QSAT sub-problem is BQP$_1$-complete

Ref. [1] attempted to show that there is a Quantum SAT problem that represents the full power of one-side error quantum computation, leading to the first natural problem that is complete for the class BQP$_1$. Here, we point out a couple of his oversights and how we deal with them successfully, as well as some redundancies that we improve which further strengthen the result. Our contributions are marked with ★.

While the circuit-to-Hamiltonian construction is commonly used to show negative results about estimating the ground state energy of local Hamiltonians with a quantum computer, here we do the contrary. We show that when the constraints of a QSAT problem are given by these Hamiltonians, there is a quantum algorithm that computes the answer with certainty. Finally, we think that our construction can be modified to yield similar QSAT problems that are complete for other quantum and classical complexity classes.

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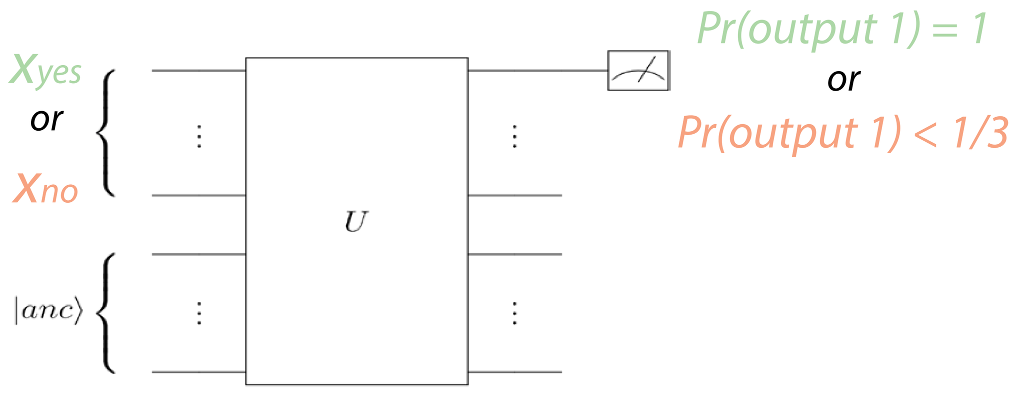
1 One-sided error?

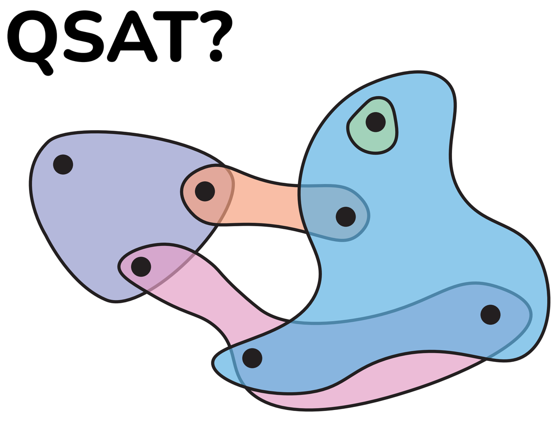
- Decision problem: decide if input x is x_{yes} or x_{no}.



BQP$_1$: Answer in $poly(n)$ time.

QMA$_1$: Verifiy a certificate in $poly(n)$ time.

2 k-QSAT?



Vertices: Qudits.

Hyperedges: Projectors Π_i acting on at most k qudits.

"Is there a global state of the qudits $|\psi\rangle$ such that $\Pi_i|\psi\rangle = 0$ for all i or $\sum_i \langle\psi|\Pi_i|\psi\rangle \geq 1/poly(n)$?"

3 Circuit-to-Hamiltonian?

- A circuit $U = U_T U_{T-1} \dots U_1$ that decides a problem can be encoded into $H = H_{init} + H_{out} + \sum_{i=1}^{T} H_{prop,i}$ acting on the space $\mathbb{C}_{logical} \otimes \mathbb{C}_{clock}$.

$H_{init} \equiv$ Initialize the logical qubits to $|0\dots0\rangle$.

$H_{prop,i} \equiv$ Apply U_i; increase clock. Apply $U_i^\dagger$; decrease clock.

$H_{out} \equiv$ At the end, answer qubit is $|1\rangle$.

