

Scale-Free Networks

Topics

- Scale-Free Networks
- Power Laws
- Hubs
- Universality
- Ultra-Small World

Scale-Free Networks

In scale-free networks, a few highly connected hubs coexist with a large number of small nodes.

Most real life networks are scale-free networks:

- WWW
- Internet
- Actor network
- Social network
- Airline Hubs

The presence of these hubs plays an important role in the system's behavior.

In random networks highly connected hubs are rare (if any).

WWW



Power Laws

Scale-Free network is a network whose degree distribution follows a power law distribution.

- Example: Airline networks
- $P_k \sim k^{-\gamma}$
- Here γ is a constant

Random networks follow a Poisson (or Binomial) distribution.

- Example: Highway networks

If a network is directed, the scale-free properties applies separately to the in- and the out-degrees.

80/20 Rule

Vilfredo Pareto, a 19th century economist, noticed that in Italy, a few wealthy individuals earned most of the money, while the majority of the population earned rather small amounts.

Pareto Rule or 80/20 is also a Power Law, suggesting:

- Roughly 80% of all money is earned by only 20% of the population.
- Roughly 80% of the links on the Web point to only 15% of the web pages.
- 80% of links in Hollywood are connected to 30% of actors.

Most quantities following a power-law distribution obey the 80/20 rule (explains hubs).



Hubs

Power Law is above Poisson for low k (a).

- Scale-Free network has a large number of small-degree nodes.

For k in the vicinity of $\langle k \rangle$ the Poisson is above Power Law (a).

- Poisson has numerous nodes with degrees $k \approx \langle k \rangle$

For very large k Power law is above Poisson (b).

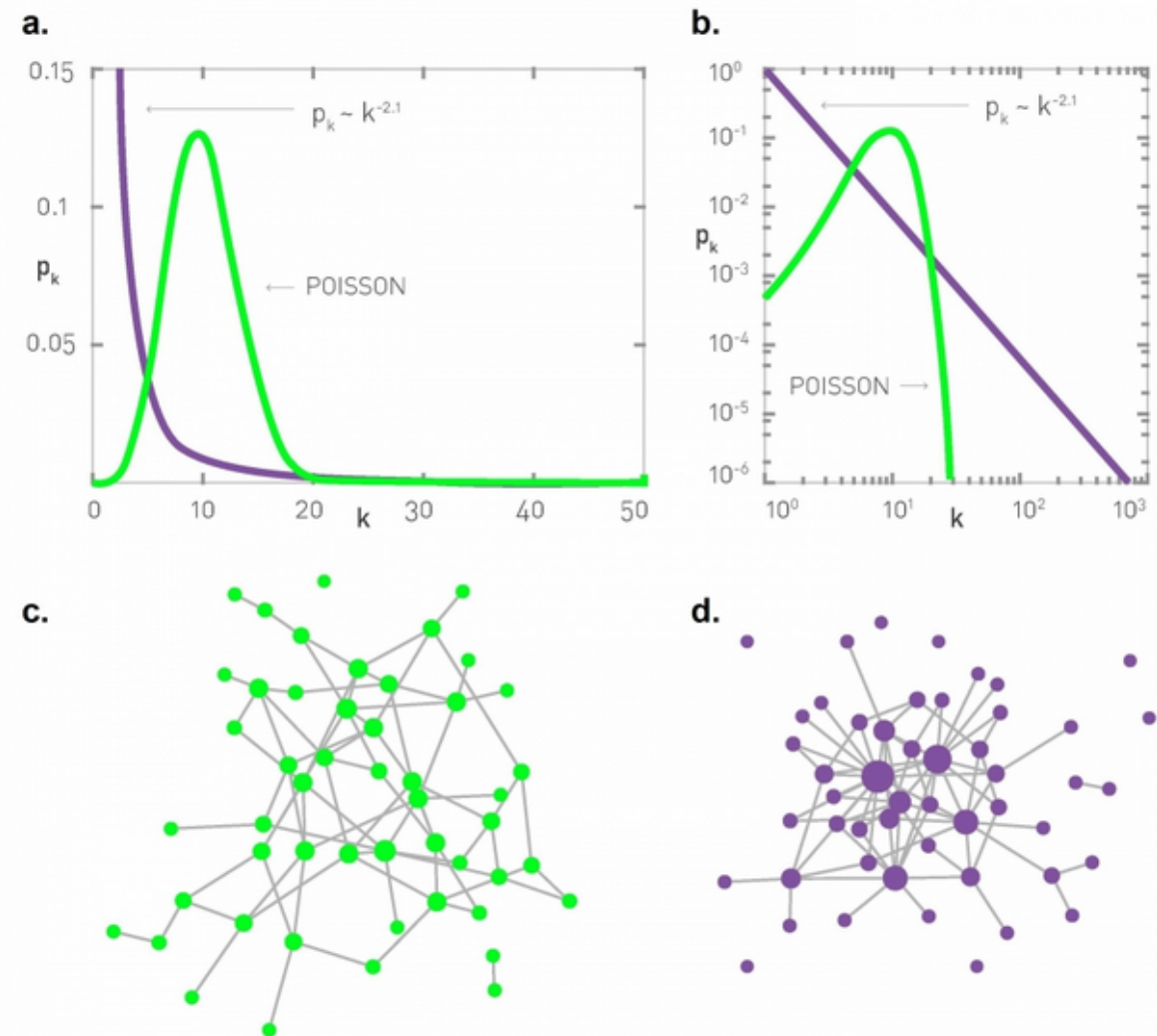
- X axis is a log-log plot.
- Probability of observing hubs (high degree nodes) is higher in scale-free networks.

