Principle of Communications

Review of Probability and Random Processes

Dr. Xiang Cheng

Wireless Communications and Signal Processing Research Center
Peking University

Outline

- Randomness in Communications
- Probability
- Random Variables
- Random Processes

Roadmap

- Randomness in Communications
- Probability
- Random Variables
- Random Processes

Randomness in Communications

Deterministic Signal: Complete certainty about their values at any instant *t*.

If signal is deterministic to the receiver, then no information can be carried over.

Random Signal: Values at certain instant *t* are not fully determined

Examples:
Transmitted message
Transmitted signal
Transmitted carrier frequency
Transmitted carrier phase
Channel gain
Channel impulse response
Interference
Channel noise
Receiver noise
Receiver carrier frequency
Receiver carrier phase

Roadmap

- Randomness in Communications
- Probability
  - Definition
  - Independence, total probability, Bayes' rule, Bernoulli trials
- Random Variables
- Random Processes

Probability

Random Experiment: An experiment is called a random experiment if its outcome cannot be predicted because the conditions under which it is performed cannot be predetermined with sufficient accuracy and completeness.

Examples: Tossing a coin, rolling a dice.
A random experiment may have several different outcomes.

Sample space S: The set of all possible outcomes. Each outcome is called an element or sample point in S.

Event A: A subset of S

Example: Rolling a dice. $S=\{1,2,3,4,5,6\}$. Event $A=\{\text{output}>3\}$
Event $B=\{\text{output is even}\}$, Event $C=\{\text{output}<3\}$

Definition of Probability

Union of events $A \cup B$

Intersection of events $A \cap B$

Complement of an event A: A^c $\quad A \cup A^c = S \quad A \cap A^c = \phi$

If $A \cap B = \phi$, then we say A and B are disjoint or mutually exclusive.

The probability of an event $\Pr[A]$ is a real number satisfying axioms:

$0 \leq \Pr[A] \leq 1$

$\Pr[S] = 1$

If $A \cap B = \phi$, then $\Pr[A \cup B] = \Pr[A] + \Pr[B]$

If $\{A_i\}$ mutually disjoint, then $\Pr[\cup A_i] = \sum_i \Pr[A_i]$

Independence

Conditional Probability: conditional probability of A given B is

$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]} \quad \Pr[B] \neq 0$$ the probability of A when it is known that event B has occurred.

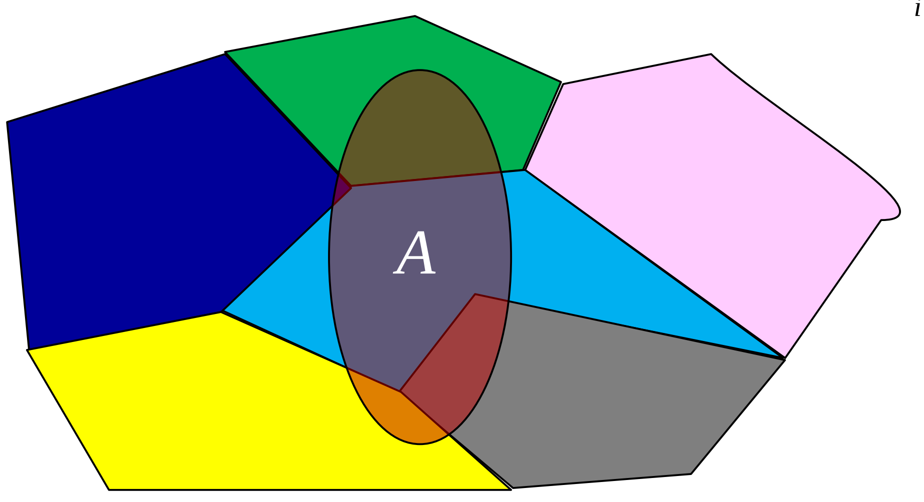
Similarly $$\Pr[B \mid A] = \frac{\Pr[A \cap B]}{\Pr[A]} \quad \Pr[A] \neq 0$$

The events A and B are independent iff $\Pr[A \mid B] = \Pr[A]$

iff $\Pr[A \cap B] = \Pr[A]\Pr[B]$

Law of Total Probability

$\{B_1,\ldots,B_n\}$ is a partition iff (1) they mutually disjoint, and (2) $\bigcup_i B_i = S$



Then

$$\Pr[A] = \Pr\left[\cup_i (A \cap B_i)\right]$$
$$= \sum_i \Pr[A \cap B_i]$$

$$\Pr[A] = \sum_i \Pr[A \mid B_i]\Pr[B_i]$$

Special case: $\Pr[A] = \Pr[A \mid B]\Pr[B] + \Pr[A \mid \overline{B}]\Pr[\overline{B}]$

Bayes' Rule

Given

a priori probabilities $\{P(B_i)\}$

conditional probabilities $\{P(A \mid B_i)\}$

How to find *a posteriori* probability $P(B_i \mid A)$?

$$P(B_k \mid A) = \frac{\Pr[B_k \cap A]}{\Pr[A]}$$

$$= \frac{\Pr[A \mid B_k]\Pr[B_k]}{\Pr[A]}$$

$$P(B_k \mid A) = \frac{\Pr[A \mid B_k]\Pr[B_k]}{\sum_i \Pr[A \mid B_i]\Pr[B_i]}$$

Example

A transmitter sends a 1 or 0 with probability 99% and 1%.

The received bit is the same as the transmitted bit with probability 80%.

If the received bit is a 1, what is the probability that the transmitted bit is a 1?

$$P(t=1)=0.99, \quad P(t=0)=0.01, \qquad P(r=t)=0.80, \quad P(r\neq t)=0.20,$$

$$P(t=1\,|\,r=1)=\frac{P(t=1,r=1)}{P(r=1)}$$

$$=\frac{P(r=1\,|\,t=1)P(t=1)}{P(r=1\,|\,t=1)P(t=1)+P(r=1\,|\,t=0)P(t=0)}$$

$$=\frac{0.80\times 0.99}{0.80\times 0.99+0.20\times 0.01}=0.9975$$

Bernoulli Trials

In Bernoulli trials, if a certain event A occurs, we call it a "success".

