Principle of Communications

Final Review

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Roadmap

- Line Coding (Transmission Coding)
  - Line coding
  - Power Spectral Density (PSD)
- Digital Baseband Transmission
- Digital Band-Pass Modulation

Line Coding

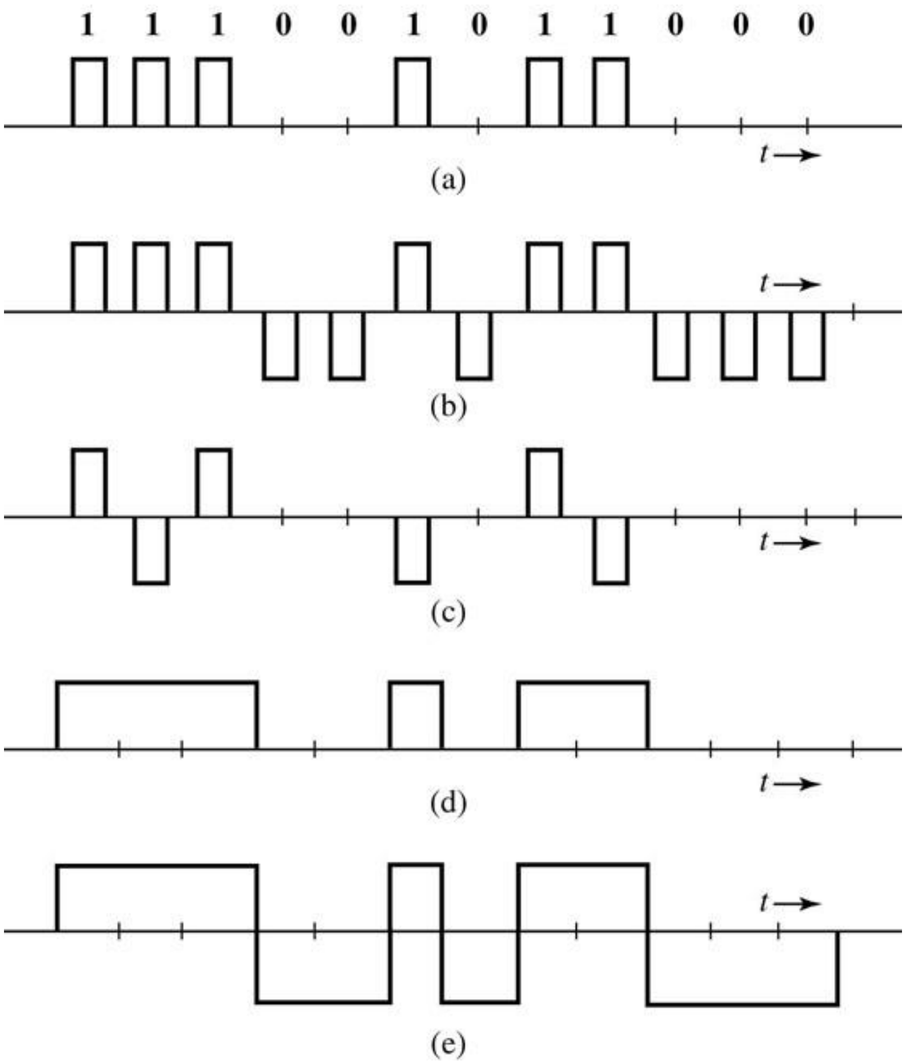


Figure 7.2 Line code examples: (a) on-off (RZ); (b) polar (RZ); (c) bipolar (RZ); (d) on-off (NRZ); (e) polar (NRZ).

Power Spectral Density (PSD)

PSD of Line Codes: Covers a large class of line codes

Assume pulses places every T_b seconds: $R_b = 1/T_b$ pulses/second

Basic pulse: $p(t) \leftrightarrow P(\omega)$

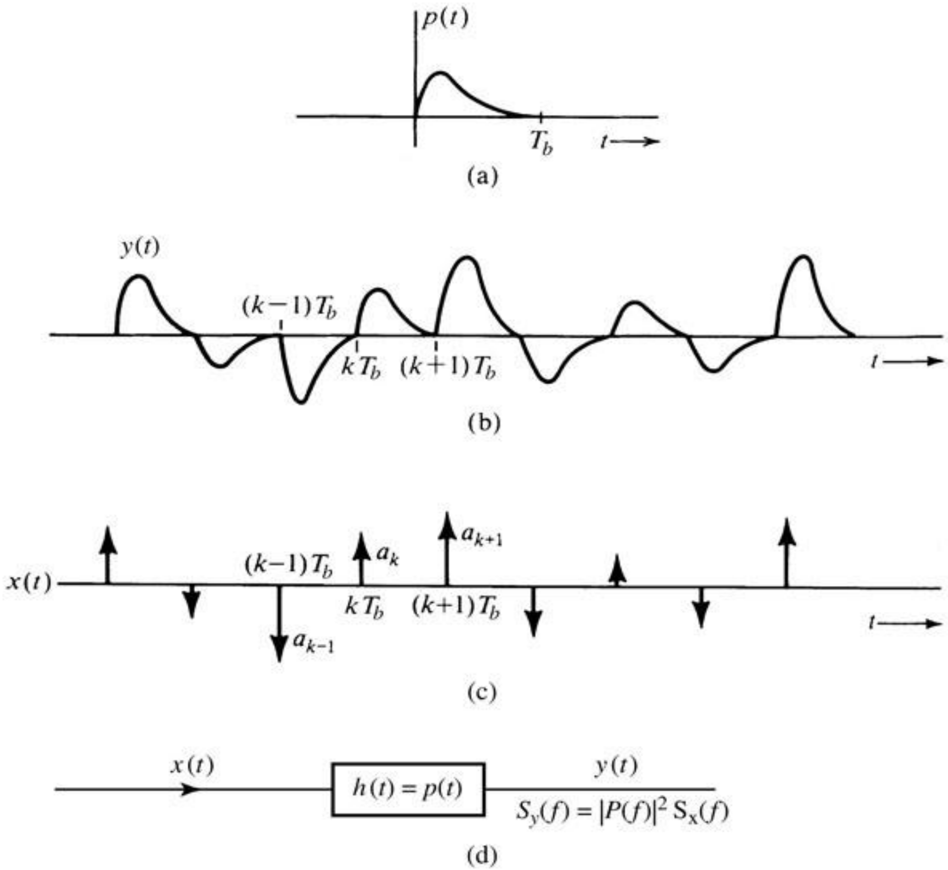


Figure 7.4 Random pulse-amplitude-modulated signal and its generation from a PAM impulse.

$$y(t) = \sum a_k p(t - kT_b)$$

$$S_y(\omega) = |P(\omega)|^2 S_x(\omega)$$

a_k does not need to be binary

PSD Computation

$$x(t) = \sum a_k \delta(t - kT_b)$$

$$R_x(\tau) = \frac{1}{T_b} \sum_{n=-\infty}^{\infty} R_n \delta(\tau - nT_b)$$

$$S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau} d\tau = \frac{1}{T_b} \sum_{n=-\infty}^{\infty} R_n e^{-jn\omega T_b}$$

For a discrete-time signal $\{a_k\}$, its autocorrelation function $\{R_n\}$ is defined as

$$R_n = E\left[a_k a_{k-n}^*\right] = \lim_{N\to\infty} \frac{1}{2N} \sum_{k=-N}^{N} a_k a_{k-n}^*$$

$$R_0 = E\left[|a_k|^2\right] = \lim_{N\to\infty} \frac{1}{2N} \sum_{k=-N}^{N} |a_k|^2 \qquad R_n = E\left[a_k a_{k-n}^*\right] = \left\{E\left[a_k a_{k+n}^*\right]\right\}^* = R_{-n}^*$$

For real-valued signal, $R_n = R_{-n}^*$

Polar Signaling

Assume binary input. $1 \rightarrow p(t)$ $\quad x_k = 1 \rightarrow a_k = 1$

$0 \rightarrow -p(t)$ $\quad x_k = 0 \rightarrow a_k = -1$

$$a_k \in \{+1, -1\} \Rightarrow a_k^{\,2} = 1 \qquad R_0 = E\left[a_k^{\,2}\right] = 1$$

if x_k are independent, x_k is equally likely to be 0 or 1, for all k,

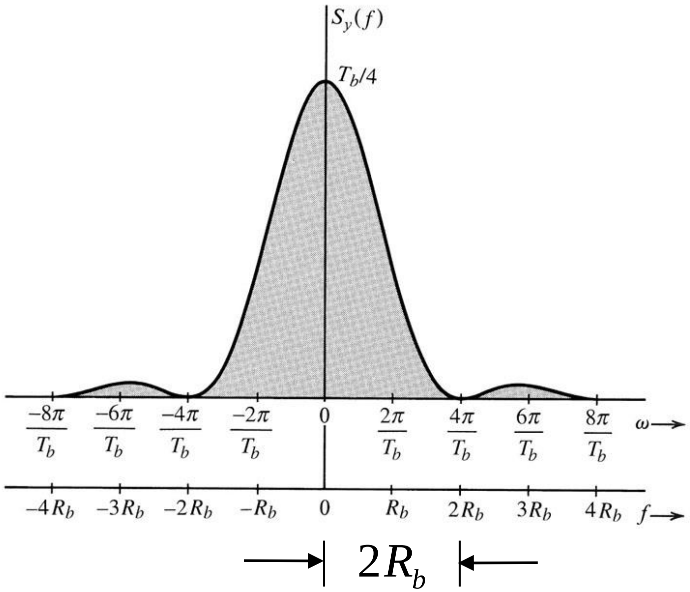
then a_k is equally likely to be 1 or -1, for all k

$$R_n = E\left[a_k a_{k+n}\right] = E\left[a_k\right] E\left[a_{k+n}\right] = 0$$

$$R_x(\tau) = \frac{1}{T_b}\delta(\tau) \qquad S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau} d\tau = \frac{1}{T_b}$$

$$\text{if } \; p(t) = \mathrm{rect}\left(\frac{t}{T_b/2}\right) \qquad P(\omega) = \frac{T_b}{2}\,\mathrm{sinc}\left(\frac{\omega T_b}{4}\right)$$

$$S_y(\omega) = |P(\omega)|^2 S_x(\omega) = \frac{T_b}{4}\,\mathrm{sinc}^2\left(\frac{\omega T_b}{4}\right)$$

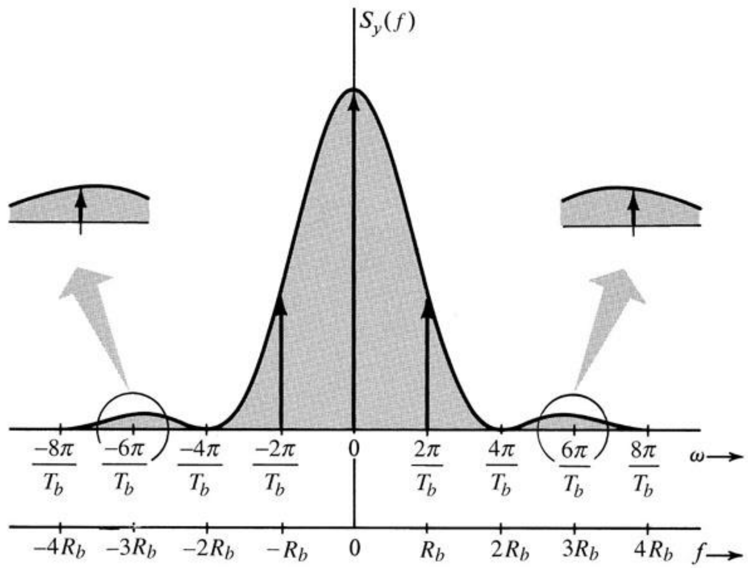


essential bandwidth

On-Off Keying

Assume rectangular pulse.

$$S_y(\omega)=\frac{|P(\omega)|^2}{4T_b}\left[1+\frac{2\pi}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi n}{T_b}\right)\right]=\frac{T_b}{16}\operatorname{sinc}^2\left(\frac{\omega T_b}{4}\right)\left[1+\frac{2\pi}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi n}{T_b}\right)\right]$$



Same spectrum as polar signaling. Samples every $2\pi n/T_b$.

Indeed, on-off keying = polar signaling + periodic square waveform.

Con: not power efficient, not transparent, not DC null.

Pro: enables non-coherent detection

Bipolar Signaling

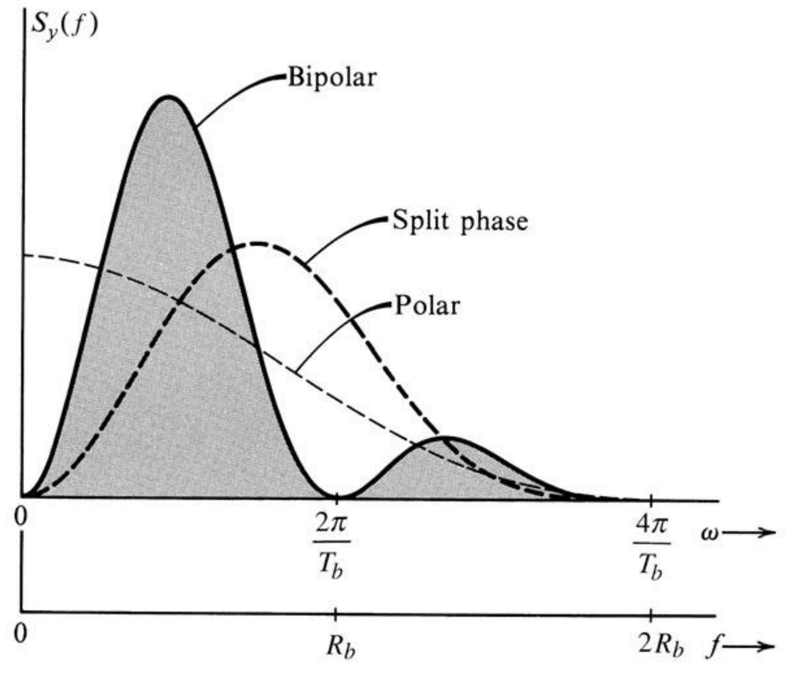
For rectangular pulse.

$$S_y(\omega) = \frac{|P(\omega)|^2}{T_b} \sin^2\left(\frac{\omega T_b}{2}\right) = \frac{T_b}{4} \operatorname{sinc}^2\left(\frac{\omega T_b}{4}\right) \sin^2\left(\frac{\omega T_b}{2}\right)$$

Essential bandwidth $= R_b = \dfrac{1}{T_b}$

Advantages: DC null, small bandwidth, single error detection

Disadvantages: power efficiency is the same as on-off, not transparent



Roadmap

- Line Coding (Transmission Coding)
- Digital Baseband Transmission
  - Pulse shaping
  - Eye diagram
  - Channel equalization
- Digital Band-Pass Modulation

Inter-Symbol Interference (ISI)

Inter-symbol interference (ISI): the kth sample contains not only a_k, but a combination of other symbols.

$$a(t)=\sum_k a_k \delta(t-kT_b)$$

Effective pulse $\quad p(t)=g(t)*h(t) \quad g(t)$ pulse waveform, $h(t)$ channel impulse