Random network (c).

- Most nodes have equal number of degrees around $\langle k \rangle$

Scale-Free network (d).

- Numerous small-degree nodes coexist with a few highly connected hubs.



Highways vs. Air-Traffic

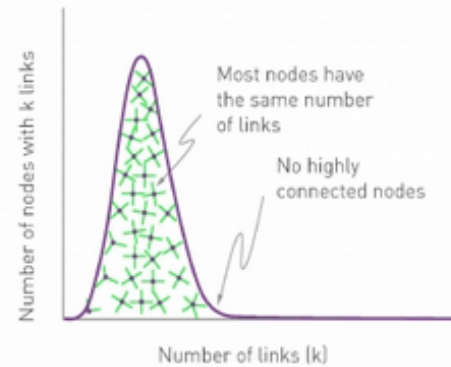
Random network looks more like a highway network.

- Boston to Green Bay via Albany, Buffalo, Cleveland, Toledo, Chicago, Milwaukee etc.

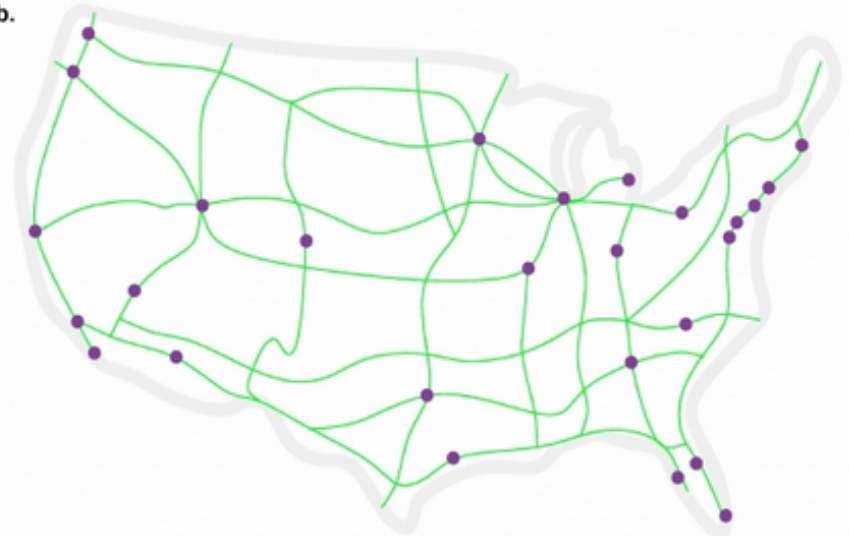
Scale-Free network looks more like air-traffic network.

- Lowers number of intermediary connecting nodes.
- Boston to Green Bay via Chicago.

