Solved Examples for Chapter 3

Example for Section 3.1

The Apex Television Company has to decide on the number of 60- and 42-inch sets to be produced at one of its factories. Market research indicates that at most 40 of the 60-inch sets and 10 of the 42-inch sets can be sold per month. The maximum number of work-hours available is 500 per month. A 60-inch set requires 20 work-hours and a 42-inch set requires 10 work-hours. Each 60-inch set sold produces a profit of $120 and each 42-inch set produces a profit of $80. A wholesaler has agreed to purchase all the television sets produced if the numbers do not exceed the maxima indicated by the market research.

(a) Formulate a linear programming model for this problem.

The decisions that need to be made are the number of 60-inch and 42-inch TV sets to be produced per month by the Apex Television Company. Therefore, the decision variables for the model are

x_1 = number of 60-inch TV sets to be produced per month,
x_2 = number of 42-inch TV sets to be produced per month.

Also let

Z = total profit per month.

The model now can be formulated in terms of these variables as follows.

The total profit per month is $Z = 120\,x_1 + 80\,x_2$.

The resource constraints are:

(1) Number of 60-inch sets sold per month: $x_1 \le 40$
(2) Number of 42-inch sets sold per month: $x_2 \le 10$
(3) Work-hours availability: $20\,x_1 + 10\,x_2 \le 500$.

Nonnegativity constraints on TV sets produced:

$x_1 \ge 0$
$x_2 \ge 0$

With the objective of maximizing the total profit per month, the LP model for this problem is

Maximize $Z = 120\,x_1 + 80\,x_2$,
subject to

$$x_1 \qquad\qquad \leq \;\; 40$$
$$x_2 \;\; \leq \;\; 10$$
$$20\,x_1 + 10\,x_2 \;\; \leq \; 500$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

(b) Use the graphical method to solve this model.

The constraint, $x_1 \leq 40$, has a constraint boundary at $x_1 = 40$. Similarly, the constraint, $x_2 \leq 10$, has a constraint boundary at $x_2 = 10$.
      The constraint boundary for the constraint, $20\,x_1 + 10\,x_2 \leq 500$, intercepts the x_1-axis at $20\,x_1 + 10(0) = 500$, so at $x_1 = 25$. Similarly, this constraint boundary intercepts the x_2-axis at $20(0) + 10\,x_2 = 500$, so at $x_2 = 50$. This constraint boundary lies well within the constraint boundary for the first constraint, $x_1 \leq 40$, so the first constraint is redundant and can be ignored.
      For a sample value of Z, say, $Z = 2{,}400$, the corresponding objective function line, $Z = 2{,}400 = 120\,x_1 + 80\,x_2$, intercepts the x_1-axis at $120\,x_1 + 80(0) = 2{,}400$, so at $x_1 = 20$. It intercepts the x_2-axis at $120\,(0) + 80\,x_2 = 2{,}400$, so at $x_2 = 30$. The corresponding objective function lines for other values of Z are parallel to this line. Pushing these lines up as much as possible while still passing through a point in the feasible region reveals that the optimal solution is $(x_1, x_2) = (20, 10)$ with $Z = 3{,}200$, as depicted in the following graph.

$$Z = 120 x_1 + 80 x_2 = 3200$$
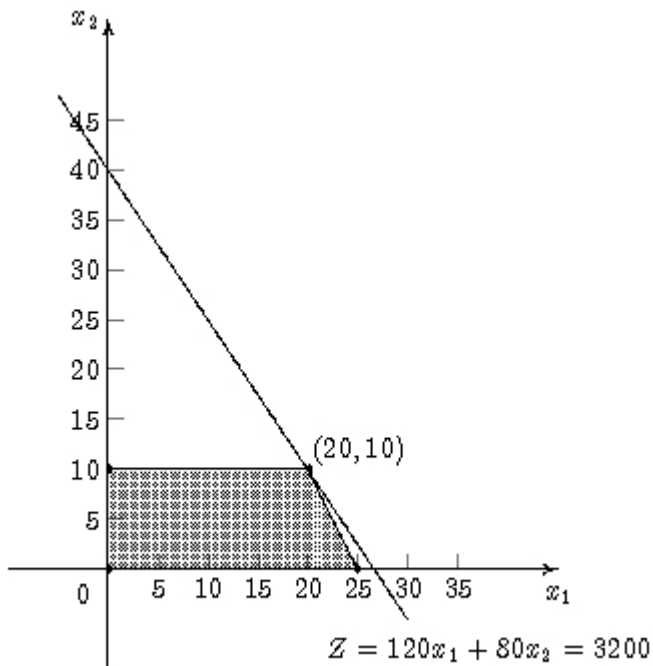
Example for Section 3.4

