CHAPTER 3: INTRODUCTION TO LINEAR PROGRAMMING

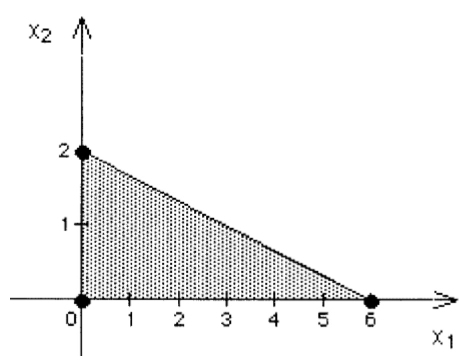
3.1-1.

Swift & Company solved a series of LP problems to identify an optimal production schedule. The first in this series is the scheduling model, which generates a shift-level schedule for a 28-day horizon. The objective is to minimize the difference of the total cost and the revenue. The total cost includes the operating costs and the penalties for shortage and capacity violation. The constraints include carcass availability, production, inventory and demand balance equations, and limits on the production and inventory. The second LP problem solved is that of capable-to-promise models. This is basically the same LP as the first one, but excludes coproduct and inventory. The third type of LP problem arises from the available-to-promise models. The objective is to maximize the total available production subject to production and inventory balance equations.

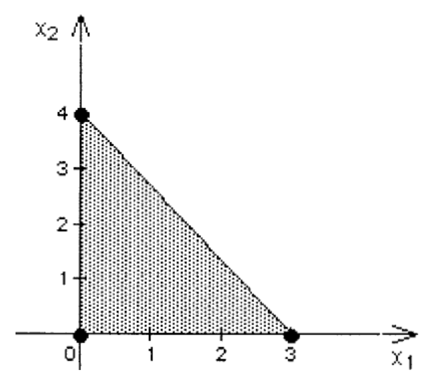
As a result of this study, the key performance measure, namely the weekly percent-sold position has increased by 22%. The company can now allocate resources to the production of required products rather than wasting them. The inventory resulting from this approach is much lower than what it used to be before. Since the resources are used effectively to satisfy the demand, the production is sold out. The company does not need to offer discounts as often as before. The customers order earlier to make sure that they can get what they want by the time they want. This in turn allows Swift to operate even more efficiently. The temporary storage costs are reduced by 90%. The customers are now more satisfied with Swift. With this study, Swift gained a considerable competitive advantage. The monetary benefits in the first years was $12.74 million, including the increase in the profit from optimizing the product mix, the decrease in the cost of lost sales, in the frequency of discount offers and in the number of lost customers. The main nonfinancial benefits are the increased reliability and a good reputation in the business.

3.1-2.

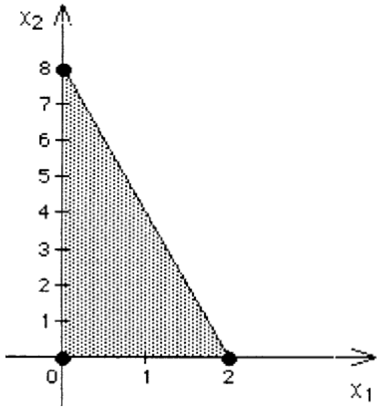
(a)



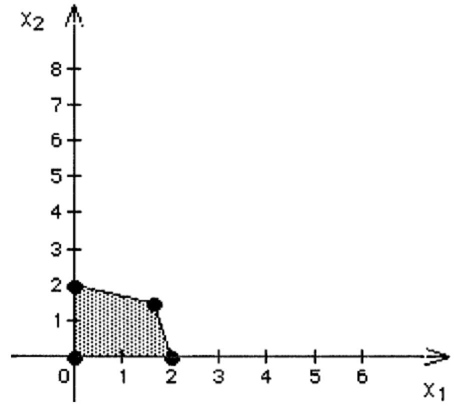
(b)



(c)

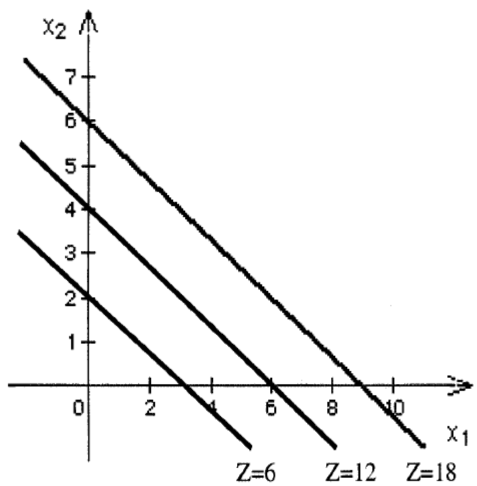


(d)



3.1-3.

