# Homework

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September 1, 2024

## Question 1

Questions are automatically numbered, starting from one. Convenient packages such as `amsmath` and `hyperref` are included by default.

Paragraphs are separated by whitespace instead of being indented.

## Question 2

Now, let's consider a mathematical example.

Suppose that $f$ and $g$ are real-valued functions such that $\forall n > 0$, $f(n) > 0$ and $g(n) > 0$.

**Definition 2.1** (Big-O) — We denote by $f(n) = O(g(n))$ if and only if

$$(\exists c > 0)(\exists n_0 > 0)(\forall n \geq n_0)(0 \leq f(n) \leq cg(n)).$$

Intuitively, the Big-O notation characterizes an *upper bound* on the asymptotic behavior of a function. In this case, the function $g$ is an upper bound on the function $f$.

**Definition 2.2** (Big-Omega) — We denote by $f(n) = \Omega(g(n))$ if and only if

$$(\exists c > 0)(\exists n_0 > 0)(\forall n \geq n_0)(0 \leq cg(n) \leq f(n)).$$

Intuitively, the Big-$\Omega$ notation characterizes a *lower bound* on the asymptotic behavior of a function. In this case, the function $g$ is a lower bound on the function $f$.

**Definition 2.3** (Big-Theta) — We denote by $f(n) = \Theta(g(n))$ if and only if

$$(\exists c_1 > 0)(\exists c_2 > 0)(\exists n_0 > 0)(\forall n \geq n_0)(c_1 g(n) \leq f(n) \leq c_2 g(n)).$$

Intuitively, the Big-$\Theta$ notation characterizes a *tight bound* on the asymptotic behavior of a function. In this case, the function $g$ is a tight bound on the function $f$.

The definitions of the various asymptotic notations are closely related to the definition of a limit. As a result, $\lim_{n\to\infty} f(n)/g(n)$ reveals a lot about the asymptotic relationship between $f$ and $g$, provided that the limit exists.

**Lemma 2.4** — *Assume that the following limit exists. We have*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \implies f(n) = O(g(n)).$$

**Lemma 2.5** — *Assume that the following limit exists. We have*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 \implies f(n) = \Omega(g(n)).$$

**Lemma 2.6** — *Assume that the following limit exists. We have*

$$0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \implies f(n) = \Theta(g(n)).$$

Note that `*` can be used instead of `\cdot`, and `\R` instead of `\mathbb{R}`. For a normal asterisk, use `\ast`. Of course, there are also macros for the natural numbers etc. Commands such as `\abs{}` and `\set{}` can be used to create (scaled) delimiters. We demonstrate them in the following proof of Lemma 2.6.

*Proof.* Suppose that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = L, \text{ for some } L \in \mathbb{R} \text{ such that } 0 < L < \infty.$$

By the definition of limit,

$$\begin{aligned} \lim_{n\to\infty} \frac{f(n)}{g(n)} = L &\iff (\forall \varepsilon > 0)(\exists n_0 > 0)(\forall n \geq n_0)\left(\left|\frac{f(n)}{g(n)} - L\right| < \varepsilon\right) \\ &\iff (\forall \varepsilon > 0)(\exists n_0 > 0)(\forall n \geq n_0)((L - \varepsilon)g(n) < f(n) < (L + \varepsilon)g(n)). \end{aligned}$$

If we choose $\varepsilon = L/2$, then

$$\begin{aligned} &(\exists n_0 > 0)(\forall n \geq n_0)(L/2 \cdot g(n) < f(n) < 3L/2 \cdot g(n)) \\ \implies &(\exists c_1 > 0)(\exists c_2 > 0)(\exists n_0 > 0)(\forall n \geq n_0)(c_1 g(n) \leq f(n) \leq c_2 g(n)) \\ \implies &f(n) = \Theta(g(n)). \end{aligned}$$

□

# Question 3

Some questions may consist of multiple parts. Let's consider the following example.

Ogden's lemma is a generalization of the pumping lemma for context-free languages.

**Theorem 3.1** (Ogden's Lemma) — *Let $L$ be a context-free language. Then there exists a constant $n \geq 0$ such that for every string $z \in L$ where $|z| \geq n$, and for every way of "marking" $n$ or more of the positions in $z$, we can break $z$ into five substrings, $z = uvwxy$, such that:*

1. *$vx$ has at least one marked position,*
2. *$vwx$ has at most $n$ marked positions, and*
3. *for all $i \geq 0$, the string $uv^i wx^i y$ is also in $L$.*

*In the special case where every position is marked, Ogden's lemma is equivalent to the pumping lemma for context-free languages.*

### Part 1

Prove Ogden's lemma.

### Part 2

There exist context-free languages for which no unambiguous context-free grammar can exist. Such languages are called *inherently ambiguous*.

Let $L_0 = \{a^n b^m c^m \mid m, n \geq 1\}$ and $L_1 = \{a^m b^n c^n \mid m, n \geq 1\}$. Using Ogden's lemma, show that the language $L = L_0 \cup L_1$ is inherently ambiguous.

# Question A

Optionally, you can fully customize the numbering of each question ...

# Question 8

... or skip a few, using the `\setcounter{question}{x}` command.