Dwight is an elementary school teacher who also raises pigs for supplemental income. He is trying to decide what to feed his pigs. He is considering using a combination of pig feeds available from local suppliers. He would like to feed the pigs at minimum cost while also making sure each pig receives an adequate supply of calories and vitamins. The cost, calorie content, and vitamin content of each feed are given in the table below.

Contents	Feed Type A	Feed Type B
Calories (per pound)	800	1,000
Vitamins (per pound)	140 units	70 units
Cost (per pound)	$0.40	$0.80

Each pig requires at least 8,000 calories per day and at least 700 units of vitamins. A further constraint is that no more than one-third of the diet (by weight) can consist of Feed Type A, since it contains an ingredient which is toxic if consumed in too large a quantity.

(a) Formulate a linear programming model for this problem.

Let A and B be the quantity (pounds) of Feed Type A and Feed Type B, respectively, used per day. Also let Z be the total daily cost of the feed per pig. Then, the daily cost is

$Z = \$0.4\,A + \$0.8\,B.$

The constraints on the minimum daily requirements of calories and vitamins are

(1) Calories requirement: $800\,A + 1000\,B \geq 8{,}000.$
(2) Vitamins requirement: $140\,A + 70\,B \geq 700.$

Also, Dwight needs to avoid using too much of Feed Type A because of the toxic ingredient in it. The toxic constraint is
$$A \leq 1/3\ (A+B),$$
which reduces to $2/3\,A - 1/3\,B \leq 0.$

Nonnegativity constraints: $A \geq 0, \quad B \geq 0.$

The resulting linear programming model for this problem is
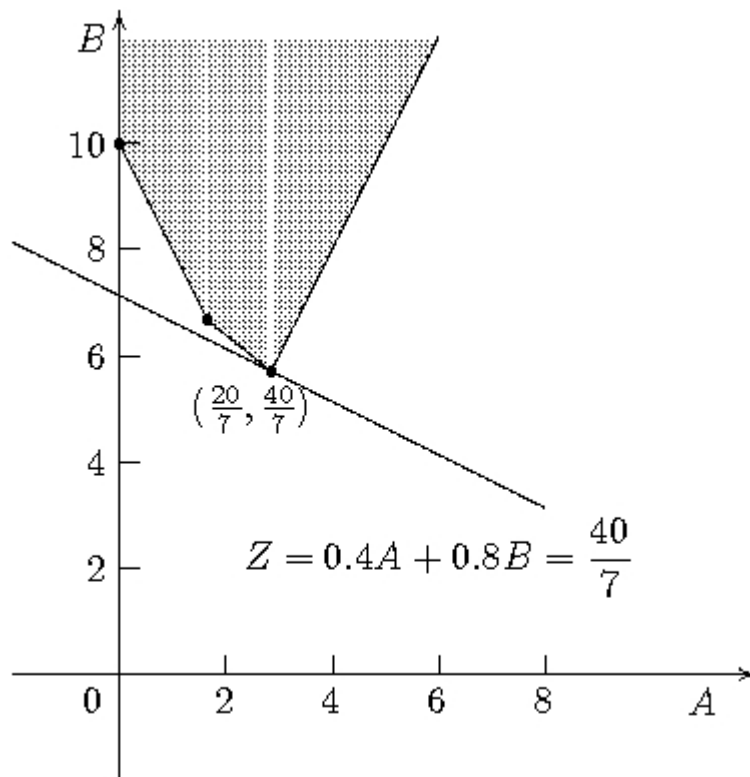
Minimize $Z = 0.4\,A + \$0.8\,B,$
subject to

