UNIVERSITY OF WISCONSIN
DATA SCIENCE

DS 775

Prescriptive Analytics

Game Theory

Two-Person, Zero-Sum Games

Problem 15.1-2 will be used to illustrate a *two-person, zero-sum* game.

The players: Two manufacturers competing for sales in two equally profitable product lines (A & B)

The objective: To get an improvement to the market first

The strategies: (for both players)
1 – normal development of both products
2 – crash development of product A
3 – crash development of product B

We'll begin by looking at exercise 15.1-2 to illustrate some basic definitions and see how to set up a two-person, zero-sum game. You'll have to read the exercise on page 687 in the textbook to get all of the details, but the basic setup is shown here.

Notes:
Turn to problem 15.1-2 in your textbook and read the details to follow along.

Problem 15.1-2 - Details

Total time to market

Strategy	Manufacturer 1	Manufacturer 2
1	12 mo. for both A & B	12 mo. for both A & B
2	10 mo. for A, 19 mo. for B	9 mo. for A, 18 mo. for B
3	10 mo. for B, 19 mo. for A	9 mo. for B, 18 mo. for A

This information is based on the second paragraph of problem 15.1-2, showing the total amount of time in months to get each product ready for sale for each of the manufacturers. Strategy one is that the manufacturer develops both product lines simultaneously and both of them would have both products on the market in 12 months.

When manufacturer one chooses strategy two, the crash development of product A, then it will have product A on the market in 10 months and product B in an additional nine months, for a total of 19 months for product B. When manufacturer one chooses strategy three, the crash development of product B, then it will have product B on the market in 10 months and product A in an additional nine months, for a total of 19 months. Likewise, when manufacturer two uses strategy one, both products will be in development in 12 months. And when strategies two or three are chosen, then the first product will be ready for market in nine months and the second one in an additional nine months.

This table, along with the description of the gain or loss in market share for manufacturer one in the third paragraph of the textbook problem, will be used to set up the payoff table. The payoff table shows the gain or loss for player one, in this case manufacturer one, for each combination of strategies chosen by the two players.

Problem 15.1-2 - Payoff Table

The entries in the payoff table are the gain or loss of total share of future sales of products A and B combined for *Manufacturer 1* (*i.e.* Player 1).

		Player 2		
	Strategy	1	2	3
Player 1	1	16	20	20
	2	8	-8	26
	3	8	26	-8

There was just a bit of work to do to get the values for the payoff table in this problem. Remember that these values represent the payoffs from the perspective of player one, or manufacturer one for this problem, and reversing the signs on them would be the payoffs for player two, since this is a zero-sum game.

In a zero-sum game, whatever player one gains player two loses, and vice versa. If both manufacturers choose strategy one, then they will both get products on the market in 12 months and the share of total future sales for manufacturer one will increase by 8% for both products, for a total of 16% increase.

When manufacturer one uses strategy two and manufacturer two uses strategy one, then manufacturer one will get product A to the market in 10 months, which is two months ahead of manufacturer two, who will have product A ready in 12 months. And so the increase in share is 20% for product A. But manufacturer one will not get product B ready until 19 months have passed, seven months after manufacturer two has product B ready, and so will lose 12% of the total future sales for product B. Therefore, the overall increase to manufacturer one is 20 minus 12, which is 8%.

Similarly, when manufacturer one uses strategy one and manufacturer two goes with strategy two, then manufacturer one will get product A to market in 12 months, which is three months behind manufacturer two, who will have product A ready in nine months, and so lose 10% for product A. However, manufacturer one will have product B ready in 12 months, six months ahead of manufacturer two for product B, and so will gain 30% of the total future sales for product B. Therefore, the overall increase to manufacturer one is 30 minus 10, or 20%. The remaining entries are filled in similarly by comparing the time for each product to get to market for each manufacturer and adding up the share of total future sales gained or lost.

Problem 15.1-2 - Are There any Dominated Strategies?

		Player 2		
	Strategy	1	2	3
Player 1	1	16	20	20
	2	8	-8	26
	3	8	26	-8

For a given player, one strategy dominates another if it is always at least as good as the other. If a strategy is dominated by another, then it can be removed from the payoff table. And reducing the payoff table will generally make the problem easier to solve. In fact, sometimes one strategy dominates all others for a player and there is no longer a question of what strategy that player should choose. If the table is reduced so that both players have only one strategy remaining, then the problem is solved, under the assumptions that both players are rational and choose strategies optimally to promote solely their own welfare.

Let's see if there are any dominated strategies in this table. From the perspective of player one, beginning with strategy one, and first comparing it to strategy two, 16 exceeds 8 and 20 provides a much better payoff to player one than the negative 8. But 20 is less than 26, so neither strategy one nor strategy two dominates the other. Comparing strategy one to strategy three, there is a similar outcome. 16 is bigger than 8, but 20 is less than 26. So, again, neither strategy is dominated. A comparison of strategy two to strategy three shows again that neither dominates the other. So for player one, which is manufacturer one in this context, all three strategies are still on the table.

