Principle of Communications

Principles of Digital Data Transmission

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Outline

- Line Coding (Transmission Coding)
- Digital Baseband Transmission
- Digital Band-Pass Modulation

Roadmap

- Line Coding (Transmission Coding)
  - Line coding
  - Power Spectral Density (PSD)
  - Polar signaling
  - On-off signaling
  - Bipolar signaling
- Digital Baseband Transmission
- Digital Band-Pass Modulation

Building Blocks of Digital Communication Systems

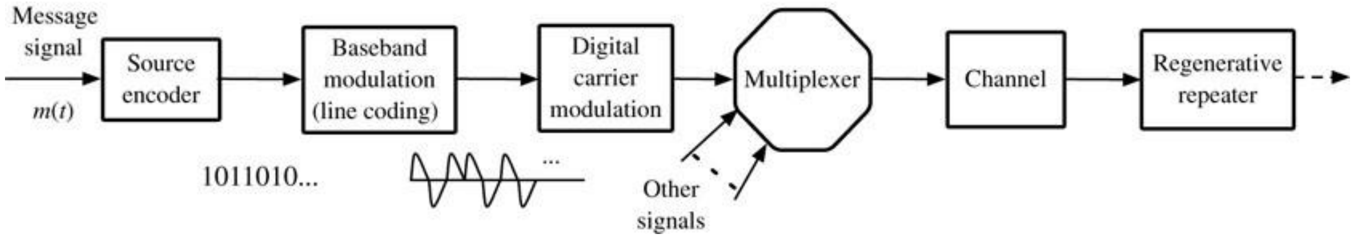


Figure 7.1 Fundamental building blocks of digital communication systems.

Input: Sequence of digits, 0 or 1, *M*-ary.

Generated from a data set, a computer, a digitized voice signal.

Binary: Two symbols

M-ary: *M* symbols

Line Codes: Digital sequences are coded into electrical pulses or waveforms

Multiplexer: Time or frequency sharing between multiple messages

Regenerative Repeater: Detect, restore, and then relay digital signals.

The diagram shows basic components.

Line Coding

Many possible ways of assigning waveforms (pulses) to digital data

Binary Line Coding (basic model)

$\cdots, x_3, x_2, x_1$ → coding → $\cdots, a_3, a_2, a_1$ → waveform assignment → $y(t) = \sum a_k p(t - kT_b)$

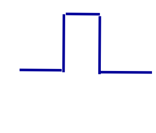
binary sequence

encoded sequence

On-Off Coding

$1 \rightarrow p(t)$

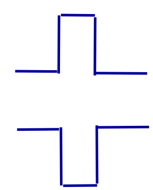
$0 \rightarrow 0$



Polar Coding

$1 \rightarrow p(t)$

$0 \rightarrow -p(t)$

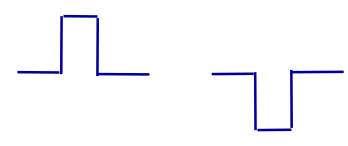


better noise tolerance
(more power efficient)

Bipolar Coding

$1 \rightarrow p(t) \text{ or } -p(0)$

$0 \rightarrow 0$



alternating

Line Coding

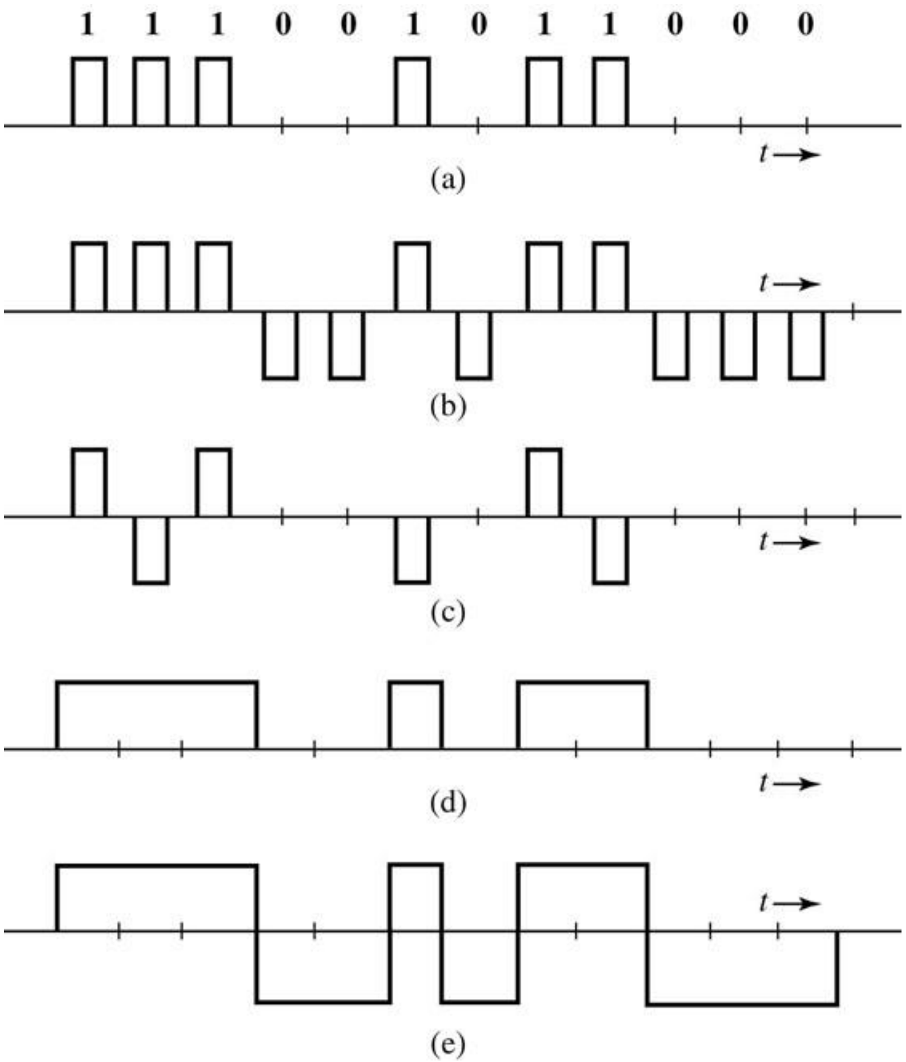


Figure 7.2 Line code examples: (a) on-off (RZ); (b) polar (RZ); (c) bipolar (RZ); (d) on-off (NRZ); (e) polar (NRZ).

Line Coding

Performance concerns in line coding design

1. Transmission Bandwidth: As small as possible
2. Power Efficiency: Minimum power required for a given bandwidth & P_e
3. Error Detection & Correction: Detect & preferably correct errors
4. Favorable Power Spectral Density: bipolar case: single error will cause bipolar violation.

   Desirable to have zero DC component. (because some system uses the DC component to transmit power that supports the operations of the regenerative repeater.)
5. Adequate Timing Content: Extract timing or clock information
6. Transparency: It should be possible to transmit a digital signal correctly regardless of the pattern of 1's and 0's.

   Polar is transparent.

   On-off and bipolar are not.

Power Spectral Density (PSD)

PSD of Line Codes: Covers a large class of line codes

Assume pulses every T_b seconds: $R_b = 1/T_b$ pulses/second

Basic pulse: $p(t) \leftrightarrow P(\omega)$

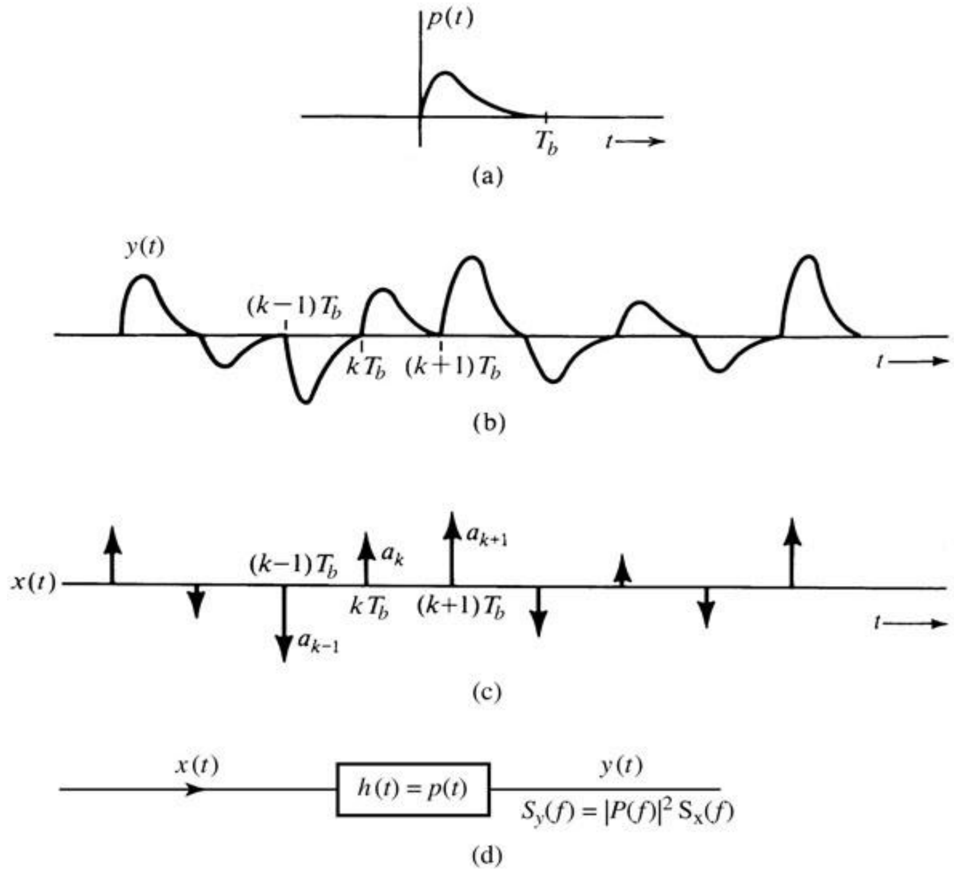


Figure 7.4 Random pulse-amplitude-modulated signal and its generation from a PAM impulse.

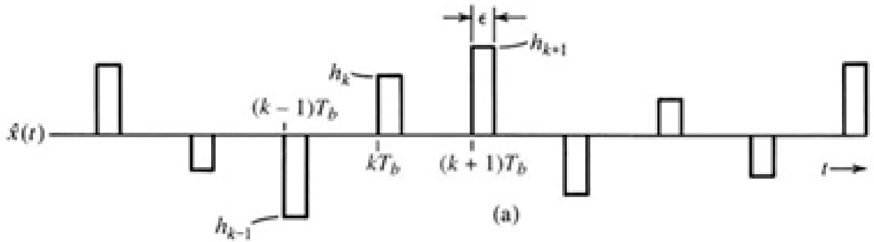
$$y(t) = \sum a_k p(t - kT_b)$$

$$S_y(\omega) = |P(\omega)|^2 S_x(\omega)$$

a_k does not need to be binary

PSD Computation (1)

In order to calculate $S_x(\omega)$, we need to calculate $R_x(\tau)$. But that involves the multiplication of delta functions. To avoid computation problem, we expand each delta function to a narrow rectangular waveform.



$$h_k = a_k / \varepsilon \qquad \hat{x}(t) = \sum h_k rec\left(\frac{t-kT_b}{\varepsilon}\right) \qquad R_{\hat{x}}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\hat{x}(t)\hat{x}(t-\tau)dt$$

$$\text{for } |\tau| \le \varepsilon,\ R_{\hat{x}}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\hat{x}(t)\hat{x}(t-\tau)dt = \lim_{T\to\infty}\frac{1}{T}\sum_k h_k{}^2(\varepsilon - |\tau|) = \lim_{T\to\infty}\frac{1}{T}\sum_k a_k{}^2\left(\frac{\varepsilon-|\tau|}{\varepsilon^2}\right)$$

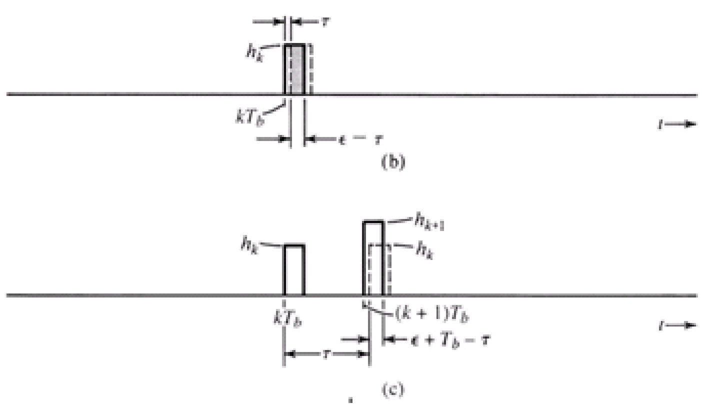
$$= E\left[a_k{}^2\right]\frac{1}{T_b}\left(\frac{\varepsilon-|\tau|}{\varepsilon^2}\right) = R_0\frac{1}{T_b}\left(\frac{\varepsilon-|\tau|}{\varepsilon^2}\right)$$

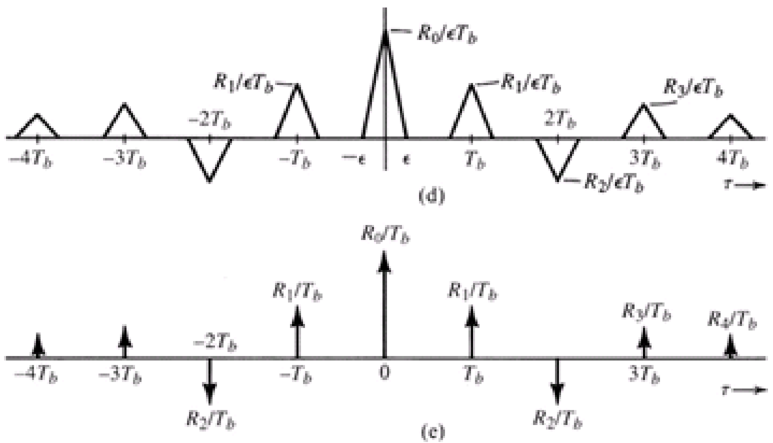
PSD Computation (2)

$$\text{for } |\tau - nT_b| \le \varepsilon,\ R_{\hat{x}}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\hat{x}(t)\hat{x}(t-\tau)dt = \lim_{T\to\infty}\frac{1}{T}\sum_k h_k h_{k+n}\left(\varepsilon - |nT_b - \tau|\right)$$

$$= \lim_{T\to\infty}\frac{1}{T}\sum_k a_k a_{k+n}\left(\frac{\varepsilon - |nT_b - \tau|}{\varepsilon^2}\right)$$

$$= E[a_k a_{k+n}]\frac{1}{T_b}\left(\frac{\varepsilon - |nT_b - \tau|}{\varepsilon^2}\right) = R_n \frac{1}{T_b}\left(\frac{\varepsilon - |nT_b - \tau|}{\varepsilon^2}\right)$$





$$R_x(\tau) = \frac{1}{T_b}\sum_{n=-\infty}^{\infty} R_n \delta(\tau - nT_b) \qquad S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)e^{-j\omega\tau}d\tau = \frac{1}{T_b}\sum_{n=-\infty}^{\infty} R_n e^{-jn\omega T_b}$$

PSD Computation (3)

For a discrete-time signal $\{a_k\}$, its autocorrelation function $\{R_n\}$ is defined as

$$R_n = E\left[a_k a_{k-n}^*\right] = \lim_{N\to\infty} \frac{1}{2N} \sum_{k=-N}^{N} a_k a_{k-n}^*$$

$$R_0 = E\left[|a_k|^2\right] = \lim_{N\to\infty} \frac{1}{2N} \sum_{k=-N}^{N} |a_k|^2$$

For real-valued signal, $R_n = R_{-n}^*$ $\qquad R_n = E\left[a_k a_{k-n}^*\right] = \left\{E\left[a_k a_{k+n}^*\right]\right\}^* = R_{-n}^*$

$$S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau} d\tau = \frac{1}{T_b} \sum_{n=-\infty}^{\infty} R_n e^{-jn\omega T_b}$$

$$S_y(\omega) = |P(\omega)|^2 S_x(\omega)$$

Polar Signaling

Assume binary input. $1 \rightarrow p(t)$ $\qquad x_k = 1 \rightarrow a_k = 1$

$0 \rightarrow -p(t)$ $\qquad x_k = 0 \rightarrow a_k = -1$

$$a_k \in \{+1, -1\} \Rightarrow a_k^{\,2} = 1 \qquad R_0 = E\left[a_k^{\,2}\right] = 1$$

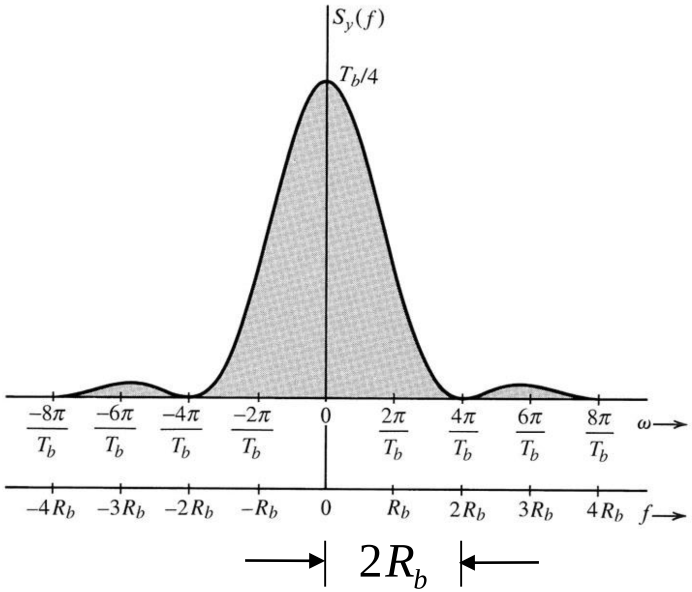
if x_k are independent, x_k is equally likely to be 0 or 1, for all k,
then a_k is equally likely to be 1 or -1, for all k

$$R_n = E\left[a_k a_{k+n}\right] = E\left[a_k\right] E\left[a_{k+n}\right] = 0$$

$$R_x(\tau) = \frac{1}{T_b}\delta(\tau) \qquad S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau} d\tau = \frac{1}{T_b}$$

$$\text{if } p(t) = \text{rect}\left(\frac{t}{T_b/2}\right) \qquad P(\omega) = \frac{T_b}{2}\,\text{sinc}\left(\frac{\omega T_b}{4}\right)$$

$$S_y(\omega) = |P(\omega)|^2 S_x(\omega) = \frac{T_b}{4}\,\text{sinc}^2\left(\frac{\omega T_b}{4}\right)$$



essential bandwidth

Polar Signaling

Pros:

1. Most power efficient: for a given power, the detection-error probability is the smallest possible.
2. Transparent: No zero

Cons:

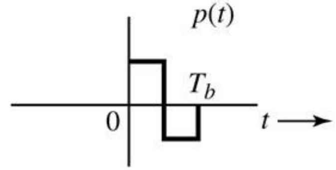
1. High bandwidth requirement
2. No error detection
3. Can have non-zero PSD at DC (ω=0)

Pulse Shaping for DC Null in PSD

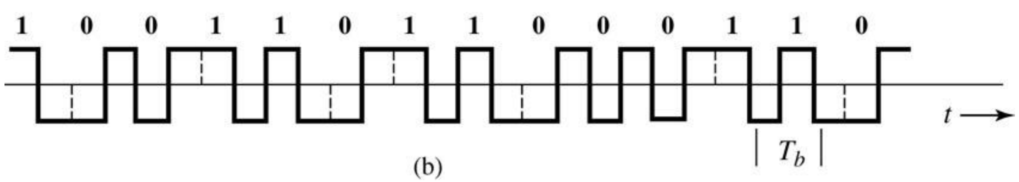
$$S_y(\omega) = |P(\omega)|^2 S_x(\omega) = \frac{|P(\omega)|^2}{T_b}$$

To have DC null in PSD, need $P(\omega) = 0$ for $\omega = 0$

$$P(\omega = 0) = \int_{-\infty}^{\infty} p(t)\,dt = 0$$



(a)



(b)

This is called Manchester coding, or phase-split coding

On-Off Keying (OOK)

Assume binary input. $1 \rightarrow p(t)$ $\quad x_k = 1 \rightarrow a_k = 1$

$0 \rightarrow 0$ $\quad x_k = 0 \rightarrow a_k = 0$

Assume x_k are independent, x_k is equally likely to be 0 or 1, for all k,

$$R_0 = E\left[a_k^{\ 2}\right] = \frac{1}{2}\times 1 + \frac{1}{2}\times 0 = \frac{1}{2}$$

$$R_n = E\left[a_k a_{k+n}\right] = \frac{1}{4}\times 1 + \frac{3}{4}\times 0 = \frac{1}{4}$$

$$S_x(\omega) = \frac{1}{T_b}\sum_{n=-\infty}^{\infty} R_n e^{-jn\omega T_b} = \frac{1}{2T_b} + \frac{1}{4T_b}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{-jn\omega T_b} = \frac{1}{4T_b} + \frac{1}{4T_b}\sum_{n=-\infty}^{\infty} e^{-jn\omega T_b}$$

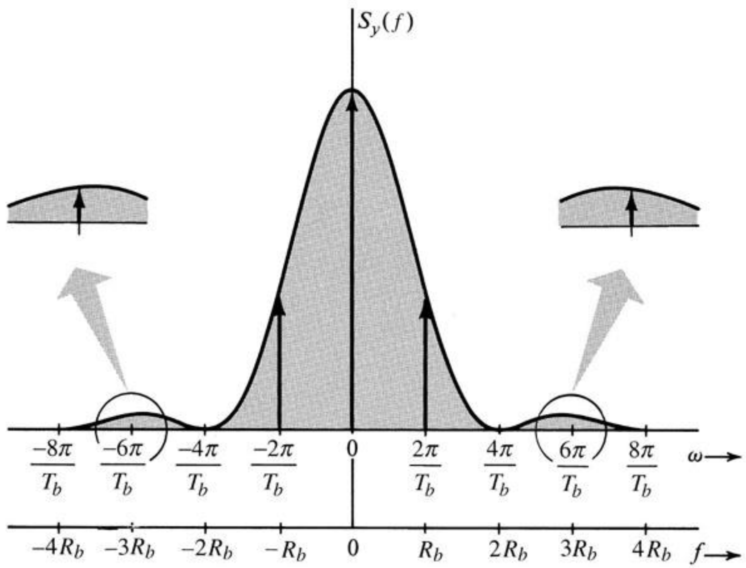
$$= \frac{1}{4T_b} + \frac{1}{4T_b}\frac{2\pi}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi n}{T_b}\right)$$

Because $\sum_{n=-\infty}^{\infty}\delta(t - nT_b) = \frac{1}{T_b}\sum_{n=-\infty}^{\infty} e^{jn\frac{2\pi}{T_b}t}$, Fourier transform $\sum_{n=-\infty}^{\infty} e^{-jn\omega T_b} = \frac{2\pi}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi n}{T_b}\right)$

On-Off Keying

Assume rectangular pulse.

$$S_y(\omega)=\frac{|P(\omega)|^2}{4T_b}\left[1+\frac{2\pi}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi n}{T_b}\right)\right]=\frac{T_b}{16}\,\mathrm{sinc}^2\left(\frac{\omega T_b}{4}\right)\left[1+\frac{2\pi}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi n}{T_b}\right)\right]$$



Same spectrum as polar signaling. Samples every $2\pi n/T_b$.

