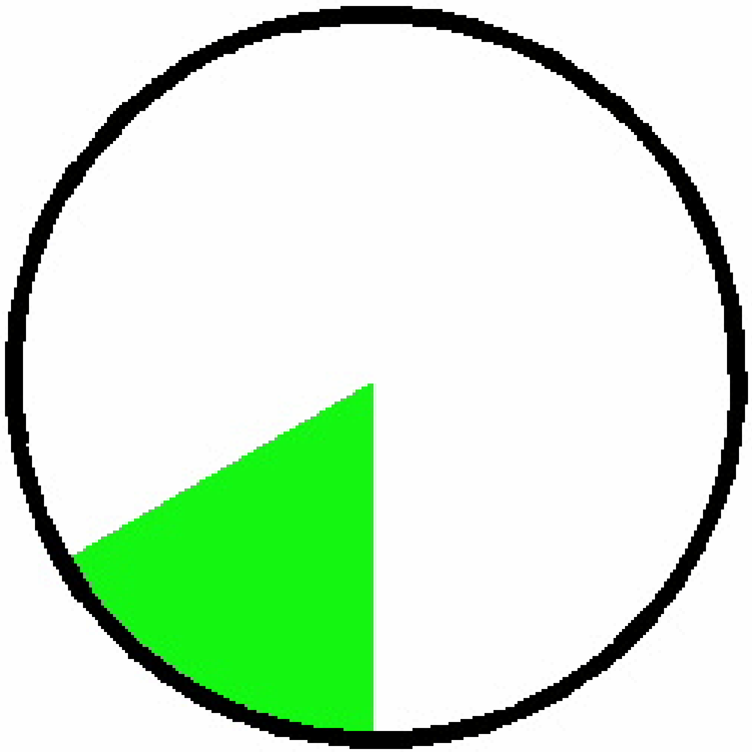
Rotating period: T/8

Sampling period: Ts=T/12

Rotating frequency: f=8/T

Sampling frequency: fs=12/T=1.5f

Aliasing Example 1: A Turning Wheel



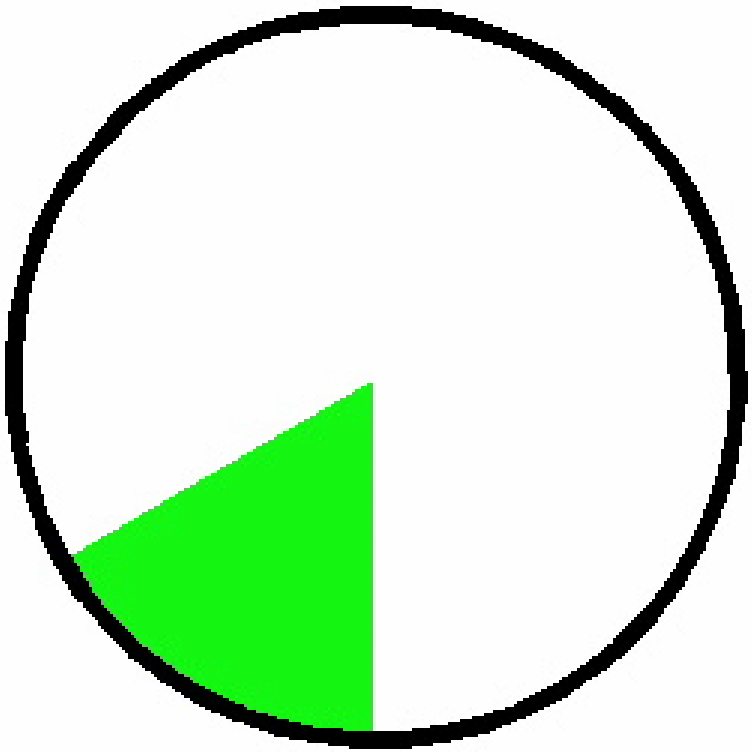
Rotating period: T/10

Sampling period: Ts=T/12

Rotating frequency: f=10/T

Sampling frequency: fs=12/T=1.2f

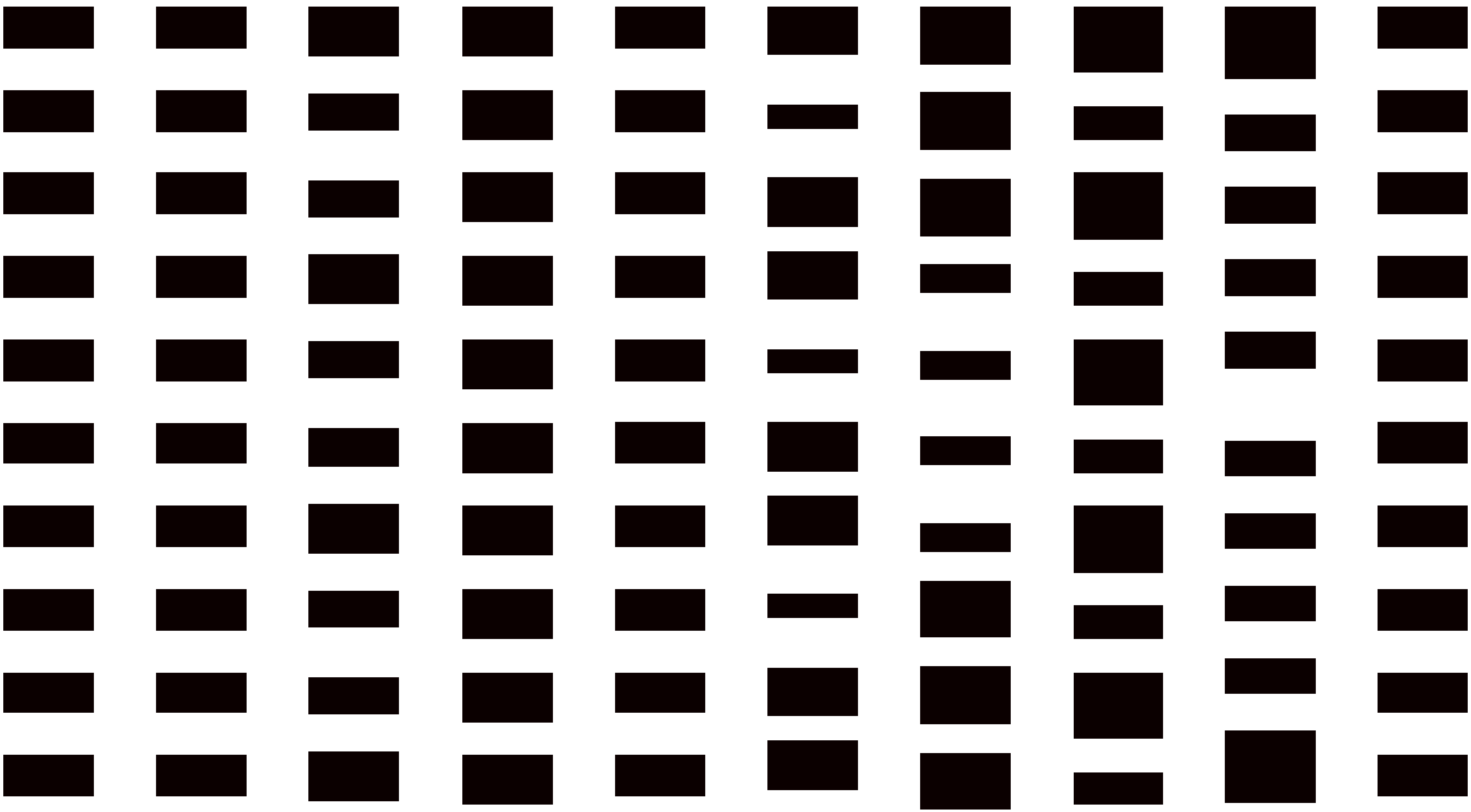
Aliasing Example 1: A Turning Wheel



Rotating period: T/12
Sampling period: Ts=T/12

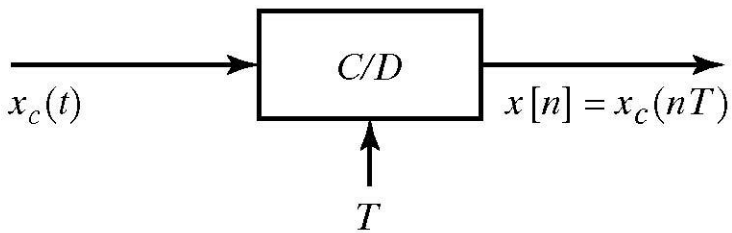
Rotating frequency: f=12/T
Sampling frequency: fs=12/T=f

Aliasing Example 2: Stripes



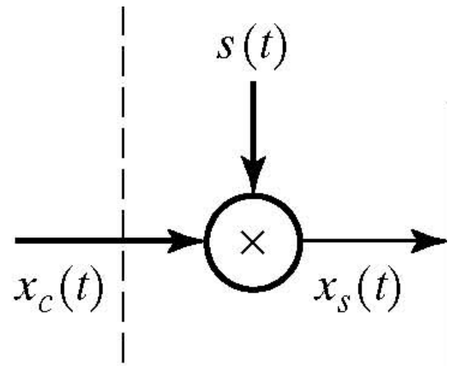
Implementation of the Sampling Process

- Ideal continuous-to-discrete-time (C/D) converter

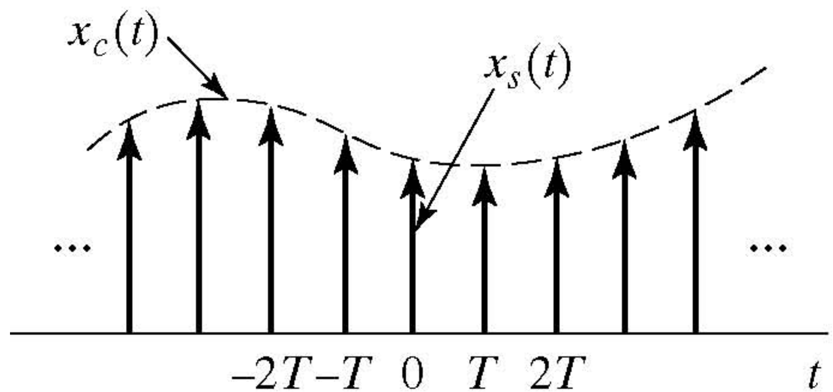


- Practical approximation of the ideal C/D converter: analog-to-digital (A/D) converter.
  - quantization
  - linearity of quantization
  - sample-and-hold circuits
  - sampling rate

Mathematical Representation



$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



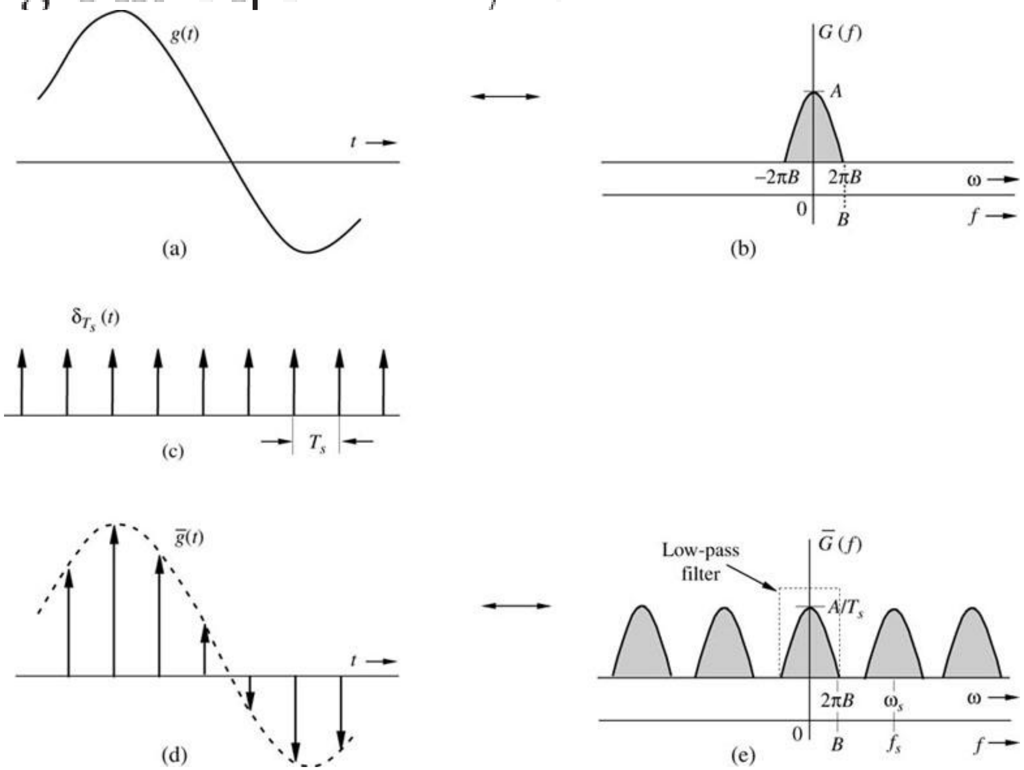
$$x_s(t) = x_c(t)s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

Sampling Theorem

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

To obtain the FT of $x_s(t)$ from the FT of $x_c(t)$

Summation of an infinite number of replicas of the given spectrum, shifted by integer multiples of $2\pi/T$.



Proof of Sampling Theorem

$$x_s(t) = s(t)x_c(t) \quad \Rightarrow \quad X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) \star S(j\Omega) \ ,$$

$$S(j\Omega) := \mathcal{F}\{s(t)\} = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j\Omega) \star \delta\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

Band-Limited Signal & Nyquist frequency

DEF

A continuous-time signal $x(t)$ is called *band limited* if its FT vanishes outside a certain frequency range; that is, if $\exists \Omega_m > 0$ such that

$$X_c(j\Omega) = 0, \quad \forall |\Omega| > \Omega_m \ .$$

DEF

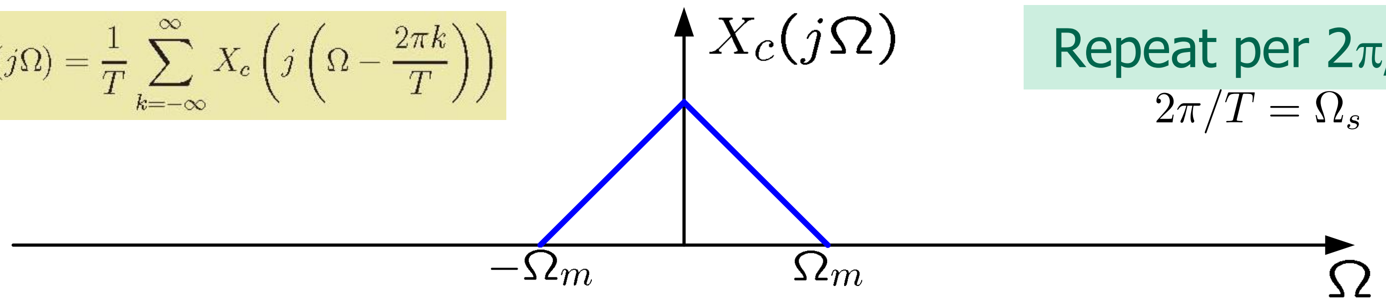
The Nyquist frequency is $\Omega_N = 2\Omega_m$.

