Principle of Communications

Review of Signals and Systems

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Outline

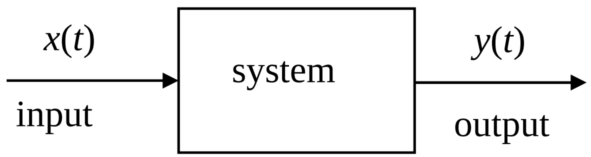
- Signals
- Signal space representation
- Fourier transform
- Signal transmission through a linear system

Roadmap

- Signals: definition, classification, operations
- Signal space representation
- Fourier transform
- Signal transmission through a linear system

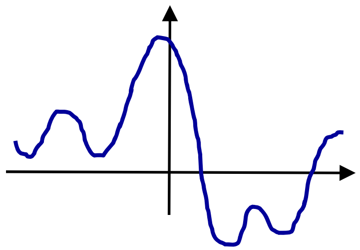
Signal and System

- Signal: The time history of some quantity, usually a voltage or current.
- System: A combination of devices and networks chosen to perform a desired function (on signals).

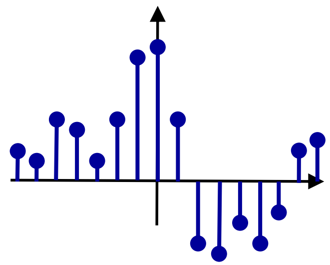


- Signal Classification
  - Deterministic vs Random
  - Periodic vs Aperiodic
  - Complex vs Real
  - Continuous-time vs Discrete-time
  - Analog vs Digital

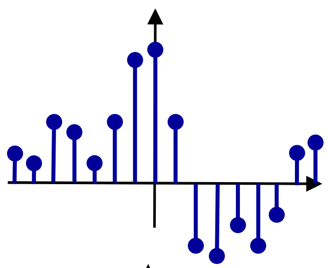
Analog and Digital Signals

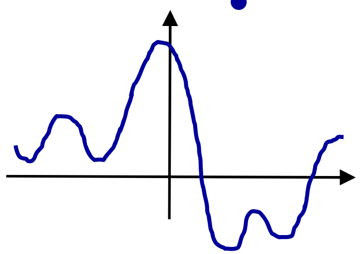


continuous-time signal



discrete-time signal





analog signal

0,1,1,1,0,1,0,0

5,4,2,-4,-3,5,1,-2

digital signal

Signal Examples

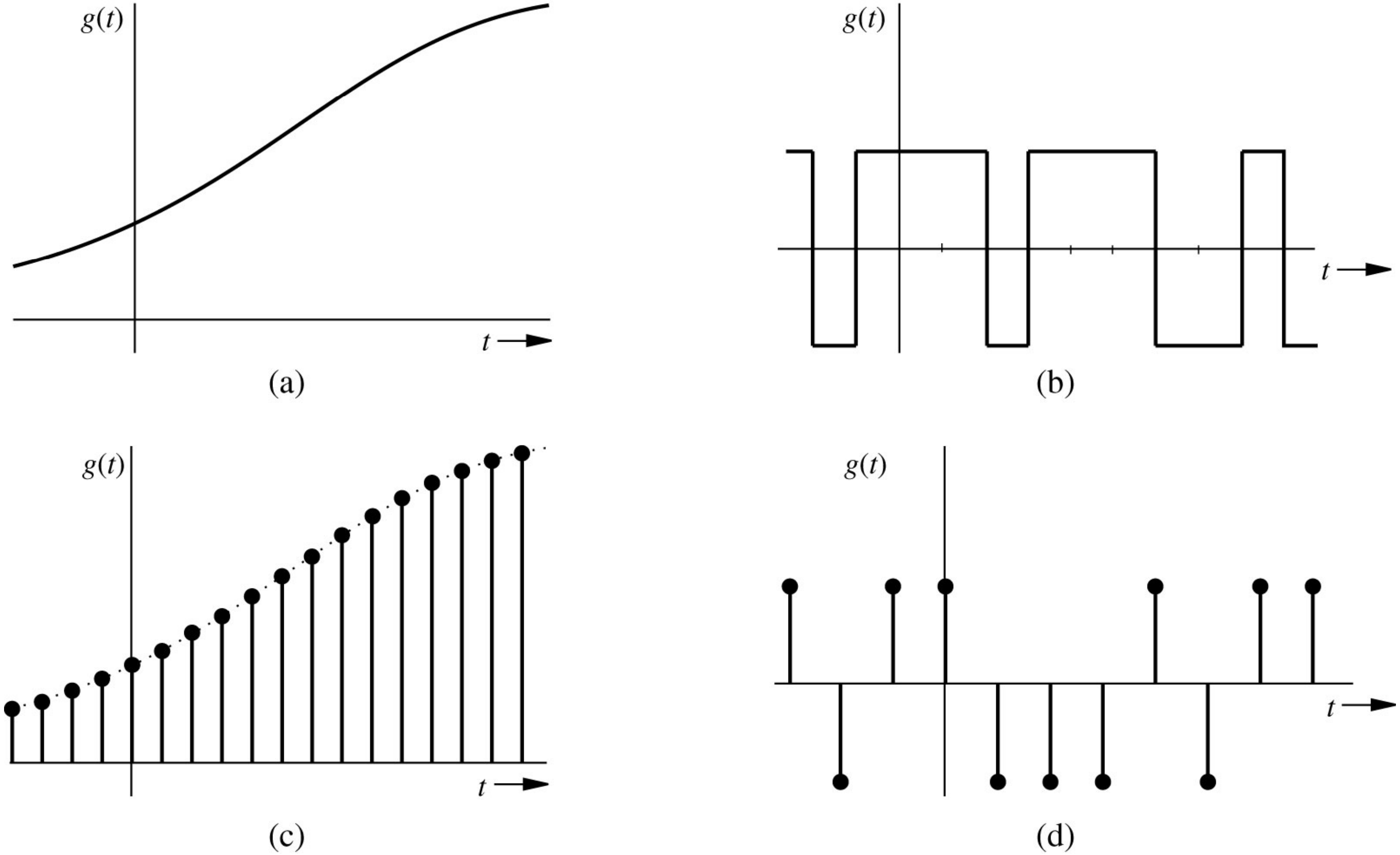
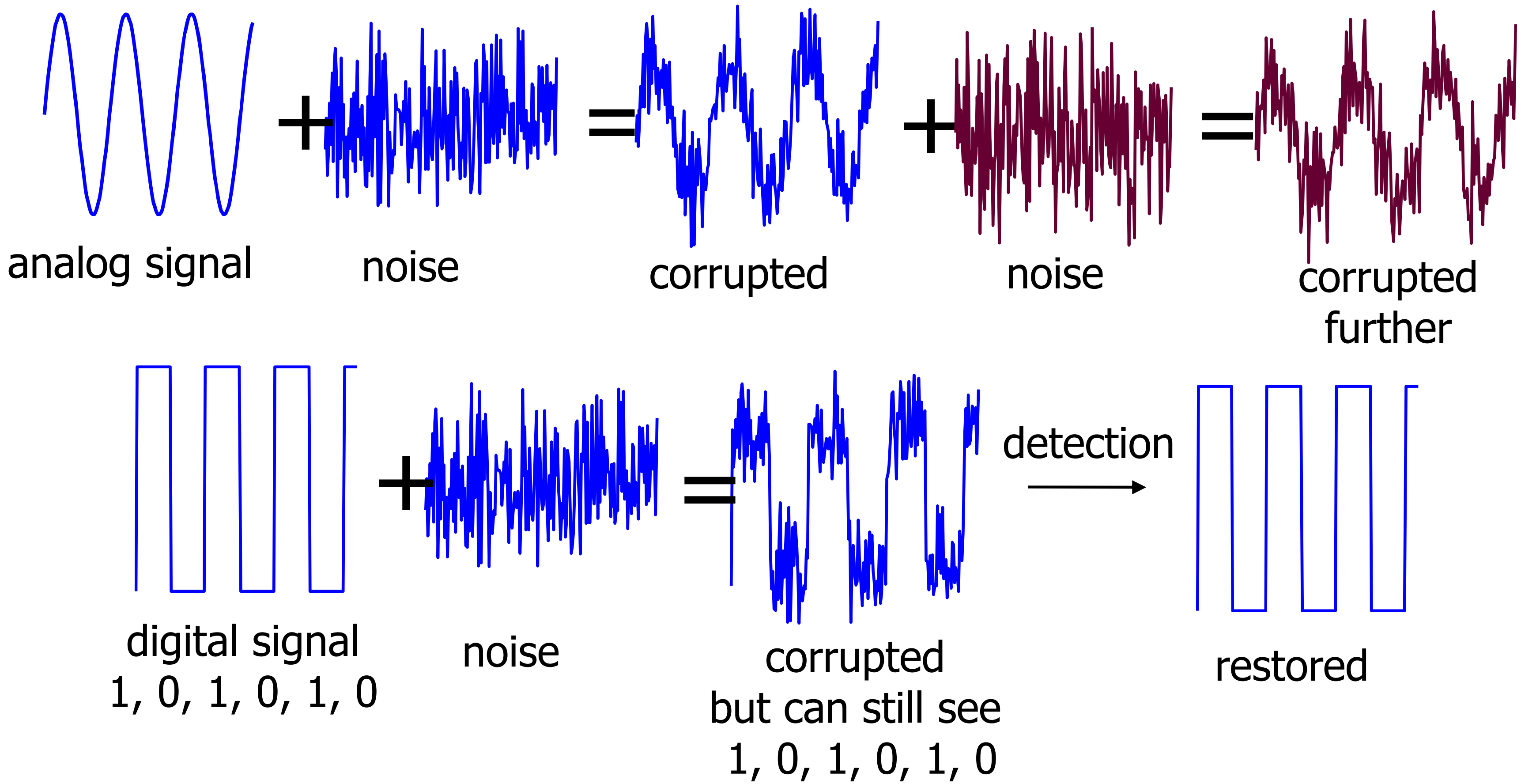


Figure 2.4 Examples of signals: (a) analog and continuous time; (b) digital and continuous time; (c) analog and discrete time; (d) digital and discrete time.

Why Digital?

Digital signal communication is more robust to noise corruption.



It is possible to restore a digital signal from its corrupted version.