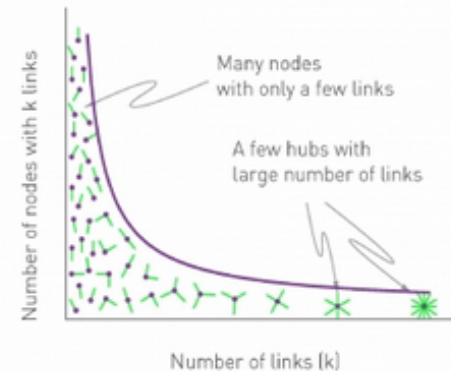
a. POISSON



b.



c. POWER LAW



d.



Power Law $P_k \sim k^{-\gamma}$

For many scale-free networks degree exponent is between 2 and 3.

$$P_k \sim k^{-\gamma}$$

Ten Reference Networks

Network	N	L	$\langle k \rangle$	$\langle k_{in}^2 \rangle$	$\langle k_{out}^2 \rangle$	$\langle k^2 \rangle$	γ_{in}	γ_{out}	γ
Internet	192,244	609,066	6.34	-	-	240.1	-	-	3.42*
WWW	325,729	1,497,134	4.60	1546.0	482.4	-	2.00	2.31	-
Power Grid	4,941	6,594	2.67	-	-	10.3	-	-	Exp.
Mobile-Phone Calls	36,595	91,826	2.51	12.0	11.7	-	4.69*	5.01*	-
Email	57,194	103,731	1.81	94.7	1163.9	-	3.43*	2.03*	-
Science Collaboration	23,133	93,437	8.08	-	-	178.2	-	-	3.35*
Actor Network	702,388	29,397,908	83.71	-	-	47,353.7	-	-	2.12*
Citation Network	449,673	4,689,479	10.43	971.5	198.8	-	3.03*	4.00*	-
E. Coli Metabolism	1,039	5,802	5.58	535.7	396.7	-	2.43*	2.90*	-
Protein Interactions	2,018	2,930	2.90	-	-	32.3	-	-	2.89*-

Scale-Free networks have relatively higher standard deviations for any given $\langle k \rangle$