Indeed, on-off keying = polar signaling + periodic square waveform.

Con: not power efficient, not transparent, not DC null.

Pro: enables non-coherent detection

Bipolar Signaling

Assume binary input. $1 \to p(t)$ or $-p(t)$ $\quad x_k = 1 \to a_k = \pm 1$ alternating

$0 \to 0 \quad x_k = 0$

$$R_0 = E\left[a_k^{\ 2}\right] = \frac{1}{2}\times 1 + \frac{1}{2}\times 0 = \frac{1}{2}$$

$$R_1 = E\left[a_k a_{k+1}\right] = \frac{1}{4}\times(-1) + \frac{3}{4}\times 0 = -\frac{1}{4} \qquad (x_k, x_{k+1}) = (1,1),(1,0),(0,1),(0,0)$$

$$R_2 = E\left[a_k a_{k+2}\right] = \frac{1}{8}\times 1 + \frac{1}{8}\times(-1) + \frac{3}{4}\times 0 = 0$$

$$(x_k, x_{k+1}, x_{k+2}) = \{(1,0,1),(1,1,1),(1,x,0),(0,x,1),(0,x,0)\}$$

Similarly, $R_n = 0$ for $n \geq 2$

$$S_x(\omega) = \frac{1}{T_b}\sum_{n=-\infty}^{\infty} R_n e^{-jn\omega T_b} = \frac{1}{2T_b}\left(1-\cos(\omega T_b)\right) = \frac{1}{T_b}\sin^2\left(\frac{\omega T_b}{2}\right)$$

$$S_y(\omega) = |P(\omega)|^2 S_x(\omega) = \frac{|P(\omega)|^2}{T_b}\sin^2\left(\frac{\omega T_b}{2}\right) \qquad S_y(\omega = 0) = 0$$

Bipolar Signaling

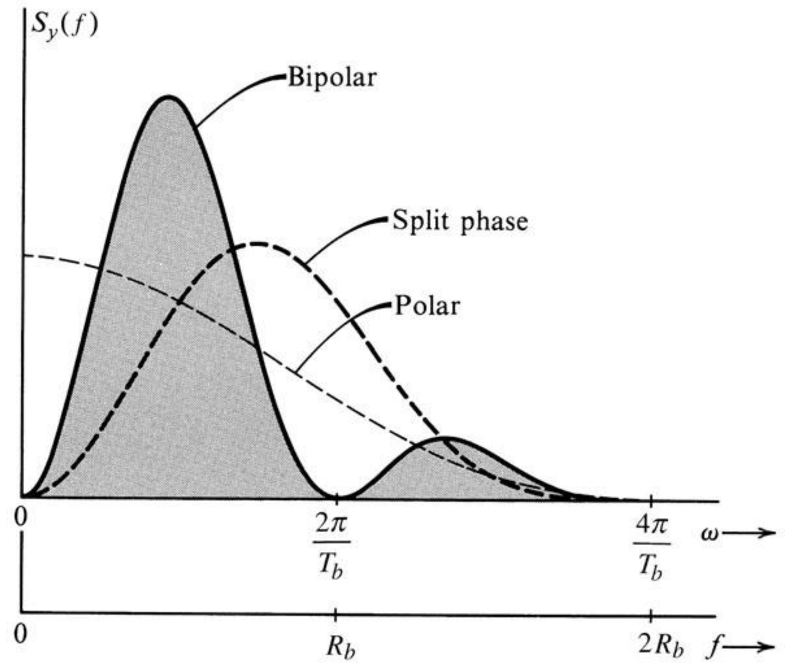
For rectangular pulse.

$$S_y(\omega) = \frac{|P(\omega)|^2}{T_b}\sin^2\left(\frac{\omega T_b}{2}\right) = \frac{T_b}{4}\operatorname{sinc}^2\left(\frac{\omega T_b}{4}\right)\sin^2\left(\frac{\omega T_b}{2}\right)$$

$$\text{Essential bandwidth} = R_b = \frac{1}{T_b}$$

Advantages: DC null, small bandwidth, single error detection

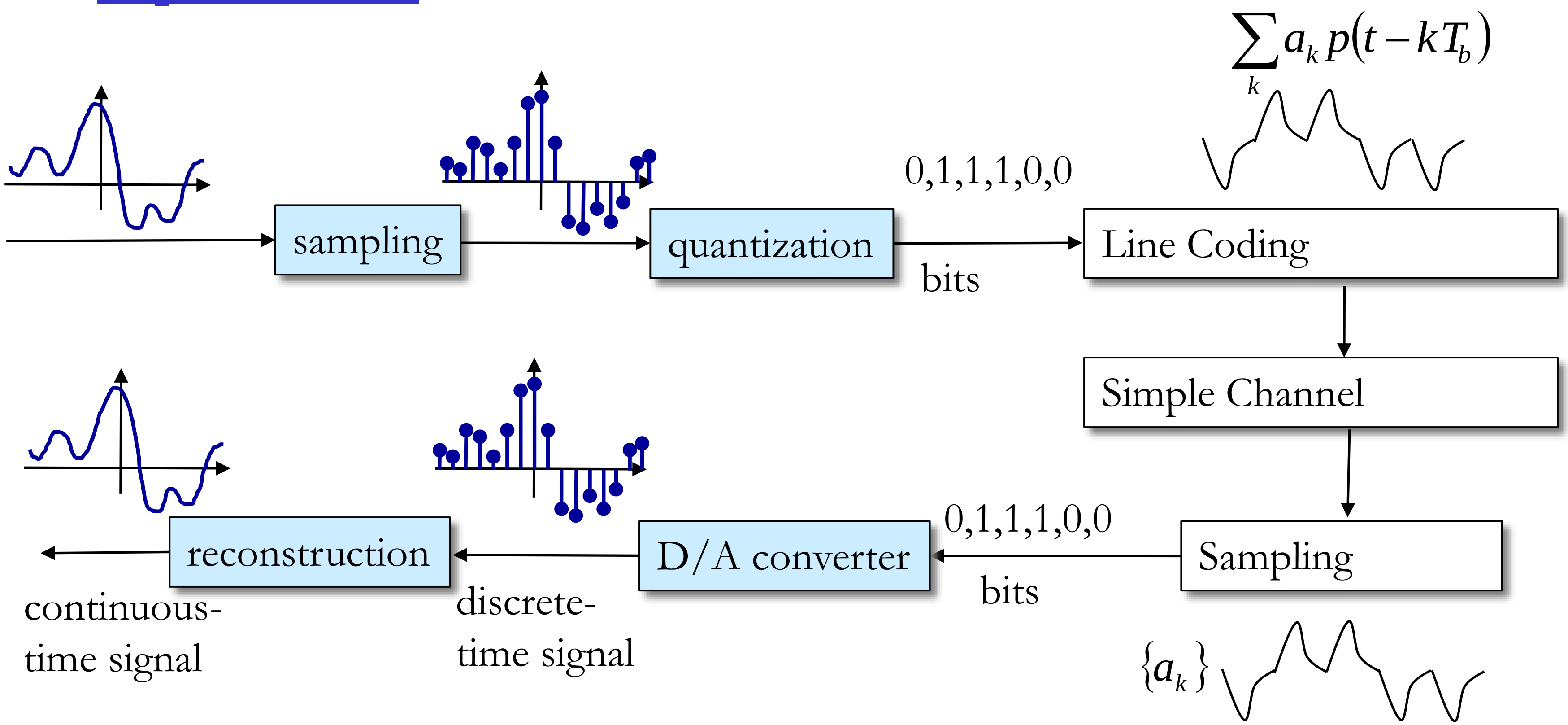
Disadvantages: power efficiency is the same as on-off, not transparent



Roadmap

- Line Coding (Transmission Coding)
- Digital Baseband Transmission
  - Pulse shaping
  - Eye diagram
  - Channel equalization and channel identification
- Digital Band-Pass Modulation

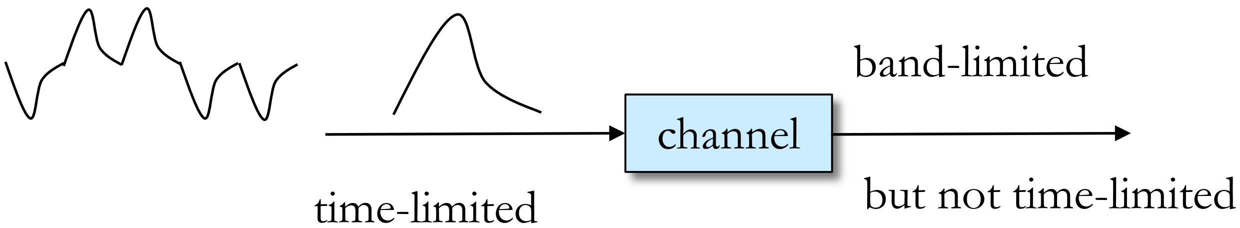
Diagram of Baseband Communication Systems



The pulse shaping problem: If $p(t)$ is not time limited, the k^{th} sample at the receiver will not be a_k.

Inter-Symbol Interference (ISI)

Even if $p(t)$ is time-limited, it is no longer so after passing the signal through a band-pass channel.



Inter-symbol interference (ISI): the kth sample contains not only a_k, but a combination of other symbols.

$$a(t)=\sum_k a_k \delta(t-kT_b)$$

*E*ffective pulse $\quad p(t)=g(t)*h(t) \quad g(t)$ pulse waveform, $h(t)$ channel impulse

$$y(t)=a(t)*p(t)=\sum_k a_k p(t-kT_b)$$

Sample $\quad y_n = y(nT_b)=\sum_k a_k p((n-k)T_b)$

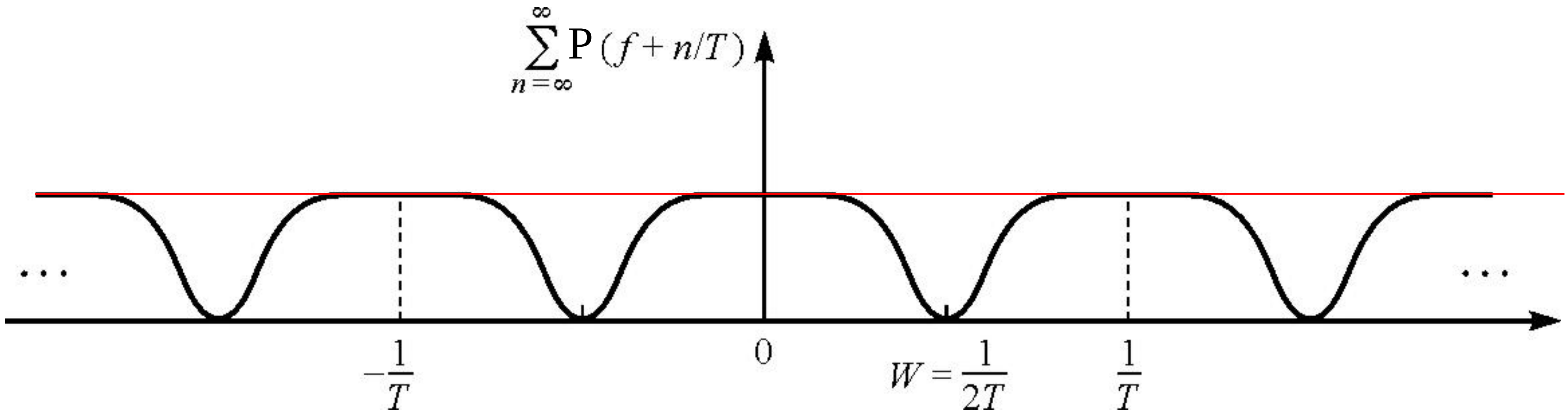
Nyquist Criterion for Zero ISI

- Define $p_i = p(iT_b)$
- Zero ISI ➔ $y_n = \sum_k a_k p((n-k)T_b) = \sum_k a_k p_{n-k} = \sum_i p_i a_{n-i} = \sqrt{E} a_n$
- Let $p_0 = \sqrt{E}$, then $y_n = a_n \sqrt{E} + \sum_{i \neq 0} p_i a_{n-i}$
- Cannot control $\{a_k\}$ ➔ $\sum_{i \neq 0} p_i a_{n-i} = 0$

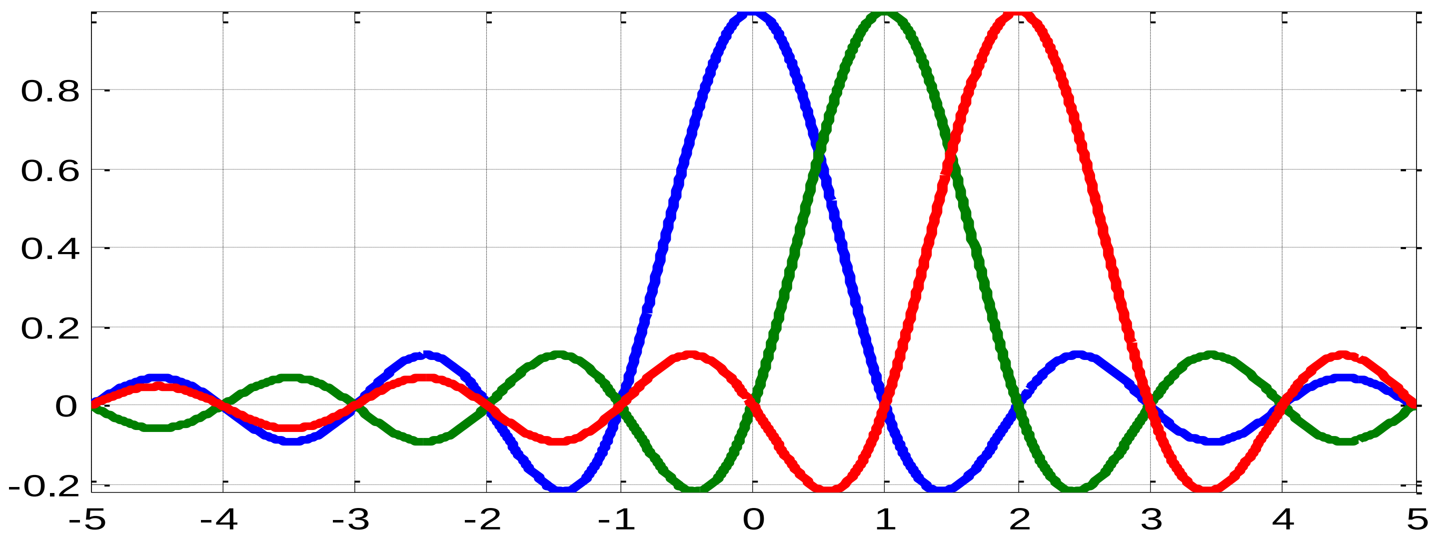
Nyquist Criterion for zero ISI

$$p_i = p(iT_b) = \begin{cases} \sqrt{E} & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

If 1/T=2W, then

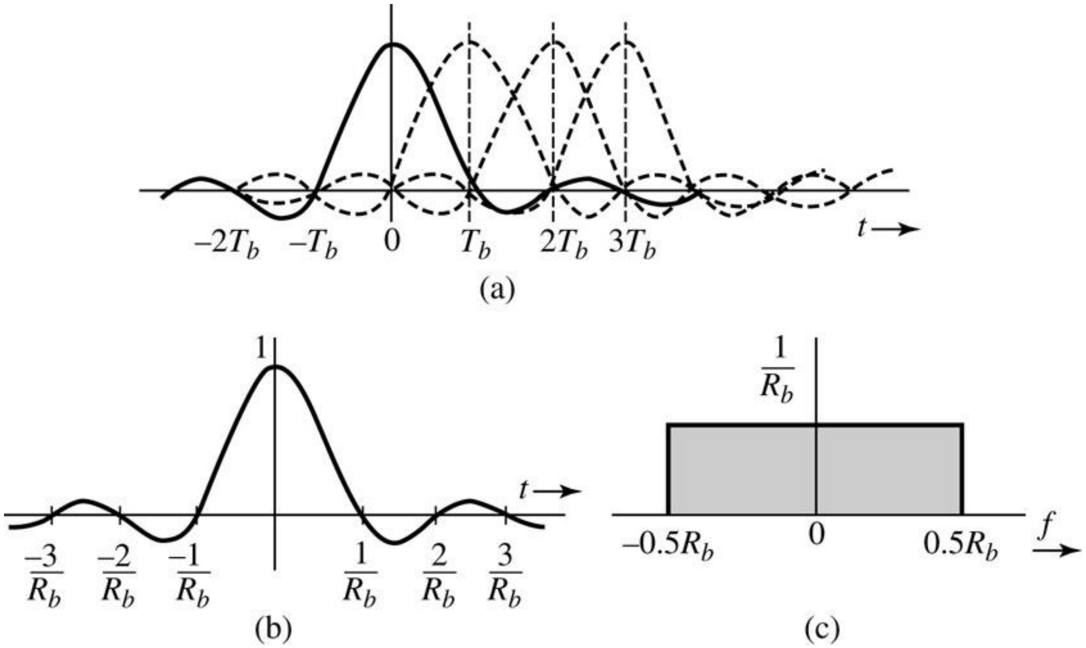


$\sum_{n=\infty}^{\infty} P(f+n/T)$
...
...
$-\frac{1}{T}$
0
$W=\frac{1}{2T}$
$\frac{1}{T}$



0.8
0.6
0.4
0.2
0
-0.2
-5
-4
-3
-2
-1
0
1
2
3
4
5

Ideal Nyquist Pulse



$$p(t) = \sqrt{E}\,\mathrm{sinc}\left(\frac{t}{T_b}\right) \qquad P(\omega) = \begin{cases} \sqrt{E} & |\omega| \le \dfrac{2\pi}{2T_b} \\ 0 & \text{otherwise} \end{cases} \qquad p_i = p(iT_b) = \begin{cases} \sqrt{E} & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

The Timing Error Problem

Should we use the optimal Nyquist pulse if feasible?

$$y(t)=a(t)*p(t)=\sum_k a_k p(t-kT_b) \qquad y_n=y(nT_b)=\sum_k a_k p((n-k)T_b)=a_n\sqrt{E}$$

