Case Study: Computer Graphics in Automotive Design

In this case study, we explore how we take a three-dimensional image and render it effectively in two dimensions. This type of operation is very important in automotive design, for it allows engineers to experiment with images on a computer screen instead of using a three-dimensional model of the automobile. We will first discuss how to convert a three-dimensional image to a two-dimensional picture using a **perspective projection**. You should read Section 2.8 before attempting this case study.

The data for this case study was derived from measurements made on the author's 1983 Toyota Corolla; all coordinates are measured in feet. The origin is placed at the center of the car.

Data Points: $(-6.5, -2, -2.5)$, $(-6.5, -2, 2.5)$, $(-6.5, .5, 2.5)$, $(-6.5, .5, -2.5)$,
$(-2.5, .5, -2.5)$, $(-2.5, .5, 2.5)$, $(-.75, 2, -2.5)$, $(-.75, 2, 2.5)$,
$(3.25, 2, -2.5)$, $(3.25, 2, 2.5)$, $(4.5, .5, -2.5)$, $(4.5, .5, 2.5)$,
$(6.5, .5, -2.5)$, $(6.5, .5, 2.5)$, $(6.5, -2, 2.5)$, $(6.5, -2, -2.5)$

We collect the data points in a data matrix D: each column contains the x, y, and z coordinates of a particular data point. Since we will be using homogeneous coordinates, we also include a fourth row containing all ones.

$$\begin{bmatrix} -6.5 & -6.5 & -6.5 & -6.5 & -2.5 & -2.5 & -.75 & -.75 & 3.25 & 3.25 & 4.5 & 4.5 & 6.5 & 6.5 & 6.5 & 6.5 \\ -2 & -2 & .5 & .5 & .5 & .5 & 2 & 2 & 2 & 2 & .5 & .5 & .5 & .5 & -2 & -2 \\ -2.5 & 2.5 & 2.5 & -2.5 & -2.5 & 2.5 & -2.5 & 2.5 & -2.5 & 2.5 & -2.5 & 2.5 & -2.5 & 2.5 & 2.5 & -2.5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

This data matrix D accompanies this case study. In addition to knowing the data points, we must also know how they are to be connected. In our case we can simply supply a list of which points connect to which others. We could also use an **adjacency matrix**. This matrix consists only of 0's and 1's; the (i, j) entry in the matrix is a 1 if points i and j are connected. Figure 1 shows both the table of connections and the adjacency matrix.

If we connect the data points as given in Figure 1, we get the picture in Figure 2, which Mathematica has rendered as a two-dimensional image.

We could well ask how it is that Mathematica has rendered the picture in Figure 2. The key is a **perspective projection**. A form of this projection is studied in the text on pages 160 through 162. In order to perform a perspective projection we will need to identify a **center of projection** (b, c, d) and a **viewing plane**. The center of projection is the position of the viewer's eye; the viewing plane is the plane onto which we shall project the image. In the book, it is assumed that the center of projection has coordinates $(0, 0, d)$ and that the viewing plane is the xy plane. In what follows we will also assume that the viewing plane is the xy plane, but will allow for any choice of center of projection (b, c, d).

Point	Coordinates	Connects to
1	$(-6.5, -2, -2.5)$	2,4,16
2	$(-6.5, -2, 2.5)$	1,3,15
3	$(-6.5, .5, 2.5)$	2,4,6
4	$(-6.5, .5, -2.5)$	1,3,5
5	$(-2.5, .5, -2.5)$	4,6,7
6	$(-2.5, .5, 2.5)$	3,5,8
7	$(-.75, 2, -2.5)$	5,8,9
8	$(-.75, 2, 2.5)$	6,7,10
9	$(3.25, 2, -2.5)$	7,10,11
10	$(3.25, 2, 2.5)$	8,9,12
11	$(4.5, .5, -2.5)$	9,12,13
12	$(4.5, .5, 2.5)$	10,11,14
13	$(6.5, .5, -2.5)$	11,14,16
14	$(6.5, .5, 2.5)$	12,13,15
15	$(6.5, -2, 2.5)$	2,14,16
16	$(6.5, -2, -2.5)$	1,13,15

$$\begin{bmatrix}
0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1\\
1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0
\end{bmatrix}$$

