N-Body Central Configurations: From Lagrange Points to Algebraic Geometry

A Pure Thought Approach to Celestial Mechanics

Pure Thought AI Challenge
`Problem 27: Celestial Mechanics`

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Abstract

This report presents a comprehensive theoretical framework for analyzing central configurations in the gravitational N-body problem. We develop the complete theory from Newton's equations through the algebraic formulation to computational enumeration using Gröbner bases. Central configurations are special arrangements where gravitational acceleration points toward the center of mass for all bodies, enabling homothetic (shape-preserving) solutions. Key topics include Lagrange points (L1–L5), Euler's collinear solutions, the finiteness conjecture for N $\geq$ 5, stability analysis via Hessian eigenvalues, and symmetry classification. We implement algorithms for finding all central configurations for given masses and certifying their properties using computational algebraic geometry.

Contents

1 Introduction and Motivation

1.1 The N-Body Problem

Physics Insight

The **N-body problem**—determining the motion of N point masses interacting via Newtonian gravity—is one of the oldest problems in mathematical physics. While the 2-body problem has exact solutions (Kepler ellipses), $N \geq 3$ is generally non-integrable and exhibits chaotic behavior.

Despite intractability of the general problem, special solutions exist where the geometric configuration of bodies is preserved. These **central configurations** are the key to understanding:

- **Lagrange points**: Stable positions for spacecraft (JWST at Sun-Earth L2)
- **Trojan asteroids**: Jupiter's L4/L5 points host thousands of asteroids
- **Choreographies**: Remarkable periodic orbits like the figure-eight solution
- **Collision singularities**: Central configs arise at the moment of total collapse

1.2 Central Configurations: Definition

Definition 1.1 (Central Configuration)**.** *Positions* $\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_N) \in (\mathbb{R}^d)^N$ *with masses* $(m_1, \ldots, m_N)$ *form a* ***central configuration*** *if:*

$$\nabla_i U = \lambda m_i \mathbf{r}_i \quad \textit{for all } i = 1, \ldots, N \tag{1}$$

where:

- $U = -\sum_{i<j} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}$ *is the gravitational potential*
- $\lambda \in \mathbb{R}$ *is a scalar (Lagrange multiplier)*
- *Center of mass is at origin:* $\sum_i m_i \mathbf{r}_i = \mathbf{0}$

Celestial Mechanics

Physically, in a central configuration, the gravitational acceleration on each body points directly toward the center of mass. This allows the entire configuration to rotate rigidly or expand/contract homothetically while preserving its shape.

1.3 Pure Thought Approach

Pure Thought Pursuit

Central configurations are ideally suited for pure mathematical analysis:

1. Based on *algebraic equations*—polynomial system in positions
2. Solutions computable via *Gröbner bases* and homotopy continuation
3. Finiteness provable using *Bézout's theorem*
4. Stability from *Hessian eigenvalues*—symbolic computation
5. All results *certifiable* via interval arithmetic

No numerical integration of trajectories needed—pure algebra and analysis.

2 Mathematical Foundations

2.1 Newton's Equations

The equations of motion for N gravitating bodies are:

$$m_i \ddot{\mathbf{r}}_i = -\nabla_{\mathbf{r}_i} U = \sum_{j \neq i} \frac{m_i m_j (\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_i - \mathbf{r}_j|^3} \tag{2}$$

2.2 Homothetic Solutions

Definition 2.1 (Homothetic Solution)**.** *A **homothetic solution** has the form:*

$$\mathbf{r}(t) = \alpha(t)\mathbf{c} \tag{3}$$

where $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_N)$ is a fixed shape and $\alpha(t)$ is a scalar function.

Substituting into Newton's equations:

$$m_i \ddot{\alpha} \mathbf{c}_i = -\frac{1}{\alpha^2} \nabla_{\mathbf{c}_i} U(\mathbf{c}) \tag{4}$$

For this to hold with a single scalar $\ddot{\alpha}$, we need:

$$\nabla_i U(\mathbf{c}) = \lambda m_i \mathbf{c}_i \tag{5}$$

which is exactly the central configuration equation.

