Principle of Communications

Performance Analysis of Digital Communication Syst

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Outline

- Probabilistic Detectors
- Probability of Error Analysis
- Direct Sequence Spread Spectrum (DSSS) Communications

Roadmap

- Probabilistic Detectors
  - Maximum A Posterior (MAP) Detector
  - Maximum Likelihood (ML) Detector
- Probability of Error Analysis

Probability Detection

An alternative to minimum-distance detection that accounts for noise statistics

Our progression — Start with simplest model:

1. $r = a_m + n$, real
2. $r = a_m + n$, complex
3. $r = s_m + n$, vector
4. $r(t) = s_m(t) + n(t)$, real waveforms
5. $r(t) = s_m(t) + n(t)$, complex-valued waveform

M-ary Detection for Scalar Channel

Given $$R = A + N$$

Where $A \in \mathcal{A}$, a finite alphabet with $|\mathcal{A}| = M$.
and given noise statistics,

$$\ldots \text{find the "best" decision } \hat{A}.$$

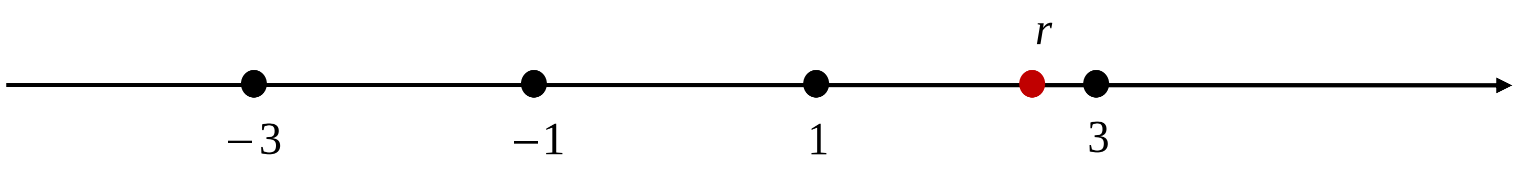
Example:

$\mathcal{A} = \{-3, -1, 1, 3\}.$

$N \sim Exp(1), \quad f(N) = \exp(-N)$

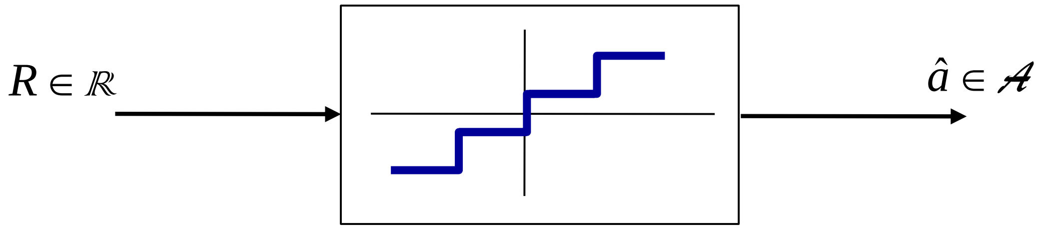
$P_A(-3) = 0.4, P_A(-1) = 0.3, P_A(1) = 0.2, P_A(3) = 0.1$

$r = 2.9$



A Scalar Detector

Threshold device=quantizer=slicer:

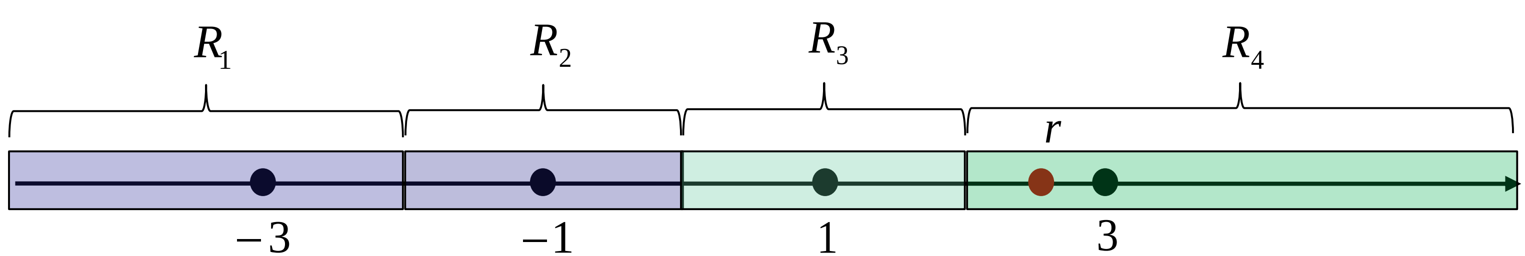


Defined by :

Decision regions $R_1 = \{r : \hat{a} = a_1\}, R_2 = \{r : \hat{a} = a_2\}, \cdots R_M$

(disjoint, cover $\mathbb{R}$)

Thresholds



Derive A Detector to Minimize Pr[Err]

As a function of the decision regions $\{R_1, R_2, \ldots R_M\}$, for any detector:

$$\Pr[\text{correct}] = \int_{-\infty}^{\infty} \Pr[\text{correct} \mid R = r] f(r) dr$$

$$= \int_{R_1} \Pr[\text{correct} \mid r] f(r) dr + \int_{R_2} \Pr[\text{correct} \mid r] f(r) dr + \int_{R_3} \Pr[\text{correct} \mid r] f(r) dr + \cdots$$

$$= \int_{R_1} \Pr[a_1 \mid r] f(r) dr + \int_{R_2} \Pr[a_2 \mid r] f(r) dr + \int_{R_3} \Pr[a_3 \mid r] f(r) dr + \cdots$$

$\Pr[a_2 \mid r]$	$\Pr[a_1 \mid r]$	$\Pr[a_1 \mid r]$
$\Pr[a_3 \mid r]$	$\Pr[a_3 \mid r]$	$\Pr[a_2 \mid r]$
$\Pr[a_4 \mid r]$	$\Pr[a_4 \mid r]$	$\Pr[a_4 \mid r]$

Step through each $r \in \mathbb{R}$ and assign to a decision region:

Which assignment maximizes Pr[correct]?

Assigning r to R_i contributes $P(a_i \mid r) f(r) dr$ to total

$\Rightarrow$ MAP contributes the most! $\Rightarrow$ MAP minimizes Pr[error].