Figure 1: Connection of Data Points: Table and Adjacency Matrix

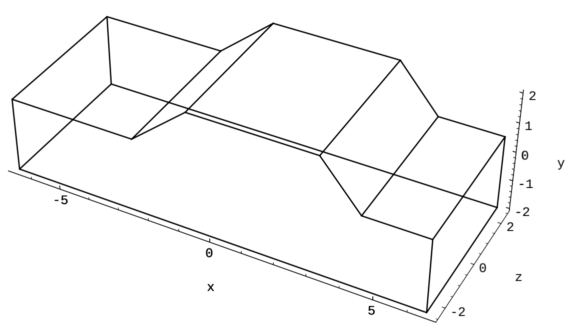

Figure 2: The Toyota data points connected

We are given the center of projection (b, c, d) and some data point (x, y, z). We wish to find the point $(x^*, y^*, 0)$ in the xy plane which lies on the same line as (b, c, d) and (x, y, z); see Figure 6 on page 160 for a picture. We will then plot the point (x^*, y^*) in two-dimensional space. To find the coordinates x^* and y^*, we will need to find the equation of the line through (b, c, d) and (x, y, z). As is noted on pages 51 and 52 of the text, we may let the vectors $\mathbf{p}$ and $\mathbf{v}$ be defined as follows:

$$\mathbf{p} = \begin{bmatrix} b \\ c \\ d \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x-b \\ y-c \\ z-d \end{bmatrix}$$

Then we are looking for the equation of the line through $\mathbf{p}$ parallel to $\mathbf{v}$, which may be written as

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} = \begin{bmatrix} b \\ c \\ d \end{bmatrix} + t \begin{bmatrix} x-b \\ y-c \\ z-d \end{bmatrix} = \begin{bmatrix} b+t(x-b) \\ c+t(y-c) \\ d+t(z-d) \end{bmatrix}$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a point on the line and t is a parameter which varies over all real numbers. Since we want $z = 0$, we find that $d + t(z - d) = 0$, or $t = \dfrac{d}{d - z}$. We can find x^* and y^* by plugging this value of t into their respective equations:

$$x^* = b + t(x - b) = b + \frac{d(x - b)}{d - z} = \frac{dx - bz}{d - z} \text{ and } y^* = c + t(y - c) = b + \frac{d(y - c)}{d - z} = \frac{dy - cz}{d - z}$$

Dividing the numerators and denominators of these fractions by d, we have

$$x^* = \frac{x - \frac{b}{d}z}{1 - \frac{1}{d}z} \text{ and } y^* = \frac{y - \frac{c}{d}z}{1 - \frac{1}{d}z}$$

As in the text, we will want to represent the perspective projection by a matrix P, and we will use homogeneous coordinates to represent the points. We want the point $(x, y, z, 1)$ to map to the point $(x^*, y^*, 0, 1)$, but we can scale these coordinates by the factor $1 - \frac{1}{d}z$, and instead map the point $(x, y, z, 1)$ to the point $(x - \frac{b}{d}z, y - \frac{c}{d}z, 0, 1 - \frac{1}{d}z)$. Now it is relatively easy to display P:

$$P \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{b}{d} & 0 \\ 0 & 1 & -\frac{c}{d} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{d} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x - \frac{b}{d}z \\ y - \frac{c}{d}z \\ 0 \\ 1 - \frac{1}{d}z \end{bmatrix}$$

For example, we will compute the perspective projection of the Toyota from two different points:

Example 1: $(b, c, d) = (0, 0, 10)$ This is the situation computed in the text. We find that

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{10} & 1 \end{bmatrix}$$

so the data matrix D is converted to PD:

$$\begin{bmatrix} -6.5 & -6.5 & -6.5 & -6.5 & -2.5 & -2.5 & -.75 & -.75 & 3.25 & 3.25 & 4.5 & 4.5 & 6.5 & 6.5 & 6.5 & 6.5 \\ -2 & -2 & .5 & .5 & .5 & .5 & 2 & 2 & 2 & 2 & .5 & .5 & .5 & .5 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.25 & .75 & .75 & 1.25 & 1.25 & .75 & 1.25 & .75 & 1.25 & .75 & 1.25 & .75 & 1.25 & .75 & .75 & 1.25 \end{bmatrix}$$

To obtain coordinates in $\mathbb{R}^3$ we divide the top three entries in each column by the corresponding entry in the fourth row, then discard the fourth row. We get

$$\begin{bmatrix} -5.2 & -8.67 & -8.67 & -5.2 & -2 & -3.33 & -.6 & -1 & 2.6 & 4.33 & 3.6 & 6 & 5.2 & 8.67 & 8.67 & 5.2 \\ -1.6 & -2.67 & .67 & .4 & .4 & .67 & 1.6 & 2.67 & 1.6 & 2.67 & .4 & 0.67 & .4 & 0.67 & -2.67 & -1.6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