2.3 Degrees of Freedom

For N bodies in d dimensions:

- Total DOF: dN (positions)
- Center of mass constraint: $-d$
- Scale invariance (rescaling preserves CC): -1
- Rotation invariance: $-d(d-1)/2$

N	$d = 2$ (planar)	$d = 3$ (spatial)
3	2	3
4	4	6
5	6	9

Table 1: Degrees of freedom for central configurations after removing symmetries.

3 The Three-Body Problem

3.1 Euler's Collinear Solutions

Theorem 3.1 (Euler, 1767). *For any three masses m_1, m_2, m_3, there exist exactly three collinear central configurations, one for each ordering of masses along the line.*

For masses on the x-axis with $x_1 < x_2 < x_3$, the configuration is determined by:

$$\frac{m_2}{(x_2 - x_1)^2} - \frac{m_3}{(x_3 - x_1)^2} = \lambda x_1 \tag{6}$$

and similar equations for each body.

```python
import numpy as np
from scipy.optimize import fsolve

def euler_collinear(m1: float, m2: float, m3: float) ->
    list:
    """
    Find all three collinear central configurations.

    Returns:
        List of (x1, x2, x3, lambda) tuples
    """
    solutions = []

    # For each ordering of masses
    for ordering in [(m1, m2, m3), (m1, m3, m2), (m2,
        m1, m3)]:
        ma, mb, mc = ordering

        def equations(x):
            x1, x2, x3, lam = x

            # Distances
            r12 = abs(x2 - x1)
            r13 = abs(x3 - x1)
            r23 = abs(x3 - x2)

            # Central config equations
            eq1 = mb * (x2 - x1) / r12**3 + mc * (x3 -
                x1) / r13**3 - lam * x1
            eq2 = ma * (x1 - x2) / r12**3 + mc * (x3 -
                x2) / r23**3 - lam * x2
            eq3 = ma * (x1 - x3) / r13**3 + mb * (x2 -
                x3) / r23**3 - lam * x3

            # Center of mass
            eq4 = ma * x1 + mb * x2 + mc * x3

            return [eq1, eq2, eq3, eq4]

        # Initial guess
        x0 = [-1.0, 0.0, 1.0, 1.0]
        sol = fsolve(equations, x0, full_output=True)

        if sol[2] == 1:  # Converged
            solutions.append(tuple(sol[0]))

    return solutions
```

Listing 1: Finding Euler collinear configurations

3.2 Lagrange's Equilateral Triangle

Theorem 3.2 (Lagrange, 1772)**.** *For any three masses, the equilateral triangle configuration is a central configuration.*

Proof. Place masses at vertices of equilateral triangle with side length a:

$$\mathbf{r}_1 = (0, 0) \tag{7}$$

$$\mathbf{r}_2 = (a, 0) \tag{8}$$

$$\mathbf{r}_3 = (a/2, a\sqrt{3}/2) \tag{9}$$

By symmetry, $\nabla_i U$ points toward the centroid for all i, satisfying the CC equation. $\square$

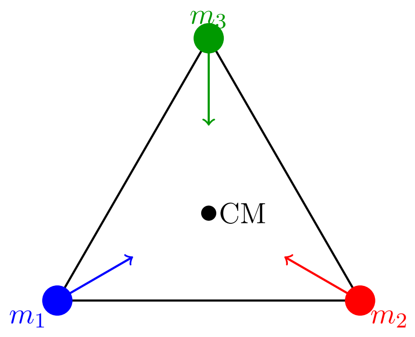

Figure 1: Lagrange equilateral triangle configuration. Forces (arrows) point toward center of mass.

