Probability
Part 2 – Conditional Probability

DASC 512

Overview

- What is Conditional Probability?
- Calculating conditional probability
- Independence
- Multiplication Rule
- Bayes’s Rule

Conditional Probability

- We often want to restrict a question to part of the sample space, or we gain extra information that allows us to refine a probability estimate.

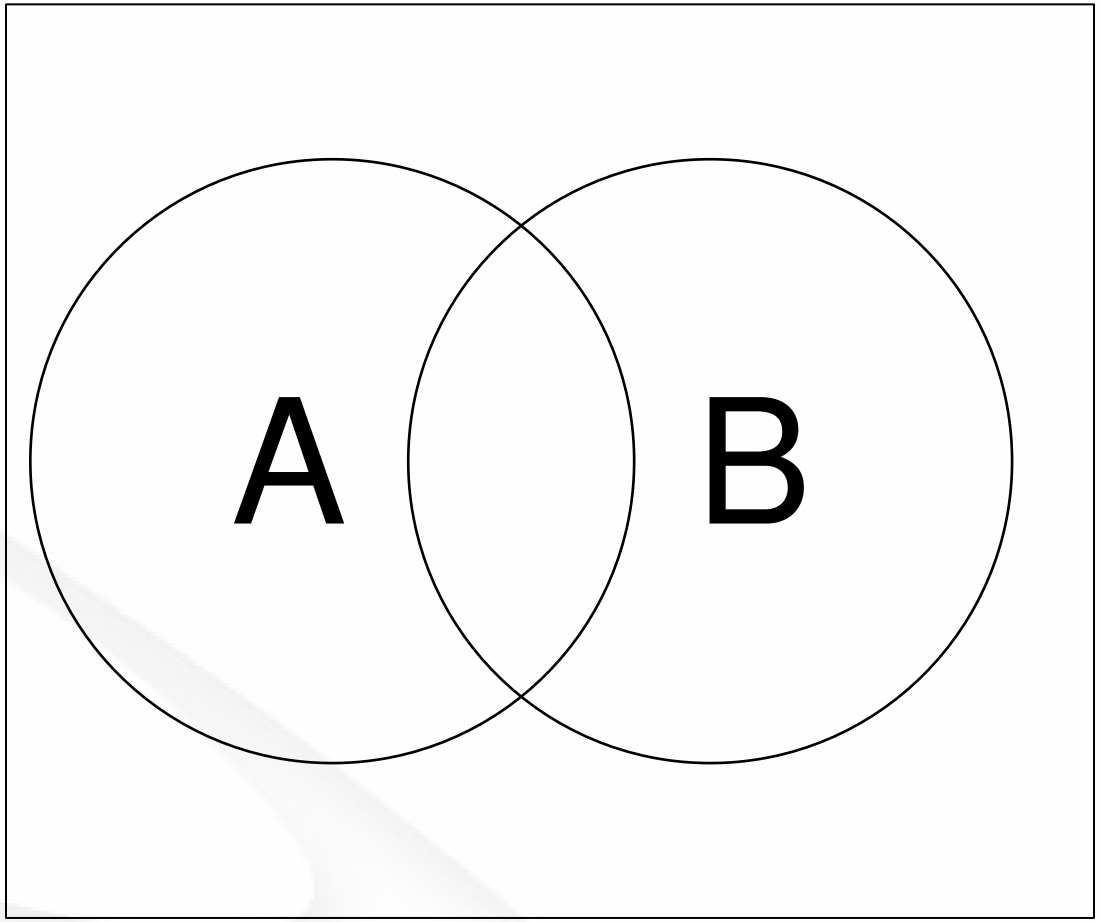
- For events *A* and *B*, the conditional probability of *A* given *B* is the probability that *A* occurs given that *B* has occurred. We write this

$$P(A|B)$$

Conditional Probability

- Recall, $P(A \cap B)$ is the area where *A* and *B* overlap
- $P(A|B)$ is the portion of the area of *B* that overlaps with *A*.
- Mathematically, for $P(B) \neq 0$,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