(a)



(b)

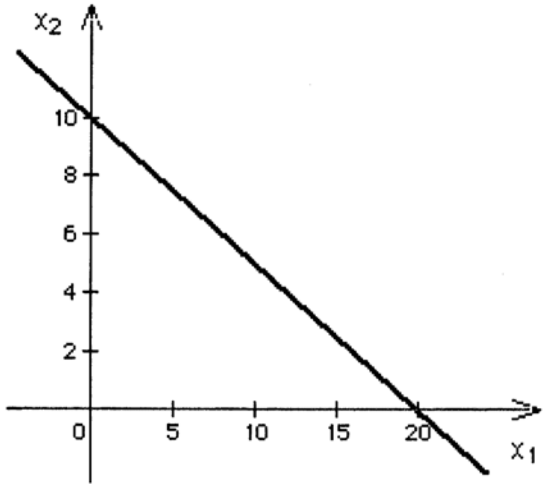
	Slope-Intercept Form	Slope	Intercept
$Z=6$	$x_2=-\frac{2}{3}x_1+2$	$-\frac{2}{3}$	2
$Z=12$	$x_2=-\frac{2}{3}x_1+4$	$-\frac{2}{3}$	4
$Z=18$	$x_2=-\frac{2}{3}x_1+6$	$-\frac{2}{3}$	6

3.1-4.

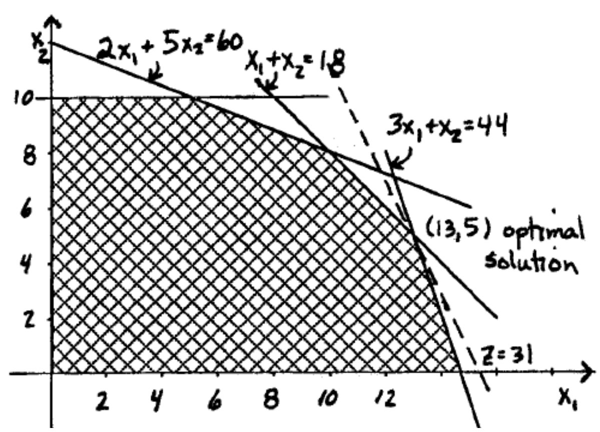
(a) $x_2 = -\frac{1}{2}x_1 + 10$

(b) The slope is $-1/2$, the x_2 intercept is 10.

(c)

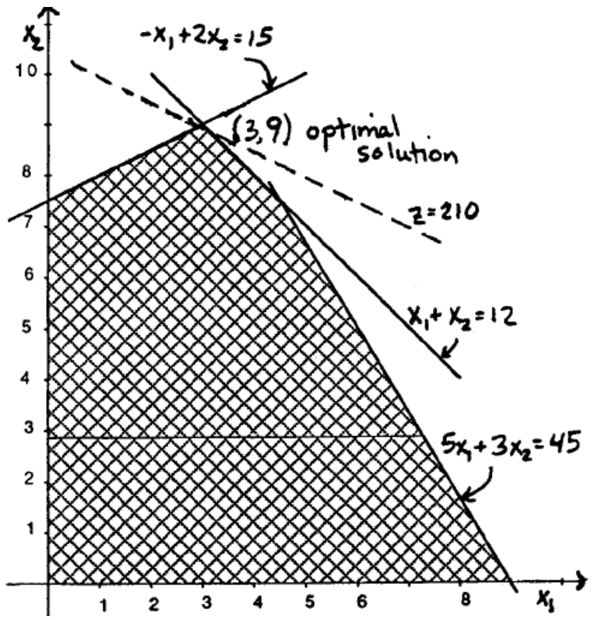


3.1-5.
Optimal Solution: $(x_1^*, x_2^*) = (13, 5)$ and $Z^* = 31$



3.1-6.