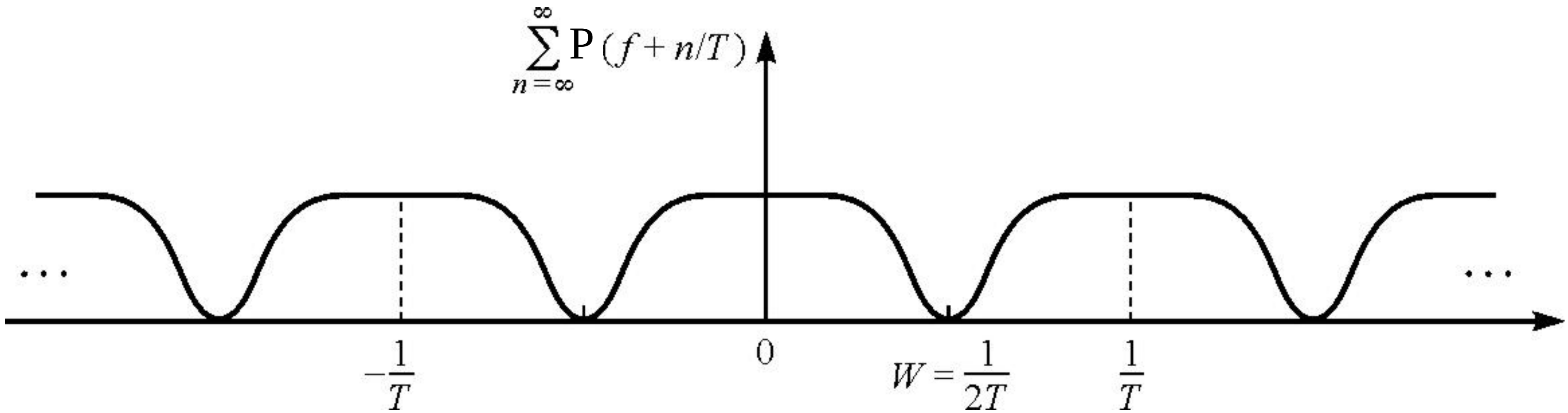
$$y(t)=a(t)*p(t)=\sum_k a_k p(t-kT_b)$$

Sample $\quad y_n=y(nT_b)=\sum_k a_k p((n-k)T_b)$

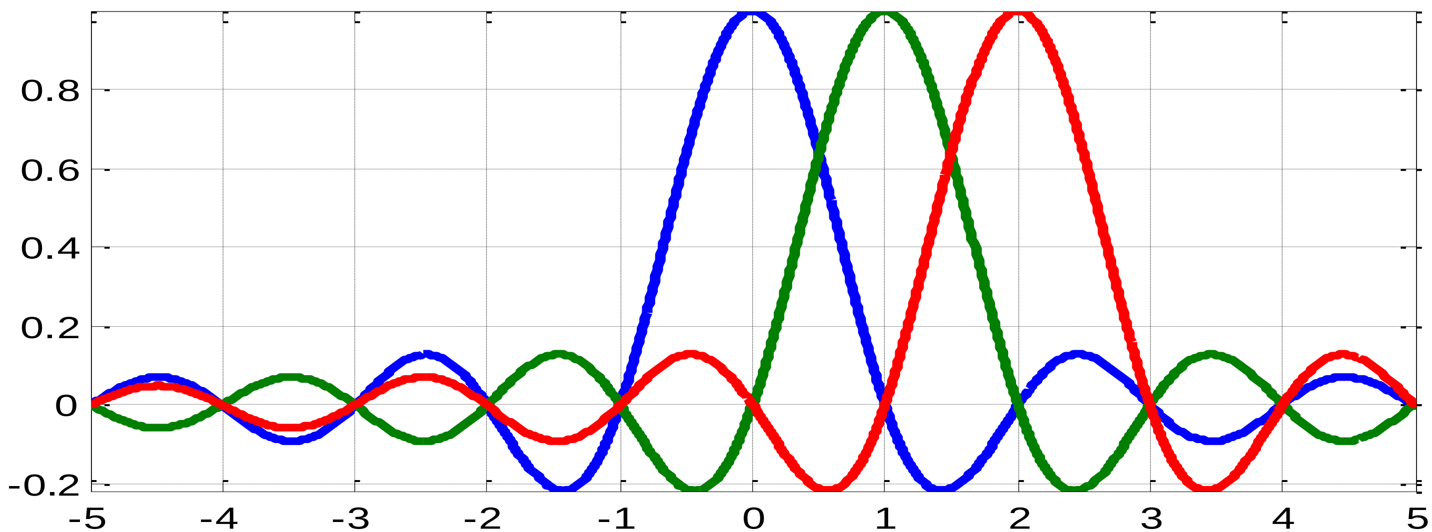
Nyquist Criterion for zero ISI

$$p_i=p(iT_b)=\begin{cases}\sqrt{E} & i=0\\ 0 & \text{otherwise}\end{cases}$$

If 1/T=2W, then



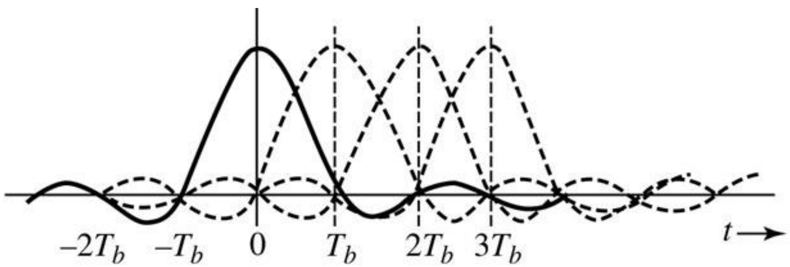
$\sum_{n=\infty}^{\infty} P(f+n/T)$
$-\frac{1}{T}$
0
$W=\frac{1}{2T}$
$\frac{1}{T}$



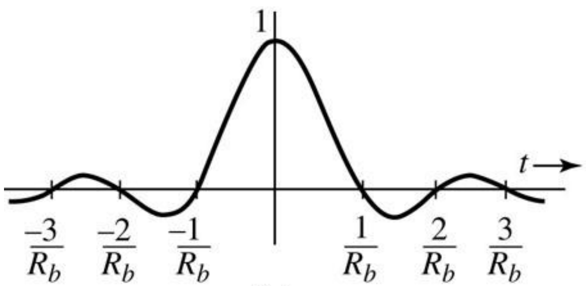
0.8
0.6
0.4
0.2
0
-0.2
-5
-4
-3
-2
-1
0
1
2
3
4
5

Ideal Nyquist Pulse

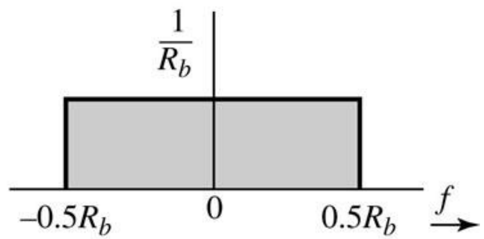
Bandwidth of $p(t) \geq 1/2T_b$. Can we achieve the minimum BW of $1/2T_b$?



(a)



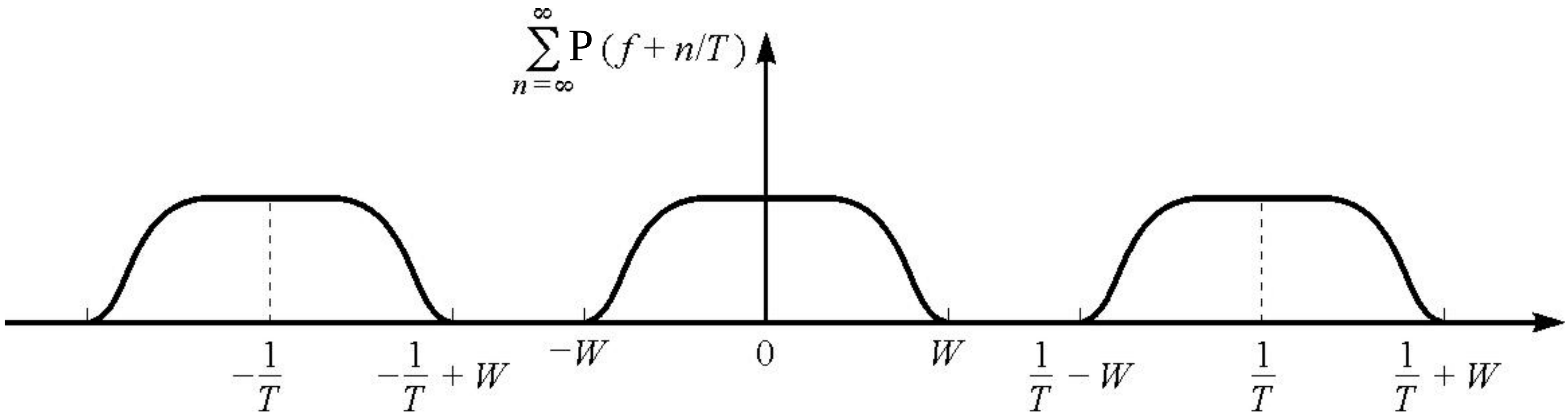
(b)



(c)

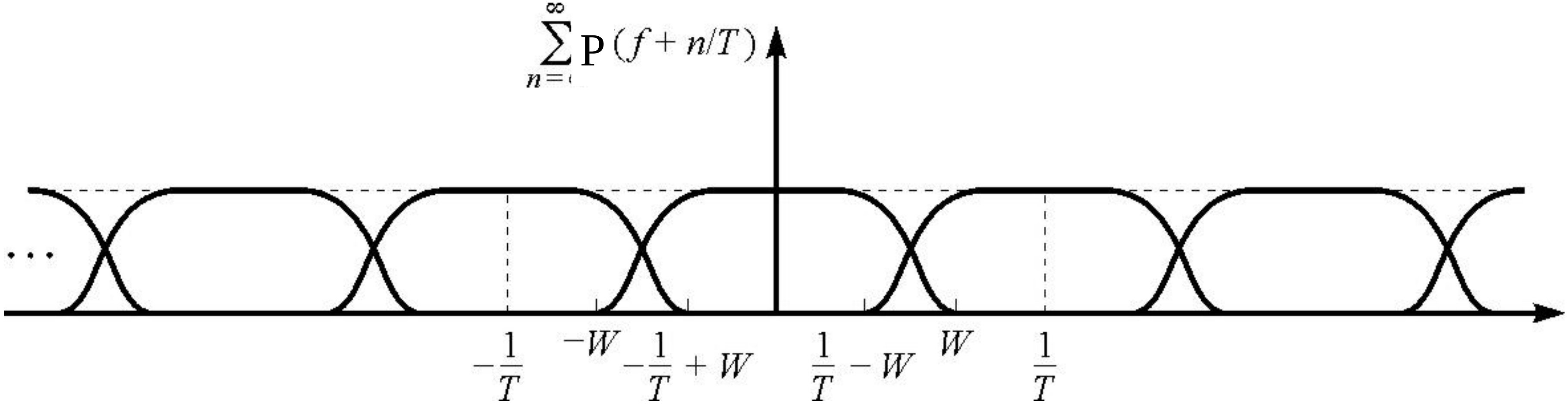
$$p(t) = \sqrt{E}\,\mathrm{sinc}\left(\frac{t}{T_b}\right) \qquad P(\omega) = \begin{cases} \sqrt{E} & |\omega| \leq \dfrac{2\pi}{2T_b} \\ 0 & \text{otherwise} \end{cases} \qquad p_i = p(iT_b) = \begin{cases} \sqrt{E} & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

If 1/T>2W, then



$\sum_{n=\infty}^{\infty} P(f+n/T)$
$-\frac{1}{T}$
$-\frac{1}{T}+W$
$-W$
0
W
$\frac{1}{T}-W$
$\frac{1}{T}$
$\frac{1}{T}+W$

If 1/T<2W, then



$\sum_{n=-\infty}^{\infty} P(f+n/T)$
$\cdots$
$-\frac{1}{T}$
$-W$
$-\frac{1}{T}+W$
$\frac{1}{T}-W$
W
$\frac{1}{T}$

Raised Cosine Spectrum

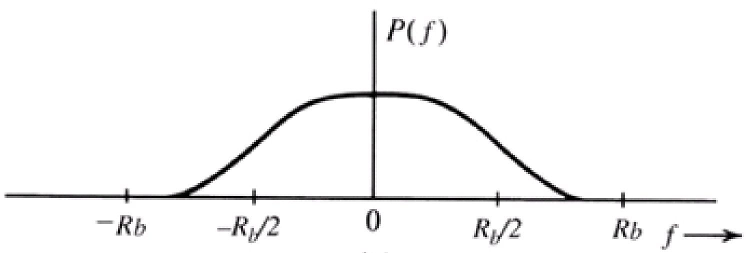
$$p(t)=\sqrt{E}\,\mathrm{sinc}\left(\frac{t}{T_b}\right)\left(\frac{\cos(\alpha t\pi/T_b)}{1-4\alpha^2t^2/T_b^{\,2}}\right)=\left[\sqrt{E}\,\mathrm{sinc}\left(\frac{t}{T_b}\right)\right]\times\left(\frac{\cos(\alpha t\pi/T_b)}{1-4\alpha^2t^2/T_b^{\,2}}\right)$$

$$p_i=p(iT_b)=\begin{cases}\sqrt{E} & i=0\\ 0 & \text{otherwise}\end{cases}$$

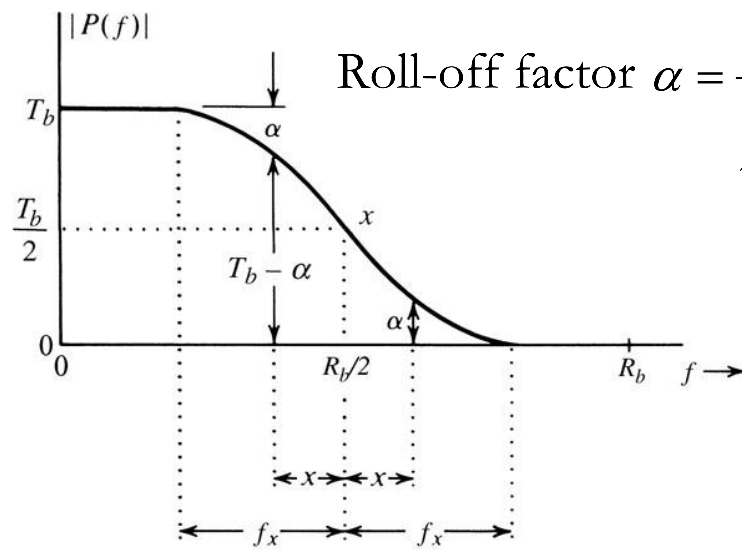
For large t, envelope of $p(t)$ decays in $1/t^3$.

Transmission bandwidth requirement

$$B_T=1/T_b-f_1=(1+\alpha)/2T_b$$



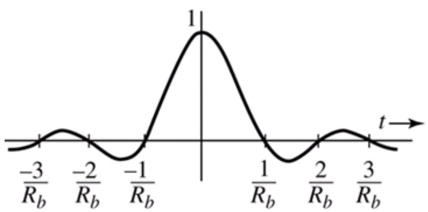
(a)



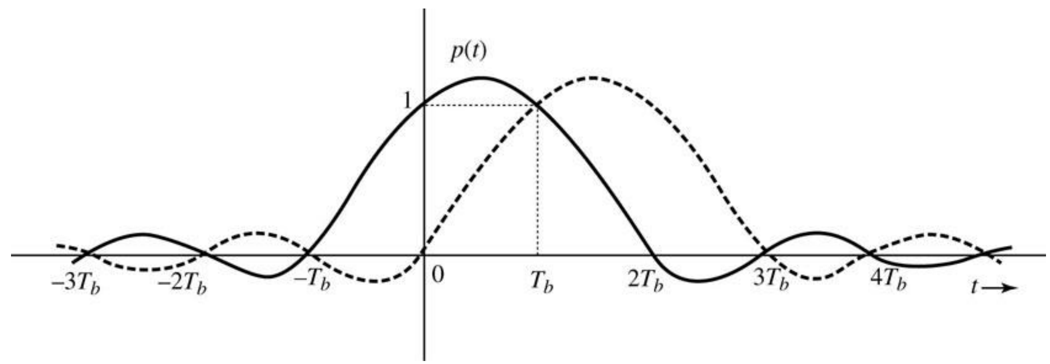
Roll-off factor $\alpha=\dfrac{f_x}{1/2T_b}$

Signaling with Controlled ISI

Basic Idea: Want to further reduce the bandwidth of *p*(*t*). Can't satisfy the Nyquist criterion. However, can achieve a controlled ISI pattern, and therefore can remove ISI in the digital signal after sampling.



(b)



$$p(nT_b)=\begin{cases}1 & n=0\\ 0 & \text{otherwise}\end{cases}$$

$$p(nT_b)=\begin{cases}1 & n=0,1\\ 0 & \text{otherwise}\end{cases}$$

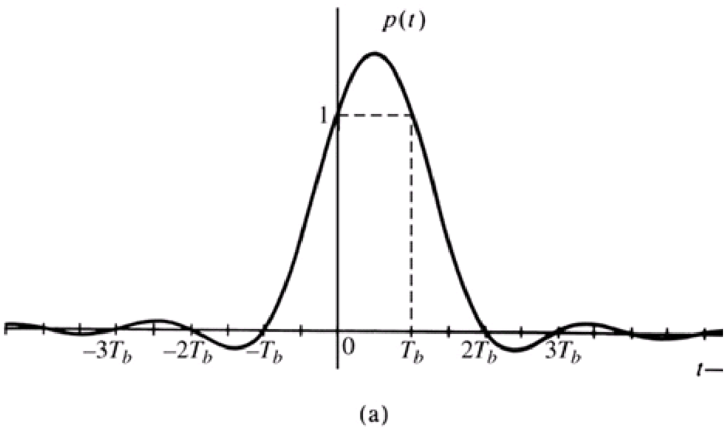
$$y_n=y(nT_b)=\sum_k a_k p((n-k)T_b)=a_n$$

$$y_n=y(nT_b)=\sum_k a_k p((n-k)T_b)=a_n+a_{n-1}$$

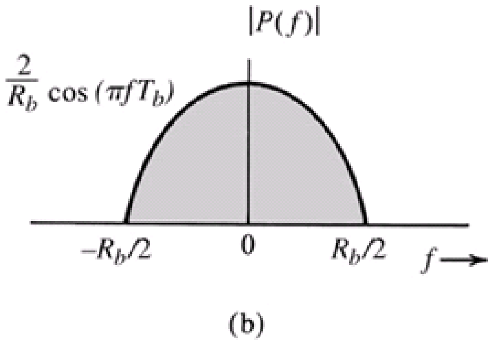
Essentially doing power and bandwidth tradeoff

Duobinary Pulse

$$p(t)=\frac{\sin(\pi R_b t)}{\pi R_b t(1-R_b t)} \qquad P(\omega)=\frac{2}{R_b}\cos\left(\frac{\omega}{2R_b}\right)\mathrm{rect}\left(\frac{\omega}{2\pi R_b}\right)e^{-j\frac{\omega}{2R_b}}$$



(a)



(b)

For large t, $p(t)$ decays in $1/t^2$. Therefore, no time error problem.