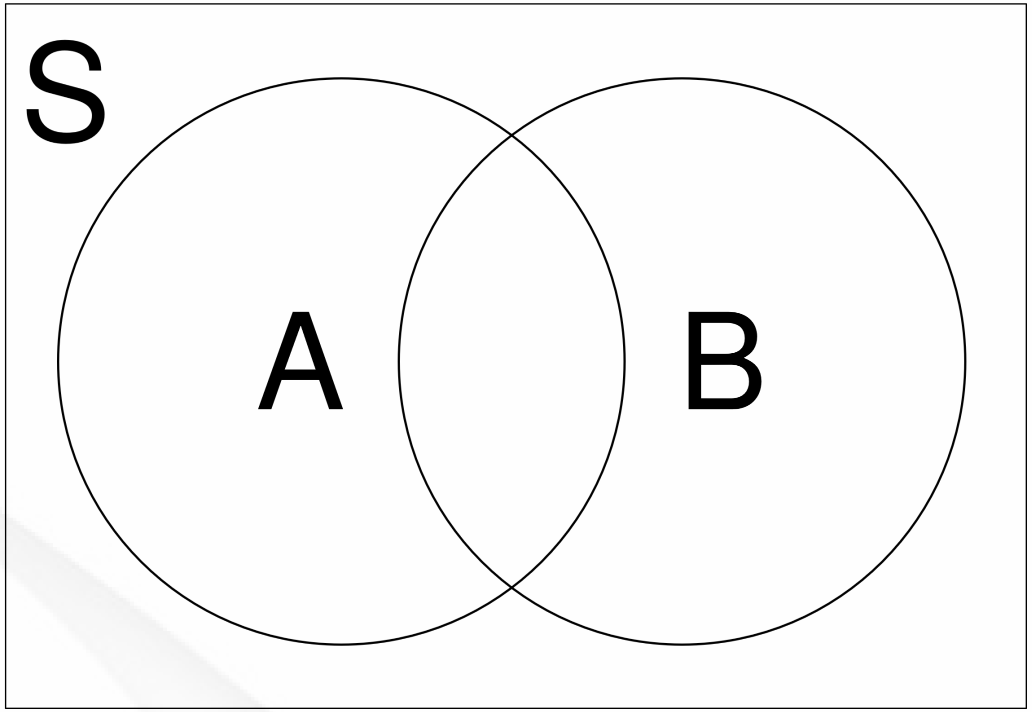


Consider this…

There is an implied conditionality in every probability statement so far…

P(A) is the fraction of S that is also in A.

