

# Homework

John Smith

January 26, 2024

## Question 1

Questions are automatically numbered, starting from one. Convenient packages such as `amsmath` and `hyperref` are included by default.

Paragraphs are separated by whitespace instead of being indented.

## Question 2

Now, let's consider a mathematical example.

Suppose that  $f$  and  $g$  are real-valued functions such that  $\forall n > 0, f(n) > 0$  and  $g(n) > 0$ .

**Definition 2.1** (Big-O) — We denote by  $f(n) = O(g(n))$  if and only if

$$(\exists c > 0)(\exists n_0 > 0)(\forall n \geq n_0)(0 \leq f(n) \leq cg(n)).$$

Intuitively, the Big-O notation characterizes an *upper bound* on the asymptotic behavior of a function. In this case, the function  $g$  is an upper bound on the function  $f$ .

**Definition 2.2** (Big-Omega) — We denote by  $f(n) = \Omega(g(n))$  if and only if

$$(\exists c > 0)(\exists n_0 > 0)(\forall n \geq n_0)(0 \leq cg(n) \leq f(n)).$$

Intuitively, the Big- $\Omega$  notation characterizes a *lower bound* on the asymptotic behavior of a function. In this case, the function  $g$  is a lower bound on the function  $f$ .

**Definition 2.3** (Big-Theta) — We denote by  $f(n) = \Theta(g(n))$  if and only if

$$(\exists c_1 > 0)(\exists c_2 > 0)(\exists n_0 > 0)(\forall n \geq n_0)(c_1 g(n) \leq f(n) \leq c_2 g(n)).$$

Intuitively, the Big- $\Theta$  notation characterizes a *tight bound* on the asymptotic behavior of a function. In this case, the function  $g$  is a tight bound on the function  $f$ .

The definitions of the various asymptotic notations are closely related to the definition of a limit. As a result,  $\lim_{n \rightarrow \infty} f(n)/g(n)$  reveals a lot about the asymptotic relationship between  $f$  and  $g$ , provided that the limit exists.

**Lemma 2.4** — If the limit exists, then

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty \implies f(n) = O(g(n)).$$

**Lemma 2.5** — *If the limit exists, then*

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} > 0 \implies f(n) = \Omega(g(n)).$$

**Lemma 2.6** — *If the limit exists, then*

$$0 < \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty \implies f(n) = \Theta(g(n)).$$

Note that `*` can be used instead of `\cdot`, and `\R` instead of `\mathbb{R}`. For a normal asterisk, use `\ast`. Of course, there are also macros for the natural numbers etc. Commands such as `\abs{}` and `\set{}` can be used to create (scaled) delimiters. We demonstrate them in the following proof of Lemma 2.6.

*Proof.* Suppose that

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = L, \text{ for some } L \in \mathbb{R} \text{ such that } 0 < L < \infty.$$

By the definition of limit,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = L &\iff (\forall \varepsilon > 0)(\exists n_0 > 0)(\forall n \geq n_0) \left( \left| \frac{f(n)}{g(n)} - L \right| < \varepsilon \right) \\ &\iff (\forall \varepsilon > 0)(\exists n_0 > 0)(\forall n \geq n_0) ((L - \varepsilon)g(n) < f(n) < (L + \varepsilon)g(n)). \end{aligned}$$

If we choose  $\varepsilon = L/2$ , then

$$\begin{aligned} &(\exists n_0 > 0)(\forall n \geq n_0)(L/2 \cdot g(n) < f(n) < 3L/2 \cdot g(n)) \\ \implies &(\exists c_1 > 0)(\exists c_2 > 0)(\exists n_0 > 0)(\forall n \geq n_0)(c_1 g(n) \leq f(n) \leq c_2 g(n)) \\ \implies &f(n) = \Theta(g(n)). \end{aligned}$$

□

## Question 3

Some questions may consist of multiple parts. Let's consider the following example.

Ogden's lemma is a generalization of the pumping lemma for context-free languages.

**Theorem 3.1** (Ogden's Lemma) — *Let  $L$  be a context-free language. Then there exists a constant  $n \geq 0$  such that for every string  $z \in L$  where  $|z| \geq n$ , and for every way of "marking"  $n$  or more of the positions in  $z$ , we can break  $z$  into five substrings,  $z = uvwxy$ , such that:*

1.  $vx$  has at least one marked position,
2.  $vwx$  has at most  $n$  marked positions, and
3. for all  $i \geq 0$ , the string  $uv^iwx^iy$  is also in  $L$ .

*In the special case where every position is marked, Ogden's lemma is equivalent to the pumping lemma for context-free languages.*

### Part 1

Prove Ogden's lemma.

### Part 2

There exist context-free languages for which no unambiguous context-free grammar can exist. Such languages are called *inherently ambiguous*.

Let  $L_0 = \{a^n b^m c^m \mid m, n \geq 1\}$  and  $L_1 = \{a^m b^n c^n \mid m, n \geq 1\}$ . Using Ogden's lemma, show that the language  $L = L_0 \cup L_1$  is inherently ambiguous.

## Question A

Optionally, you can fully customize the numbering of each question ...

## Question 8

... or skip a few, using the `\setcounter{question}{x}` command.