$P(A)=p$ probability of success is p
$P(A^c)=1-p$ probability of failure is $(1-p)$

Probability of k successes in n Bernoulli trials

$$\binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

Probability of k successes in n Bernoulli trials in a specific order

$$p^k (1-p)^{n-k}$$

Example 1

Rx bit = Tx bit with probability 80%

Tx transmits 20 bits, what is the probability that the Rx receives at least 65% bits correctly?

$$P_l = \sum_{k=0}^{7} \frac{20!}{(20-k)!k!} 0.8^{(20-k)} 0.2^k = 0.9679$$

Tx transmits 20 bits, what is the probability that the Rx receives less than 95% bits correctly?

$$P_u = 1 - \sum_{k=0}^{1} \frac{20!}{(20-k)!k!} 0.8^{(20-k)} 0.2^k = 0.9308$$

What if the Tx transmits 40 bits?

$$P_l = \sum_{k=0}^{14} \frac{40!}{(40-k)!k!} 0.8^{(40-k)} 0.2^k = 0.9921$$

$$P_u = 1 - \sum_{k=0}^{2} \frac{40!}{(40-k)!k!} 0.8^{(40-k)} 0.2^k = 0.9921$$

Example 2

In binary PCM, regenerative repeaters are used to detect pulses (before they are lost in noise) and retransmit new, clean pulses.

A certain PCM channel consists of n identical links in series. The pulses are detected at the end of each link and clean new pulses are transmitted over the next link.

If p_e is the probability of error in detecting a pulse over any one link, what is the probability of error in detecting one pulse at the end of the entire channel (n links)?

Since the pulse is binary, even number of errors = no error at the end

$$P_E = P(\text{odd number of error}) = P(1\,\text{error}) + P(3\,\text{error}) + \cdots$$

$$= \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{n!}{(2k+1)!(n-2k-1)!} p_e^{\,2k+1} (1-p_e)^{n-2k-1}$$

When p_e is extremely small

$$P_E \approx \frac{n!}{1!(n-1)!} p_e (1-p_e)^{n-1} \approx n p_e$$

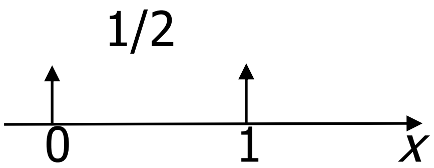
Roadmap

- Randomness in Communications
- Probability
- Random Variables
  - Discrete RVs and Continuous RVs
  - Gaussian distribution and Q function
  - Functions of an RV
  - Joint distribution and independence
  - Moment generation function
  - Central limit theorem
- Random Processes

Discrete Random Variables

Characterized by Point Mass Function (pmf), $P(x)=\Pr[X=x]$

Example: $X \sim$ Uniform{0, 1}



Moments:

Means: $\mu = E[X] = \Sigma_i x_i P(x_i)$

2nd moment: $E[X^2] = \Sigma_i x_i^2 P(x_i)$

Variance: $\sigma^2 = E[(X - \mu)^2] = E[X^2] - (E[X])^2$

Expectation of a function of a RV:

$$E[g(X)] = \Sigma_i g(x_i) P(x_i)$$

Continuous RV

Characterized by Probability Density Function (pdf), $f(x)$: $\Pr[a < X \le b] = \int_a^b f(x)dx$

Equivalently, by Cumulative Distribution Function (cdf): $F(x) = \Pr[X \le x] = \int_{-\infty}^x f(x)dx$

$X \sim \text{Uniform}(a, b) \Rightarrow f(x) = \dfrac{1}{b-a}$, for $x \in (a,b)$ only.

$$f(x) = \frac{dF(x)}{dx}$$

$X \sim \text{Exp}(\lambda) \Rightarrow f(x) = \lambda e^{-\lambda x}$, for $x > 0$ only.

Moments:

Mean: $\mu = E[X] = \int_{-\infty}^{\infty} xf(x)dx$

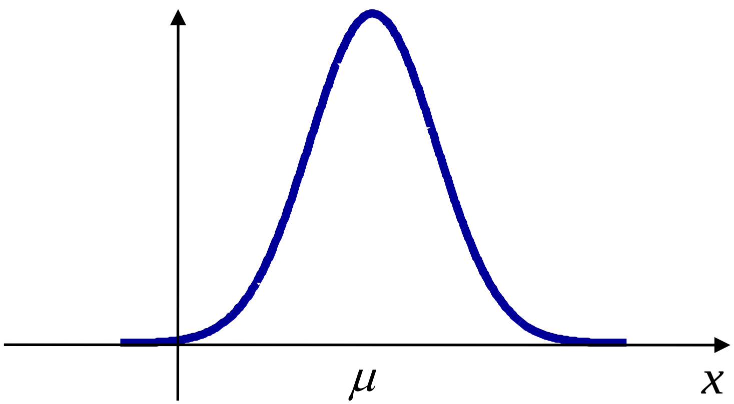
2nd moment: $E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx$

Variance: $\sigma^2 = E[(X-\mu)^2] = E[X^2] - (E[X])^2$

Expectation of a function of a RV: $E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$

Gaussian Distribution

$$X \sim N\left(\mu,\sigma^2\right) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



"normal"

$X \sim N(0,1) \Rightarrow$ "standard normal"

Properties:

mean = μ variance = σ^2

MGF $\varphi(s) = \exp\left(\mu s + \sigma^2 s^2 / 2\right)$

$$\varphi(s) = \frac{1}{\sqrt{2\pi\sigma^2}} \int e^{sx} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

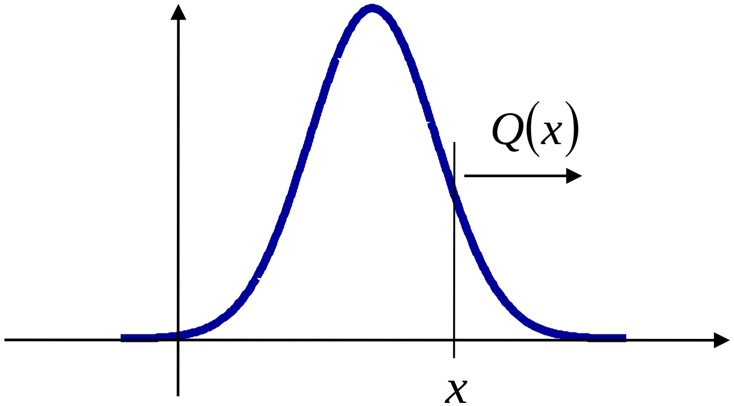
$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int e^{-\frac{x^2 - 2\mu x + \mu^2 + 2s\sigma^2 x}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int e^{-\frac{x^2 - 2\left(\mu - s\sigma^2\right)x + \mu^2}{2\sigma^2}} dx$$