Sampling Case 1 & 2: $\Omega_s \geq \Omega_N = 2\Omega_m$

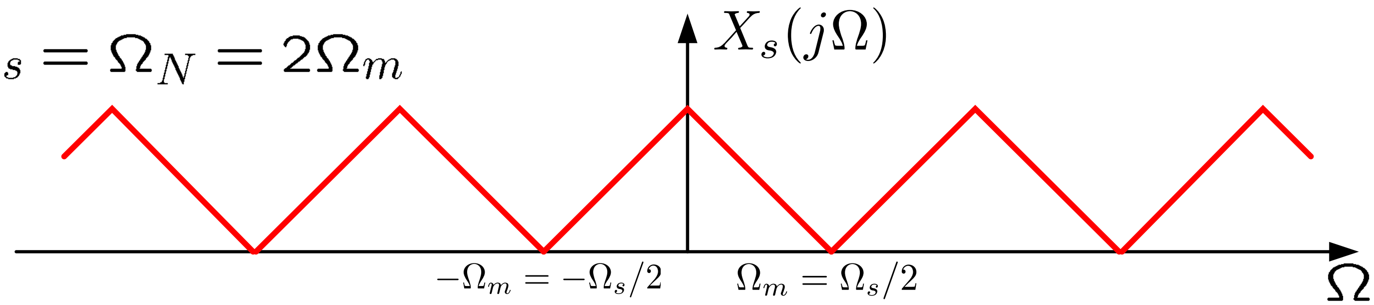
$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

Repeat per 2π/T

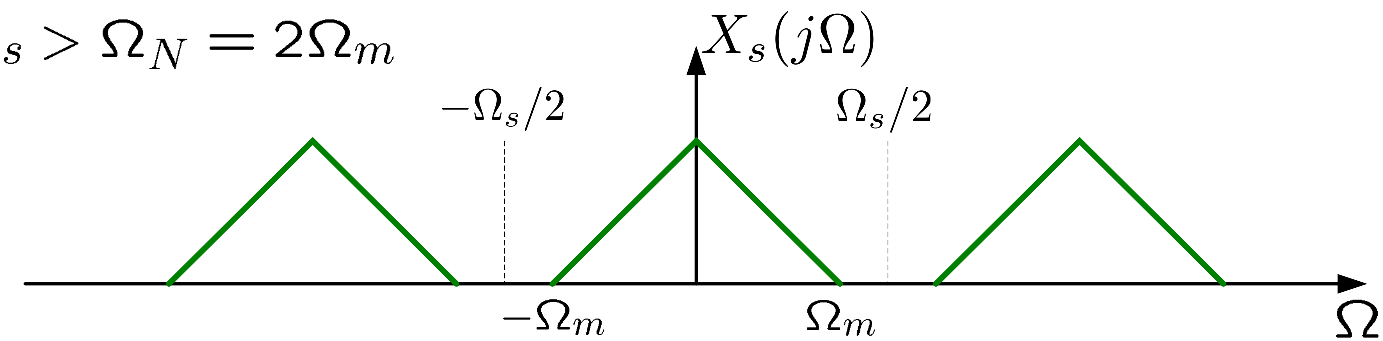
$2\pi/T = \Omega_s$



$\Omega_s = \Omega_N = 2\Omega_m$



$\Omega_s > \Omega_N = 2\Omega_m$

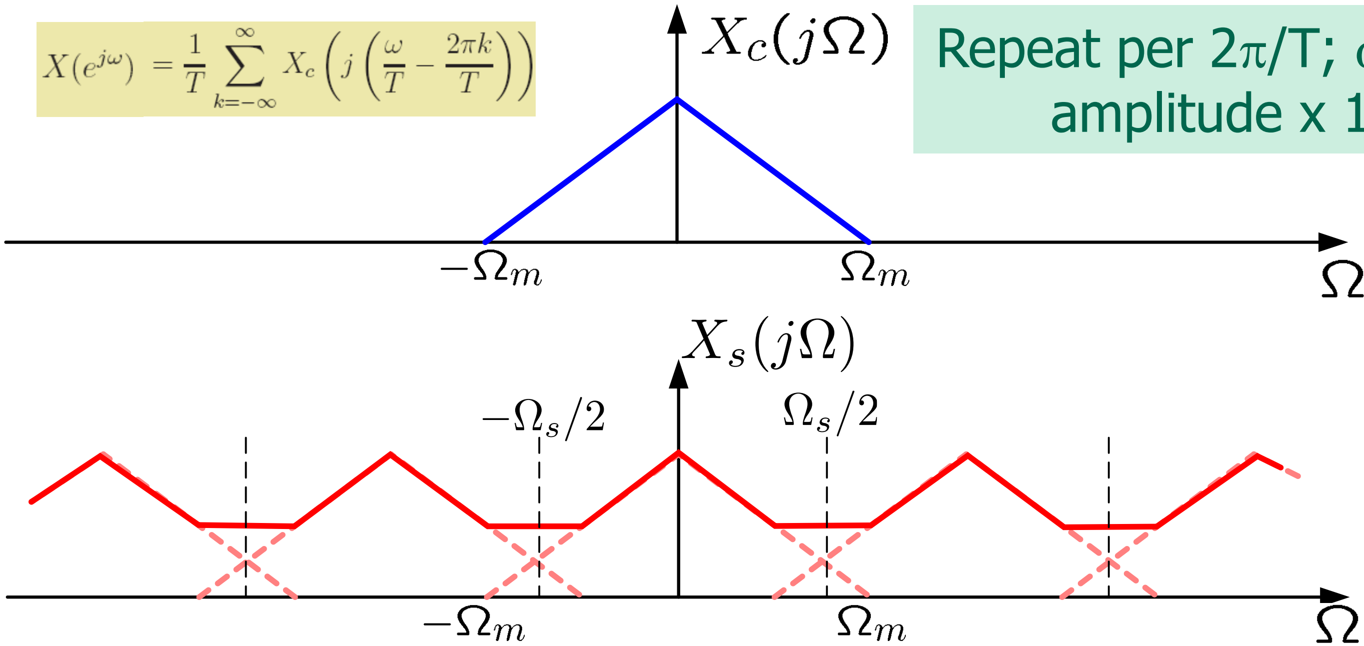


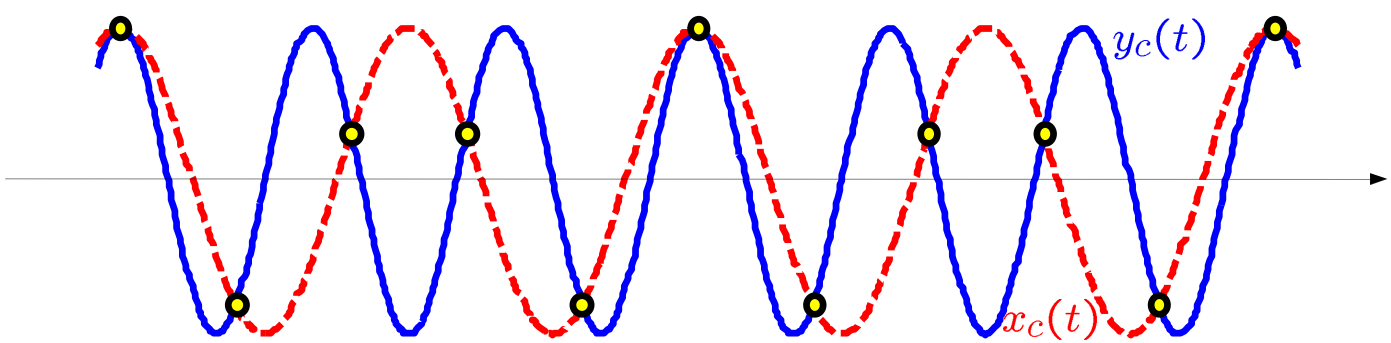
Sampling Case 3:

$$\Omega_s < \Omega_N = 2\Omega_m$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

Repeat per 2π/T; ω=ΩT; amplitude x 1/T





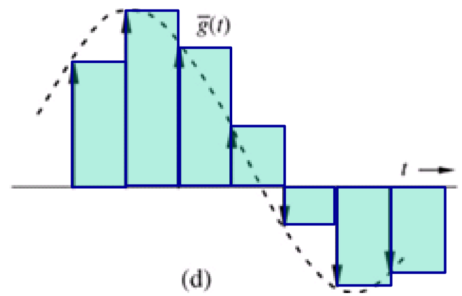
The 3 Cases of Sampling

Sampling of an infinite bandwidth signal always gives rise to aliasing, no matter how high the sampling rate.

- Problem: A famous theorem in the theory of FT asserts that a signal of finite duration must have an infinite bandwidth. Real-life signals always have finite duration, so their bandwidth is always infinite.
- Solution: Practical solutions are

1. The bandwidth of a real-life signal is always practically finite (i.e., the percentage of energy outside a certain frequency range is negligibly small).
2. Filter the signal before it is sampled. (Antialiasing filter: an analog low-pass filter whose cutoff frequency is not larger than half the sampling frequency.)

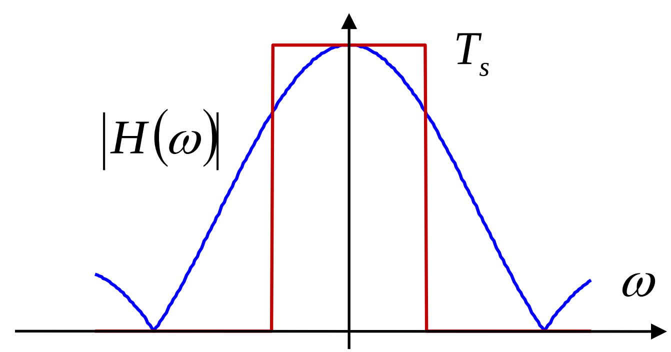
Signal Reconstruction



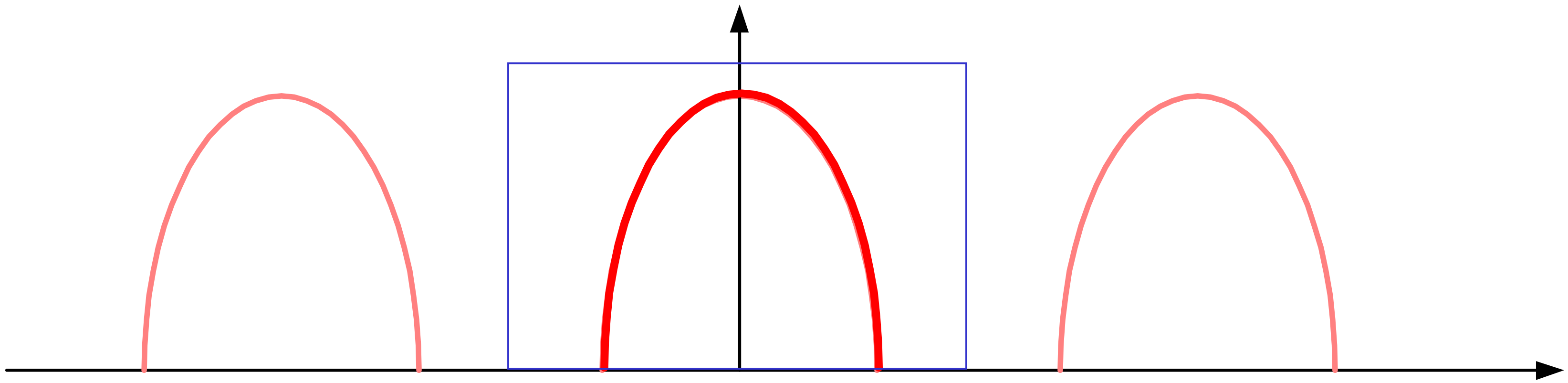
Passing $g_s(t)$ through a filter whose impulse response is $h(t) = \text{rect}\left(\frac{t}{T_s}\right)$
(This is actually a sample holding device that extend each sample into a rectangular waveform)

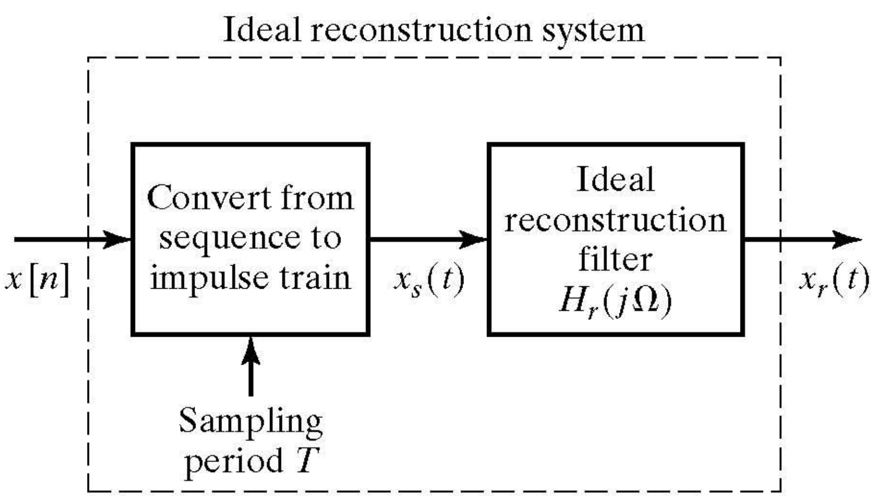
$$y(t) = \sum_{k=-\infty}^{\infty} g(kT_s)\,\text{rect}\left(\frac{t-kT_s}{T_s}\right)$$

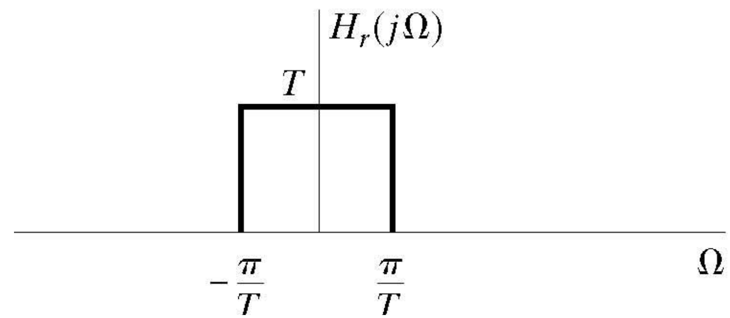
$$H(\omega) = T_s \text{sinc}\left(\frac{\omega}{2\omega_s}\right)$$