If U accepts perfectly, the g.s. with 0 energy is: $|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} U_t \dots U_1 |0\dots0\rangle \otimes |t\rangle$

- H_{clock} can be added to use a local encoding of the clock, e.g. $|11110\dots0\rangle$. $t=4$
- Estimating the ground state energy of 2-local Hamiltonians is QMA-complete [2]. k-QSAT with $k > 2$ is QMA$_1$-complete [3],[4].

Our 4-QSAT problem

I. Gates from the universal set $\mathcal{G} = \{H, HT, (H \otimes H)CNOT\}$.

II. Qudits with logical component $\{|0\rangle, |1\rangle, |U\rangle\}$ and clock component $\{|u\rangle, |a_1\rangle, |a_2\rangle, |d\rangle\}$. ★

Clock evolves as:
$$\begin{array}{c} |a_1, u, u, \dots, u\rangle \\ |a_2, u, u, \dots, u\rangle \\ |d, a_1, u, \dots, u\rangle \\ \vdots \\ |d, d, \dots, d, a_2\rangle \end{array}$$

III. Any following projectors:

★ $H'_{prop,\mathcal{G}} \equiv H'_{clock} + \begin{cases} |a_2\rangle \to |d\rangle\text{; next clock qudit } |u\rangle \to |a_1\rangle. \\ |a_1\rangle \to |a_2\rangle\text{; apply } U. \text{ If } U_i \text{ acts on } |U\rangle\text{, stop.} \end{cases}$

$H'_{init} \equiv$ If $|u\rangle$, initialize to $|0\rangle$.

$H'_{out} \equiv$ If $|a_2\rangle$, logical qudit is $|1\rangle$.

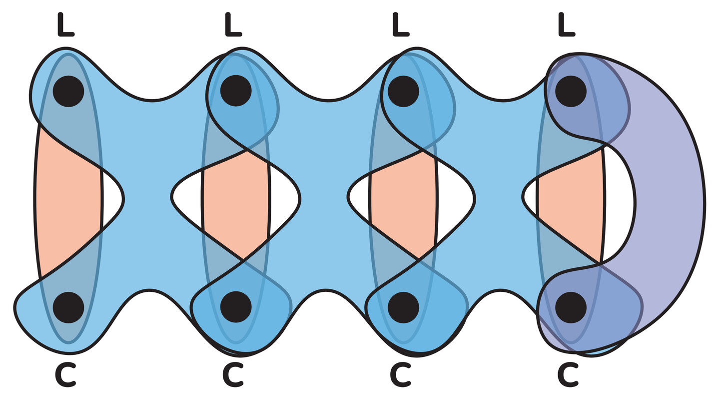
4 BQP$_1$-hard

Not so "hard"! . . . We have control over the QSAT instance setup

If $U = U_T U_{T-1} \dots U_1$ with $U_i \in \mathcal{G}$ decides a problem in BQP$_1$, create the instance where:

- Logical qudits are properly initialized by relevant clock qudits.
- There is a linear timeline and at time t, U_t is applied.
- At the end of time the answer qubit is measured.

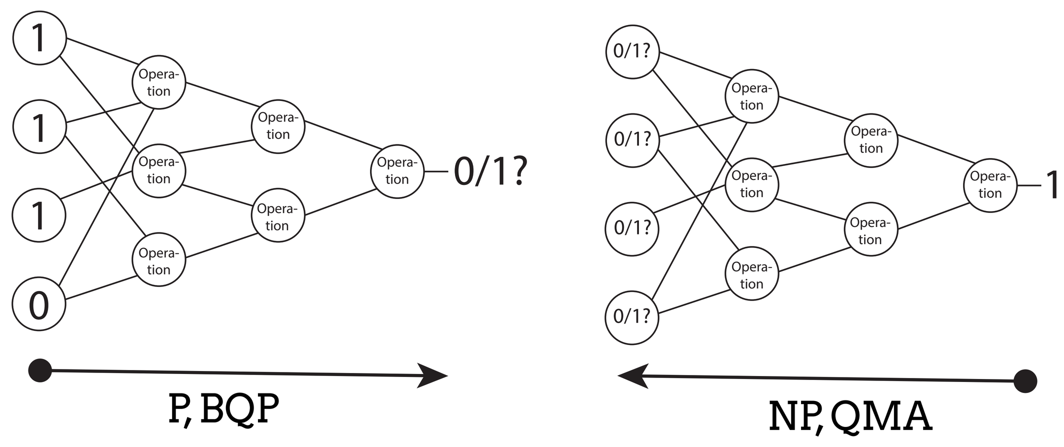
Is $U(x) = x_{yes}$?
$\Updownarrow$
Is the QSAT instance satisfiable?