The same process should be undertaken from the perspective of player two, but remember that the lower values in the table are better for player two since the table contains payoffs to player one. For example, a comparison of strategy one to strategy two for player two shows that 16 is better than 20 because it is a smaller loss should player one choose strategy one. And 8 is also a smaller loss than 26 for player two if player one chooses strategy three.

But player two would gain 8 using strategy two if player one also chose strategy two, as opposed to losing 8 when using strategy one. In other words, neither strategy one nor strategy two dominates the other. Comparing the other pairs of strategies for player two shows that none of the strategies are dominated.

If player two happened to have a dominated strategy, it would be beneficial to recheck the strategies for player one again after the table is reduced. So this payoff table cannot be reduced as there are no dominated strategies. So what's next?

Problem 15.1-2 - Choosing a Strategy
The Minimax Criterion

		Player 2			
	Strategy	1	2	3	Row Min
Player 1	1	16	20	20	16
	2	8	-8	26	-8
	3	8	26	-8	-8
	Col Max	16	26	26	

Each player can select an optimal strategy by minimizing their maximum losses. Since the payoff table shows the payoffs to player one will include the row minimums, which are basically the worst case scenarios under each strategy for player one, however, the column maximums represent the worst case scenario for player two. For player one, strategy one provides the maximum of all the minimum gains, which, in this case, is an overall gain in the share of all total future sales for products A and B. 16 is called the maximin value for player one.

Since the payoff table contains losses for player two, the best strategy for player two is the one that produces the minimum of the column maximums because this represents the smallest loss that player two will suffer to player one under the worst case scenario. 16 is called the minimax value for player two.

Notice in this table that the optimal strategies for both players result in the same outcome, a 16% increase in the share of total future sales for manufacturer one. Since this is a zero-sum game, that means that the best manufacturer two can do is to lose 16% of the share of total future sales for these products. 16 is called the value of the game. A fair game would have a value of 0, so this is not a fair game.

Since the maximin and minimax values for both players result in the same payoff value, this game is said to have a stable solution. And that combination of strategies is called a saddle point in the table.

Another Example

Consider the following payoff table for a two-player, zero-sum game.

		Player 2		
	Strategy	1	2	3
Player 1	1	1	-1	3
	2	0	4	1
	3	3	-2	5
	4	-3	6	-2

Take a look at this payoff table. A perusal of the strategies for player one shows that there are no dominated strategies for player one, but that every payoff to player one under strategy one for player two is less than the payoff to player one under strategy three. Since smaller payoffs to player one are a better outcome for player two, strategy three is dominated by strategy one. So this table can be reduced by removing strategy three for player two.

Dominated Strategies and the Reduced Table

		Player 2		
	Strategy	1	2	Row Min
Player 1	1	1	-1	-1
	2	0	4	0
	3	3	-2	-2
	4	-3	6	-3
	Col Max	3	6	

A quick check shows that even with strategy three removed for player two, none of the strategies for player one are dominated.

Next, the row minimums and column maximums have been added to apply the minimax criterion in order to find the optimal solution in the value of the game. For player one, the optimal strategy, according to the maximin value 0, is strategy two. For player two, strategy one should be selected using the minimax value, 3.

Unlike the previous example, the maximin value for player one and the minimax value for player two do not result in the same value. It's 0 for player one and 3 for player two. Therefore, this table does not have a saddle point and this game does not have a stable solution. It's unstable because, even though the game is only played once, when each player considers the other player's options, which they are aware of in the standard two-person, zero-sum game, any tentative choice of strategy leaves the player with the motive to consider changing strategies.

Until now, we have only used what are called pure strategies. With an unstable solution, each player can assign probabilities to each strategy in such a way as to minimize their expected maximum loss. Strategies with the probability distribution assigned to them are called mixed strategies. Proceed to the next slide to see how to work with mixed strategies.

Graphical Solution for a Game with Mixed Strategies

			Player 2	
	Probability		y_1	y_2
		Strategy	1	2
Player 1	x_1	1	1	-1
	x_2	2	0	4
	x_3	3	3	-2
	x_4	4	-3	6

The probabilities that will be assigned to player one's strategies are designated as x1 through x4. Player two has only two strategies, and we'll call those y1 and y2. Since the x's and y's will be valid probability distributions, the values chosen for the x sub i's must be non-negative and add up to 1, and the same goes for y1 and y2.