Two Probabilistic Detectors

Notation:

$$P_A(a) = \Pr[A = a] = a \; priori \; \text{probabilit y that } A = a$$

$$P_{A|R}(a \mid r) = \Pr[A = a \mid R = r] = a \; posteriori \; \text{probabilit y that } A = a$$

Related by Bayes rule :

$$P_{A|R}(a \mid r) = \frac{f_{R|A}(r \mid a) P_A(a)}{f_R(r)}$$

Two probabilistic detectors:

1. The maximum a posteriori (MAP) detector:

$$\begin{aligned} \hat{a}_{\mathrm{MAP}} &= \arg\max \{ P_{A|R}(a \mid r) \} \\ &= \arg\max \{ f_{R|A}(r \mid a) P_A(a) \} \end{aligned}$$

2. The maximum likelihood (ML) detector:

$$\hat{a}_{\mathrm{ML}} = \arg\max \{ f_{R|A}(r \mid a) \}$$

MAP vs. ML

MAP

minimizes probability of error

exploits (and thus requires) knowledge of *a priori* probabilities

ML

equivalent to MAP when all inputs are equally likely

"assumes" all inputs are equally likely

notation: $f_{R|A}(r|a)$ with r fixed is called the "likelihood" of a

The likelihood is especially simple for *additive* noise:

If $R = A + N$ then $f_{R|A}(r \mid a) = f_N(r - a)$

MAP Example

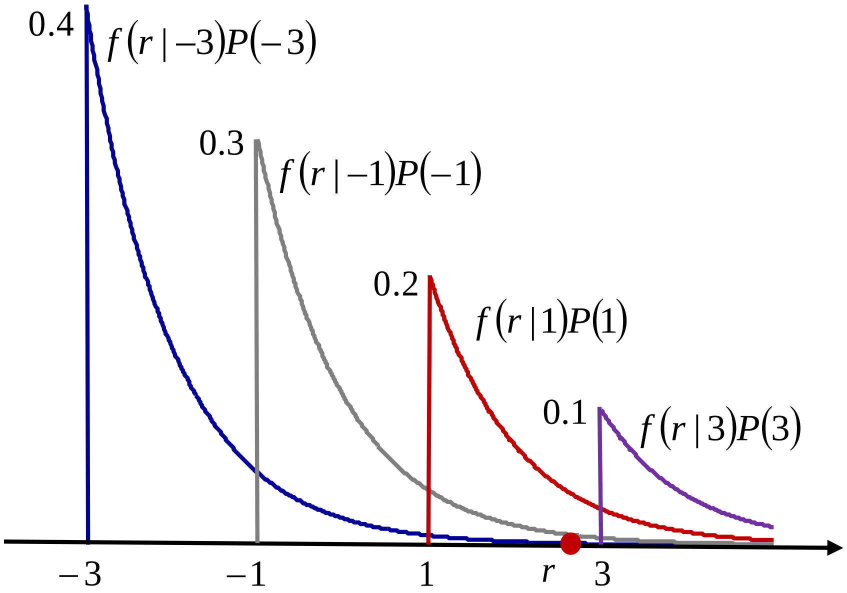
Example:

$A=\{-3,-1,1,3\}.$

$N \sim Exp(1), \quad f(N)=\exp(-N)$

$P_A(-3)=0.4, P_A(-1)=0.3, P_A(1)=0.2, P_A(3)=0.1$

$r=2.9$



The MAP detector maximizes

$$f_{R|A}(r|a)P_A(a)=f_N(r-a)P_A(a)=e^{-(r-a)}u(r-a)P_A(a)$$

Same Example, Different A Priori

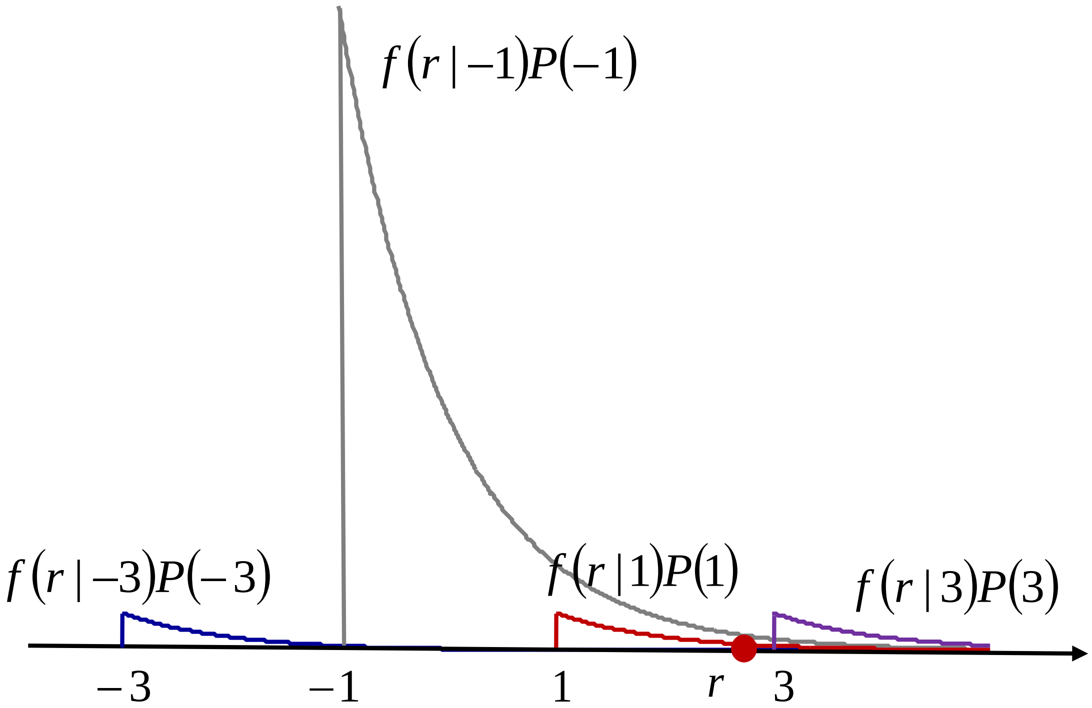
Example:

$\boldsymbol{A} = \{-3,-1,1,3\}.$

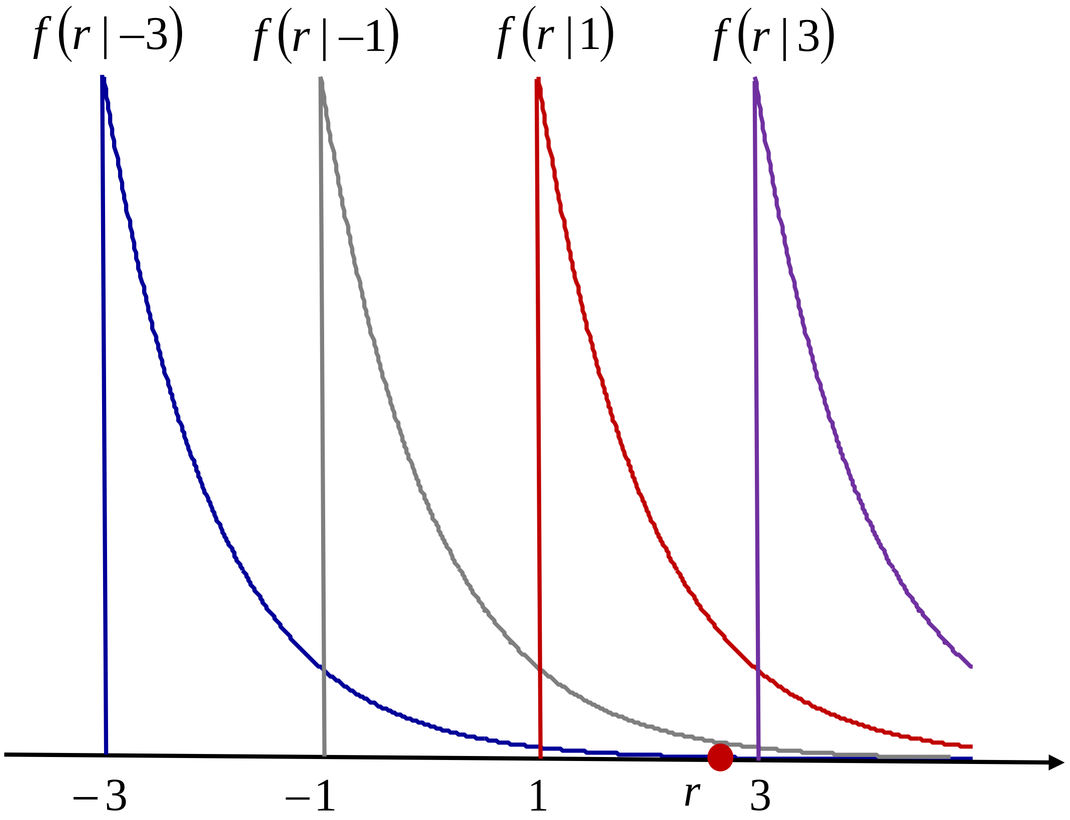
$N \sim Exp(1), \quad f(N) = \exp(-N)$

$P_A(-3) = 0.1, P_A(-1) = 0.7, P_A(1) = 0.1, P_A(3) = 0.1$

$r = 2.9$



Maximum Likelihood



Three Different Answers

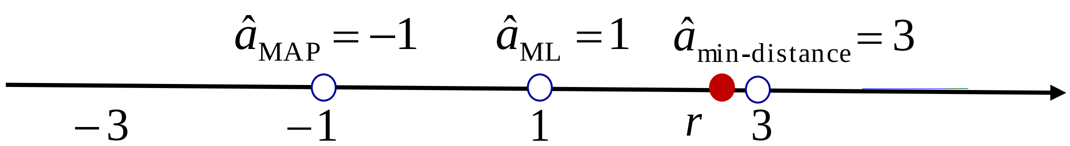
Example:

$\boldsymbol{A} = \{-3,-1,1,3\}.$

$N \sim Exp(1), \quad f(N) = \exp(-N)$

$P_A(-3) = 0.1, P_A(-1) = 0.7, P_A(1) = 0.1, P_A(3) = 0.1$

$r = 2.9$



ML vs. MD

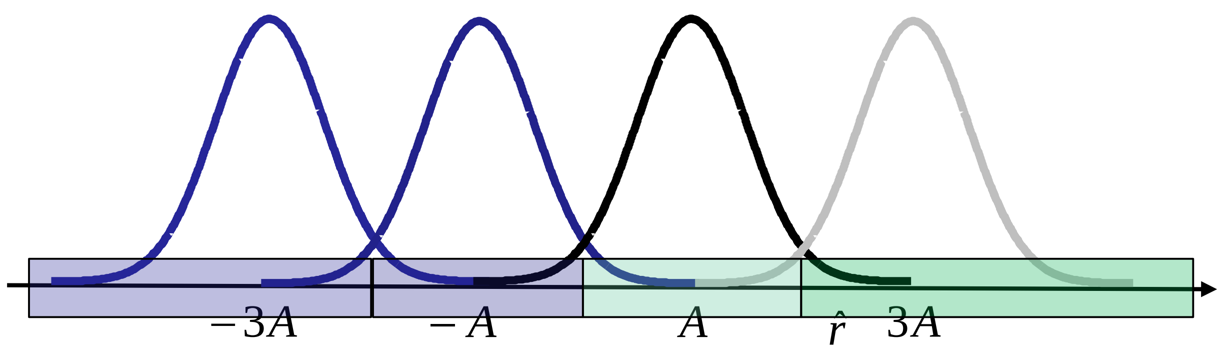
Minimum Distance Detection $\hat{s}_k = \arg\min_{s_k} (\hat{r} - s_k)^2$

Maximum Likelihood Detection $\hat{s}_k = \arg\max_{s_k} P(\hat{r} \mid s_k)$

$$P(\hat{r} \mid s_k) = \frac{1}{\sqrt{2\pi N_0}} \exp\left(-\frac{(\hat{r} - s_k)^2}{2N_0} \right)$$

Maximum Likelihood Detection $\hat{s}_k = \arg\max_{s_k} P(\hat{r} \mid s_k) = \arg\min_{s_k} (\hat{r} - s_k)^2$

Assume the symbols are equally likely



Roadmap

- Probabilistic Detectors
- Probability of Error Analysis
  - BASK
  - MASK
  - OOK
  - MPSK
  - General Signal Constellation

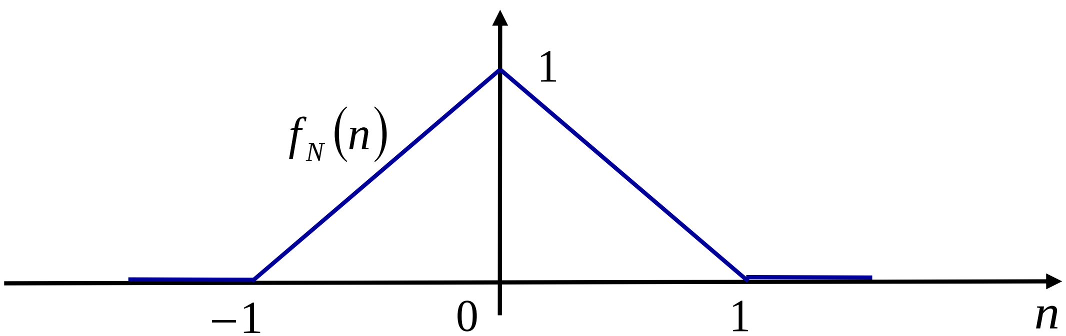
Triangular Noise Example

$R = A + N$, where

$A \in \{0,1\}$

$P_A(0) = 0.2,\ \ P_A(+1) = 0.8$

$N \sim \text{Unif}\,(-0.5, 0.5) + \text{Unif}\,(-0.5, 0.5)$

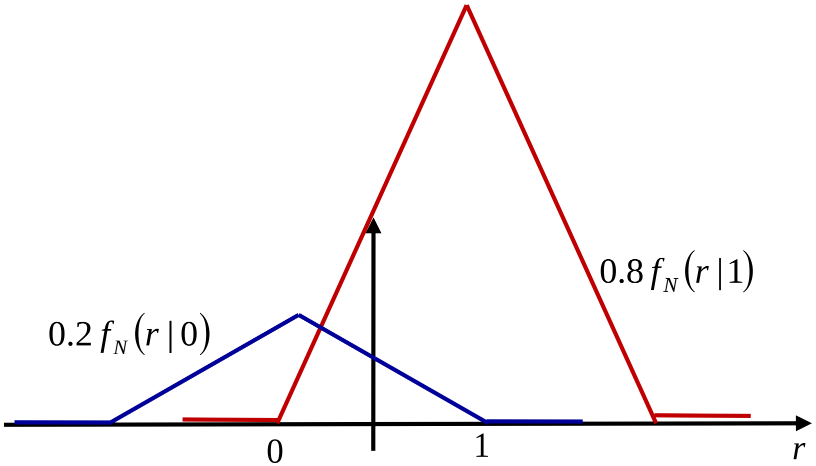


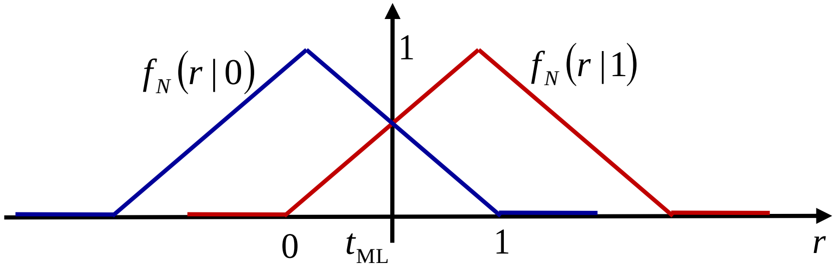
(a) Compare the ML, MAP, and minimum-distance detectors.