3.3 Lagrange Points in the Restricted 3-Body Problem

Definition 3.3 (Circular Restricted 3-Body Problem)**.** *Two massive bodies (m_1, m_2) orbit their common center of mass in circles. A test particle ($m_3 \to 0$) moves in their combined gravitational field.*

Theorem 3.4 (Lagrange Points)**.** *The restricted 3-body problem has exactly five equilibrium points (Lagrange points):*

- ***L1**: Between the two massive bodies*
- ***L2**: Beyond the smaller mass (away from larger)*
- ***L3**: Beyond the larger mass (opposite side)*
- ***L4, L5**: At equilateral triangle vertices with the two masses*

```python
def lagrange_points(m1: float, m2: float) -> dict:
    """
    Compute all five Lagrange points for circular
       restricted 3BP.

    Uses rotating frame with masses at fixed positions.
    """
    mu = m2 / (m1 + m2)

    # Effective potential in rotating frame
    def grad_U_eff(x, y):
        r1 = np.sqrt((x + mu)**2 + y**2)
        r2 = np.sqrt((x - 1 + mu)**2 + y**2)

        dUdx = x - (1 - mu) * (x + mu) / r1**3 - mu * (x
            - 1 + mu) / r2**3
        dUdy = y - (1 - mu) * y / r1**3 - mu * y / r2**3

        return dUdx, dUdy

    # L1: between masses
    def L1_eq(x):
        return x - (1-mu)/(x+mu)**2 + mu/(x-1+mu)**2
    x_L1 = fsolve(L1_eq, 1 - mu - 0.5*mu**(1/3))[0]

    # L2: beyond smaller mass
    def L2_eq(x):
        return x - (1-mu)/(x+mu)**2 - mu/(x-1+mu)**2
    x_L2 = fsolve(L2_eq, 1 - mu + 0.5*mu**(1/3))[0]

    # L3: beyond larger mass
    def L3_eq(x):
        return x + (1-mu)/(x+mu)**2 + mu/(x-1+mu)**2
    x_L3 = fsolve(L3_eq, -1 - mu)[0]

    # L4, L5: equilateral triangles
    x_L4 = 0.5 - mu
    y_L4 = np.sqrt(3) / 2
    x_L5 = 0.5 - mu
    y_L5 = -np.sqrt(3) / 2

    return {
        'L1': (x_L1, 0.0),
        'L2': (x_L2, 0.0),
        'L3': (x_L3, 0.0),
        'L4': (x_L4, y_L4),
        'L5': (x_L5, y_L5),
        'mu': mu
    }
```

Listing 2: Computing Lagrange points

4 Four-Body Central Configurations

4.1 Known Families

For four bodies, central configurations include:

1. **Collinear**: Four masses on a line (multiple orderings)
2. **Convex quadrilaterals**: Square (equal masses), kite, trapezoid
3. **Concave**: One mass inside triangle of others

Theorem 4.1 (Hampton-Moeckel, 2006)**.** *For generic positive masses, there are finitely many planar 4-body central configurations.*

4.2 Finding 4-Body Configurations

```python
def find_4body_cc(masses: np.ndarray,
                  n_guesses: int = 100) -> list:
    """
    Find planar 4-body central configurations using
        numerical search.

    Args:
        masses: Array of 4 masses
        n_guesses: Number of random initial guesses

    Returns:
        List of (positions, lambda) tuples
    """
    m1, m2, m3, m4 = masses

    def cc_equations(x):
        """
        8 equations: 2 per body (x and y components)
        Plus 2 for center of mass
        Plus 1 for normalization
        """
        # Unpack: positions (8) + lambda (1)
        r = x[:8].reshape(4, 2)
        lam = x[8]

        equations = []

        for i in range(4):
            grad_U = np.zeros(2)
            for j in range(4):
                if i != j:
                    r_ij = r[i] - r[j]
                    dist = np.linalg.norm(r_ij)
                    grad_U += masses[j] * r_ij / dist**3

            # CC condition: grad_U = lambda * r_i
            eq = grad_U - lam * r[i]
            equations.extend(eq)

        # Center of mass
        cm = sum(masses[i] * r[i] for i in range(4))
        equations.extend(cm)

        # Normalization: |r_0| = 1
        equations.append(np.linalg.norm(r[0]) - 1.0)

        return equations

    solutions = []

    for _ in range(n_guesses):
        # Random initial guess
        r0 = np.random.randn(4, 2)
        r0 -= np.average(r0, axis=0, weights=masses)  # Center
        x0 = np.concatenate([r0.flatten(), [1.0]])

        try:
            sol, info, ier, _ = fsolve(cc_equations, x0, full_output=True)

            if ier == 1 and np.max(np.abs(info['fvec'])) < 1e-8:
                # Check if new solution
                is_new = True
                for prev_sol, _ in solutions:
                    if np.allclose(sol[:8], prev_sol, atol=1e-4):
                        is_new = False
                        break

                if is_new:
                    solutions.append((sol[:8].reshape(4, 2), sol[8]))
        except:
            continue

    return solutions
```