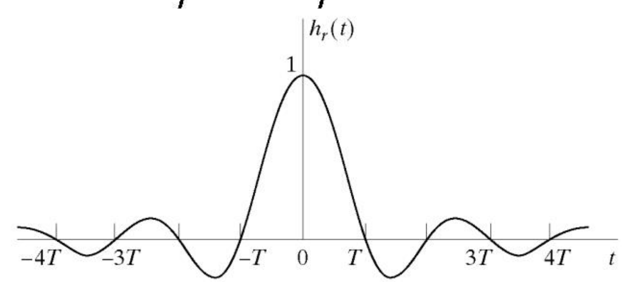


Signal Reconstruction (1)

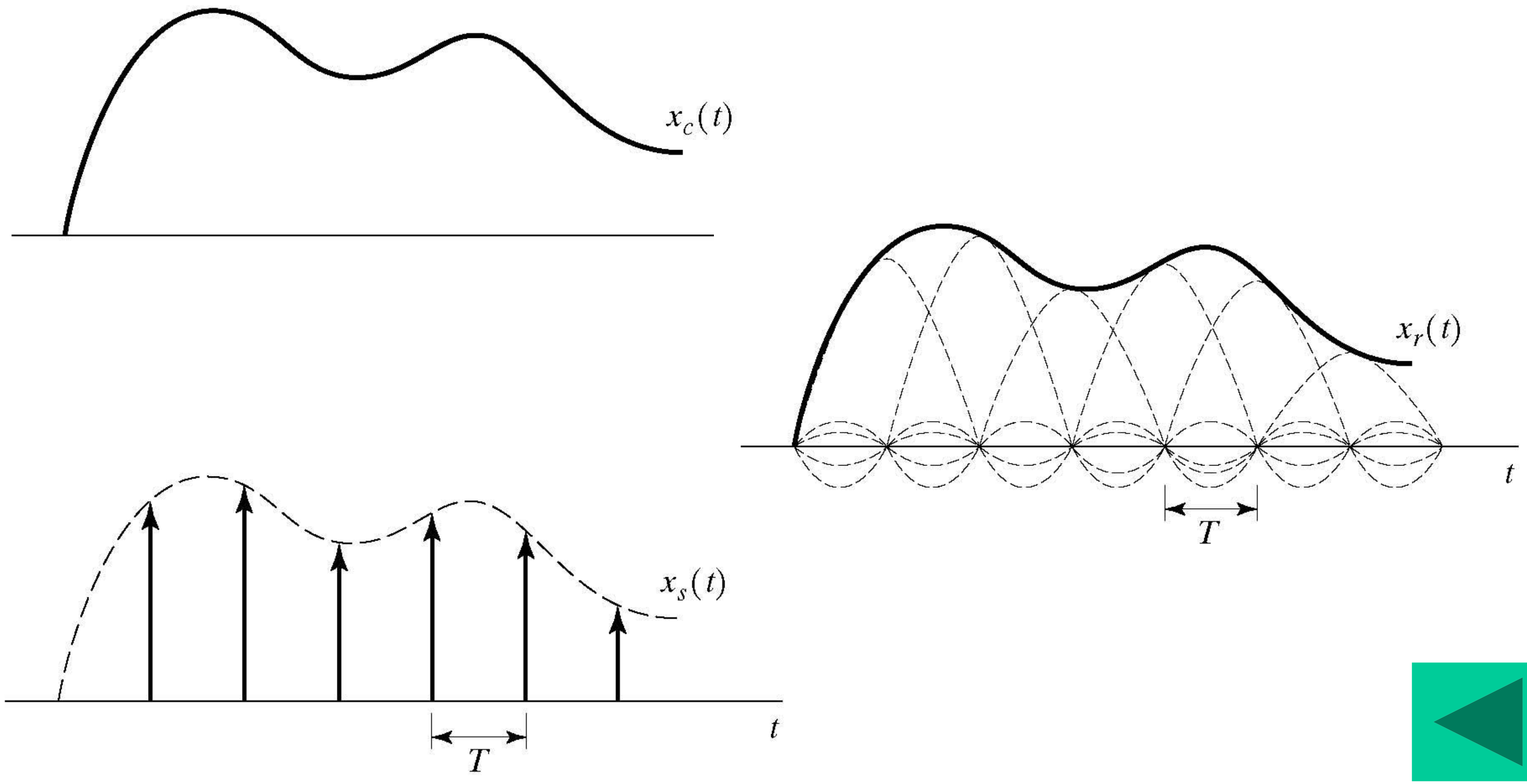








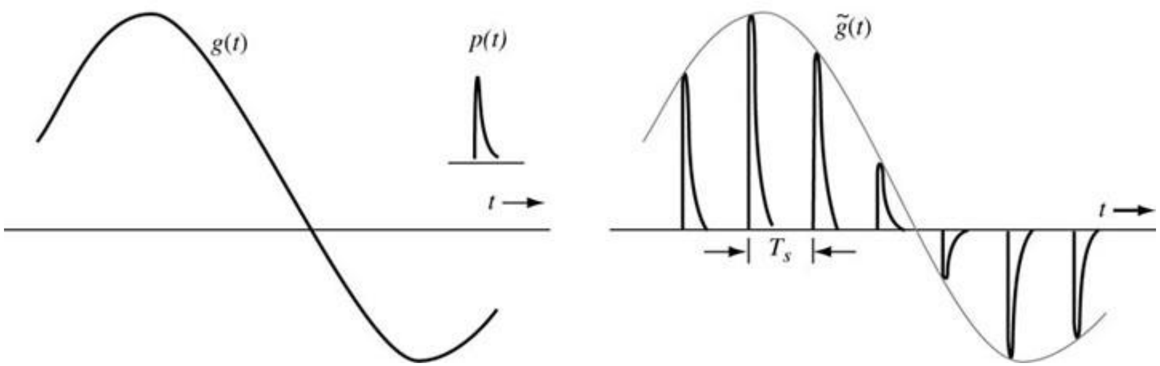
Signal Reconstruction (2)



Practical Sampling

Ideal sampling $g_s(t)=\sum_{n=-\infty}^{\infty} g(nT_s)\delta(t-nT_s)$: $\delta(t)$ has infinite bandwidth

Practical sampling $g_s(t)=\sum_{n=-\infty}^{\infty} g(nT_s)p(t-nT_s)$: $p(t)$ has finite essential bandwidth



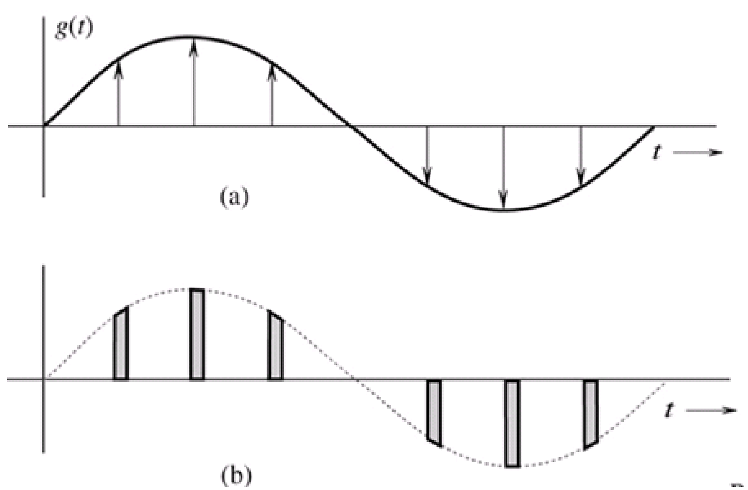
If sampling rate >2B, can be viewed as passing the ideal sampled signal through a filter. Similar to the signal reconstruction problem

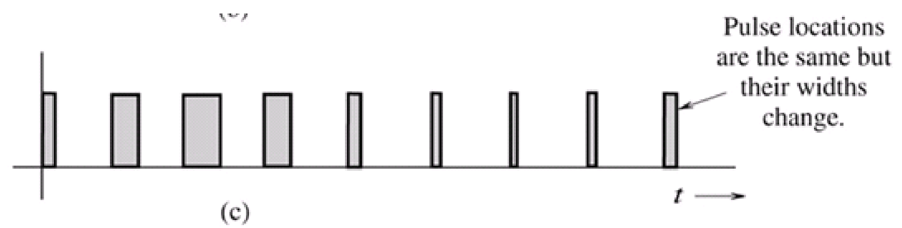
Roadmap

- Sampling Theorem
- Pulse Code Modulation (PCM)
  - Pulse modulation
  - Quantization: Uniform vs Nonuniform
  - Transmission bandwidth of PCM
  - Time division multiplexing
- Delta Modulation & Differential Pulse Code Modulation
- Vocoders and Video Compression

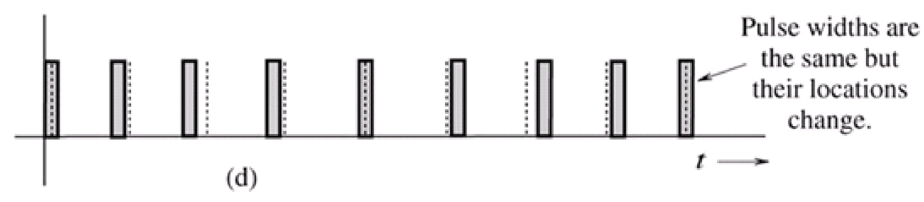
Pulse Modulation

- Sampling: a continuous-time signal ➔ a discrete-time sequence
- Pulse modulation:
  - Tx: convert the sequence into a modulated pulse train
  - Rx: reconstruct the signal by tracking certain parameter of the pulse train
  - Examples: PAM, PWM, PPM, PCM,





Pulse width modulation (PWM)



Pulse amplitude modulation (PAM)

Pulse position modulation (PPM)

Benefit of Pulse Modulation

- Enables simultaneous transmission of multiple signals using time-division multiplexing (TDM)

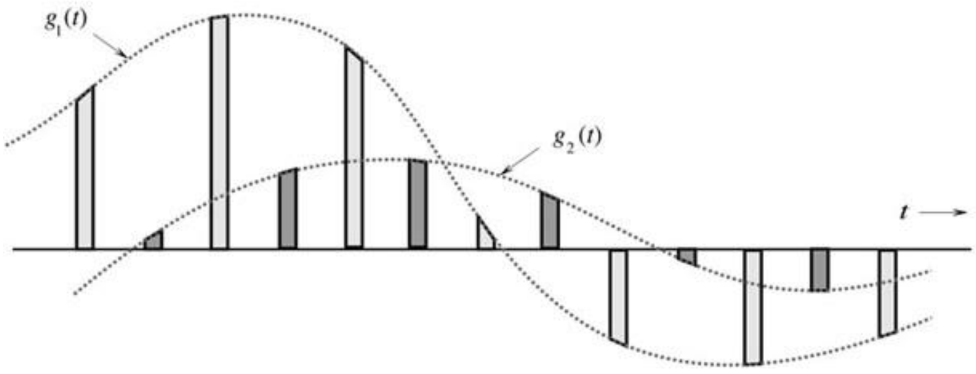
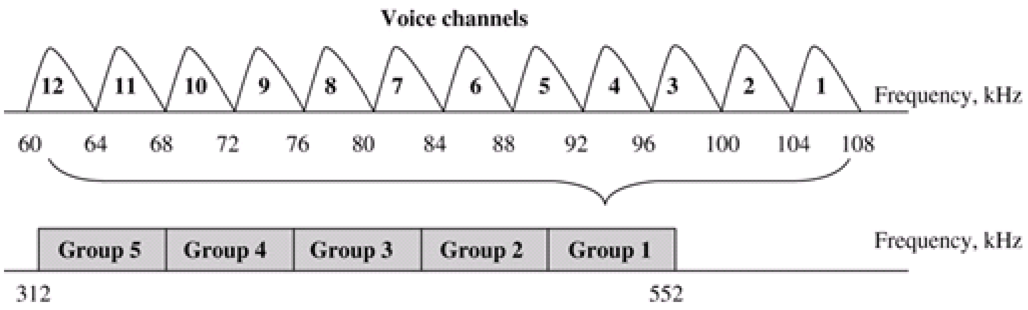


Figure 6.12 Time division multiplexing of two signals.

- This is the counterpart of frequency-division multiplexing (FDM)



Pulse Code Modulation (PCM)

- One of the most important forms used today
- Key steps of PCM: sample, quantize, and code

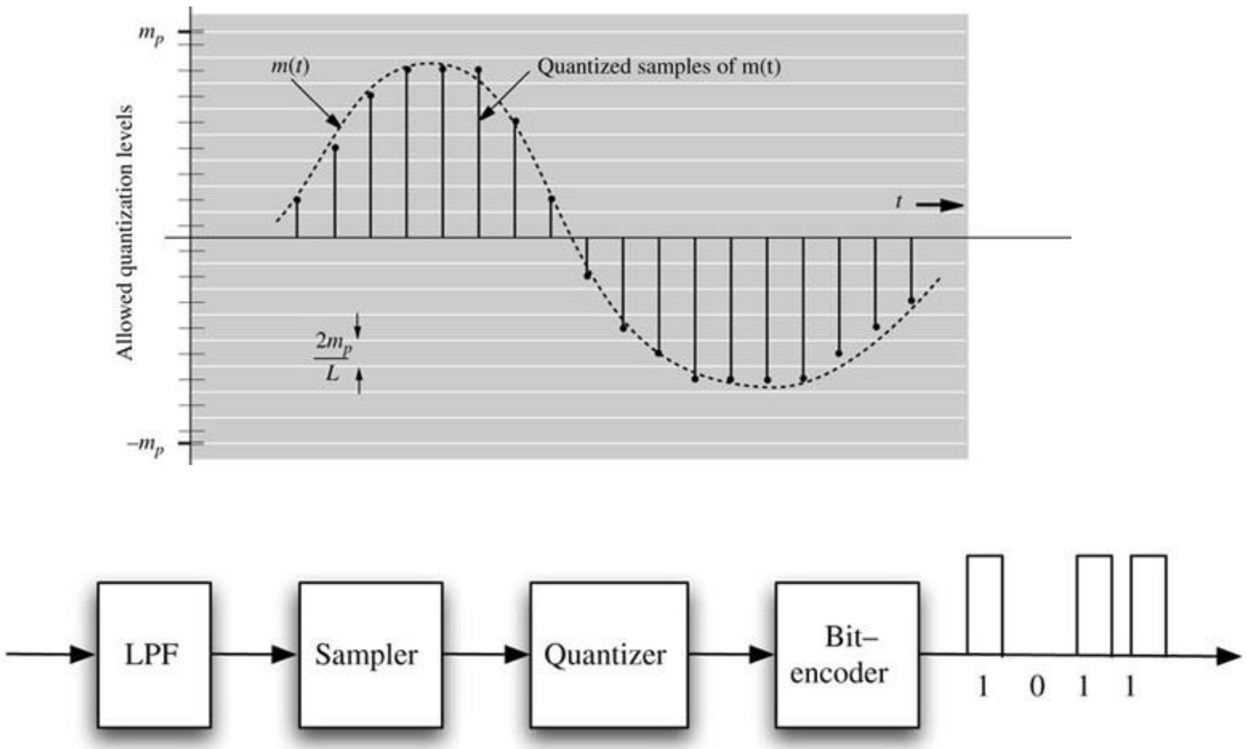


Figure 6.13 PCM system diagram.