$$P(A) = P(A|S) = \frac{\mathrm{P}(\mathrm{A} \cap S)}{P(S)} = \frac{P(A)}{1}$$

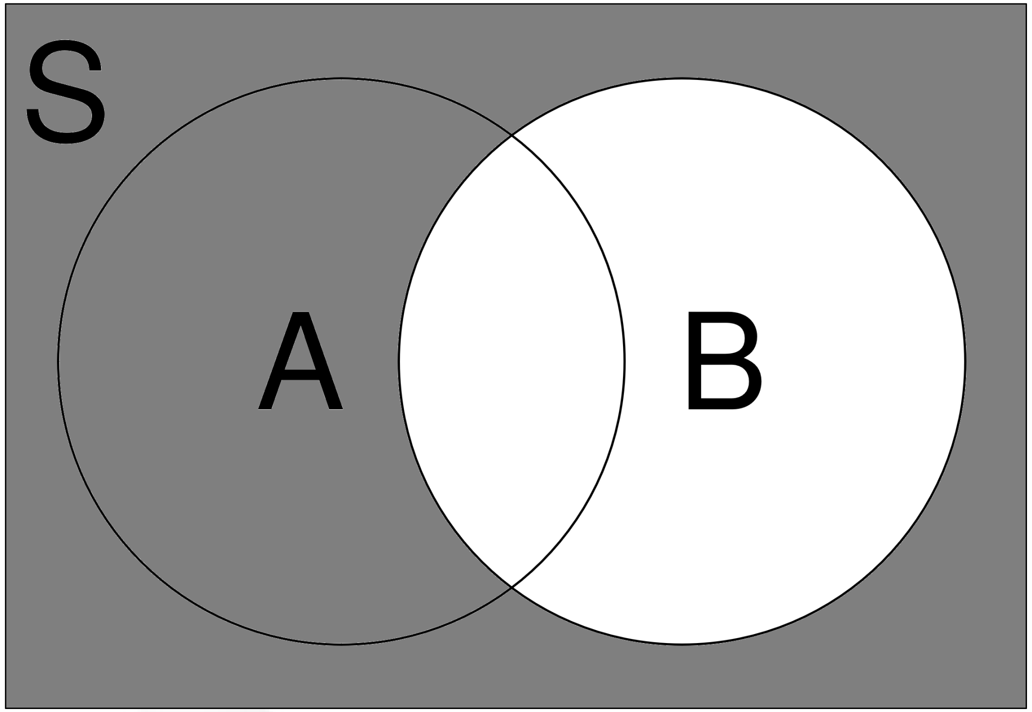


Consider this…

By conditioning on B, we restrict the sample space to B.

P(A|B) is the fraction of B that is also in A.

$$P(A|B) = \frac{\mathrm{P}(\mathrm{A} \cap B)}{P(B)}$$



Example

- A sample of Google Play Store apps in 2019 had the following attributes:

	Everyone	Ages 10+	Teen 13+ (A)	Mature 17+	Total
Free	0.7402	0.0351	0.1067	0.0442	**0.9262**
Paid (B)	0.0641	0.0030	0.0048	0.0018	**0.0738**
Total	**0.8043**	**0.0381**	**0.1115**	**0.0461**	**1**

- What is the probability that the app is paid, given it is rated Teen 13+?

https://www.kaggle.com/lava18/google-play-store-apps

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- What is the probability that the app is paid, given it is rated Teen 13+?

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{0.0048}{0.1115} = 0.0430$$

https://www.kaggle.com/lava18/google-play-store-apps

Probability Rules
Multiplication Rule

From the definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

we can derive the multiplication rule as long as $P(B) \neq 0$.