Another Example

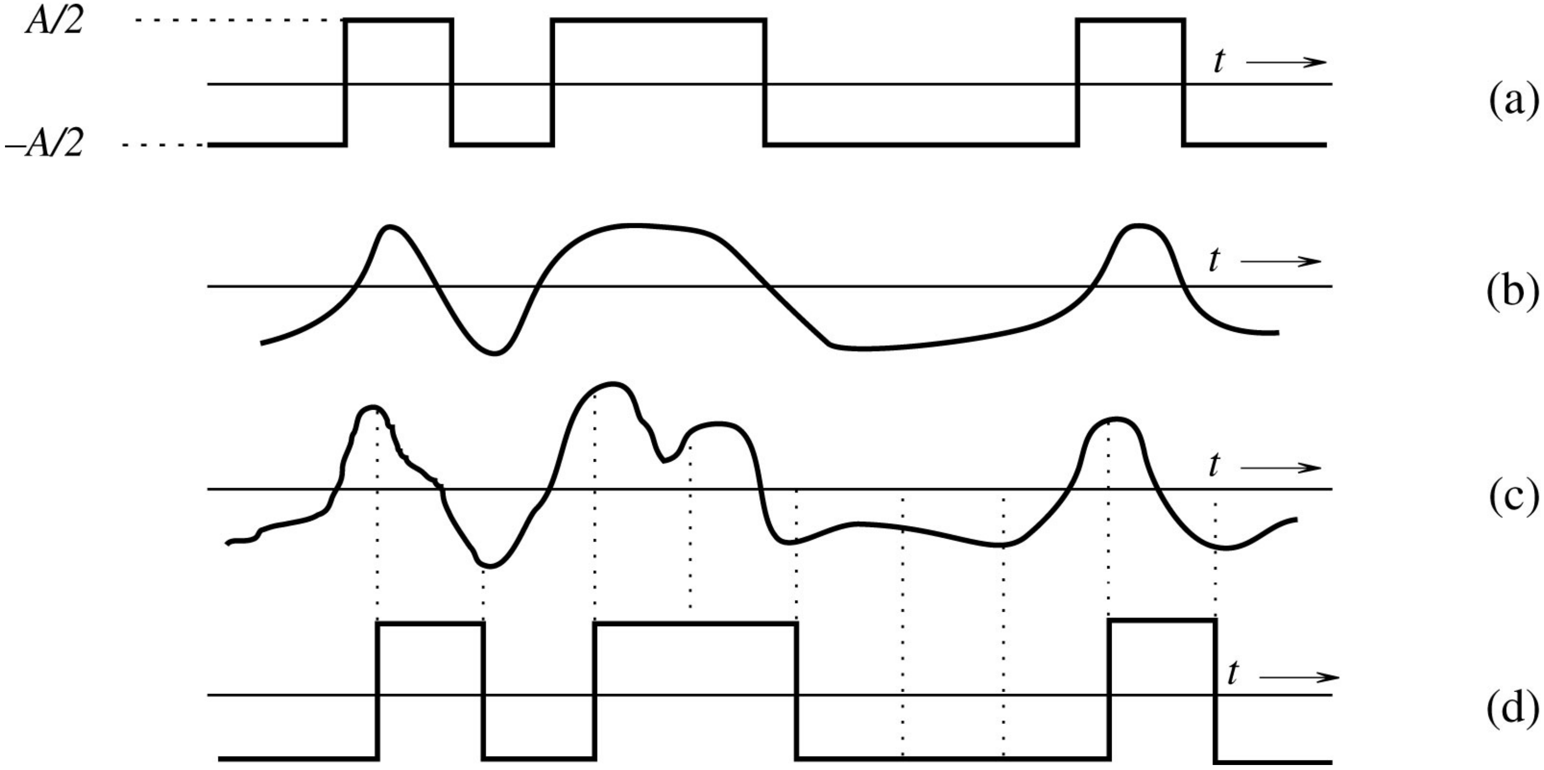


Figure 1.3 (a) Transmitted signal. (b) Received distorted signal (without noise). (c) Received distorted signal (with noise). (d) Regenerated signal (delayed).

Energy & Power

- Signal energy: $E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt$ (Joule)
  - Energy Signal: if $E_g < \infty$.

  Example : $g(t) = e^{-t}$ energy signal, $g(t) = \sin(t)$ not an energy signal
- Signal power: $P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt$ (Watt)
  - Power Signal: if $P_g < \infty$.

  Example : $g(t) = \sin(t)$ power signal, $g(t) = 5t$ not a power signal
- Sometime uses $10\log_{10}E_g$ (dBJ) and $10\log_{10}P_g$ (dBW)

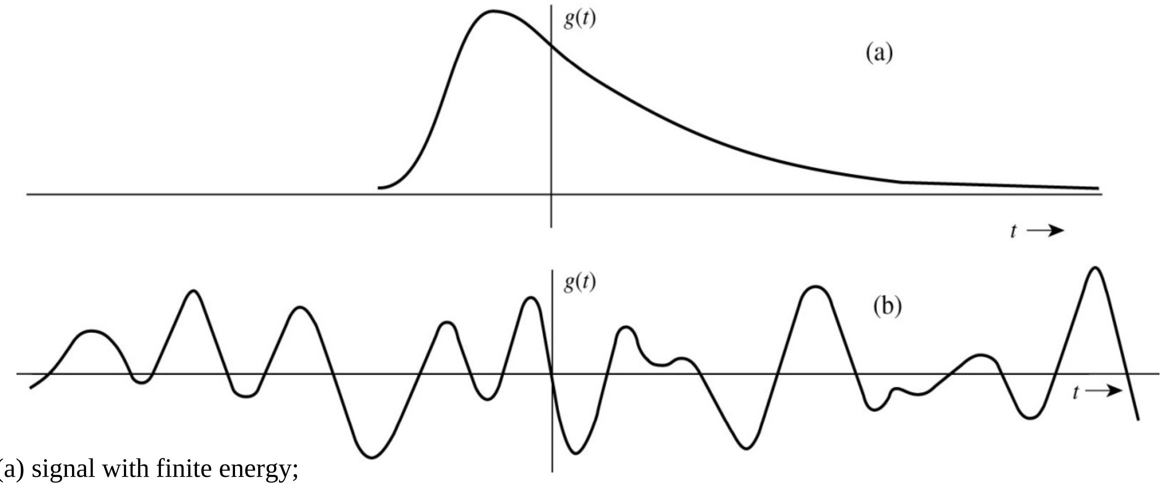


Figure 2.1 Examples of signals: (a) signal with finite energy; (b) signal with finite power.

Periodic vs. Aperiodic

- Periodic Signal:
  - Exists a T_0 such that $g(t) = g(t+T_0)$ for all t.
  - The minimum T_0 is defined as the fundamental period. $f_0 = 1/T_0$

Example : $g(t) = \sin(t)$ periodic, $T_0 = 2\pi$

Example : $g(t) = \sin(t) + \cos(0.5t)$, periodic, $T_0 = 4\pi$

Example : $g(t) = \begin{cases} \sin(t) & t > 0 \\ 0 & t \le 0 \end{cases}$, aperiodic

Example : $g(t) = \sin(t) + \sin(\pi t)$ aperiodic

NOTE: Summation of periodic signals is not necessarily periodic!

Examples

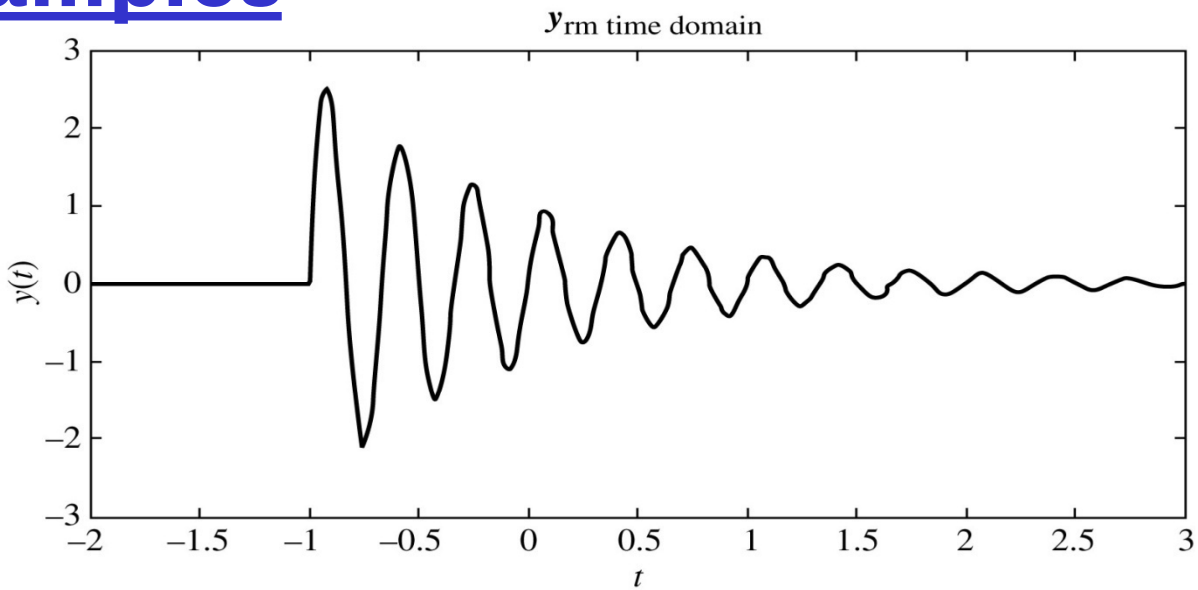


Figure 2.27 Graphing a signal.

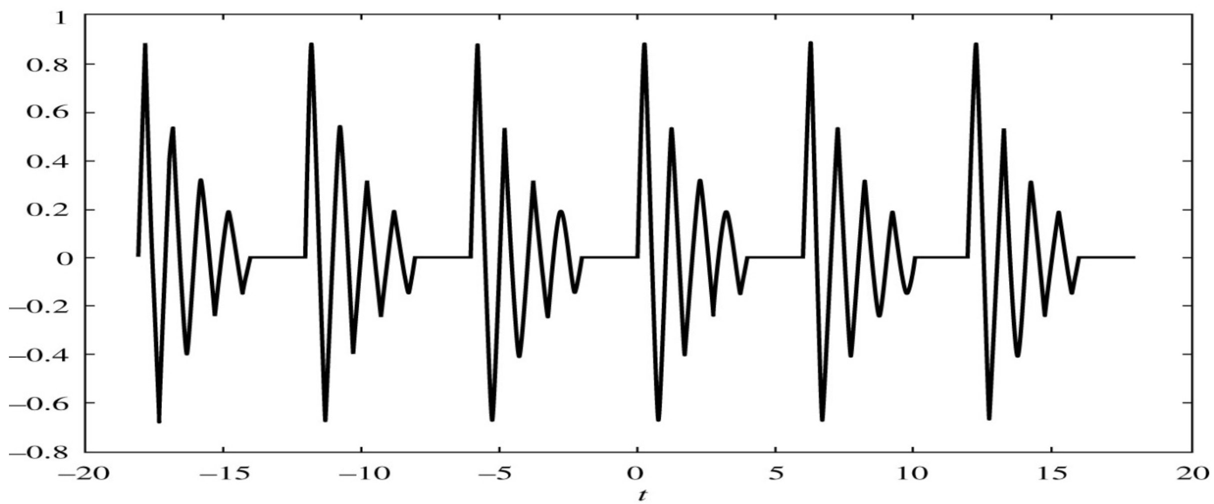


Figure 2.29 Generating a periodic signal.