Three Steps of PCM

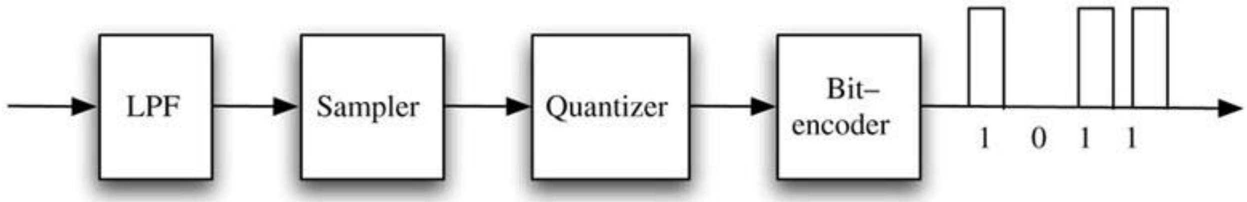


Figure 6.13 PCM system diagram.

Sampling: Convert a continuous time signal into a discrete-time signal

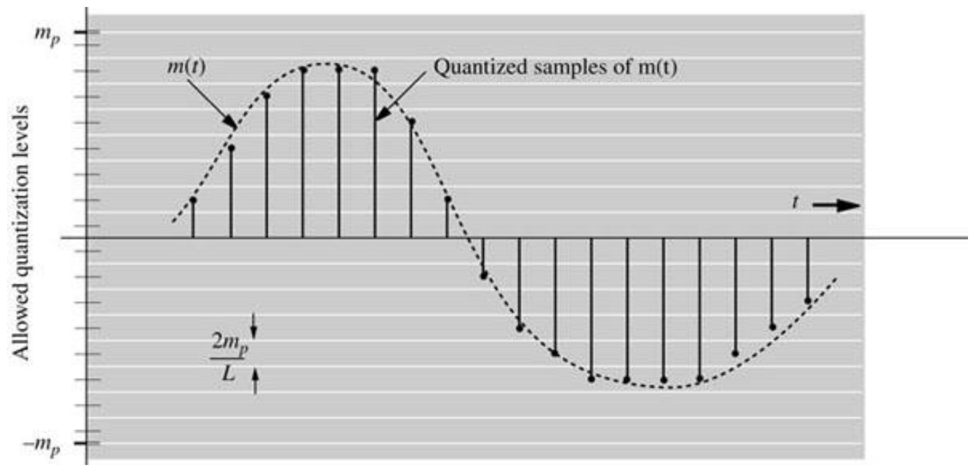
Quantization: Round off the sample values to the closest level (so the quantized samples only take finite number of possible values)

Coding: Index the finite number of levels and use bit streams to represent the indices of the samples.

- how many samples?
- how much distortion caused by quantization?
- how many bits to transmit the signal?
- What is the tradeoff between these parameters?

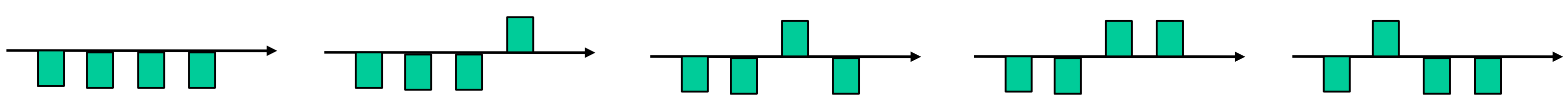
Uniform Quantization

Assume the message signal m(t) is between $\lfloor -m_p, m_p \rfloor$.
Divide $\lfloor -m_p, m_p \rfloor$ uniformly into L intervals, each of width $2m_p/L$



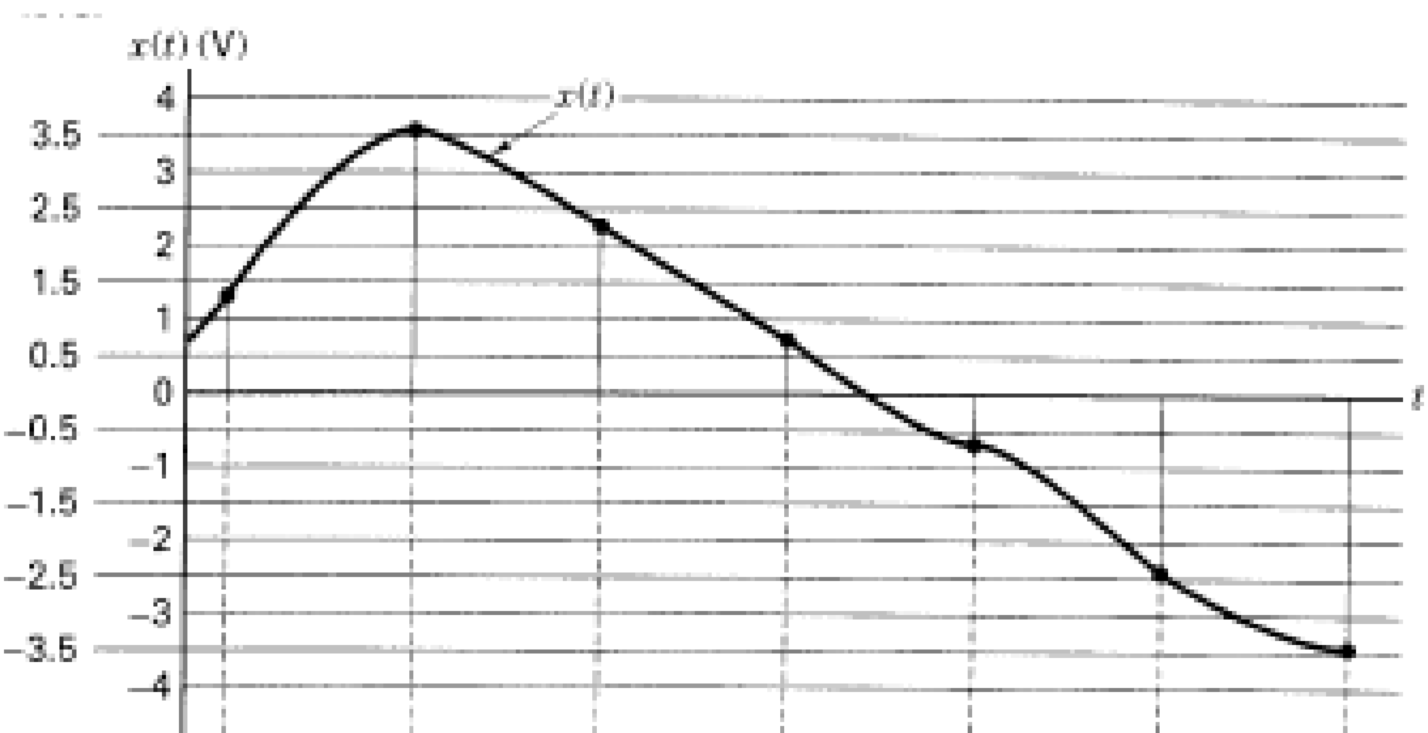
Approximate each sample using the mid-point of the interval where it falls.
The signal is now "digitized" since its amplitude now can only take L possible values. This is known as an L-ary digital signal.

Can then convert the index to binary signals using "pulse coding"



0=0000 0=0001 0=0010 3=0011 4=0100

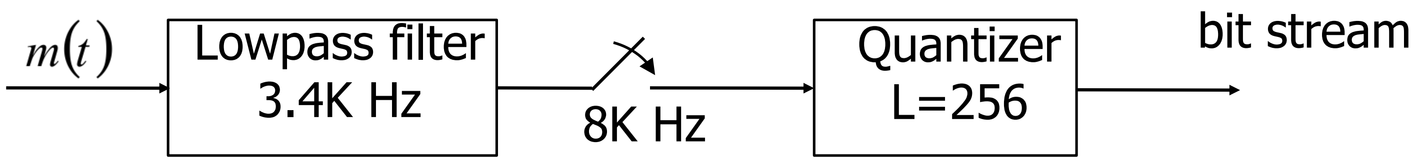
PCM Mapping



al sample value	1.3	3.6	2.3	0.7	–0.7	–2.4	–3.4
d sample value	1.5	3.5	2.5	0.5	–0.5	–2.5	–3.5
Code number	5	7	6	4	3	1	0
PCM sequence	101	111	110	100	011	001	000

Examples

Digital Telephone System



Anti-aliasing filter has cut-off frequency at 3.4KHz (for voice signal)
Sample at 8KHz, 8000 samples per second.
Use 256 level uniform quantizer. Needs 8 bits to represent each sample.
Therefore, output bitstream has data rate 64k bit/second.

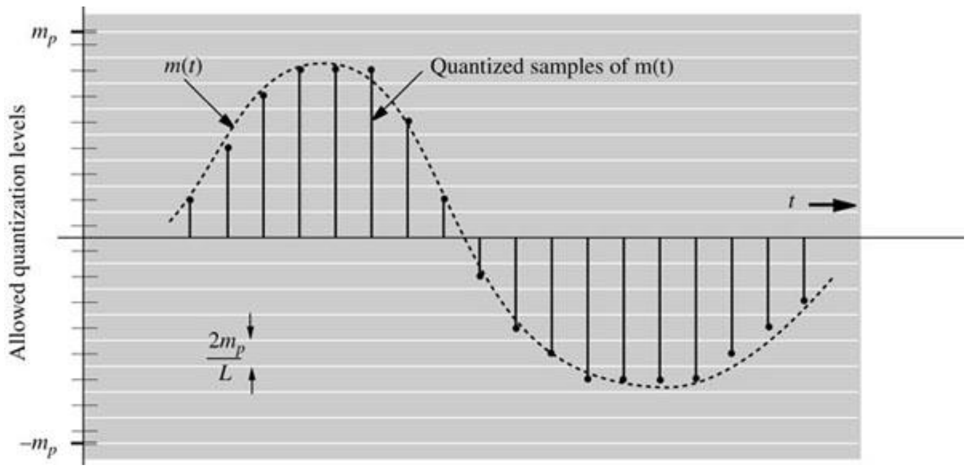
Compact CD

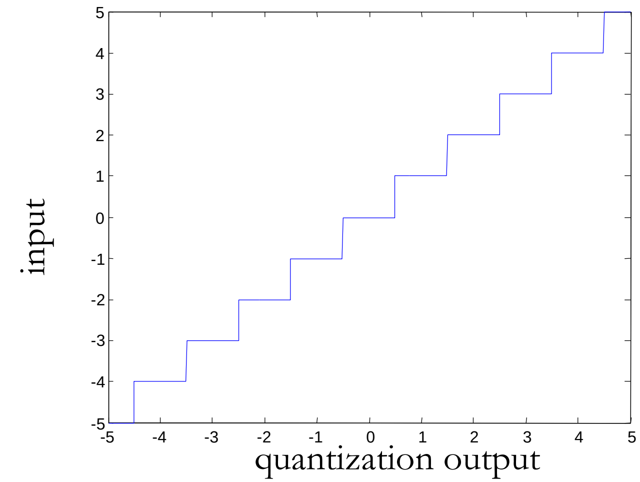
Need to record Hi-Fi music (0~15kHz)

Sample at 44.1KHz. Quantize with L=65536. Need 16 bits to represent each Sample. Hence data rate=706k bit/second

Note: In digital communication, we do not transmit these bits directly using square waveforms. There are advanced modulation schemes to save bandwidth

Quantization Noise (1)





Each sample is approximated using the mid-point of the interval where it falls.

Input $\{m[kT_s]\}$, Output $\{\hat{m}[kT_s]\}$. Quantization error, $\{q[kT_s]\} = \{m[kT_s] - \hat{m}[kT_s]\}$

Interpolation $m(t) = \sum_k m[kT_s]\,\mathrm{sinc}(2\pi Bt - k\pi)$ $\hat{m}(t) = \sum_k \hat{m}[kT_s]\,\mathrm{sinc}(2\pi Bt - k\pi)$

Distortion $q(t) = \sum_k (m[kT_s] - \hat{m}[kT_s])\,\mathrm{sinc}(2\pi Bt - k\pi)$ Quantization Noise

Quantization Noise (2)

$$q(t) = \sum_k \left(m[kT_s] - \hat{m}[kT_s]\right)\mathrm{sinc}(2\pi Bt\text{-}k\pi)$$

Power $$P_q = \lim_{T \to \infty} \frac{1}{T}\int_{-T/2}^{T/2} q^2(t)\,dt = \lim_{T \to \infty} \frac{1}{T}\int_{-T/2}^{T/2} \left[\sum_k \left(m[kT_s] - \hat{m}[kT_s]\right)\mathrm{sinc}(2\pi Bt\text{-}k\pi)\right]^2 dt$$

The signals $\mathrm{sinc}(2\pi Bt\text{-}m\pi)$ and $\mathrm{sinc}(2\pi Bt\text{-}n\pi)$ are orthogonal (can show)

$$\int_{-\infty}^{\infty} \left[\mathrm{sinc}(2\pi Bt\text{-}m\pi)\,\mathrm{sinc}(2\pi Bt\text{-}n\pi)\right] dt = \begin{cases} 0 & m \neq n \\ \dfrac{1}{2B} & m = n \end{cases}$$

Hence $$P_q = \lim_{T \to \infty} \frac{1}{T}\int_{-T/2}^{T/2} q^2(t)\,dt = \lim_{T \to \infty} \frac{1}{T}\int_{-T/2}^{T/2} \left[\sum_k \left(m[kT_s] - \hat{m}[kT_s]\right)\mathrm{sinc}(2\pi Bt\text{-}k\pi)\right]^2 dt$$

$$= \lim_{T \to \infty} \frac{1}{T}\int_{-T/2}^{T/2} \sum_k \left(m[kT_s] - \hat{m}[kT_s]\right)^2 \mathrm{sinc}^2(2\pi Bt\text{-}k\pi)\,dt$$

$$= \lim_{T \to \infty} \frac{1}{T}\sum_k \left(q[kT_s]\right)^2 \int_{-T/2}^{T/2} \mathrm{sinc}^2(2\pi Bt\text{-}k\pi)\,dt = \lim_{T \to \infty} \frac{1}{2BT}\sum_k q^2[kT_s]$$