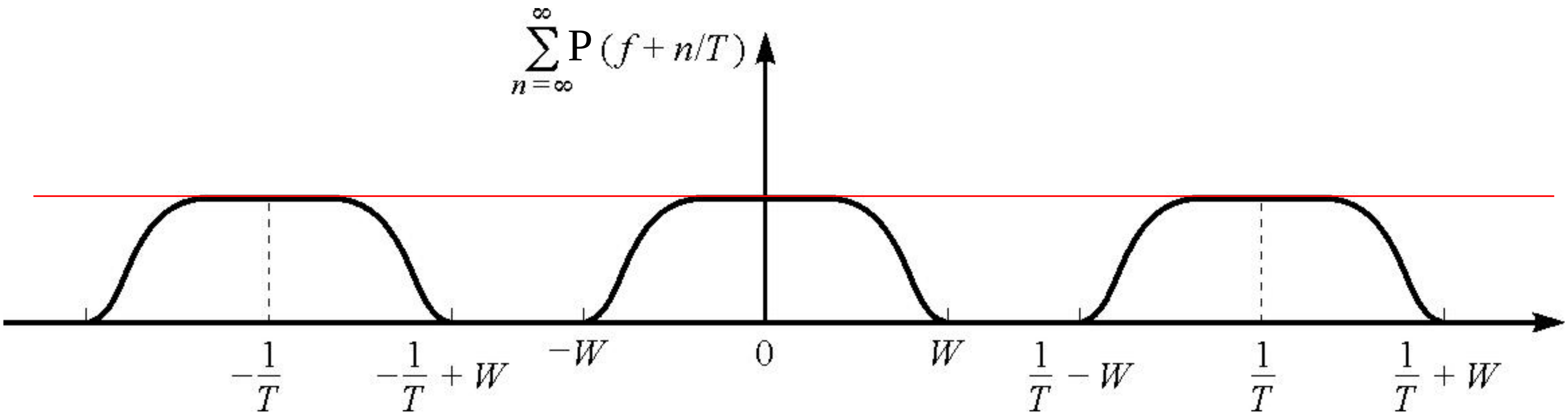
What if we sample at the receiver with a tiny offset? $\tilde{y}_n=y(nT_b+\Delta)=?$

$$p(t)=\sqrt{E}\operatorname{sinc}\left(\frac{t}{T_b}\right) \qquad \tilde{y}_0=y(\Delta)=\sum_k a_k p(\Delta-kT_b)=\sqrt{E}\sum_k a_k\operatorname{sinc}\left(\frac{\Delta-kT_b}{T_b}\right)$$

$$y(\Delta)=\sqrt{E}a_0\operatorname{sinc}\left(\frac{\Delta}{T_b}\right)+\sqrt{E}\frac{\sin\left(\frac{\pi\Delta}{T_b}\right)}{\pi}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{(-1)^k a_k}{\frac{\Delta}{T_b}-k}$$

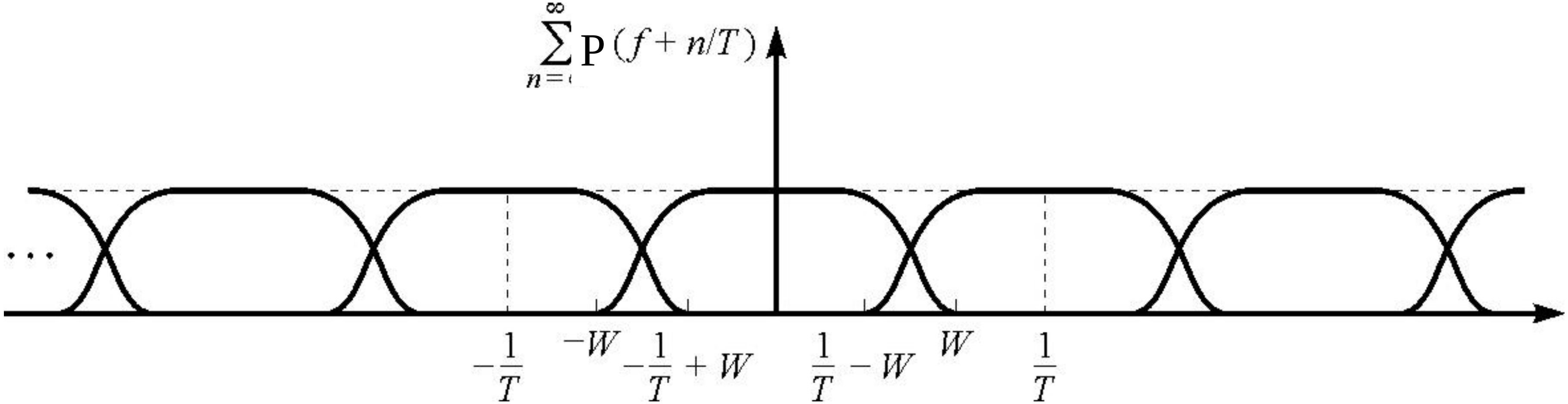
$$\approx\sqrt{E}a_0\operatorname{sinc}\left(\frac{\Delta}{T_b}\right)-\sqrt{E}\frac{\sin\left(\frac{\pi\Delta}{T_b}\right)}{\pi}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{(-1)^k a_k}{k}\approx\sqrt{E}a_0\operatorname{sinc}\left(\frac{\Delta}{T_b}\right)-\sqrt{E}\frac{\Delta}{T_b}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{(-1)^k a_k}{k}$$

If 1/T>2W, then



$\sum_{n=\infty}^{\infty} P(f+n/T)$
$-\frac{1}{T}$
$-\frac{1}{T}+W$
$-W$
0
W
$\frac{1}{T}-W$
$\frac{1}{T}$
$\frac{1}{T}+W$

If 1/T<2W, then


$\sum_{n=-\infty}^{\infty} P(f+n/T)$
$\cdots$
$-\frac{1}{T}$
$-W$
$-\frac{1}{T}+W$
$\frac{1}{T}-W$
W
$\frac{1}{T}$

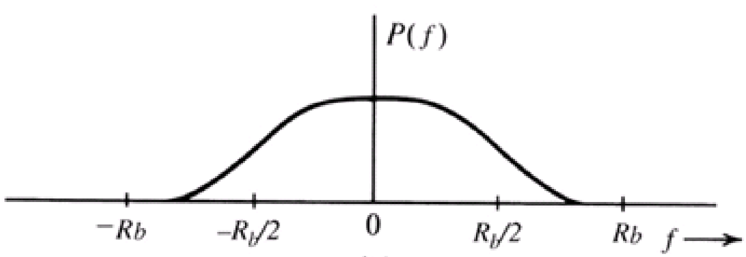
Raised Cosine Spectrum

$$p(t)=\sqrt{E}\operatorname{sinc}\left(\frac{t}{T_b}\right)\left(\frac{\cos(\alpha t\pi/T_b)}{1-4\alpha^2t^2/T_b^{\,2}}\right)=\left[\sqrt{E}\operatorname{sinc}\left(\frac{t}{T_b}\right)\right]\times\left(\frac{\cos(\alpha t\pi/T_b)}{1-4\alpha^2t^2/T_b^{\,2}}\right)$$

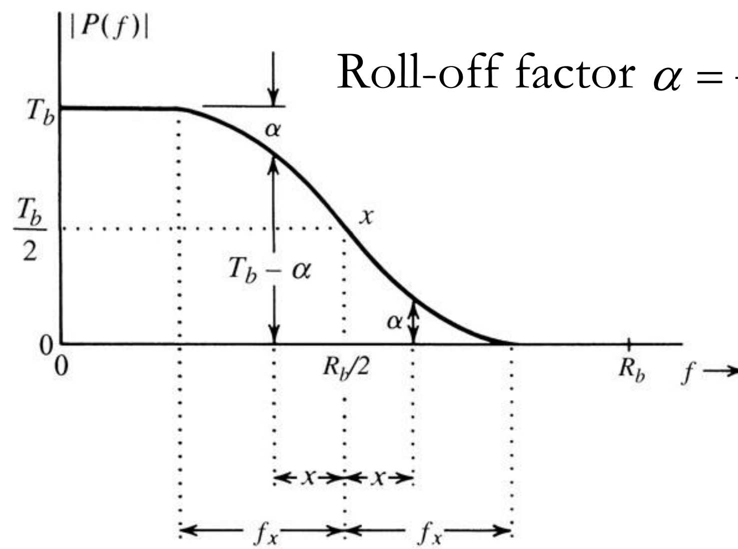
$$p_i=p(iT_b)=\begin{cases}\sqrt{E} & i=0\\ 0 & \text{otherwise}\end{cases}$$

For large t, envelope of $p(t)$ decays in $1/t^3$.

Transmission bandwidth requirement $B_T=(1+a)/2T_b$



(a)



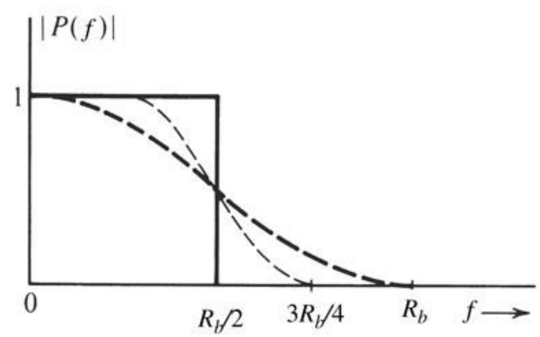
Roll-off factor $\alpha=\dfrac{f_x}{1/2T_b}$

Properties of the Raised Cosine Pulse

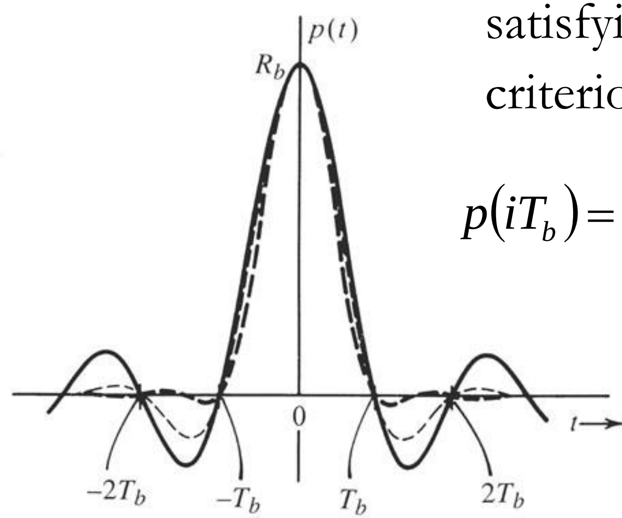
Properties:

1. When $\alpha = 1$, at $t = \pm\frac{T_b}{2}$, $p(t) = 0.5\sqrt{E}$
2. $p(t) = 0$ for $t = \pm T_b, \pm 2T_b, \cdots$
3. The roll-off portion of $P(\omega)$ is symmetric around $\omega = \frac{\pi}{T_b}$
4. $\frac{1}{T_b}\sum_k P(\omega - 2\pi k / T_b) = \sqrt{E}$ ⟵ Nyquist 2nd Criterion. All waveforms satisfying this criterion has

$$p(iT_b) = \begin{cases} \sqrt{E} & i = 0 \\ 0 & \text{otherwise} \end{cases}$$



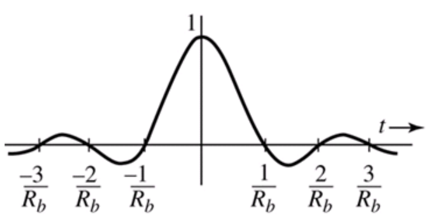
(a)



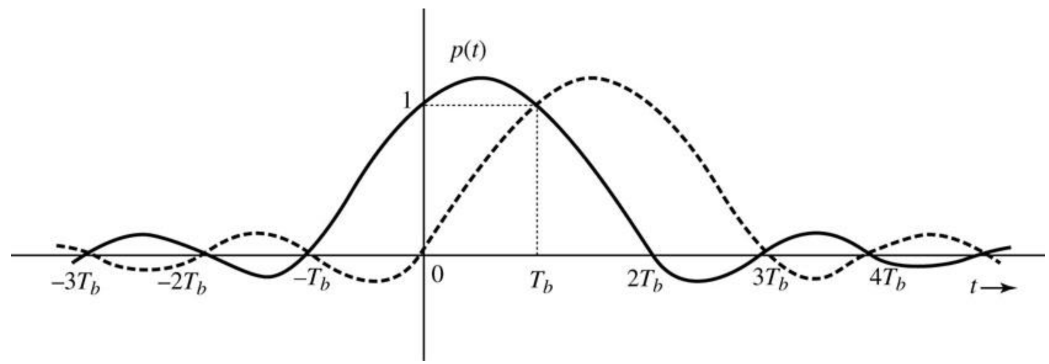
(b)

Signaling with Controlled ISI

Basic Idea: Want to further reduce the bandwidth of *p*(*t*). Can't satisfy the Nyquist criterion. However, can achieve a controlled ISI pattern, and therefore can remove ISI in the digital signal after sampling.



(b)



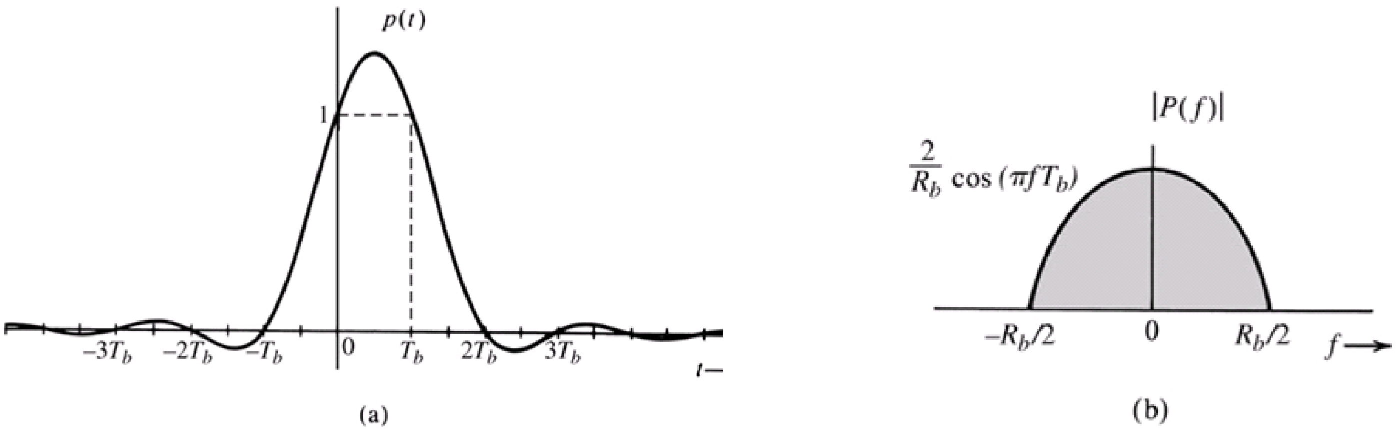
$$p(nT_b)=\begin{cases}1 & n=0\\ 0 & \text{otherwise}\end{cases} \qquad\qquad p(nT_b)=\begin{cases}1 & n=0,1\\ 0 & \text{otherwise}\end{cases}$$

$$y_n=y(nT_b)=\sum_k a_k\,p((n-k)T_b)=a_n \qquad\qquad y_n=y(nT_b)=\sum_k a_k\,p((n-k)T_b)=a_n+a_{n-1}$$

Essentially doing power and bandwidth tradeoff

Duobinary Pulse

$$p(t)=\frac{\sin(\pi R_b t)}{\pi R_b t(1-R_b t)} \qquad P(\omega)=\frac{2}{R_b}\cos\left(\frac{\omega}{2R_b}\right)\text{rect}\left(\frac{\omega}{2\pi R_b}\right)e^{-j\frac{\omega}{2R_b}}$$



For large t, $p(t)$ decays in $1/t^2$. Therefore, time error problem alleviated.

50% wider than the optimal Nyquist pulse.

$$p(nT_b)=\begin{cases}1 & n=0,1\\ 0 & \text{otherwise}\end{cases}$$

Nyquist Pulse vs. Duobinary Pulse

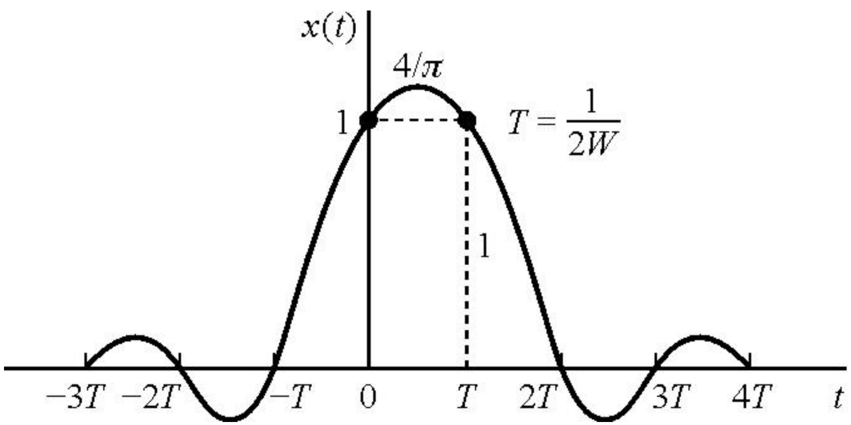
$p_a(t)$ Nyquist pulse, satisfying $p_a(nT_b) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$

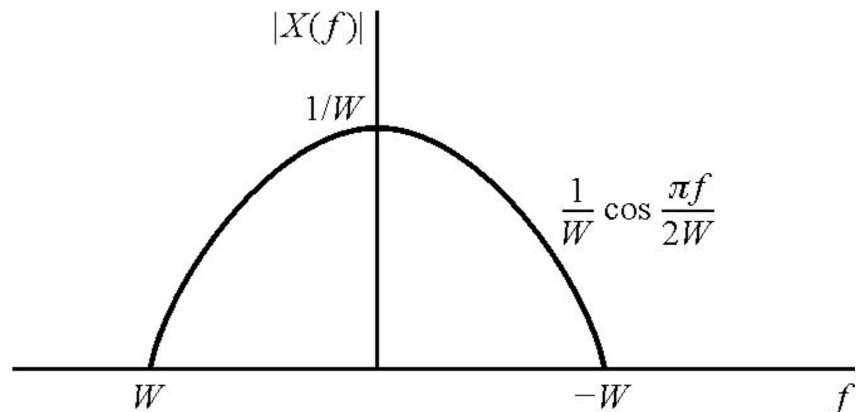
$p_b(t)$ duobinary pulse, satisfying $p_b(nT_b) = \begin{cases} 1 & n = 0,1 \\ 0 & \text{otherwise} \end{cases}$

Hence, from $p_a(t)$, can form $p_b(t) = p_a(t) + p_a(t - T_b)$

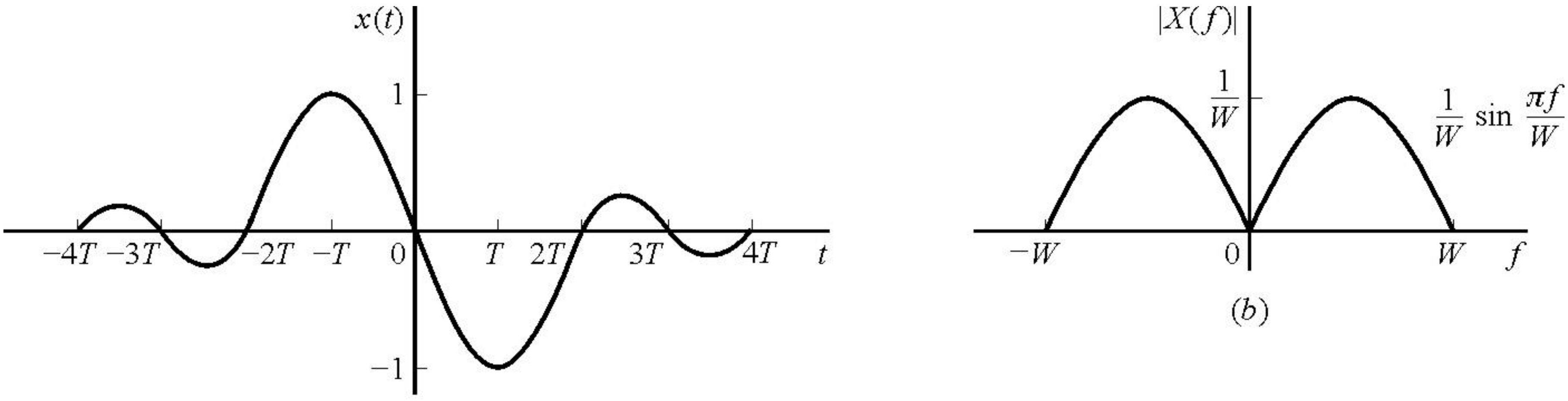
$$P_b(\omega) = P_a(\omega)\left(1 + e^{-j\omega T_b}\right)$$

$$|P_b(\omega)| = |P_a(\omega)|\sqrt{[1+\cos(\omega T_b)]^2 + \sin^2(\omega T_b)} = |P_a(\omega)|\sqrt{2[1+\cos(\omega T_b)]}$$





Modified Doubinary Pulse



(a) (b)

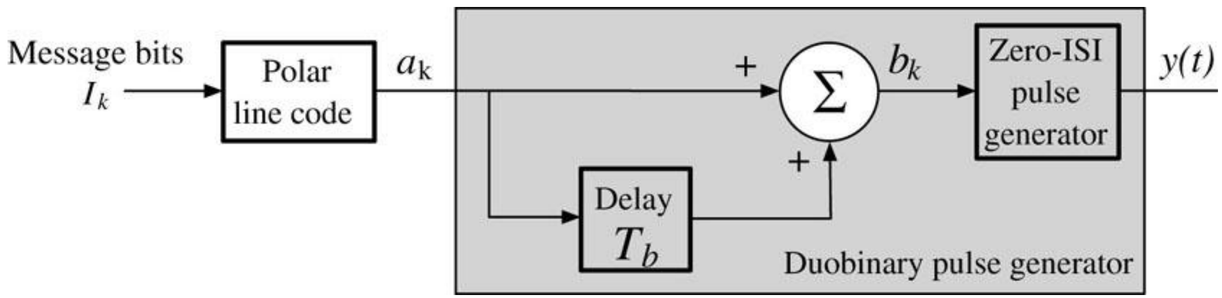
Modified duobinary pulse, requiring $p_c(nT_b) = \begin{cases} 1 & n=-1 \\ -1 & n=1 \\ 0 & \text{otherwise} \end{cases}$

$$p_c(t) = p_a(t+T_b) - p_a(t-T_b)$$

$$P_c(\omega) = P_a(\omega)\left(e^{j\omega T_b} - e^{-j\omega T_b}\right) = 2jP_a(\omega)\sin(\omega T_b)$$

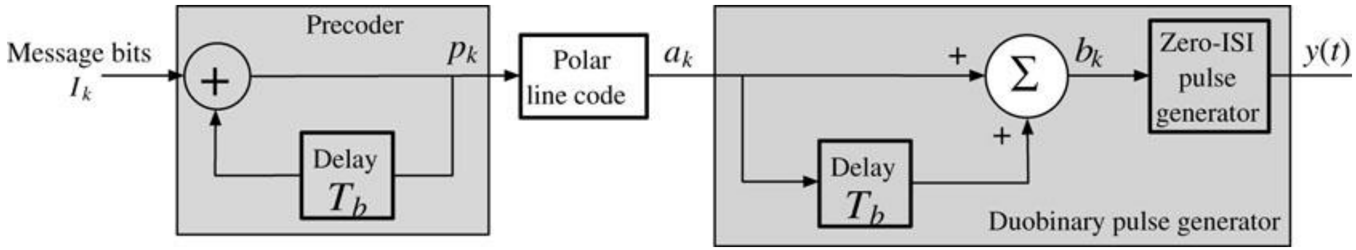
Generation of Duobinary Pulse

$$p_b(t) = p_a(t) + p_a(t - T_b)$$



If $a_k = \pm 1$, one erroneous bit can cause bit flipping in all subsequential bits.