Complex vs. Real

- A complex signal consists of two real signals

$$g(t) = x(t) + jy(t)$$

- Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$
- Amplitude and phase representation $g(t) = A(t)e^{j\theta(t)}$
- Transforms

$$A(t) = \sqrt{x^2(t) + y^2(t)} \qquad \theta(t) = \tan^{-1} y(t)/x(t)$$

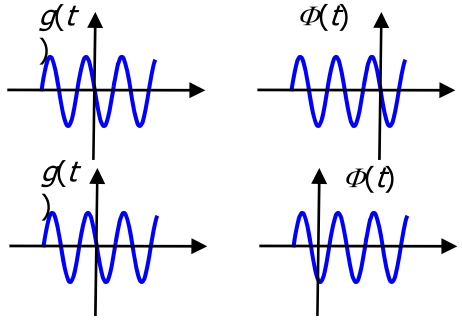
$$x(t) = A(t)\cos\theta(t) \qquad y(t) = A(t)\sin\theta(t)$$

Time-Domain Operations

- Shifting (phase change)

$$\phi(t) = g(t+T)$$ move $g(t)$ to the left

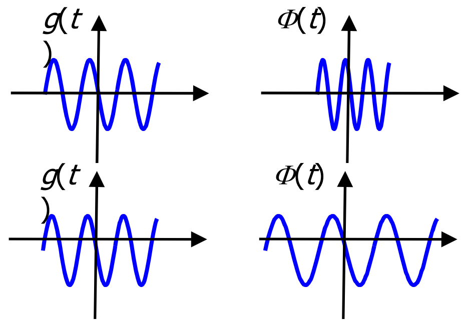
$$\phi(t) = g(t-T)$$ move $g(t)$ to the right



- Scaling (frequency change)

$$\phi(t) = g(at)$$ $|a|>1$, compress in time

$$\phi(t) = g(t/a)$$ $|a|>1$, extend in time



- Time Inversion (phase change)

$$\varphi(t) = -g(t)$$

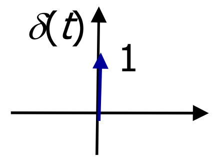
- General Time and Amplitude Transformation

$$\phi(t) = cg(at+b) + d$$

Impulse and Step Functions

- Unit Impulse Signal

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1$$



- Properties

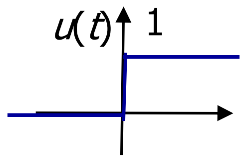
$$\phi(t)\delta(t) = \phi(0)\delta(t) \qquad \phi(t)\delta(t-T) = \phi(T)\delta(t-T)$$

$$\int_{-\infty}^{\infty} \phi(t)\delta(t-T)dt = \phi(T)\int_{-\infty}^{\infty} \delta(t-T)dt = \phi(T)$$

Sample at $t = T$

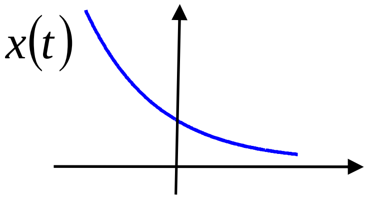
- Unit Step Signal

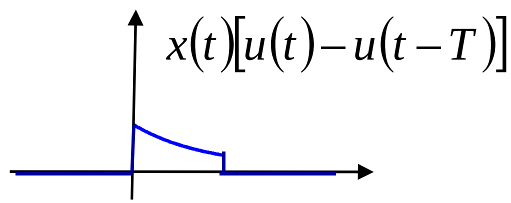
$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \qquad \delta(t) = \frac{du(t)}{dt}$$

  - Often used to form “windows”.





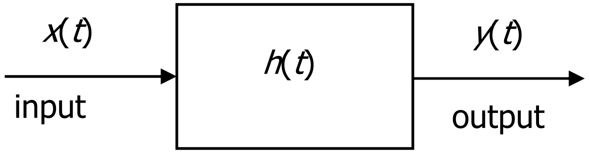
Multiplication, Correlation & Convolution

- Signal Multiplication: $\Phi(t)=x(t)c(t)$ Example: AM modulation $\Phi(t)=x(t)\cos(\omega_c t)$

  multiplication in time $\leftrightarrow$ convolution in frequency

- Signal Convolution: $y(t)=x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$

  Example: passing signal through a linear time-invariant system



$$h(t)=\delta(t)*h(t)=\int_{-\infty}^{\infty}\delta(\tau)h(t-\tau)d\tau$$

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$$

  convolution in time $\leftrightarrow$ multiplication in frequency

- Signal Correlation: $r_{xy}(\tau)=\int_{-\infty}^{\infty}x(t)y^*(t-\tau)dt$ $r(t)=x(t)*y^*(-t)$

  Example: autocorrelation $r_x(\tau)=\int_{-\infty}^{\infty}x(t)x^*(t-\tau)dt=x(\tau)*x^*(-\tau)$

  cross-correlation $r_{xy}(\tau)=\int_{-\infty}^{\infty}x(t)y^*(t-\tau)dt=x(\tau)*y^*(-\tau)$

  correlation in time $\leftrightarrow$ conjugate multiplication in frequency

Roadmap

- Signals
- Signal Space Representation
  - Vectors and vector space
  - Signal space representation
- Fourier transform
- Signal transmission through a linear system

2D Plane, Points and Vectors

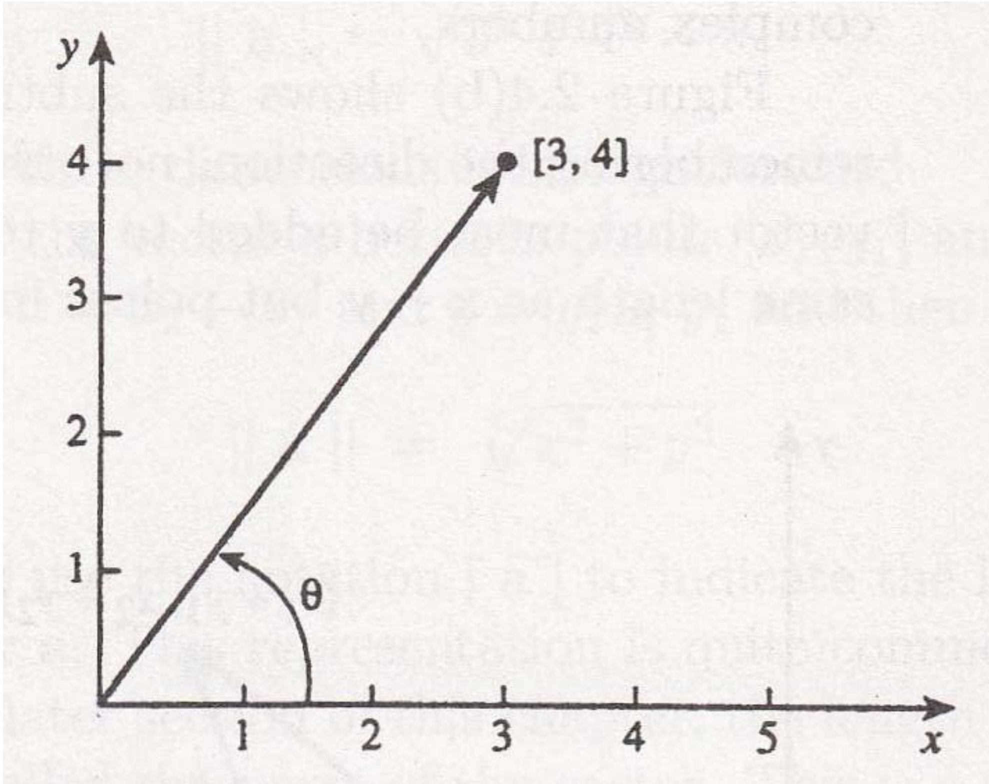
- A 2D plane is a 2-dimensional plane
  - e.g., the complex plane

- A point on a 2D plane is represented by its x and y coordinates
  - e.g., point $(3, 4)$

- A vector on a 2D plane connects any two points with a direction
  - e.g., vector $a = [3, 4]$ is the vector from the origin to the point $(3, 4)$

Length of A Vector

- The length of a 2D vector $\boldsymbol{x} = [x_1, x_2]$ follows from the Pythagorean theorem:
  - $\|\boldsymbol{x}\| = \sqrt{x_1^2 + x_2^2}$



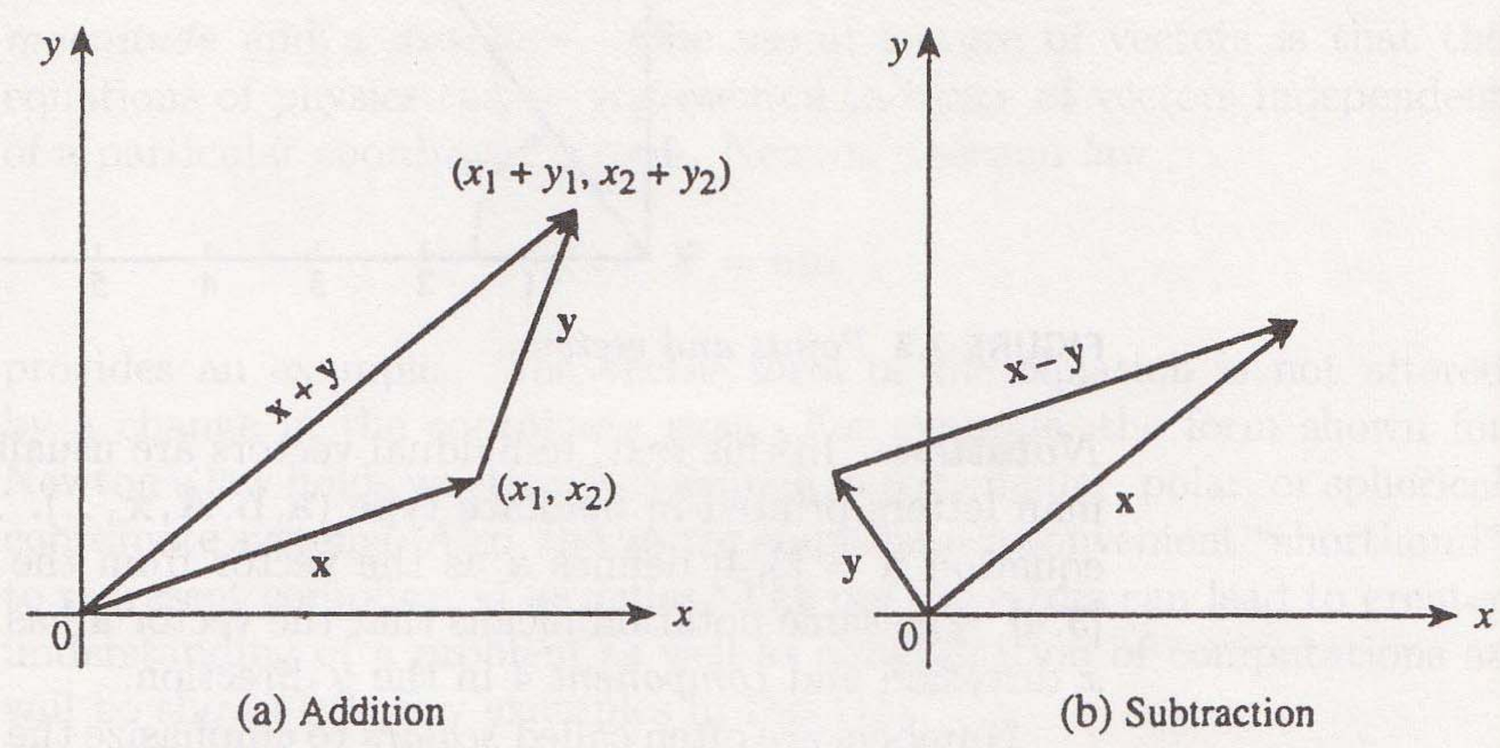
  - e.g., length of vector $[3, 4]$ is 5.
- In higher dimension, $\|\boldsymbol{x}\|$ is called the norm

Unit Vectors

- Unit vectors are vectors of norm 1
- For a vector $\boldsymbol{x}$, the unit vector $\boldsymbol{n}$ in the direction of $\boldsymbol{x}$ is $\boldsymbol{n} = \boldsymbol{x}/\|\boldsymbol{x}\|$
  - e.g., the unit vector in the direction of vector $[3, 4]$ is $[0.6, 0.8]$
  - The operation $\boldsymbol{x}/\|\boldsymbol{x}\|$ is called normalization.
- Standard unit vectors
  - In 2D: $[1, 0]$ and $[0, 1]$
  - In 3D: $\underset{\boldsymbol{i}}{[1, 0, 0]}$, $\underset{\boldsymbol{j}}{[0, 1, 0]}$, $\underset{\boldsymbol{k}}{[0, 0, 1]}$