(b) Find the probability of error for the ML detector.

(c) Find the probability of error for the MAP detector.

MAP vs. ML





$$\text{threshold} = t_{\mathrm{ML}} = 1/2 \Rightarrow \hat{a}_{\mathrm{ML}} = 1_{\{r > 1/2\}}$$

Curves intersect when $0.2 - 0.2r = 0.8r$

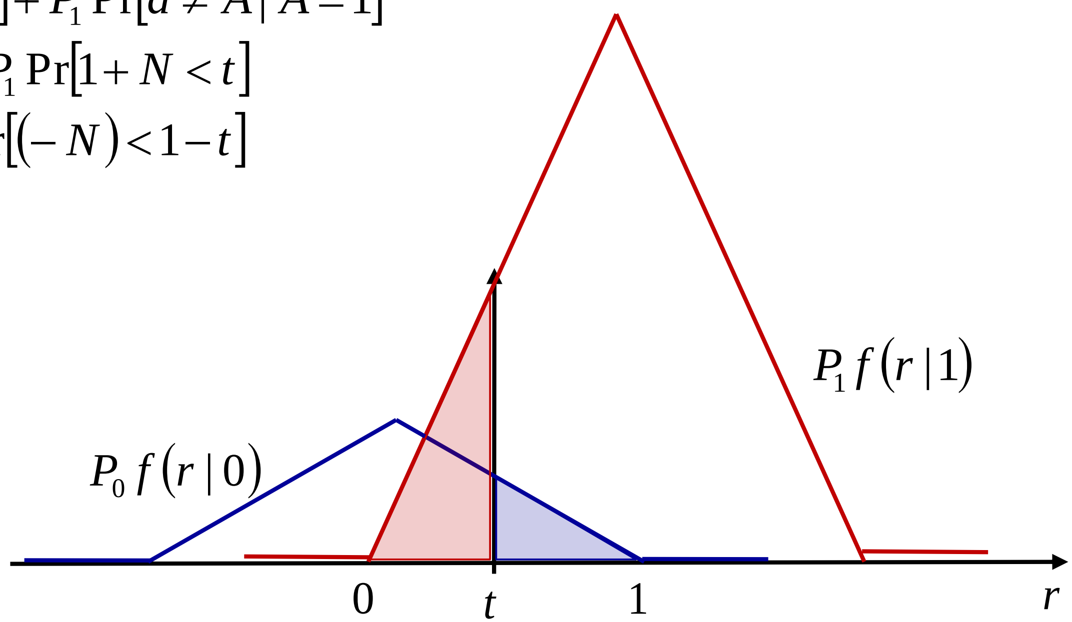
$$\Rightarrow t_{\mathrm{MAP}} = 1/5 = P_0$$

$$\Rightarrow \hat{a}_{\mathrm{ML}} = 1_{\{r > 1/5\}}$$

General Expressions of the Error Probability

For any values of $P_0 = \Pr[A=0], P_1 = \Pr[A=1]$, and threshold t:

$$\begin{aligned}\Pr[\text{error}] &= \Pr[\hat{a} \neq A] \\ &= P_0 \Pr[\hat{a} \neq A \mid A=0] + P_1 \Pr[\hat{a} \neq A \mid A=1] \\ &= P_0 \Pr[0+N>t] + P_1 \Pr[1+N<t] \\ &= P_0 \Pr[N>t] + P_1 \Pr[(-N)<1-t]\end{aligned}$$



$$\begin{aligned}\Pr[\text{error}] &= \frac{1}{2}(1-t)^2 P_0 + \frac{1}{2} t^2 P_1 \\ &= \frac{1}{2}(t-P_0)^2 + \frac{1}{2} P_0 P_1\end{aligned}$$

Special Cases

$$\text{General expression}: \Pr[\text{error}] = \frac{1}{2}(t - P_0)^2 + \frac{1}{2}P_0P_1$$

ML :

$$t_{\text{ML}} = \tfrac{1}{2} \Rightarrow \Pr[\text{error}] = \frac{1}{2}0.09 + \frac{1}{2}0.16 = 0.125$$

MAP :

$$t_{\text{MAP}} = P_0 \Rightarrow \Pr[\text{error}] = \frac{1}{2}P_0P_1 = 0.08 \le 0.125$$

What threshold minimizes $\Pr[\text{error}]$?

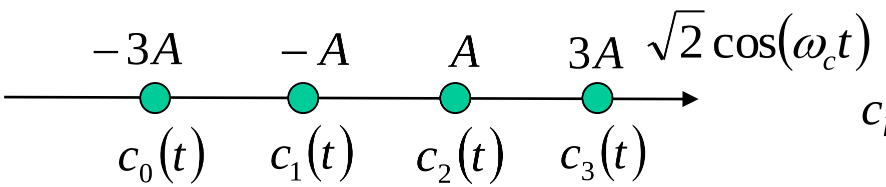
$$\frac{d}{dt}\Pr[\text{error}] = 0 \Rightarrow t_{\text{opt}} = P_0 = t_{\text{MAP}}$$

MASK

$$c_k(t) = A(2a_k - M + 1)\sqrt{2}\cos(\omega_c t) \qquad a_k \in \{0, \cdots, M-1\}$$

Let $s_k = A(2a_k - M + 1)$ $\quad c_k(t) = s_k\sqrt{2}\cos(\omega_c t)$ $\quad s_k \in \{-AM + A, \cdots, AM - 3A, AM - A\}$

One-dimensional space

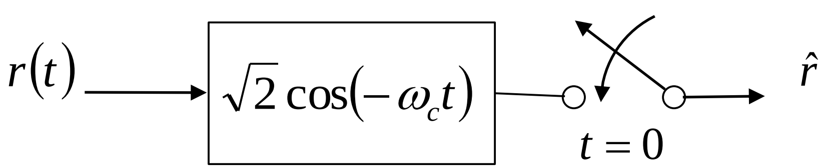


$$c_k(t) \to A(2k - M + 1) \qquad k \in \{0, \cdots, M-1\}$$

Assume white Gaussian noise, the received signal is

$$r(t) = s_k\sqrt{2}\cos(\omega_c t) + n(t) \qquad n(t)\text{ is Gaussian} \qquad R_n(\tau) = E[n(t)n(t-\tau)] = N_0\delta(\tau)$$