5 In BQP$_1$

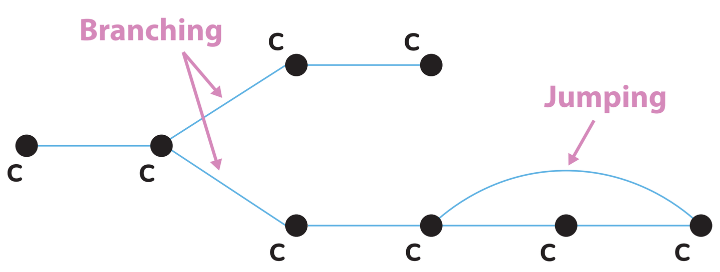
Quite troublesome! . . . Many wacky instances to consider

Non-initialized qudits

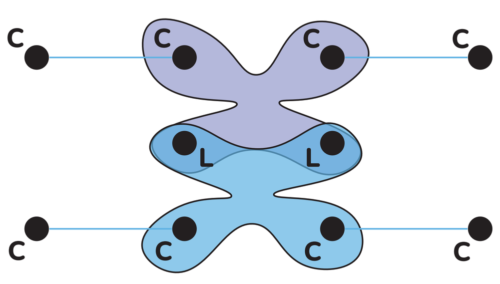


- No guarantee that all logical qudits are initialized to $|0\rangle$.
- Assume that these are *undefined* $|U\rangle$ making all outputs undefined too. Undefined answer is always accepted. ✓

★ *Clock-path acrobatics*

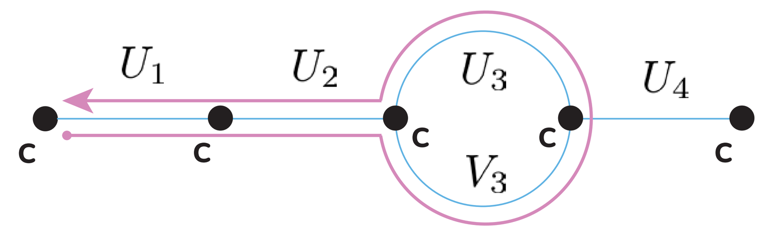


- Branching, jumping and cycles infringe H'_{clock}. ✗



- Separate clock paths acting on mutual logical qudits creates frustration (I). ?

★ *Simultaneous gates* ?



- History state has computational path $U_1^\dagger U_2^\dagger U_3^\dagger V_3 U_2 U_1 |0\dots0\rangle_L \otimes |0\rangle_C$.
- If V_3 and U_3 act differently on $U_2 U_1 |0\dots0\rangle$ then the state above infringes H'_{init}.
- Wrong paths with small amplitudes are hard to detect. The promise allows us to ignore these instances.

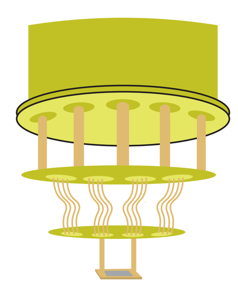
Other

- Multiple measurements. ✓
- No measurements. ✓
- Qudit is labeled both **L** and **C**. ✗

Algorithm:

Accept trivial instances. ✓

Reject instances with the wrong form. ✗



?

For every clock path, execute the specified gates and measure the required logical qudits. If a clock path has simultaneous gates, choose any arbitrary path. Accept if all clock paths can be satisfied.

[1].Meiburg A. Quantum Constraint Problems can be complete for BQP, QCMA, and more. arXiv preprint arXiv:2101.08381. 2021 Jan 21.
[2]. Kitaev AY, Shen A, Vyalyi MN. Classical and quantum computation. American Mathematical Soc.; 2002.
[3].Bravyi S. Efficient algorithm for a quantum analogue of 2-SAT. Contemporary Mathematics. 2011 Feb 14;536:33-48.
[4].Gosset D, Nagaj D. Quantum 3-SAT is QMA _1-complete. SIAM Journal on Computing. 2016;45(3):1080-128.