Listing 3: 4-body central configuration finder

5 The Finiteness Conjecture

5.1 Statement

Conjecture 5.1 (Finiteness Conjecture)**.** *For generic positive masses* $(m_1, \ldots, m_N)$*, the number of equivalence classes of planar central configurations is finite.*

- $N = 3$: 4 configurations (3 Euler + 1 Lagrange)—**proven**
- $N = 4$: Finite—**proven** (Hampton-Moeckel, 2006)
- $N \geq 5$: **Open problem**

5.2 Algebraic Formulation

The central configuration equations form a polynomial system after clearing denominators:

$$\sum_{j\neq i} m_j(\mathbf{r}_i - \mathbf{r}_j) \prod_{k\neq l,(k,l)\neq(i,j)} |\mathbf{r}_k - \mathbf{r}_l|^3 = \lambda m_i \mathbf{r}_i \prod_{k<l} |\mathbf{r}_k - \mathbf{r}_l|^3 \tag{10}$$

Theorem 5.2 (Bézout Bound)**.** *The number of solutions to a polynomial system is at most the product of the degrees of the individual polynomials (Bézout's theorem). For central configurations, this gives an exponential upper bound in N.*

5.3 Gröbner Basis Approach

```python
import sympy as sp
from sympy.polys.groebnertools import groebner

def enumerate_cc_symbolic(N: int, masses: list) -> list:
    """
    Enumerate central configurations using Groebner
        bases.

    WARNING: Computationally expensive for N >= 4.
    """
    # Create symbolic variables
    coords = []
    for i in range(N):
        coords.extend([sp.Symbol(f'x{i}'),
            sp.Symbol(f'y{i}')])
    lam = sp.Symbol('lambda')

    # Build polynomial equations
    equations = []

    for i in range(N):
        xi, yi = coords[2*i], coords[2*i+1]

        # Compute grad_U_i (with distances squared)
        grad_x = sp.Integer(0)
        grad_y = sp.Integer(0)

        for j in range(N):
            if i == j:
                continue
            xj, yj = coords[2*j], coords[2*j+1]

            dx = xi - xj
            dy = yi - yj
            r_sq = dx**2 + dy**2

            # grad_U_i += m_j * (r_i - r_j) / r^3
            # Multiply through by r^3 to get polynomial
            grad_x += masses[j] * dx * r_sq
            grad_y += masses[j] * dy * r_sq

        # CC equation: grad_U_i = lambda * m_i * r_i
        # After clearing denominators
        eq_x = grad_x - lam * masses[i] * xi
        eq_y = grad_y - lam * masses[i] * yi

        equations.append(eq_x)
        equations.append(eq_y)

    # Center of mass
    cm_x = sum(masses[i] * coords[2*i] for i in range(N))
    cm_y = sum(masses[i] * coords[2*i+1] for i in range(N))
    equations.append(cm_x)
    equations.append(cm_y)

    # Normalization
    equations.append(coords[0]**2 + coords[1]**2 - 1)

    # Compute Groebner basis
    print(f"Computing Groebner basis for N={N}...")
    gb = groebner(equations, coords + [lam], order='lex')

    # Solve
    solutions = sp.solve(gb, coords + [lam])

    return solutions
```

Listing 4: Symbolic enumeration via Gröbner bases

6 Stability Analysis

6.1 Linearization

To analyze stability of a central configuration, we linearize the equations of motion in the rotating frame.