50% wider than the optimal Nyquist pulse.

$$p(nT_b)=\begin{cases}1 & n=0,1\\ 0 & \text{otherwise}\end{cases}$$ is often called the second Nyquist criterion

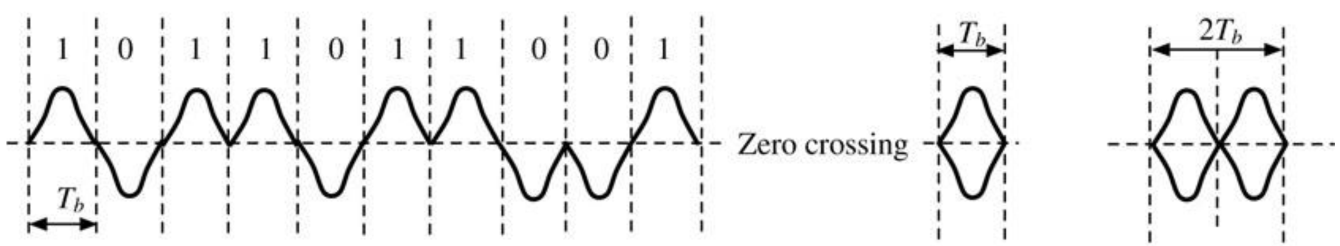
The Eye Diagram

$$y(t) = a(t) * p(t) = \sum_{k=-\infty}^{\infty} a_k p(t - kT_b)$$

If $p(t)$ is well designed, we should have $y(iT_b) = \sum_{k=-\infty}^{\infty} a_k p(iT_b - kT_b) = \sqrt{E} a_i$

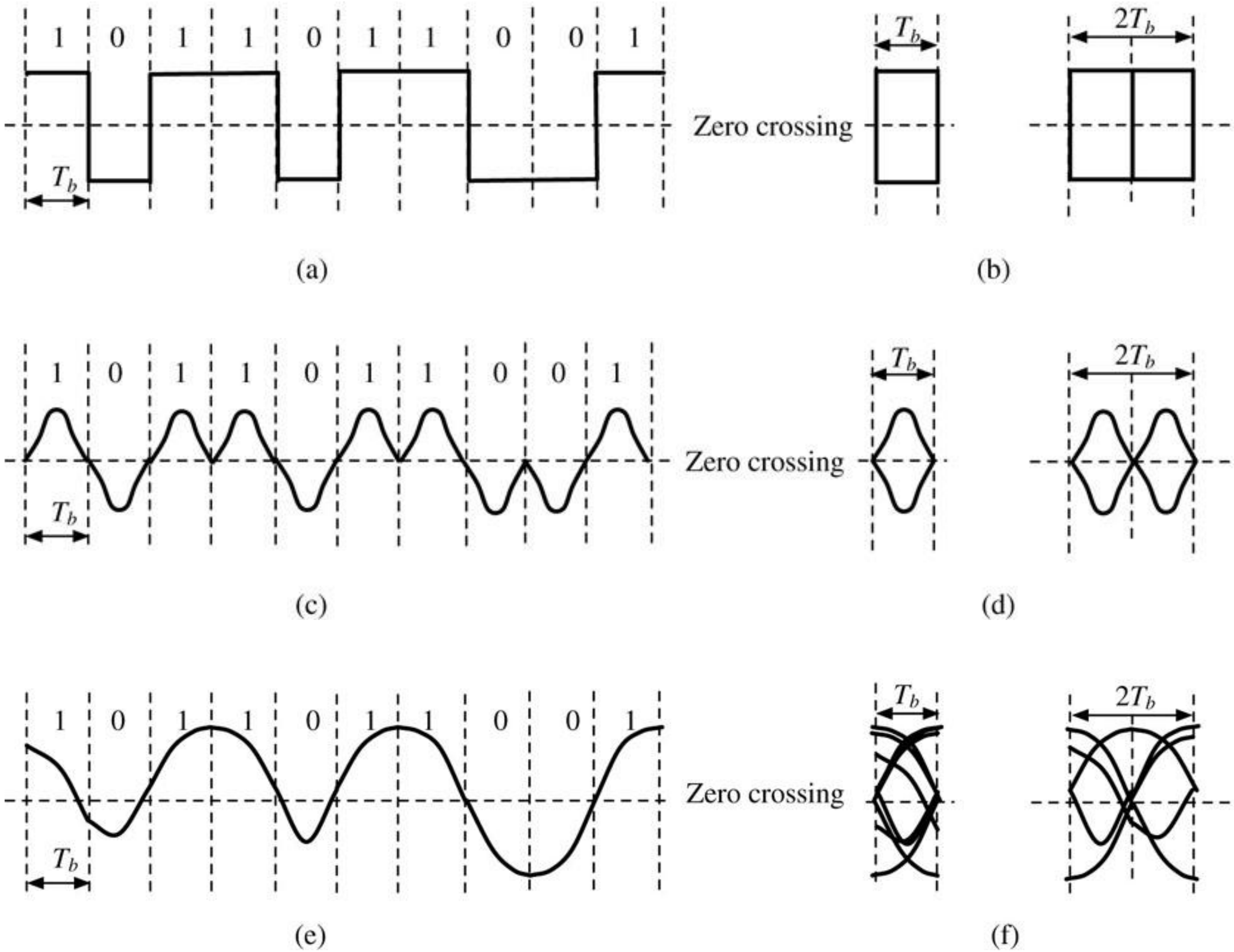
Now construct a time function defined on $-\frac{T_b}{2} \le t \le \frac{T_b}{2}$

$$eye(t) = \frac{1}{\sqrt{E}} \sum_{i=-\infty}^{\infty} y(t - iT_b) = \frac{1}{\sqrt{E}} \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_k p(t - (i+k)T_b)$$

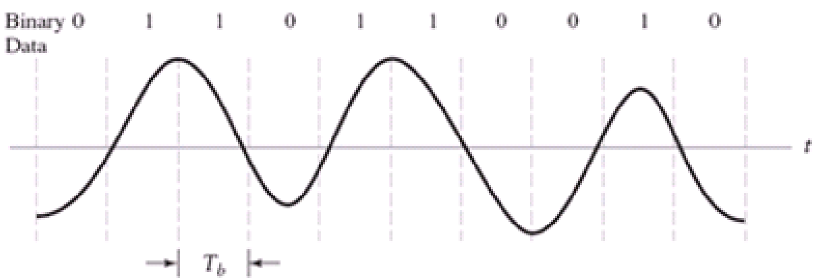


(c) (d)

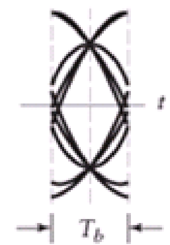
The Eye Diagram



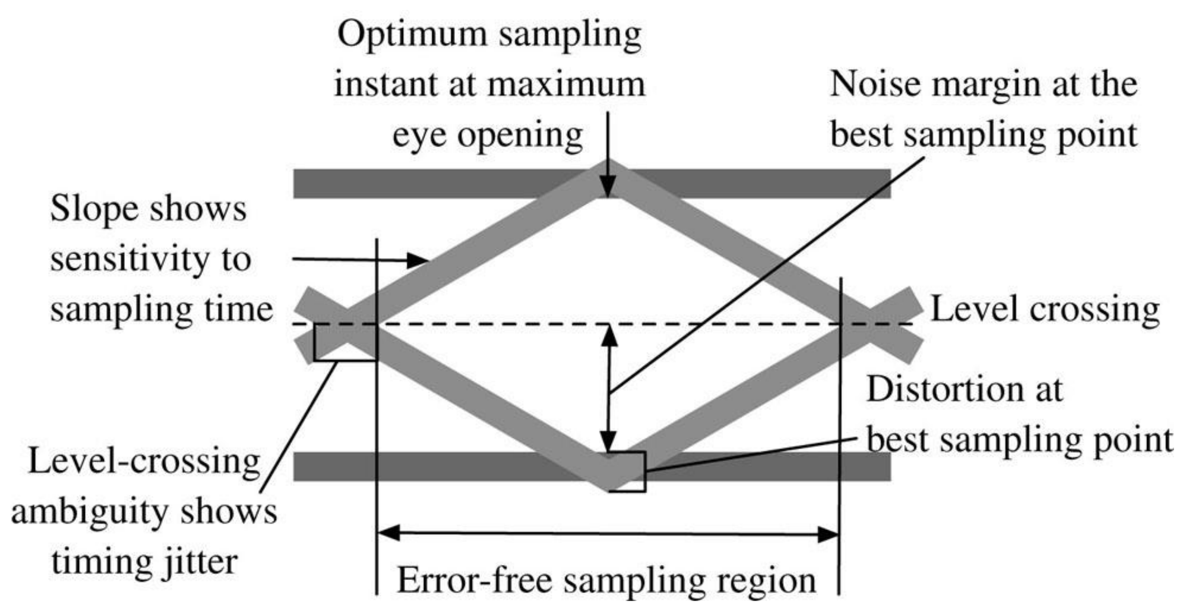
The Eye Diagram



(a)

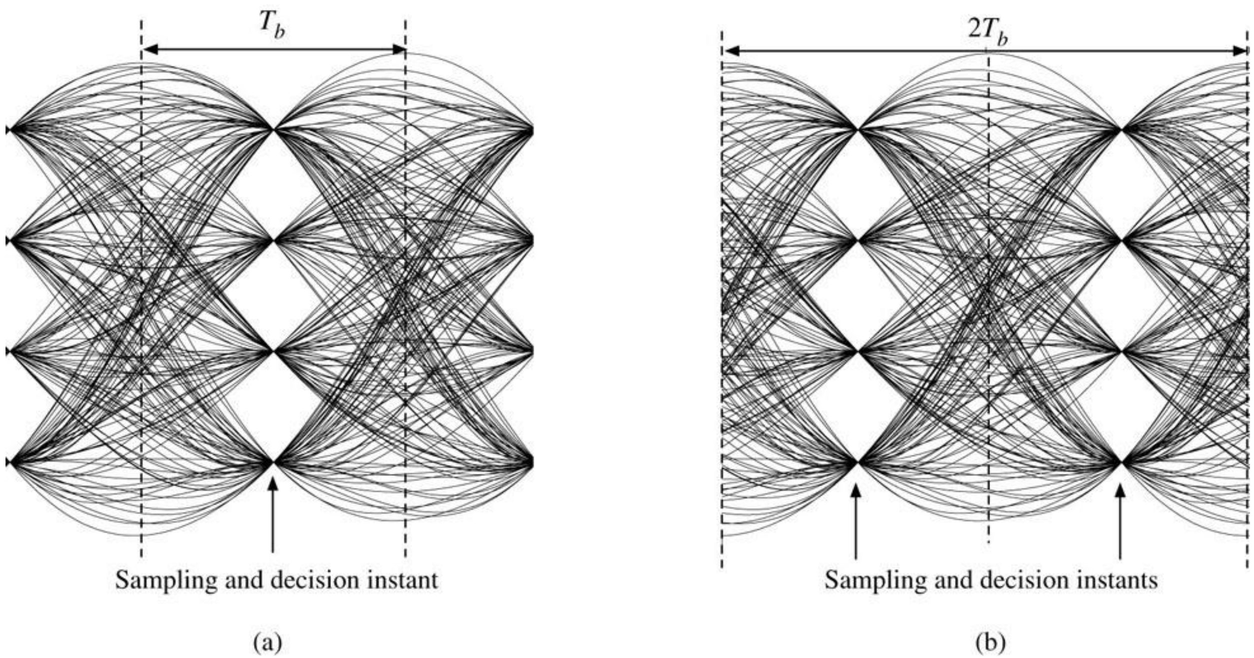


(b)



Eye Diagram for M-ary Modulation

If a_k can take M possible values instead of two, the eye pattern looks like the following



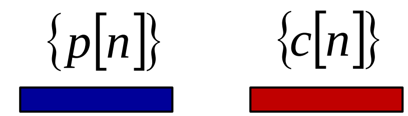
(a) (b)

$$D_{peak} = \frac{1}{\sqrt{E}} \sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} a_k \, p(iT_b - kT_b) = \left(\max_k |a_k|\right) \left[\frac{1}{\sqrt{E}} \sum_{k \neq 0} |p_k| + \left| \frac{p(0)}{\sqrt{E}} - 1 \right| \right]$$

Zero-Forcing Equalization

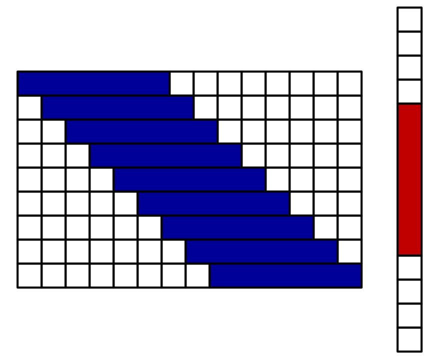
Zero-Forcing Equalization

$$C(z) = \sum_{k=-N}^{N} c[k] z^{-k} \qquad P(z)C(z) = 1 + o\left(z^{-N}\right) + o\left(z^{N}\right)$$



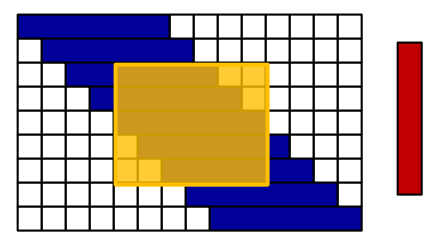
Convolution

Choose $\left[c_{-N}, c_{-N+1}, \cdots, c_0, \cdots, c_N\right]$ to satisfy



$$\begin{bmatrix} p_0 & \cdots & p_{-N+1} & p_{-N} & p_{-N-1} & \cdots & p_{-2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{N-1} & \cdots & p_0 & p_{-1} & p_{-2} & \cdots & p_{-N-1} \\ p_N & \cdots & p_1 & p_0 & p_{-1} & \cdots & p_{-N} \\ p_{N+1} & \cdots & p_2 & p_1 & p_0 & \cdots & p_{-N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{2N} & \cdots & p_{N+1} & p_N & p_{N-1} & \cdots & p_0 \end{bmatrix} \begin{bmatrix} c_{-N} \\ \vdots \\ c_{-1} \\ c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sqrt{E} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Truncated version



Channel Equalization

Example: $P(z) = 0.05z^2 - 0.2z + 1 - 0.3z^{-1} + 0.1z^{-2}$

Design a 3-tap zero-forcing equalizer $[c_{-1}, c_0, c_1]$

$$\begin{bmatrix} 1 & -0.2 & 0.05 \\ -0.3 & 1 & -0.2 \\ 0.1 & -0.3 & 1 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & -0.2 & 0.05 \\ -0.3 & 1 & -0.2 \\ 0.1 & -0.3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2094 \\ 1.1262 \\ 0.3169 \end{bmatrix}$$

$$C(z) = 0.2094\,z + 1.1262 + 0.3169\,z^{-1}$$

$$P(z)C(z) = \left(0.05z^2 - 0.2z + 1 - 0.3z^{-1} + 0.1z^{-2}\right)\left(0.2094\,z + 1.1262 + 0.3169\,z^{-1}\right)$$
$$= 0.0105\,z^3 + 0.0144\,z^2 + 1 + 0.0175\,z^{-2} + 0.0317\,z^{-3}$$

Channel Equalization

Example: $P(z) = 0.05z^2 - 0.2z + 1 - 0.3z^{-1} + 0.1z^{-2}$

Design a 5-tap zero-forcing equalizer $[c_{-2}, c_{-1}, c_0, c_1, c_2]$

$$\begin{bmatrix} 1 & -0.2 & 0.05 & 0 & 0 \\ -0.3 & 1 & -0.2 & 0.05 & 0 \\ 0.1 & -0.3 & 1 & -0.2 & 0.05 \\ 0 & 0.1 & -0.3 & 1 & -0.2 \\ 0 & 0 & 0.1 & -0.3 & 1 \end{bmatrix} \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -0.0153 \\ 0.2051 \\ 1.1267 \\ 0.3138 \\ -0.0185 \end{bmatrix}$$

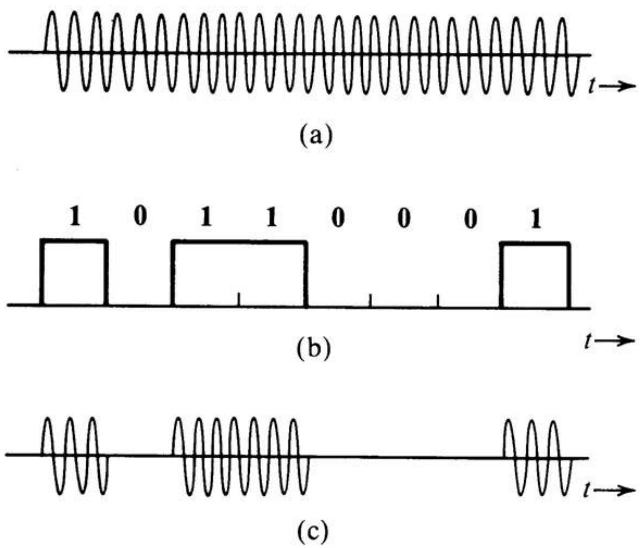
$$C(z) = -0.0153z^2 + 0.2051z + 1.1267 + 0.3138z^{-1} - 0.0185z^{-2}$$

$$P(z)C(z) = \left(0.05z^2 - 0.2z + 1 - 0.3z^{-1} + 0.1z^{-2}\right)$$
$$\times\left(-0.0153z^2 + 0.2051z + 1.1267 + 0.3138z^{-1} - 0.0185z^{-2}\right)$$
$$= -0.008z^4 + 0.0133z^3 + 1 + 0.0369z^{-3} - 0.0019z^{-4}$$