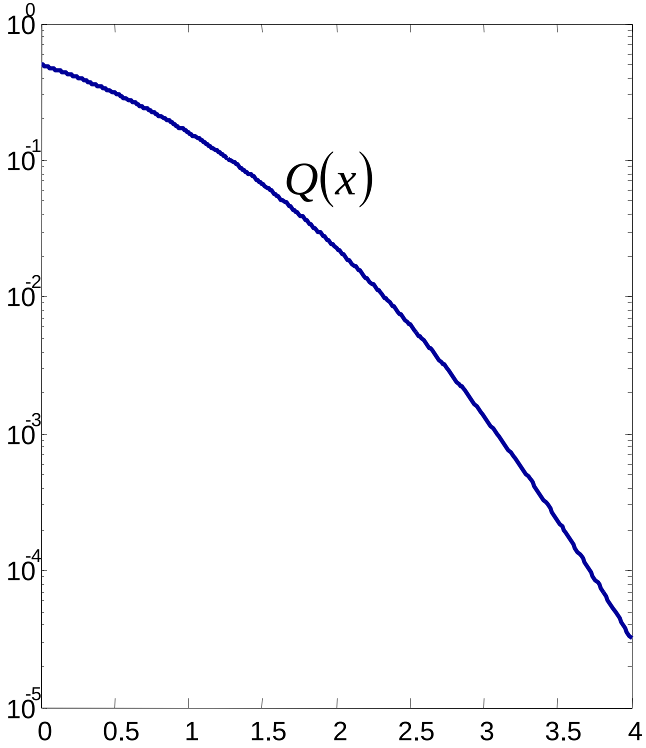
$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int e^{-\frac{x^2 - 2\left(\mu - s\sigma^2\right)x + \left(\mu - s\sigma^2\right)^2}{2\sigma^2} - \frac{s^2\sigma^4 - 2s\sigma^2}{2\sigma^2}} dx = e^{\mu s - \frac{s^2\sigma^2}{2}}$$

Q Function

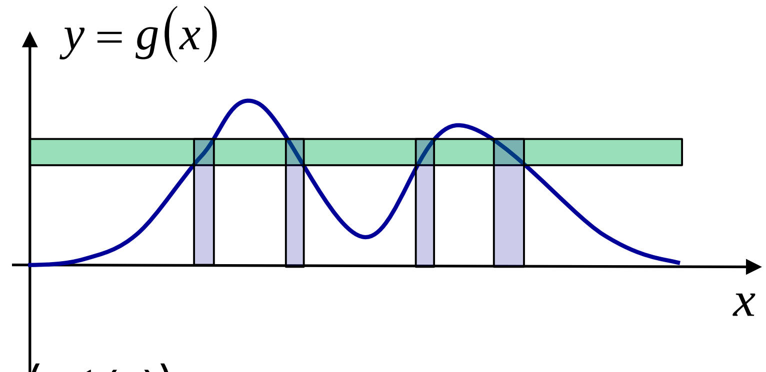
Let $U \sim N(0, 1)$ be standard normal.

$$Q(x) = \Pr[U > x]$$
$$= \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt$$





Functions of an RV



The pdf of $Y = X + a$ is $f_Y(y) = f_X(y-a)$

The pdf of $Y = aX$ is $f_Y(y) = \dfrac{1}{|a|} f_X\left(\dfrac{y}{a}\right)$

The pdf of $Y = g(X)$ is $f_Y(y) = \sum_i \dfrac{1}{\left|g'\left(g_i^{-1}(y)\right)\right|} f_X\left(g_i^{-1}(y)\right)$

Proof ? $f_Y(y)|\Delta y| = \sum_i f_X(x_i)|\Delta x_i| \Rightarrow f_Y(y) = \sum_i f_X(x_i)\left|\dfrac{\Delta x_i}{\Delta y}\right|$

Example : Suppose $X \sim \text{Uniform}(0, 1)$ Find the pdf of $Y = -\log(1-X)$

Answer: $Y \sim 1-\exp(-y) = \text{Exp}(1)$

Let x be a RV cdf $F(X)$. Define $y = F(x)$. Then y is uniformly distributed in $[0,1]$

Proof :

$$f_Y(y) = \frac{1}{\left|F'\left(F^{-1}(y)\right)\right|} f_X\left(F^{-1}(y)\right) = \frac{f_X(x)}{|f_X(x)|} = 1$$

Joint PDF

A *pair* of RV's is characterized by the joint pdf $f(x, y)$

$$\Pr\left[a < X \le b, c < Y \le d\right] = \int_c^d \int_a^b f(x, y)\,dxdy$$

Conditional Probability

Two types of conditional pdf's:

One given an event: $f(x \mid \text{event } A)$

One given another random variable: $f(x \mid y) = \dfrac{f(x,y)}{f(y)}$

Conditional mean: $E(X \mid Y = y) = \int x f(x \mid y) dx$

It is a function of y, hence is still a random variable!

The double-expectation theorem:

$$E[E[Y \mid X]] = E[Y]$$

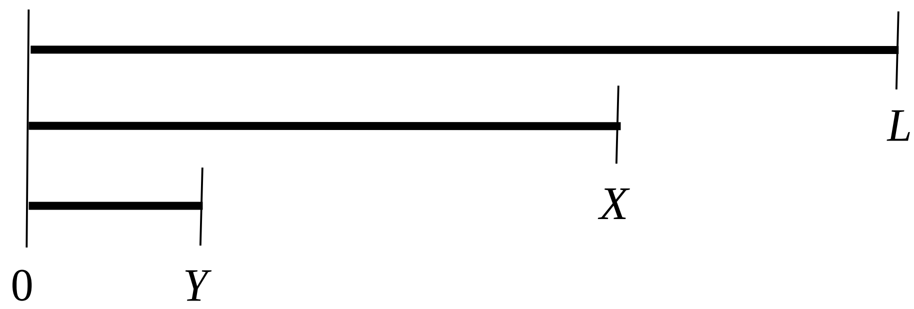
Proof : $E[Y \mid X] = \int y f(y \mid x) dy$

$$E[E[Y \mid X]] = \int E[Y \mid X] f(x) dx = \int y f(y \mid x) dy f(x) dx$$

$$= \int y f(y \mid x) f(x) dx dy = \int y f(x,y) dx dy = E[Y]$$

Break A Stick, Twice

A stick of length L is broken at a random point with equal probability. One piece is discarded. The other piece of length X is broken again at a random point with equal probability. One piece is discarded. The remaining piece has length Y. Find $E[Y]$