Addition and Subtraction

- Let $\boldsymbol{x} = [x_1, x_2]$ and $\boldsymbol{y} = [y_1, y_2]$ be two arbitrary 2D vectors
  - Addition: $\boldsymbol{x} + \boldsymbol{y} = [x_1 + y_1, x_2 + y_2]$
  - Subtraction: $\boldsymbol{x} - \boldsymbol{y} = [x_1 - y_1, x_2 - y_2]$



Inner Product of Vectors

- Definition: $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$

- Let θ be the angle between the direction of $\boldsymbol{x}$ and the direction of $\boldsymbol{y}$, then

$$\boldsymbol{x} \cdot \boldsymbol{y} = \|\boldsymbol{x}\| \, \|\boldsymbol{y}\| \cos\theta$$

- Given vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, their relative angle can be computed as:

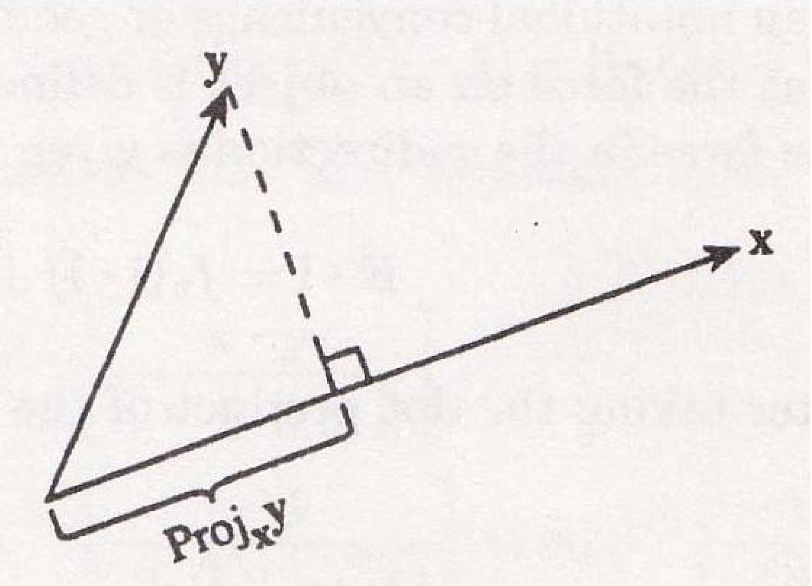
$$\cos\theta = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\| \, \|\boldsymbol{y}\|} = \frac{x_1 y_1 + x_2 y_2 + x_3 y_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}\sqrt{y_1^2 + y_2^2 + y_3^2}}$$

- Note: $\|\boldsymbol{x}\|^2 = \boldsymbol{x} \cdot \boldsymbol{x}$

Cauchy Schwartz inequality: $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \le \|\boldsymbol{x}\| \|\boldsymbol{y}\|$

Projection

- Projection of $\boldsymbol{y}$ along the direction of $\boldsymbol{x}$



  - Direction: $\boldsymbol{x}/\|\boldsymbol{x}\|$
  - Length: $\|\boldsymbol{y}\| \cos\theta$
  - Recall that $\cos\theta = (\boldsymbol{x}\cdot\boldsymbol{y})/(\|\boldsymbol{x}\|\,\|\boldsymbol{y}\|)$

The Projection of vector $\boldsymbol{y}$ along the direction of $\boldsymbol{x}$ is:

$$\mathrm{proj}_{\boldsymbol{x}}\boldsymbol{y} = \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{\|\boldsymbol{x}\|^2}\boldsymbol{x} \ .$$

Orthogonal Vectors

- Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two nonzero vectors

- If $\text{proj}_{\boldsymbol{x}}\boldsymbol{y} = 0$ (or equivalently $\text{proj}_{\boldsymbol{y}}\boldsymbol{x} = 0$), then we say $\boldsymbol{x}$ is orthogonal to $\boldsymbol{y}$, or $\boldsymbol{x} \perp \boldsymbol{y}$.

- Recall that

$$\text{proj}_{\boldsymbol{x}}\boldsymbol{y} = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\|^2}\boldsymbol{x}$$

  - $\text{proj}_{\boldsymbol{x}}\boldsymbol{y} = 0$ implies that $\boldsymbol{x} \cdot \boldsymbol{y} = \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$

Vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are said to be perpendicular ($\boldsymbol{x} \perp \boldsymbol{y}$) if and only if their inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x} \cdot \boldsymbol{y} = 0$.

Vectors in Higher Dimension

- $\boldsymbol{R}^n$: the set of all vectors with n real components.
  - e.g., $\boldsymbol{R}^2$ for 2D and $\boldsymbol{R}^3$ for 3D
- If $\boldsymbol{x}$ is one of the vectors in the set $\boldsymbol{R}^n$, we say $\boldsymbol{x} \in \boldsymbol{R}^n$
- Vector $\boldsymbol{x} \in \boldsymbol{R}^n$ can be specified as $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]$
- The quantity $\|\boldsymbol{x}\| = \sqrt{\sum_{k=1}^{n} x_k^2}$ is called the norm

Properties of Vectors

- Let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^n$ and $\alpha, \beta \in \boldsymbol{R}$
- Then $\alpha\boldsymbol{x} + \beta\boldsymbol{y} \in \boldsymbol{R}^n$

- Vector addition properties:
  - $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$
  - $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$
  - $\boldsymbol{x} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{x} = \boldsymbol{x}$
  - $\boldsymbol{x} + (-\boldsymbol{x}) = \boldsymbol{0}$
- Scalar multiplication properties:
  - $\alpha(\boldsymbol{x} + \boldsymbol{y}) = \alpha\boldsymbol{x} + \alpha\boldsymbol{y}$
  - $(\alpha + \beta)\boldsymbol{x} = \alpha\boldsymbol{x} + \beta\boldsymbol{x}$
  - $(\alpha\beta)\boldsymbol{x} = \alpha(\beta\boldsymbol{x})$
  - $\boldsymbol{x} = 1 \cdot \boldsymbol{x}$

Vector Space and Subspace

- The vector space is a set of vectors satisfying all properties of vectors.
  - e.g., $\boldsymbol{R}$, $\boldsymbol{R}^2$, $\boldsymbol{R}^3$, $\ldots$
- The subspace is a subset of $\boldsymbol{R}^n$ satisfying all properties of vectors.
  - $\mathbf{0} = [0, 0, \ldots, 0]$ is a subset of $\boldsymbol{R}^n$. It forms a subspace since $\mathbf{0} + \mathbf{0} = \mathbf{0}$, $\alpha \cdot \mathbf{0} = \mathbf{0}$ and $\mathbf{0} = 1 \cdot \mathbf{0}$.
  - $\mathbf{1} = [1, 1, \ldots, 1]$ is also a subset of $\boldsymbol{R}^n$. However, it does not form a subspace since $\mathbf{1} + \mathbf{1} = \mathbf{2} \notin \{\mathbf{1}\}$.

Conditions to check for the validity of a subspace $\mathcal{S}$. If $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$, then: 1) $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{S}$; 2) $\alpha \boldsymbol{x} \in \mathcal{S}$; 3) $\mathbf{0} \in \mathcal{S}$; and 4) $-\boldsymbol{x} \in \mathcal{S}$.

Linear Independence and Basis

- If $\boldsymbol{z} = \alpha\boldsymbol{x} + \beta\boldsymbol{y}$, then $\boldsymbol{z}$ is called a linear combination of $\boldsymbol{x}$ and $\boldsymbol{y}$.
- A set of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ is called linearly independent if none of them is a linear combination of others.
  - In other words, $\sum_{k=1}^{n} \alpha_k \boldsymbol{x}_k = \boldsymbol{0}$ can only happen if all α_ks are zeros.
- A set of n linearly independent vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ forms a basis for the vector space $\boldsymbol{R}^n$.
  - As a result, any $\boldsymbol{x} \in \boldsymbol{R}^n$ can be written as a linear combination of $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$.

Orthonormal Vectors

- A set of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ is called mutually orthogonal if any pair of them are orthogonal to each other ($\boldsymbol{u}_k \perp \boldsymbol{u}_l$, $\forall k \neq l$).
- A set of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ is called orthonormal if
  - they are mutually orthogonal; and
  - they all have unit norm
  - in other words,

$$\langle \boldsymbol{u}_k, \boldsymbol{u}_l \rangle = \delta_{kl}$$

where δ_{kl} is the Kronecker Delta function

$$\delta_{kl} = \begin{cases} 1, & \text{if } k = l, \\ 0, & \text{if } k \neq l. \end{cases}$$

- Although any set of n independent vectors consists of a basis for $\boldsymbol{R}^n$, it is more convenient to use n orthonormal vectors.

Questions

- Can you find an example of a set of orthogonal vectors that are not linearly independent?
- Can you find an example of a set of linearly independent vectors that are not mutually orthogonal?
- Suppose $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ are four vectors in $\mathbf{R}^3$.
  - Can these vectors be linearly independent?
  - Can these vectors be a basis for $\mathbf{R}^3$?

Orthonormal Basis for Vector Spaces

- Let $\boldsymbol{x}$ be an arbitrary vector in $\boldsymbol{R}^n$
- Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ be an orthonormal basis for $\boldsymbol{R}^n$
- Then

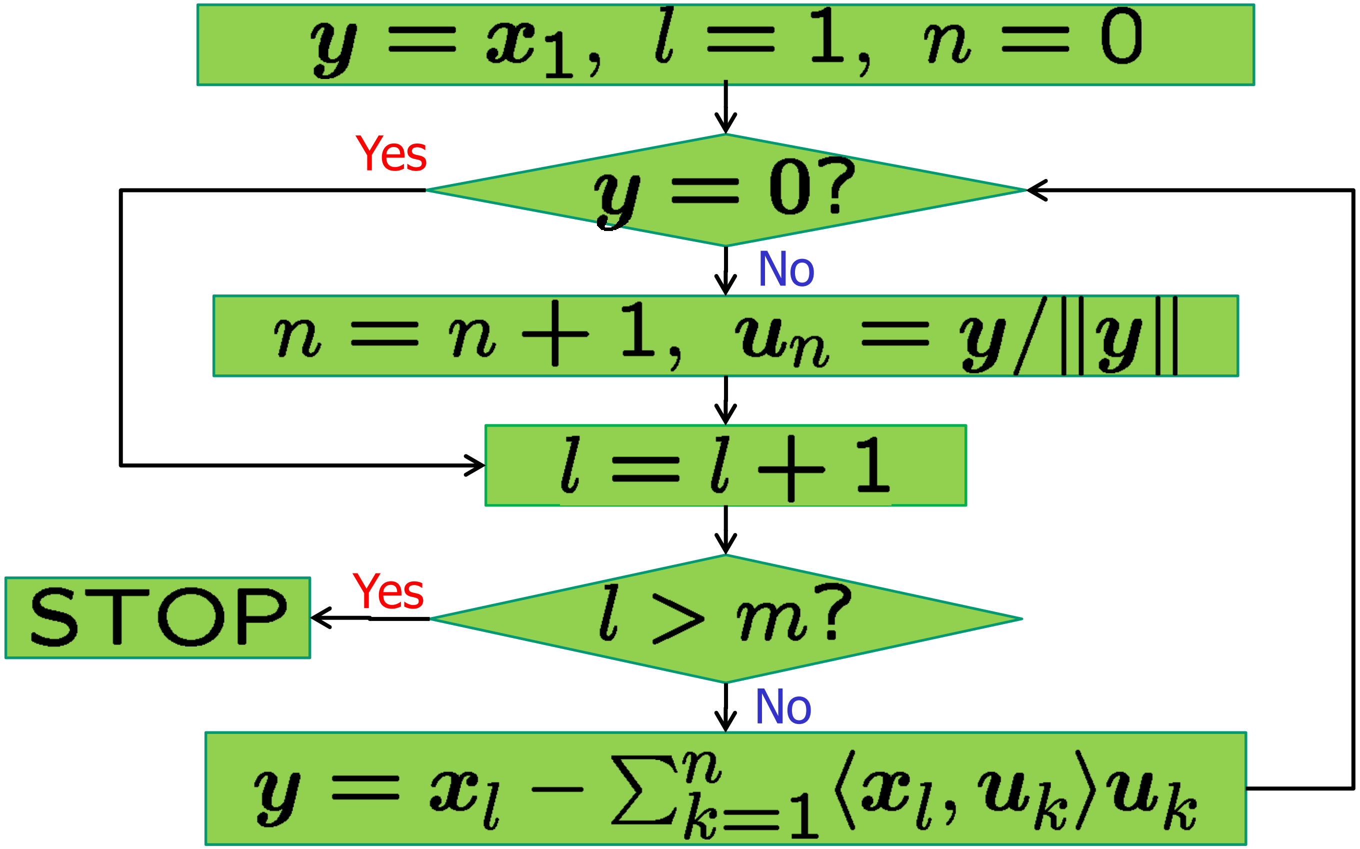
$$\boldsymbol{x} = \sum_{k=1}^{n} c_k \boldsymbol{u}_1, \quad \text{where } c_k = \langle \boldsymbol{x}, \boldsymbol{u}_k \rangle$$