Project to signal space

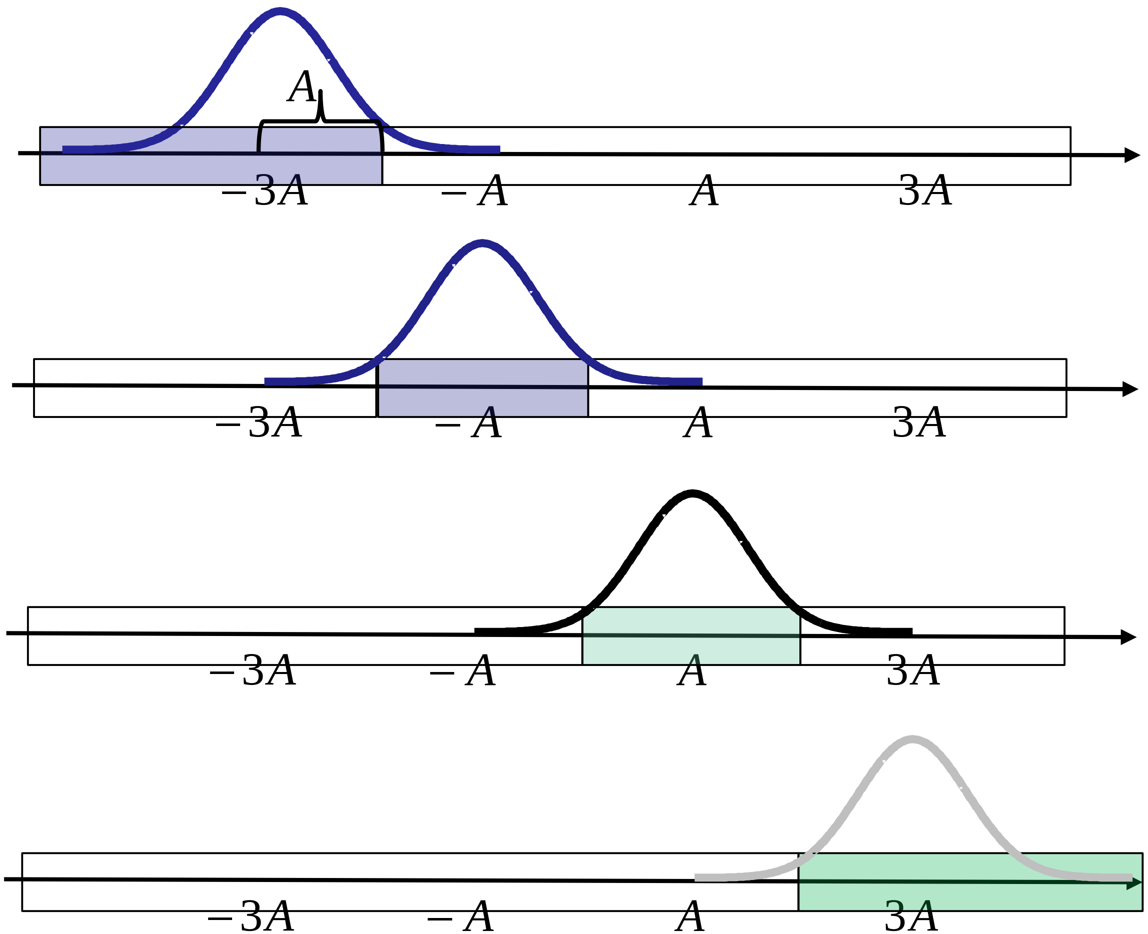


$$\hat{r} = \int_{-\infty}^{\infty} r(t)\sqrt{2}\cos(\omega_c t)dt = s_k + \int_{-\infty}^{\infty} n(t)\sqrt{2}\cos(\omega_c t)dt = s_k + n \qquad n \sim N(0, N_0)$$

Error Probability Analysis

$$P(\text{error}) = \frac{1}{M}\sum_k P\left(\hat{s}_k \neq s_k \middle| s_k\right)$$

M terms



$$\frac{1}{M}P\left(\hat{r}-(-3) > A \middle| s_k = -3A\right)$$
$$= \frac{1}{M}P(n > A) = \frac{1}{M}Q\left(\frac{A}{\sqrt{N_0}}\right)$$

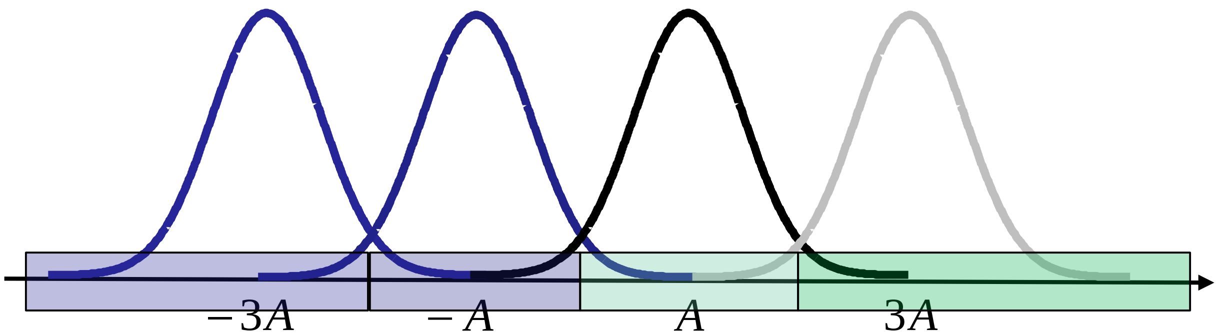
$$\frac{1}{M}P\left(\left|\hat{r}-(-1)\right| > A \middle| s_k = -A\right)$$
$$= \frac{1}{M}P(|n| > A) = \frac{2}{M}Q\left(\frac{A}{\sqrt{N_0}}\right)$$

$$\frac{1}{M}P\left(\left|\hat{r}-1\right| > A \middle| s_k = A\right)$$
$$= \frac{1}{M}P(|n| > A) = \frac{2}{M}Q\left(\frac{A}{\sqrt{N_0}}\right)$$

$$\frac{1}{M}P\left(\hat{r}-3 < -A \middle| s_k = 3A\right)$$
$$= \frac{1}{M}P(n < -A) = \frac{1}{M}Q\left(\frac{A}{\sqrt{N_0}}\right)$$

Power-Error Tradeoff

$$P(\text{error}) = \frac{1}{M}\sum_k P(\hat{s}_k \neq s_k | s_k) = \frac{2}{M} Q\left(\frac{A}{\sqrt{N_0}}\right) + \frac{2(M-2)}{M} Q\left(\frac{A}{\sqrt{N_0}}\right)$$



$$M = 2$$

$$P(\text{error}) = Q\left(\frac{A}{\sqrt{N_0}}\right)$$

Expected Energy per Symbol

$$E\left[\|s_k\|^2\right] = \frac{A^2}{M}\sum_{k=0}^{M-1}(2k - M + 1)^2 = A^2(M-1)^2 - 4A^2\frac{M-1}{M}\sum_{k=0}^{M-1}k + \frac{4}{M}A^2\sum_{k=0}^{M-1}k^2$$

$$= A^2(M-1)^2 - 4A^2\frac{M-1}{M}\frac{1}{2}(M-1)M + \frac{4}{M}A^2\frac{1}{6}(M-1)M(2M-1)$$

$$= A^2(M-1)^2 - 2A^2(M-1)^2 + \frac{2}{3}A^2(M-1)(2M-1) = \frac{1}{3}A^2(M^2-1)$$

$$M = 2$$

$$E\left[\|s_k\|^2\right] = A^2$$

BPSK: Power-Error Tradeoff

$$M = 2 \qquad P(\text{error}) = Q\left(\frac{A}{\sqrt{N_0}}\right) \qquad E\left[\|s_k\|^2\right] = A^2$$

