UNIVERSITY OF WISCONSIN
DATA SCIENCE

DS 775

Prescriptive Analytics

Optimization One: Nonlinear Programming

Course So Far

Linear Programming

- Linear objective function
- Linear constraints
- Continuous decision variables

Integer Programming

- linear objective function
- linear boundaries on feasible region
- integer or mixed decision variables

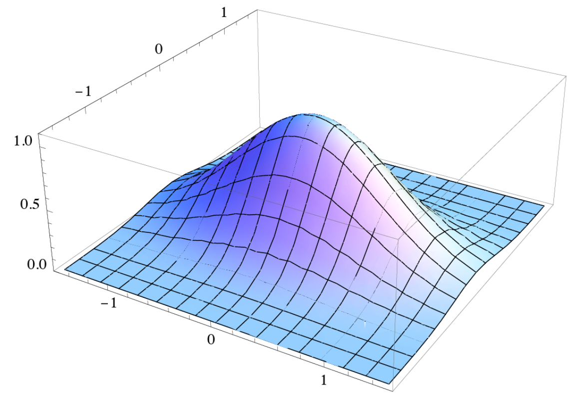
Combinatorial Optimization

- any objective function
- many different constraints (with CP)

This course has largely been about optimization. But so far, our objective functions have either been linear, or we have been in scenarios where we simply tried all combinations of feasible solutions, and evaluated the objective for each. Now we will briefly survey what happens when our objective function, and/or possibly the feasible region become a non-linear.

This Unit – Nonlinear Programming

- Problems simple enough to guarantee an optimal solution
- Algorithms often based on multivariate calculus
- Different types of nonlinear function require different algorithms



In this class, we will roughly divide nonlinear problems into problems that are simple enough mathematically to still guarantee that we can find the overall optimal solution, and those that are so complex that we can offer no guarantee. The simpler problems fall under the domain of non-linear programming, where, by restricting our attention to certain kinds of functions, we can still help define the overall optimal solution.

To really dive into the details of this kind of optimization requires background in multivariate calculus and linear algebra. But we can still get an idea about how things work, and see how we might still solve these problems using software. The presentation this week is pretty short because the text is fairly straightforward. You'll be using spreadsheets and OPL to solve a few problems this week, and you'll find many examples in the download packet.

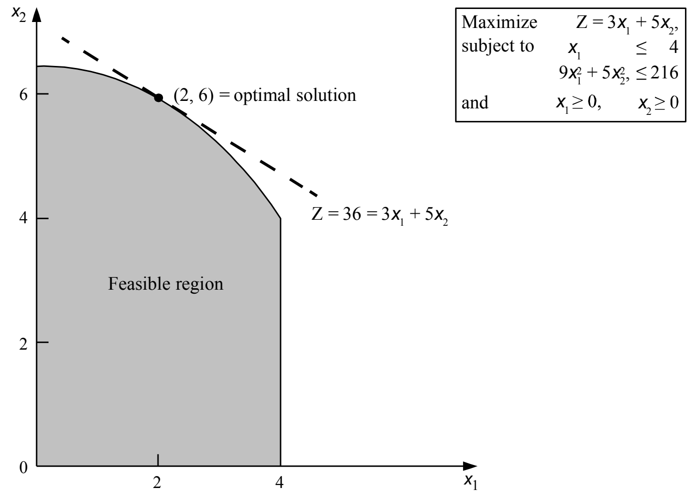
Next Unit - Metaheuristics

- Problems so complex that we cannot guarantee an optimal solution
  - Complex nonlinearities leading to many potentially optimal points
  - Combinatorial optimization with too many possible solutions
- **Goal**: Find a good feasible solution
- Algorithms are often iterative

We use metaheuristics to tackle those optimization problems that are so complex we cannot offer any guarantee of finding an optimal solution. Instead, we are forced to settle for just a good solution. These algorithms are called heuristic because they rely on search patterns which usually work, but are not guaranteed to always work.

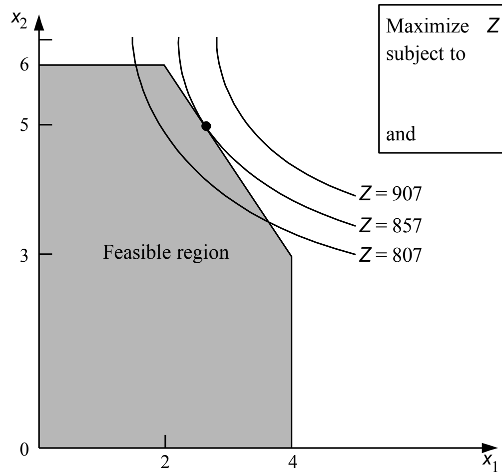
For every optimization algorithm, there is always a problem for which the algorithm will fail to find the best solution.