Optimal Solution: $(x_1^*, x_2^*) = (3, 9)$ and $Z^* = 210$



3.1-7.

(a) As in the Wyndor Glass Co. problem, we want to find the optimal levels of two activities that compete for limited resources. Let W be the number of wood-framed windows to produce and A be the number of aluminum-framed windows to produce. The data of the problem is summarized in the table below.

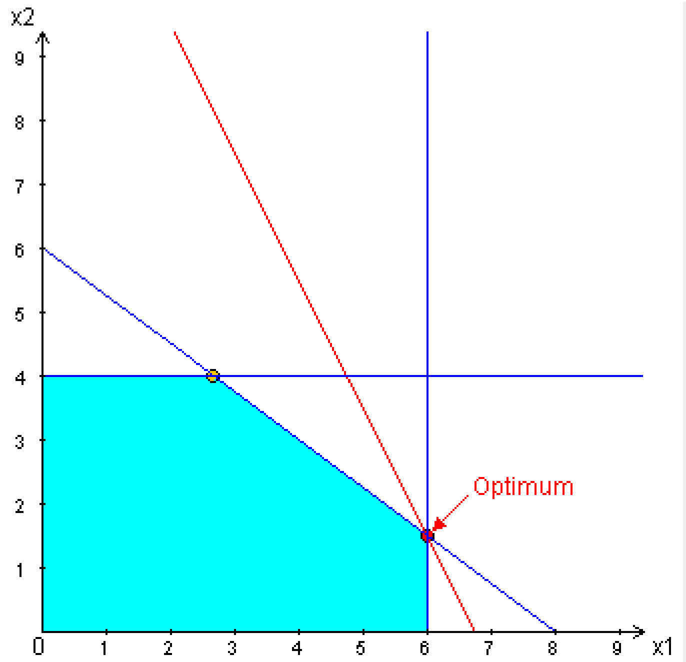
	Resource Usage per Unit of Activity		
Resource	**Wood-framed**	**Aluminum-framed**	**Available Amount**
Glass	6	8	48
Aluminum	0	1	4
Wood	1	0	6
Unit Profit	\$300	\$150	

(b) maximize $P = 300W + 150A$

subject to

$$\begin{aligned} 6W + 8A &\leq 48 \\ W &\leq 6 \\ A &\leq 4 \\ W, A &\geq 0 \end{aligned}$$

(c) Optimal Solution: $(W, A) = (x_1^*, x_2^*) = (6, 1.5)$ and $P^* = 2025$



(d) From Sensitivity Analysis in IOR Tutorial, the allowable range for the profit per wood-framed window is between 112.5 and infinity. As long as all the other parameters are fixed and the profit per wood-framed window is larger than \$112.50, the solution found in (c) stays optimal. Hence, when it is \$200 instead of \$300, it is still optimal to produce 6 wood-framed and 1.5 aluminum-framed windows and this results in a total profit of \$1425. However, when it is decreased to \$100, the optimal solution is to make 2.67 wood-framed and 4 aluminum-framed windows. The total profit in this case is \$866.67.

(e)

$$\begin{array}{llrl}
\text{maximize} & P = & 180W + 90A & \\
\text{subject to} & & 6W + \ 8A & \leq 48 \\
& & W \qquad & \leq 5 \\
& & A & \leq 4 \\
& & W, A \qquad & \geq 0
\end{array}$$

The optimal production schedule consists of 5 wood-framed and 2.25 aluminum-framed windows, with a total profit of \$1837.50.

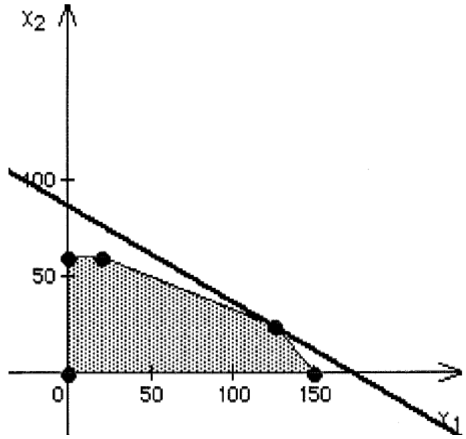
3.1-8.