(a) Easy way: $E[Y]=E[E[Y\,|\,X]]=E[X/2]=L/4$

(b) Hard way: $P(y\ge Y)=\int_Y^L P(y\ge Y\,|\,x)f(x)dx=\int_Y^L\left(1-\frac{Y}{x}\right)\frac{1}{L}dx=\frac{L-Y}{L}-\frac{Y}{L}\ln\frac{L}{Y}$

$$f(y)=-\frac{dP(y\ge Y)}{dy}=\frac{1}{L}+\frac{1}{L}\ln\left(\frac{L}{y}\right)-\frac{1}{L}=\frac{1}{L}\ln\left(\frac{L}{y}\right)$$

$$E[y]=\int_0^L yf(y)dy=\int_0^L\frac{y}{L}\ln\left(\frac{L}{y}\right)dy=-L\int_0^1\tilde{y}\ln(\tilde{y})d\tilde{y}=-L\frac{\tilde{y}^2}{2}\ln(\tilde{y})\Big|_0^1+L\int_0^1\frac{\tilde{y}^2}{2}\frac{1}{\tilde{y}}d\tilde{y}=\frac{L}{4}$$

Independence

The conditional pdf for X given Y is $f(x|y) = \dfrac{f(x,y)}{f(y)}$

The RV's X and Y are *independent* iff $f(x|y) = f(x)$

iff $f(x,y) = f(x)f(y)$

Correlation

$$\sigma_{xy} = E[(x-\mu_x)(y-\mu_y)] = E[xy] - \mu_x\mu_y$$

The RV's X and Y are *uncorrelated* iff $E[XY]=E[X]E[Y]$ (or $\sigma_{xy} = 0$)

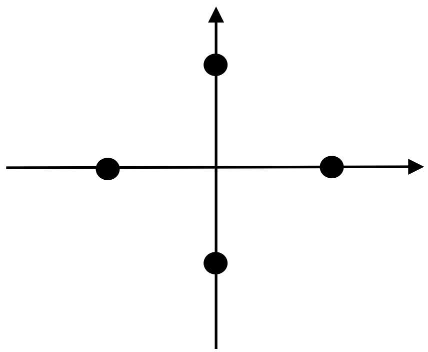
Independence implies uncorrelated,

But

Uncorrelated does not imply independence. (Except if the RV's are Gaussian)

Example

Let $\begin{bmatrix} X \\ Y \end{bmatrix}$ be a random vector uniformly distributed over $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$



$$p(X) = p(Y) = \begin{cases} 1 & \frac{1}{4} \\ 0 & \frac{1}{2} \\ -1 & \frac{1}{4} \end{cases}$$

Are X and Y uncorrelated?

Yes. $E[XY] = 0 = E[X]E[Y]$

Are X and Y independent?

No. $p(X = 1, Y = 1) = 0 \neq p(X = 1)p(Y = 1)$

The Moment Generation Function

The *moment-generating function* (MGF) of a RV is

$$\varphi(s) = E\left[e^{sX}\right] = \int_{-\infty}^{\infty} f(x) e^{sx} dx$$

$$e^{sx} = 1 + xs + \frac{1}{2}x^2 s^2 + \cdots$$

$$\varphi(s) = \int_{-\infty}^{\infty} \left[1 + xs + \frac{1}{2}x^2 s^2 + \cdots\right] f(x) dx$$

Generally *s* is real-valued.

Application 1: Finding the Moment

The hard way: $E\left[X^k\right]=\int_{-\infty}^{\infty} x^k f(x)dx$

The easy way: $E\left[X^k\right]=\frac{d^k}{ds^k}\varphi(s)|_{s=0}$

Example 2: $X \sim \text{Exp}(\lambda) \Rightarrow f(x)=\lambda e^{-\lambda x}, \text{ for } x>0$

$$\Rightarrow \varphi(s)=\frac{\lambda}{\lambda-s}$$

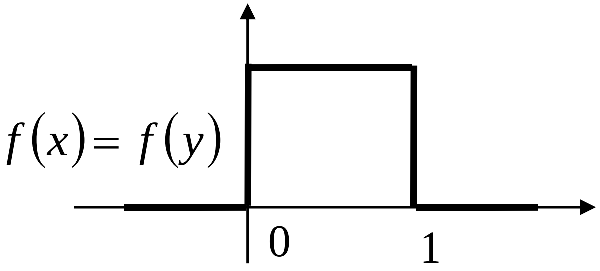
$$\Rightarrow \varphi'(s)=\frac{\lambda}{(\lambda-s)^2} \Rightarrow \varphi'(0)=E[X]=\frac{1}{\lambda}$$

$$\Rightarrow \varphi''(s)=\frac{2\lambda}{(\lambda-s)^3} \Rightarrow E\left[X^2\right]=\frac{2}{\lambda^2}$$

$$\Rightarrow \varphi^{(3)}(s)=\frac{6\lambda}{(\lambda-s)^4} \Rightarrow E\left[X^3\right]=\frac{6}{\lambda^3}$$

Application 2: Adding two RVs

Suppose X and Y are i.i.d. Uniform(0, 1)



What is the pdf of the sum $Z=X+Y$?

$$F(z) = P(Z \le z)$$

$$\text{if } z \le 1, F(z) = \int_0^z \int_0^{z-x} f(y) f(x) dy dx = \int_0^z (z-x) dx = zx - \frac{x^2}{2}\Big|_0^z = \frac{z^2}{2}$$

$$\text{if } 1 < z \le 2, F(z) = \int_{z-1}^1 \int_0^{z-x} f(y) f(x) dy dx = \int_{z-1}^1 (z-x) dx = zx - \frac{x^2}{2}\Big|_{z-1}^1 = 1 - \frac{1}{2} + \frac{(z-1)^2}{2}$$

Method of Moment

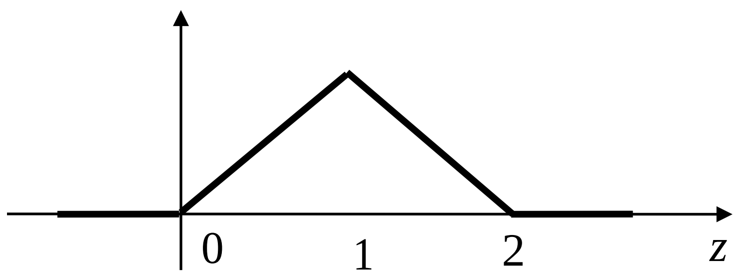
Finding the pdf of the sum of independent random variables, $Z=X+Y$

$$\begin{aligned}\varphi_s(s) &= E\left[e^{sZ}\right] \\ &= E\left[e^{s(X+Y)}\right] \\ &= E\left[e^{sX}e^{sY}\right] \\ &= E\left[e^{sX}\right]E\left[e^{sY}\right] \\ &= \varphi_X(s)\varphi_Y(s)\end{aligned}$$

$$\begin{aligned}E[g(X)h(Y)] &= \int_x\int_y g(x)h(y)f_x(x)f_y(y)dxdy \\ &= \int_x g(x)f_x(x)dx\int_y h(y)f_y(y)dy \\ &= E[g(X)]E[h(Y)]\end{aligned}$$