$$800\,A + 1000\,B \geq 8000$$
$$140\,A + 70\,B \geq 700$$
$$2/3\,A - 1/3\,B \leq 0$$

and

$$A \geq 0, \quad B \geq 0.$$

(b) Use the graphical method to solve this model. What is the resulting daily cost per pig?

As shown below, the optimal solution is $(A, B) = (20/7, 40/7)$. The resulting daily cost per pig is $Z = 40/7 = \$5.71$.

Another Example for Section 3.4

The Fagersta Steelworks currently is working two mines to obtain its iron ore. This iron ore is shipped to either of two storage facilities. When needed, it then is shipped on to the company's steel plant. The diagram below depicts this distribution network, where M1 and M2 are the two mines, S1 and S2 are the two storage facilities, and P is the steel plant. The diagram also shows the monthly amounts produced at the mines and needed at the plant, as well as the shipping cost and the maximum amount that can be shipped per month through each shipping lane.
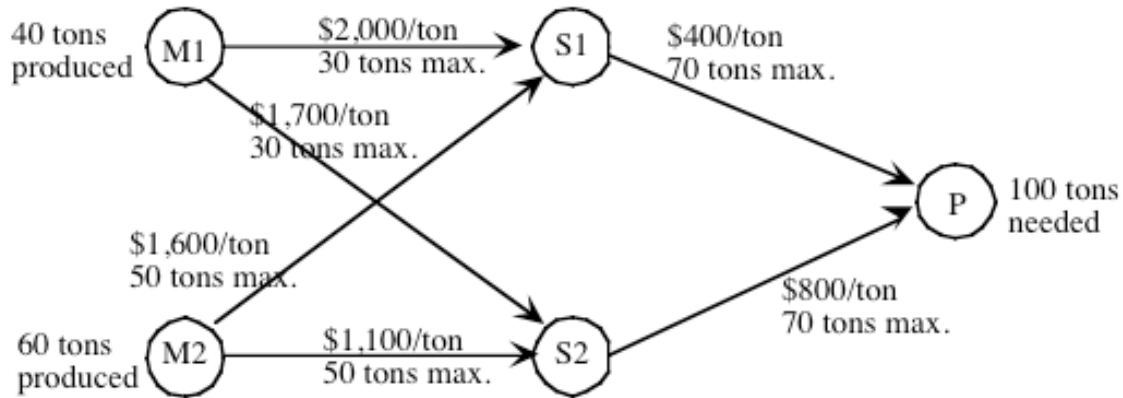
Management now wants to determine the most economic plan for shipping the iron ore from the mines through the distribution network to the steel plant.

(a) Formulate a linear programming model for this problem.

The decision variables are defined as follows:

$x_{m1\text{-}s1}$: number of units (tons) shipped from Mine 1 to Storage Facility 1,
$x_{m1\text{-}s2}$: number of units (tons) shipped from Mine 1 to Storage Facility 2,
$x_{m2\text{-}s1}$: number of units (tons) shipped from Mine 2 to Storage Facility 1,
$x_{m2\text{-}s2}$: number of units (tons) shipped from Mine 1 to Storage Facility 2,
$x_{s1\text{-}p}$: number of units (tons) shipped from Storage Facility 1 to the Plant,
$x_{s2\text{-}p}$: number of units (tons) shipped from Storage Facility 2 to the Plant.