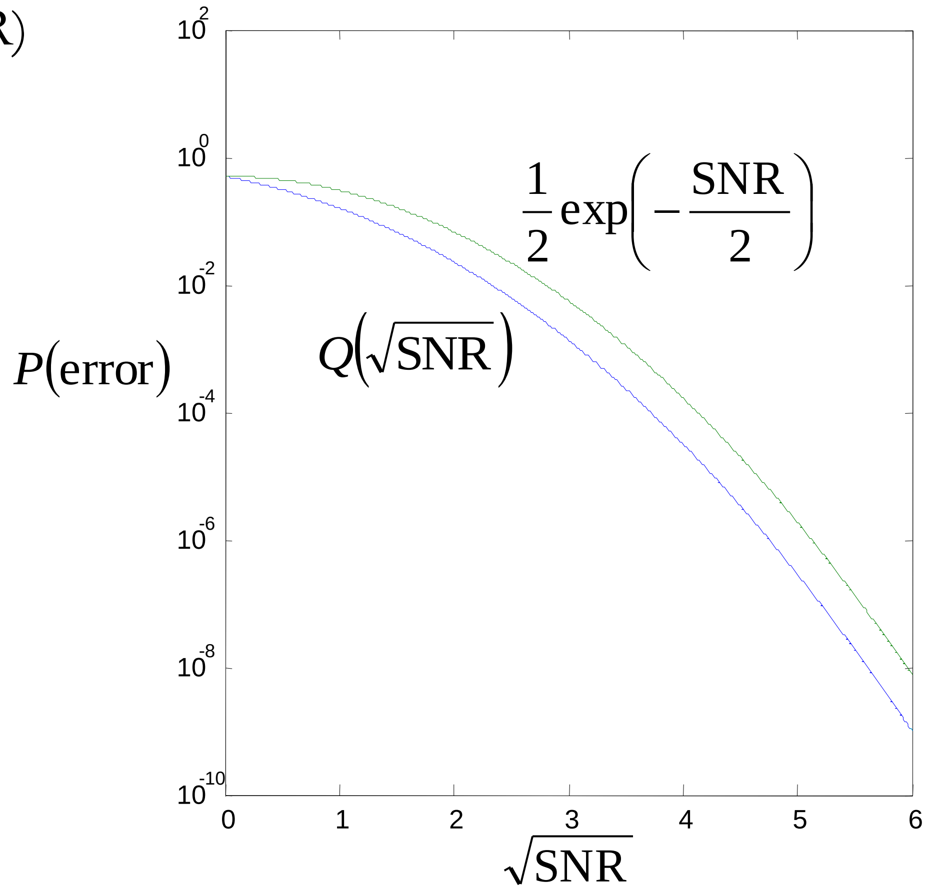
Define Signal-to-Noise-Ratio (SNR)

$$\text{SNR} = \frac{E\left[\|s_k\|^2\right]}{N_0} = \frac{A^2}{N_0}$$

$$P(\text{error}) = Q\left(\sqrt{\text{SNR}}\right)$$

$$n \sim N(0, N_0)$$

$$E\left[n^2\right] = N_0$$

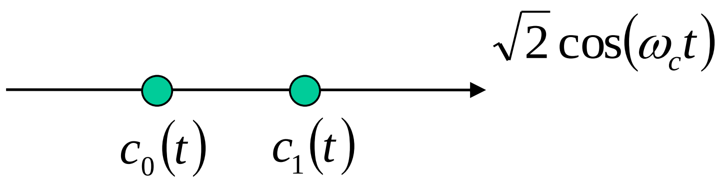


OOK

$$c_k(t) = A a_k \sqrt{2}\cos(\omega_c t) \qquad a_k \in \{0,1\}$$

Let $\quad s_k = A a_k \qquad\qquad c_k(t) = s_k \sqrt{2}\cos(\omega_c t)$

One-dimensional space



$$c_k(t) \to Ak \qquad k \in \{0,1\}$$

Assume white Gaussian noise, the received signal is

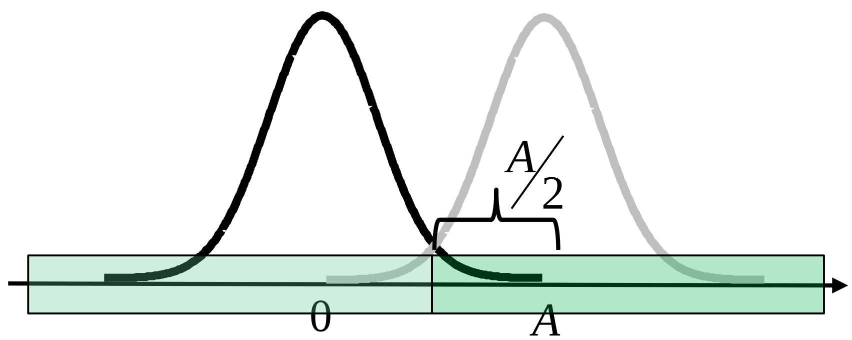
$$r(t) = s_k \sqrt{2}\cos(\omega_c t) + n(t) \qquad n(t) \text{ is Gaussian} \qquad R_n(\tau) = E[n(t)n(t-\tau)] = N_0 \delta(\tau)$$

$$\hat{r} = \int_{-\infty}^{\infty} r(t)\sqrt{2}\cos(\omega_c t)dt = s_k + \int_{-\infty}^{\infty} n(t)\sqrt{2}\cos(\omega_c t)dt = s_k + n \qquad n \sim N(0, N_0)$$