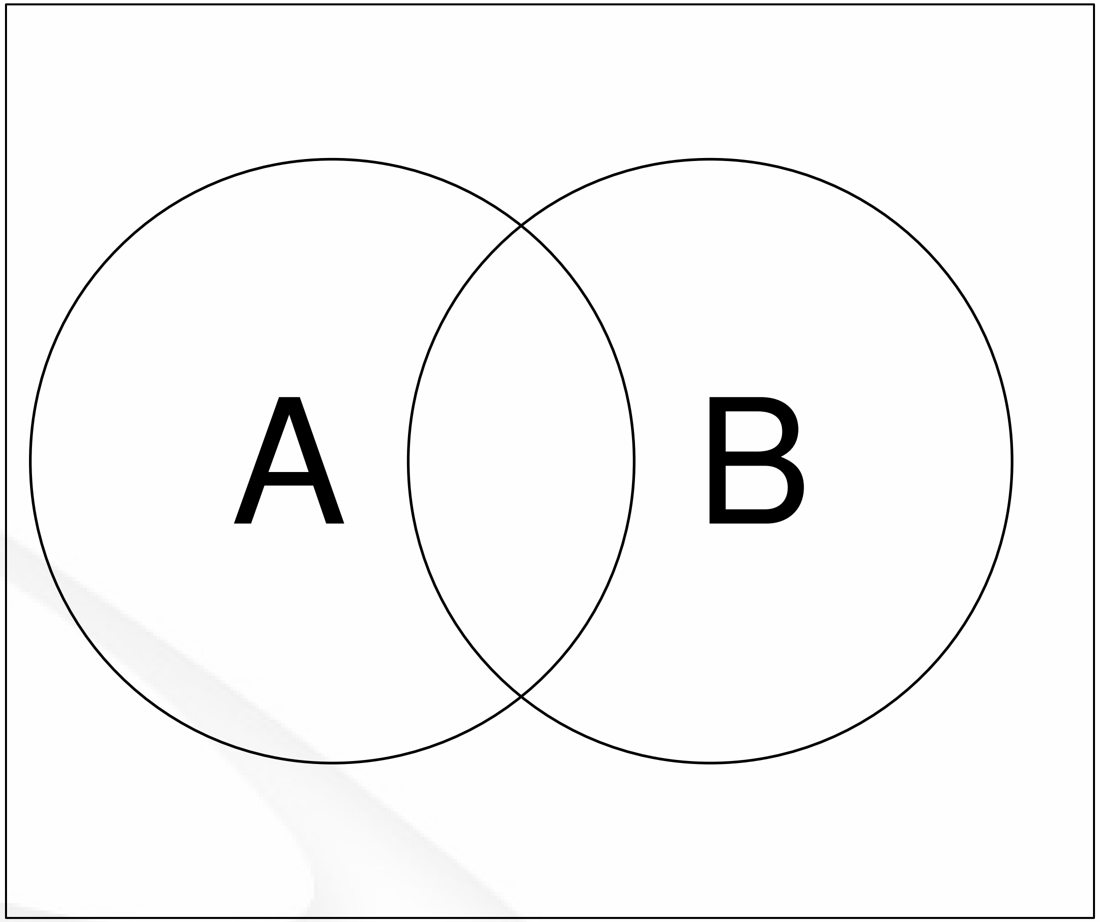
$$P(A|B)P(B) = P(A \cap B)$$

Independence of Events

- If, and only if, *A* and *B* are <u>independent events</u>,

  $$P(A|B) = P(A)$$
  $$P(B|A) = P(B)$$
  $$P(A \cap B) = P(A) \times P(B)$$

- Note: independent events **are not** mutually exclusive.



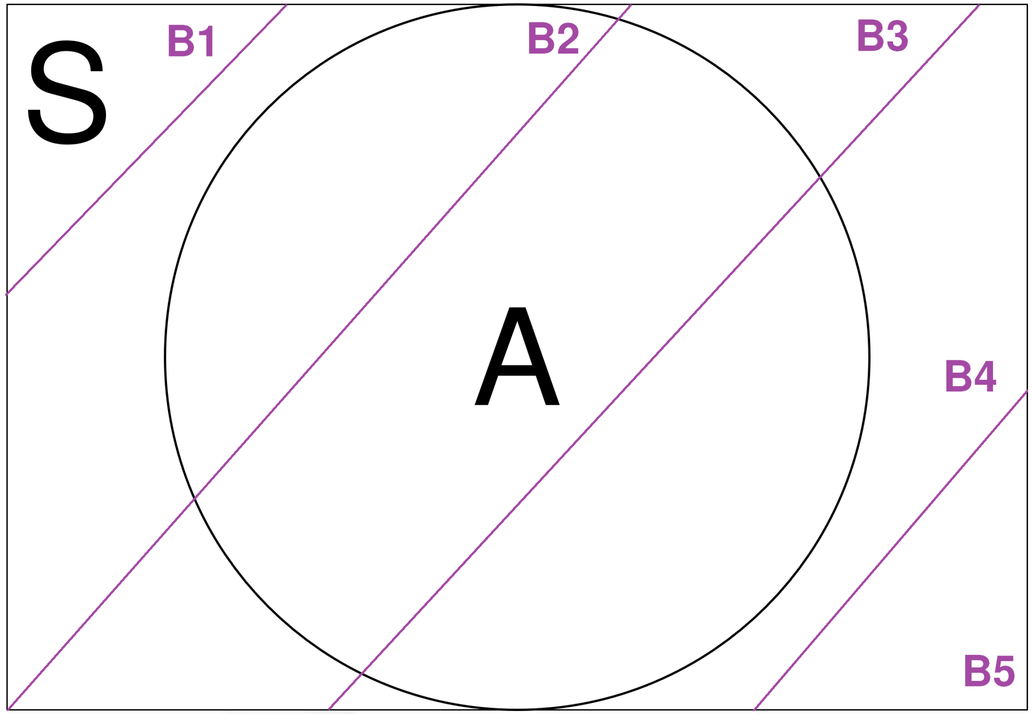
Bayes's Rule

Let $B_1, \ldots, B_k$ be a partition of S.

Let A be an observed event.