Solution: encode $a_k \oplus a_{k\text{-}1}$



Baseband Transmission of M-ary Data

- Duobinary signaling:
  - Supports Tx of a bit-stream with less bandwidth than the Nyquist pulses.
  - Plays the power-bandwidth tradeoff game: use more power but less bandwidth

- More directly: Compress several binary symbols into one *M*-ary symbol

Example: 4-ary PAM

$$b_k = \begin{cases} -3 & a_{2k} = -1, a_{2k+1} = -1 \\ -1 & a_{2k} = 1, a_{2k+1} = -1 \\ 1 & a_{2k} = -1, a_{2k+1} = 1 \\ 3 & a_{2k} = 1, a_{2k+1} = 1 \end{cases}$$

Example: Use 8kHz sampling rate, each sample is quantized into 4bits. Use 4-ary PAM, and use raised-cosine pulse with α=1. What is the required channel bandwidth?

Bit rate = $8k \times 4 = 32k\ bit/s$ Symbol rate = $32k/2 = 16k$ symbol / s

Bandwidth = $(1+\alpha)/2T_b = 1/T_b = 16kHz$

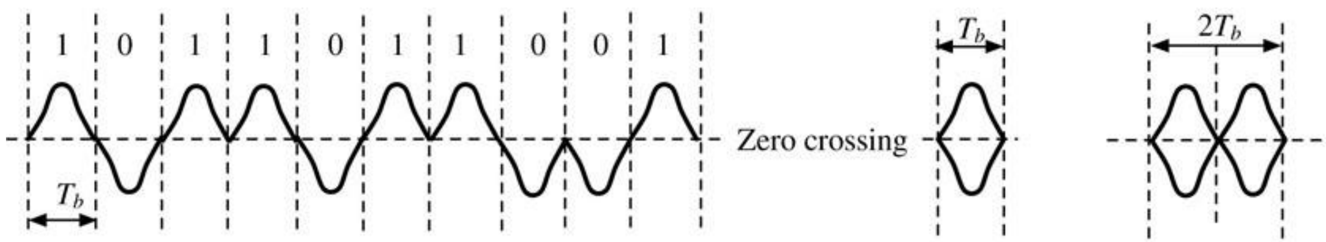
The Eye Diagram

$$y(t) = a(t) * p(t) = \sum_{k=-\infty}^{\infty} a_k p(t - kT_b)$$

If $p(t)$ is well designed, we should have $y(iT_b) = \sum_{k=-\infty}^{\infty} a_k p(iT_b - kT_b) = \sqrt{E} a_i$

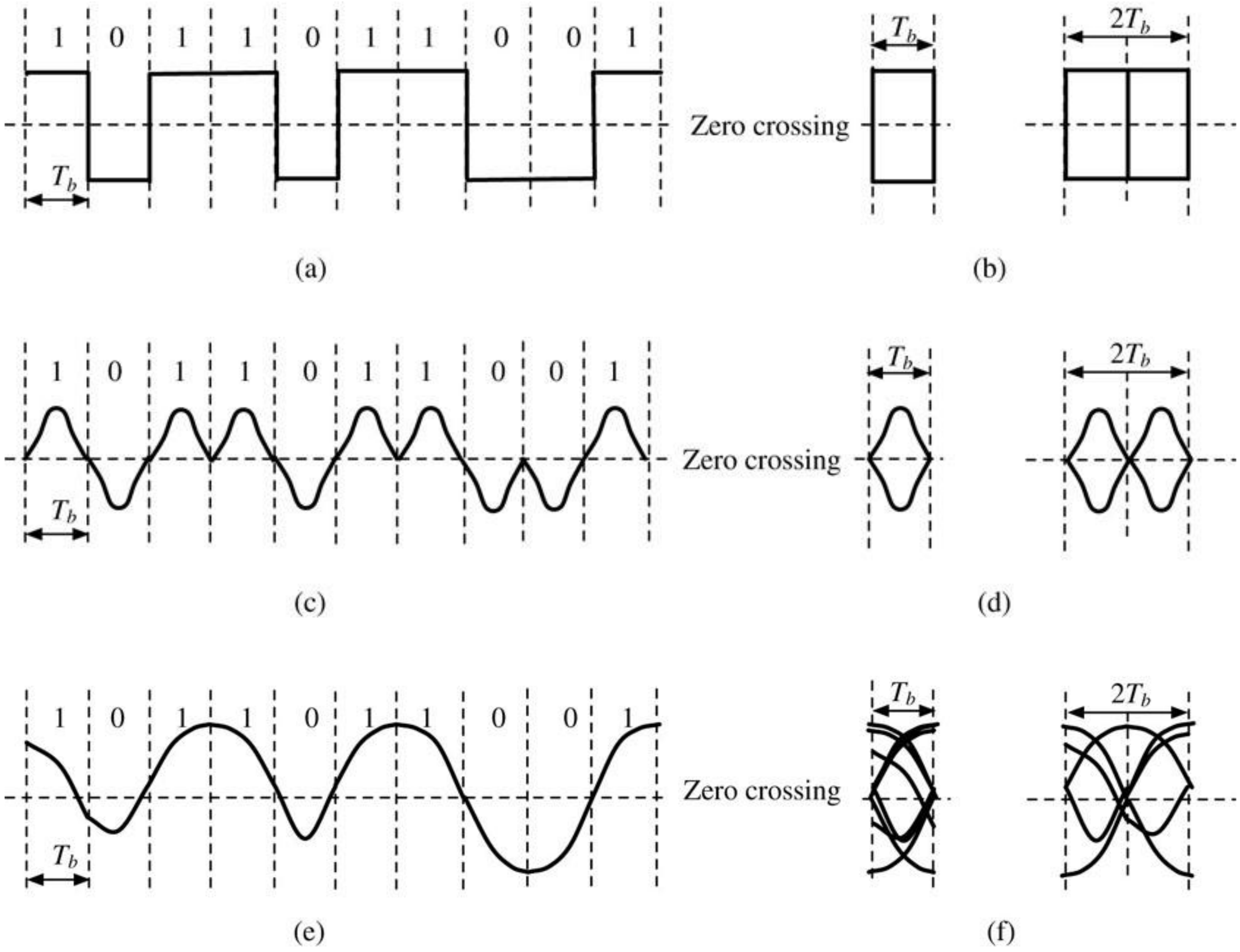
Now construct a time function defined on $-\frac{T_b}{2} \le t \le \frac{T_b}{2}$

$$eye(t) = \frac{1}{\sqrt{E}} \sum_{i=-\infty}^{\infty} y(t - iT_b) = \frac{1}{\sqrt{E}} \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_k p(t - (i+k)T_b)$$

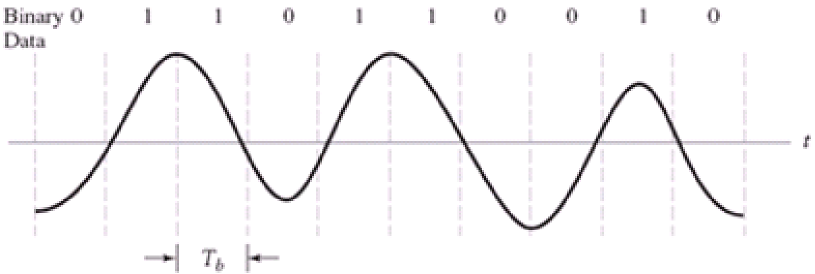


(c) (d)

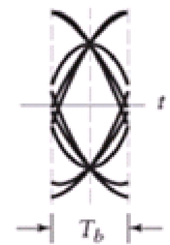
The Eye Diagram



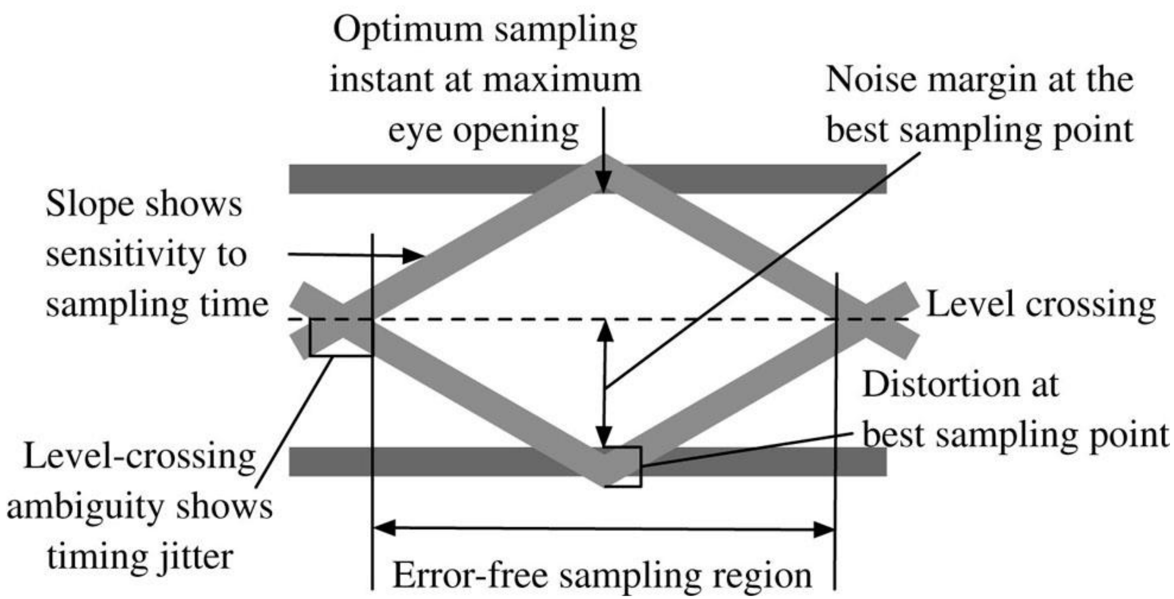
The Eye Diagram



(a)

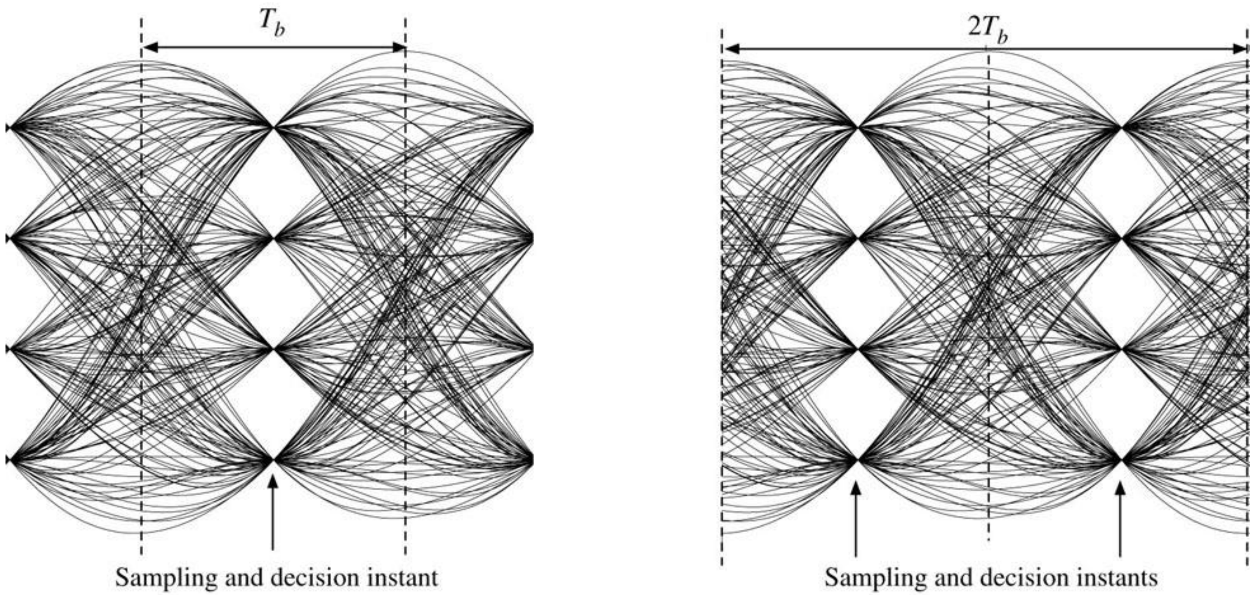


(b)



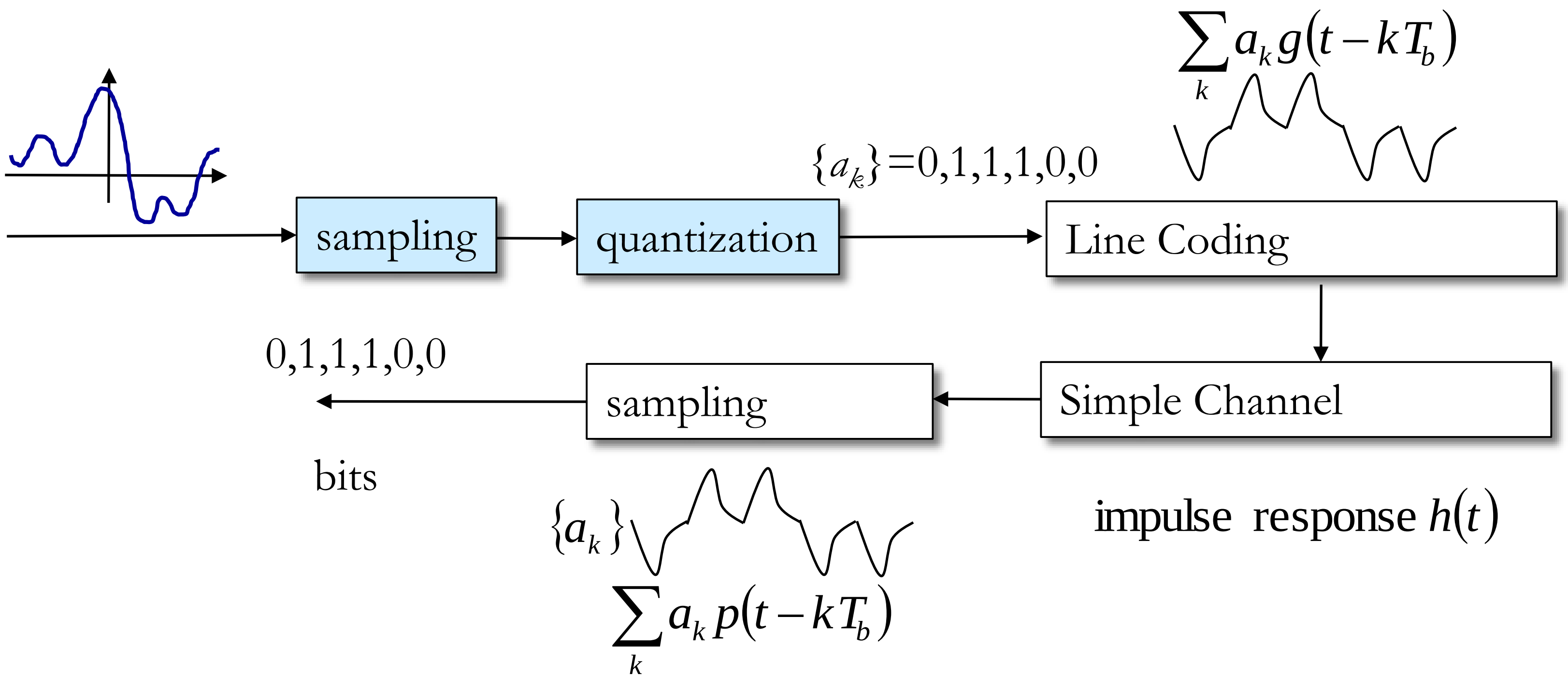
Eye Diagram for M-ary Modulation

If a_k can take M possible values instead of two, the eye pattern looks like the following



(a) (b)

Channel Equalization



We have very limited circuit tools to generate waveforms.

We do not really control the channel.

Therefore, the effective $p(t)$ experienced by the receiver often does not satisfy the Nyquist criterion.

