Homework

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Question 1

Questions are automatically numbered, starting from one. Convenient packages such as `amsmath` and `hyperref` are included by default.

Paragraphs are separated by whitespace instead of being indented.

Question 2

Now, let's consider a mathematical example.

Suppose that f and g are real-valued functions such that $\forall n > 0$, $f(n) > 0$ and $g(n) > 0$.

Definition 2.1 (Big-O) — We denote by $f(n) = O(g(n))$ if and only if

$$(\exists c > 0)(\exists n_0 > 0)(\forall n \geq n_0)(0 \leq f(n) \leq cg(n)).$$

Intuitively, the Big-O notation characterizes an *upper bound* on the asymptotic behavior of a function. In this case, the function g is an upper bound on the function f.

Definition 2.2 (Big-Omega) — We denote by $f(n) = \Omega(g(n))$ if and only if

$$(\exists c > 0)(\exists n_0 > 0)(\forall n \geq n_0)(0 \leq cg(n) \leq f(n)).$$

Intuitively, the Big-Ω notation characterizes a *lower bound* on the asymptotic behavior of a function. In this case, the function g is a lower bound on the function f.

Definition 2.3 (Big-Theta) — We denote by $f(n) = \Theta(g(n))$ if and only if

$$(\exists c_1 > 0)(\exists c_2 > 0)(\exists n_0 > 0)(\forall n \geq n_0)(c_1 g(n) \leq f(n) \leq c_2 g(n)).$$

Intuitively, the Big-Θ notation characterizes a *tight bound* on the asymptotic behavior of a function. In this case, the function g is a tight bound on the function f.

The definitions of the various asymptotic notations are closely related to the definition of a limit. As a result, $\lim_{n\to\infty} f(n)/g(n)$ reveals a lot about the asymptotic relationship between f and g, provided that the limit exists.

Lemma 2.4 — *If the limit exists, then*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \implies f(n) = O(g(n)).$$

Lemma 2.5 — *If the limit exists, then*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 \implies f(n) = \Omega(g(n)).$$

Lemma 2.6 — *If the limit exists, then*

$$0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \implies f(n) = \Theta(g(n)).$$

Note that `*` can be used instead of `\cdot`, and `\R` instead of `\mathbb{R}`. For a normal asterisk, use `\ast`. Of course, there are also macros for the natural numbers etc. Commands such as `\abs{}` and `\set{}` can be used to create (scaled) delimiters. We demonstrate them in the following proof of Lemma 2.6.

Proof. Suppose that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = L, \text{ for some } L \in \mathbb{R} \text{ such that } 0 < L < \infty.$$

By the definition of limit,

$$\begin{aligned}\lim_{n\to\infty} \frac{f(n)}{g(n)} = L &\iff (\forall \varepsilon > 0)(\exists n_0 > 0)(\forall n \geq n_0)\left(\left|\frac{f(n)}{g(n)} - L\right| < \varepsilon\right) \\ &\iff (\forall \varepsilon > 0)(\exists n_0 > 0)(\forall n \geq n_0)((L - \varepsilon)g(n) < f(n) < (L + \varepsilon)g(n)).\end{aligned}$$

If we choose $\varepsilon = L/2$, then

$$\begin{aligned}&(\exists n_0 > 0)(\forall n \geq n_0)(L/2 \cdot g(n) < f(n) < 3L/2 \cdot g(n)) \\ \implies &(\exists c_1 > 0)(\exists c_2 > 0)(\exists n_0 > 0)(\forall n \geq n_0)(c_1 g(n) \leq f(n) \leq c_2 g(n)) \\ \implies &f(n) = \Theta(g(n)).\end{aligned}$$

□

Question 3

Some questions may consist of multiple parts. Let's consider the following example.

Ogden's lemma is a generalization of the pumping lemma for context-free languages.

Theorem 3.1 (Ogden's Lemma) — *Let L be a context-free language. Then there exists a constant $n \geq 0$ such that for every string $z \in L$ where $|z| \geq n$, and for every way of "marking" n or more of the positions in z, we can break z into five substrings, $z = uvwxy$, such that:*

1. *vx has at least one marked position,*
2. *vwx has at most n marked positions, and*
3. *for all $i \geq 0$, the string uv^iwx^iy is also in L.*

In the special case where every position is marked, Ogden's lemma is equivalent to the pumping lemma for context-free languages.

Part 1

Prove Ogden's lemma.

Part 2

There exist context-free languages for which no unambiguous context-free grammar can exist. Such languages are called *inherently ambiguous*.

Let $L_0 = \{a^n b^m c^m \mid m, n \geq 1\}$ and $L_1 = \{a^m b^n c^n \mid m, n \geq 1\}$. Using Ogden's lemma, show that the language $L = L_0 \cup L_1$ is inherently ambiguous.

Question A

Optionally, you can fully customize the numbering of each question ...

Question 8

... or skip a few, using the `\setcounter{question}{x}` command.