Roadmap

- Line Coding (Transmission Coding)
- Digital Baseband Transmission
- Digital Band-Pass Modulation
  - Binary band-pass modulation
  - M-ary band-pass modulation
  - Signal space diagram
  - Detector options

Binary Amplitude Shift Keying



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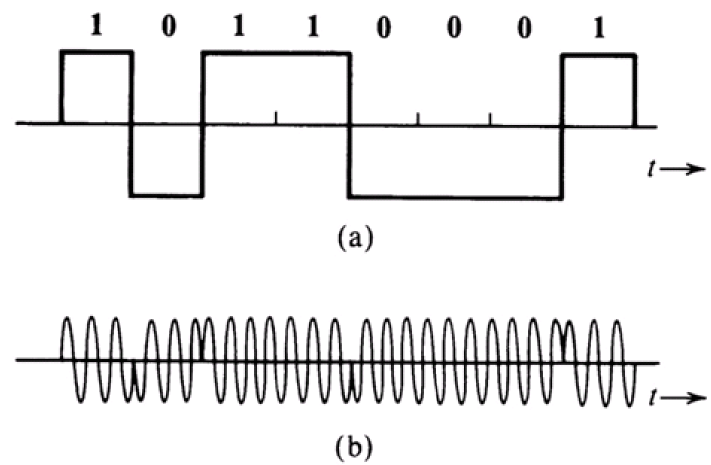
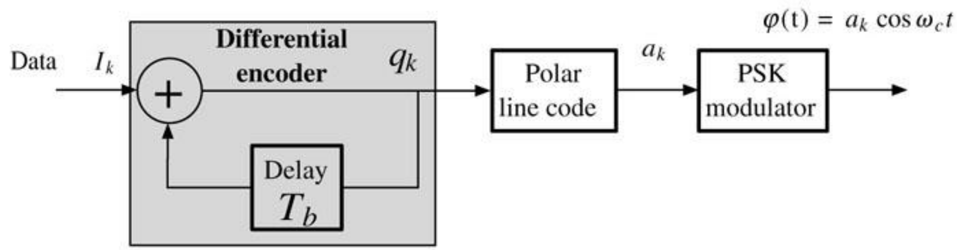


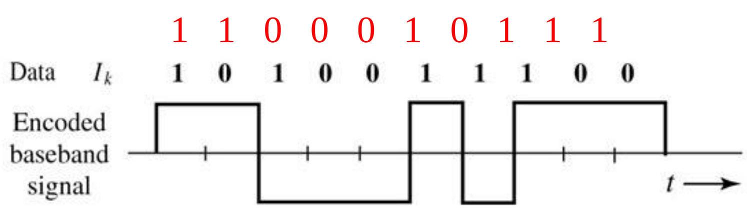
BASK

Differential Phase Shift Keying (DPSK)

User PSK to modulate $a_k \oplus a_{k-1}$



(a)



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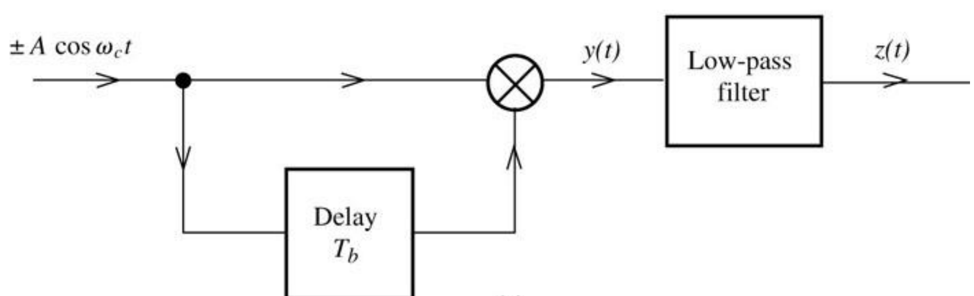
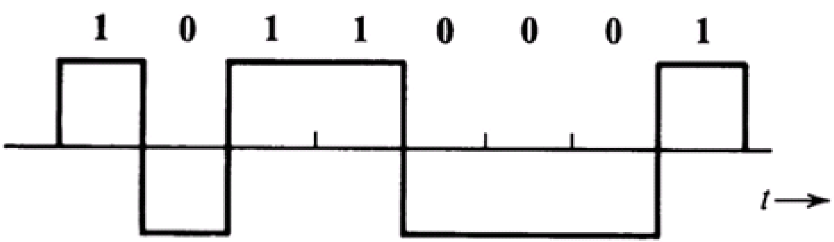


(c)

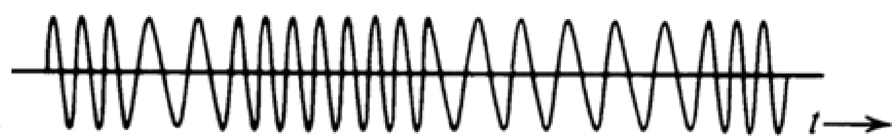
Binary Frequency Shift Keying (BFSK)

Modulation

$$c_0(t)=\sqrt{\frac{2}{T_b}}\cos(\omega_{c0}t) \qquad c_1(t)=\sqrt{\frac{2}{T_b}}\cos(\omega_{c1}t) \qquad \Phi(t)=\begin{cases}\sqrt{\frac{2}{T_b}}\cos(\omega_{c1}t) & \text{for symbol 1}\\ \sqrt{\frac{2}{T_b}}\cos(\omega_{c0}t) & \text{for symbol 0}\end{cases}$$



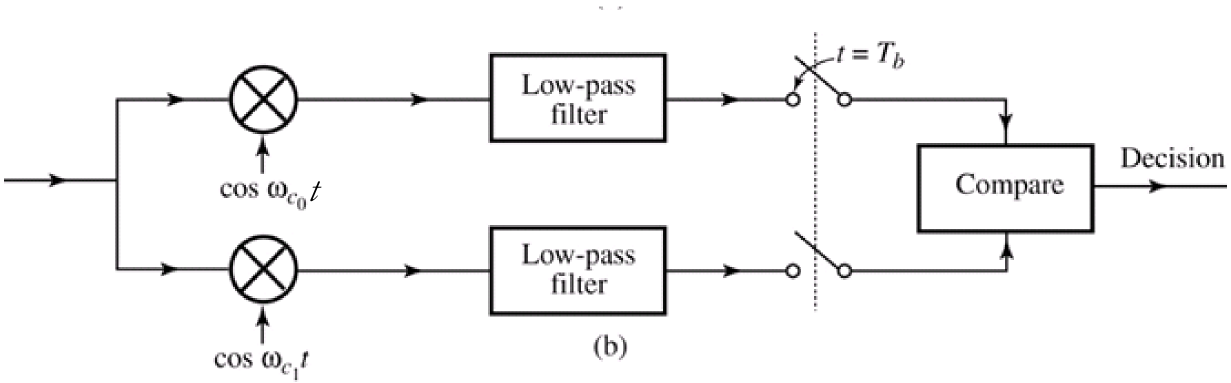
(a)



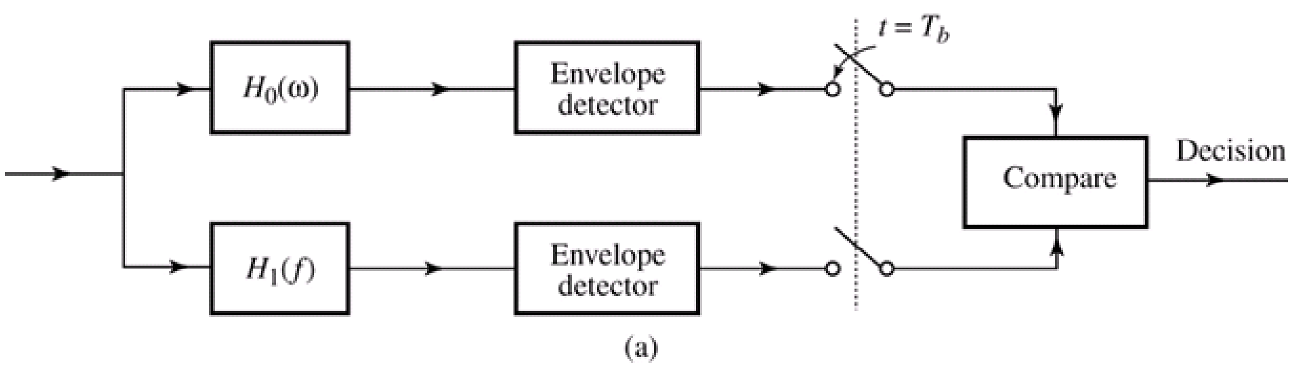
(c)

BFSK Detection

Coherent Detection (need to know the phase of the carrier)



Non-coherent Detection (no phase information)

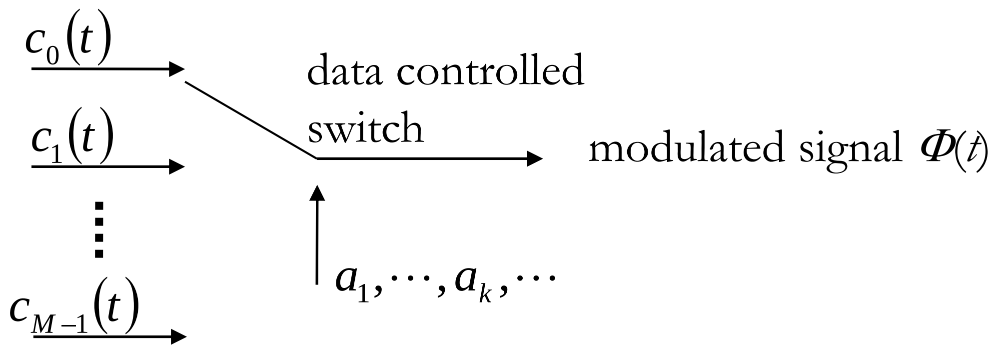


No need to synchronize phase

M-ary Digital Carrier Modulation

Input data $\{a_k\}$ $a_k \in \{0, \cdots, M-1\}$

Basic Diagram:



M-ary Amplitude-Shift Keying (MASK)

$$\Phi(t) = a_k \sqrt{\frac{2}{T_b}} \cos(\omega_c t)$$

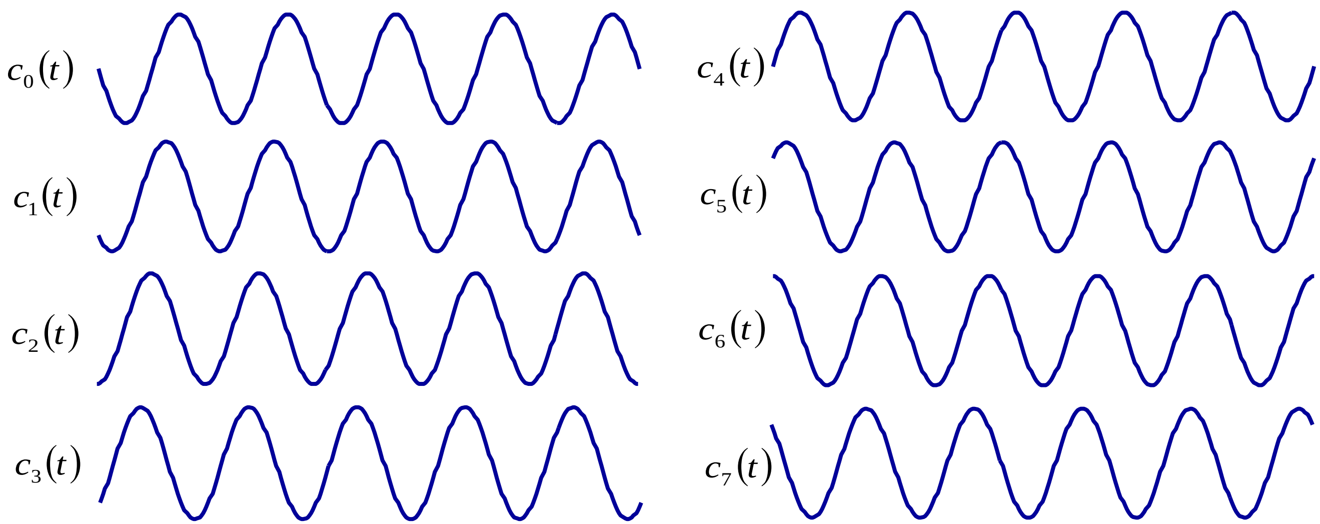
Variation

$$\Phi(t) = (2a_k - M + 1)\sqrt{\frac{2}{T_b}} \cos(\omega_c t)$$

MPSK and MFSK

M-ary Phase-Shift Keying

$$c_i(t) = \sqrt{\frac{2}{T_b}} \cos\left(\omega_c t + \frac{2\pi i}{M}\right) \qquad i = 0, \cdots, M-1 \qquad 8-PSK$$



M-ary Frequency-Shift Keying

$$c_i(t) = \sqrt{\frac{2}{T_b}} \cos(\omega_i t) \qquad i = 0, \cdots, M-1$$

Signal Space Diagram

Some basis signals

$$\sqrt{2}\cos(\omega_1 t) \qquad \sqrt{2}\sin(\omega_1 t) \qquad \sqrt{2}\cos(\omega_2 t) \qquad \sqrt{2}\sin(\omega_2 t)$$

Inner product

$$\langle x(t), y(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)y^*(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)y(t)dt$$

$$\left\langle \sqrt{2}\cos(\omega_1 t), \sqrt{2}\cos(\omega_1 t)\right\rangle = 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\cos^2(\omega_1 t)dt = 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{1+\cos(2\omega_1 t)}{2}dt = 1$$

$$\left\langle \sqrt{2}\sin(\omega_1 t), \sqrt{2}\sin(\omega_1 t)\right\rangle = 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sin^2(\omega_1 t)dt = 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{1-\cos(2\omega_1 t)}{2}dt = 1$$

$$\left\langle \sqrt{2}\sin(\omega_1 t), \sqrt{2}\cos(\omega_1 t)\right\rangle = 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sin(\omega_1 t)\cos(\omega_1 t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sin(2\omega_1 t)dt = 0$$

$$\left\langle \sqrt{2}\cos(\omega_1 t), \sqrt{2}\cos(\omega_2 t)\right\rangle = 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\cos(\omega_1 t)\cos(\omega_1 t)dt$$

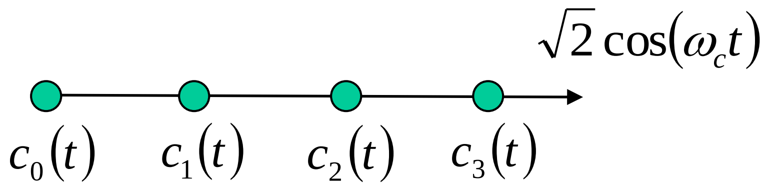
$$= 2\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{\cos((\omega_1+\omega_2)t)+\cos((\omega_1-\omega_2)t)}{2}dt = 0$$

MASK

First version

$$c_k(t) = a_k\sqrt{2}\cos(\omega_c t) \qquad a_k \in \{0,\cdots,M-1\}$$

One-dimensional space

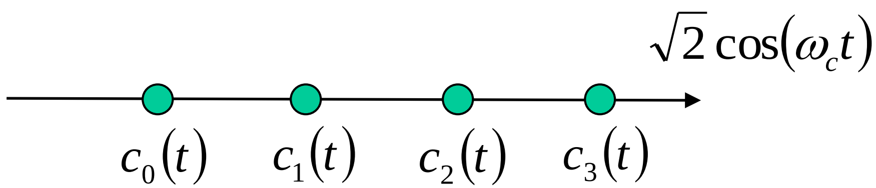


$$c_k(t) \to k \qquad k \in \{0,\cdots,M-1\}$$

Second version

$$c_k(t) = (2a_k - M + 1)\sqrt{2}\cos(\omega_c t) \qquad a_k \in \{0,\cdots,M-1\}$$