Zero-Forcing Channel Equalization

Basic Idea: $p(t)$ does not satisfy the Nyquist criterion. Can find an $c[n]$, such that after sampling $p(t)$ to get $p[n]$, we have $p[n]*c[n]=\delta[n]$

Not always possible.

$$y_i = y(iT_b) = \sum_{k=-\infty}^{\infty} a_k p(iT_b - kT_b) = \sum_{k=-\infty}^{\infty} a_k p_{i-k} \qquad y[n]=a[n]*p[n]$$

If $p[n] \neq \delta[n]$, find $c[n]$ to achieve $p[n]*c[n]=\delta[n-n_0]$

In Z-domain

$$Y(z)=A(z)P(z) \qquad P(z)C(z)=z^{-n_0}$$

Example:

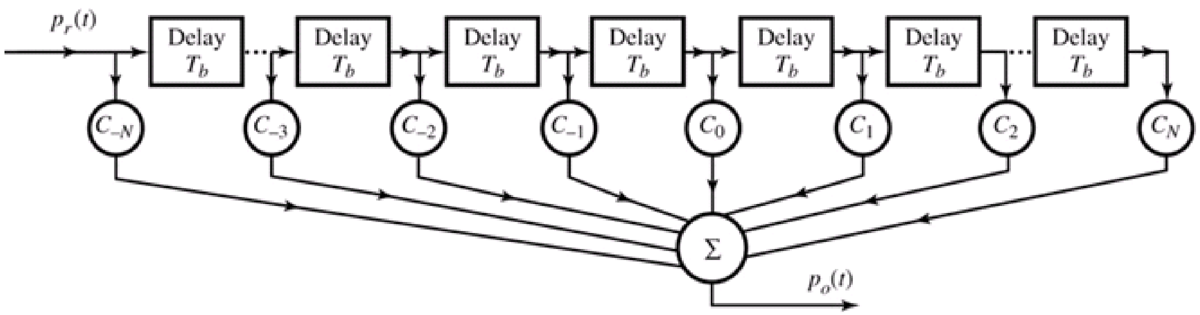
$$P(z)=1-0.5z^{-1} \qquad P(z)C(z)=z^{-n_0} \Rightarrow C(z)=\frac{z^{-n_0}}{1-0.5z^{-1}}=z^{-n_0}\left(1+0.5z^{-1}+\cdots\right)$$

Channel Equalization

Example:

$$P(z)=1-0.5z^{-1} \qquad P(z)C(z)=z^{-n_0} \Rightarrow C(z)=\frac{z^{-n_0}}{1-0.5z^{-1}}=z^{-n_0}\left(1+0.5z^{-1}+\cdots\right)$$

Often prefer FIR filter for equalization



Therefore, need to design FIR filter such that $P(z)C(z)\approx 1$

Zero-Forcing Equalization

$$C(z)=\sum_{k=-N}^{N} c[k]z^{-k} \qquad P(z)C(z)=o\left(z^{N}\right)+0\times z^{N-1}+\cdots+1+\cdots+0\times z^{-N+1}+o\left(z^{-N}\right)$$

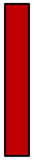
Zero-Forcing Equalization

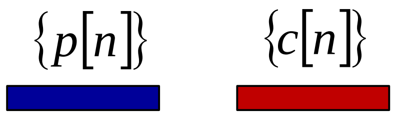
Zero-Forcing Equalization

$$C(z)=\sum_{k=-N}^{N} c[k]z^{-k} \qquad P(z)C(z)=1+o\left(z^{-N}\right)+o\left(z^{N}\right)$$

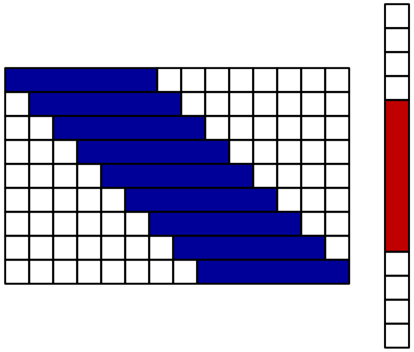
Choose $\left[c_{-N}, c_{-N+1}, \cdots, c_0, \cdots, c_N\right]$ to satisfy

$$\begin{bmatrix} p_0 & \cdots & p_{-N+1} & p_{-N} & p_{-N-1} & \cdots & p_{-2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{N-1} & \cdots & p_0 & p_{-1} & p_{-2} & \cdots & p_{-N-1} \\ p_N & \cdots & p_1 & p_0 & p_{-1} & \cdots & p_{-N} \\ p_{N+1} & \cdots & p_2 & p_1 & p_0 & \cdots & p_{-N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{2N} & \cdots & p_{N+1} & p_N & p_{N-1} & \cdots & p_0 \end{bmatrix} \begin{bmatrix} c_{-N} \\ \vdots \\ c_{-1} \\ c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sqrt{E} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

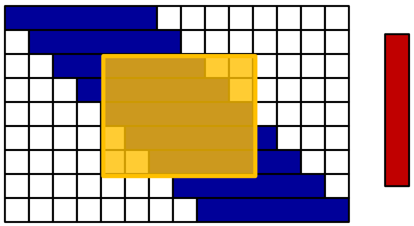




Convolution



Truncated version



Channel Equalization

Example: $P(z) = 0.05z^{2} - 0.2z + 1 - 0.3z^{-1} + 0.1z^{-2}$

Design a 3-tap zero-forcing equalizer $[c_{-1}, c_0, c_1]$

$$\begin{bmatrix} 1 & -0.2 & 0.05 \\ -0.3 & 1 & -0.2 \\ 0.1 & -0.3 & 1 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & -0.2 & 0.05 \\ -0.3 & 1 & -0.2 \\ 0.1 & -0.3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2094 \\ 1.1262 \\ 0.3169 \end{bmatrix}$$

$$C(z) = 0.2094\, z + 1.1262 + 0.3169\, z^{-1}$$

$$P(z)C(z) = \left(0.05z^{2} - 0.2z + 1 - 0.3z^{-1} + 0.1z^{-2}\right)\left(0.2094\, z + 1.1262 + 0.3169\, z^{-1}\right)$$
$$= 0.0105\, z^{3} + 0.0144\, z^{2} + 1 + 0.0175\, z^{-2} + 0.0317\, z^{-3}$$

Channel Equalization

Example: $P(z)=0.05z^{2}-0.2z+1-0.3z^{-1}+0.1z^{-2}$

Design a 5-tap zero-forcing equalizer $[c_{-2},c_{-1},c_{0},c_{1},c_{2}]$

$$\begin{bmatrix} 1 & -0.2 & 0.05 & 0 & 0 \\ -0.3 & 1 & -0.2 & 0.05 & 0 \\ 0.1 & -0.3 & 1 & -0.2 & 0.05 \\ 0 & 0.1 & -0.3 & 1 & -0.2 \\ 0 & 0 & 0.1 & -0.3 & 1 \end{bmatrix}\begin{bmatrix} c_{-2} \\ c_{-1} \\ c_{0} \\ c_{1} \\ c_{2} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_{0} \\ c_{1} \\ c_{2} \end{bmatrix}=\begin{bmatrix} -0.0153 \\ 0.2051 \\ 1.1267 \\ 0.3138 \\ -0.0185 \end{bmatrix}$$

$$C(z)=-0.0153z^{2}+0.2051z+1.1267+0.3138z^{-1}-0.0185z^{-2}$$

$$\begin{aligned} P(z)C(z)&=\left(0.05z^{2}-0.2z+1-0.3z^{-1}+0.1z^{-2}\right)\\ &\quad\times\left(-0.0153z^{2}+0.2051z+1.1267+0.3138z^{-1}-0.0185z^{-2}\right)\\ &=-0.008z^{4}+0.0133z^{3}+1+0.0369z^{-3}-0.0019z^{-4} \end{aligned}$$

Channel Identification

$$y_i = y(iT_b) = \sum_{k=-\infty}^{\infty} a_k\, p(iT_b - kT_b) = \sum_{k=-\infty}^{\infty} a_k\, p_{i-k}$$

Since $p(t)$ is the joint effect of pulse shaping and channel distortion it may not be known at the transmitter or at the receiver. How can we identify $\{p_k\}$?

Approach 1: transmitting pilot sequence, i.e., known $\{a_k\}$

$$\begin{bmatrix} a_0 & \cdots & a_{-N+1} & a_{-N} & a_{-N-1} & \cdots & a_{-2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{N-1} & \cdots & a_0 & a_{-1} & a_{-2} & \cdots & a_{-N-1} \\ a_N & \cdots & a_1 & a_0 & a_{-1} & \cdots & a_{-N} \\ a_{N+1} & \cdots & a_2 & a_1 & a_0 & \cdots & a_{-N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2N} & \cdots & a_{N+1} & a_N & a_{N-1} & \cdots & a_0 \end{bmatrix} \begin{bmatrix} p_{-N} \\ \vdots \\ p_{-1} \\ p_0 \\ p_1 \\ \vdots \\ p_N \end{bmatrix} = \begin{bmatrix} y_{-N} \\ \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix}$$

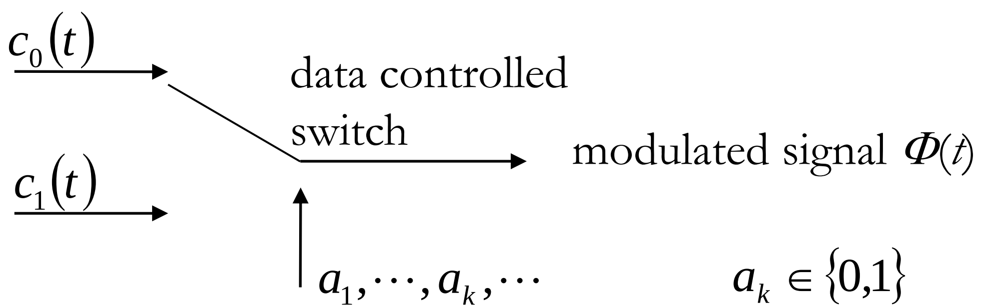
Roadmap

- Line Coding (Transmission Coding)
- Digital Baseband Transmission
- Digital Band-Pass Modulation
  - Binary band-pass modulation
  - M-ary band-pass modulation
  - Signal space diagram
  - Detector options

Digital Band-Pass Modulation Techniques

Requirement: transmit digital signal through a band-pass channel.
Key Idea: Use binary (or digital) sequence to modulate a carrier sinusoid signal $c(t) = A_c \cos(\omega_c t + \phi_c)$

Basic Diagram:



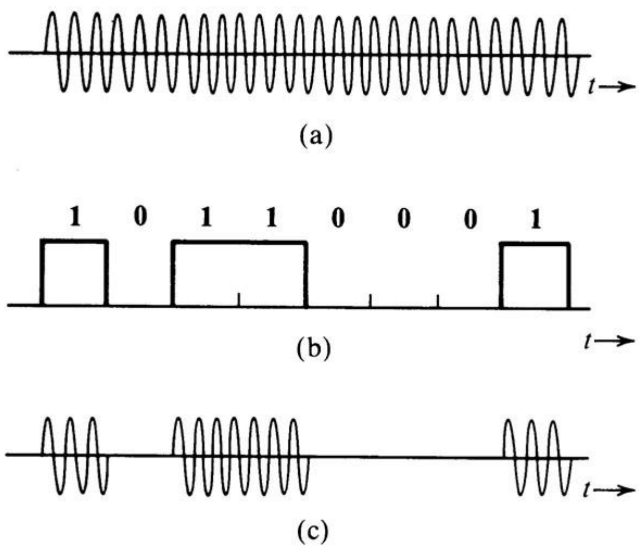
$a_k \in \{0,1\}$

Usually normalize the carrier signal power

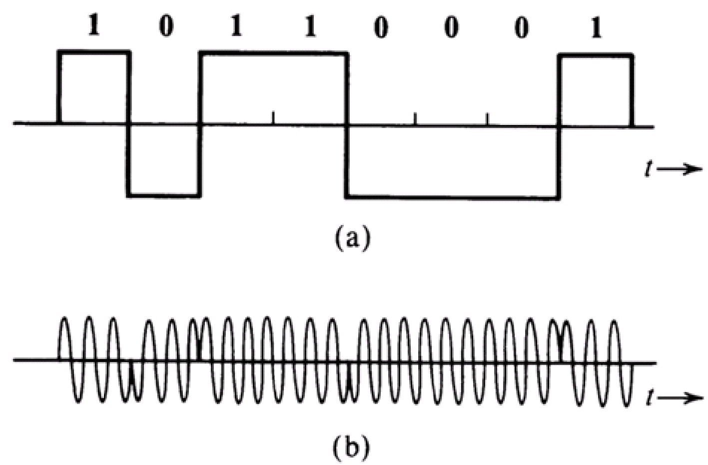
$$c(t) = \sqrt{\frac{2}{T_b}} \cos(\omega_c t + \phi_c) \qquad P_c = \frac{1}{2}\left(\sqrt{\frac{2}{T_b}}\right)^2 = \frac{1}{T_b}$$

Unit energy in one symbol duration

Binary Amplitude Shift Keying



OOK



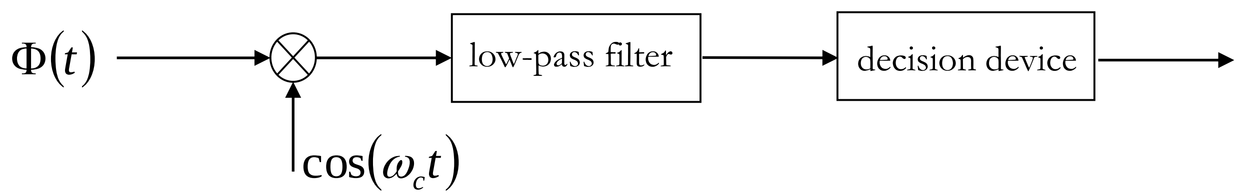
BASK

On-Off Keying

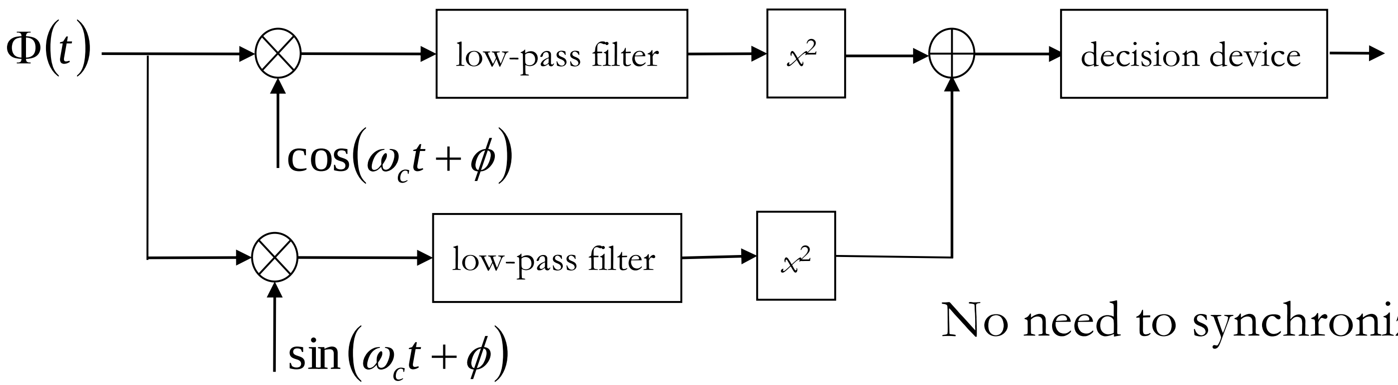
Modulation

$$c_0(t) = 0 \qquad c_1(t) = \sqrt{\frac{2}{T_b}}\cos(\omega_c t) \qquad \Phi(t) = \begin{cases} \sqrt{\frac{2}{T_b}}\cos(\omega_c t) & \text{for symbol 1} \\ 0 & \text{for symbol 0} \end{cases}$$

Coherent Detection (need to know the phase of the carrier)



Non-coherent Detection (no phase information)



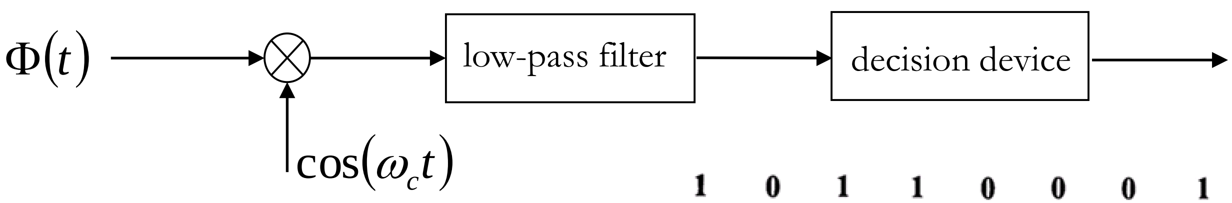
No need to synchronize phase

Binary Phase Shift Keying (BPSK)

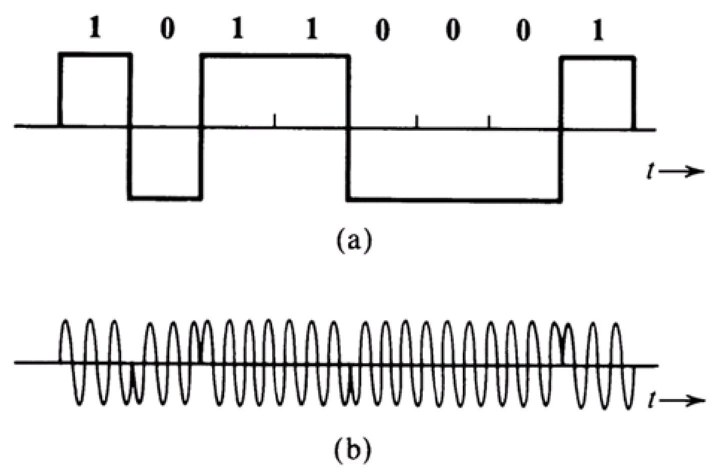
Modulation

$$c_0(t)=\sqrt{\frac{2}{T_b}}\cos(\omega_c t+\pi) \quad c_1(t)=\sqrt{\frac{2}{T_b}}\cos(\omega_c t) \quad \Phi(t)=\begin{cases}\sqrt{\frac{2}{T_b}}\cos(\omega_c t) & \text{for symbol 1}\\ -\sqrt{\frac{2}{T_b}}\cos(\omega_c t) & \text{for symbol 0}\end{cases}$$

Coherent Detection (need to know the phase of the carrier)

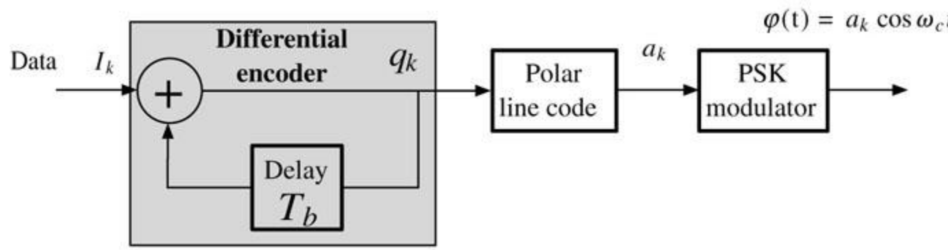


The same as BASK

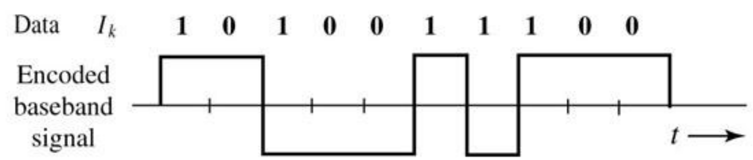


Differential Phase Shift Keying (DPSK)

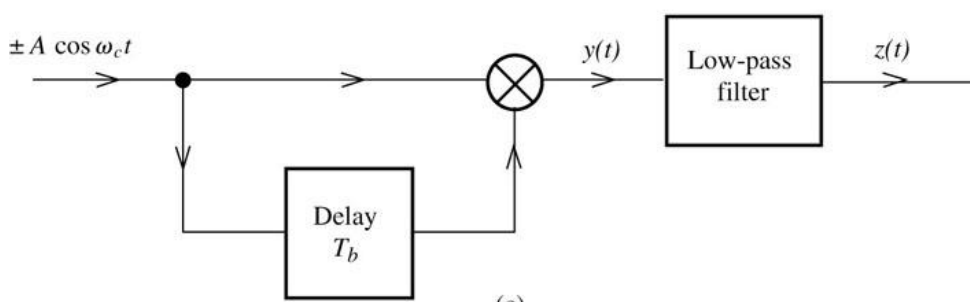
User PSK to modulate $a_k \oplus a_{k-1}$



(a)



(b)

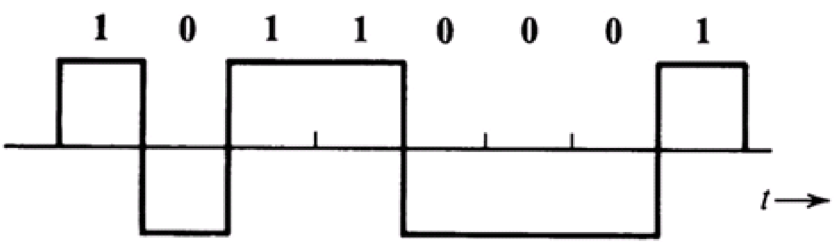


(c)

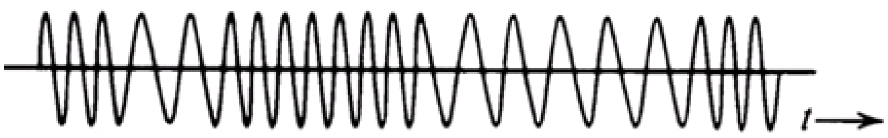
Binary Frequency Shift Keying (BFSK)

Modulation

$$c_0(t)=\sqrt{\frac{2}{T_b}}\cos(\omega_{c0}t) \qquad c_1(t)=\sqrt{\frac{2}{T_b}}\cos(\omega_{c1}t) \qquad \Phi(t)=\begin{cases}\sqrt{\frac{2}{T_b}}\cos(\omega_{c1}t) & \text{for symbol 1}\\ \sqrt{\frac{2}{T_b}}\cos(\omega_{c0}t) & \text{for symbol 0}\end{cases}$$



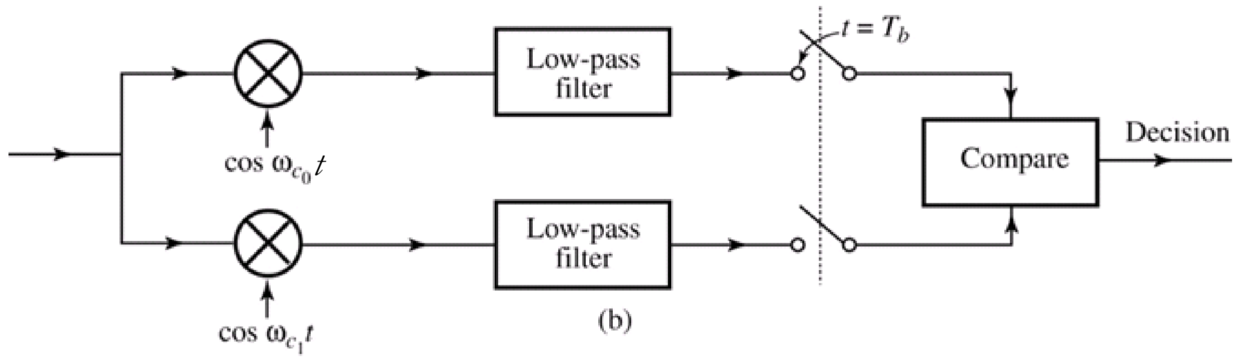
(a)



(c)