Expected Payoffs

Since the payoff table contains payoffs to Player 1, the expected payoff to Player 1 when mixed stategies are used is:

$$\text{Expected payoff to Player 1} = \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}x_i y_j$$

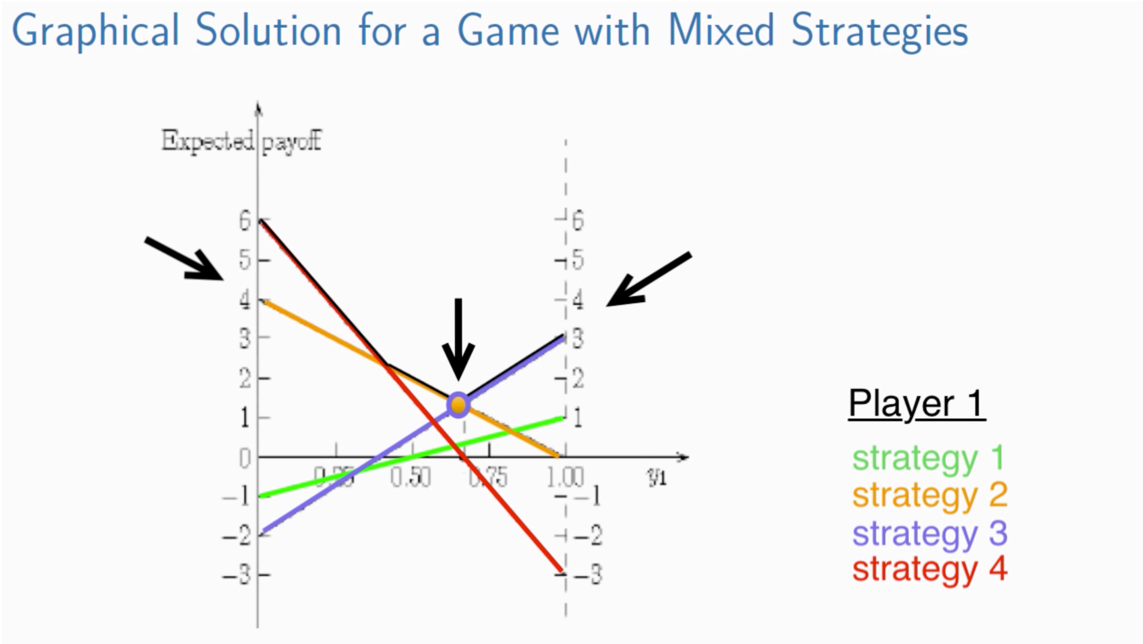
The p sub ij in this notation refers to the payoff to player one when player one chooses strategy i and player two uses strategy j. According to probability theory, the expected payoff to player one is found by multiplying the probability that player one chooses strategy i by the probability that player two chooses strategy j times the payoff for that combination of strategies for all possible combinations, and then adding them together.

Expected Payoffs for Player 2

			Player 2	
	Probability		y_1	$1 - y_1$
		Strategy	1	2
Player 1	x_1	1	1	-1
	x_2	2	0	4
	x_3	3	3	-2
	x_4	4	-3	6

(x_1, x_2, x_3, x_4)	Computation	Expected Payoff
(1, 0, 0, 0)	$1 \cdot x_1 \cdot y_1 + (-1) \cdot x_1 \cdot (1 - y_1) = y_1 - (1 - y_1)$	$2y_1 - 1$
(0, 1, 0, 0)	$0 \cdot x_2 \cdot y_1 + 4 \cdot x_2 \cdot (1 - y_1) = 4(1 - y_1)$	$-4y_1 + 4$
(0, 0, 1, 0)	$3 \cdot x_3 \cdot y_1 + (-2) \cdot x_3 \cdot (1 - y_1) = 3y_1 - 2(1 - y_1)$	$5y_1 - 2$
(0, 0, 0, 1)	$-3 \cdot x_4 \cdot y_1 + 6 \cdot x_4 \cdot (1 - y_1) = -3y_1 + 6(1 - y_1)$	$-9y_1 + 6$

Since there are only two strategies for player two, the probabilities can be expressed using only one variable, y1. And equations for expected payoffs for player two can be written for each pure strategy used by player one. The pure strategies are the ones where all of the probability is loaded onto one strategy. Remember that the payoffs are to player one.