One-dimensional space



$$c_k(t) \to 2k - M + 1 \qquad k \in \{0,\cdots,M-1\}$$

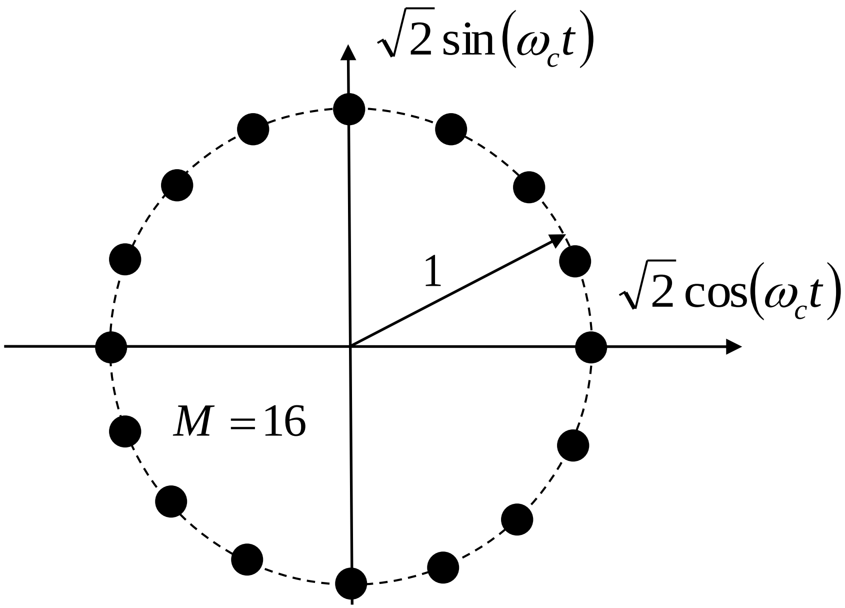
MPSK

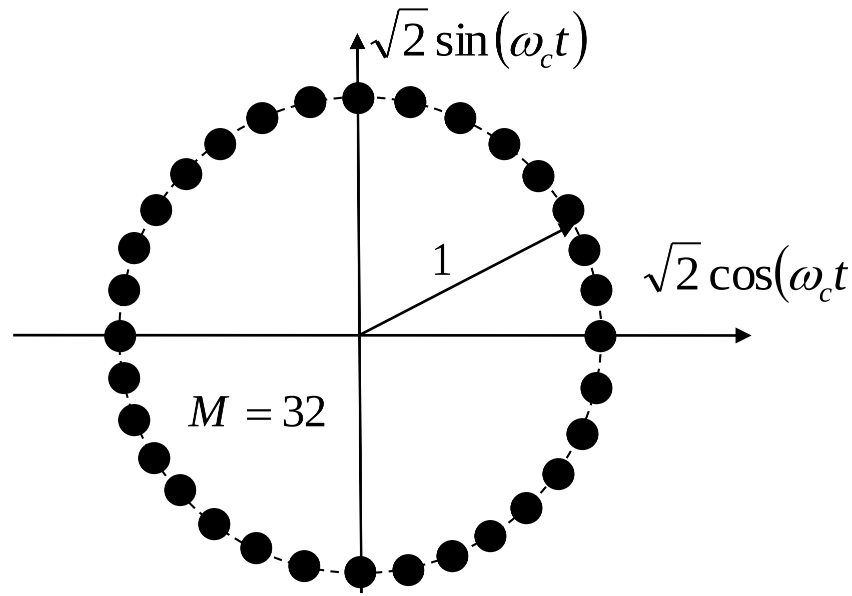
$$c_k(t)=\sqrt{2}\cos\left(\omega_c t+\frac{2\pi k}{M}\right)=\left(\cos\frac{2\pi k}{M}\right)\sqrt{2}\cos(\omega_c t)-\left(\sin\frac{2\pi k}{M}\right)\sqrt{2}\sin(\omega_c t)$$

$$k=0,\cdots,M-1$$

Two-dimensional space

$$c_k(t)\to\begin{bmatrix}\cos\frac{2\pi k}{M}\\ \sin\frac{2\pi k}{M}\end{bmatrix}$$





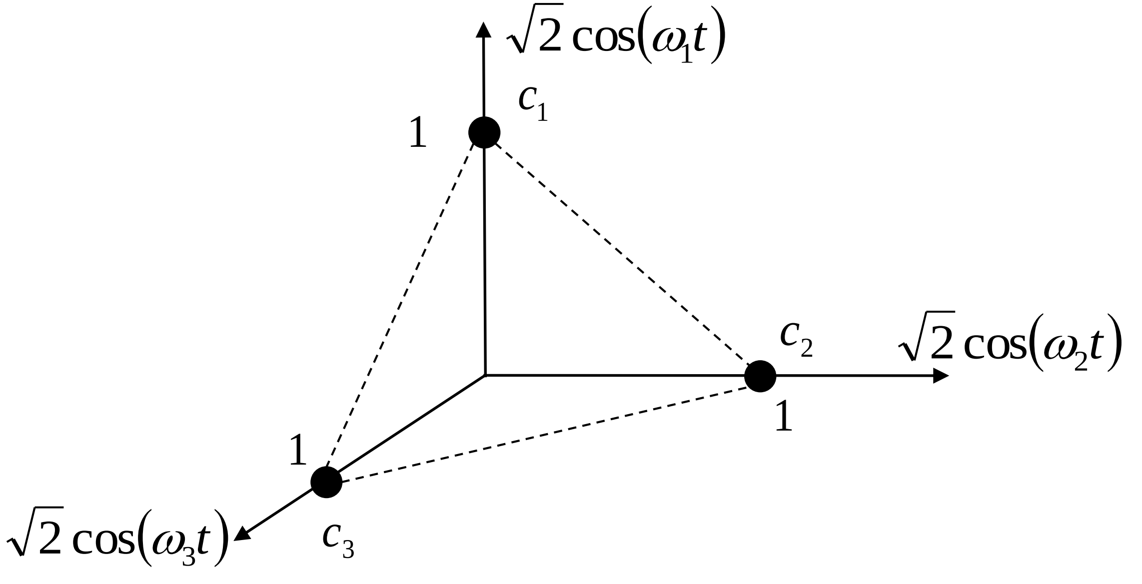
MFSK

$$c_k(t) = \sqrt{2}\cos(\omega_k t) \qquad k = 0, \cdots, M-1$$

M-dimensional Space $\qquad c_k(t) \to [0 \quad \cdots \quad 1 \quad \cdots \quad 0]^T$

the k^{th} element

3-FSK



Signal Constellation

Let the modulated waveform be

$$\Phi_k(t) = x_1(k)\phi_1(t) + x_2(k)\phi_2(t) + \cdots + x_K(k)\phi_K(t)$$

where $\phi_1(t), \phi_2(t), \cdots, \phi_K(t)$ are K carrier signals. Assume the digital system transmits one symbol every T seconds. We say T is the symbol duration.

Assume

$$\int_0^T \phi_i^{\,2}(t)dt = 1 \text{ for all } i. \qquad \text{unit energy}$$

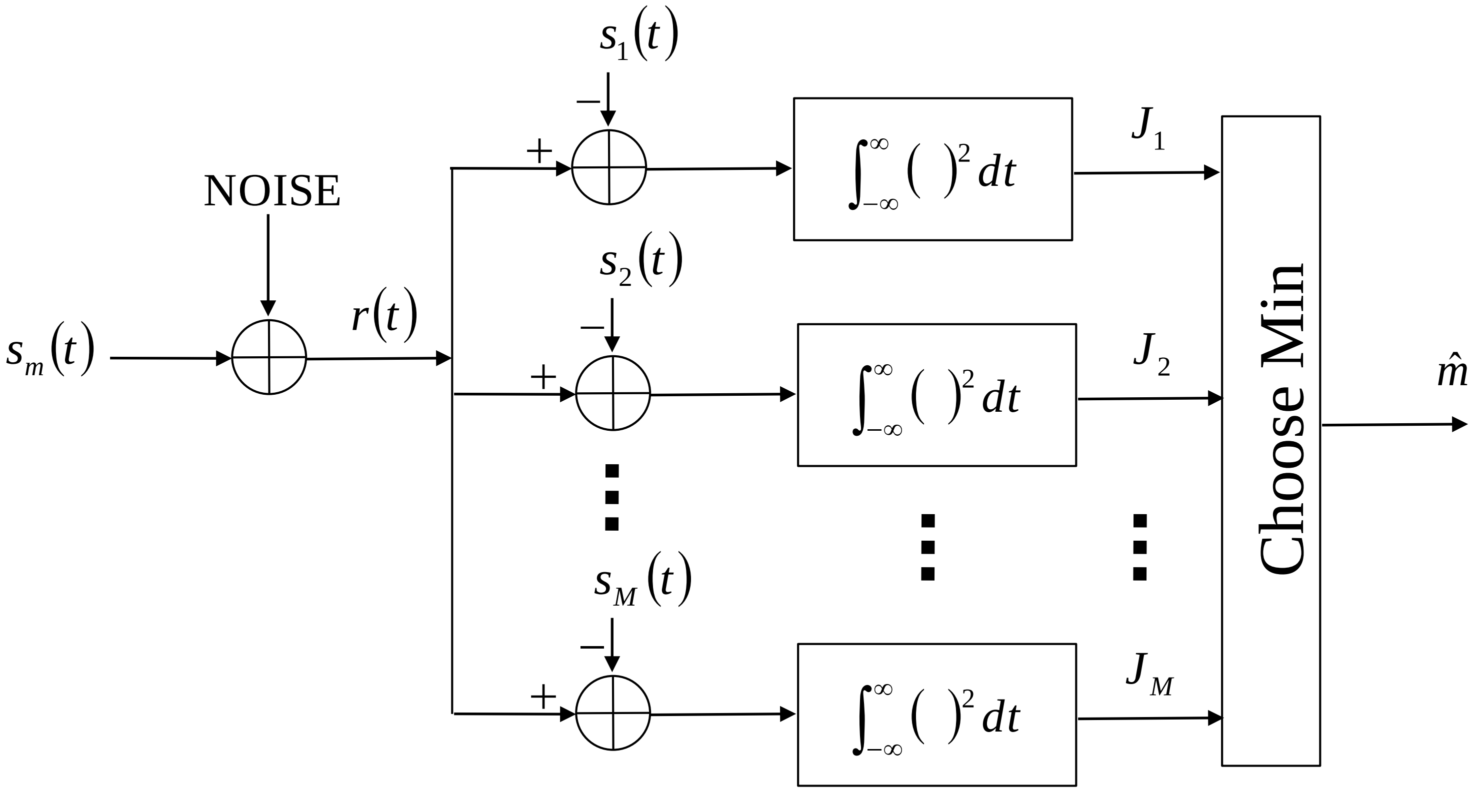
$$\int_0^T \phi_i(t)\phi_j(t)dt = 0 \text{ for all } i \neq j. \qquad \text{orthogonal}$$

We can map the modulated signal corresponding to the k^{th} symbol to a point with coordinate $[x_1(k), x_2(k), \cdots, x_K(k)]^T$ in the K-dimensional space. This is called the signal constellation.

Minimum Distance Detector

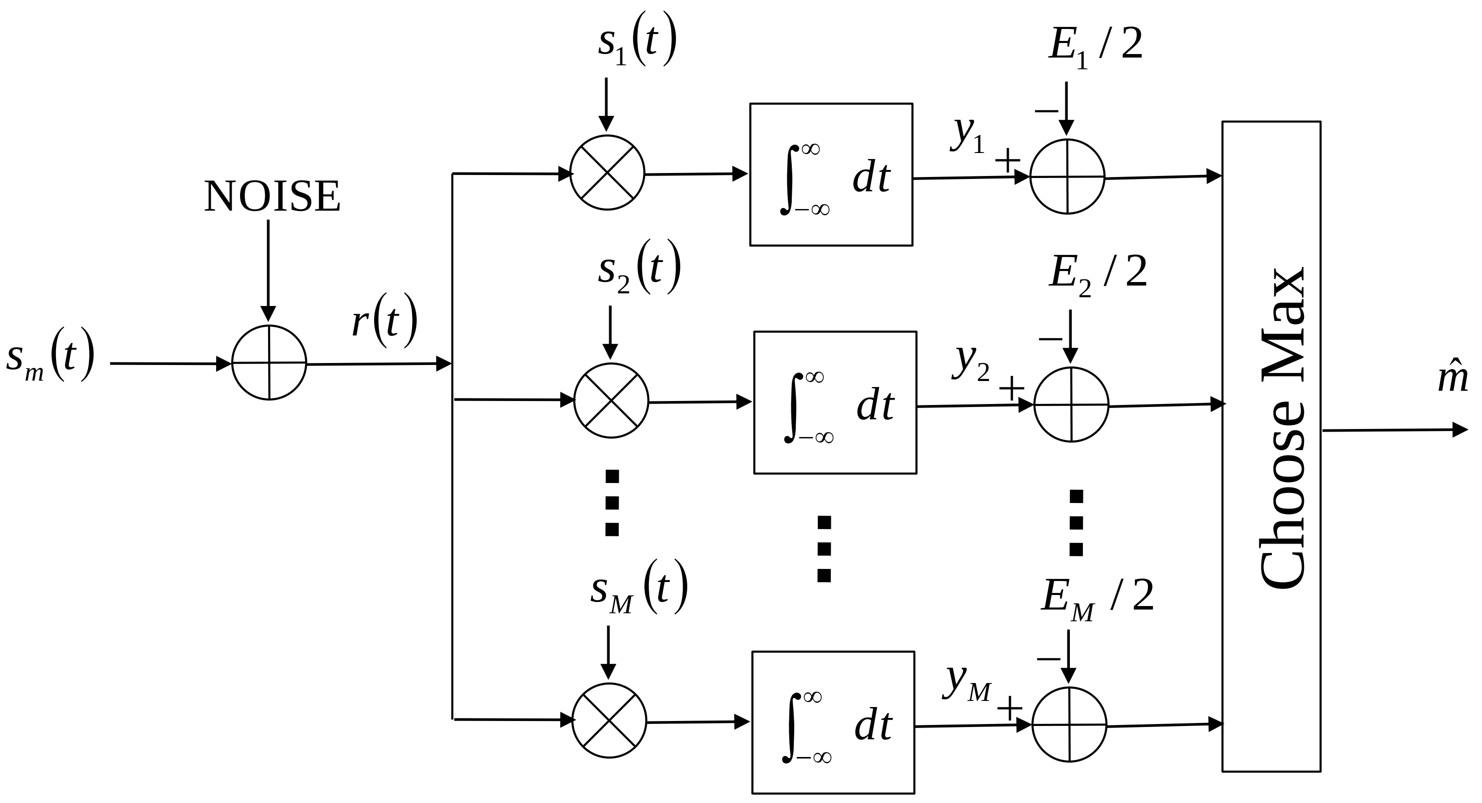
Receiver decision is $s_{\hat{m}}(t)$, where $\hat{m}$ minimizes :

$$J_i = \int_{-\infty}^{\infty} \left(r(t) - s_i(t)\right)^2 dt$$

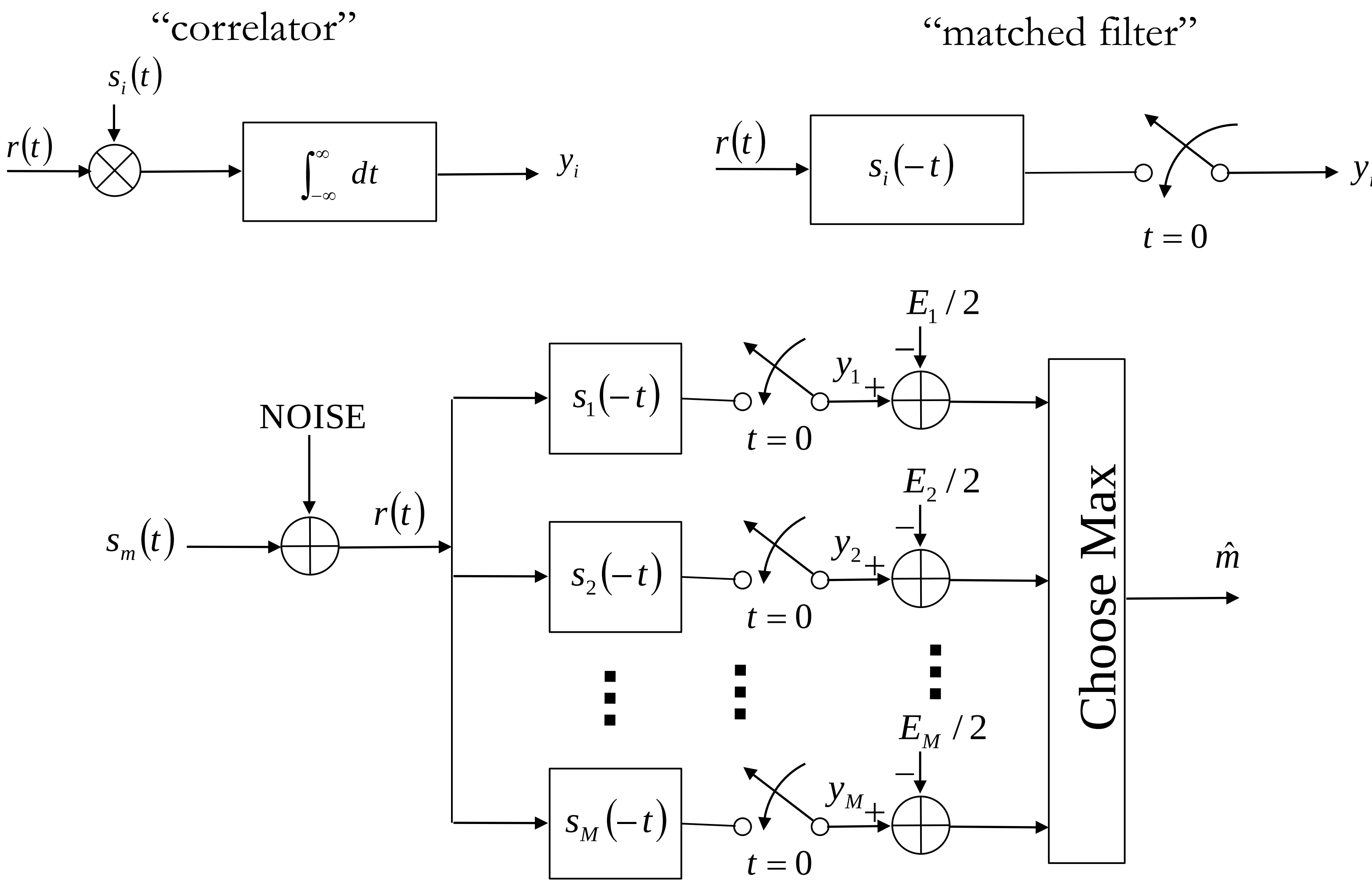


Brute-force implementation

Correlation Receiver

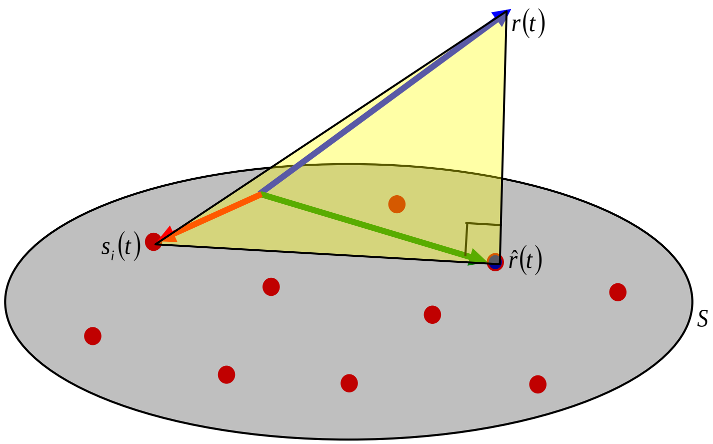


Alternative Implementation



“Minimum Distance” Revisited

$$J_i = \int_{-\infty}^{\infty} \left(r(t) - s_i(t)\right)^2 dt$$



Let $S = \text{Span}\{s_1(t), \cdots, s_M(t)\}$ be the "signal space"