Nonlinear Boundaries



One of the complications that arise in nonlinear programming is that the boundaries of the feasible region may no longer be linear, and the optimal values may not occur at [? corner ?] feasible points. We'll often assume that the feasible region is convex, meaning that all points lying on a line segment between two feasible points are also feasible points. So any line segment joining two points stays entirely inside the feasible region.

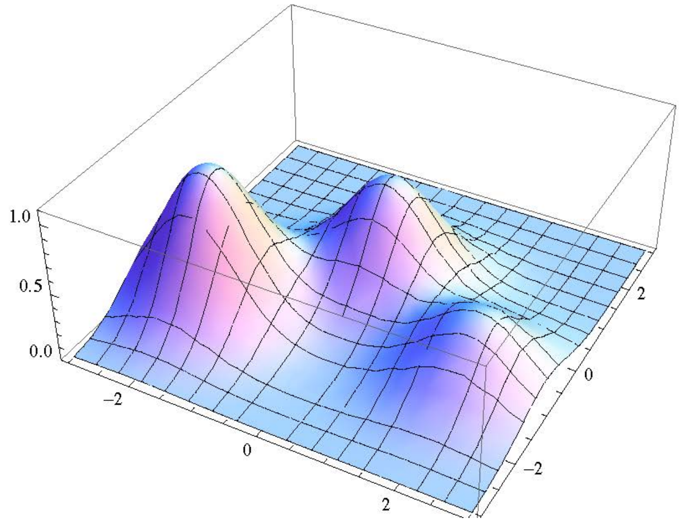
Nonlinear Objective



$$
\begin{array}{llcl}
\text{Maximize} & Z = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2, & & \\
\text{subject to} & x_1 & \leq & 4 \\
 & 2x_2 & \leq & 12 \\
 & 3x_1 + 2x_2 & \leq & 18 \\
\text{and} & x_1 \geq 0, \quad x_2 \geq 0 & &
\end{array}
$$

The objective function can also be nonlinear, which means that the optimal solution may no longer even be on the boundary of the feasible region. If the nonlinear function is simple enough, like quadratic, or bowl-shaped, or mound-shaped, then we can still guarantee that the optimal value can be found using nonlinear programming.

Local Max or Global Max?



One of the problems that arises with nonlinear objective functions is that there may be multiple peaks and valleys. Such functions are called multimodal. A nonlinear programming algorithm might climb a hill and find a maximum, but it may not be the highest hill around, as shown in the picture. There are multiple local maxima in this picture, and one of those is the global maximum.

Quadratic Programming

- Linear constraints
- Objective function is quadratic and concave (mound-shaped)
- Guarantee that max is either at a single critical point inside the feasible region or on the boundary
- Solve with modified simplex method (Solver/ASPE and OPL)

One of the most frequently-encountered nonlinear programming types is called quadratic programming. The feasible region has linear boundaries and we are trying to maximize a concave-- or, that is, a mound-shaped-- function. Or we are trying to minimize a convex function-- that is, a bowl-shape function.

Either way, the optimal solution will occur either at a single point inside the feasible region, called a critical point, which is essentially the vertex of a parabola, or the optimal solution occurs at a point along the boundary. A modified simplex method, implemented in both ASPE and OPL, can be used to solve these problems.

In ASPE, one can simply select the quadratic solver, and the CPLEX engine in OPL can automatically detect if we have an appropriate quadratic function.

Convex Programming

- Feasible region is convex (boundaries are linear or bend outward)
- Find the maximum of a convcave objective function or the minimum of a convex objective function
- GRG Nonlinear method in Solver/ASPE

Convex programming utilizes multivariable calculus to find appropriate directions to climb the hill, or to descend into a valley. The concavity of the objective function and the convexity of the boundaries guarantee that a maximum will occur at the top of a single hill, or on the boundary of the region. ASPE includes an algorithm called the generalized reduced gradient method for solving these problems. Just select the [? GRG ?] nonlinear method as the solver.