(a) Let x_1 be the number of units of product 1 to produce and x_2 be the number of units of product 2 to produce. Then the problem can be formulated as follows:

$$\text{maximize } P = x_1 + 2x_2$$

$$\begin{array}{lrl}
\text{subject to} & 2x_1 + 3x_2 & \leq 200 \\
& 2x_1 + 2x_2 & \leq 300 \\
& x_2 & \leq 60 \\
& x_1, x_2 & \geq 0
\end{array}$$

(b) Optimal Solution: $(x_1^*, x_2^*) = (125, 25)$ and $P^* = 175$

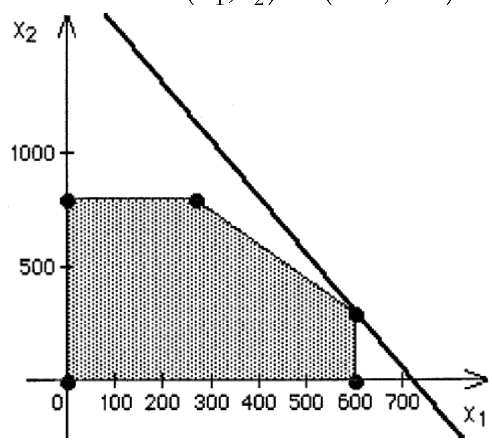


3.1-9.

(a) Let x_1 be the number of units on special risk insurance and x_2 be the number of units on mortgages.

$$\begin{array}{lrcl} \text{maximize} & z = 5x_1 + 2x_2 & & \\ \text{subject to} & 3x_1 + 2x_2 & \leq & 2400 \\ & x_2 & \leq & 800 \\ & 2x_1 & \leq & 1200 \\ & x_1 \geq 0, x_2 \geq 0 & & \end{array}$$

(b) Optimal Solution: $(x_1^*, x_2^*) = (600, 300)$ and $Z^* = 3600$



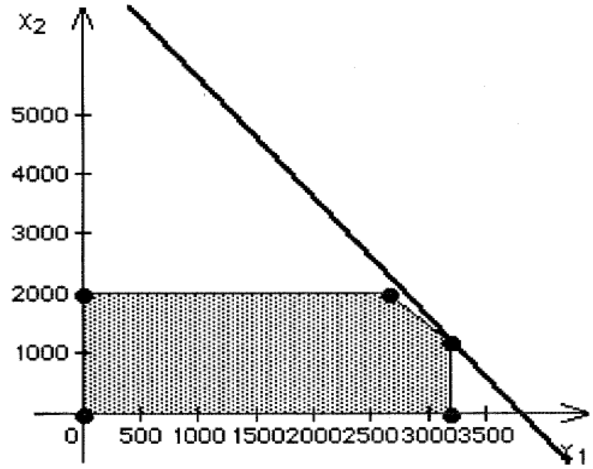
(c) The relevant two equations are $3x_1 + 2x_2 = 2400$ and $2x_1 = 1200$, so $x_1 = 600$ and $x_2 = \frac{1}{2}(2400 - 3x_1) = 300$, $z = 5x_1 + 2x_2 = 3600$.

3.1-10.

(a)

$$\begin{array}{ll} \text{maximize} & P = 0.88H + 0.33B \\ \text{subject to} & 0.1B \leq 200 \\ & 0.25H \leq 800 \\ & 3H + 2B \leq 12,000 \\ & H, B \geq 0 \end{array}$$

(b) Optimal Solution: $(x_1^*, x_2^*) = (3200, 1200)$ and $P^* = 3212$



3.1-11.