Let $\hat{r}(t) =$ projection of $r(t)$ onto S

Rewrite cost :

$$J_i = \langle r(t) - s_i(t), r(t) - s_i(t) \rangle = \|r(t) - s_i(t)\|^2$$

$$= \|r(t) - \hat{r}(t) + \hat{r}(t) - s_i(t)\|^2$$

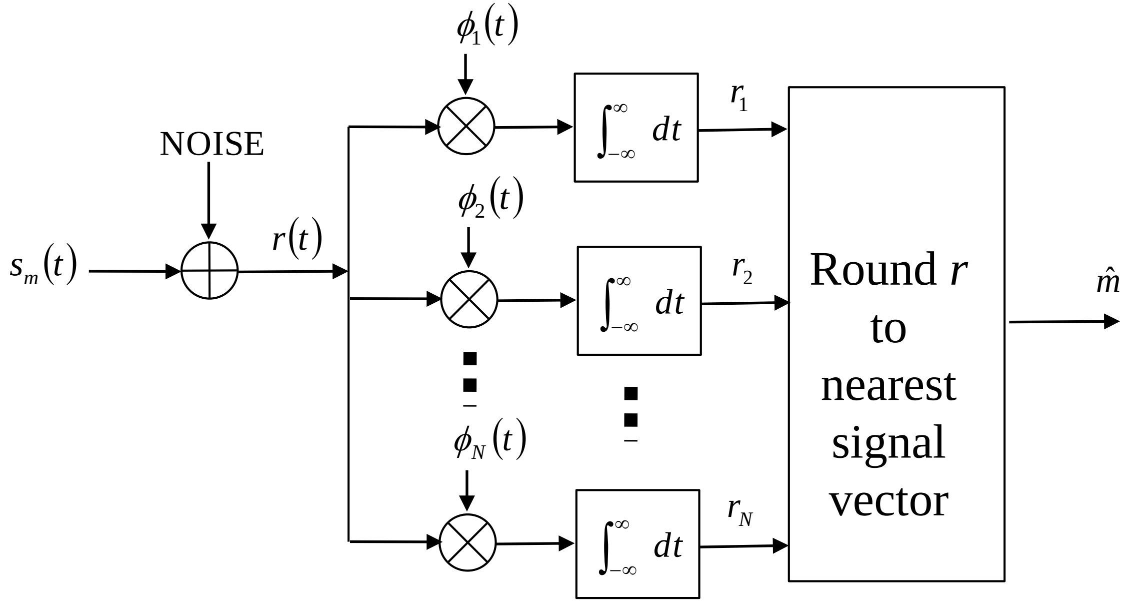
$$= \|r(t) - \hat{r}(t)\|^2 + \|\hat{r}(t) - s_i(t)\|^2 + 2\langle r(t) - \hat{r}(t), \hat{r}(t) - s_i(t) \rangle$$

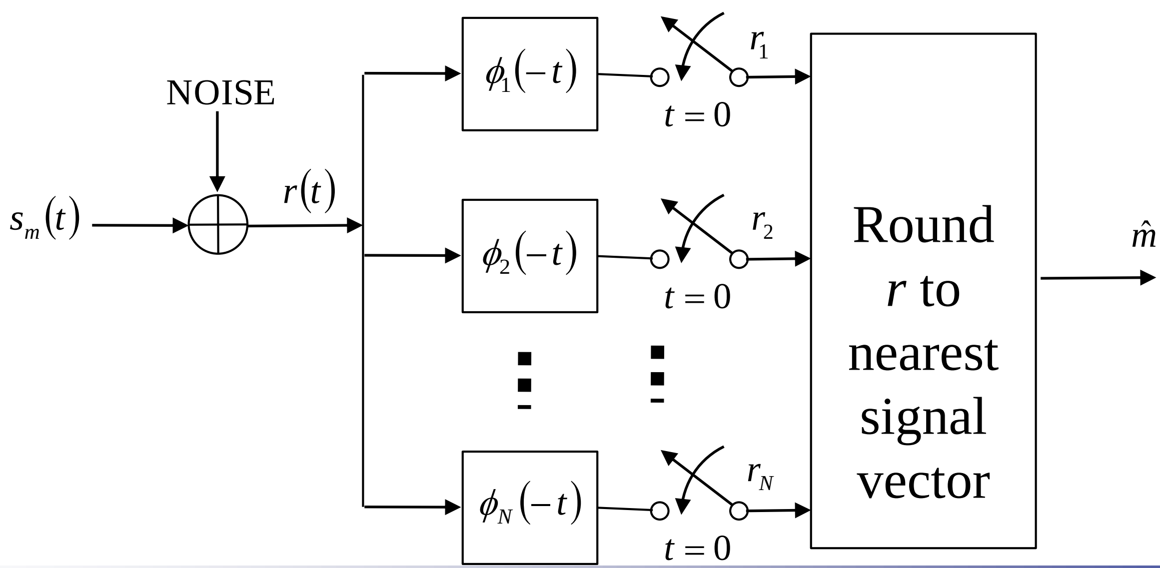
First term: independent of i

Last term: zero, because of orthorgonality principle

$\Rightarrow$The minimum-distance receiver minimizes second term only.

The Projection Receiver



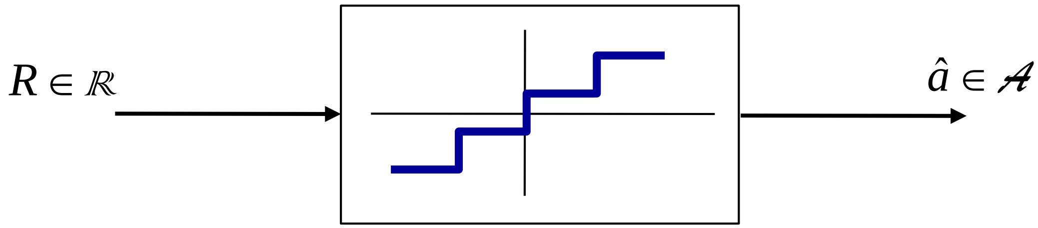


Roadmap

- Probabilistic Detectors
  - Maximum A Posterior (MAP) Detector
  - Maximum Likelihood (ML) Detector
- Probability of Error Analysis
- Direct Sequence Spread Spectrum (DSSS) Communications

A Scalar Detector

Threshold device=quantizer=slicer:

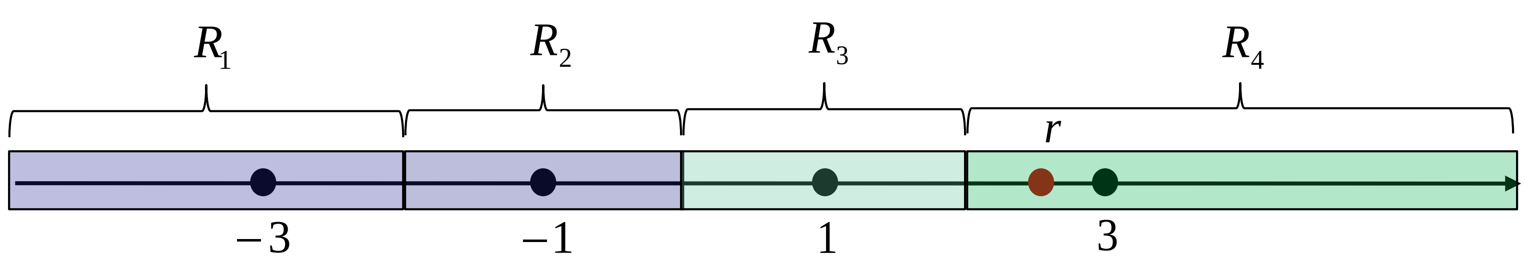


Defined by :

Decision regions $R_1 = \{r : \hat{a} = a_1\}, R_2 = \{r : \hat{a} = a_2\}, \cdots R_M$

(disjoint, cover $\mathbb{R}$)

Thresholds



Derive A Detector to Minimize Pr[Err]

As a function of the decision regions $\{R_1, R_2, \ldots R_M\}$, for any detector:

$$\Pr[\text{correct}] = \int_{-\infty}^{\infty} \Pr[\text{correct} \mid R = r] f(r) dr$$
$$= \int_{R_1} \Pr[\text{correct} \mid r] f(r) dr + \int_{R_2} \Pr[\text{correct} \mid r] f(r) dr + \int_{R_3} \Pr[\text{correct} \mid r] f(r) dr + \cdots$$
$$= \int_{R_1} \Pr[a_1 \mid r] f(r) dr + \int_{R_2} \Pr[a_2 \mid r] f(r) dr + \int_{R_3} \Pr[a_3 \mid r] f(r) dr + \cdots$$

$\Pr[a_2 \mid r]$	$\Pr[a_1 \mid r]$	$\Pr[a_1 \mid r]$
$\Pr[a_3 \mid r]$	$\Pr[a_3 \mid r]$	$\Pr[a_2 \mid r]$
$\Pr[a_4 \mid r]$	$\Pr[a_4 \mid r]$	$\Pr[a_4 \mid r]$

Step through each $r \in \mathbb{R}$ and assign to a decision region:

Which assignment maximizes Pr[correct]?

Assigning r to R_i contributes $P(a_i \mid r) f(r) dr$ to total

$\Rightarrow$ MAP contributes the most! $\Rightarrow$ MAP minimizes Pr[error].

Two Probabilistic Detectors

Notation:

$$P_A(a) = \Pr[A = a] = a \textit{ priori } \text{probability that } A = a$$

$$P_{A|R}(a \mid r) = \Pr[A = a \mid R = r] = a \textit{ posteriori } \text{probability that } A = a$$

Related by Bayes rule:

$$P_{A|R}(a \mid r) = \frac{f_{R|A}(r \mid a) P_A(a)}{f_R(r)}$$

Two probabilistic detectors:

1. The maximum a posteriori (MAP) detector:

$$\begin{aligned} \hat{a}_{\mathrm{MAP}} &= \arg\max \{ P_{A|R}(a \mid r) \} \\ &= \arg\max \{ f_{R|A}(r \mid a) P_A(a) \} \end{aligned}$$

2. The maximum likelihood (ML) detector:

$$\hat{a}_{\mathrm{ML}} = \arg\max \{ f_{R|A}(r \mid a) \}$$

MAP Example

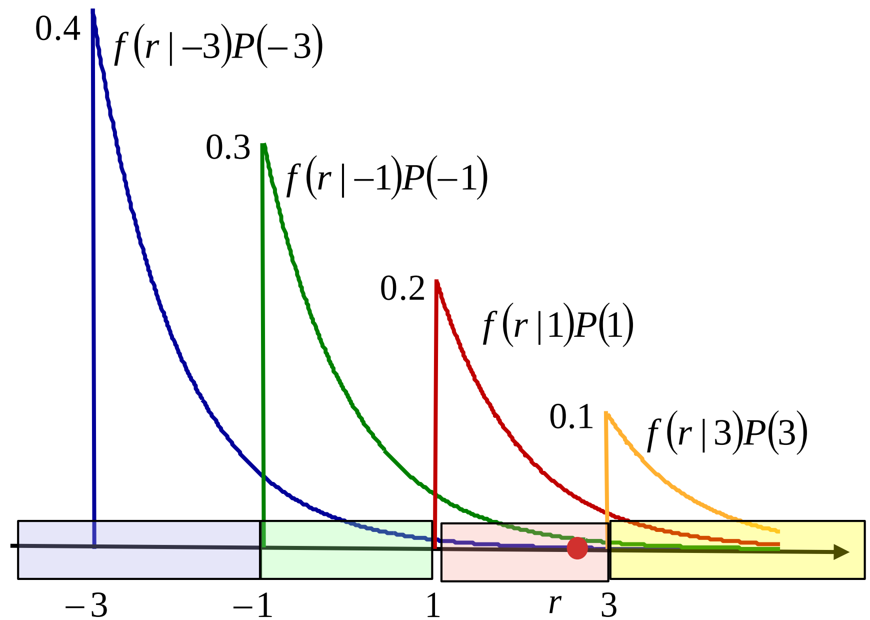
Example:

$A = \{-3,-1,1,3\}$.

$N \sim Exp(1), \quad f(N) = \exp(-N)$

$P_A(-3) = 0.4, P_A(-1) = 0.3, P_A(1) = 0.2, P_A(3) = 0.1$

$r = 2.9$



$f_{R|A}(r\,|\,a)P_A(a) = f_N(r-a)P_A(a) = e^{-(r-a)}u(r-a)P_A(a)$

The MAP detector maximizes

$$f_{R|A}(r\,|\,a)P_A(a) = f_N(r-a)P_A(a) = e^{-(r-a)}u(r-a)P_A(a)$$

Same Example, Different A Priori

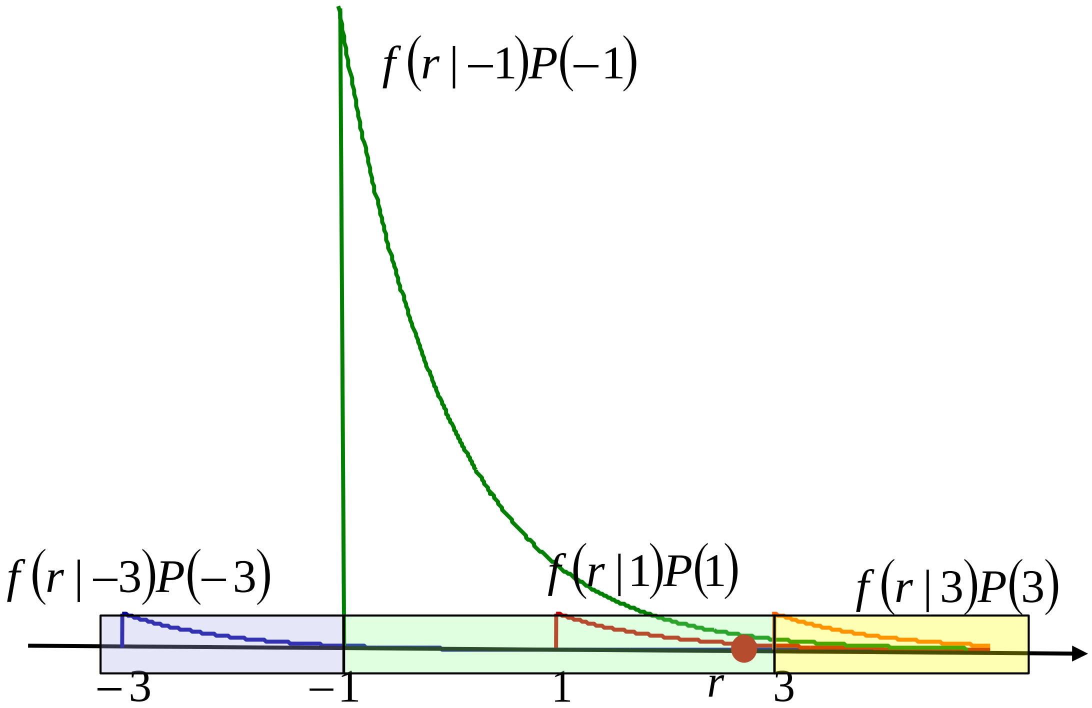
Example :