Quantization Noise (3)

Quantization interval is $2m_p/L$, and the sample values are approximated by the mid-points. We have

$$\left|q(kT_s)\right| < m_p/L$$

Therefore

$$P_q = \lim_{T \to \infty} \frac{1}{2BT} \sum_k q^2[kT_s] \le (m_p/L)^2$$

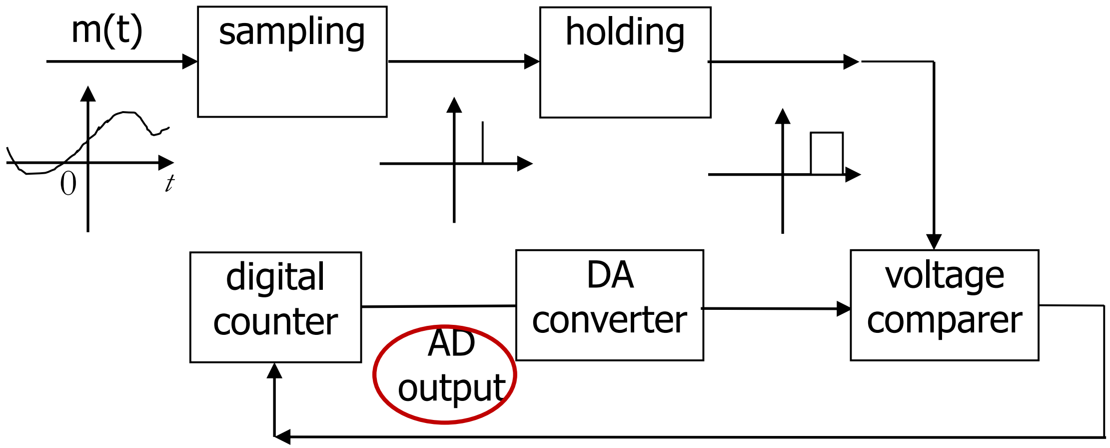
Alternatively, if we assume error takes value in $\left[-m_p/L, m_p/L\right]$ with uniform probability . We have

$$P_q = \lim_{T \to \infty} \frac{1}{2BT} \sum_k q^2[kT_s] = E\left[q^2[kT_s]\right] = \int_{-m_p/L}^{m_p/L} q^2 \frac{1}{2m_p/L} dq = \frac{1}{6m_p/L} q^3 \Big|_{-m_p/L}^{m_p/L} = \frac{1}{3} \frac{m_p^2}{L^2}$$

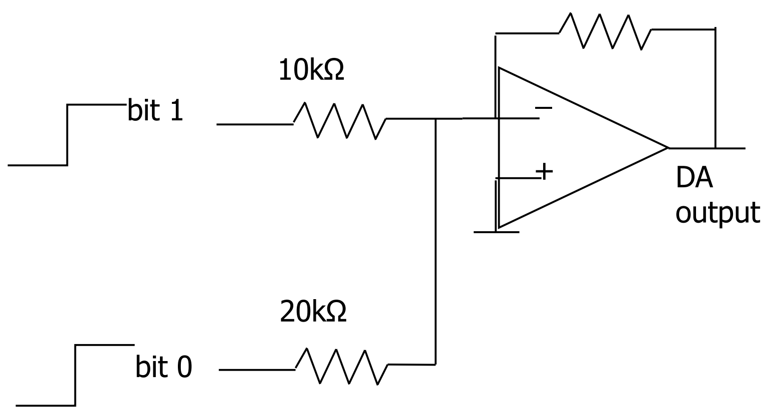
If we define $\Delta = 2m_p/L$, which is the width of each interval, $P_q = \frac{1}{12}\Delta^2$

Signal to noise ratio $SNR = \frac{P_m}{P_q} = \frac{P_m}{\Delta^2/12}$

Analog-to-Digital Conversion



A/D convertor

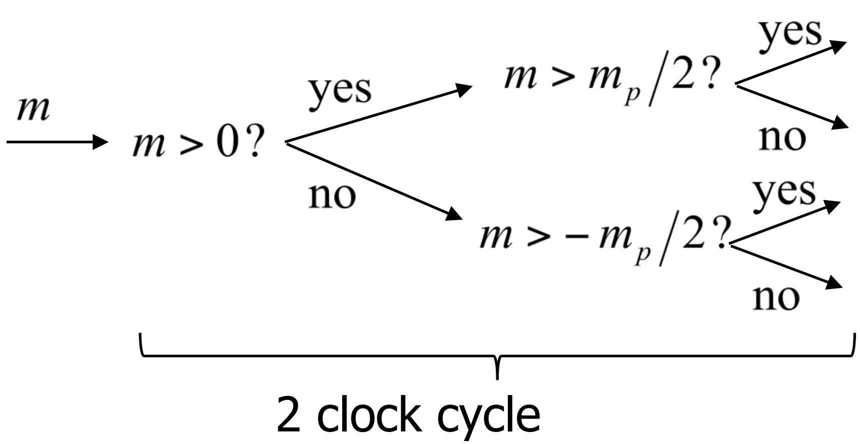


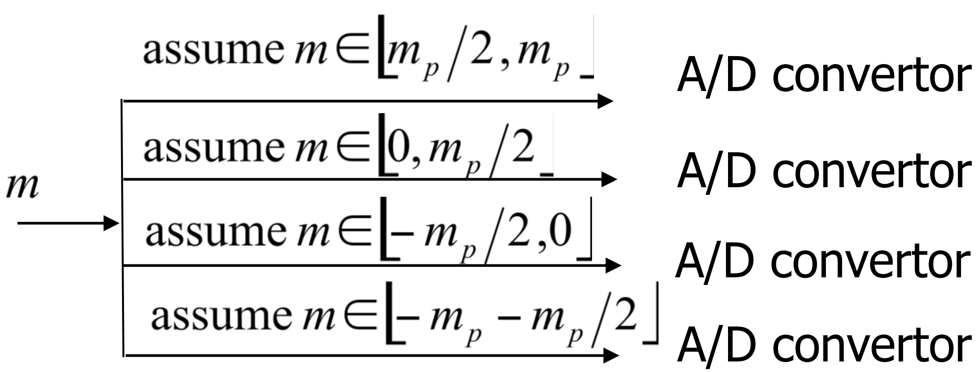
A 2bit D/A convertor

It is difficult to do A/D conversion with a large number of quantization levels.

Expensive High-Speed ADC

Single A/D convertor





Can use 4 parallel A/D convertors to save 2 clock cycles (expensive!)

Nonuniform Quantization

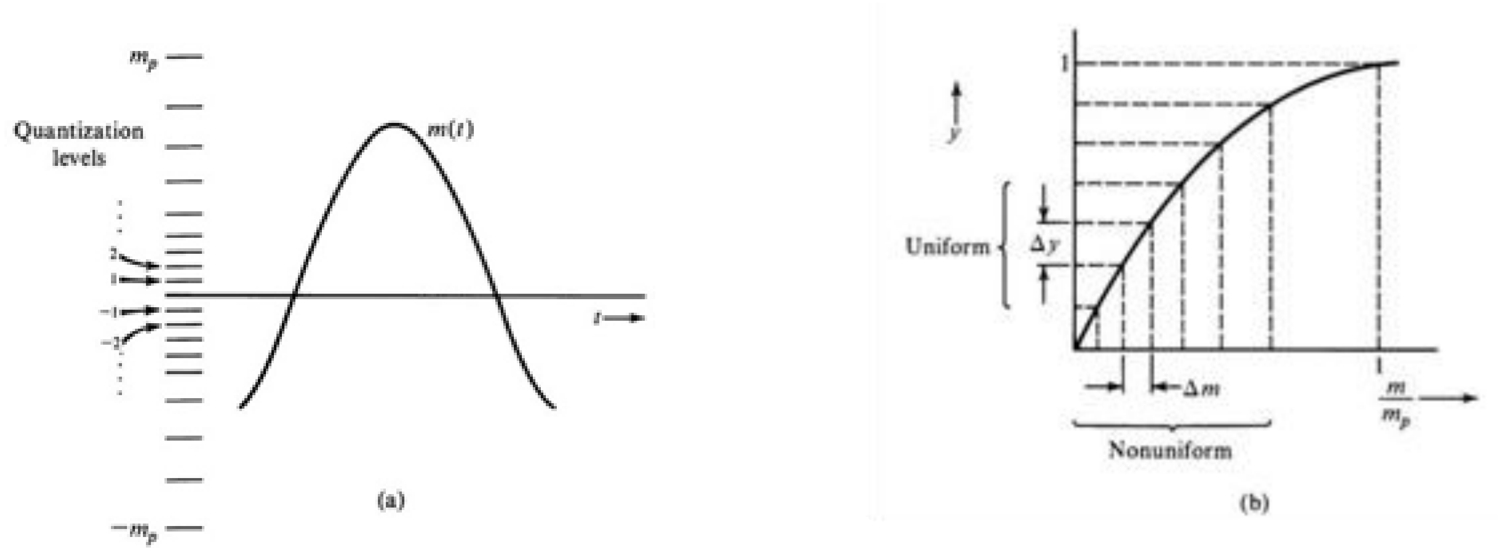


Figure 6.15 Nonuniform quantization.

Can be interpreted as uniform quantization on a compressed (transformed) signal

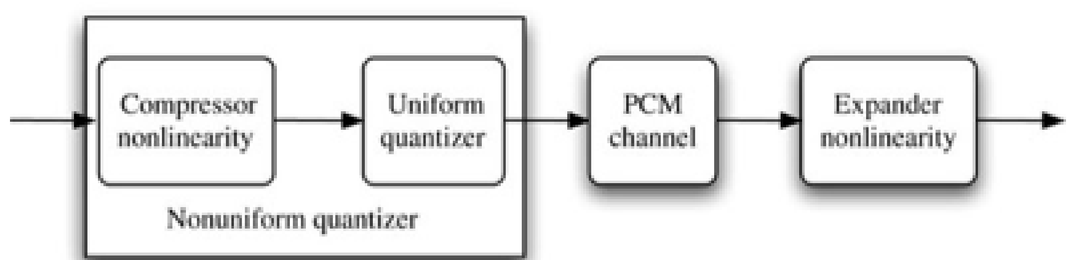


Figure 6.17 Utilization of compressor and expander for nonuniform quantization.

Compressor

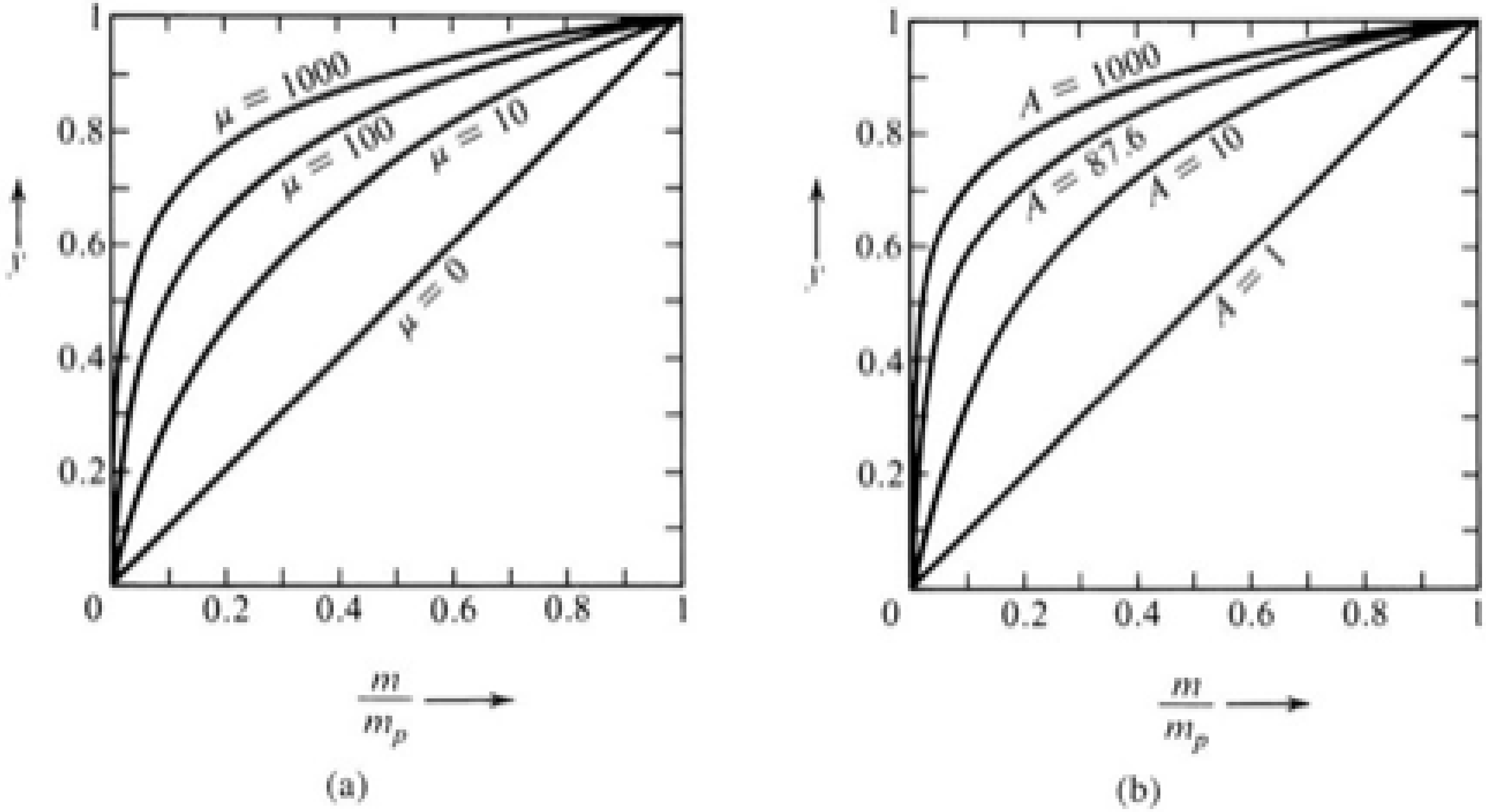


Figure 6.16 (a) μ-Law characteristic. (b) A-Law characteristic.