Multiplication in freq domain $\Rightarrow$ *convolution*:

$$f_z(z) = f_X(z) * f_Y(z)$$

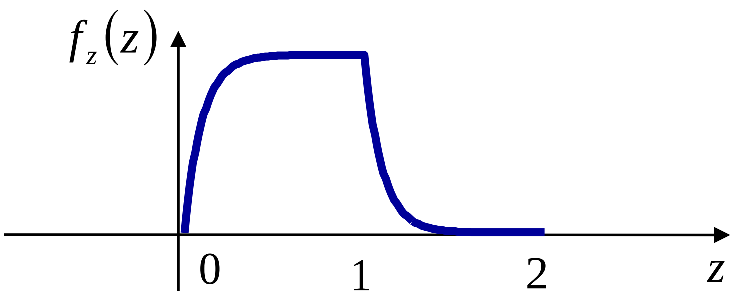


Another Example

Let

$X \sim \text{Uniform}(0, 1)$
$Y \sim \text{Exp}(10)$ be independent

$\Rightarrow$ the sum $Z = X + Y$ has the following pdf.



MGF of Gaussian

$$X \sim N\left(\mu,\sigma^2\right) \Rightarrow f\left(x\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

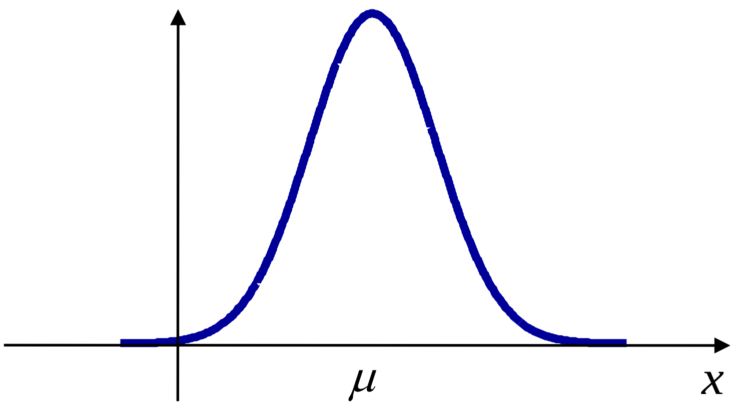
"normal"

$X \sim N(0,1) \Rightarrow$ "standard normal"

Properties:

mean = μ variance = σ^2

MGF $\varphi\left(s\right) = \exp\left(\mu s + \sigma^2 s^2 / 2\right)$



A Combo of Independent Gaussians

Let $\{X_1, X_2, \cdots, X_n\}$ be independent with $X_i \sim N\left(\mu_i, \sigma_i^2\right)$

Let $Y = X_1 + X_2 + \cdots X_n$

$$
\begin{aligned}
\Rightarrow \varphi_Y(s) &= E\left[e^{s(X_1 + X_2 + \cdots X_n)}\right] \\
&= E\left[e^{sX_1} e^{sX_2} \cdots e^{sX_n}\right] \\
&= \varphi_{X_1}(s)\varphi_{X_2}(s)\cdots\varphi_{X_n}(s) \\
&= \exp\left(\mu_1 s + \sigma_1^2 s^2 / 2\right)\exp\left(\mu_2 s + \sigma_2^2 s^2 / 2\right)\cdots\exp\left(\mu_n s + \sigma_n^2 s^2 / 2\right) \\
&= \exp\left(\mu s + \sigma^2 s^2 / 2\right)
\end{aligned}
$$

where $\mu = \sum_i \mu_i$

$\sigma^2 = \sum_i \sigma_i^2$

$\Rightarrow Y$ is normal

Joint Gaussian

Define $X_1, X_2, \cdots, X_n$ as *jointly Gaussian* iff $\boldsymbol{x} = \boldsymbol{\mu} + \boldsymbol{G}\boldsymbol{v}$

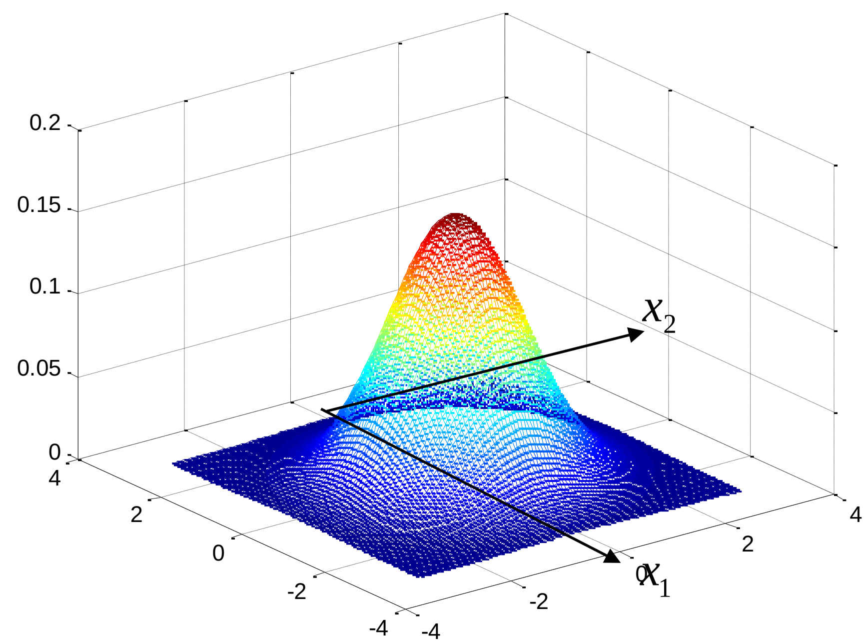
where $\{v_i\} \sim i.i.d.\ N(0,1)$. In other words, iff :

$$f(x_1, x_2, \cdots, x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{|\boldsymbol{G}\boldsymbol{G}^T|}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T (\boldsymbol{G}\boldsymbol{G}^T)^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}$$

$$covariance\ E\left[(\boldsymbol{x}-\boldsymbol{\mu})(\boldsymbol{x}-\boldsymbol{\mu})^T\right] = \boldsymbol{G}E\left[\boldsymbol{v}\boldsymbol{v}^T\right]\boldsymbol{G}^T = \boldsymbol{G}\boldsymbol{G}^T = \boldsymbol{\Omega}$$