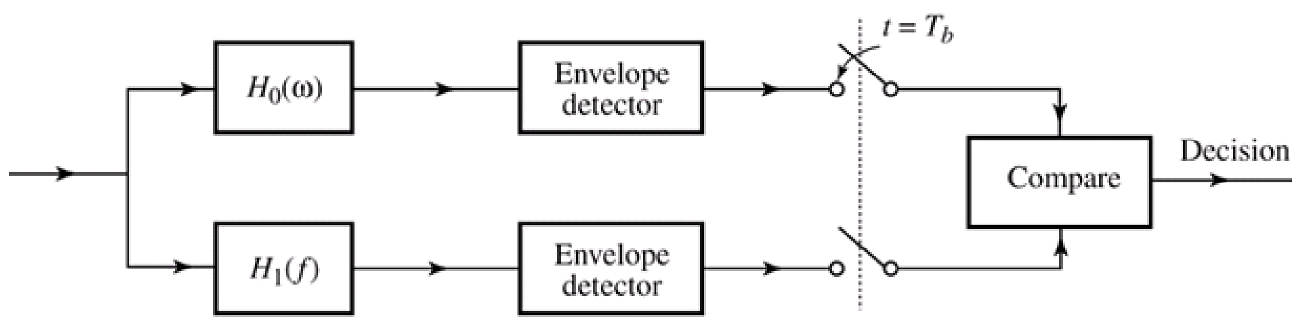
BFSK Detection

Coherent Detection (need to know the phase of the carrier)



(b)

Non-coherent Detection (no phase information)



(a)

No need to synchronize phase

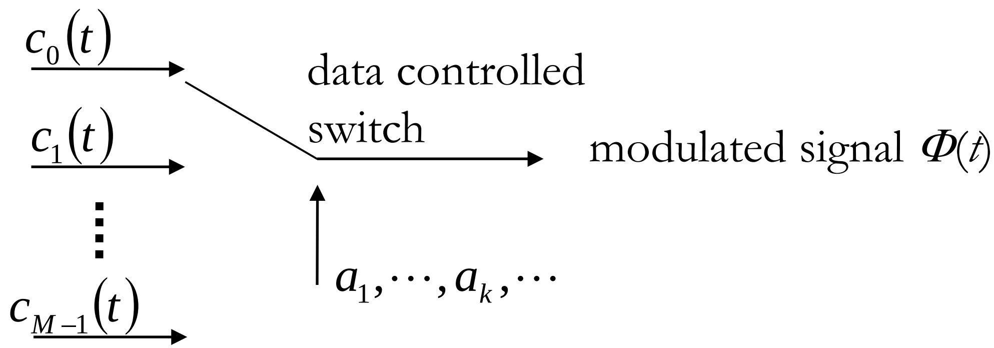
Roadmap

- Line Coding (Transmission Coding)
- Digital Baseband Transmission
- Digital Band-Pass Modulation
  - Binary band-pass modulation
  - M-ary band-pass modulation
  - Signal space diagram
  - Detector options

M-ary Digital Carrier Modulation

Input data $\{a_k\}$ $\quad a_k \in \{0, \cdots, M-1\}$

Basic Diagram:



M-ary Amplitude-Shift Keying (MASK)

$$\Phi(t) = a_k \sqrt{\frac{2}{T_b}} \cos(\omega_c t)$$

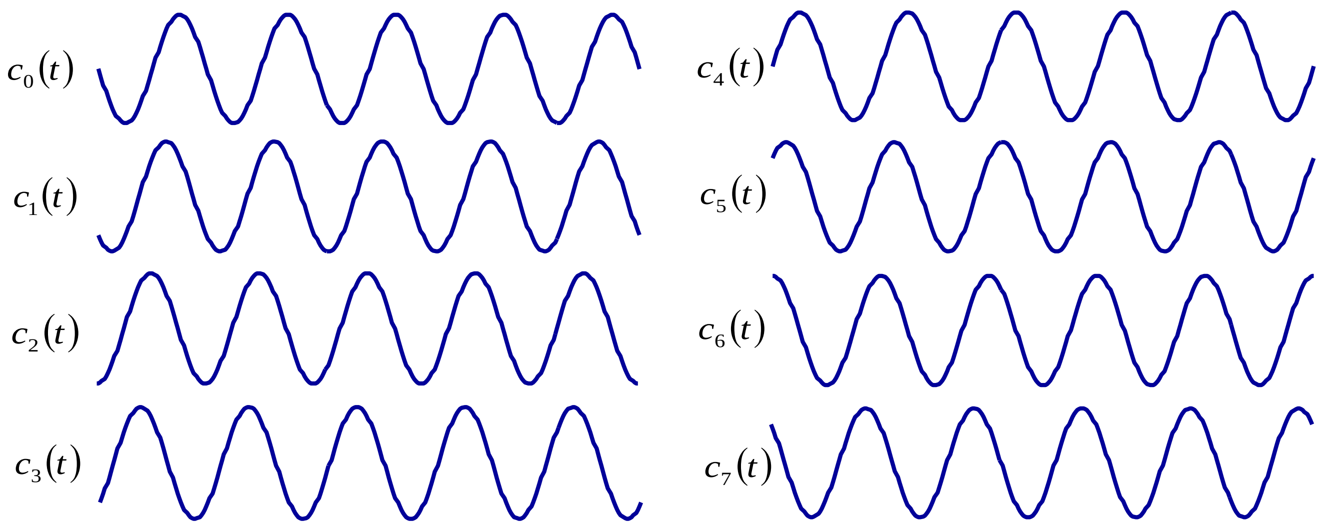
Variation

$$\Phi(t) = (2a_k - M + 1)\sqrt{\frac{2}{T_b}} \cos(\omega_c t)$$

MPSK and MFSK

M-ary Phase-Shift Keying

$$c_i(t) = \sqrt{\frac{2}{T_b}} \cos\left(\omega_c t + \frac{2\pi i}{M}\right) \qquad i = 0, \cdots, M-1 \qquad 8-PSK$$



M-ary Frequency-Shift Keying

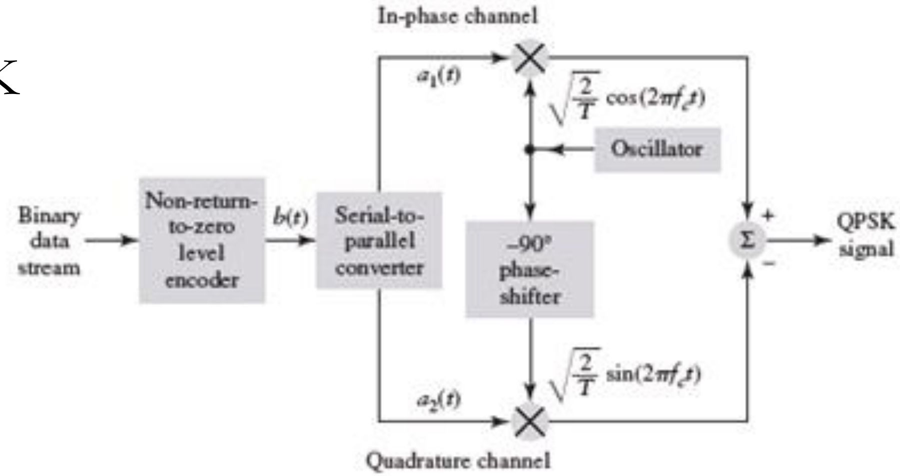
$$c_i(t) = \sqrt{\frac{2}{T_b}} \cos(\omega_i t) \qquad i = 0, \cdots, M-1$$

QPSK

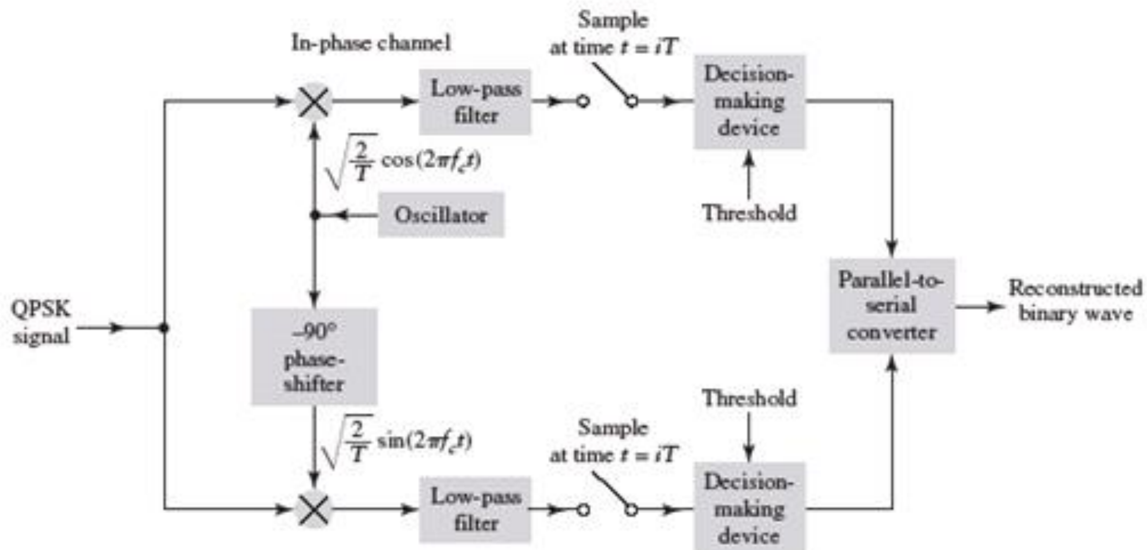
4-ary Phase-Shift Keying = QPSK

$$c_i(t) = \sqrt{\frac{2}{T_b}} \cos\left(\omega_c t + \frac{\pi i}{2}\right)$$

$$i = 0, \cdots, 3$$



(a)



(b)

Signal Space Diagram

Some basis signals

$$\sqrt{2}\cos(\omega_1 t) \quad \sqrt{2}\sin(\omega_1 t) \quad \sqrt{2}\cos(\omega_2 t) \quad \sqrt{2}\sin(\omega_2 t)$$

Inner product

$$\langle x(t), y(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)y^*(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)y(t)dt$$

$$\left\langle\sqrt{2}\cos(\omega_1 t), \sqrt{2}\cos(\omega_1 t)\right\rangle = 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\cos^2(\omega_1 t)dt = 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{1+\cos(2\omega_1 t)}{2}dt = 1$$

$$\left\langle\sqrt{2}\sin(\omega_1 t), \sqrt{2}\sin(\omega_1 t)\right\rangle = 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sin^2(\omega_1 t)dt = 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{1-\cos(2\omega_1 t)}{2}dt = 1$$

$$\left\langle\sqrt{2}\sin(\omega_1 t), \sqrt{2}\cos(\omega_1 t)\right\rangle = 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sin(\omega_1 t)\cos(\omega_1 t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sin(2\omega_1 t)dt = 0$$

$$\left\langle\sqrt{2}\cos(\omega_1 t), \sqrt{2}\cos(\omega_2 t)\right\rangle = 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\cos(\omega_1 t)\cos(\omega_1 t)dt$$

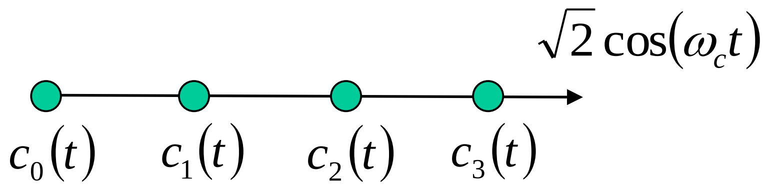
$$= 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{\cos((\omega_1+\omega_2)t)+\cos((\omega_1-\omega_2)t)}{2}dt = 0$$

MASK

First version

$$c_k(t) = a_k \sqrt{2}\cos(\omega_c t) \qquad a_k \in \{0, \cdots, M-1\}$$

One-dimensional space

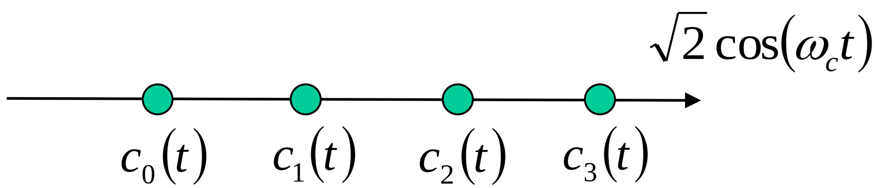


$$c_k(t) \to k \qquad k \in \{0, \cdots, M-1\}$$

Second version

$$c_k(t) = (2a_k - M + 1)\sqrt{2}\cos(\omega_c t) \qquad a_k \in \{0, \cdots, M-1\}$$

One-dimensional space



$$c_k(t) \to 2k - M + 1 \qquad k \in \{0, \cdots, M-1\}$$

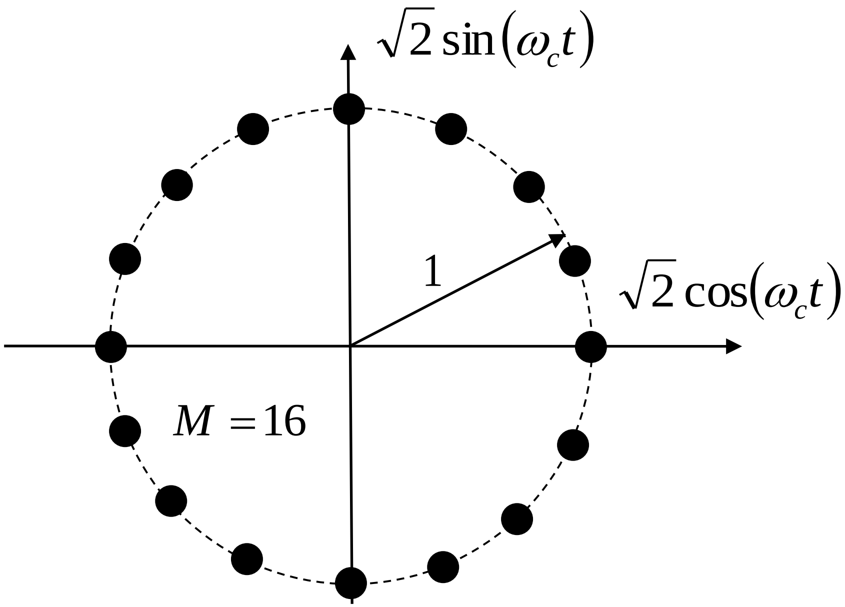
MPSK

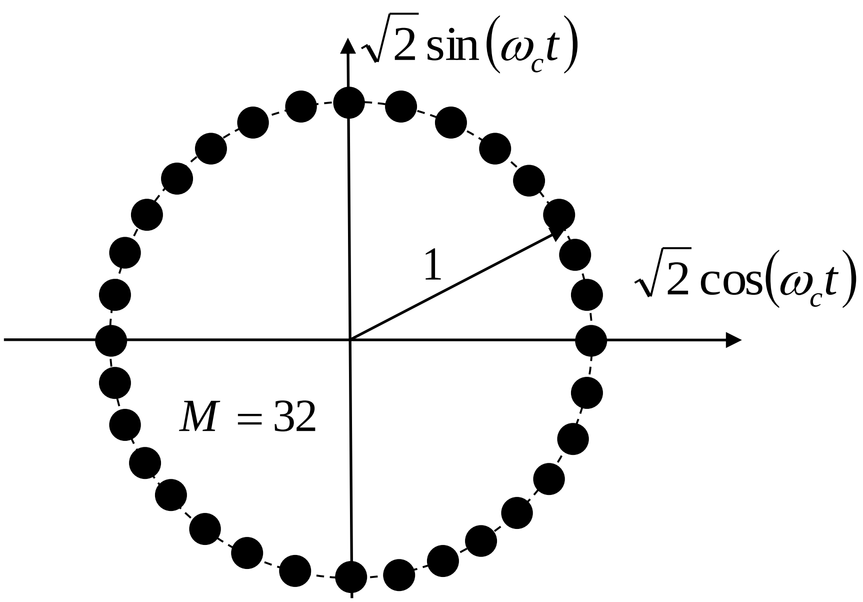
$$c_k(t) = \sqrt{2}\cos\left(\omega_c t + \frac{2\pi k}{M}\right) = \left(\cos\frac{2\pi k}{M}\right)\sqrt{2}\cos(\omega_c t) - \left(\sin\frac{2\pi k}{M}\right)\sqrt{2}\sin(\omega_c t)$$

$$k = 0, \cdots, M-1$$

Two-dimensional space

$$c_k(t) \rightarrow \begin{bmatrix} \cos\frac{2\pi k}{M} \\ \sin\frac{2\pi k}{M} \end{bmatrix}$$





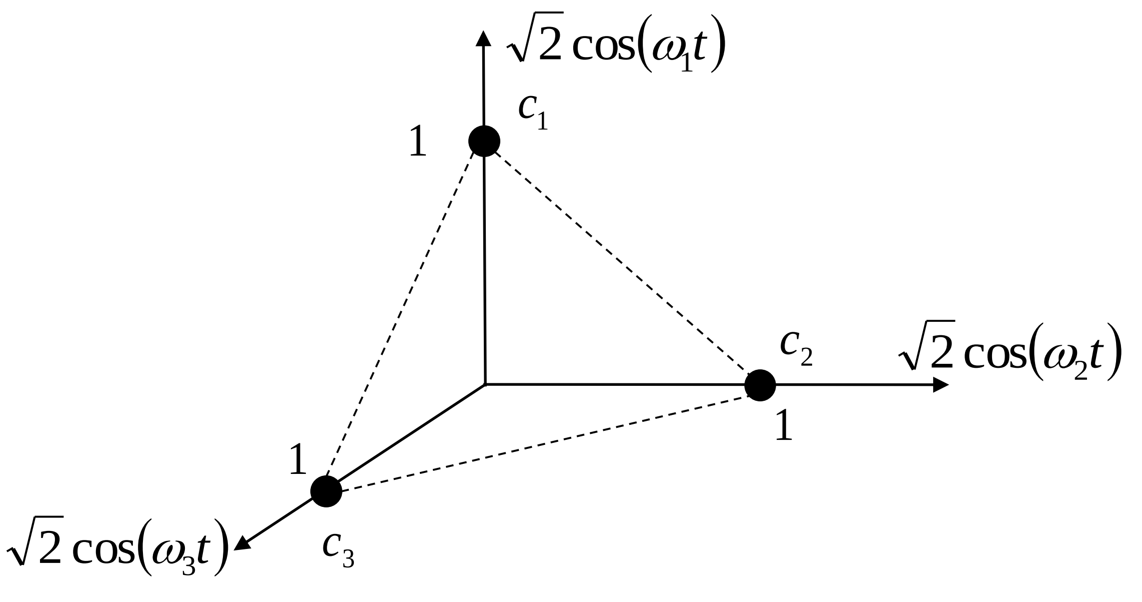
MFSK

$$c_k(t) = \sqrt{2}\cos(\omega_k t) \qquad k = 0, \cdots, M-1$$

M-dimensional Space $\qquad c_k(t) \to [0 \quad \cdots \quad 1 \quad \cdots \quad 0]^T$

the k^{th} element

3-FSK



Roadmap

- Line Coding (Transmission Coding)
- Digital Baseband Transmission
- Digital Band-Pass Modulation
  - Binary band-pass modulation
  - M-ary band-pass modulation
  - Signal space diagram
  - Detector options

Signal Constellation

Let the modulated waveform be

$$\Phi_k(t) = x_1(k)\phi_1(t) + x_2(k)\phi_2(t) + \cdots + x_K(k)\phi_K(t)$$

where $\phi_1(t), \phi_2(t), \cdots, \phi_K(t)$ are K basis waveforms. Assume the digital system transmits one symbol every T seconds. We say T is the symbol duration.