By the definition of conditional probability, we know

$$P(B_i|A) = \frac{P(B_i \cap A)}{P(A)}$$



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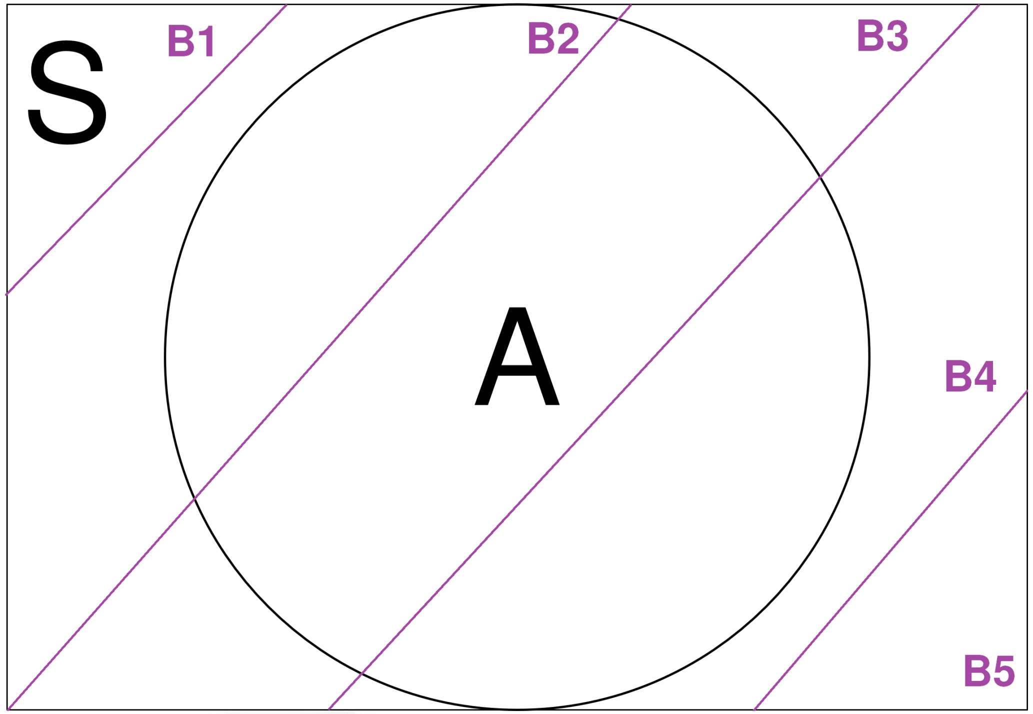


Recall from the multiplication rule,

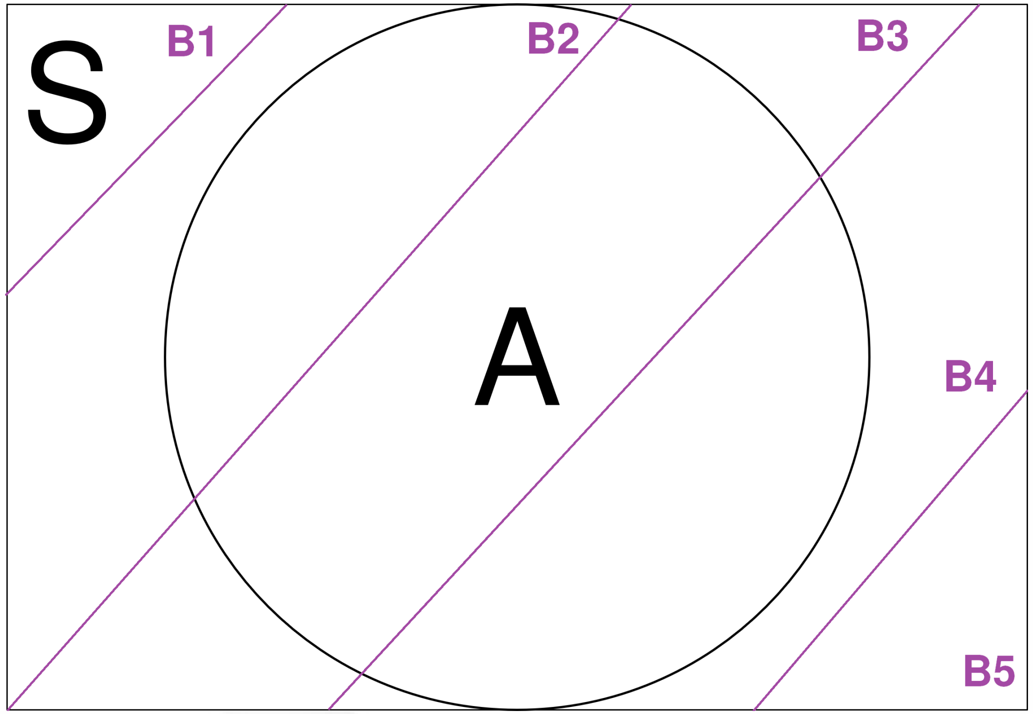
$$\begin{aligned} P(B_i \cap A) &= P(A \cap B_i) \\ &= P(A|B_i)P(B_i) \end{aligned}$$