$\boldsymbol{\Omega} = \boldsymbol{G}\boldsymbol{G}^T$ is the *covariance matrix*

Example : $n = 2,\ \mu = 0,\ \boldsymbol{\Omega} = \boldsymbol{I}$



Uncorrelated Gaussians

Suppose $X_1, X_2, \cdots, X_n$ are jointly Gaussian and are mutually *uncorrelated*

$$\Rightarrow \boldsymbol{\Omega} \text{ is diagonal } \boldsymbol{\Omega} = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix} \Rightarrow \boldsymbol{\Omega}^{-1} \text{ is diagonal}$$

$$f(x_1, x_2, \cdots, x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{|\boldsymbol{\Omega}|}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Omega}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}$$

$$= \frac{1}{(2\pi\sigma_1^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma_1^2}(x_1-\mu_1)^2\right\} \cdots \frac{1}{(2\pi\sigma_n^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma_n^2}(x_n-\mu_n)^2\right\}$$

$$= f_1(x_1) f_2(x_2) \cdots f_n(x_n)$$

$$\Rightarrow X_1, \cdots, X_n \text{ are } \textit{independent}$$

For any RV's: Independent $\Rightarrow$ Uncorrelated

For Gaussian RV's only: Uncorrelated $\Rightarrow$ Independent

Central Limit Theorem

Let $\{X_1, X_2, \cdots, X_n\}$ be i.i.d. ANYTHING with mean μ, variance σ^2

Let $$\bar{X} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu)$$

Clearly

$$E[\bar{X}] = 0 \qquad Var[\bar{X}] = \sigma^2$$

Central Limit Theorem : As $n \to \infty$,

$$\bar{X} \to N(0, \sigma^2)$$

Proof:

$$\varphi_{\bar{X}}(s) = \left[\varphi_{X/n}(s)\right]^n = \left(1 + \frac{\sigma^2}{2n} s^2 + o(n^{-1})\right)^n$$

$$\ln \varphi_{\bar{X}}(s) = n \ln \varphi_{(X-\mu)/\sqrt{n}}(s) = n \ln\left(1 + \frac{\sigma^2}{2n} s^2 + o(n^{-1})\right) = n\left[\frac{\sigma^2}{2n} s^2 + o(n^{-1})\right]$$

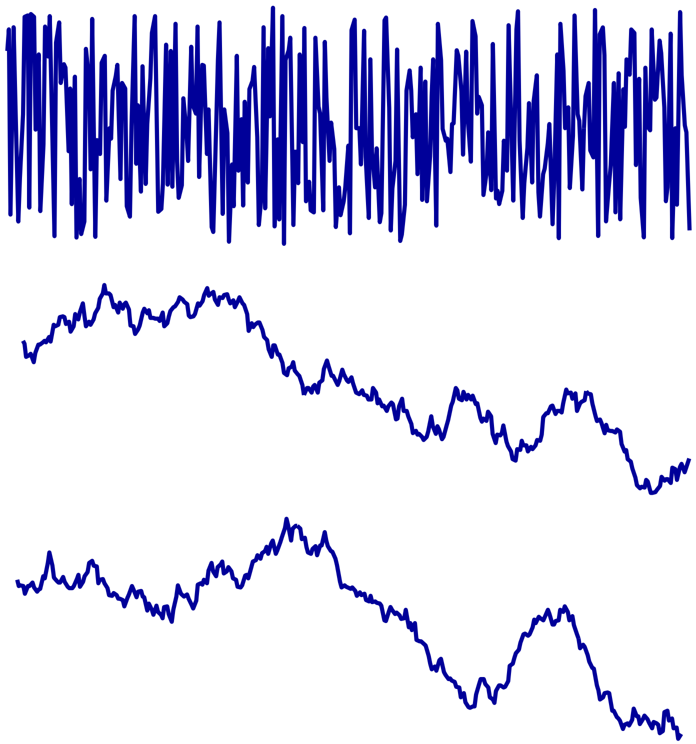
$$= \frac{\sigma^2}{2} s^2 + o(1)$$

Roadmap

- Randomness in Communications
- Probability
- Random Variables
- Random Processes
  - Ergodicity
  - Stationarity
  - Power Spectral Density
  - Filtering

Random Processes

A continuous-time random process $X(t)$ is indexed by a real variable t.



Each with an associated probability.

Two Types of Averages

The *ensemble* average: $E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\,dx$

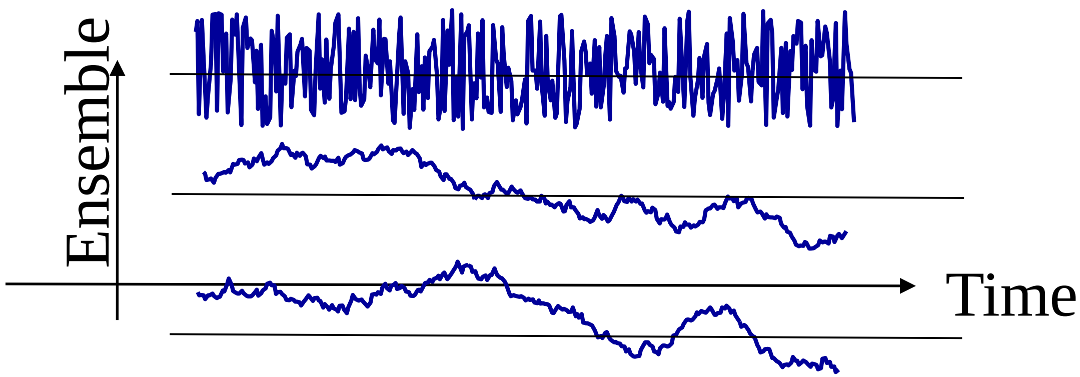
The expected value of the RV $X(t)$.
Generally a function of time.