- Example: In 3D vector space $\boldsymbol{R}^3$,
  - $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ is an orthonormal basis
  - Any vector $\boldsymbol{x} = [x_1, x_2, x_3]$ can be expressed as $\boldsymbol{x} = x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$
  - Note: $\langle \boldsymbol{x}, \boldsymbol{i} \rangle = x_1$ and likewise for x_2 and x_3

Gram Schmidt Procedure

Q: Given an arbitrary set of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$, how to form a set of orthonormal vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$?

A: Gram-Schmidt Procedure:



Vector Space of Functions

- Inner product:

$$\langle f, g \rangle = \int f(x) g(x) dx$$

- Norm:

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int [f(x)]^2 dx}$$

- Orthonormal functions:

$$\langle \phi_k(x), \phi_l(x) \rangle = \int \phi_k(x) \phi_l(x) dx = \delta_{kl}$$

- Orthonormal basis $\phi_1(x), \phi_2(x), \ldots, \phi_n(x)$ and an arbitrary function $f(x)$:

$$f(x) = \sum_k \langle f(x), \phi_k(x) \rangle \phi_k(x)$$

Typical Inner Product Definitions

- For energy signals

$$\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t) y^*(t) dt \qquad \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \qquad \int_{-\infty}^{\infty} |y(t)|^2 dt < \infty$$

- For power signals

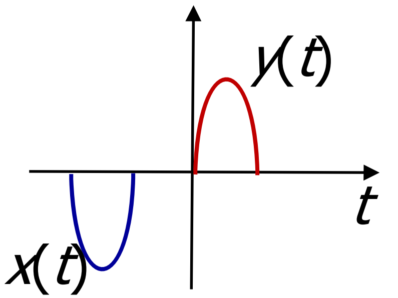
$$\langle x(t), y(t)\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) y^*(t) dt \qquad \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt < \infty \qquad \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |y(t)|^2 dt < \infty$$

- For periodic signals

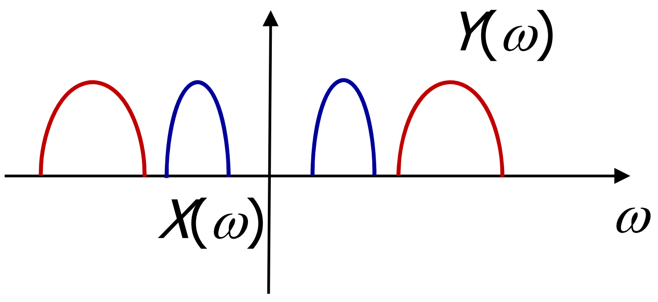
$$\langle x(t), y(t)\rangle = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) y^*(t) dt \qquad x(t) = x(t + T_0) \qquad y(t) = y(t + T_0)$$

Time and Frequency Basis

- $\left\{e^{j\omega t}\right\}$, for all ω, forms a complete orthonormal basis for all signals
- $\left\{\delta(t-\tau)\right\}$, for all τ, forms a complete orthogonal basis for all signals
- Signals in orthogonal spaces do not interfere each other.



Do not overlap in time

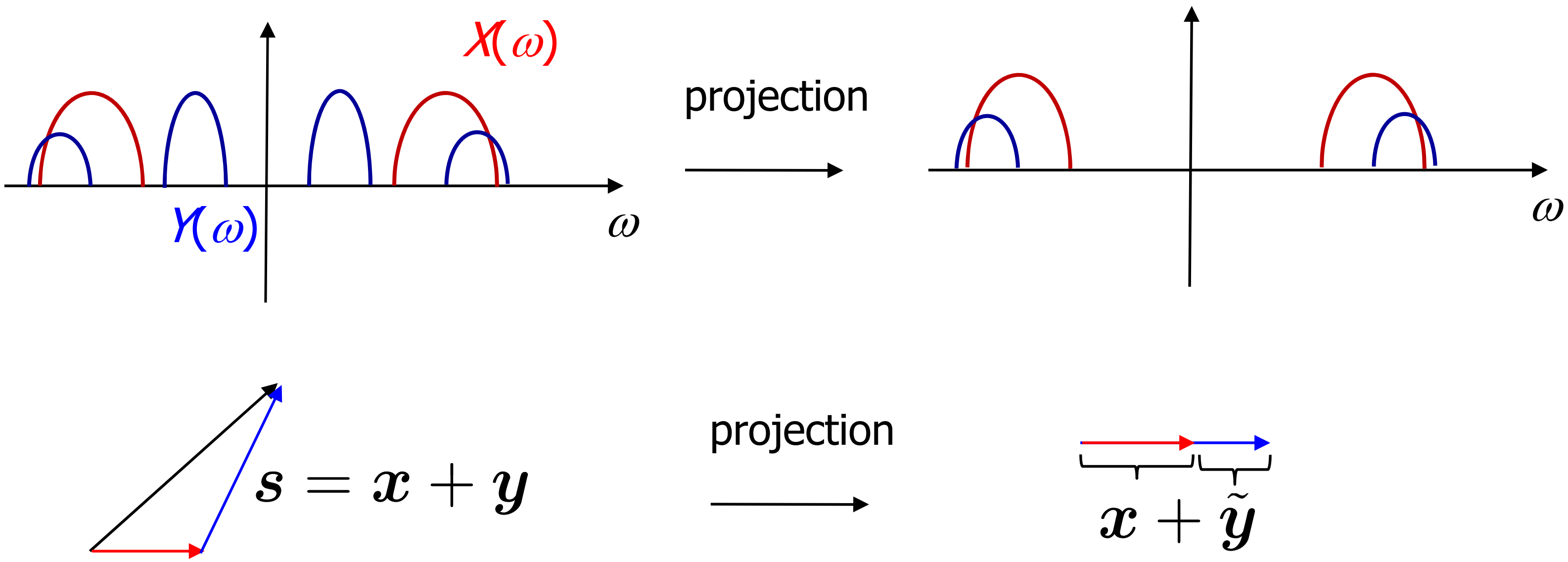


Do not overlap in frequency

Signal Space Representation Ex1

Q: We receive $s(t) = x(t) + y(t)$, and know that $X(\omega) = 0$ for $|\omega| \in [\omega_1, \omega_2]$.
How to extract x(t) from s(t)?

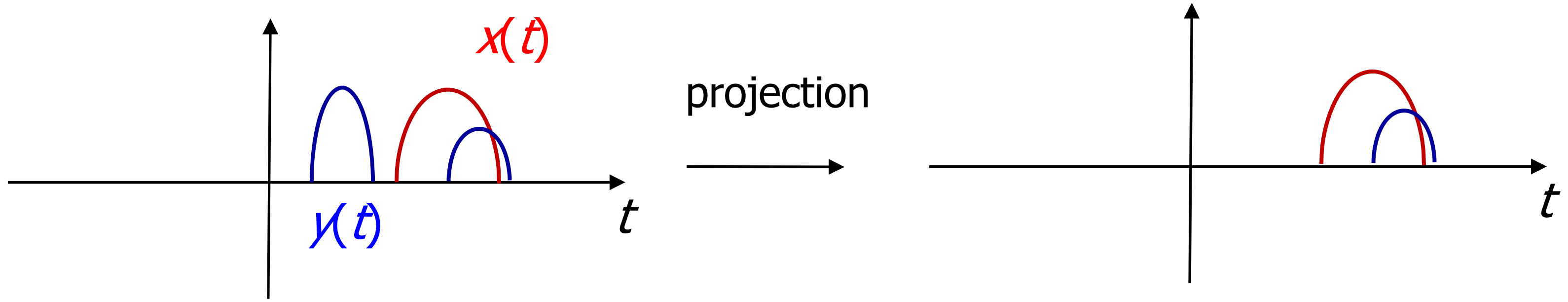
A: Bandstop filter s(t).



Signal Space Representation Ex2

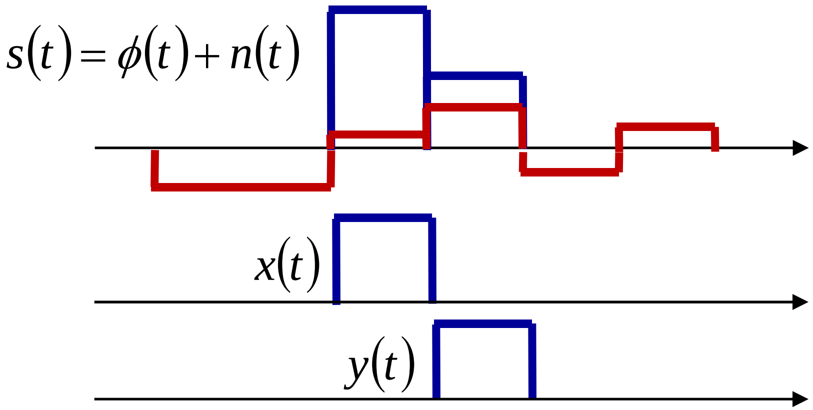
Q: We receive $s(t)=x(t)+y(t)$, and know that $x(t)=0$ for $t \in [t_1, t_2]$.
How to extract x(t) from s(t)?

A: Set $s(t)=0$ for $t \in [t_1, t_2]$.