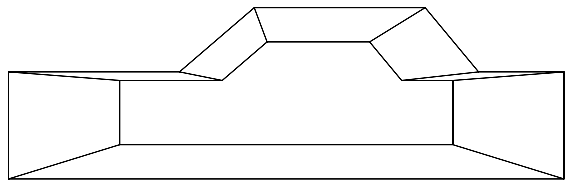

Figure 3: The Toyota from point $(0, 0, 10)$

We can then plot the points (x^*, y^*) given in the first two rows of this matrix and connect them using the connection data in Figure 1. We get the picture in Figure 3.

Example 2: $(b, c, d) = (10, 5, 10)$ In this case

$$P = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{10} & 1 \end{bmatrix}$$

so the data matrix D is converted to PD:

$$\left[\begin{array}{cccccccccccccccc} -4 & -9 & -9 & -4 & 0 & -5 & 1.75 & -3.25 & 5.75 & .75 & 7 & 2 & 9 & 4 & 4 & 9 \\ -.75 & -3.25 & -.75 & 1.75 & 1.75 & -.75 & 3.25 & .75 & 3.25 & .75 & 1.75 & -.75 & 1.75 & -.75 & -3.25 & -.75 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.25 & .75 & .75 & 1.25 & 1.25 & .75 & 1.25 & .75 & 1.25 & .75 & 1.25 & .75 & 1.25 & .75 & .75 & 1.25 \end{array}\right]$$

After rescaling by the elements in the fourth row, we have the $\mathbb{R}^3$ coordinates:

$$\left[\begin{array}{cccccccccccccccc} -3.2 & -12 & -12 & -3.2 & 0 & -6.67 & 1.4 & -4.33 & 4.6 & 1 & 5.6 & 2.67 & 7.2 & 5.33 & 5.33 & 7.2 \\ -.6 & -4.33 & -1 & 1.4 & 1.4 & -1 & 2.6 & 1 & 2.6 & 1 & 1.4 & -1 & 1.4 & -1 & -4.33 & -0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Again plotting the points (x^*, y^*) in the first two rows of this matrix and connecting them using the connection data in Figure 1, we get the picture in Figure 4.

To give the effect of moving around while viewing the Toyota, we could construct a sequence of pictures from a varying set of centers of projection. Alternatively, we could first rotate or "zoom" in on the original data then take a perspective projection from a fixed point. We do examples of each of these next.

Rotations: Suppose that we wish to rotate the three-dimensional object about the y-axis by an angle φ. By convention, a positive angle is the counterclockwise direction when looking

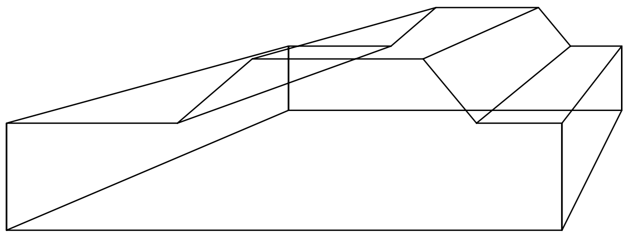

Figure 4: The Toyota from point $(10, 5, 10)$

toward the origin from the positive half of the axis of rotation. In the text (pages 159 and 160) we find that the rotation matrix using homogeneous coordinates for such a rotation is

$$A_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Likewise the matrices using homogeneous coordinates for rotations about the x and z axes are

$$A_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } A_z = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To rotate the Toyota by 45° (or $\frac{\pi}{4}$ radians) about the x-axis. We find

$$A_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and we apply this matrix to the data matrix D, getting $A_x D$:

$$\begin{bmatrix} -6.5 & -6.5 & -6.5 & -6.5 & -2.5 & -2.5 & -.75 & -.75 & 3.25 & 3.25 & 4.5 & 4.5 & 6.5 & 6.5 & 6.5 & 6.5 \\ .35 & -3.18 & -1.41 & 2.12 & 2.12 & -1.41 & 3.18 & -.35 & 3.18 & -.35 & 2.12 & -1.41 & 2.12 & -1.41 & -3.18 & .35 \\ -3.18 & .35 & 2.12 & -1.41 & -1.41 & 2.12 & -.35 & 3.18 & -.35 & 3.18 & -1.41 & 2.12 & -1.41 & 2.12 & .35 & -3.18 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