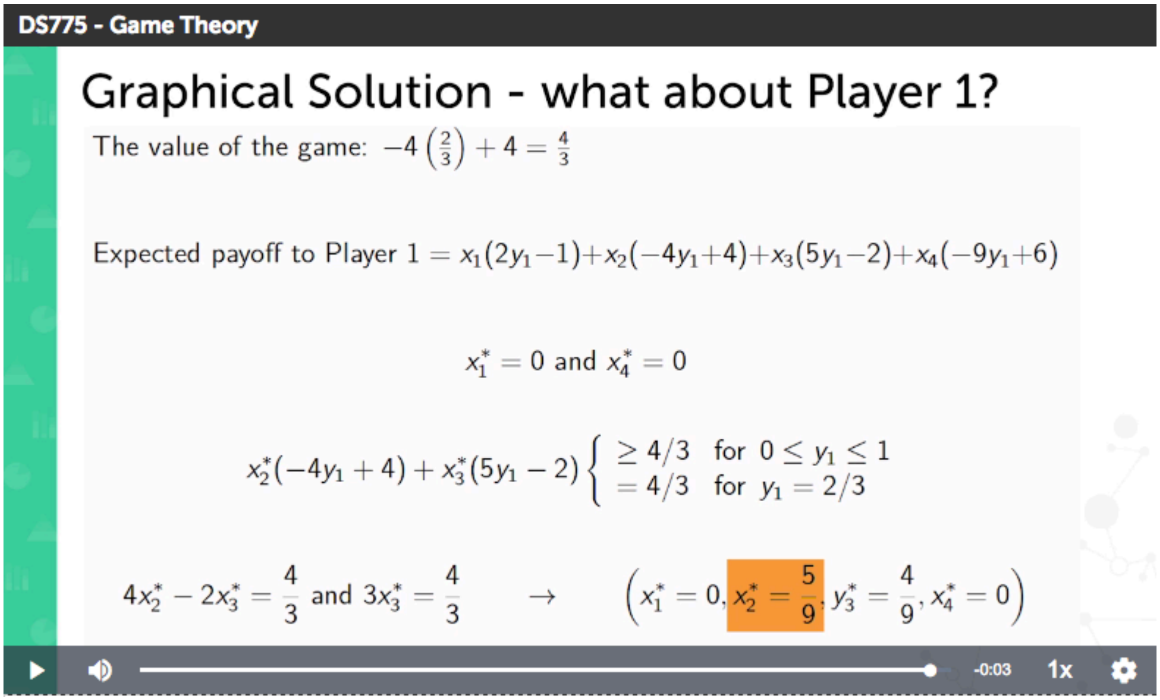
Since y1 is the probability, it can be any real number between zero and one, including zero and one. When each expected payoff to player one is graphed over the interval zero to one for y1, the minimax theorem can be applied to find a stable solution to game. The minimax theorem basically says that when mixed strategies are used, a stable solution can be obtained by finding the probabilities for the mixed strategies such that the maximin value for player one is equal to the minimax value for player two. The resulting value is also the value of the game. At this point, neither player can improve by changing strategies, and the game is stable.

In this graph, each of the four expected payoff equations for the pure strategies of player one are shown plotted against the possible probabilities assigned to y1. When player one uses pure strategy one, here is the line representing the expected payoffs to player one for values of y1 between zero and one. 2y1 minus one.

This line, negative 4y1 plus four, shows the relationship between the expected payoff to player one when player one chooses strategy two for the various values of y1. Likewise, when player one uses strategy three, this line 5y1 minus two, and for strategy four, the line negative 9y1 plus six, shows the expected payoff to player one for the various values of y1.

Player one wants to maximize this expected payoff. Given y1, player one can do this by choosing the pure strategy that corresponds to the top line for that value of y1 in the graph. Since the expected payoff for player one is the expected loss for player two, the minimax criterion says that player two should minimize his maximum expected loss by selecting the value of y1 where the top line reaches its lowest point. This lowest point is shown in the graph as the dot where the negative 4y1 plus 4 and 5y1 minus two lines intersect.

To solve for the value of y1 where these two lines intersect, we set negative 4y1 plus four equal to 5y1 minus two, which yields y1 equals 2/3. Thus, the optimal mixed strategy for player two is to choose strategy one with probability y1 star equals 2/3 and strategy two with probability y2 star equal to 1/3.



The value of the game is the expected payoff to player one when player two assigns a probability of 2/3 to strategy one and can be computed using the equation for the expected payoff to player one when player one uses either strategy two or three since these are the ones that intersect at the optimal point for player two. Here, the equation negative 4y 1 plus 4 is being used, the one from strategy two, to compute the value of the game as 4/3.

To find the corresponding optimal mixed strategy for player one, the probability distribution for player one's strategies must be such that player one's expected payoff is equal to the value of the game, 4/3, when player two uses the optimal strategy. This requires having zero weight on the lines in the graph that corresponds to pure strategies one and four for player one, since they don't pass through the point that is optimal for player two.

So the probabilities x1 and x4 can immediately be set to 0. Furthermore, if player two were to choose any other mixed strategy than y1 equals 2/3, player one must be able to achieve an expected payoff at least as large as the value of the game. Since x2 and x3 are numbers, the left hand side gives the equation of a line as y1 goes from 0 to 1, which must be a horizontal line to satisfy the greater than or equal to 4/3 condition on both sides of Y1 equals 2/3.