Bayes's Rule

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)}$$

Recall from the law of total probability,

$$P(A) = \sum_{j=1}^{k} P(A \cap B_j)$$

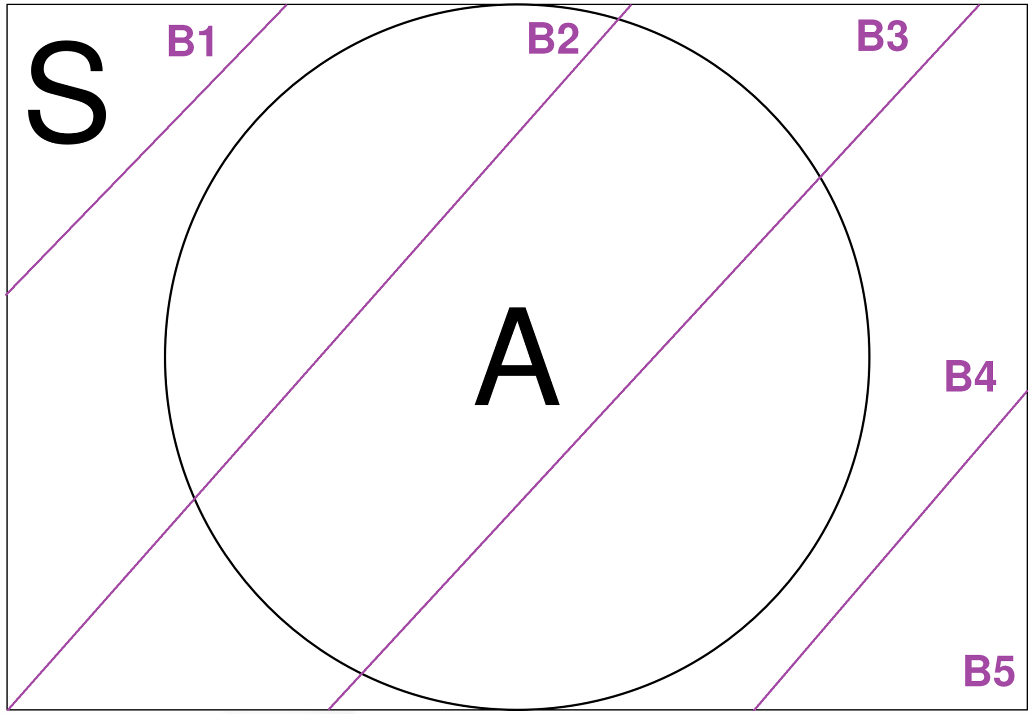


Bayes's Rule

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{k} P(A \cap B_j)}$$

Applying the definition of conditional probability again

$$P(A \cap B_j) = P(B_j)P(A|B_j)$$

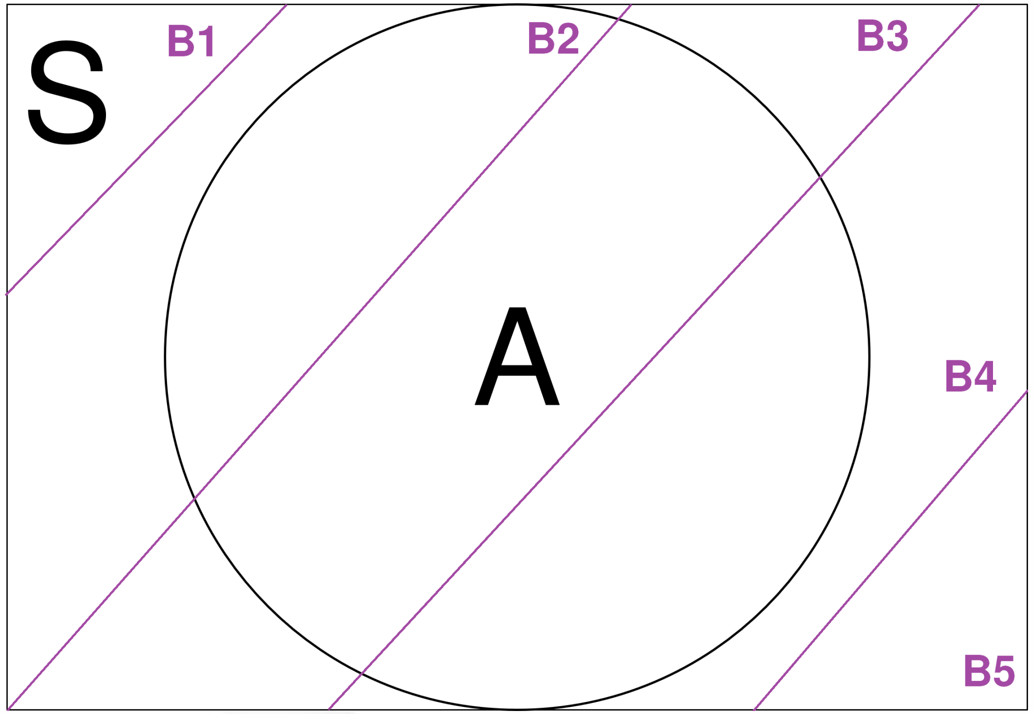


Bayes's Rule

Let $B_1, \ldots, B_k$ be a partition of S.

Let A be an observed event.

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{j=1}^{k} P(B_j)P(A|B_j)}$$



Bayes’s Rule Example

- Consider an automated target classification system.
  - Event *T*: The object is actually a target.
  - Event *A*: The system classifies the object as a target.

- Let the system have the following probabilities:
  - Probability of true positive $P(A|T) = 0.9$
  - Probability of true negative $P(A^c|T^c) = 0.9$

- What is the probability that an object that the system identifies as a target is actually a target? That is, what is $P(T|A)$?

Bayes's Rule Example

If we assume that 80% of objects are actually targets,

$$P(T|A) = \frac{P(A|T)P(T)}{P(A|T)P(T) + P(A|T^c)P(T^c)}$$

$$= \frac{0.9 \times 0.8}{(0.9 \times 0.8) + (0.1 \times 0.2)} = 0.97$$

Bayes's Rule Example