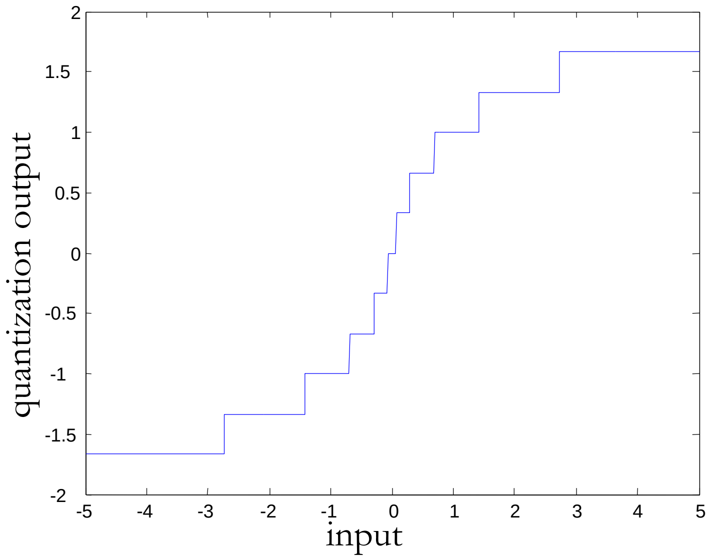
μ law:

$$|v| = \frac{\log(1+\mu|m|)}{\log(1+\mu)}$$

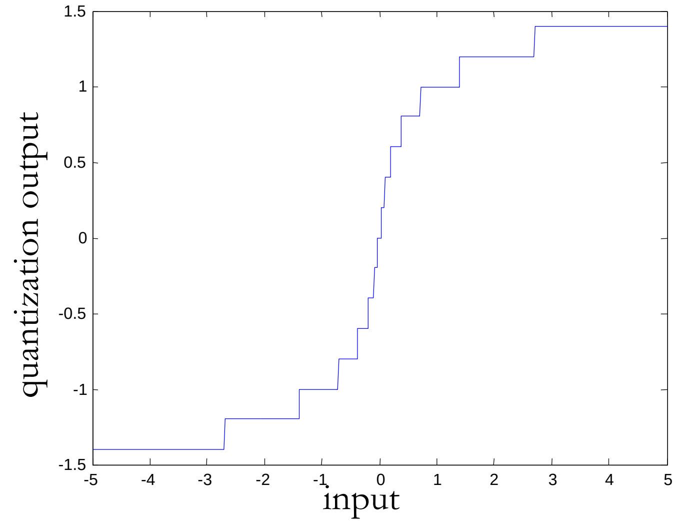
A law:

$$|v| = \begin{cases} \dfrac{A|m|}{1+\log A} & 0 \le |m| \le \dfrac{1}{A} \\ \dfrac{1+\log(A|m|)}{1+\log A} & \dfrac{1}{A} \le |m| \le 1 \end{cases}$$

Another Illustration



An illustration of μ law



An illustration of A law

μ law:

$$|v| = \frac{\log(1 + \mu|m|)}{\log(1 + \mu)}$$

A law:

$$|v| = \begin{cases} \dfrac{A|m|}{1 + \log A} & 0 \le |m| \le \dfrac{1}{A} \\[2ex] \dfrac{1 + \log(A|m|)}{1 + \log A} & \dfrac{1}{A} \le |m| \le 1 \end{cases}$$

Uniform vs. Nonuniform

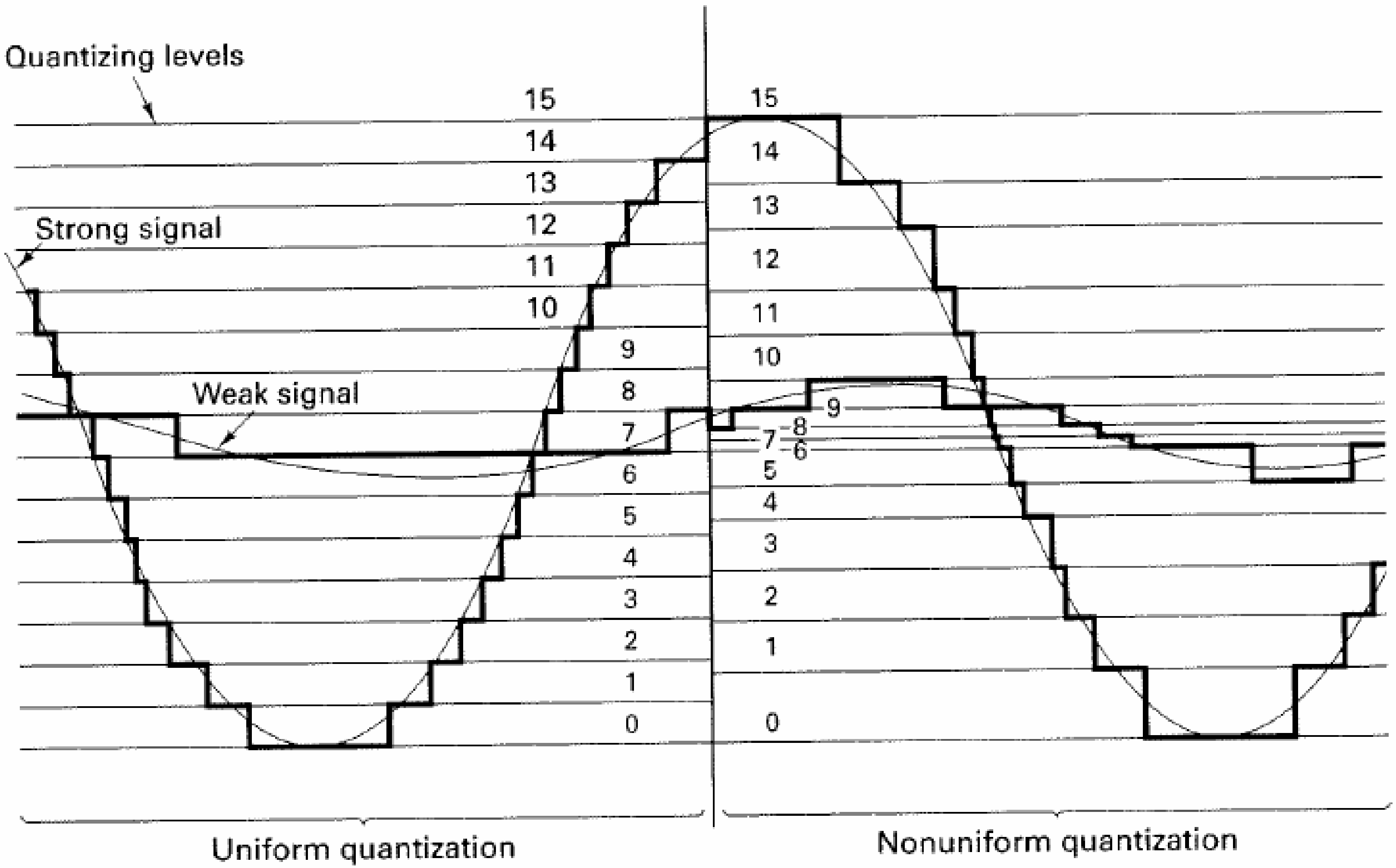


Figure 2.18 Uniform and nonuniform quantization of signals.

Benefit of Nonuniform Quantization

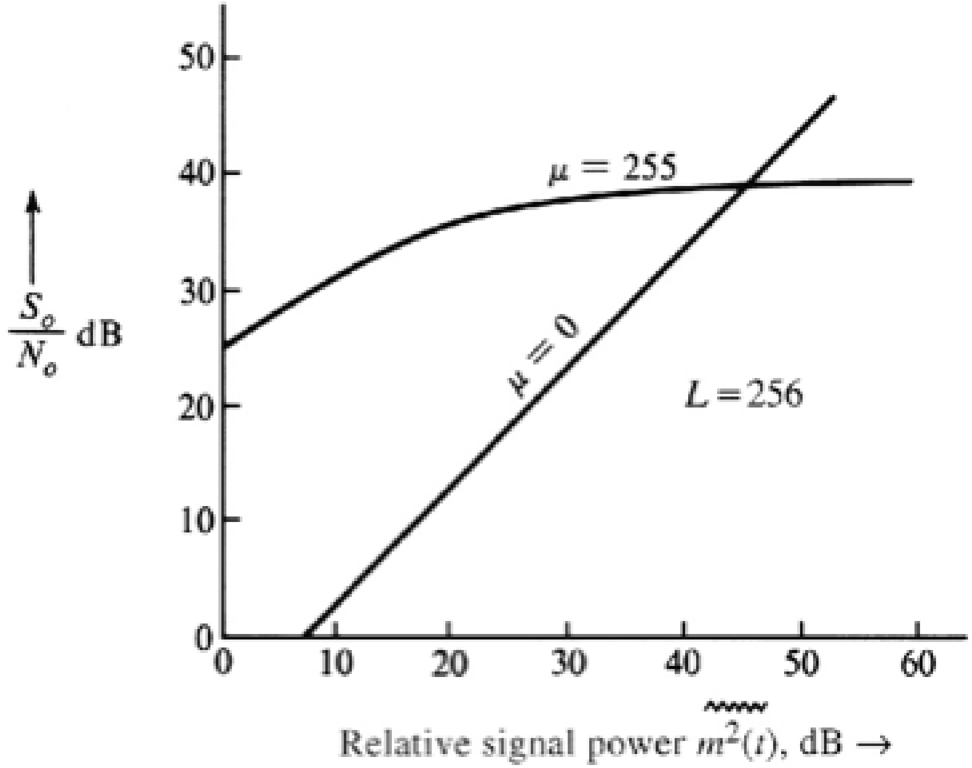


Figure 6.18 Ratio of signal to quantization noise in PCM with and without compression.

Transmission Bandwidth of PCM (1)

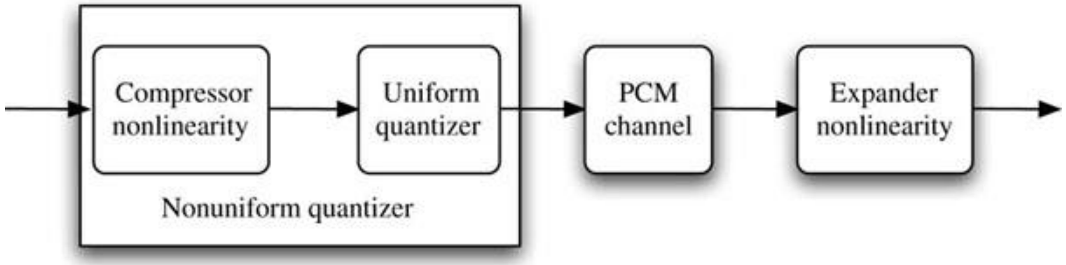


Figure 6.17 Utilization of compressor and expander for nonuniform quantization.

Binary PCM

Signal is transmitted using binary waveforms, $m(t)=\sum_k m(kT_s)p(t-kT_s)$

that is, $m(kT_s)=0,1$ or $m(kT_s)=\pm 1$

If no noise, a channel of bandwidth *B* can at most support 2*B* bps.

Example 1: With sampling rate f_s and *L*-level uniform quantization, determine the minimum channel bandwidth for binary PCM.

The output data contains $f_s\lceil \log_2 L\rceil$ bits per second. To transmit these bits using binary PCM, the minimum channel bandwidth is $f_s\lceil \log_2 L\rceil/2$ Hz

Transmission Bandwidth of PCM (2)

Example 2:
Use binary PCM, the channel bandwidth is *B*. Use *L*-level uniform quantization. Determine the maximum bandwidth of the message signal.

Each sample contains $\lceil \log_2 L \rceil$ bits. Channel can support $2B$ sym/sec
Therefore, we can at most have $2B/\lceil \log_2 L \rceil$ samples per second. According to Nyquist sampling theorem, the maximum bandwidth of the message signal equals $2B/\lceil \log_2 L \rceil/2$

Example 3:
Use binary PCM, the channel bandwidth is *B*. The message signal has bandwidth *W*. Determine the maximum number of the uniform quantization levels.

Need $2W$ samples per second. Can transmit at most $2B$ bits per second.
Therefore each sample can have at most $\lfloor B/W \rfloor$ bits. This means the maximum number of uniform quantization levels equals $2^{\lfloor B/W \rfloor}$

Roadmap

- Sampling Theorem
- Pulse Code Modulation (PCM)
- Delta Modulation & Differential Pulse Code Modulation
  - Delta modulation
  - PCM vs. Delta modulation
  - Differential PCM
  - Delta modulation vs. Differential PCM
- Vocoders and Video Compression

Delta Modulation

Motivations:

- To save channel bandwidth, we want to use as small number of bits as possible to represent a (quantized) sample.
- It is difficult to implement high speed A/D converter with many levels.

If message signal vary slowly in time:

- Do NOT represent each quantized sample independently
- DO represent the difference between the amplitudes of successive samples.

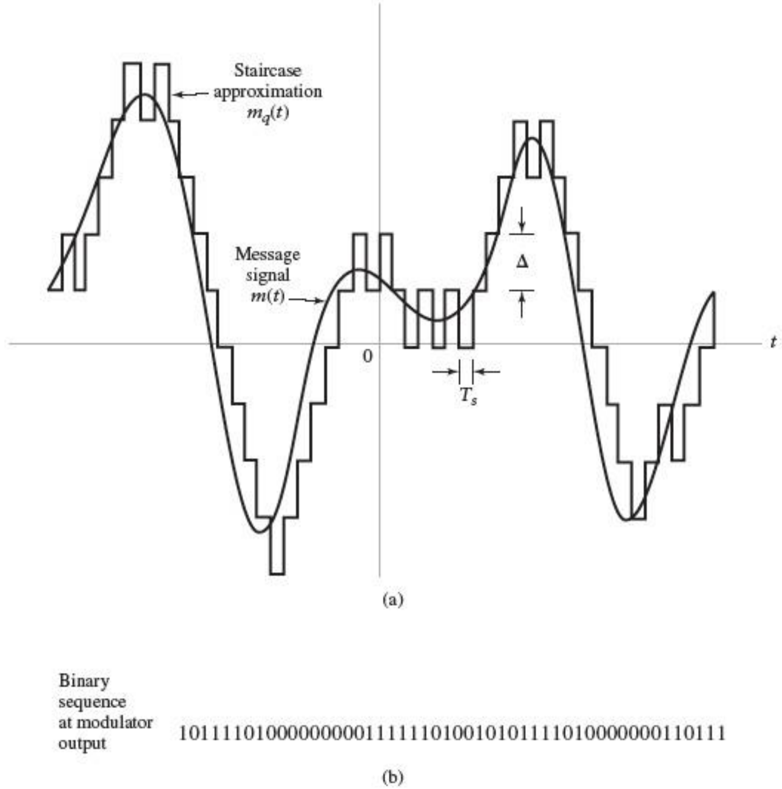
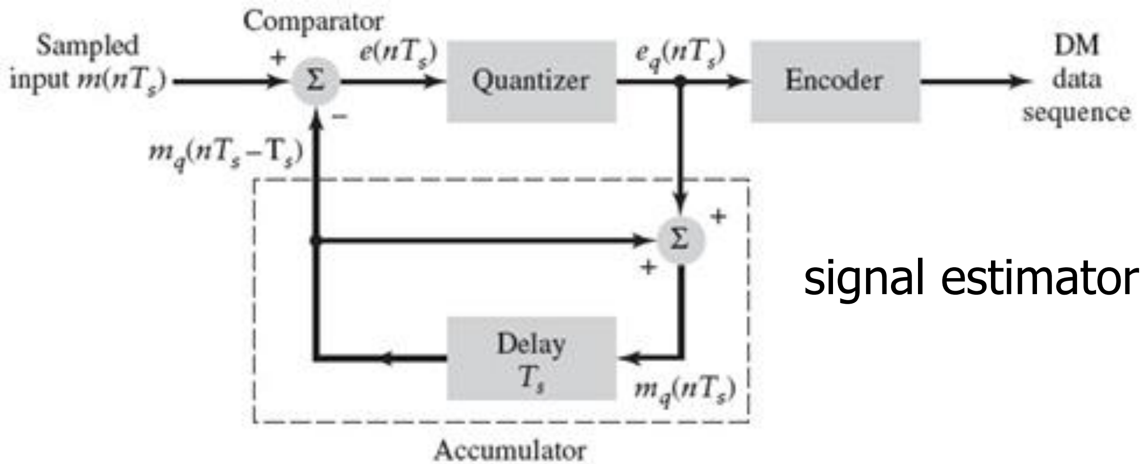


FIGURE 5.14 Illustration of delta modulation. (*a*) Analog waveform $m(t)$ and its staircase approximation $m_q(t)$. (*b*) Binary sequence at the modulator output.