The *time* average: $\langle X(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t)\,dt$

Generally a RV

Example:

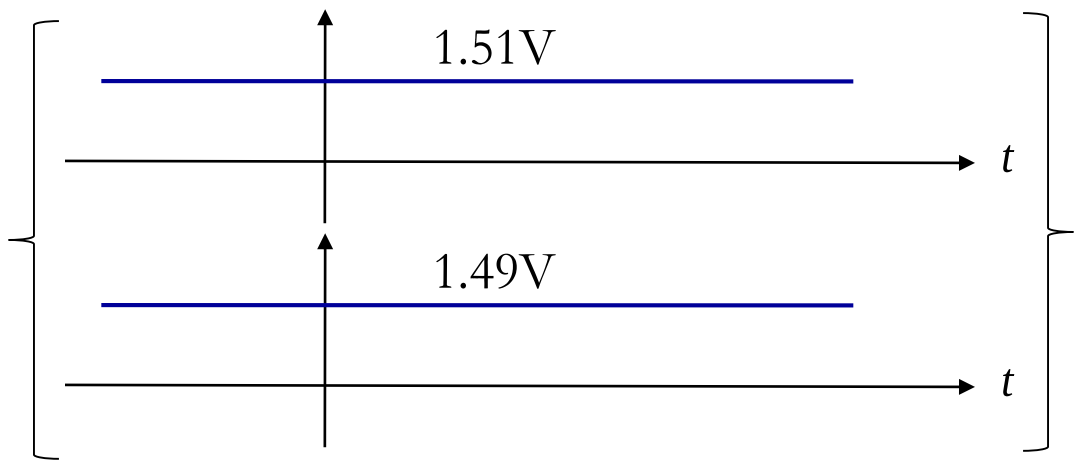
An ensemble of 3 signals, <$x(t)$> can take 3 possible values.



Example

Two batteries in a box.
Choose one at random.
Let $X(t)$ be resulting voltage.

The ensemble:



The ensemble average is $E[X(t)] = 1.5$

The time average is a RV: $\langle X(t) \rangle \sim \text{Uniform}\{1.49, 1.51\}$

Ergodicity

A r.p. is ergodic if *time* average = *ensemble* averages.

(Require this rule to apply to whatever function you evaluate)

Ergodic in the mean:

$$\langle X(t) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t) dt = E[X(t)]$$

Implication:

Each realization displays statistics of entire ensemble

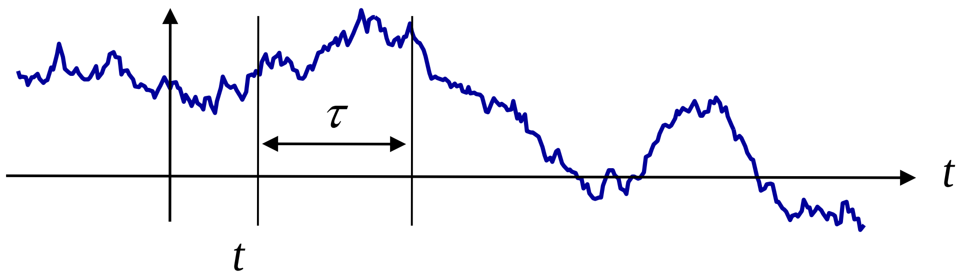
Can substitute time averages to determine moments, etc.

$$E[X^*(t) X(t+\tau)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X^*(t) X(t+\tau) dt$$

Moments

mean: $\mu(t) = E[X(t)]$

autocorrelation: $R_{XX}(t, t-\tau) = E[X(t)X^*(t-\tau)]$

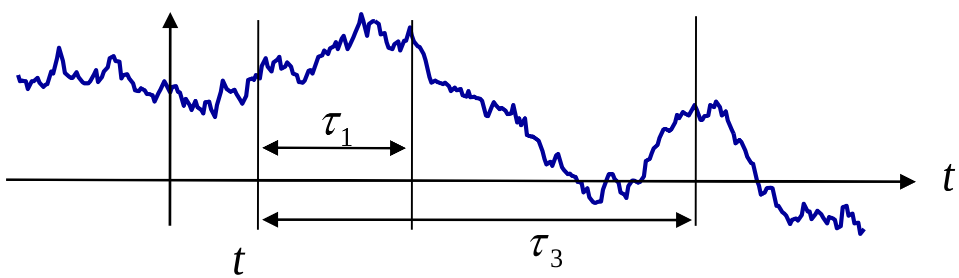


Strict Sense Stationary (SSS)

A r.p. is strict-sense stationary (SSS) if its statistics are invariant to a time shift;
i.e., the joint pdf for $\{X(t), X(t+\tau_1), \ldots, X(t+\tau_n)\}$ is independent of t.

for any $n>0$

for any set of lags $\{\tau_1,\ldots, \tau_n\}$

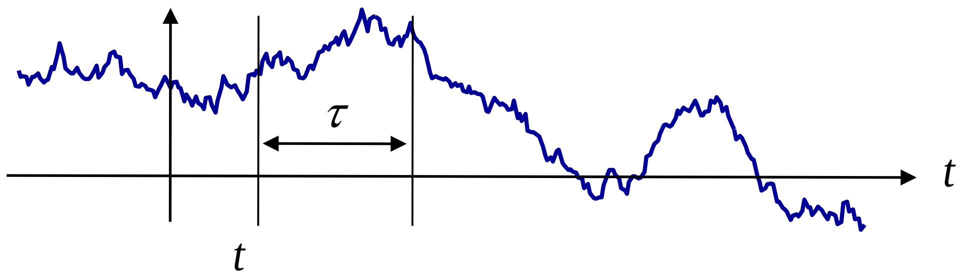


Special case: $n=1 \Rightarrow E[X(t)]=\mu$, independent of t.
Special case: $n=2 \Rightarrow E[X(t)\, X^*(t-\tau)]$ is independent of t.

Fact: Ergodic $\Rightarrow$SSS. But does SSS $\Rightarrow$ Ergodic? (Recall the battery example)

Wide Sense Stationary (WSS)

$$\left.\begin{aligned} &E[X(t)] = \mu \\ &E[X(t)X^*(t-\tau)] = R(\tau) \end{aligned}\right\} \text{independent of time } t$$



Compare

Wide-sense stationary $\Rightarrow$ 1st and 2nd order statistics are independent of time.

Strict-sense stationary $\Rightarrow$ *all* statistics are independent of time *t*.

SSS $\Rightarrow$ WSS

WSS $\not\Rightarrow$ SSS

Example

$$X(t) = \cos(440\pi t + \Theta), \text{ where } \Theta \sim \text{Uniform}[0, 2\pi)$$

(a) Is it WSS?