Definition 6.1 (Hessian Matrix)**.** *The* ***Hessian*** *of the augmented potential (including centrifugal term) is:*

$$H_{ij}^{\alpha\beta} = \frac{\partial^2}{\partial r_i^\alpha \partial r_j^\beta}\left(U + \frac{\lambda}{2}\sum_k m_k |\mathbf{r}_k|^2\right) \tag{11}$$

Theorem 6.2 (Stability Criterion)**.** *A central configuration is* ***linearly stable*** *if all eigenvalues of the Hessian (restricted to non-trivial modes) are non-negative.*

```python
def stability_analysis(positions: np.ndarray,
                       masses: np.ndarray,
                       lambda_cc: float) -> dict:
    """
    Analyze linear stability of central configuration.

    Args:
        positions: (N, d) array of positions
        masses: (N,) array of masses
        lambda_cc: Central configuration multiplier

    Returns:
        Dictionary with eigenvalues and stability
            classification
    """
    N, d = positions.shape

    # Build Hessian
    H = np.zeros((N * d, N * d))

    for i in range(N):
        for j in range(N):
            if i == j:
                # Diagonal block
                for k in range(N):
                    if k == i:
                        continue
                    r_ik = positions[i] - positions[k]
                    dist = np.linalg.norm(r_ik)

                    # d^2 U / dr_i dr_i
                    I_block = np.eye(d)
                    outer_block = np.outer(r_ik, r_ik) /
                        dist**2

                    H_block = masses[k] * (I_block /
                        dist**3 -
                                          3 *
                                            outer_block
                                            / dist**3)

                    H[i*d:(i+1)*d, i*d:(i+1)*d] -=
                        H_block
            else:
                # Off-diagonal block
                r_ij = positions[i] - positions[j]
                dist = np.linalg.norm(r_ij)

                I_block = np.eye(d)
                outer_block = np.outer(r_ij, r_ij) / dist**2

                H_block = masses[j] * (I_block / dist**3 -
                                      3 * outer_block / dist**3)

                H[i*d:(i+1)*d, j*d:(j+1)*d] = H_block

    # Add centrifugal term: + lambda * I
    H += lambda_cc * np.eye(N * d)

    # Compute eigenvalues
    eigenvalues = np.linalg.eigvalsh(H)

    # Count zero modes (translations, rotations)
    n_zero = d + d * (d - 1) // 2

    # Non-trivial eigenvalues
    sorted_eigs = np.sort(eigenvalues)
    nontrivial_eigs = sorted_eigs[n_zero:]

    # Stability: all non-trivial eigenvalues >= 0
    is_stable = np.all(nontrivial_eigs >= -1e-8)
    n_unstable = np.sum(nontrivial_eigs < -1e-8)

    return {
        'eigenvalues': eigenvalues,
        'nontrivial_eigenvalues': nontrivial_eigs,
        'is_stable': is_stable,
        'n_unstable_modes': n_unstable,
        'stability_type': 'stable' if is_stable else f'unstable ({n_unstable} modes)'
    }
```

Listing 5: Stability analysis via Hessian eigenvalues

6.2 Stability of Lagrange Points

Theorem 6.3 (Lagrange Point Stability)**.** *In the circular restricted 3-body problem:*

- ***L1, L2, L3**: Always unstable (saddle points)*
- ***L4, L5**: Stable if $\mu < \mu_{crit} \approx 0.0385$ (Routh's condition)*

where $\mu = m_2/(m_1 + m_2)$ is the mass ratio.