Delta Modulation: Transmitter



(a)

signal estimator

$$e(nT_s)=m(nT_s)-m_q(nT_s-T_s)$$

$$m_q(nT_s)=\sum_{k=0}^{n}e_q(kT_s) \qquad m_q(z)=e_q(z)\frac{1}{1-z^{-1}}$$

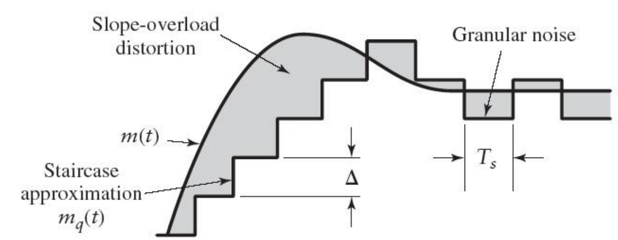


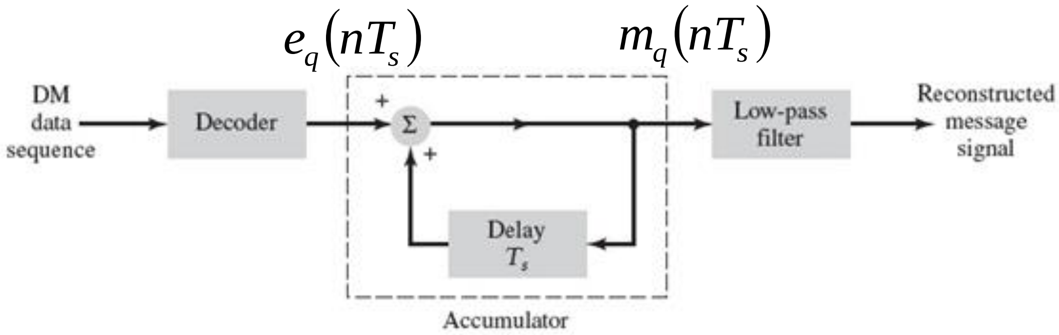
FIGURE 5.16 Illustration of quantization errors, slope-overload distortion and granular noise, in delta modulation.

Conventionally, use one bit to represent sample amplitude difference. Can have slope overload distortion.

If no overload distortion

$$\left|m(nT_s)-m_q(nT_s)\right|=\left|e(nT_s)-e_q(nT_s)\right|\leq\frac{\Delta}{2}$$

Delta Modulation: Receiver

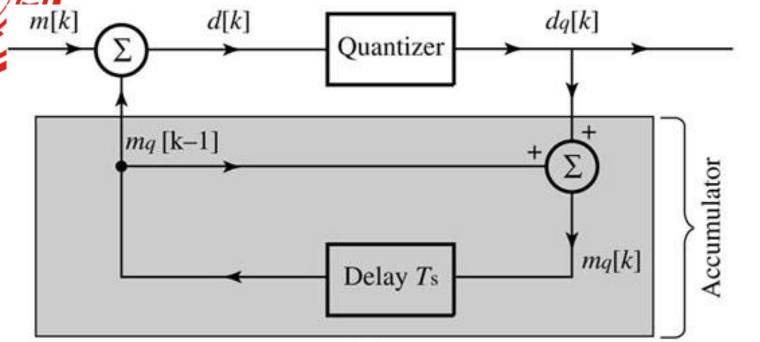


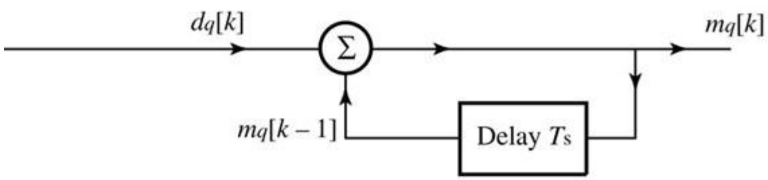
(b)

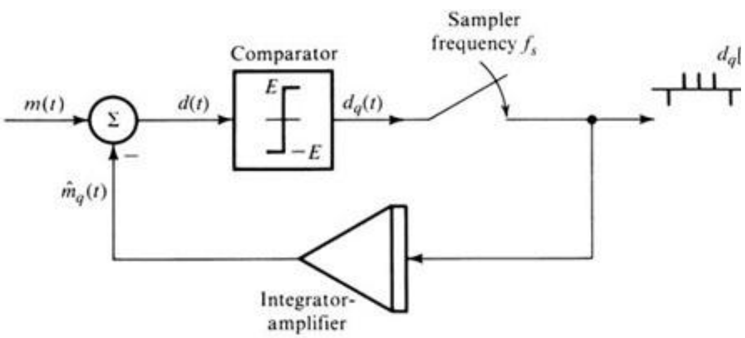
$$m_q(nT_s) = m_q(nT_s - T_s) + e_q(nT_s)$$

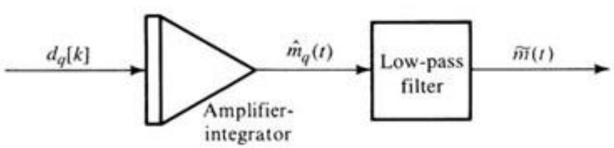
$$m_q(z) = m_q(z)z^{-1} + e_q(z) \qquad m_q(z) = e_q(z)\frac{1}{1-z^{-1}}$$

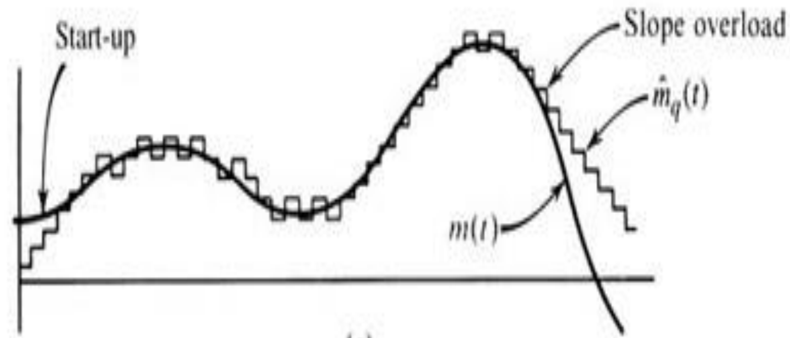
$$m_q(nT_s) = \sum_{k=0}^{n} e_q(kT_s) \qquad e(nT_s) = m(nT_s) - m_q(nT_s - T_s)$$

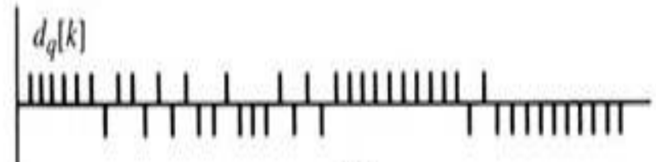

m[k]
d[k]
Quantizer
dq[k]
mq [k–1]
Σ
Delay Ts
mq[k]
Accumulator
(a)

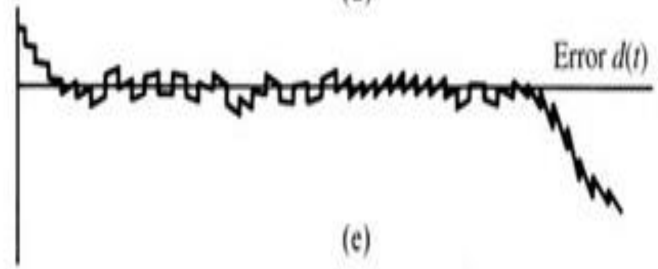

dq[k]
Σ
mq[k]
mq[k – 1]
Delay Ts
(b)


Sampler
frequency f_s
Comparator
$m(t)$
$d(t)$
E
$-E$
$d_q(t)$
$\hat{m}_q(t)$
Integrator-
amplifier
(a)


$d_q[k]$
$\hat{m}_q(t)$
Low-pass
filter
$\tilde{m}(t)$
Amplifier-
integrator
(b)


Start-up
Slope overload
$\hat{m}_q(t)$
$m(t)$
(c)


$d_q[k]$
(d)


Error $d(t)$
(e)

PCM vs. Delta Modulation

PCM:

- \# of bits per sample determines # of quantization levels of samples

➔ Bounds on the sample amplitude

Delata-modulation:

- \# of bits per sample determines # of quantization levels of sample differences

➔ Bounds on sample amplitude differences

➔ Works well for low-frequency data

Σ-Δ Modulation:

- Δ -modulation essentially samples $dm(t)/dt$

➔ Can apply Δ -modulation to $\int_{-\infty}^{t} m(\tau)d\tau$

Differential PCM

Why Δ -modulation works for low frequency signal?

- $m(kT_s)$ and $m((k+1)T_s)$ are highly dependent.
- $m((k+1)T_s)$ can be "partially" predicted by $m(kT_s)$
- Only need to quantize the non-predictable part.

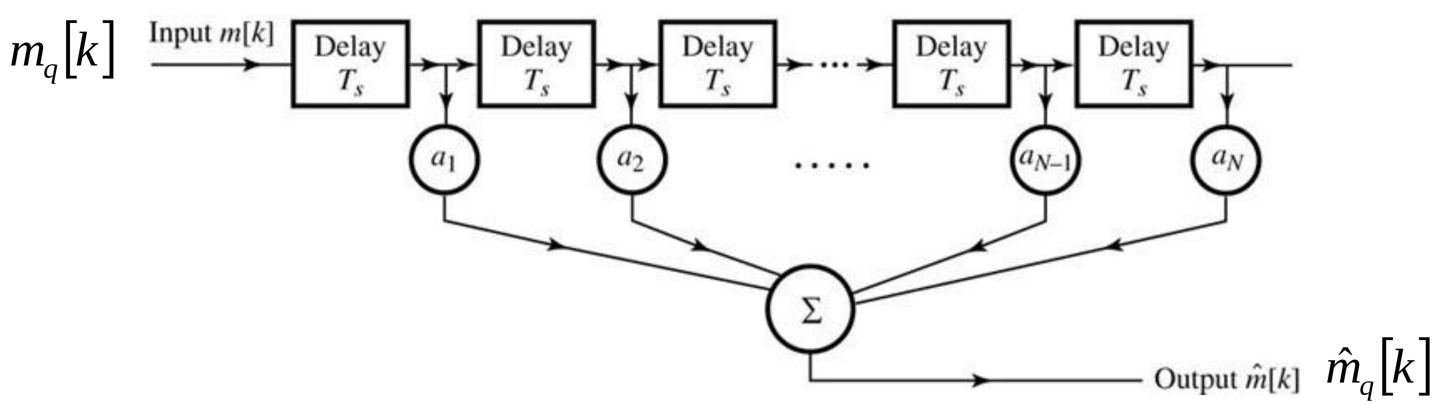
Generalization:

- Write $m(kT_s)$ as the summation of a predictable part and an independent noise:

$$m(kT_s) = \hat{m}(kT_s) + d(kT_s)$$

$$\hat{m}_q(kT_s) = a_1 m_q((k-1)T_s) + a_2 m_q((k-2)T_s) + \cdots + a_N m_q((k-N)T_s)$$

- Only quantize the non-predictable part $d(kT_s)$.



DPCM: Transmitter & Receiver

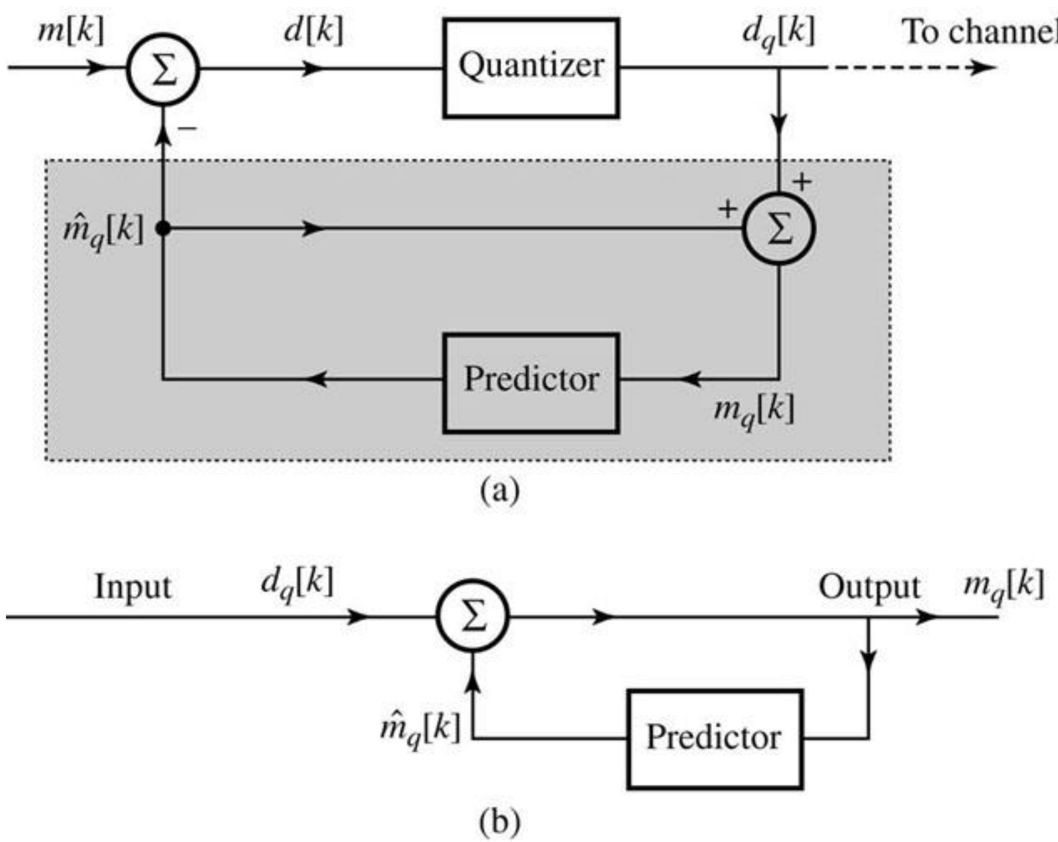
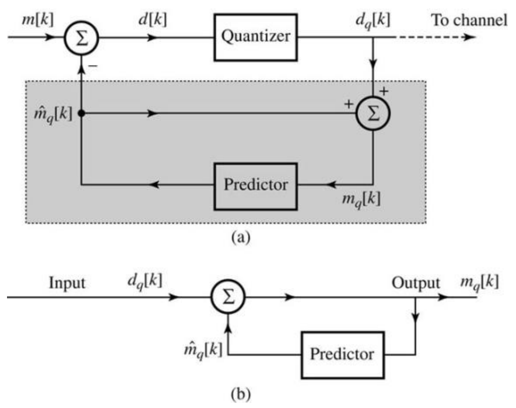


Figure 6.28 DPCM system: (a) transmitter; (b) receiver.