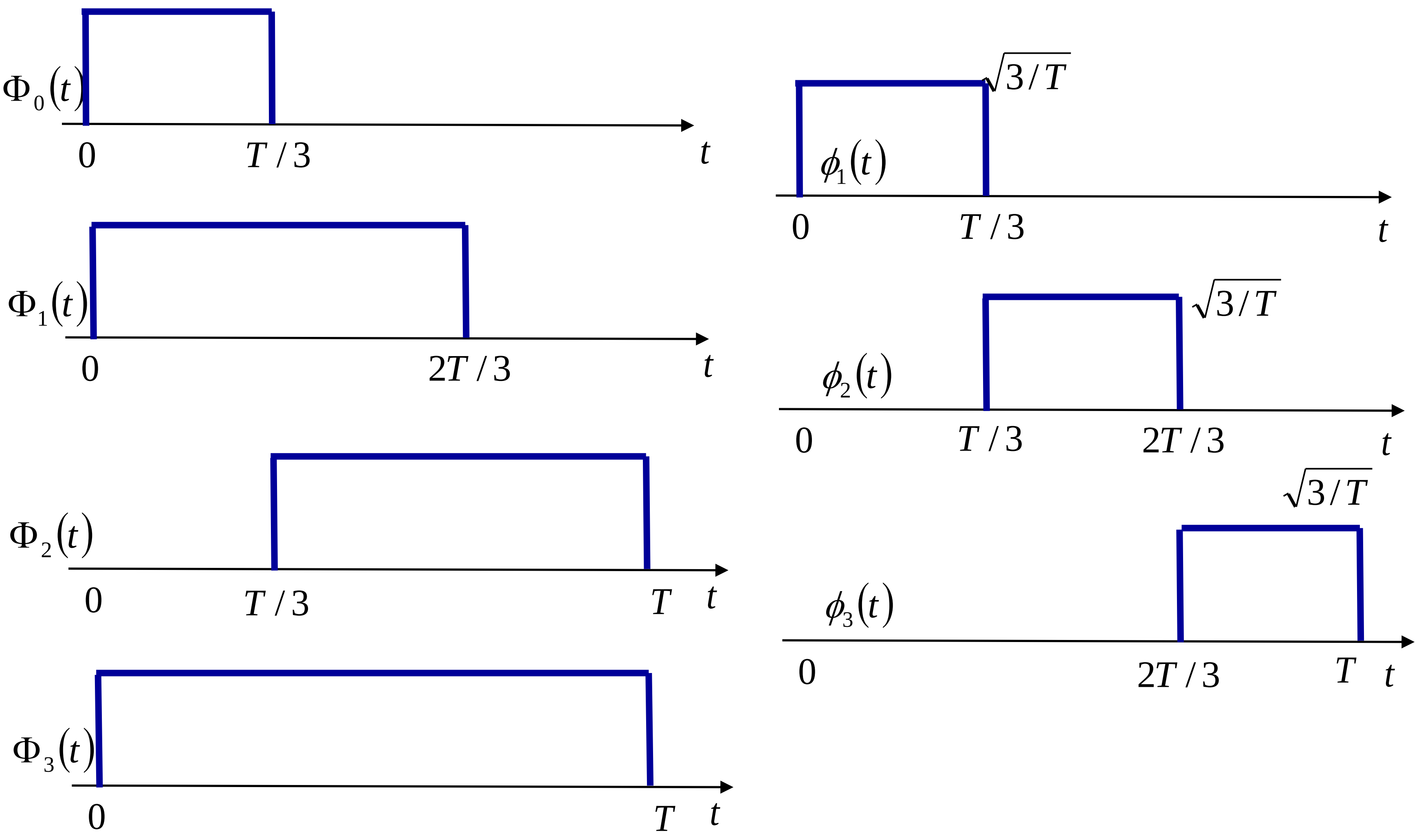
Assume

$$\int_0^T \phi_i^{\,2}(t)dt = 1 \text{ for all } i. \qquad \text{unit energy}$$

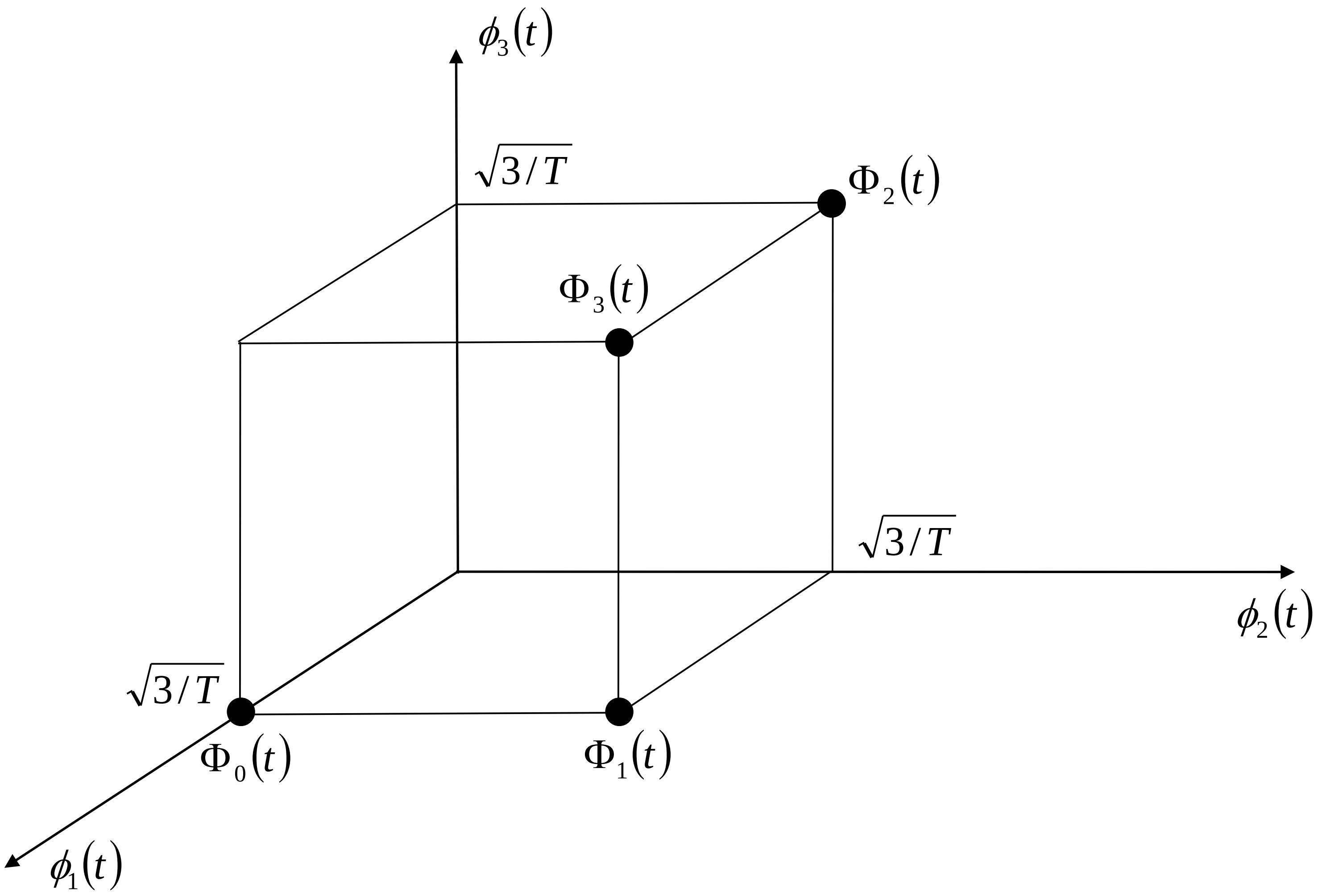
$$\int_0^T \phi_i(t)\phi_j(t)dt = 0 \text{ for all } i \neq j. \qquad \text{orthogonal}$$

We can map the modulated signal corresponding to the k^{th} symbol to a point with coordinate $[x_1(k), x_2(k), \cdots, x_K(k)]^T$ in the K-dimensional space. This is called the signal constellation.

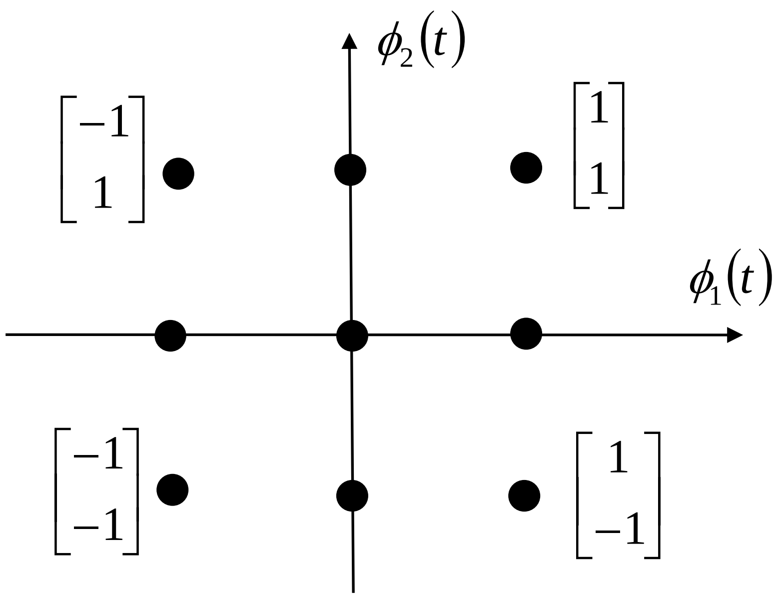
Signal Constellation



Signal Constellation



Example: 9-point QAM



If symbols are equally probable, average energy per symbol is

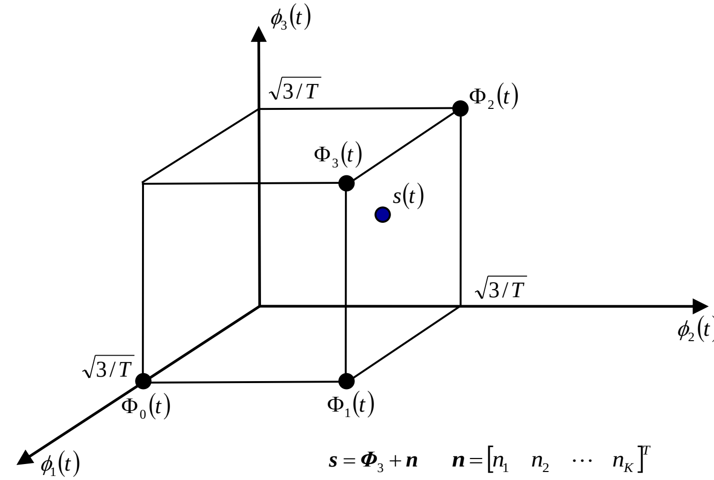
$$E_s = \frac{1}{9}(2+1+2+1+0+1+2+1+2) = \frac{4}{3}$$

One symbol represents $\lceil \log_2 9 \rceil = 4$ bits.

Average energy per bit is

$$E_b = \frac{E_s}{4} = \frac{1}{3}$$

Noise Resistance



Let the modulated waveform be

$$\Phi_k(t) = x_1(k)\phi_1(t) + x_2(k)\phi_2(t) + \cdots + x_K(k)\phi_K(t)$$

where $\phi_1(t), \phi_2(t), \cdots, \phi_K(t)$ forms an orthonormal basis .

Suppose the noise signal is $n(t)$. The received signal is

$$s(t) = x_1(k)\phi_1(t) + x_2(k)\phi_2(t) + \cdots + x_K(k)\phi_K(t) + n(t)$$

Let

$$n(t) = n_1\phi_1(t) + n_2\phi_2(t) + \cdots + n_K\phi_K(t) + \tilde{n}(t)$$

$$n_j = \int_0^T n(t)\phi_j(t)dt \qquad \int_0^T \tilde{n}(t)\phi_j(t)dt = 0 \qquad \text{for all } j$$

$$\int_0^T \tilde{n}(t)\phi_j(t)dt = \int_0^T \left[n(t) - \sum_{k=1}^{K} n_k\phi_k(t) \right] \phi_j(t)dt = \int_0^T n(t)\phi_j(t)dt - n_j \int_0^T \phi_j(t)^2 dt = 0$$

Roadmap

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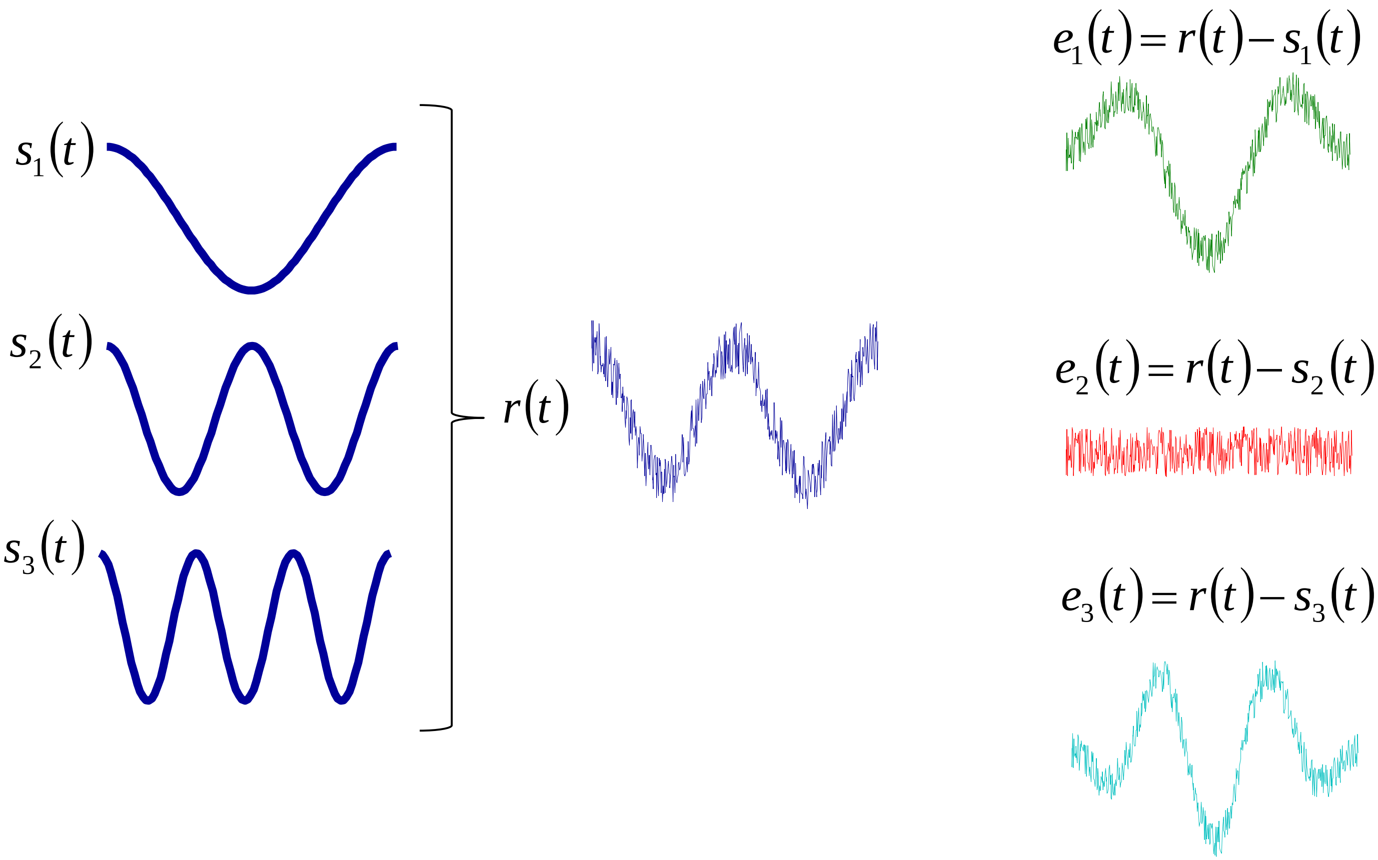
Automated Detection?

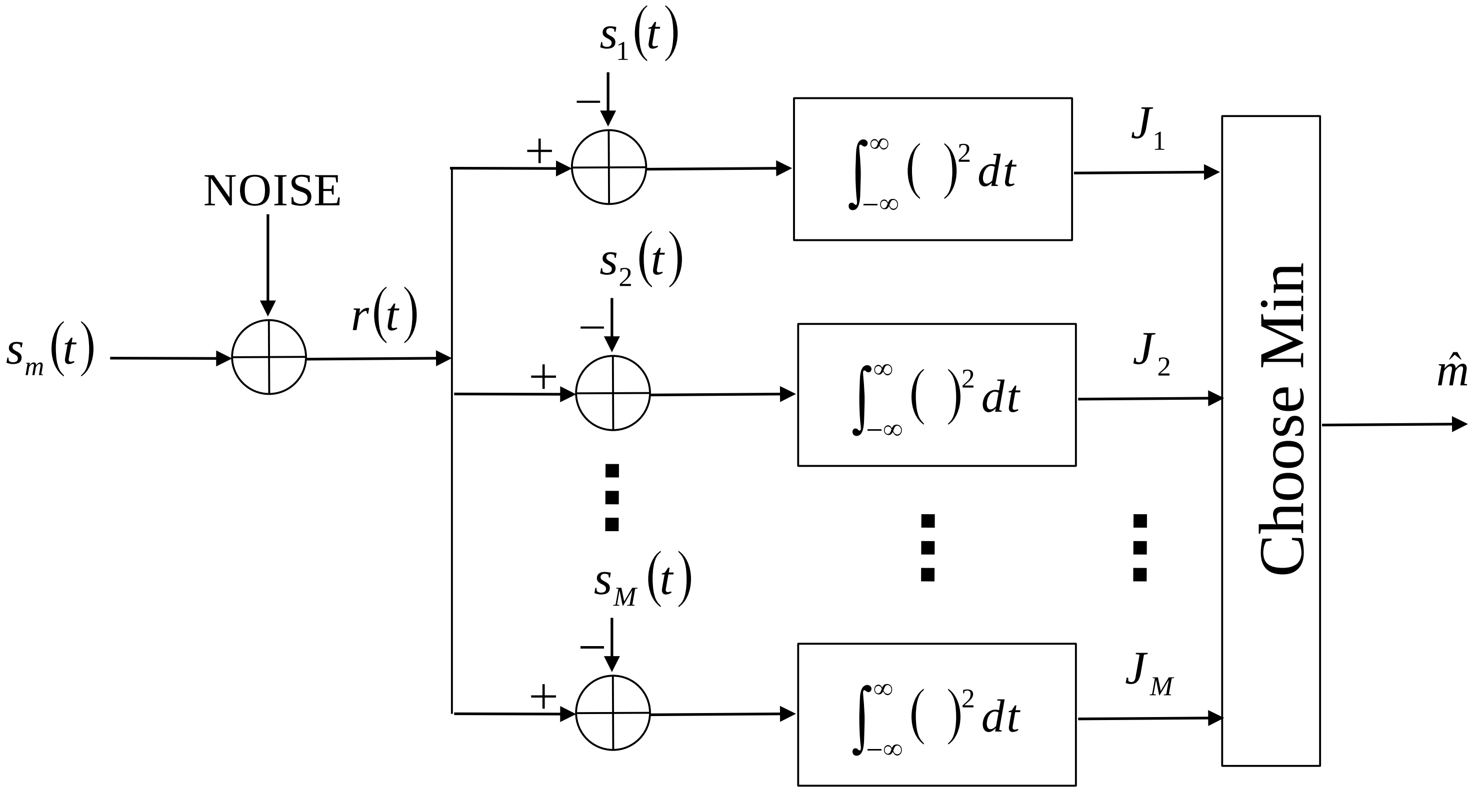


Choose signal "closest" to $r(t)$ $\Rightarrow$ minimum distance detection.

Minimum Distance Detector

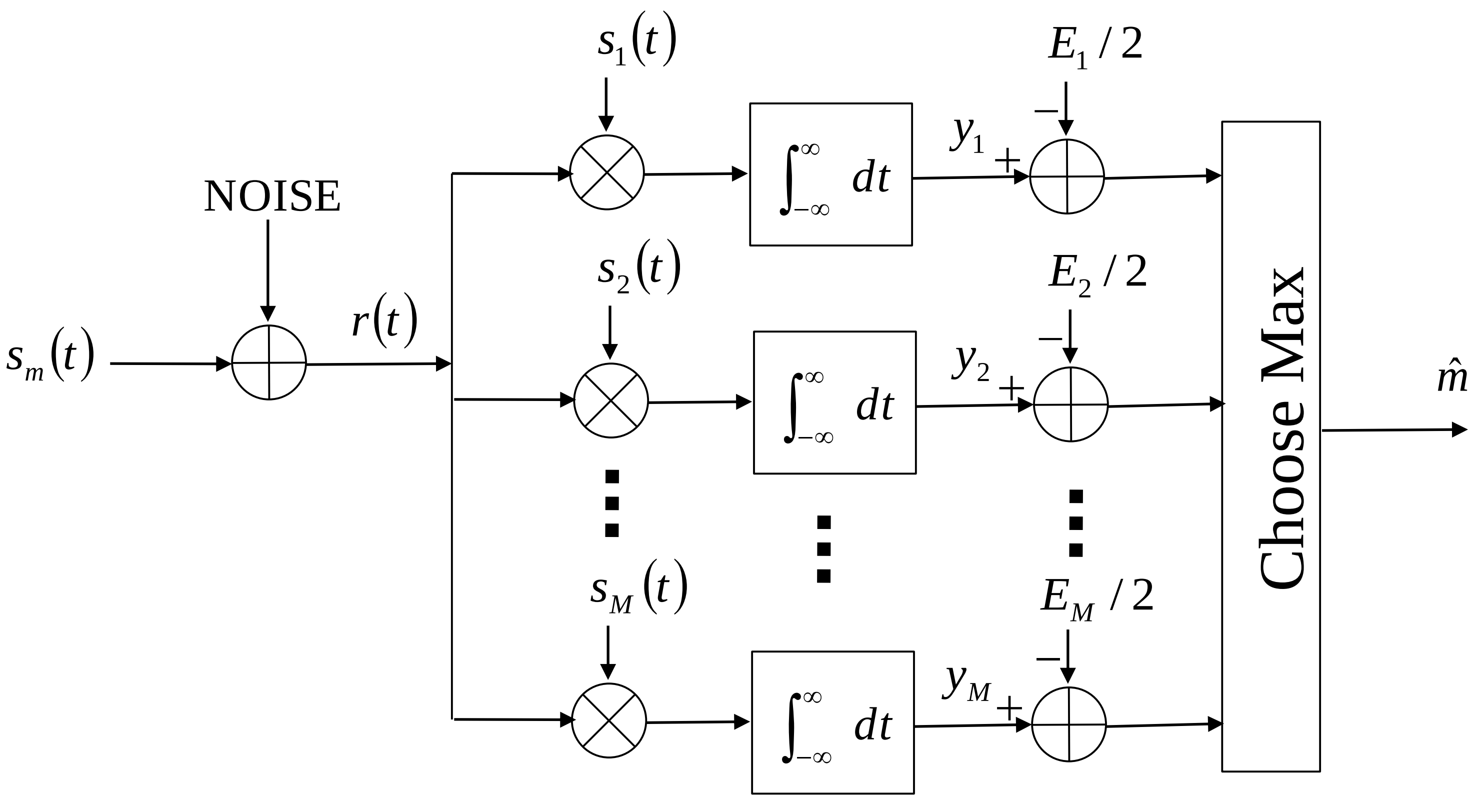
Receiver decision is $s_{\hat{m}}(t)$, where $\hat{m}$ minimizes :