We now use this set of data and apply a perspective projection to it. If we let the center of projection be $(0, 0, 10)$ as in our first example, we can compute $PA_x D$, then divide by the fourth row to get the matrix

$$\begin{bmatrix} -4.93 & -6.74 & -8.25 & -5.69 & -2.19 & -3.17 & -.72 & -1.1 & 3.14 & 4.77 & 3.94 & 5.71 & 5.69 & 8.25 & 6.74 & 4.93 \\ .27 & -3.3 & -1.79 & 1.86 & 1.86 & -1.79 & 3.07 & -.52 & 3.07 & -.52 & 1.86 & -1.79 & 1.86 & -1.79 & -3.3 & .27 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

When we plot these points we get the picture in Figure 5; notice how the image is rotated from that in Figure 1.

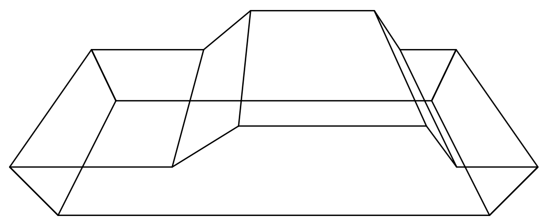

Figure 5: The rotated Toyota

"Zooming": Imaging software packages offer the ability to zoom in (or out) from a picture. Usually 100% zoom indicates the original size of the graphics image. A zoom percentage of 200% would be double that of the original, while a percentage of 50% would be half the size of the original. To make this into a transformation, we note that this is a special case of scaling where each of the x, y and z axes is scaled by the same factor p. For a 200% zoom, $p = 2$; for a 50% zoom, $p = .5$. As on page 157, we could show that the matrix which performs this scaling in homogeneous coordinates is

$$A = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, for example, if we wished to zoom in by 200% on the image of the Toyota, we could first find that

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and thus that AD is the matrix

$$\begin{bmatrix} -13 & -13 & -13 & -13 & -5 & -5 & -1.5 & -1.5 & 6.5 & 6.5 & 9 & 9 & 13 & 13 & 13 & 13 \\ -4 & -4 & 1 & 1 & 1 & 1 & 4 & 4 & 4 & 4 & 1 & 1 & 1 & 1 & -4 & -4 \\ -5 & 5 & 5 & -5 & -5 & 5 & -5 & 5 & -5 & 5 & -5 & 5 & -5 & 5 & 5 & -5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

We can now compute PAD (given center of projection $(0, 0, 10)$), then divide by the fourth row to get the matrix

$$\begin{bmatrix} -8.67 & -26 & -26 & -8.67 & -3.33 & -10 & -1 & -3 & 4.33 & 13 & 6 & 18 & 8.67 & 26 & 26 & 8.67 \\ -2.67 & -8 & 2 & .67 & .67 & 2 & 2.67 & 8 & 2.67 & 8 & .67 & 2 & .67 & 2 & -8 & -2.67 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

When these points are plotted, we get the picture in Figure 6. The original picture is provided for comparison. Notice that the size of the zoomed image is not exactly twice that of the original; this has happened because we are viewing each image from the same point, and near objects are "magnified" more by the perspective transformation. If we want the zoomed image to be exactly twice the size of the original, we could apply the scaling transformation **after** the perspective projection. Of course, this will seem to change the center of projection of the zoomed image.

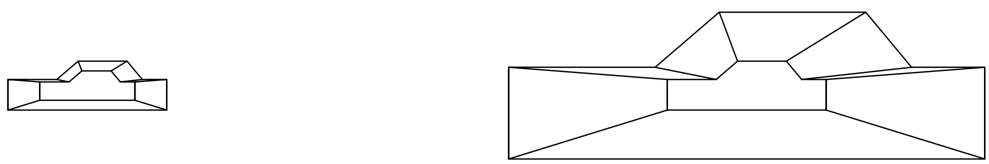

Figure 6: The Toyota: original picture and zoom of 200%

References:

1. Foley, James D., Andries van Dam, Steven K. Feiner, and John F. Hughes. *Computer Graphics: Principles and Practice*. Second Edition. Reading: Addison-Wesley, 1996.

   Chapters 5 and 6 contain a wealth of information about transformations and viewing in two and three dimensions.