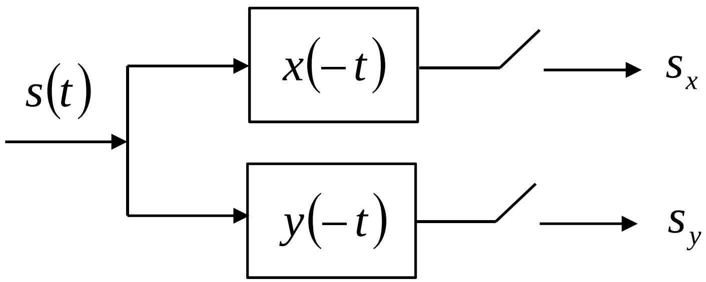


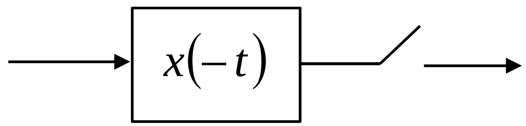
Signal Space Representation Ex3

Q: We receive $s(t)=\phi(t)+n(t)$, and we know $\phi(t)=ax(t)+by(t)$.
How to extract $\varphi(t)$ from s(t)?



$$s(t)\to\begin{bmatrix} s_x \\ s_y \end{bmatrix}\qquad \begin{aligned} s_x &= \langle s(t), x(t)\rangle = \int_{-\infty}^{\infty} s(t)x(t)dt = \int_{-\infty}^{\infty} s(t)x(-(-t))dt = [s(t)*x(-t)]\big|_{t=0} \\ s_y &= \langle s(t), y(t)\rangle = \int_{-\infty}^{\infty} s(t)y(t)dt = \int_{-\infty}^{\infty} s(t)y(-(-t))dt = [s(t)*y(-t)]\big|_{t=0} \end{aligned}$$





Matched filter

Homework

- 2.5-5, 2.6-1, 2.7-2, 2.9-1, 3.6-1

Roadmap

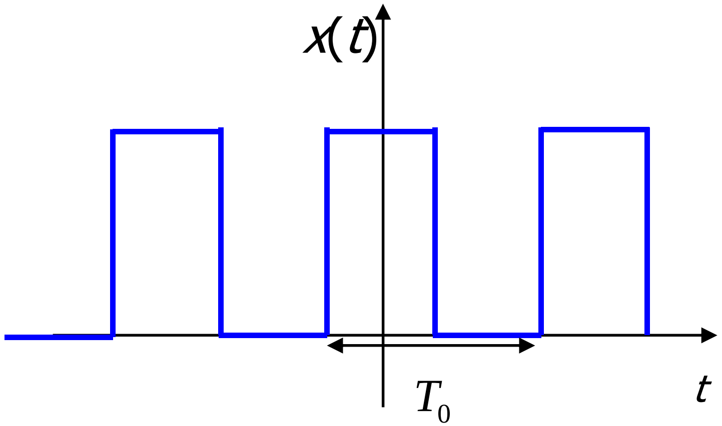
- Signals
- Signal space representation
- Fourier transform
- Signal transmission through a linear system

Fourier Series

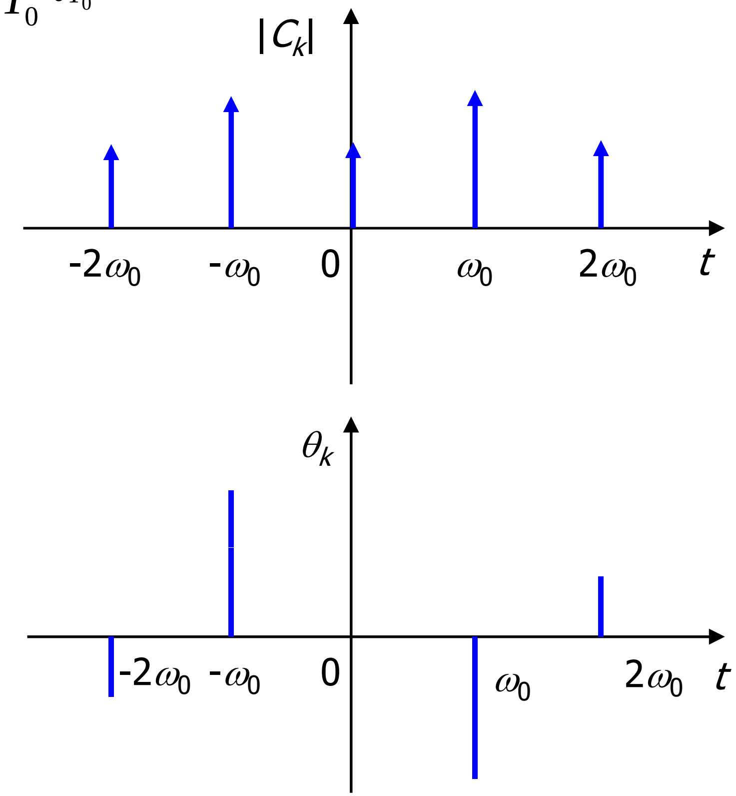
Every periodic signal $x(t)$ with fundamental period T_0 can be decomposed as

$$x(t) \leftrightarrow \{C_k, k\omega_0\}$$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \qquad C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$$



$$\omega_0 = \frac{2\pi}{T_0}$$



Parseval's Theorem

- Fourier series = project a signal onto the orthonormal basis
  - Projection to obtain the coefficients

$$C_k = \left\langle x(t), e^{jk\omega_0 t} \right\rangle = \frac{1}{T_0}\int_{T_0} x(t)\left(e^{jk\omega_0 t}\right)^* dt = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

  - Use coefficients to synthesize $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

- Parseval's Theorem

$$\frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |C_k|^2$$

  - Signal space interpretation:

$$\|x(t)\|^2 = \langle x(t), x(t)\rangle = \frac{1}{T_0}\int_{T_0} x(t) x^*(t) dt = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt$$

$$\|x(t)\|^2 = \sum_{k=-\infty}^{\infty} |C_k|^2$$

Fourier Transform

- Definition: $f(t) \overset{F}{\leftrightarrow} F(\omega)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Assumption $\int_{-\infty}^{\infty} |f(t)| dt < \infty$, i.e., the signal is absolutely integratable $\Rightarrow$ aperiodic

- Signal Space Interpretation:

$\{e^{j\omega t}\}$, for all ω, forms a complete orthonormal basis for all signals

$$\langle e^{j\omega t}, e^{j\omega t} \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{j\omega t} \left(e^{j\omega t}\right)^* dt = 1$$

$$\langle e^{j\omega_1 t}, e^{j\omega_2 t} \rangle = 0 \text{ for } \omega_1 \neq \omega_2$$

Properties of Fourier Transform

- Linear

$$f_1(t)\overset{F}{\leftrightarrow}F_1(\omega) \quad f_2(t)\overset{F}{\leftrightarrow}F_2(\omega) \Rightarrow af_1(t)+bf_2(t)\overset{F}{\leftrightarrow}aF_1(\omega)+bF_2(\omega)$$

- Scaling in Time

$$f(t)\overset{F}{\leftrightarrow}F(\omega) \Rightarrow f(at)\overset{F}{\leftrightarrow}\frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

extend in time = compress in frequency
compress in time = extend in frequency

- Scaling in Frequency

$$f(t)\overset{F}{\leftrightarrow}F(\omega) \Rightarrow \frac{1}{|a|}f\left(\frac{t}{a}\right)\overset{F}{\leftrightarrow}F(a\omega)$$

- Shifting in Time

$$f(t)\overset{F}{\leftrightarrow}F(\omega) \Rightarrow f(t-t_0)\overset{F}{\leftrightarrow}F(\omega)e^{-j\omega t_0}$$

shift in time = phase change in frequency

- Shifting in Frequency

$$f(t)\overset{F}{\leftrightarrow}F(\omega) \Rightarrow f(t)e^{j\omega_0 t}\overset{F}{\leftrightarrow}F(\omega-\omega_0)$$

shift in frequency = modulation in time

Properties of Fourier Transform

- Convolution

$$f_1(t)*f_2(t)=\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau \overset{F}{\leftrightarrow} F_1(\omega)F_2(\omega)$$

convolution in time = multiplication in frequency

- Multiplication

$$f_1(t)f_2(t) \overset{F}{\leftrightarrow} \frac{1}{2\pi}F_1(\omega)*F_2(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\tilde{\omega})F_2(\omega-\tilde{\omega})d\tilde{\omega}$$

multiplication in time = multiplication in time

Properties of Fourier Transform

- Differentiation in Time

$$f(t) \overset{F}{\leftrightarrow} F(\omega) \quad \Rightarrow \frac{df(t)}{dt} \overset{F}{\leftrightarrow} j\omega F(\omega) \qquad \frac{d^n f(t)}{dt^n} \overset{F}{\leftrightarrow} (j\omega)^n F(\omega)$$

differentiation in time = multiplying $j\omega$ in frequency

- Frequency Differentiation

$$f(t) \overset{F}{\leftrightarrow} F(\omega) \quad \Rightarrow (-jt) f(t) \overset{F}{\leftrightarrow} \frac{dF(\omega)}{d\omega} \qquad (-jt)^n f(t) \overset{F}{\leftrightarrow} \frac{d^n F(\omega)}{d\omega^n}$$

differentiation in frequency = multiplying $-jt$ in time

- Integration in Time

$$f(t) \overset{F}{\leftrightarrow} F(\omega) \quad \Rightarrow \int_{-\infty}^{t} f(\tau) d\tau \overset{F}{\leftrightarrow} \frac{1}{j\omega} F(\omega) + \pi F(0)\delta(\omega)$$

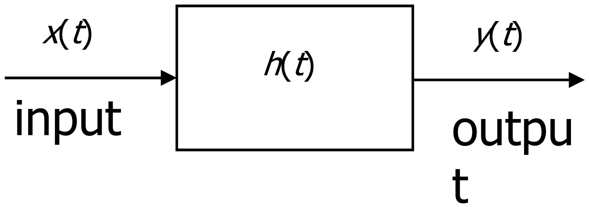
$$\text{If } F(0) = 0 \quad \Rightarrow \int_{-\infty}^{t} f(\tau) d\tau \overset{F}{\leftrightarrow} \frac{1}{j\omega} F(\omega)$$

integration in time = dividing $j\omega$ in frequency

Roadmap

- Signals
- Signal space representation
- Fourier transform
- Signal transmission through a communication channel
  - Signal transmission through a linear system
  - Filters: ideal vs. practical
  - Signal transmission through a communication channel

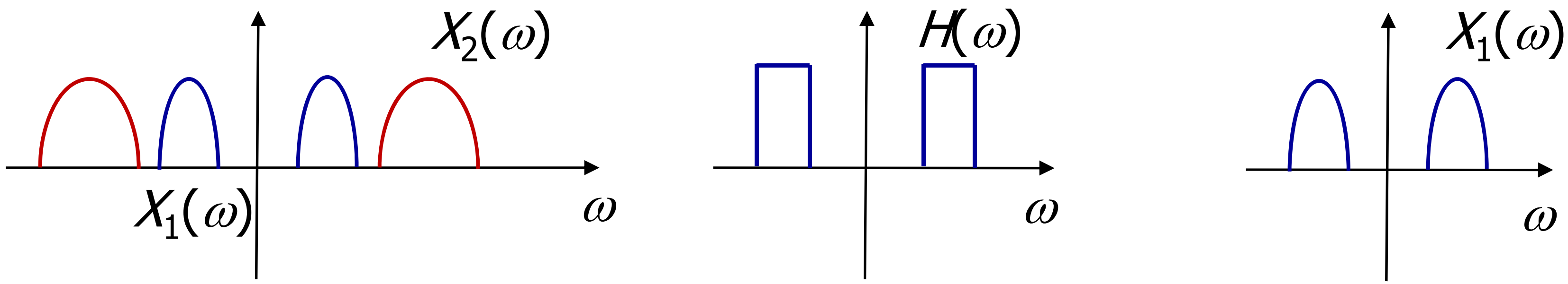
Signal Transmission through a Linear System



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$X(\omega)$ $\quad H(\omega)$ $\quad Y(\omega) = H(\omega) X(\omega)$

Example:



Use a bandpass filter to pick up signal from a specific bandwidth.

Magnitude Spectrum

$$|Y(\omega)| = |H(\omega)||X(\omega)|$$

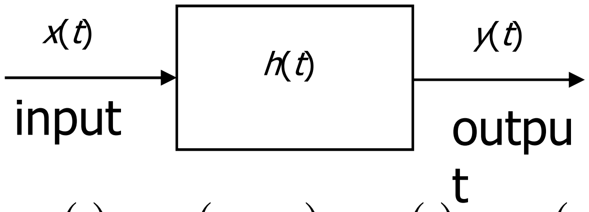
Phase Spectrum

$$\angle Y(\omega) = \angle H(\omega) + \angle X(\omega)$$

Distortion-Less Transmission

- A system is distortion-less if it only scales and delays the input signal.