Minimum Distance Detection $\qquad \hat{s}_k = \arg\min_{s_k} (\hat{r} - s_k)^2$

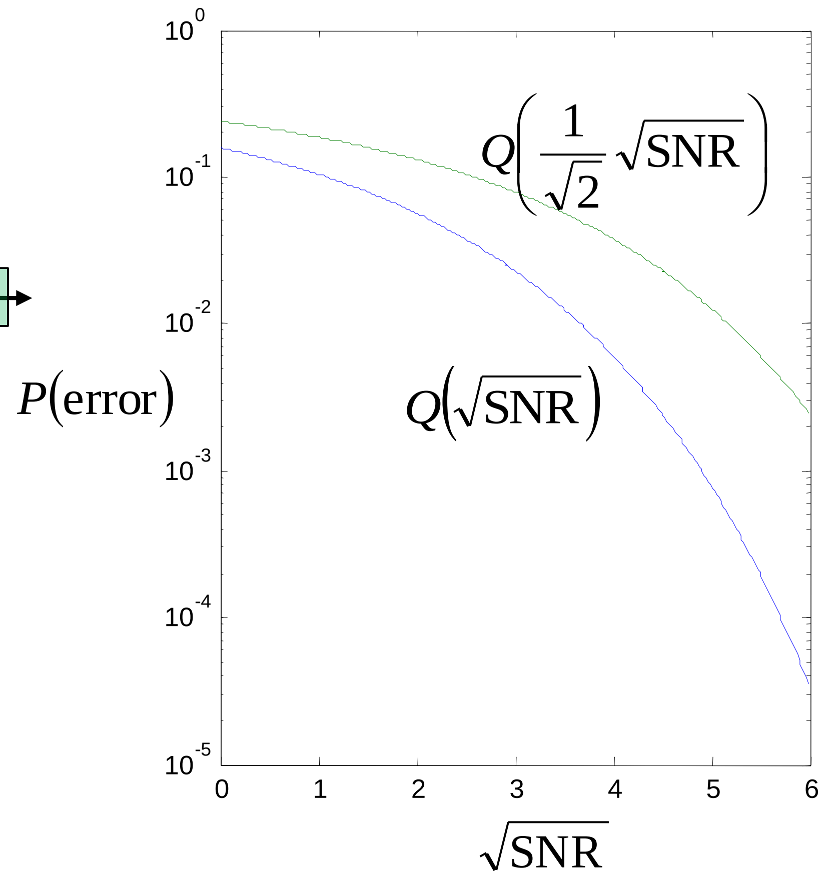
OOK: Power-Error Tradeoff

$$P(\text{error}) = Q\left(\frac{A}{2\sqrt{N_0}}\right) \qquad E\left[\|s_k\|^2\right] = \frac{1}{2}A^2 \qquad \text{SNR} = \frac{E\left[\|s_k\|^2\right]}{N_0} = \frac{A^2}{2N_0}$$



$$P(\text{error}) = Q\left(\frac{1}{\sqrt{2}}\sqrt{\text{SNR}}\right)$$

$$n \sim N(0, N_0)$$



MASK: Power-Error Tradeoff

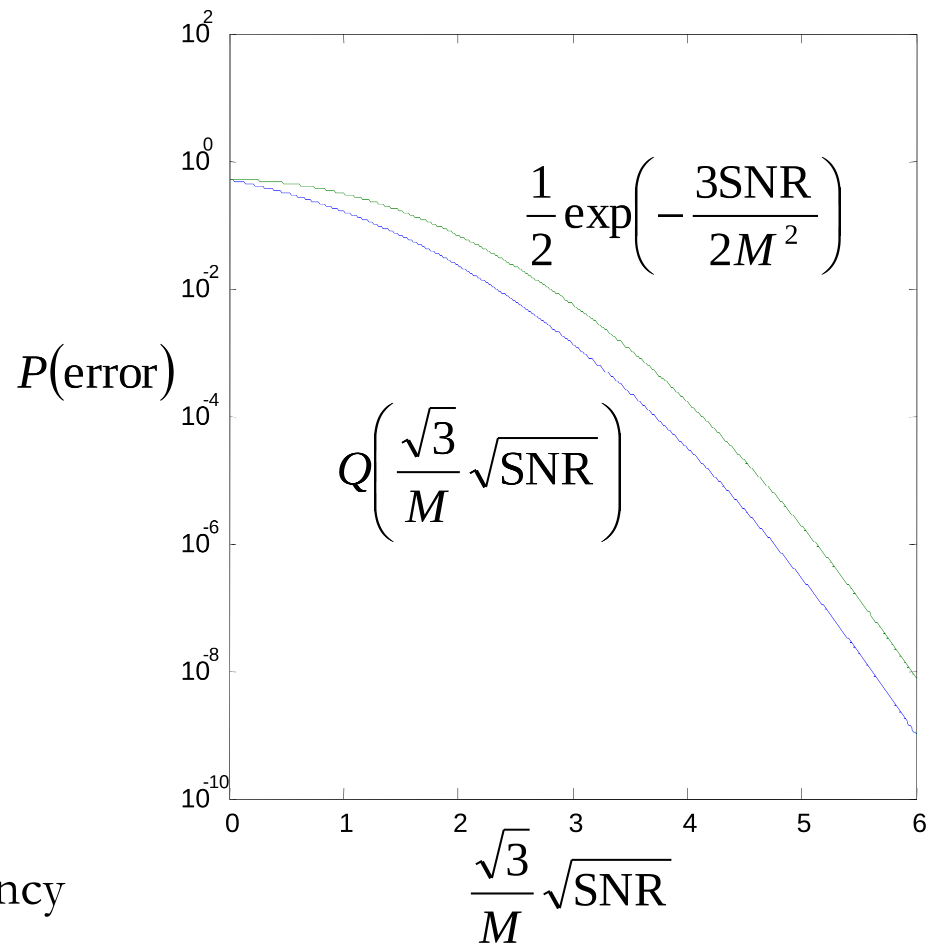
$$P(\text{error}) = \frac{1}{M}\sum_k P(\hat{s}_k \neq s_k | s_k) = \frac{2}{M}Q\left(\frac{A}{\sqrt{N_0}}\right) + \frac{2(M-2)}{M}Q\left(\frac{A}{\sqrt{N_0}}\right)$$

$$\approx \frac{2(M-1)}{M}Q\left(\frac{A}{\sqrt{N_0}}\right) \approx Q\left(\frac{A}{\sqrt{N_0}}\right)$$

$$E\left[\|s_k\|^2\right] = \frac{1}{3}A^2(M^2-1)$$

$$\text{SNR} = \frac{E\left[\|s_k\|^2\right]}{N_0} \approx \frac{M^2}{3}\frac{A^2}{N_0}$$

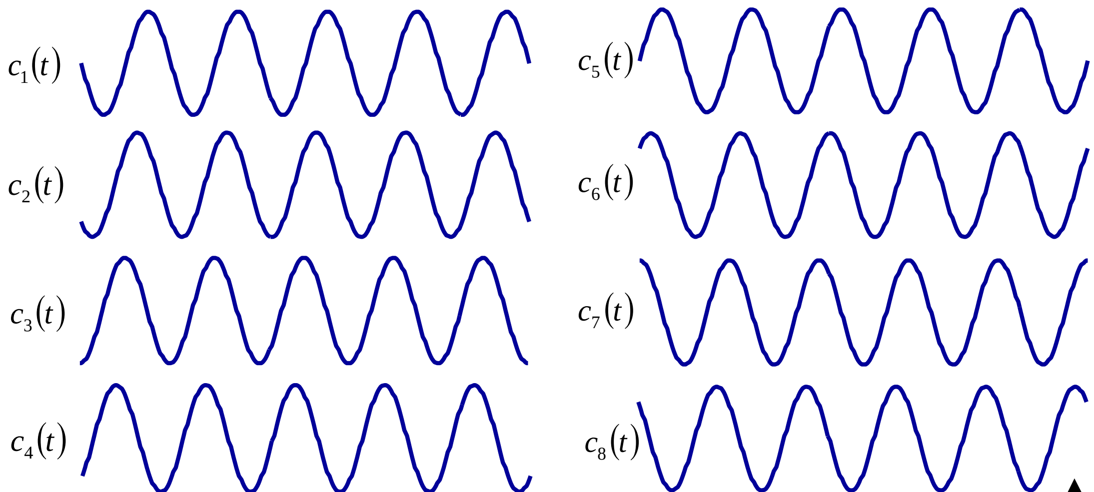
$$P(\text{error}) = Q\left(\frac{\sqrt{3}}{M}\sqrt{\text{SNR}}\right)$$