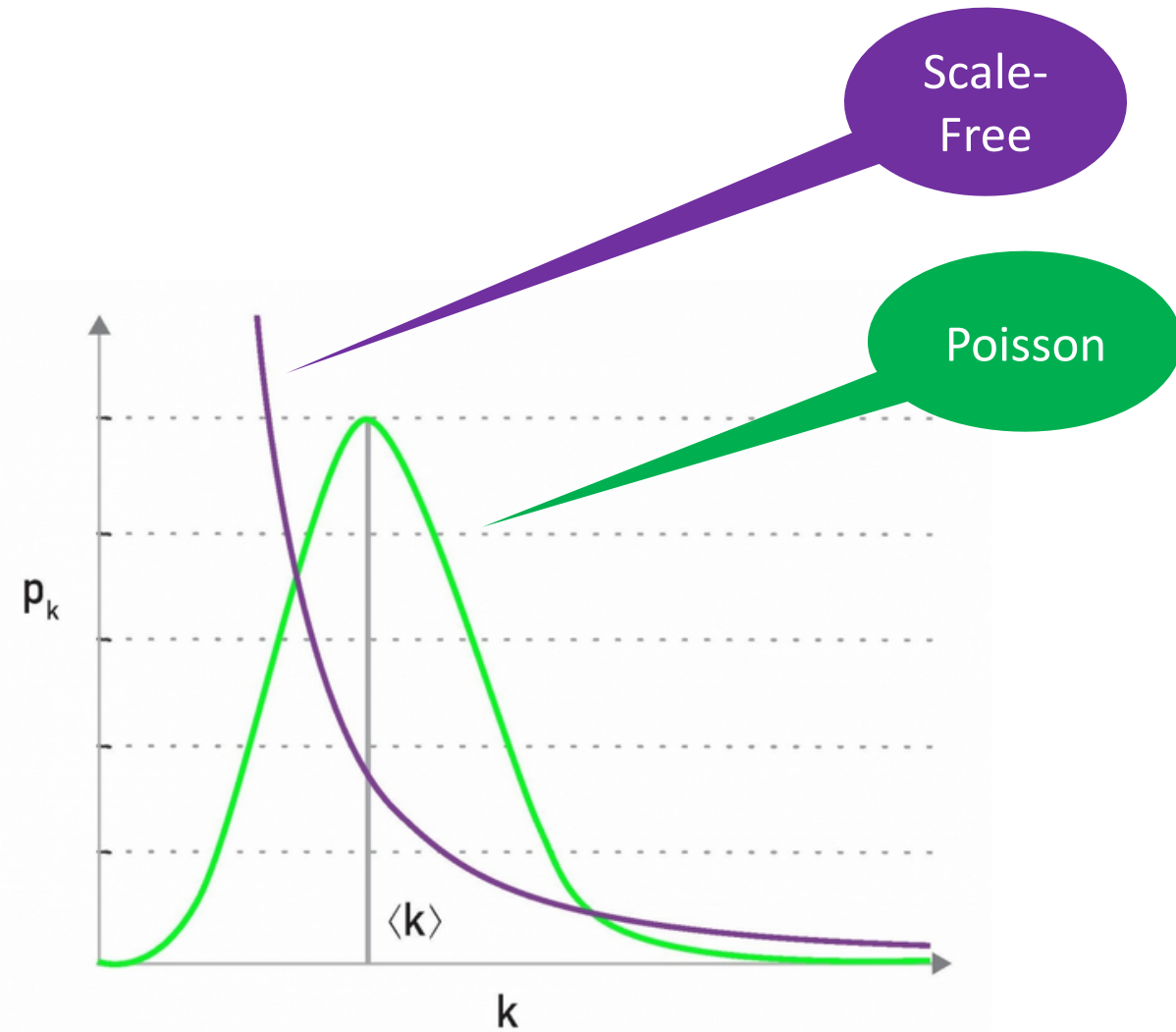
Power grids have relatively small standard deviation hence it does not qualify as a scale-free network.

Phone network is a scale-free network, but since it has high γ it is well approximated by a random graph.

Scale-Free v. Random

Random networks are approximated by Poisson who have very high probability of having degree of a randomly chosen node in the vicinity of $\langle k \rangle$. Hence $\langle k \rangle$ serves as the network's scale.

Power law distributions can have a degree of a randomly selected node significantly different from $\langle k \rangle$. Hence, $\langle k \rangle$ does not serve as an intrinsic scale. Hence, they are called scale-free.



Universality

Many real networks are scale-free:

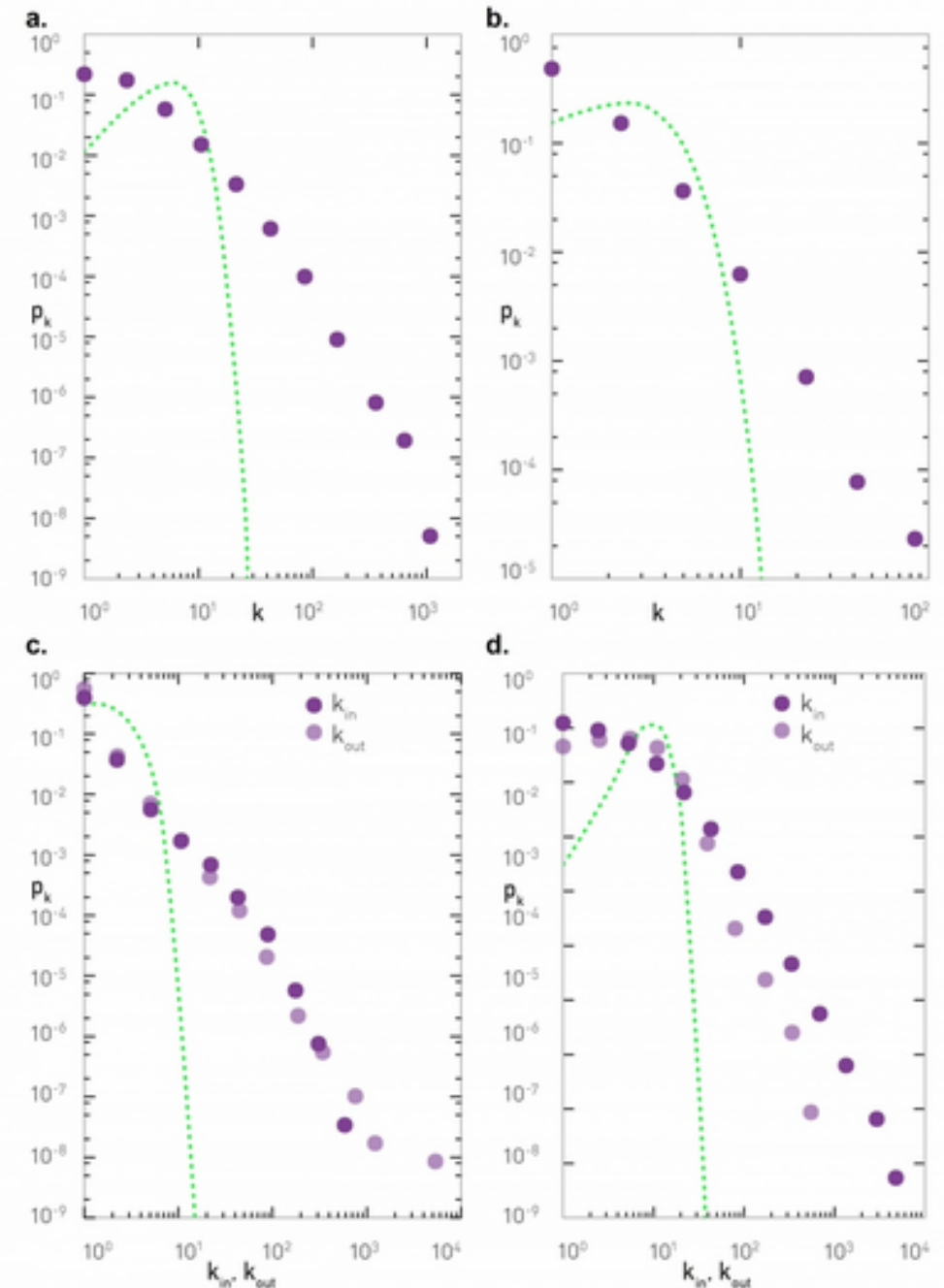
- a. Internet
- b. Protein-protein interaction
- c. Email network
- d. Citation network

X-axis is double logarithmic scale, called a log-log plot.

Green dotted lines shows Poisson distribution with same $\langle k \rangle$, showing real networks are not Poisson.

Degree exponent can be estimated by plotting fitting a straight line to p_k on the log-log plot.

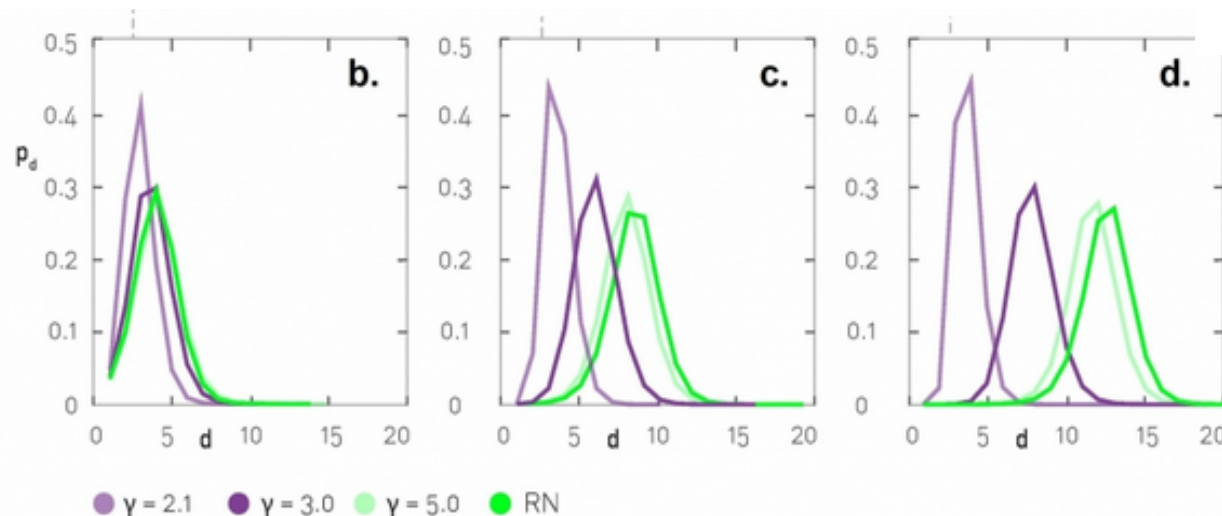
Not all real networks are scale-free e.g. power grid network, and molecular network in materials such as water molecules.