$\mathcal{A} = \{-3, -1, 1, 3\}.$

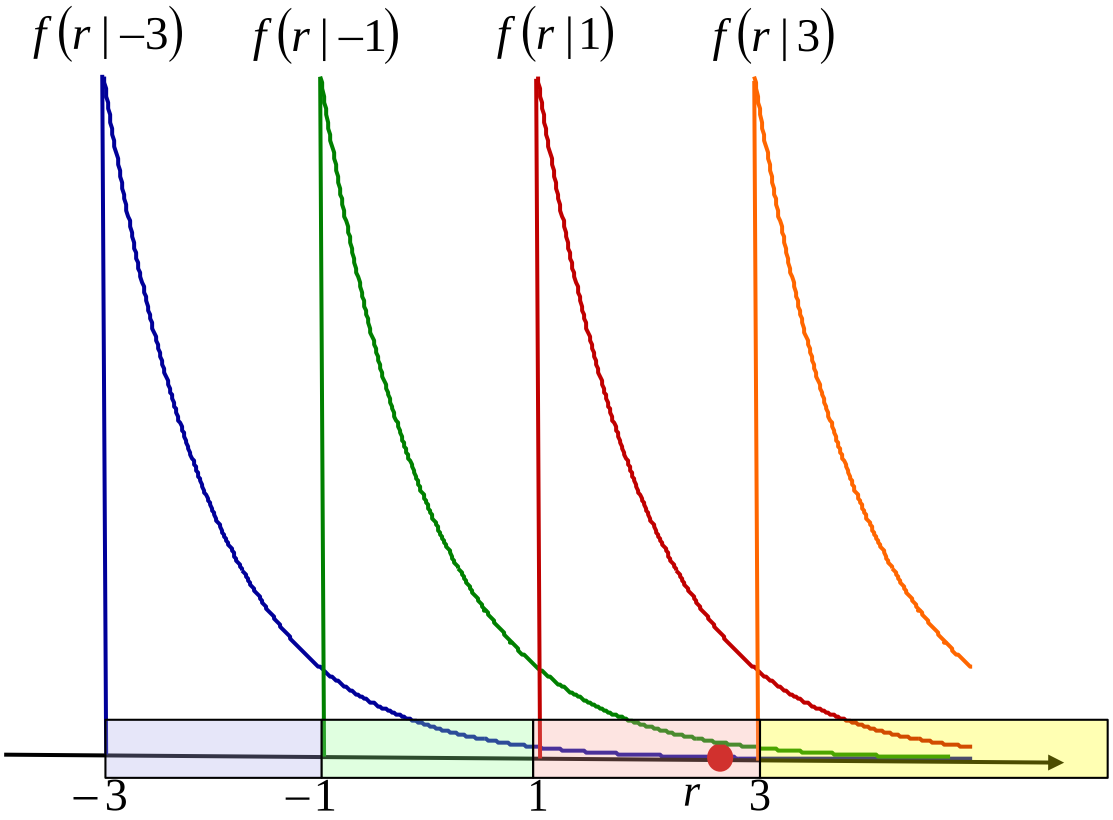
$N \sim \text{Exp}(1), \qquad f(N) = \exp(-N)$

$P_A(-3) = 0.05, P_A(-1) = 0.85, P_A(1) = 0.05, P_A(3) = 0.05$

$R = 2.9$



Maximum Likelihood



Three Different Answers

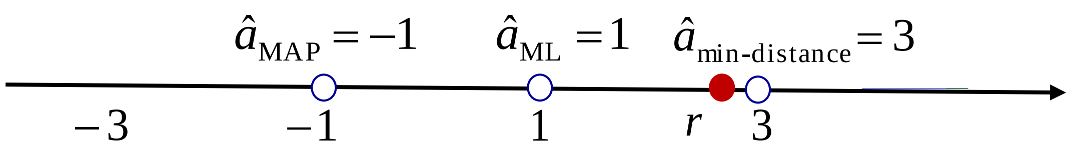
Example :

$\mathcal{A} = \{-3, -1, 1, 3\}.$

$N \sim \mathrm{Exp}(1), \qquad f(N) = \exp(-N)$

$P_A(-3) = 0.05, P_A(-1) = 0.85, P_A(1) = 0.05, P_A(3) = 0.05$

$R = 2.9$



ML vs. MD

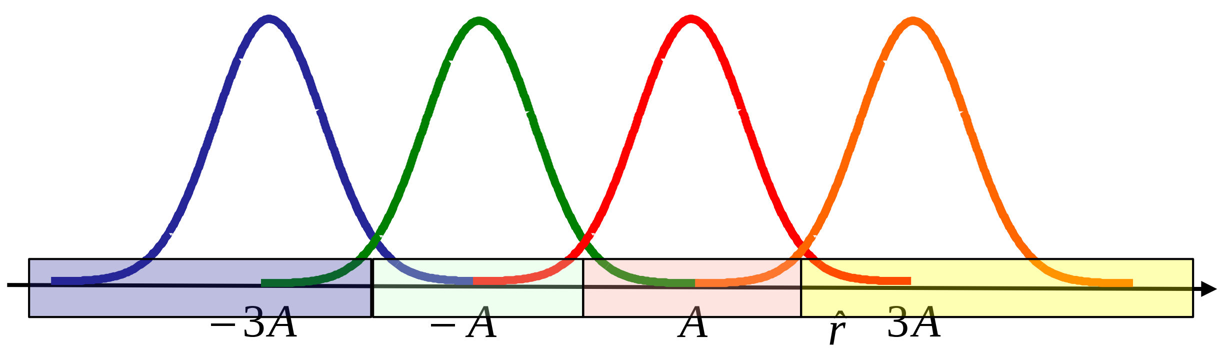
Minimum Distance Detection $\hat{s}_k = \arg\min_{s_k} (\hat{r} - s_k)^2$

Maximum Likelihood Detection $\hat{s}_k = \arg\max_{s_k} P(\hat{r} \mid s_k)$

$$P(\hat{r} \mid s_k) = \frac{1}{\sqrt{2\pi N_0}} \exp\left(-\frac{(\hat{r} - s_k)^2}{2N_0} \right)$$

Maximum Likelihood Detection $\hat{s}_k = \arg\max_{s_k} P(\hat{r} \mid s_k) = \arg\min_{s_k} (\hat{r} - s_k)^2$

Assume the symbols are equally likely



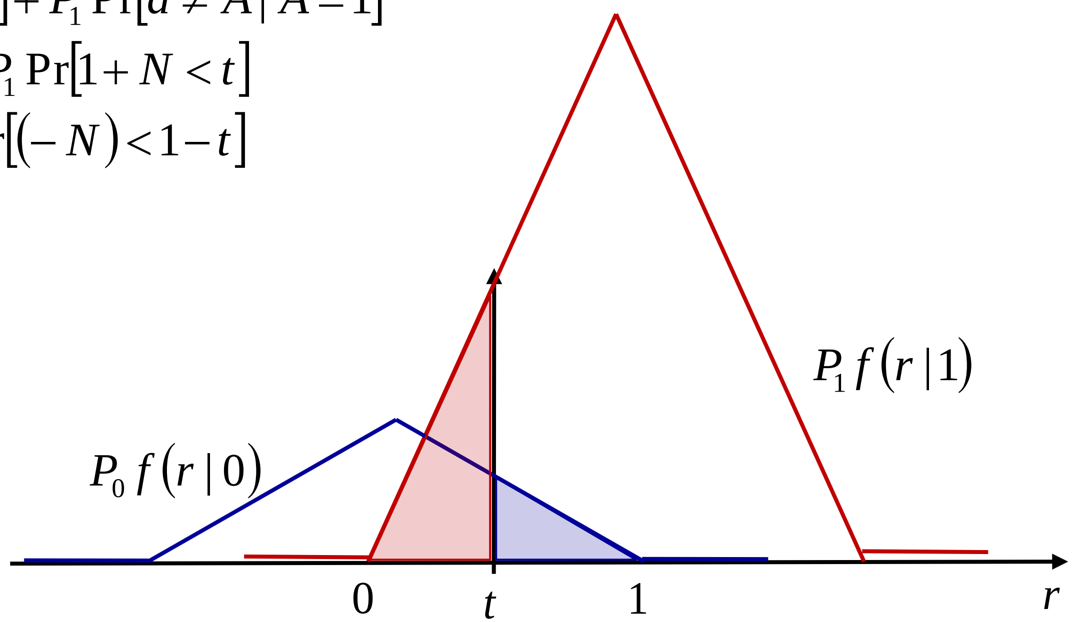
Roadmap

- Probabilistic Detectors
- Probability of Error Analysis
  - BASK
  - MASK
  - OOK
  - MPSK
- Direct Sequence Spread Spectrum (DSSS) Communications

General Expressions of the Error Probability

For any values of $P_0 = \Pr[A=0], P_1 = \Pr[A=1]$, and threshold t:

$$\begin{aligned}\Pr[\text{error}] &= \Pr[\hat{a} \neq A] \\ &= P_0 \Pr[\hat{a} \neq A \mid A=0] + P_1 \Pr[\hat{a} \neq A \mid A=1] \\ &= P_0 \Pr[0+N > t] + P_1 \Pr[1+N < t] \\ &= P_0 \Pr[N > t] + P_1 \Pr[(-N) < 1-t]\end{aligned}$$



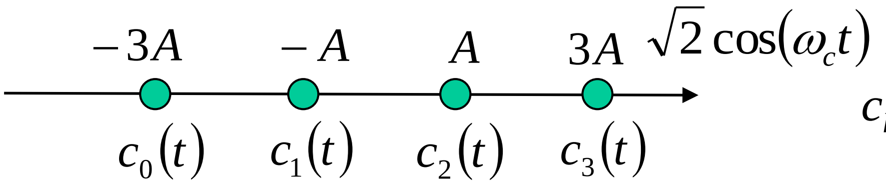
$$\begin{aligned}\Pr[\text{error}] &= \frac{1}{2}(1-t)^2 P_0 + \frac{1}{2} t^2 P_1 \\ &= \frac{1}{2}(t-P_0)^2 + \frac{1}{2} P_0 P_1\end{aligned}$$

MASK

$$c_k(t) = A(2a_k - M + 1)\sqrt{2}\cos(\omega_c t) \qquad a_k \in \{0, \cdots, M-1\}$$

Let $s_k = A(2a_k - M + 1)$ $\quad c_k(t) = s_k\sqrt{2}\cos(\omega_c t)$ $\quad s_k \in \{-AM + A, \cdots, AM - 3A, AM - A\}$

One-dimensional space

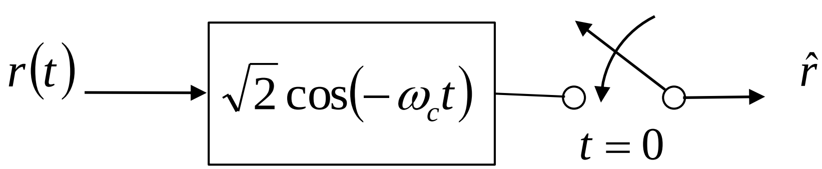


$$c_k(t) \to A(2k - M + 1) \qquad k \in \{0, \cdots, M-1\}$$

Assume white Gaussian noise, the received signal is

$$r(t) = s_k\sqrt{2}\cos(\omega_c t) + n(t) \qquad n(t) \text{ is Gaussian} \qquad R_n(\tau) = E[n(t)n(t-\tau)] = N_0\delta(\tau)$$

Project to signal space

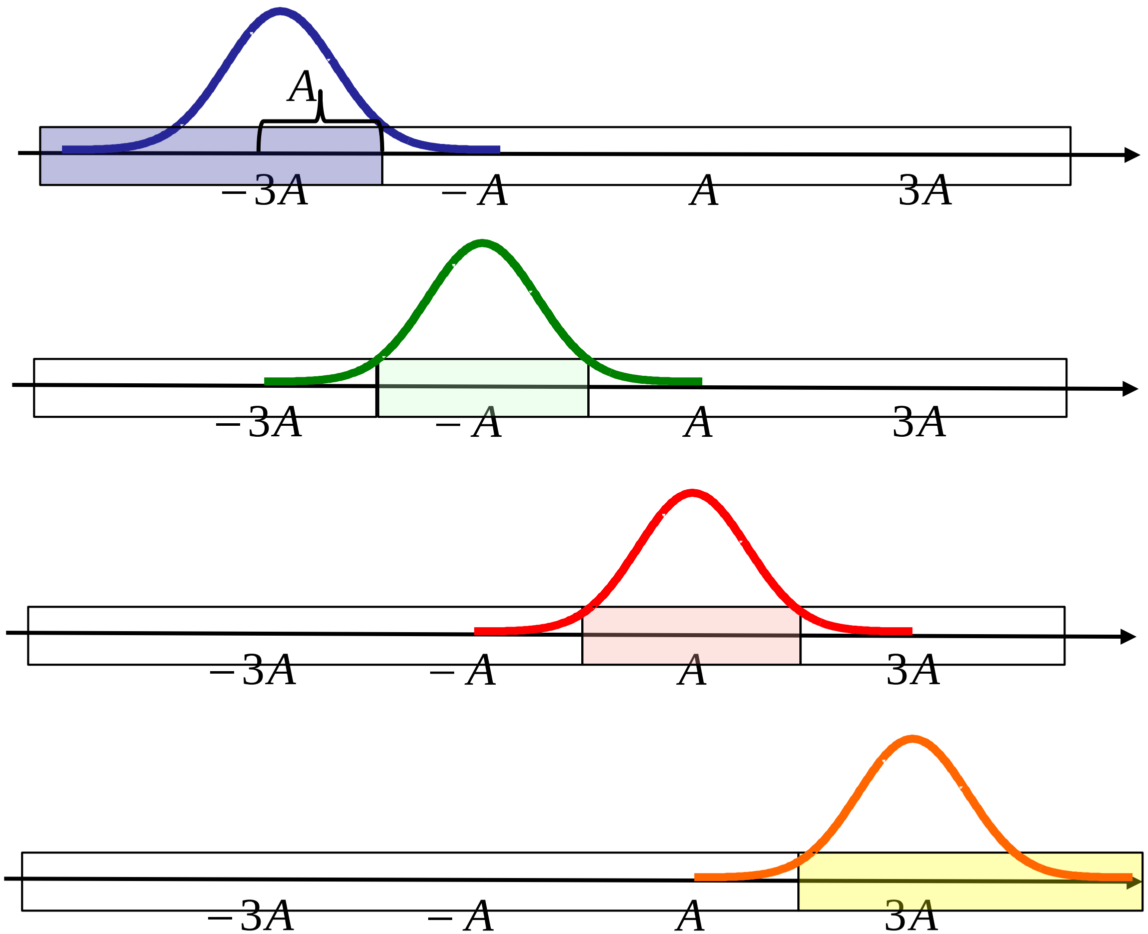


$$\hat{r} = \int_{-\infty}^{\infty} r(t)\sqrt{2}\cos(\omega_c t)dt = s_k + \int_{-\infty}^{\infty} n(t)\sqrt{2}\cos(\omega_c t)dt = s_k + n \qquad n \sim N(0, N_0)$$

Error Probability Analysis

$$P(\text{error}) = \frac{1}{M}\sum_k P\left(\hat{s}_k \neq s_k \middle| s_k\right)$$

M terms



$$\frac{1}{M} P\left(\hat{r} - (-3) > A \middle| s_k = -3A\right)$$

$$= \frac{1}{M} P(n > A) = \frac{1}{M} Q\left(\frac{A}{\sqrt{N_0}}\right)$$

$$\frac{1}{M} P\left(\left|\hat{r} - (-1)\right| > A \middle| s_k = -A\right)$$

$$= \frac{1}{M} P(|n| > A) = \frac{2}{M} Q\left(\frac{A}{\sqrt{N_0}}\right)$$

$$\frac{1}{M} P\left(\left|\hat{r} - 1\right| > A \middle| s_k = A\right)$$

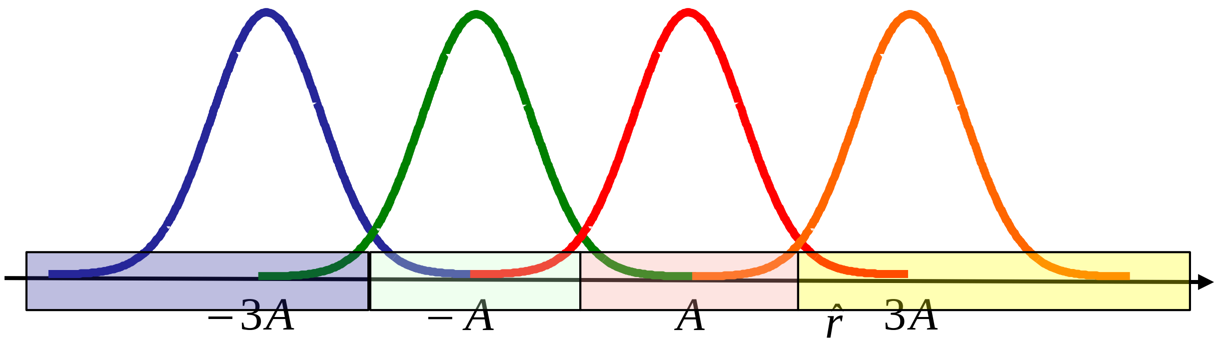
$$= \frac{1}{M} P(|n| > A) = \frac{2}{M} Q\left(\frac{A}{\sqrt{N_0}}\right)$$

$$\frac{1}{M} P\left(\hat{r} - 3 < -A \middle| s_k = 3A\right)$$

$$= \frac{1}{M} P(n < -A) = \frac{1}{M} Q\left(\frac{A}{\sqrt{N_0}}\right)$$

Power-Error Tradeoff

$$P(\text{error}) = \frac{1}{M}\sum_k P(\hat{s}_k \neq s_k | s_k) = \frac{2}{M} Q\left(\frac{A}{\sqrt{N_0}}\right) + \frac{2(M-2)}{M} Q\left(\frac{A}{\sqrt{N_0}}\right)$$



$M = 2$

$$P(\text{error}) = Q\left(\frac{A}{\sqrt{N_0}}\right)$$

Expected Energy per Symbol

$$E\left[\|s_k\|^2\right] = \frac{A^2}{M}\sum_{k=0}^{M-1}(2k - M + 1)^2 = A^2(M-1)^2 - 4A^2\frac{M-1}{M}\sum_{k=0}^{M-1} k + \frac{4}{M}A^2\sum_{k=0}^{M-1} k^2$$

$$= A^2(M-1)^2 - 4A^2\frac{M-1}{M}\frac{1}{2}(M-1)M + \frac{4}{M}A^2\frac{1}{6}(M-1)M(2M-1)$$

$$= A^2(M-1)^2 - 2A^2(M-1)^2 + \frac{2}{3}A^2(M-1)(2M-1) = \frac{1}{3}A^2(M^2-1)$$