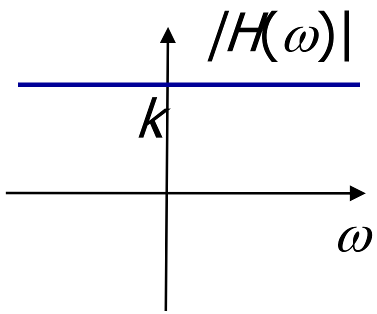
$$y(t) = kx(t - t_d)$$

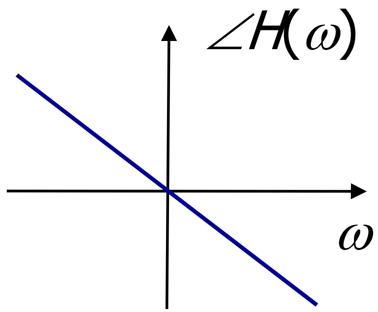


$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$

$$y(t) = kx(t - t_d) \Rightarrow h(t) = k\delta(t - t_d)$$

$$H(\omega) = ke^{-j\omega t_d}$$



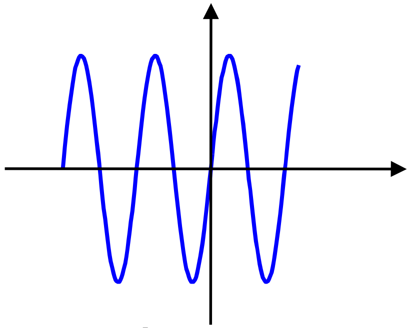


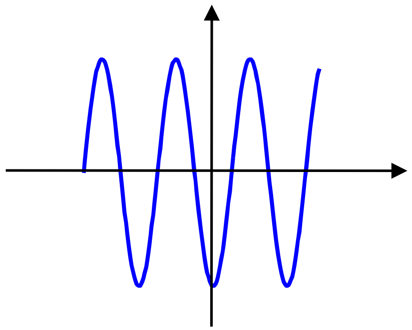
Constant time delay
= linear phase shift
≠ constant phase shift

- Flat magnitude spectrum and linear angle spectrum.
  Or, a constant scaling and a constant time delay.
- Note: All pass filter = flat magnitude spectrum.
  All pass filter + linear phase spectrum = distortion-less

Natural Distortion in Audio & Video

Human ear is sensitive to audio amplitude distortion, but insensitive to phase distortion





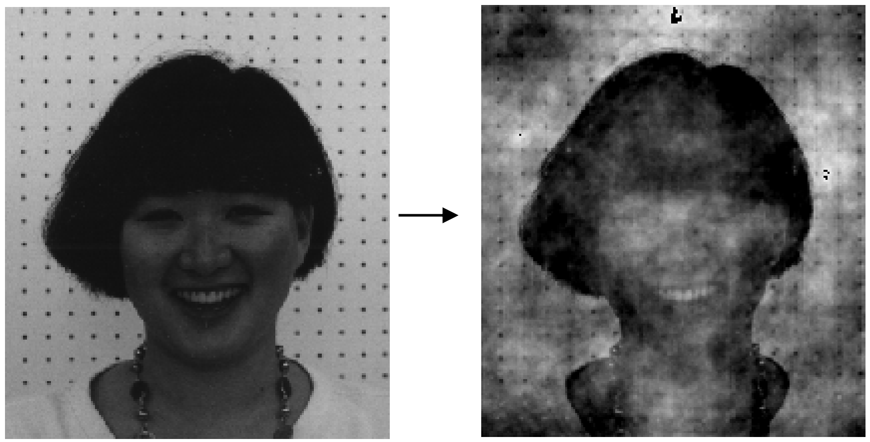
Hear the same tone

Human eye is relatively sensitive to video phase distortion, but insensitive to amplitude distortion.

2-D Fourier Transform

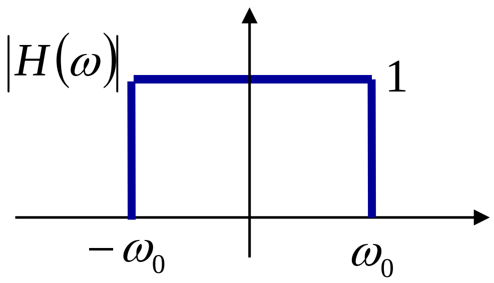
$$F(\sigma,\tau)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j\left(\frac{2\pi\sigma x}{M}+\frac{2\pi\tau y}{N}\right)}$$

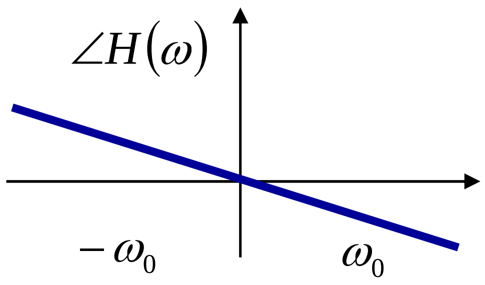
Change phase, and then take inverse transform



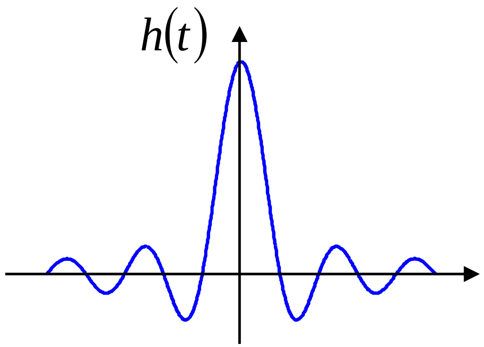
Ideal Filters - Impractical

Practical systems are causal





Ideal lowpass filter



$h(t) \neq 0$ for some $t < 0$

Non-causal

Given a constant t_0, $h(t - t_0) \neq 0$ for some $t < 0$

Can't be viewed as a delayed version of a causal filter

Ideal filters are not implementable.

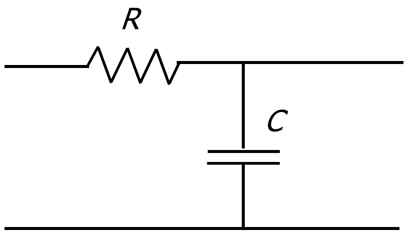
Paley-Wiener Criterion

Paley-Wiener Theorem: $H(\omega)$ is realizable if and only if $\int_{-\infty}^{\infty} \frac{\left|\ln\left|H(\omega)\right|\right|}{1+\omega^2} d\omega < \infty$

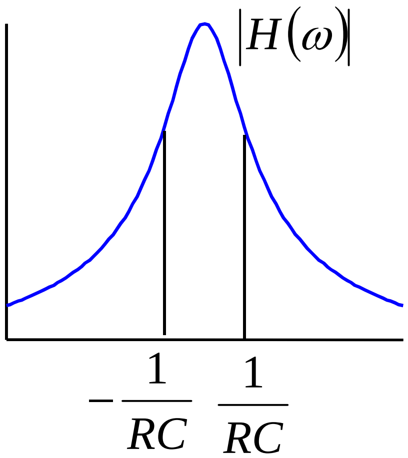
Corollary:

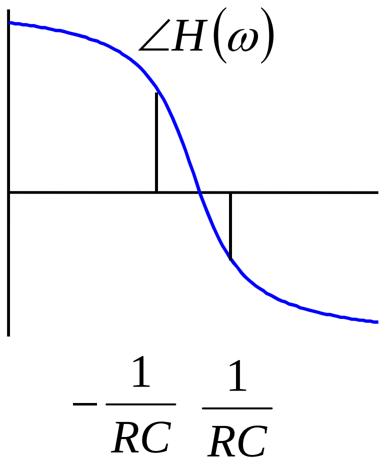
$H(\omega)$ is not realizable if $|H(\omega)|=0$ over a non-zero length frequency interval

- In other words, a practical filter can only "suppress" signals in certain frequency band, but cannot block them completely.
- A simple practical low pass filter



$$H(\omega) = \frac{1}{1 + j\omega RC}$$

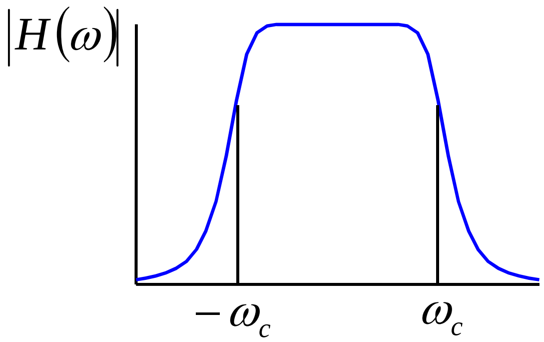




Practical Filters

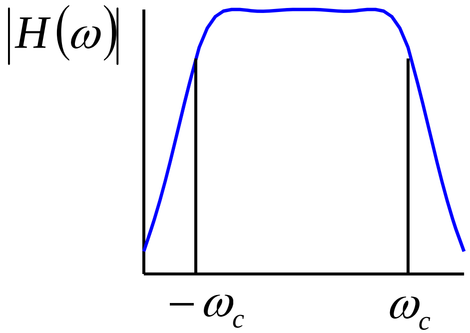
N th order Butterworth lowpass filter $|H(\omega)|^2 = H(\omega)H^*(\omega) = \dfrac{1}{1+\left(\dfrac{\omega}{\omega_c}\right)^{2n}}$

6th order $|H(\omega)|^2 = \dfrac{1}{1+\left(\dfrac{\omega}{\omega_c}\right)^{12}}$



N th order Chebyshev lowpass filter $|H(\omega)|^2 = \dfrac{1}{1+\varepsilon^2 T_n^2\left(\dfrac{\omega}{\omega_c}\right)}$

3rd order



Can uniquely determine $H(\omega)$ by choosing poles and zeros on the left half-plane