The total shipping cost is:

$$Z = 2000\, x_{m1\text{-}s1} + 1700\, x_{m1\text{-}s2} + 1600\, x_{m2\text{-}s1} + 1100\, x_{m2\text{-}s2} + 400\, x_{s1\text{-}p} + 800\, x_{s2\text{-}p}$$

The constraints we need to consider are:

(1) Supply constraint on M1 and M2:

$$x_{m1\text{-}s1} + x_{m1\text{-}s2} = 40$$
$$x_{m2\text{-}s1} + x_{m2\text{-}s2} = 60$$

(2) Conservation-of-flow constraint on S1 and S2:

$$x_{m1\text{-}s1} + x_{m2\text{-}s1} - x_{s1\text{-}p} = 0$$
$$x_{m1\text{-}s2} + x_{m2\text{-}s2} - x_{s2\text{-}p} = 0$$

(3) Demand constraint on P:

$$x_{s1\text{-}p} + x_{s2\text{-}p} = 100$$

(4) Capacity constraints:

$$x_{\text{m1-s1}} \le 30, \quad x_{\text{m1-s2}} \le 30$$
$$x_{\text{m2-s1}} \le 50, \quad x_{\text{m2-s2}} \le 50$$
$$x_{\text{s1-p}} \le 70, \quad x_{\text{s2-p}} \le 70$$

(5) Nonnegativity constraints:

$$x_{\text{m1-s1}} \ge 0, \quad x_{\text{m1-s2}} \ge 0$$
$$x_{\text{m2-s1}} \ge 0, \quad x_{\text{m2-s2}} \ge 0$$
$$x_{\text{s1-p}} \ge 0, \quad x_{\text{s2-p}} \ge 0$$

The resulting linear programming model for this problem is:

Minimize $\quad Z = 2000\, x_{\text{m1-s1}} + 1700\, x_{\text{m1-s2}} + 1600\, x_{\text{m2-s1}} + 1100\, x_{\text{m2-s2}}$
$$+ \, 400\, x_{\text{s1-p}} + 800\, x_{\text{s2-p}},$$

subject to

$$x_{\text{m1-s1}} + x_{\text{m1-s2}} = 40$$
$$x_{\text{m2-s1}} + x_{\text{m2-s2}} = 60$$
$$x_{\text{m1-s1}} + x_{\text{m2-s1}} - x_{\text{s1-p}} = 0$$
$$x_{\text{m1-s2}} + x_{\text{m2-s2}} - x_{\text{s2-p}} = 0$$
$$x_{\text{s1-p}} + x_{\text{s2-p}} = 100$$
$$x_{\text{m1-s1}} \le 30, \quad x_{\text{m1-s2}} \le 30$$
$$x_{\text{m2-s1}} \le 50, \quad x_{\text{m2-s2}} \le 50$$
$$x_{\text{s1-p}} \le 70, \quad x_{\text{s2-p}} \le 70$$

and

$$x_{\text{m1-s1}} \ge 0, \quad x_{\text{m1-s2}} \ge 0$$
$$x_{\text{m2-s1}} \ge 0, \quad x_{\text{m2-s2}} \ge 0$$
$$x_{\text{s1-p}} \ge 0, \quad x_{\text{s2-p}} \ge 0$$

(b) Solve this model by the simplex method.

This model can be solved by using Solver, which employs the simplex method. This spreadsheet formulation and solution of the problem is shown next.

	M1-S1	M1-S2	M2-S1	M2-S2	S1-P	S2-P				
Unit Cost	$2,000	$1,700	$1,600	$1,100	$400	$800				
										Required
		Contribution Toward Required Amount						Totals		Amount
M1	1	1						40	=	40
M2			1	1				60	=	60
S1	1		1		-1			0	=	0
S2		1		1		-1		0	=	0
P					1	1		100	=	100
										Total Cost
Amount Shipped	30	10	10	50	40	60				$212,000
	<=	<=	<=	<=	<=	<=				
Capacity	30	30	50	50	70	70				

Solver Parameters

Set Objective Cell: TotalCost
To: Min
By Changing Variable Cells:
    AmountShipped
Subject to the Constraints:
    TotalShipped = Required
    AmountShipped <= Capacity
Solver Options:
    Make Variables Nonnegative
    Solving Method: Simplex LP