higher rate, lower power efficiency

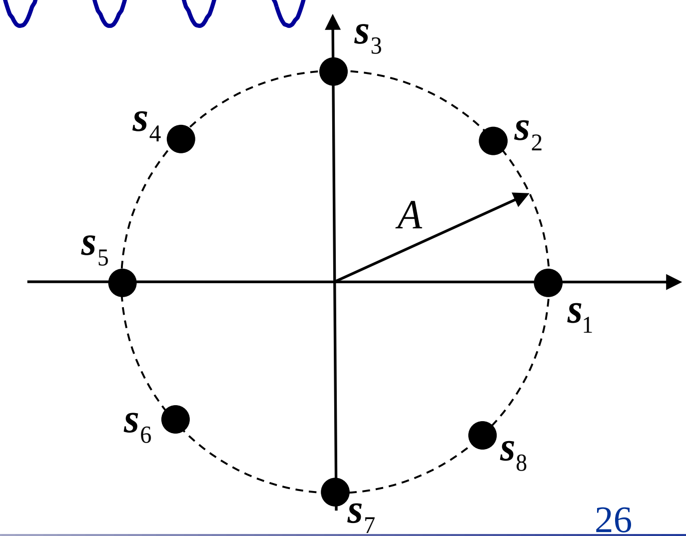
MPSK

$$c_k(t) = A\sqrt{2}\cos\left(\omega_c t + (k-1)\frac{2\pi}{M}\right) = A\cos\left((k-1)\frac{2\pi}{M}\right)\sqrt{2}\cos(\omega_c t) - A\sin\left((k-1)\frac{2\pi}{M}\right)\sqrt{2}\sin(\omega_c t)$$



$$\phi_1(t) = \sqrt{2}\cos(\omega_c t),\ \phi_2(t) = -\sqrt{2}\sin(\omega_c t)$$

$$\mathbf{s}_k = \begin{bmatrix} s_{k1} \\ s_{k2} \end{bmatrix} = A\begin{bmatrix} \cos\left(\frac{(k-1)2\pi}{M}\right) \\ \sin\left(\frac{(k-1)2\pi}{M}\right) \end{bmatrix}$$



MPSK Detection

Assume white Gaussian noise, the received signal is

$$r(t) = c_k(t) + n(t)$$

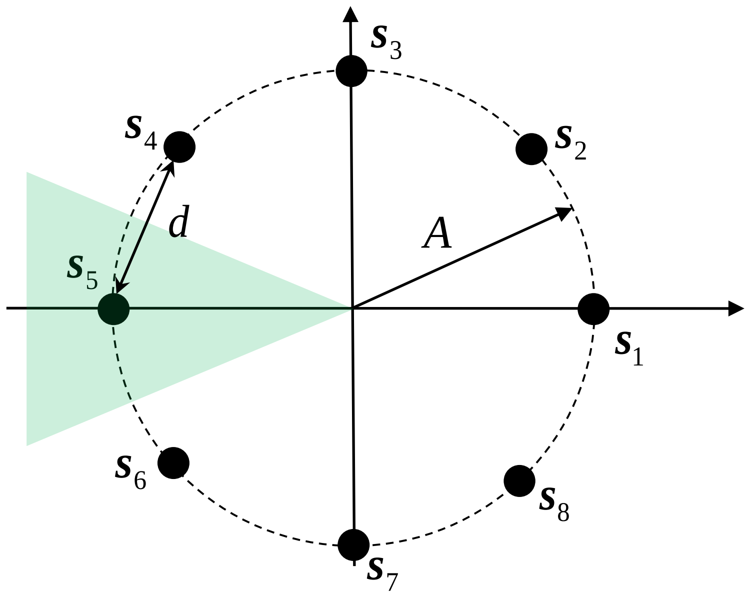
$n(t)$ is Gaussian

$$R_n(\tau) = E[n(t)n(t-\tau)] = N_0\delta(\tau)$$

Project $r(t)$ onto the signal space, we get

$$\boldsymbol{r} = \boldsymbol{s}_k + \boldsymbol{n} \qquad \boldsymbol{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

where n_1 and n_2 are i.i.d. $\sim N(0, N_0)$



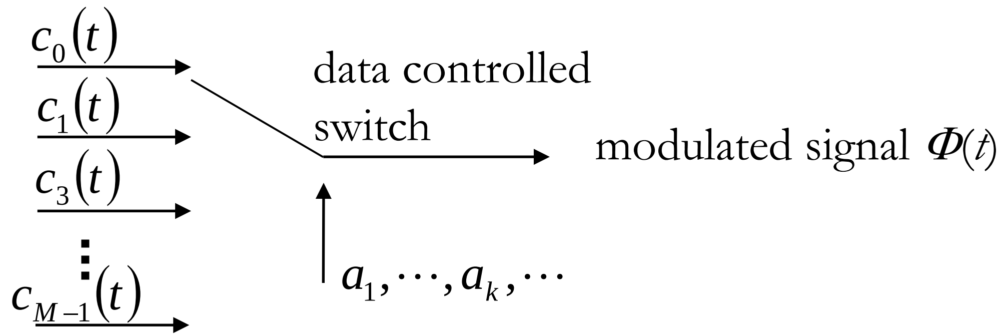
Maximum Likelihood Detection

$$\hat{\boldsymbol{s}}_k = \arg\min_{\boldsymbol{s}_k} \|\boldsymbol{r} - \boldsymbol{s}_k\|^2$$

Error probability $\quad P(\text{error}) \approx Q\left(\dfrac{d}{2\sqrt{N_0}}\right) = Q\left(0.3827\dfrac{A}{\sqrt{N_0}}\right)$ $\qquad \text{SNR} = A^2/N_0$

General Constellation

Basic Diagram (for M-ary signal):



Let the modulated waveform be $c_k(t) = s_{k1}\phi_1(t) + s_{k2}\phi_2(t) + \cdots + s_{kK}\phi_K(t)$

Signal points

$$\mathbf{s}_k = \begin{bmatrix} s_{k1} \\ \vdots \\ s_{kK} \end{bmatrix}$$

Minimum Distance

$$d_{\min} = \min_{\mathbf{s}_i, \mathbf{s}_j} \left\| \mathbf{s}_i - \mathbf{s}_j \right\|$$

Expected Symbol Energy

$$E\left[\left\| \mathbf{s}_k \right\|^2\right] = \frac{1}{M} \sum_{k=1}^{M} \left\| \mathbf{s}_k \right\|^2$$

Assume white Gaussian noise, $r(t) = c_k(t) + n(t)$ $\quad R_n(\tau) = E[n(t)n(t-\tau)] = N_0\delta(\tau)$

Error Probability $P(\text{error}) \approx Q\left(\dfrac{d_{\min}}{2\sqrt{N_0}}\right)$