$M = 2$

$$E\left[\|s_k\|^2\right] = A^2$$

BPSK: Power-Error Tradeoff

$$M = 2 \qquad P(\text{error}) = Q\left(\frac{A}{\sqrt{N_0}}\right) \qquad E\left[\|s_k\|^2\right] = A^2$$

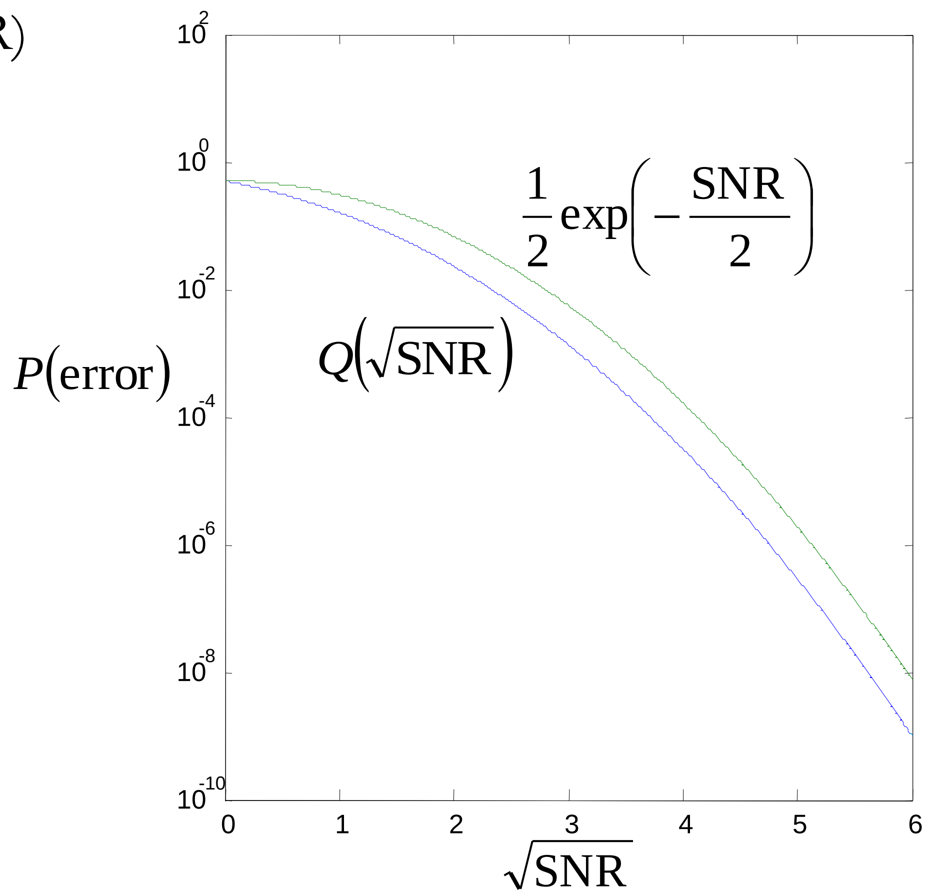
Define Signal-to-Noise-Ratio (SNR)

$$\text{SNR} = \frac{E\left[\|s_k\|^2\right]}{N_0} = \frac{A^2}{N_0}$$

$$P(\text{error}) = Q\left(\sqrt{\text{SNR}}\right)$$

$$n \sim N(0, N_0)$$

$$E\left[n^2\right] = N_0$$

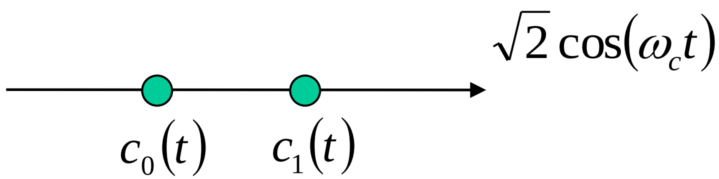


OOK

$$c_k(t) = Aa_k\sqrt{2}\cos(\omega_c t) \qquad a_k \in \{0,1\}$$

Let $s_k = Aa_k$ $\qquad c_k(t) = s_k\sqrt{2}\cos(\omega_c t)$

One-dimensional space



$$c_k(t) \to Ak \qquad k \in \{0,1\}$$

Assume white Gaussian noise, the received signal is

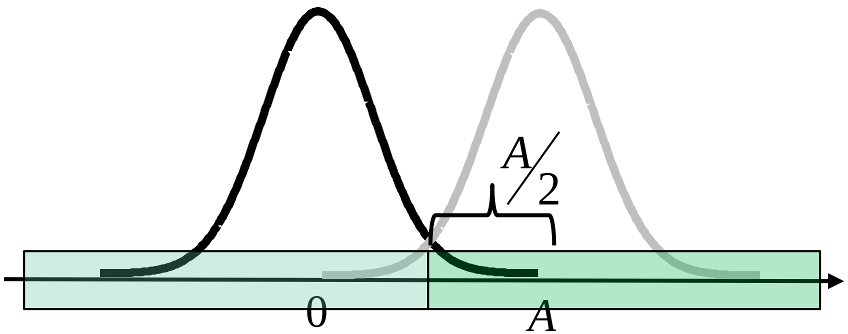
$$r(t) = s_k\sqrt{2}\cos(\omega_c t) + n(t) \qquad n(t)\ \text{is Gaussian} \qquad R_n(\tau) = E[n(t)n(t-\tau)] = N_0\delta(\tau)$$

$$\hat{r} = \int_{-\infty}^{\infty} r(t)\sqrt{2}\cos(\omega_c t)dt = s_k + \int_{-\infty}^{\infty} n(t)\sqrt{2}\cos(\omega_c t)dt = s_k + n \qquad n \sim N(0, N_0)$$

Minimum Distance Detection $\qquad \hat{s}_k = \arg\min_{s_k}(\hat{r} - s_k)^2$

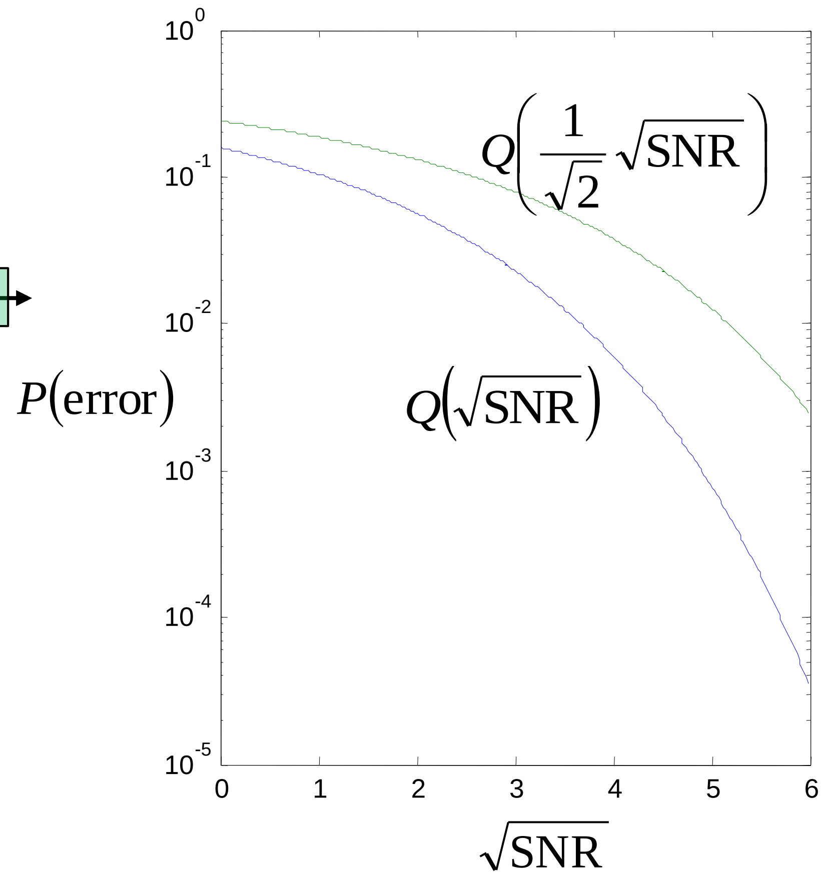
OOK: Power-Error Tradeoff

$$P(\text{error}) = Q\left(\frac{A}{2\sqrt{N_0}}\right) \qquad E\left[\|s_k\|^2\right] = \frac{1}{2}A^2 \qquad \text{SNR} = \frac{E\left[\|s_k\|^2\right]}{N_0} = \frac{A^2}{2N_0}$$



$$P(\text{error}) = Q\left(\frac{1}{\sqrt{2}}\sqrt{\text{SNR}}\right)$$

$$n \sim N(0, N_0)$$



MASK: Power-Error Tradeoff

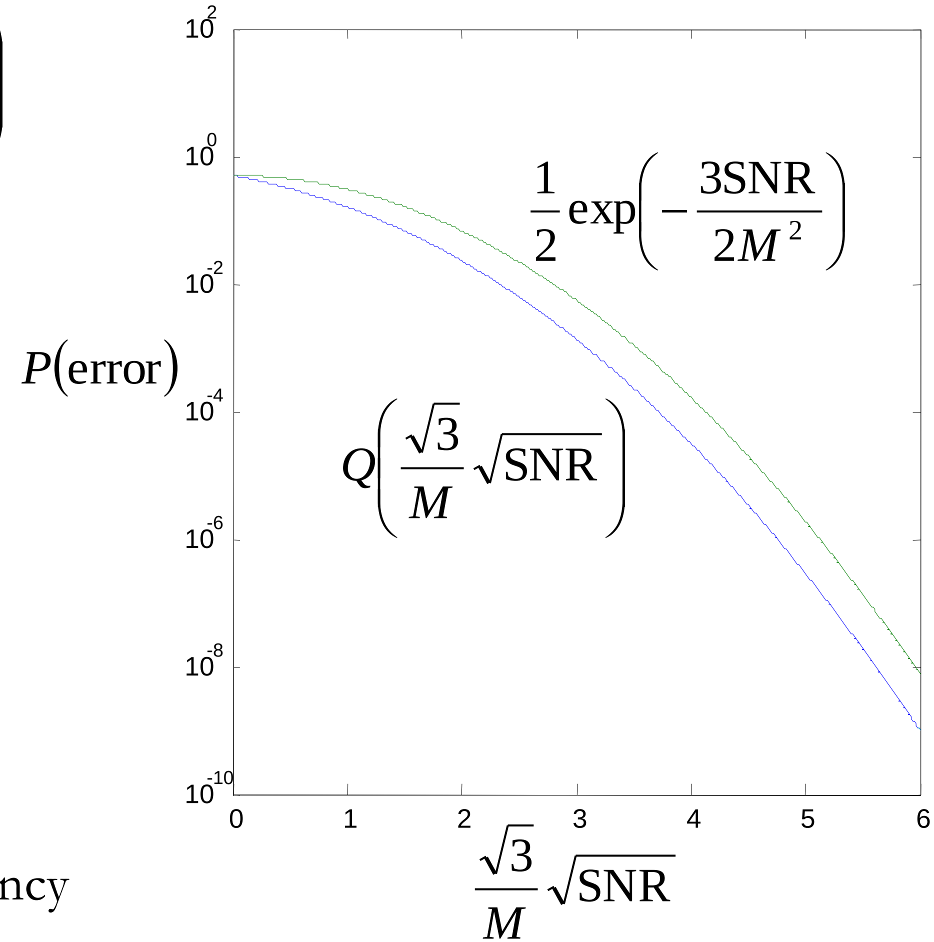
$$P(\text{error}) = \frac{1}{M}\sum_k P(\hat{s}_k \neq s_k | s_k) = \frac{2}{M} Q\left(\frac{A}{\sqrt{N_0}}\right) + \frac{2(M-2)}{M} Q\left(\frac{A}{\sqrt{N_0}}\right)$$

$$\approx \frac{2(M-1)}{M} Q\left(\frac{A}{\sqrt{N_0}}\right) \approx Q\left(\frac{A}{\sqrt{N_0}}\right)$$

$$E\left[\|s_k\|^2\right] = \frac{1}{3}A^2(M^2-1)$$

$$\text{SNR} = \frac{E\left[\|s_k\|^2\right]}{N_0} \approx \frac{M^2}{3}\frac{A^2}{N_0}$$

$$P(\text{error}) = Q\left(\frac{\sqrt{3}}{M}\sqrt{\text{SNR}}\right)$$



higher rate, lower power efficiency

MPSK Detection

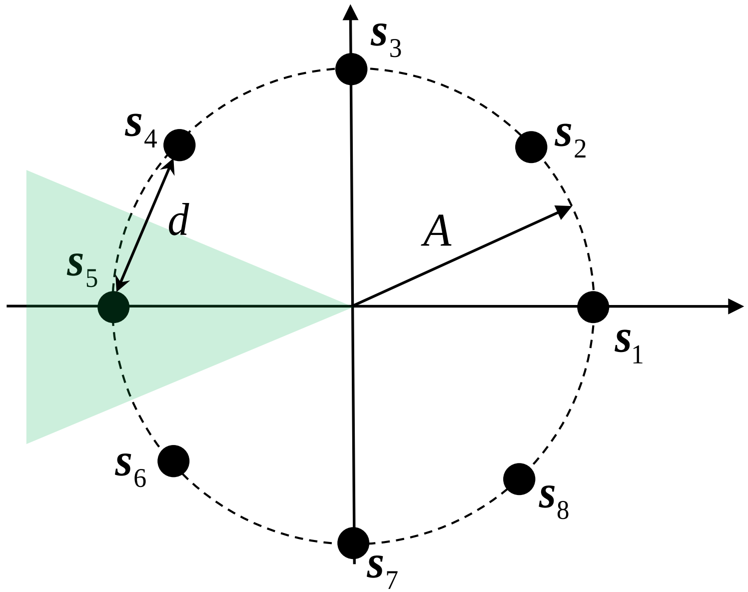
Assume white Gaussian noise, the received signal is

$$r(t) = c_k(t) + n(t) \qquad n(t) \text{ is Gaussian} \qquad R_n(\tau) = E[n(t)n(t-\tau)] = N_0\delta(\tau)$$

Project $r(t)$ onto the signal space, we get

$$\boldsymbol{r} = \boldsymbol{s}_k + \boldsymbol{n} \qquad \boldsymbol{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

where n_1 and n_2 are i.i.d. $\sim N(0, N_0)$



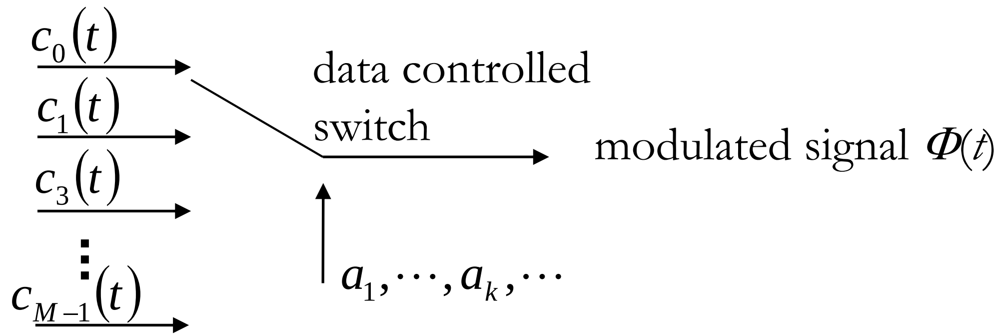
Maximum Likelihood Detection

$$\hat{\boldsymbol{s}}_k = \arg\min_{\boldsymbol{s}_k} \|\boldsymbol{r} - \boldsymbol{s}_k\|^2$$

Error probability $\quad P(\text{error}) \approx Q\left(\dfrac{d}{2\sqrt{N_0}}\right) = Q\left(0.3827\dfrac{A}{\sqrt{N_0}}\right) \qquad \text{SNR} = A^2/N_0$

General Constellation

Basic Diagram (for *M*-ary signal):



Let the modulated waveform be $c_k(t) = s_{k1}\phi_1(t) + s_{k2}\phi_2(t) + \cdots + s_{kK}\phi_K(t)$

Signal points

$$\boldsymbol{s}_k = \begin{bmatrix} s_{k1} \\ \vdots \\ s_{kK} \end{bmatrix}$$

Minimum Distance

$$d_{\min} = \min_{\boldsymbol{s}_i, \boldsymbol{s}_j} \left\| \boldsymbol{s}_i - \boldsymbol{s}_j \right\|$$

Expected Symbol Energy

$$E\left[\left\|\boldsymbol{s}_k\right\|^2\right] = \frac{1}{M}\sum_{k=1}^{M}\left\|\boldsymbol{s}_k\right\|^2$$

Assume white Gaussian noise, $r(t) = c_k(t) + n(t)$ $\quad R_n(\tau) = E[n(t)n(t-\tau)] = N_0\delta(\tau)$

Error Probability $\quad P(\text{error}) \approx Q\left(\dfrac{d_{\min}}{2\sqrt{N_0}}\right)$