So the greater than or equal to 4/3 can be replaced by equals 4/3 for all values of Y1. Choosing any two values of Y1, such as 0 and 1, to make it easy, the expected payoff to player one reduces to 4x2 minus 2x3 equals 4/3 when Y1 equals 0 and 3x3 equals 4/3 when Y1 is 1. These two equations yield x2 star equals 5/9 and x3 star is 4/9. Therefore, the optimal mixed strategy for player one is to assign a probability of 0 to strategies one and four, choose strategy two with probability 5/9, and strategy three with probability 4/9.

The Linear Programming Solution: Player 1 Perspective

Maximize x_5

subject to

$$
\begin{aligned}
x_1 \quad\quad\quad + 3x_3 - 3x_4 - x_5 &\geq 0 \\
-x_1 + 4x_2 - 2x_3 + 6x_4 - x_5 &\geq 0 \\
x_1 + x_2 + x_3 + x_4 \quad\quad &= 1
\end{aligned}
$$

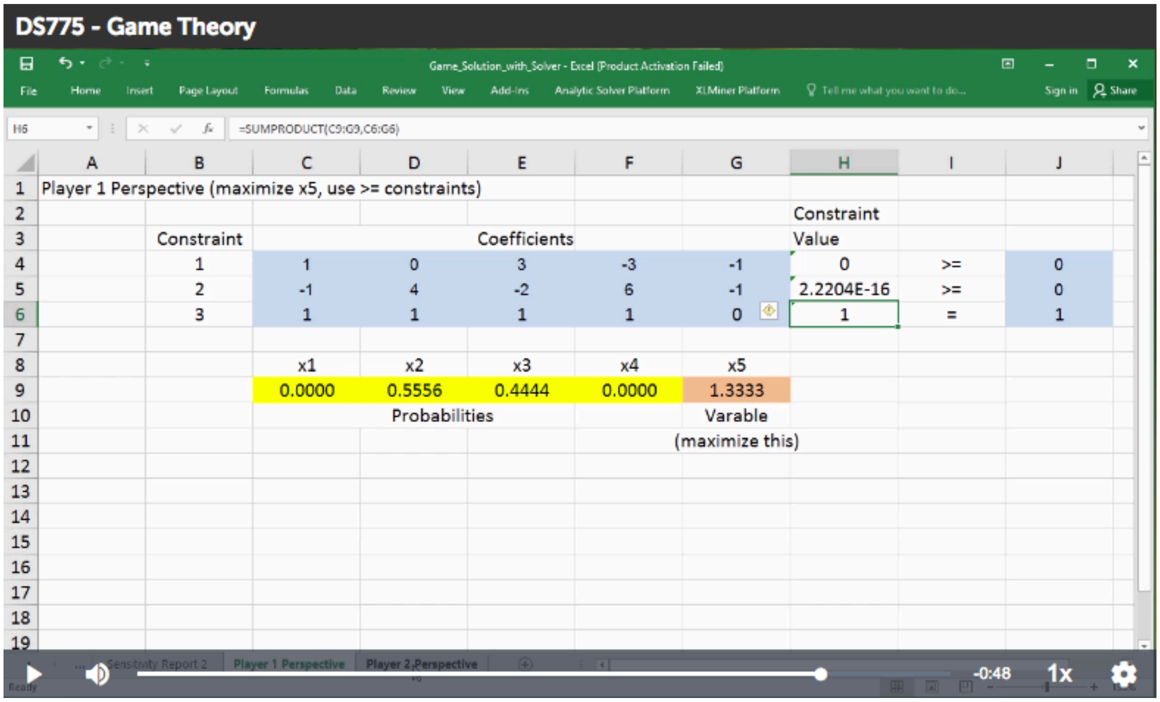
and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.$

Two-person, zero-sum games can also be solved by structuring them as linear programming problems. The introduction of a fifth variable is needed to do this. The maximum value of the variable x5 will be the value of the game.

The optimal mixed strategies for players one and two can be found through linear programming, too. And, unlike the graphical solution, we're not restricted to one of the players having only two strategies to choose from. The details of how to get from the payoff table to the linear programming problem are given in 15.5 of the textbook.

The coefficients for this part of the constraints are from the reduced payoff table. And these constraints are to ensure that the probabilities chosen for player one's mixed strategies constitute a legitimate probability distribution.

This is just one more example for you to look at. The next slide will show you how this is solved using the Excel Solver and will take a look at how probabilities for player two's optimal mixed strategy are obtained through the sensitivity report.



