Prof. Dr. Markus Bibinger
AG Quantitatives Risiko und hochdimensionale Finanzdaten
Wintersemester 2024/25
Julius-Maximilians-Universität Würzburg

Project covariance estimation: shrinkage

Implement a Monte Carlo simulation with $n = 100$ observations of multivariate normal random vectors $X \in \mathbb{R}^5$, which have expectation 0, and variance-covariance matrix

```
Sigma<-matrix(c(1,rep(.1,4),.1,1,rep(.1,3),.1,.1,1,.1,.1,
              rep(.1,3),1,.1,rep(.1,4),1),5,5).
```

(a) Based on the Monte Carlo simulation with $M = 1{,}000$ iterations, determine the sorted eigenvalues of the sample covariance matrix and compare them to the true eigenvalues of Σ in a plot. Compare the sum of the eigenvalues of the sample covariance matrices to the sum of the eigenvalues of Σ by looking at the mean and the empirical distribution in a histogram.
Hint: For some suitable matrix A, the command eigen(A)$\$values$ gives its eigenvalues.

(b) Analyse for weights $w = l/100, 0 \leqslant l \leqslant 100$, the squared bias, variance and mean squared error, defined with the Frobenius norm as in the lecture, of shrinkage estimators

$$w \cdot \gamma \cdot I_5 + (1 - w) \cdot S\,,$$

with S the sample covariance matrix and $\gamma \cdot I_5$ the optimal shrinkage target using the true Σ, with $M = 1{,}000$ iterations. Illustrate the dependence on w in a plot and determine an empirically optimal w.
What do you expect how the plot changes for different n? Rerun the simulation with $n = 1{,}000$ observations.

(c) Implement a feasible (non-oracle) shrinkage estimator based on estimates of the optimal weight and γ and perform a Monte Carlo simulation with $M = 1{,}000$ and $n = 100$. Determine the eigenvalues of this estimator and compare the sorted eigenvalues to that of the sample covariance matrix and the true eigenvalues of Σ in a plot. Compare the estimated optimal weight to your results in (b).

(d) Consider Σ with entries 1 on the diagonal and all entries off the diagonal equal to 0.1. Compare for dimension $p = l/10, 1 \leqslant l \leqslant 10$, the squared norms of estimation errors of Σ^{-1}, estimated with the inverted sample covariance matrix and the inverted shrinkage estimator from (c). Illustrate the ratios of these squared norms of estimation errors in a plot.

(e) Download the data Stock_Bond_2004_to_2006.csv from the WueCampus site and extract the returns, i.e. the differences of log-prices for each of the 8 stocks. We analyse the plug-in portfolio built upon the optimal weights and estimating Σ^{-1} with the sample covariance matrix and compare to a portfolio for that we estimate Σ^{-1} with the feasible shrinkage estimator. Moreover, we compare to a benchmark portfolio with constantly equal weight 1/8 on each asset. Use the data over 336 days backwards as a training sample to estimate Σ^{-1}, assuming it is constant and i.i.d. observations. We can then compute and compare the three portfolios for 336 days. Compare the overall returns of the three portfolios.

The project results are discussed in a presentation and a concise report is submitted.