Probability
Part 3 – Counting Problems

DASC 512

Overview

- What is a counting problem?
- Replacement
- Order
- Counting rules

What is a counting problem?

- Counting problems are defined by calculating the number of possible outcomes of some process
- Examples:
    - There are 24 students in the class, and I split the class into groups of 4. How many different groups could you be a part of?
    - A raffle has 3 different prizes. 213 people bought tickets. How many ways can the prizes be divided up?

Why counting problems

- There are often discrete probability problems with large sample spaces of equally likely outcomes.
- Counting rules, or combinatorics, help us solve these problems

Formulation

- Let the population of possibilities be n
  - Example: The size of the class, the number of raffle tickets, etc.

- Let the selection be of size r
  - Example: The number in a group, the number of prizes, etc.

- We are looking for the total possible arrangements of size r from n possibilities

- Selection of a technique depends on replacement and order

Is there replacement?

- With replacement: Any of the n possibilities may be chosen repeatedly
  - Example: 3-digit numbers, raffles where names are re-entered after winning

- Without replacement: Any one possibility may be chosen only once
  - Examples: group selection, typical raffles

Does order matter?

- Order matters: The outcome is defined by the order of selection
  - Example: Raffle with different prizes, horse race trifectas

- Order does not matter: The outcome is defined by being selected or not
  - Example: Raffle with the same prizes, group assignment

Counting rules

- Recall: $x! = 1 \times 2 \times \cdots \times x$

	Without Replacement	With Replacement
Order Matters	$\frac{n!}{(n-r)!}$	n^r
Order Doesn't Matter	$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$	$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!\,(n-1)!}$

- Definitions: (Without Replacement rules)
  - Permutation: $\frac{n!}{(n-r)!}$
  - Combination: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Counting Example

- 8 candidates are up for 5 promotions: (A, B, C, D, E, F, G, H)
- What is the probability that A, B, C, D, and E are promoted?

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- What is the probability that A, B, C, D, and E are promoted?

- Intuitive Solution:

The probability that the first promotion goes to one of those 5 is 5/8. The next is 4/7, then 3/6, 2/5, and 1/4. Thus, the probability is:

$$\frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4} = \frac{1}{56}$$

Counting Example

- 8 candidates are up for 5 promotions: (A, B, C, D, E, F, G, H)
- What is the probability that A, B, C, D, and E are promoted?

- Using counting rules: how many possible pools of 5 are there?

Order is irrelevant, and candidates are not replaced → Combination

$$\binom{8}{5} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(5 \times 4 \times 3 \times 2 \times 1) \times (3 \times 2 \times 1)} = \frac{8 \times 7 \times 6}{3 \times 2} = 56$$

$$P(\{A, B, C, D, E\}) = \frac{1}{56}$$

Counting Example

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- What if they were promoted in the order they were selected? What is the probability that they are promoted in the order (A, B, C, D, E)?

- Intuitive Solution:

A is promoted first with probability 1/8. Then B with probability 1/7. C: 1/6, D: 1/5, and E: 1/4.

$$\frac{1}{8} \times \frac{1}{7} \times \frac{1}{6} \times \frac{1}{5} \times \frac{1}{4} = \frac{1}{6720}$$

Counting Example

- What if they were promoted in the order they were selected? What is the probability that they are promoted in the order (A, B, C, D, E)?

- How many possible ordered promotion lists of 5?
  - Order now matters, still no replacement → Permutation

$$\frac{8!}{3!} = (8 \times 7 \times 6 \times 5 \times 4) = 6720$$

$$P(\{A, B, C, D, E\}) = \frac{1}{6720}$$

Recap

- What is a counting problem?
- Replacement
- Order
- Counting rules