$$J_i = \int_{-\infty}^{\infty} \left(r(t) - s_i(t)\right)^2 dt$$

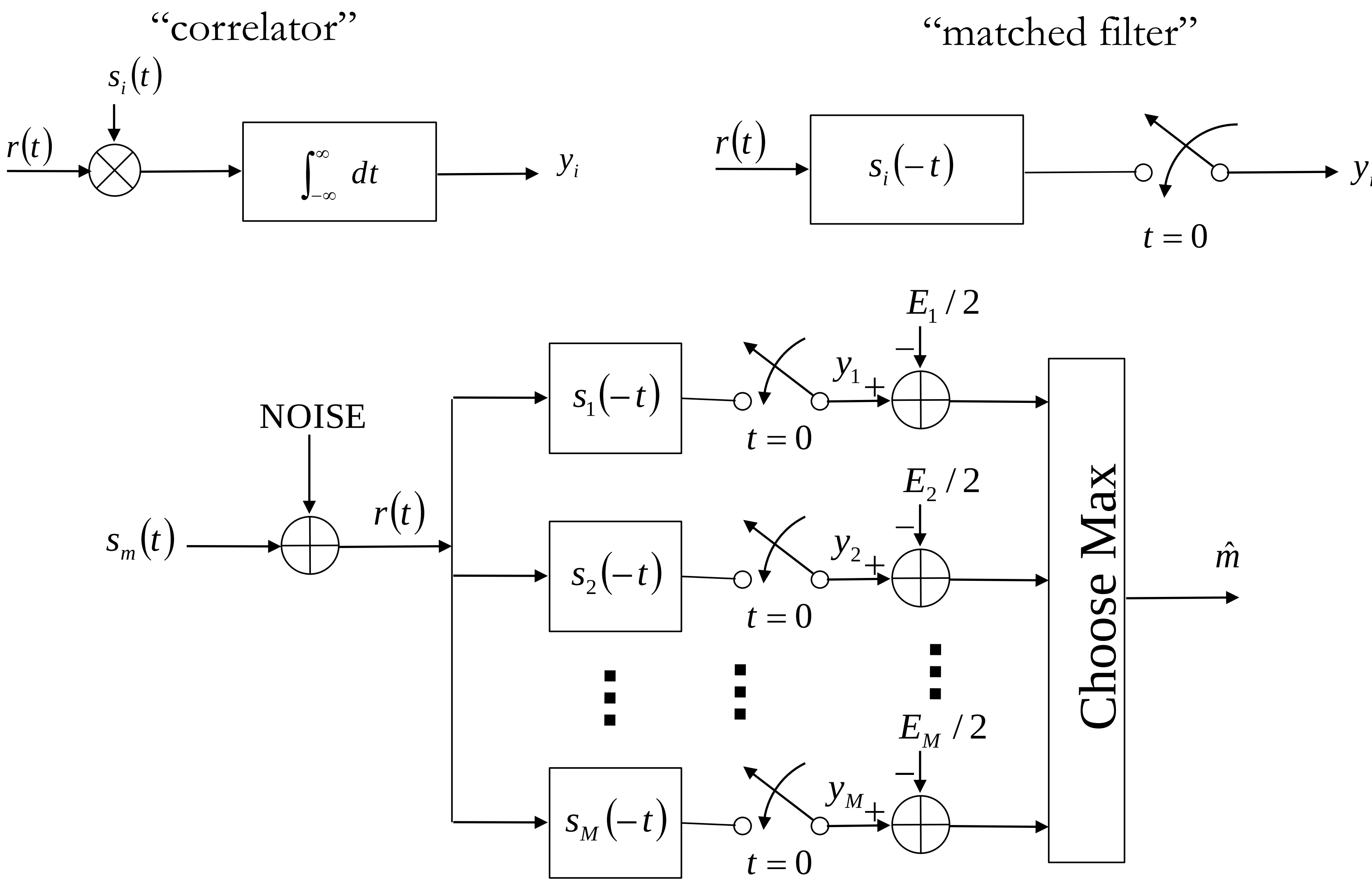


Brute-force implementation

Correlation Receiver

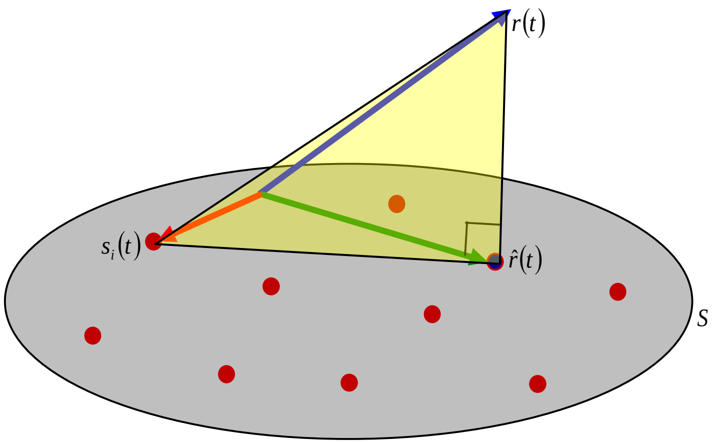


Alternative Implementation



“Minimum Distance” Revisited

$$J_i = \int_{-\infty}^{\infty} (r(t) - s_i(t))^2 \, dt$$



Let $S = \text{Span}\{s_1(t), \cdots, s_M(t)\}$ be the "signal space"

Let $\hat{r}(t) =$ projection of $r(t)$ onto S

Rewrite cost :

$$J_i = \langle r(t) - s_i(t), r(t) - s_i(t) \rangle = \| r(t) - s_i(t) \|^2$$

$$= \| r(t) - \hat{r}(t) + \hat{r}(t) - s_i(t) \|^2$$

$$= \| r(t) - \hat{r}(t) \|^2 + \| \hat{r}(t) - s_i(t) \|^2 + 2 \langle r(t) - \hat{r}(t), \hat{r}(t) - s_i(t) \rangle$$

First term: independent of i

Last term: zero, because of orthorgonality principle

$\Rightarrow$The minimum-distance receiver minimizes second term only.

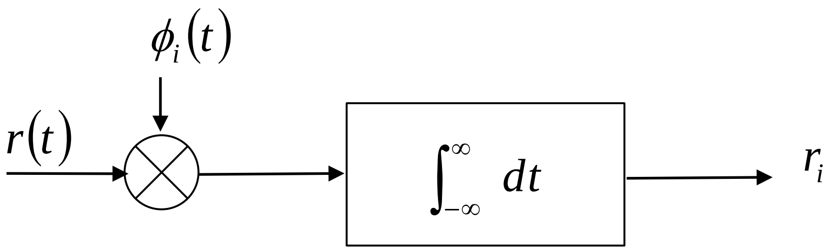
How to Compute Projection?

In terms of an orthonormal basis $\{\phi_1(t),\cdots,\phi_N(t)\}$ for S, the projection is:

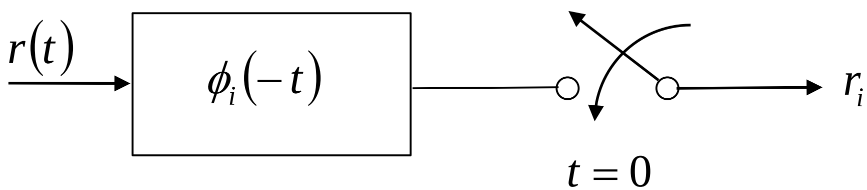
$$\hat{r}(t)=\sum_{i=1}^{N} r_i\phi_i(t) \qquad \text{where} \qquad r_i=\langle r(t),\phi_i(t)\rangle$$

Also, since $s_m(t)\in S$, we have $s_m(t)=\sum_{i=1}^{N} s_{mi}\phi_i(t)$

"correlator" implementation



"matched filter" implementation



Vector Definitions

Let $\hat{\boldsymbol{r}} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_N \end{bmatrix}$ be the *received vector*, where $r_i = \langle \hat{r}(t), \phi_i(t) \rangle = \langle r(t), \phi_i(t) \rangle$

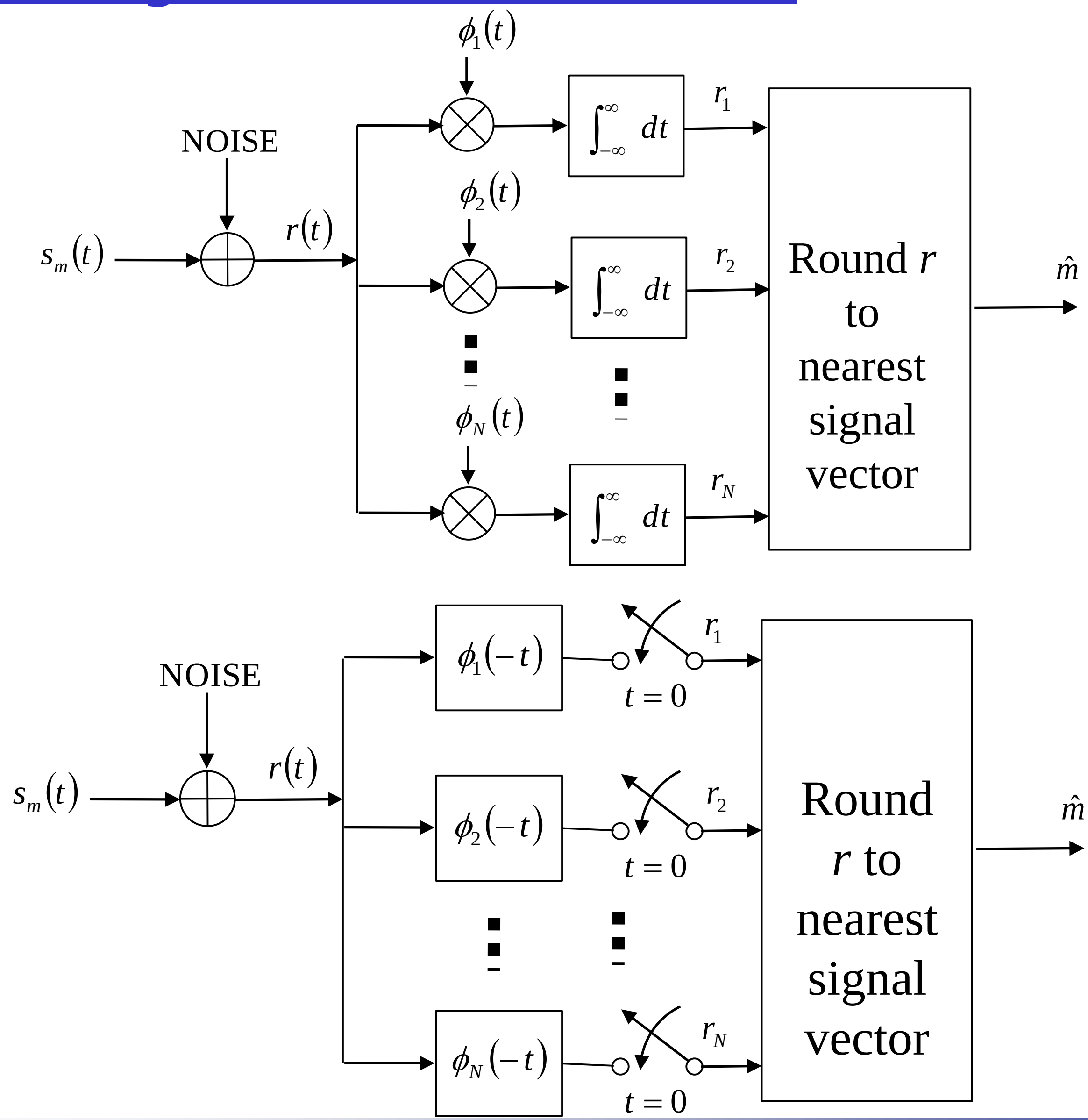
Let $\boldsymbol{s}_m = \begin{bmatrix} s_{m1} \\ s_{m2} \\ s_{m3} \\ \vdots \\ s_{mN} \end{bmatrix}$ be the m-th *signal vector*, where $s_{mi} = \langle s_m(t), \phi_i(t) \rangle$

The projection receiver minimizes

$$J_i'' = \left\| \hat{r}(t) - s_i(t) \right\|^2 = \left\| \hat{\boldsymbol{r}} - \boldsymbol{s}_i \right\|^2 \Leftrightarrow \left\| \boldsymbol{r} - \boldsymbol{s}_i \right\|^2$$

Hence the name "minimum distance".

The Projection Receiver



Complexity Comparison

Direct implementation: M integrators

Correlation receiver: M correlator (or matched filters)

Projection receiver: N correlator (or matched filter)

But $N \leq M$

Sometimes, $N << M$.

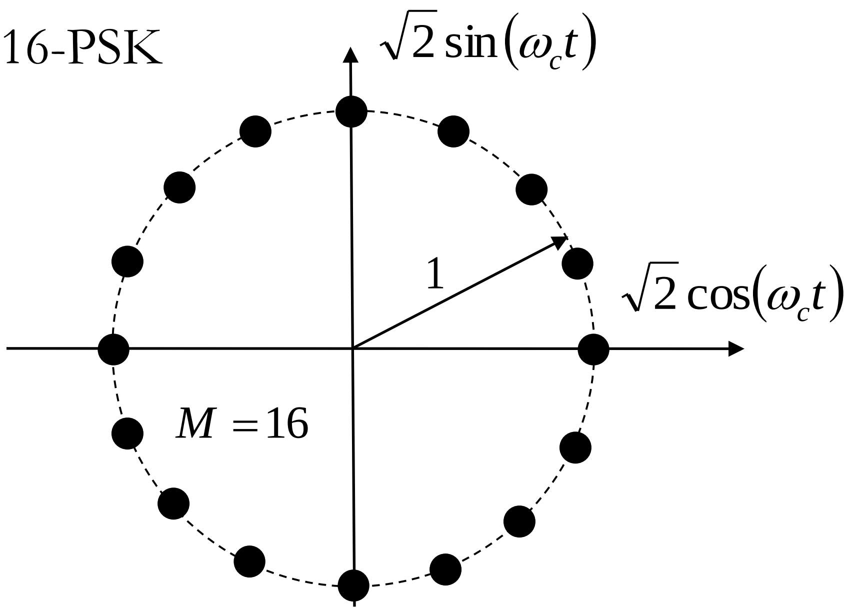
Example: 16-PSK

M=16

N=2

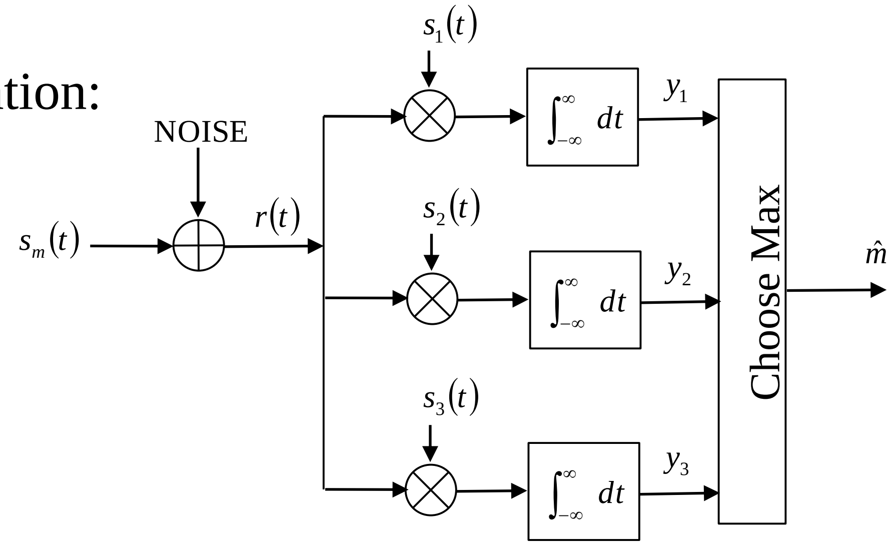
$\Rightarrow$ projection receiver preferred.

16-PSK



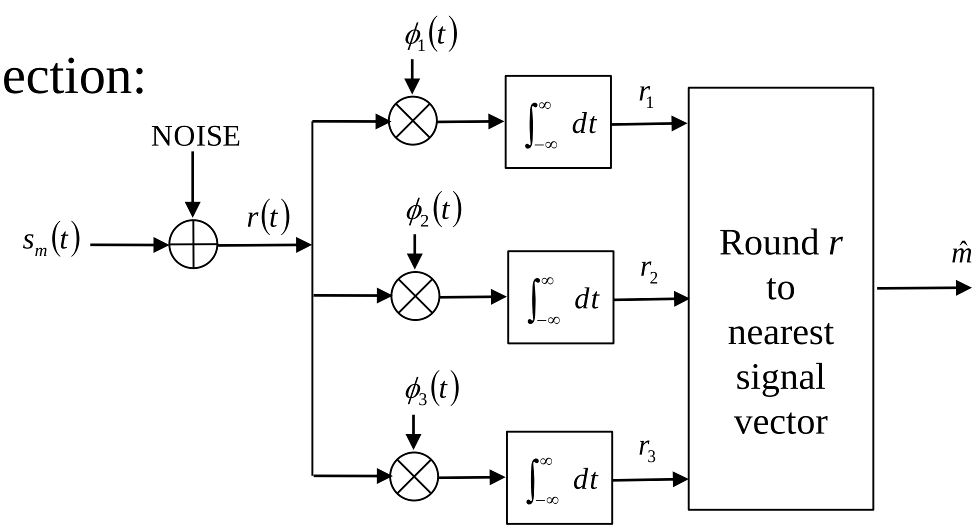
Example: Receiver for 3-FSK

Correlation:



Subtraction of $E_i/2$ is not needed, since E_1=E_2=E_3

Projection:



But $s_i(t) = \sqrt{E}\phi_i(t) \Rightarrow y_i = \sqrt{E}r_i$

Looks identical to the correlation receiver. But why 'round' instead of choose max?

Round to Nearest Vector

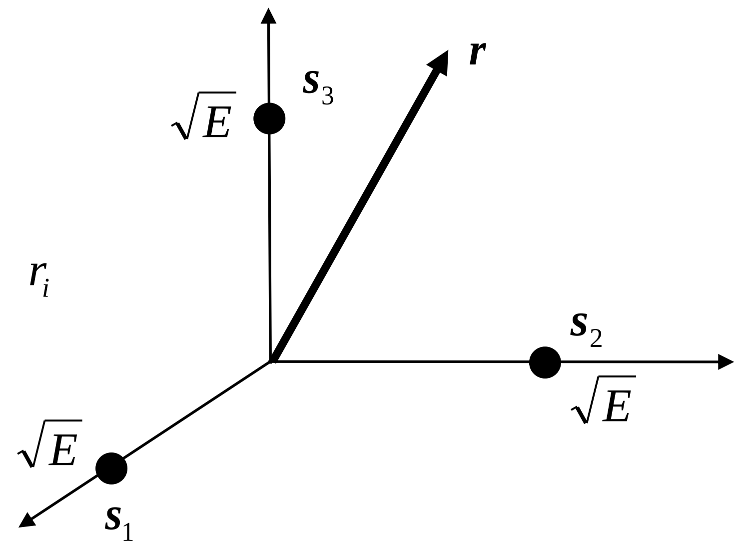
= Choose Max for 3-FSK

$$\|\boldsymbol{r}-\boldsymbol{s}_1\|^2 = \left(r_1-\sqrt{E}\right)^2 + r_2^2 + r_3^2 = \|\boldsymbol{r}\|^2 + E - 2\sqrt{E}r_1$$

$$\|\boldsymbol{r}-\boldsymbol{s}_2\|^2 = r_1^2 + \left(r_2-\sqrt{E}\right)^2 + r_3^2 = \|\boldsymbol{r}\|^2 + E - 2\sqrt{E}r_2$$

$$\|\boldsymbol{r}-\boldsymbol{s}_3\|^2 = r_1^2 + r_2^2 + \left(r_3-\sqrt{E}\right)^2 = \|\boldsymbol{r}\|^2 + E - 2\sqrt{E}r_3$$

$$\Rightarrow \text{minimizing } \|\boldsymbol{r}-\boldsymbol{s}_i\|^2 \text{ is equivalent to maximizing } r_i$$



$\boldsymbol{r}$ is closest to $\boldsymbol{s}_3$ iff $r_3 > r_1, r_2$

Summary

- Line Coding (Transmission Coding)
  - Line coding
  - Power Spectral Density (PSD)
  - Polar signaling
  - On-off signaling
  - Bipolar signaling
- Digital Baseband Transmission
  - Pulse shaping
  - Eye diagram
  - Channel equalization and channel identification
- Digital Band-Pass Modulation
  - Binary band-pass modulation
  - M-ary band-pass modulation
  - Signal space diagram
  - Detector options