(b) Find its autocorrelation function.

$$E[X(t)] = E[\cos(440\pi t + \Theta)] = 0$$

$$R_X(\tau) = E[X(t)X^*(t-\tau)] = E[\cos(440\pi t + \Theta)\cos(440\pi(t-\tau) + \Theta)]$$

$$= \frac{1}{2}E[\cos(880\pi t - 440\pi\tau + 2\Theta) + \cos(440\pi\tau)]$$

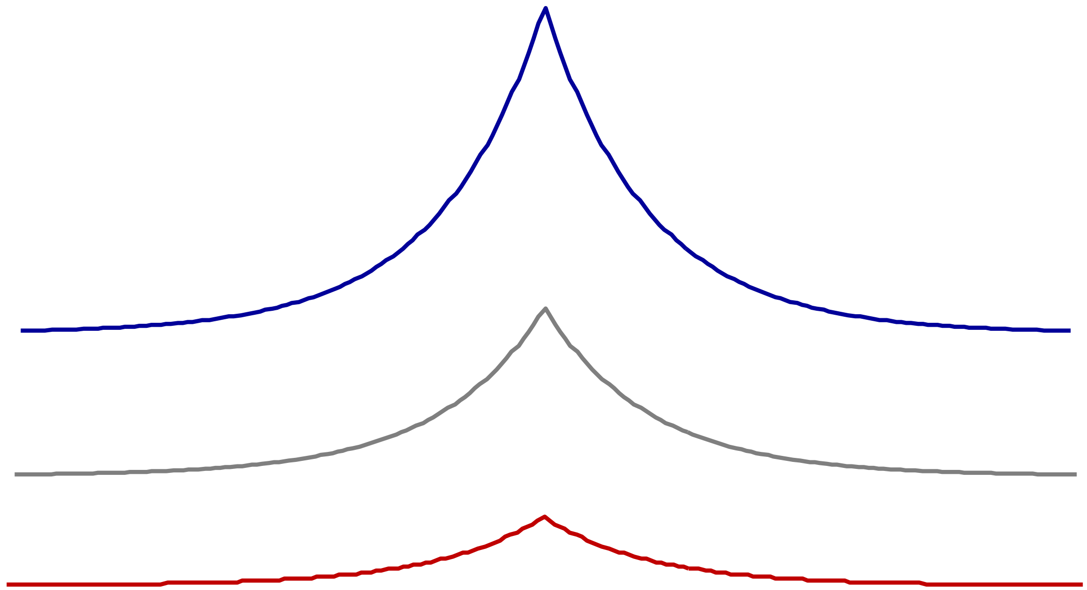
$$= \frac{1}{2}\cos(440\pi\tau)$$

The random process is indeed SSS, hence it is also WSS.

But it is not ergodic.

Example

Let $X(t) = Ae^{-|t|}$, where $A \sim N(2,1)$



(a) Ergodic?
(b) SSS?
(b) WSS?

No for all these questions. $E[X(t)]=2e^{-|t|}$, depends on time.

Gaussian

A r.p. is Gaussian $\Leftrightarrow$

all sets of samples $\{X(t), X(t+\tau_1), \ldots, X(t+\tau_n)\}$ are jointly Gaussian

WSS+Gaussian $\Rightarrow$ SSS

Properties $R(\tau)=E[X(t)X^*(t-\tau)]$ (WSS)

1. $R_X(0) = E\left[|X(t)|^2\right] = \text{power}$

2. $R_X(-\tau) = R_X^*(\tau) \Rightarrow$ Hermitian symmetry

3. $|R_X(\tau)| \le R_X(0) \Rightarrow$ max magnitude at lag 0.

   Why? Consider $E\left[|X(t) - \lambda X(t+\tau)|^2\right]$ with $\lambda = R_X^*(\tau) \big/ R_X(0)$

   If equality, then $E\left[|X(t) - \lambda X(t+\tau)|^2\right] = 0$

4. If $R_X(\tau_0) = R_X(0)$ then $R_X(m\tau_0) = R_X(0)$ $\quad \left(R_X(\tau_0) = R_X(0) \Rightarrow X(t+\tau_0) = X(t)e^{j\theta}\right)$

5. $X(t)$ ergodic, and not periodic $\Rightarrow R_X(\pm\infty) = \mu^2$

6. $F\{R_X(\tau)\} \ge 0$ for all frequencies ($F\{\ \}$ denotes the Fourier transform)

7. $R_X(\tau)$ says nothing about the pdf of $X(t)$

Power Spectral Density for WSS

The PSD is the Fourier transform of the autocorrelation function:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$

Inverse Fourier transform:

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} df$$

$$\Rightarrow R_X(0) = \int_{-\infty}^{\infty} S_X(f) df$$

$$\Rightarrow \text{power} = \int_{-\infty}^{\infty} \text{power spectral density.}$$

<u>Remark:</u>

$$\underset{\text{(random sense)}}{\text{"power"}} = E\left[\left|X(t)\right|^2\right] \underset{\text{if ergodic}}{=} \left\langle \left|X(t)\right|^2 \right\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|X(t)\right|^2 dt = \underset{\text{(deterministic sense)}}{\text{"power"}}$$

Filtering

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

$$
\begin{aligned}
E\left[Y^*(t)Y(t+\tau)\right] &= E\left[\int_{-\infty}^{\infty} h^*(u)X^*(t-u)du\int_{-\infty}^{\infty} X(v)h(t+\tau-v)dv\right] \\
&= \int_{-\infty}^{\infty} h^*(u)\int_{-\infty}^{\infty} E\left[X^*(t-u)X(v)\right]h(t+\tau-v)dvdu \\
&= \int_{-\infty}^{\infty} h^*(u)\int_{-\infty}^{\infty} R_X(v+u-t)h(t+\tau-v)dvdu, \qquad w=v+u-t \\
&= \int_{-\infty}^{\infty} h^*(u)\left[\int_{-\infty}^{\infty} R_X(w)h(\tau+u-w)dw\right]du \\
&= \int_{-\infty}^{\infty} h^*(u)\left[g(\tau+u)\right]du \qquad \text{where } g(\tau)=h(\tau)*R_X(\tau) \\
&= \int_{-\infty}^{\infty} h^*(-u)\left[g(\tau-u)\right]du = h^*(-\tau)*g(\tau) \\
&= h^*(-\tau)*h(\tau)*R_X(\tau)
\end{aligned}
$$

$$E\left[Y(t)\right] = E\left[\int_{-\infty}^{\infty} X(t-\tau)h(\tau)d\tau\right] = \int_{-\infty}^{\infty} E\left[X(t-\tau)\right]h(\tau)d\tau = \mu\int_{-\infty}^{\infty} h(\tau)d\tau = \mu H(0)$$

Summary

- Randomness in Communications
- Random Variables
  - Discrete RVs and Continuous RVs
  - Gaussian distribution and Q function
  - Functions of an RV
  - Joint distribution and independence
  - Moment generation function
  - Central limit theorem
- Random Processes
  - Ergodicity
  - Stationarity
  - Power Spectral Density
  - Filtering