Ultra-Small World Property

Hubs in scale-free networks decrease the number of hops needed to connect two nodes.

Distances in scale-free ($\sim \ln \ln N$) are smaller than the distances observed in an equivalent random network ($\sim \ln N$).



$$\langle d \rangle \sim \begin{cases} \text{const.} \\ \ln \ln N \\ \frac{\ln N}{\ln \ln N} \\ \ln N \end{cases}$$

$$2 < \gamma < 3$$

$$\gamma > 3$$

Scale-Free

Random
(small
world)

γ values between 2 and 3 have higher probability of smaller d (ultra small world).

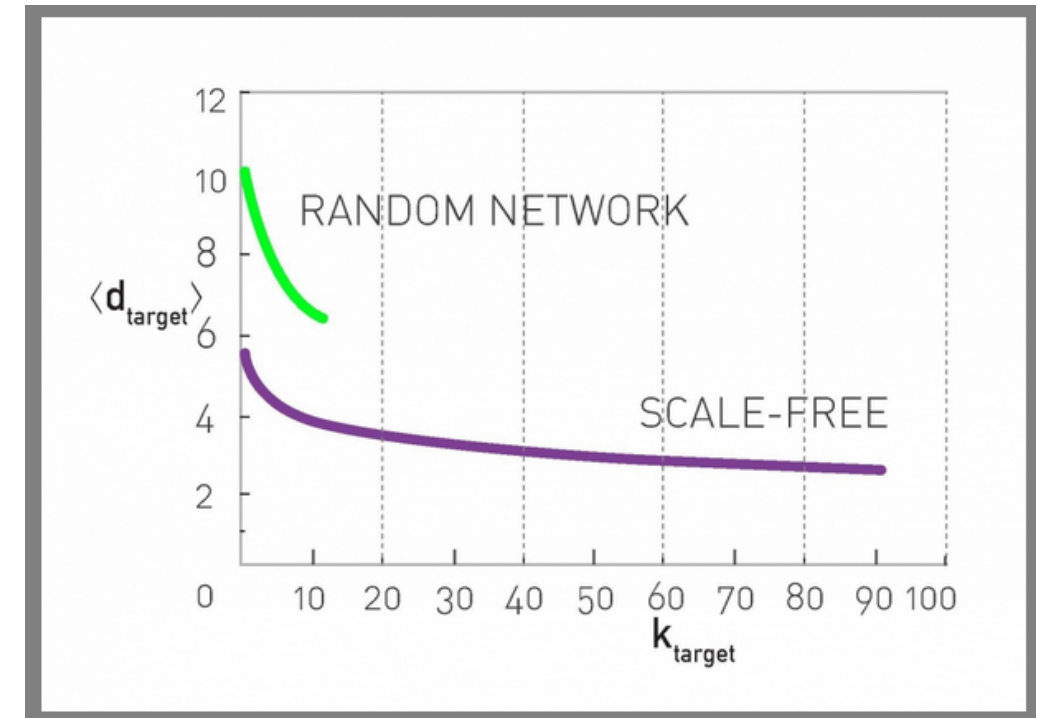
We are Always Closer to the Hubs

“Its always easier to find someone who knows a famous or popular figure than some run-of-the-mill, insignificant person.” – Frigyes Karinthy (1929)

There are always short paths linking us to famous individuals.

Scale-Free portrays this better than Random networks.

The smaller is the γ , shorter are the distances between two nodes.





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