	A	B	C	D	E	F	G	H	I	J
1	Player 1 Perspective (maximize x5, use >= constraints)									
2								Constraint		
3		Constraint			Coefficients			Value		
4		1	1	0	3	-3	-1	0	>=	0
5		2	-1	4	-2	6	-1	2.2204E-16	>=	0
6		3	1	1	1	1	0	1	=	1
7										
8			x1	x2	x3	x4	x5			
9			0.0000	0.5556	0.4444	0.0000	1.3333			
10				Probabilities			Varable			
11							(maximize this)			

The Linear Programming Solution: Player 2 Perspective

$$\text{Minimize } y_3$$

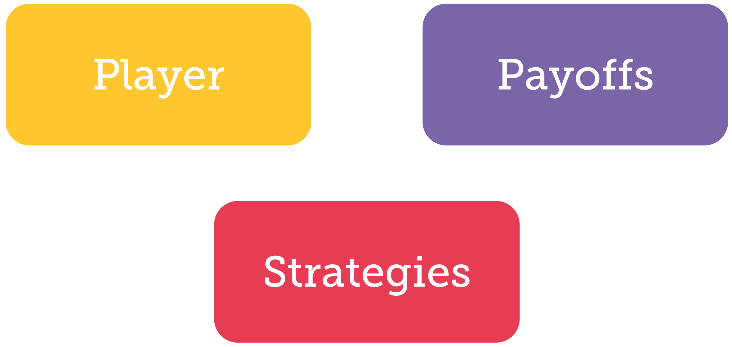
$$\begin{array}{lrcrcrl} \text{subject to} & y_1 & - & y_2 & - & y_3 & \leq 0 \\ & & & 4y_2 & - & y_3 & \leq 0 \\ & 3y_1 & - & 2y_2 & - & y_3 & \leq 0 \\ & -3y_1 & + & 6y_2 & - & y_3 & \leq 0 \\ & y_1 & + & y_2 & & & = 1 \end{array}$$

and $y_1 \geq 0, y_1 \geq 0$

This problem could have also been set up and solved from the perspective of player two. Once again, these coefficients were taken from the reduced payoff table for the example we've been working with and these constraints are to ensure that we get legitimate probabilities. The Excel spreadsheet called Player Two Perspective in the file Game Solution with Solver contains this version.

The sensitivity report for this one will contain the dual solution with shadow prices indicating the optimal mixed strategy for player one.

Applying Game Theory



In order to make use of game theory in your work as a data scientist, you need to be able to look at a given situation and identify the players, determine the payoffs under various outcomes, and delineate all of the strategies that each player might use. Section 15.6 in the textbook discusses some of the more complicated scenarios, such as two-person, constant-sum games, n-person games, non-zero sum games, and more. This is just an introduction to game theory as a field of study. So we're not going into detail about these advanced methods. But in your career as a data scientist, the learning never stops.

Question 1

DS775 - Game Theory

Question for Self Assessment: Multiple Choice

A _________________________ shows the gain (positive or negative) for one of the players that would result from each combination of strategies for the two players.

- payoff table.
- strategy
- dominated strategy.
- decision rule.
- rational criteria.

SUBMIT

Answer is at the end of the transcript

Question 2

DS775 - Game Theory

Question for Self Assessment: Multiple Choice

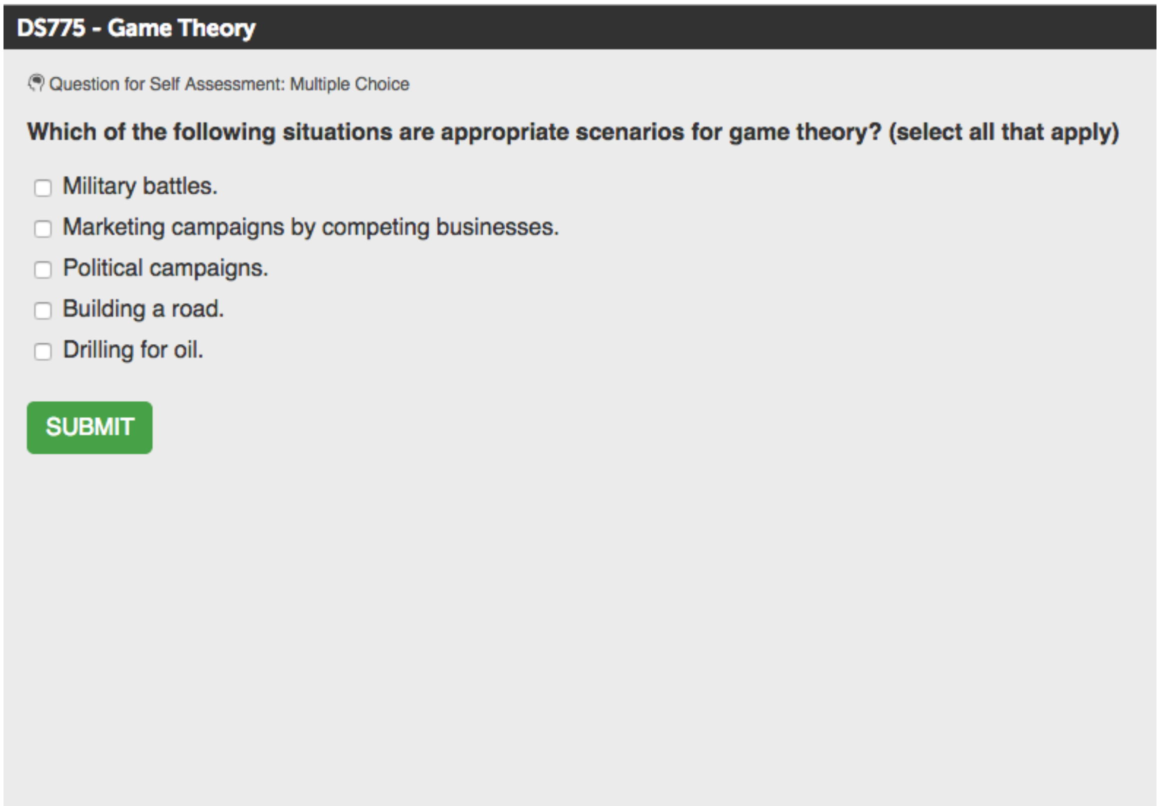
In game theory the opponent is passive and chooses strategies at random, while in decision analysis the opponent is rational and actively trying to win.