If we assume that only 10% of objects are actually targets,

$$P(T|A) = \frac{P(A|T)P(T)}{P(A|T)P(T) + P(A|T^c)P(T^c)}$$

$$= \frac{0.9 \times 0.1}{(0.9 \times 0.1) + (0.1 \times 0.9)} = 0.50$$

Bayes's Rule Example 2

Consider a medical screening test for a particular disease.

- Event *A*: You have the disease.
- Event *B*: The test indicates that you have the disease.

The test is calibrated to have a low false negative rate.

- Probability of true positive $P(B|A) = 0.99$
- Probability of a false positive $P(B|A^c) = 0.10$

This disease is not currently prevalent. $P(A) = 0.00125$

The test indicates you have the disease. Should you be worried?

Bayes's Rule Example 2

$$P(B|A) = 0.99$$
$$P(B|A^c) = 0.10$$
$$P(A) = 0.00125$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

$$= \frac{0.99 \times 0.00125}{(0.99 \times 0.00125) + (0.10 \times 0.99875)}$$

$$= 0.01224$$

Should you be worried? Is this test useful? What if the test was negative?

Bayes's Rule Example 2

$$P(B|A) = 0.99$$
$$P(B|A^c) = 0.10$$
$$P(A) = 0.00125$$

$$P(A^c|B^c) = \frac{P(B^c|A^c)P(A^c)}{P(B^c|A^c)P(A^c) + P(B^c|A)P(A)}$$

$$= \frac{0.90 \times 0.99875}{(0.90 \times 0.99875) + (0.01 \times 0.00125)}$$

$$= 0.999986$$

Recap

- What is Conditional Probability?
- Calculating conditional probability
- Independence
- Multiplication Rule
- Bayes’s Rule