Lowpass filter can be converted to highpass and bandpass filters

Practical vs. Ideal Filters

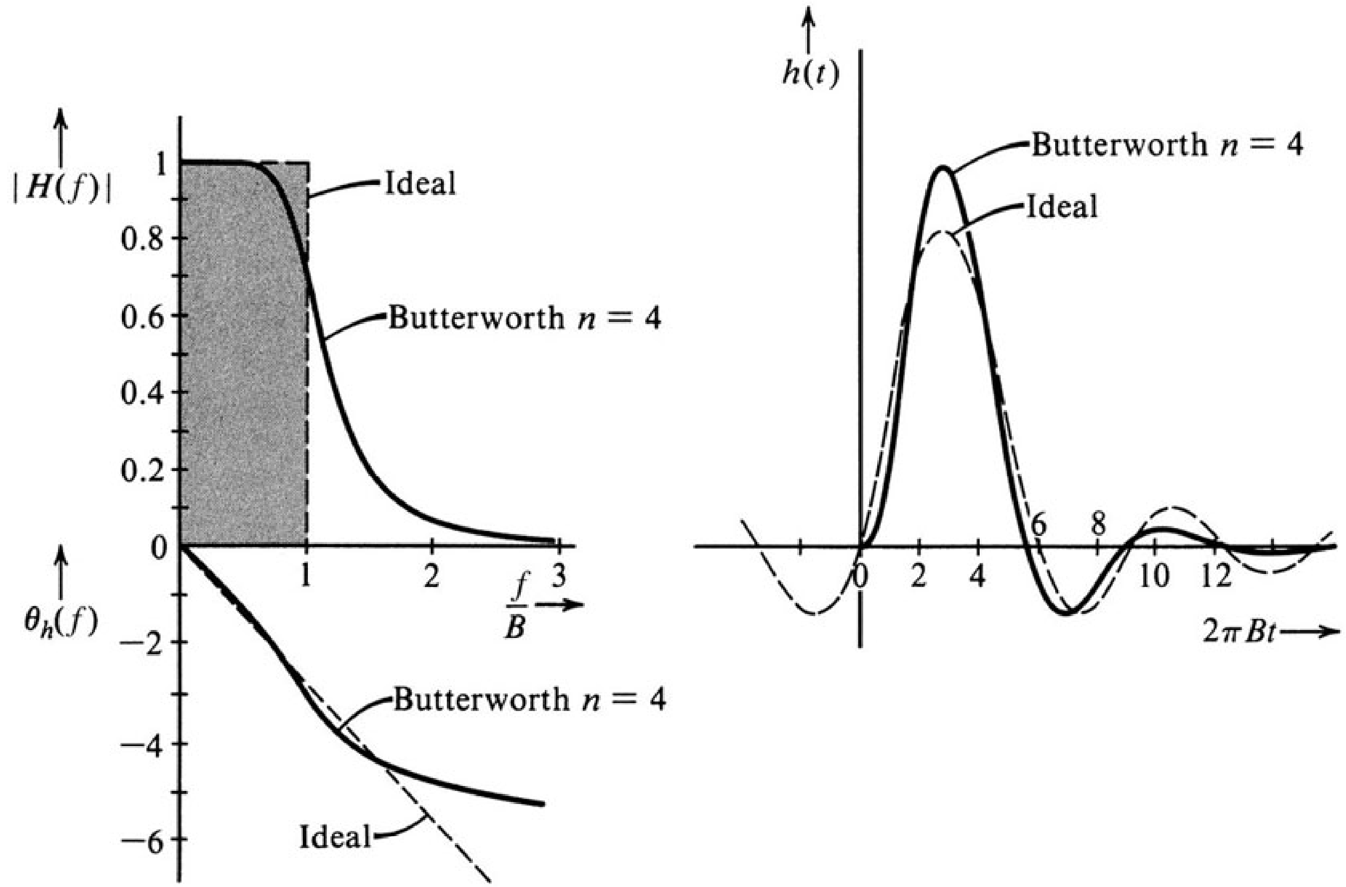
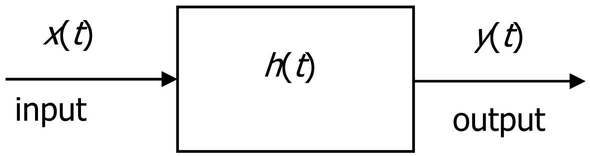


Figure 3.33 Comparison of Butterworth filter (n = 4) and an ideal filter.

Signal Distortion through a Comm. Channel

- Signal transmitted through a channel can be distorted due to channel imperfection and other uncontrollable factors.
- Linear distortion: Distortion caused by linear time-invariant channel (other than the distortion-less channel)



$$Y(\omega) = H(\omega)X(\omega)$$

  - Can cause magnitude suppression and/or phase change
  - If phase change is not linear, $\angle H(\omega) \neq -\omega t_0$, then different frequency components will experience different delays

Linear Distortion

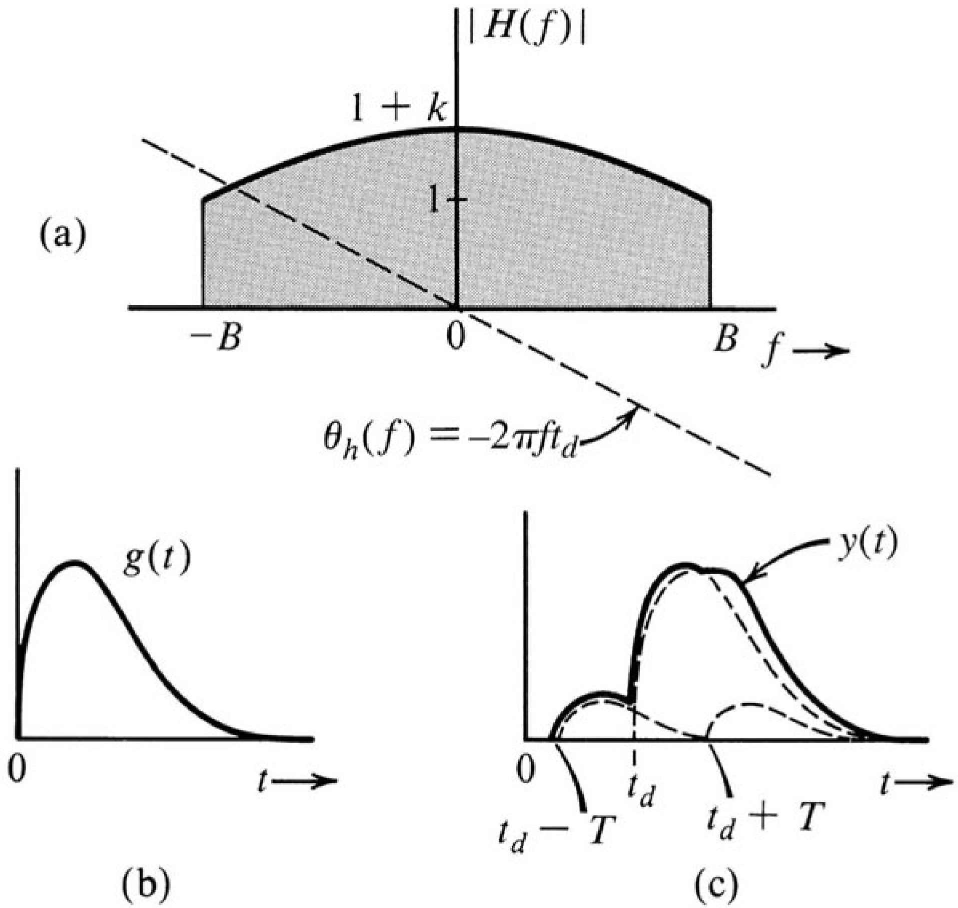
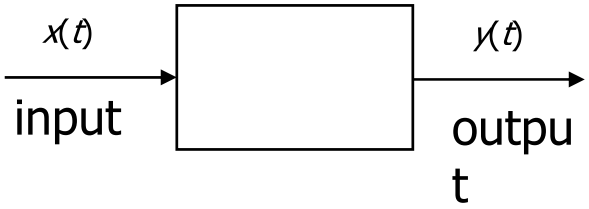


Figure 3.35 Pulse is dispersed when it passes through a system that is not distortion-less.

Signal extension due to distortion can cause problem in time-division systems

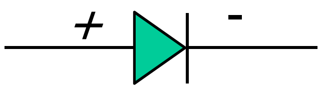
Non-Linear Distortion

- Sometime useful for special signal processing objectives



$$y(t) = G(x(t))$$

- Example:



$x(t)$: voltage over the diode

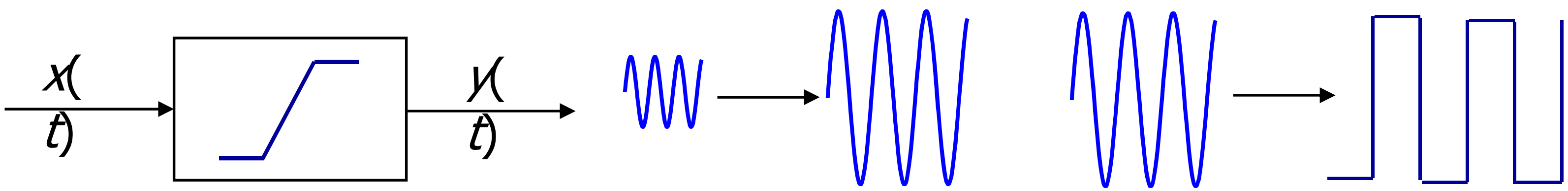
$y(t)$: current of the diode

$$y(t) = I_s\left(e^{x(t)/nV_T} - 1\right)$$

$$y(t) = G(x(t)) = G(0) + G'(0)x(t) + \frac{G''(0)}{2}x^2(t) + \frac{G'''(0)}{6}x^3(t) + \cdots$$

If processed carefully, this can be used to calculate $x^2(t)$ in an "analog" computer

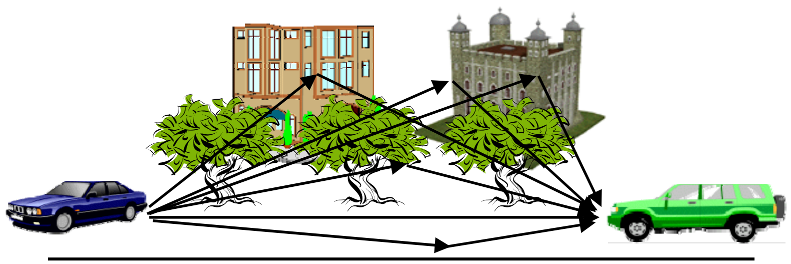
- Example: Saturation of amplifier



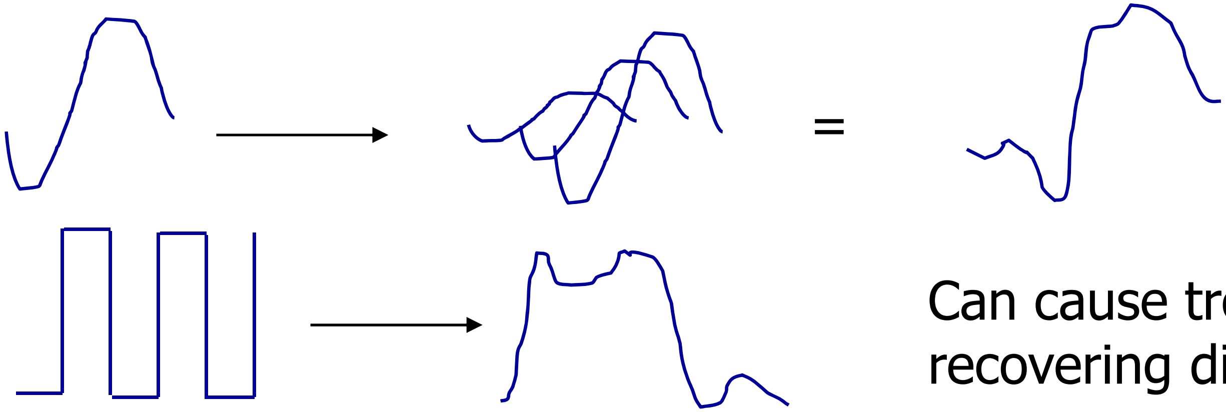
If $x(t)$ has bandwidth B, then $x^2(t)$ has bandwidth $2B$, $x^3(t)$ has bandwidth $3B$, etc.
Bandwidth extension can cause significant problem in frequency division systems

Signal Distortion through a Comm. Channel

Multipath Effect: Commonly seen in wireless communication



Received signal is a summation of signals coming from different paths



Can cause trouble in recovering digital signals

Frequency-flat fading: all signals copies arrive with roughly the same delay

Frequency-selective fading: signals copies arrive with significant delay differences

Time-invariant vs. time-varying

Summary

- Signals: definition, classification, operations
- Signal Space Representation
  - Vectors and vector space
  - Signal space representation
- Fourier transform
- Signal transmission through a communication channel
  - Signal transmission through a linear system
  - Filters: ideal vs. practical
  - Signal transmission through a communication channel