- True
- False

SUBMIT

Answer is at the end of the transcript

Question 3


DS775 - Game Theory

Question for Self Assessment: Multiple Choice

Which of the following situations are appropriate scenarios for game theory? (select all that apply)

- [ ] Military battles.
- [ ] Marketing campaigns by competing businesses.
- [ ] Political campaigns.
- [ ] Building a road.
- [ ] Drilling for oil.

SUBMIT

Answer is at the end of the transcript

Question 4

DS775 - Game Theory

Question for Self Assessment: Multiple Choice

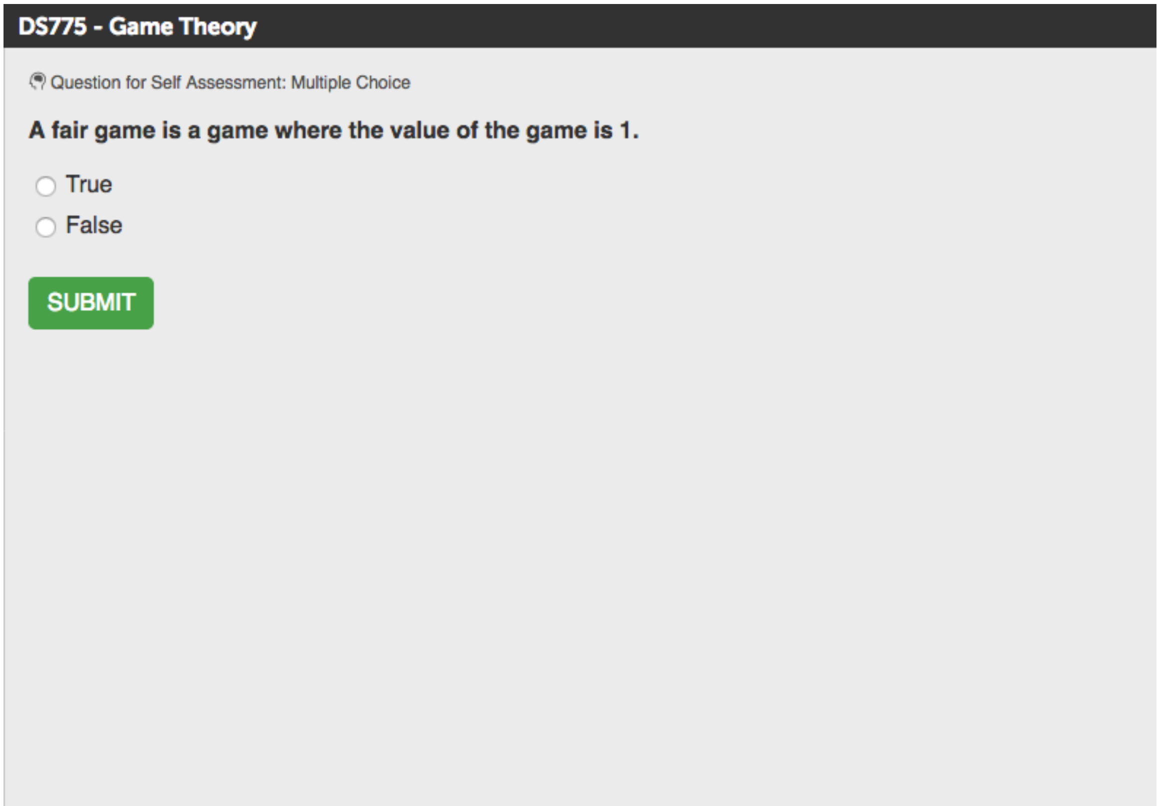
A ________________ strategy is one that another strategy is always at least as good as regardless of what the opponent does.