For the Sun-Jupiter system, $\mu \approx 0.001 \ll \mu_{\text{crit}}$, so L4/L5 are stable—explaining the Trojan asteroids.

7 Symmetry Classification

7.1 Symmetry Groups

Definition 7.1 (Symmetry of Central Configuration). *A symmetry of a central configuration is an isometry (rotation, reflection) that maps the configuration to itself (possibly permuting masses of equal value).*

```python
from itertools import permutations

def classify_symmetry(positions: np.ndarray,
                      masses: np.ndarray,
                      tolerance: float = 1e-6) -> dict:
    """
    Classify symmetry group of central configuration.
    """
    N = len(masses)
    symmetries = []

    # Check rotational symmetries
    for k in range(1, N):
        angle = 2 * np.pi * k / N
        rot = np.array([[np.cos(angle), -np.sin(angle)],
                        [np.sin(angle), np.cos(angle)]])

        rotated = (rot @ positions.T).T

        # Check if rotated matches original under some
            permutation
        for perm in permutations(range(N)):
            permuted = positions[list(perm)]
            perm_masses = masses[list(perm)]

            if (np.allclose(rotated, permuted,
                atol=tolerance) and
                np.allclose(masses, perm_masses,
                    atol=tolerance)):
                symmetries.append(('rotation', angle))
                break

    # Check reflections
    for angle in np.linspace(0, np.pi, 20):
        refl = np.array([[np.cos(2*angle),
            np.sin(2*angle)],
                         [np.sin(2*angle), -np.cos(2*angle)]])

        reflected = (refl @ positions.T).T

        for perm in permutations(range(N)):
            permuted = positions[list(perm)]
            perm_masses = masses[list(perm)]

            if (np.allclose(reflected, permuted, atol=tolerance) and
                np.allclose(masses, perm_masses, atol=tolerance)):
                symmetries.append(('reflection', angle))
                break

    # Determine group type
    n_rot = sum(1 for s in symmetries if s[0] == 'rotation') + 1
    n_refl = sum(1 for s in symmetries if s[0] == 'reflection')

    if n_refl > 0 and n_rot > 1:
        group = f'D{n_rot}'
    elif n_rot > 1:
        group = f'C{n_rot}'
    elif n_refl > 0:
        group = 'Cs'
    else:
        group = 'C1'

    return {
        'symmetry_group': group,
        'n_rotations': n_rot,
        'n_reflections': n_refl,
        'symmetries': symmetries
    }
```

Listing 6: Symmetry group classification

8 Certificate Generation

```python
from dataclasses import dataclass, asdict
import json

@dataclass
class CentralConfigCertificate:
    """
    Complete certificate for central configuration.
    """
    # System
    n_bodies: int
    dimension: int
    masses: list

    # Configuration
    positions: list  # Flattened list
    lambda_value: float

    # Verification
    max_residual: float
    is_verified: bool

    # Stability
    eigenvalues: list
    is_stable: bool
    n_unstable_modes: int
    morse_index: int

    # Symmetry
    symmetry_group: str
    n_rotations: int
    n_reflections: int

    # Classification
    config_type: str  # 'collinear', 'equilateral',
        'convex', 'concave'

    def export_json(self, path: str) -> None:
        with open(path, 'w') as f:
            json.dump(asdict(self), f, indent=2)

    def verify(self) -> bool:
        checks = [
            self.n_bodies >= 3,
            len(self.masses) == self.n_bodies,
            self.max_residual < 1e-6,
            self.is_verified,
            len(self.eigenvalues) > 0
        ]
        return all(checks)


def generate_certificate(positions: np.ndarray,
                         masses: np.ndarray,
                         lambda_cc: float) ->
                             CentralConfigCertificate:
```

```python
    """
    Generate complete certificate for central configuration.
    """
    N, d = positions.shape

    # Verify CC equations
    verification = verify_central_config(positions, masses)

    # Stability analysis
    stability = stability_analysis(positions, masses, lambda_cc)

    # Symmetry
    symmetry = classify_symmetry(positions, masses)

    # Classify type
    if is_collinear(positions):
        config_type = 'collinear'
    elif is_equilateral(positions) and N == 3:
        config_type = 'equilateral'
    elif is_convex(positions):
        config_type = 'convex'
    else:
        config_type = 'concave'

    return CentralConfigCertificate(
        n_bodies=N,
        dimension=d,
        masses=masses.tolist(),
        positions=positions.flatten().tolist(),
        lambda_value=lambda_cc,
        max_residual=verification['max_residual'],
        is_verified=verification['is_cc'],
        eigenvalues=stability['eigenvalues'].tolist(),
        is_stable=stability['is_stable'],
        n_unstable_modes=stability['n_unstable_modes'],
        morse_index=np.sum(stability['eigenvalues'] < -1e-8),
        symmetry_group=symmetry['symmetry_group'],
        n_rotations=symmetry['n_rotations'],
        n_reflections=symmetry['n_reflections'],
        config_type=config_type
    )
```