DPCM



$$m_q(z) = \hat{m}(z) + d_q(z) = W(z) m_q(z) + d_q(z)$$

$$m_q(z) = \frac{d_q(z)}{1 - W(z)}$$

$$\hat{m}(nT_s) = w_1 m_q((n-1)T_s) + w_2 m_q((n-2)T_s) + \cdots + w_p m_q((n-p)T_s)$$

$$\hat{m}(z) = W(z) m_q(z) \qquad W(z) = w_1 z^{-1} + w_2 z^{-2} + \cdots + w_N z^{-N}$$

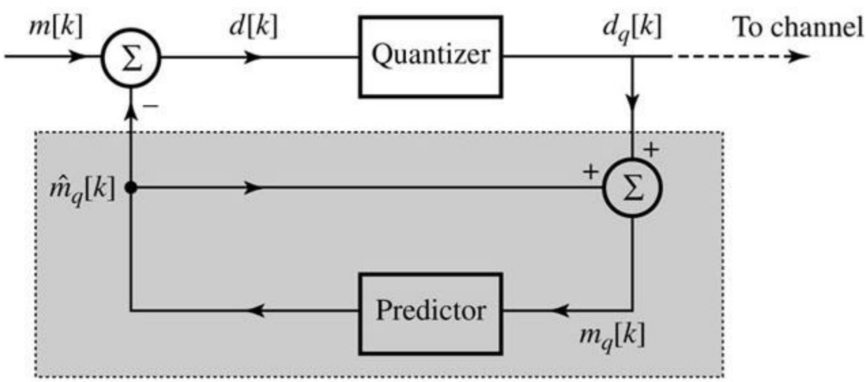
$$d(z) = m(z) - \hat{m}(z) \qquad d_q(z) = \text{quantize}(d(z))$$

If no overload distortion $\left| d(z) - d_q(z) \right| \le \frac{\Delta}{2}$

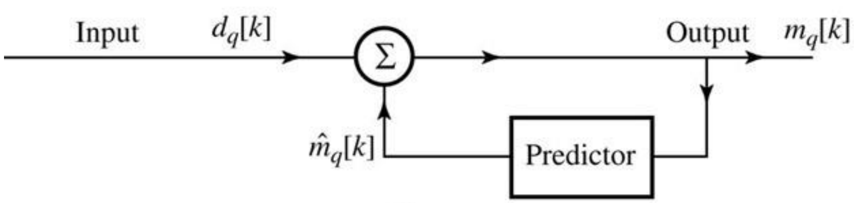
$$\left| m(z) - \hat{m}(z) - d_q(z) \right| = \left| m(z) - \left(\frac{W(z) d_q(z)}{1 - W(z)} + d_q(z) \right) \right| = \left| m(z) - \frac{d_q(z)}{1 - W(z)} \right| \le \frac{\Delta}{2}$$

Delta Modulation Revisited

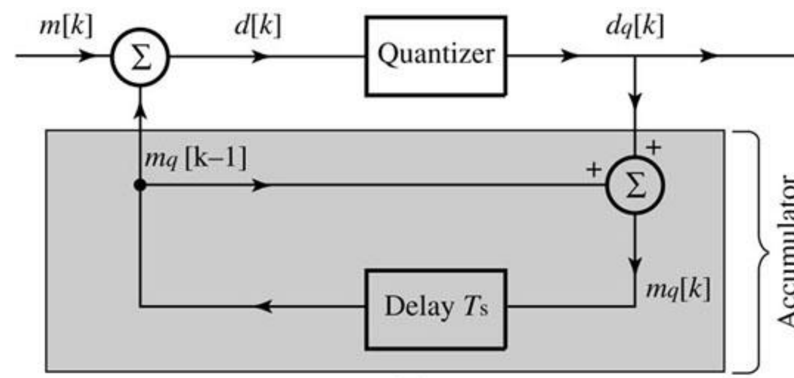
- DPCM
- Delta Modulation



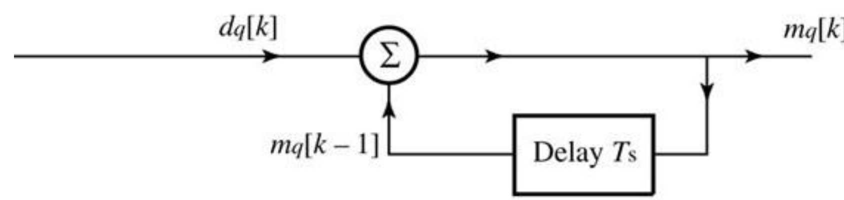
(a)



(b)



(a)



(b)

Delta Modulation is a special case of DPCM

Adaptive DPCM

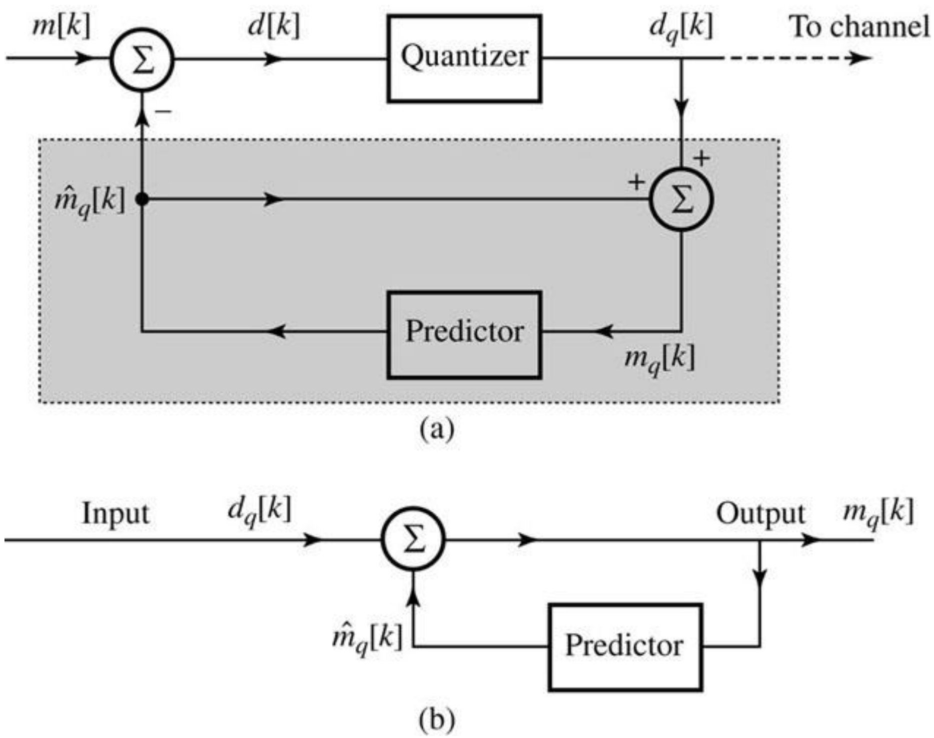


Figure 6.28 DPCM system: (a) transmitter; (b) receiver.

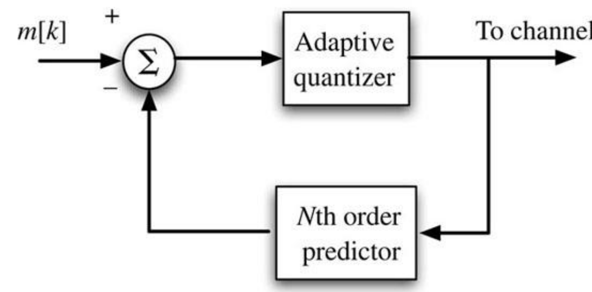


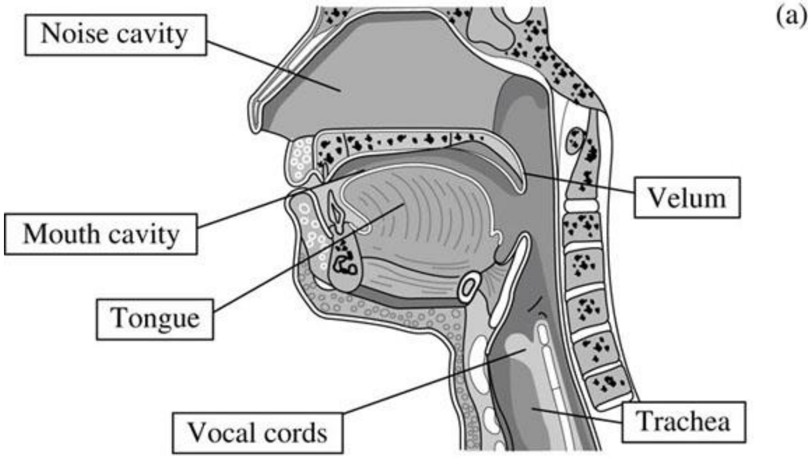
Figure 6.29 ADPCM encoder uses an adaptive quantizer controlled only by the encoder output bits.

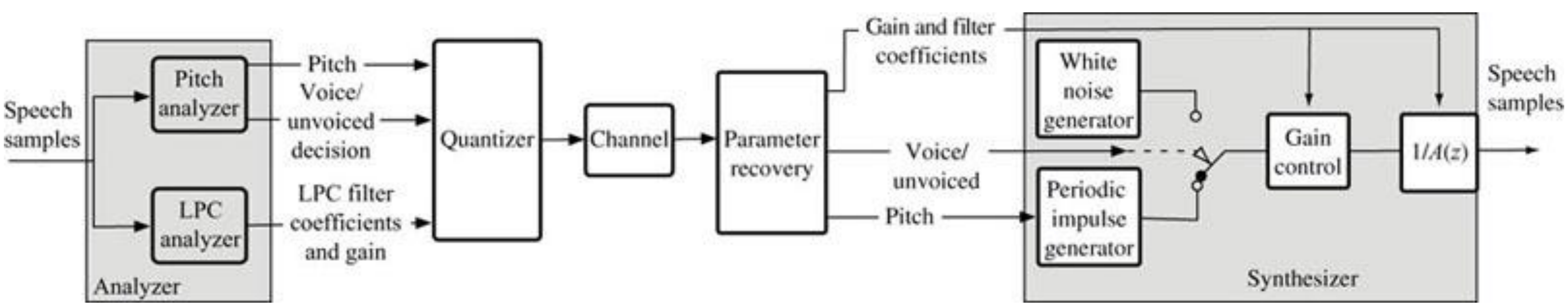
- Fix quantization level, but adjust the quantization interval.
- Tx and Rx need to agree upon the quantization interval.
- In the area of waveform source coding.

Roadmap

- Sampling Theorem
- Pulse Code Modulation (PCM)
- Delta Modulation & Differential Pulse Code Modulation
- Vocoders and Video Compression

Source Coding: Vocoder





Model-based Vocoder:

- Use certain parameters to model voice signals
- Quantize the parameters
- Reconstruct (artificially simulate) voice signals at the receiver

Source Coding: Video Compression

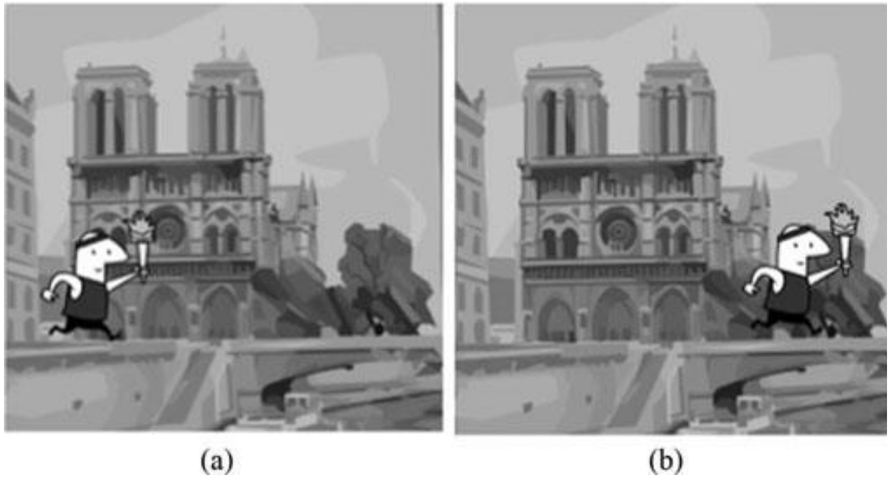


Figure 6.36 (a) Frame 1. (b) Frame 2. (c) Information transferred between frames 1 and 2.

MPEG Video compression:

- Use bidirectional interpolation and advanced lossy compression algorithms to minimize the number of bits used to describe each frame.

Note:

- lossless compression, zip, rar, etc.
- lossy compression, MP3, MPEG, etc.

Summary

- Sampling Theorem
  - Sampling
  - Reconstruction
- Pulse Code Modulation (PCM)
  - Pulse modulation
  - Quantization: Uniform vs Nonuniform
  - Transmission bandwidth of PCM
  - Time division multiplexing
- Delta Modulation & Differential Pulse Code Modulation
  - Delta modulation
  - Differential pulse code modulation
- Vocoders and Video Compression