- dominated
- pure
- mixed
- preferred
- rational

SUBMIT

Answer is at the end of the transcript

Question 5



Answer is at the end of the transcript

Question 6

DS775 - Game Theory

Question for Self Assessment: Multiple Choice

In game theory, each player knows the strategies available to the other player and also knows the payoff table before the game begins.

- True
- False

SUBMIT

Answer is at the end of the transcript

Question 7

DS775 - Game Theory

Question for Self Assessment: Multiple Choice

Smaller game theory problems cannot be solved graphically by plotting the expected payoff for one of the players against the probability of that player selecting a given strategy.

- True
- False

SUBMIT

Answer is at the end of the transcript

Question 8

DS775 - Game Theory

Question for Self Assessment: Multiple Choice

Games with mixed strategies can be solved by transforming the problem to a linear programming problem.

- True
- False

SUBMIT

Answer is at the end of the transcript

Question 9

DS775 - Game Theory

Question for Self Assessment: Multiple Choice

While game theory provides the conceptual tools to deal with competitive situations, it usually plays just a supplementary role due to the complexity of most situations.

- True
- False

SUBMIT

Answer is at the end of the transcript

Question 10

DS775 - Game Theory

Question for Self Assessment: Multiple Choice

Which of the following are extensions of game theory mentioned in section 15.6?

- [ ] two-person, constant sum
- [ ] n-person
- [ ] non-zero sum
- [ ] cooperative
- [ ] infinite (infinitely many pure strategies available)

SUBMIT

Answer is at the end of the transcript

Question 1 answer

DS775 - Game Theory

Feedback for Self Assessment

✓ Correct!

A ________________________ shows the gain (positive or negative) for one of the players that would result from each combination of strategies for the two players.

Your answer:
payoff table.

Correct answer:
payoff table.

Question 2 answer

DS775 - Game Theory

Feedback for Self Assessment

✓ Correct!

In game theory the opponent is passive and chooses strategies at random, while in decision analysis the opponent is rational and actively trying to win.

Your answer:
False

Correct answer:
False

Question 3 answer

DS775 - Game Theory

Feedback for Self Assessment

✓ Correct!

Which of the following situations are appropriate scenarios for game theory? (select all that apply)

Your answer:

- Military battles.
- Marketing campaigns by competing businesses.
- Political campaigns.

Correct answer:

- Military battles.
- Marketing campaigns by competing businesses.
- Political campaigns.

Question 4 answer

DS775 - Game Theory

Feedback for Self Assessment

✓ Correct!

A ________________ strategy is one that another strategy is always at least as good as regardless of what the opponent does.

Your answer:
dominated

Correct answer:
dominated

Question 5 answer

DS775 - Game Theory

Feedback for Self Assessment

✓ Correct!

A fair game is a game where the value of the game is 1.

Your answer:
False

Correct answer:
False

Feedback:
Right. The value of the game is 0 in a fair game.

Question 6 answer

DS775 - Game Theory

Feedback for Self Assessment

✓ Correct!

In game theory, each player knows the strategies available to the other player and also knows the payoff table before the game begins.

Your answer:
True

Correct answer:
True

Feedback:
Yes, but the actual play of the game consists of each player simultaneously choosing a strategy without knowing the opponent's choice.

Question 6 answer

DS775 - Game Theory

Feedback for Self Assessment

✓ Correct!

In game theory, each player knows the strategies available to the other player and also knows the payoff table before the game begins.

Your answer:
True

Correct answer:
True

Feedback:
Yes, but the actual play of the game consists of each player simultaneously choosing a strategy without knowing the opponent's choice.

Question 7 answer

DS775 - Game Theory

Feedback for Self Assessment

✓ Correct!

Smaller game theory problems cannot be solved graphically by plotting the expected payoff for one of the players against the probability of that player selecting a given strategy.

Your answer:
False

Correct answer:
False

Question 8 answer

DS775 - Game Theory

Feedback for Self Assessment

✓ Correct!

Games with mixed strategies can be solved by transforming the problem to a linear programming problem.

Your answer:
True

Correct answer:
True

Question 9 answer

DS775 - Game Theory

Feedback for Self Assessment

✓ Correct!

While game theory provides the conceptual tools to deal with competitive situations, it usually plays just a supplementary role due to the complexity of most situations.

Your answer:
True

Correct answer:
True

Feedback:
Yes, and research continues in this area to extend in to more and more complex scenarios.

Question 10 answer

DS775 - Game Theory

Feedback for Self Assessment

✓ Correct!

Which of the following are extensions of game theory mentioned in section 15.6?

Your answer:

- two-person, constant sum
- n-person
- non-zero sum
- cooperative
- infinite (infinitely many pure strategies available)

Correct answer:

- two-person, constant sum
- n-person
- non-zero sum
- cooperative
- infinite (infinitely many pure strategies available)