Least Squares Estimation

DASC 512

Least Squares Estimation

Goal: Be able to use the method of least squares to fit a hypothesized model to fit sample data

We'll be assuming a first-order linear model as the deterministic component for now.

$$y = \beta_0 + \beta_1 x + \epsilon$$

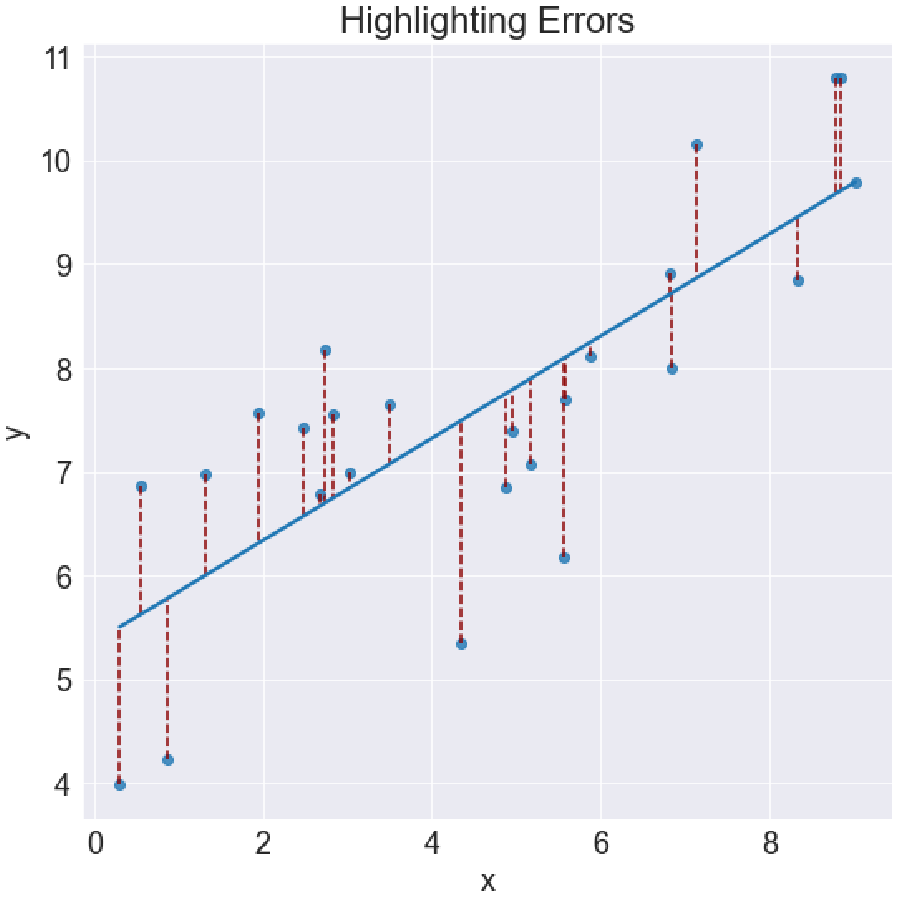
The method of least squares is one method to find values of β_0 and β_1. It minimizes the squared errors in prediction – hence the name!

What is Least Squares?

Least Squares Estimation finds the coefficient values that minimize the sum of squared errors

Thinking back on ANOVA, minimizing SS Error also maximizes F

Why not just minimize sum of error?



Parameter Estimation

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

Least squares estimates for the slope and intercept are

$$\hat{\beta}_1 = \frac{SS_{xy}}{SS_{xx}}, \qquad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$SS_{xy} = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

$$SS_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

Ordinary Least Squares Model

The Ordinary Least Squares (OLS) model is the model produced by the least squares estimates, $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$

It has the following properties:

- The sum of all errors equals zero $\sum_{i=1}^{n} y_i - \hat{y} = 0 \;\Rightarrow\; \mu_\epsilon = 0$
- The Sum of Squared Errors (SSE) is smaller than for any other straight-line model (i.e., the σ_ϵ^2 is minimized)

$$SSE = \sum_{i=1}^{n} (y_i - \hat{y})^2$$

The fitted least squares line is only valid on the range of data that was used to create it. Outside of that range, we are extrapolating.

Interpreting the Betas

β_0 is the y-intercept

- May not have practical meaning if $x = 0$ is outside range of data
- If it does, it is the expected value of y when $x = 0$

β_1 is the slope

- Remember back to high school: “rise over run”
- Represents the expected change in y for every one-unit increase in x
- This relationship often will not hold outside range of sample data

Example: Rocket Propellant

Pretend we are all rocket scientists!

We have gathered a sample of rocket propellants, noted their age in weeks, and measured their sheer strength in pounds per square inch (psi)

What is the relationship between age and sheer strength?

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We now have our hypothesized, parameterized model!

$$y = 2628 - 37.15x + \epsilon$$

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Breaking this model down, we expect:

- Brand new propellant to have sheer strength of 2628 psi
  - Note: Lowest observed value is 2 weeks, so this may be extrapolation.
- Propellant will lose 37.15 psi of sheer strength per week
  - Note: Up to 25 weeks. Beyond that, the linear model may not hold.

And what about that epsilon?

Because we now have the model

$$y = \hat{\beta}_0 + \hat{\beta}_1 x + \epsilon = 2628 - 37.15x + \epsilon$$

we can also calculate the observed errors

$$\epsilon_i = y_i - \hat{\beta}_1 x_i - \hat{\beta}_0 = y_i + 37.15x_i - 2628$$

These are called the residuals, and characterizing them is the focus of our next lesson on checking model assumptions

Equations Recap

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

$$\hat{\beta}_1 = \frac{SS_{xy}}{SS_{xx}}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$SS_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$SS_{xy} = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

Next time…

Checking Assumptions of an OLS Model