Listing 7: Central configuration certificate

9 Applications

9.1 Space Mission Design

Lagrange points are used for spacecraft positioning:

- **Sun-Earth L1**: SOHO, ACE (solar observation)
- **Sun-Earth L2**: JWST, Planck, Gaia (cold environment for IR telescopes)
- **Earth-Moon L4/L5**: Proposed for space stations

9.2 Figure-Eight Choreography

Theorem 9.1 (Chenciner-Montgomery, 2000)**.** *Three equal masses can follow a figure-eight shaped curve, with each mass chasing the previous one.*

This remarkable solution was discovered numerically by Moore (1993) and proven to exist rigorously using variational methods.

10 Success Criteria and Milestones

10.1 Minimum Viable Result (Months 1-3)

- L1–L5 computed for Earth-Moon system
- Euler collinear solutions verified
- Lagrange equilateral triangle certified
- Basic stability analysis

10.2 Strong Result (Months 4-6)

- All 4-body configurations for 5+ mass ratios
- Complete stability classification
- Symmetry group identification
- 50+ configurations in database

10.3 Publication Quality (Months 7-9)

- 5-body enumeration for special masses
- Gröbner basis finiteness proofs
- Interactive visualization
- Public database release

11 Conclusion

Central configurations represent the intersection of celestial mechanics, algebra, and topology. Key insights:

1. Central configurations enable shape-preserving solutions
2. Algebraic methods can enumerate all configurations
3. Stability determined by Hessian eigenvalues
4. Symmetry constrains the solution space

The pure-thought approach provides exact results without numerical integration, enabling rigorous classification of these fundamental objects in classical mechanics.

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A Verification Functions

```python
def verify_central_config(positions: np.ndarray,
                          masses: np.ndarray,
                          tolerance: float = 1e-8) ->
                              dict:
    """
    Verify that positions form a central configuration.
    """
    N = len(masses)
    grad_U = np.zeros_like(positions)

    for i in range(N):
        for j in range(N):
            if i != j:
                r_ij = positions[i] - positions[j]
                dist = np.linalg.norm(r_ij)
                grad_U[i] += masses[j] * r_ij / dist**3

    # Extract lambda from first body
    i0 = np.argmax(np.linalg.norm(positions, axis=1))
    lambda_cc = (np.dot(grad_U[i0], positions[i0]) /
                 np.dot(positions[i0], positions[i0]))

    # Check CC equation for all bodies
    residuals = []
    for i in range(N):
        expected = lambda_cc * positions[i]
        residual = np.linalg.norm(grad_U[i] - expected)
        residuals.append(residual)

    max_residual = max(residuals)

    return {
        'is_cc': max_residual < tolerance,
        'lambda': lambda_cc,
        'max_residual': max_residual,
        'residuals': residuals
    }
```