(a) Let x_i be the number of units of product i produced for $i = 1, 2, 3$.

$$\begin{array}{ll} \text{maximize} & Z = 50x_1 + 20x_2 + 25x_3 \\ \text{subject to} & 9x_1 + 3x_2 + 5x_3 \leq 500 \\ & 5x_1 + 4x_2 \leq 350 \\ & 3x_1 + 2x_3 \leq 150 \\ & x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(b)

Solve Automatically by the Simplex Method:

Optimal Solution

Value of the
Objective Function: Z = 2904.7619

Variable	Value
X_1	26.1905
X_2	54.7619
X_3	20

Constraint	Slack or Surplus	Shadow Price
1	0	4.7619
2	0	1.42857
3	31.4286	0
4	0	1.19048

Sensitivity Analysis

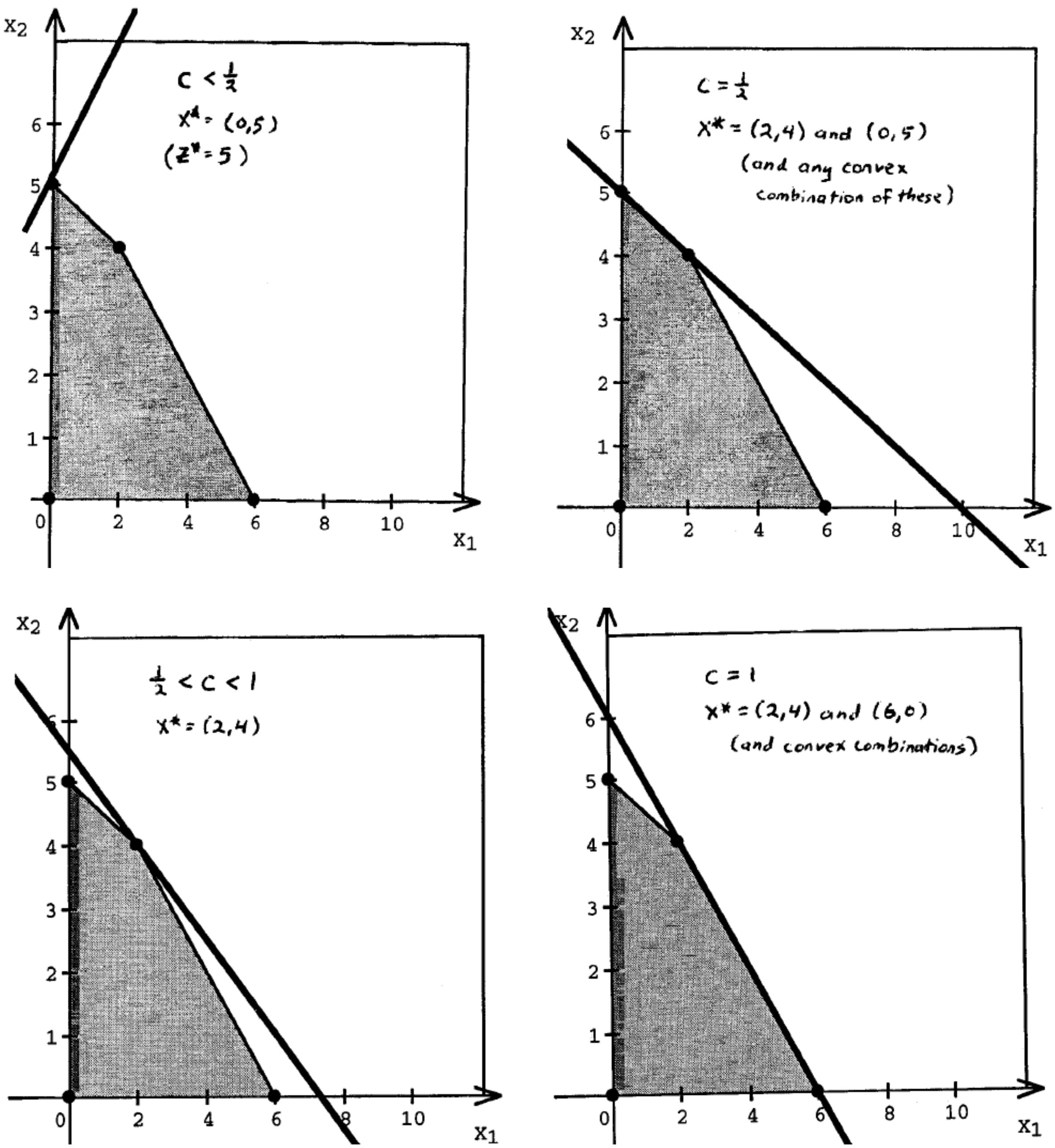
Objective Function Coefficient

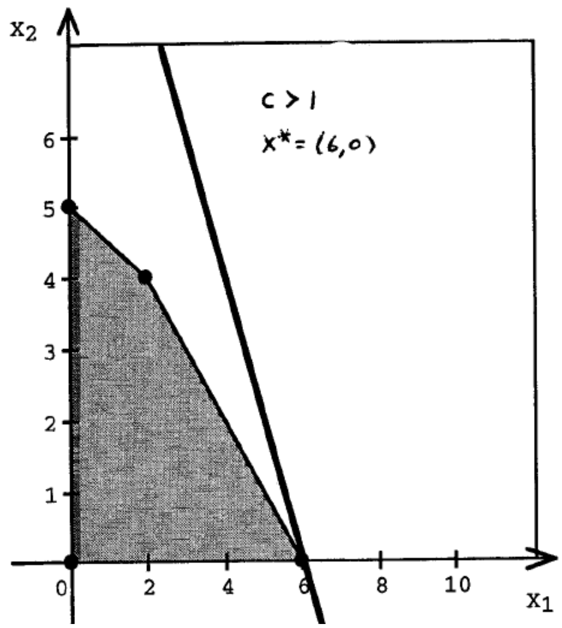
Current Value	Allowable Range To Stay Optimal Minimum	Maximum
50	25	51.25
20	19	40
25	23.8095	$+\infty$

Right Hand Sides

Current Value	Allowable Range To Stay Feasible Minimum	Maximum
500	362.5	555
350	276.667	533.333
150	118.571	$+\infty$
20	0	47.5

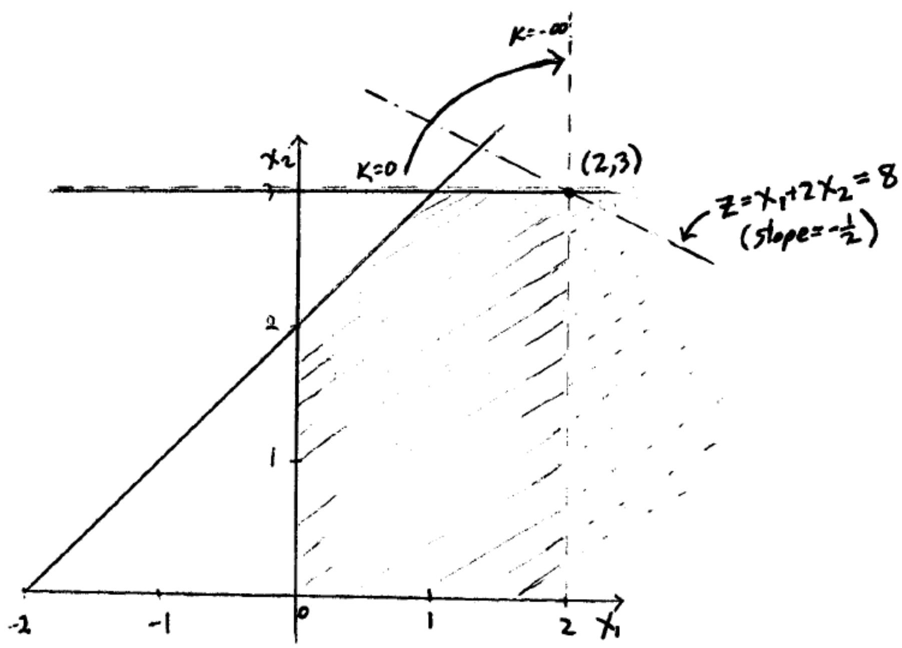
3.1-12.


x_2
x_1
$c < \frac{1}{2}$
$x^* = (0,5)$
$(Z^* = 5)$
$c = \frac{1}{2}$
$x^* = (2,4)$ and $(0,5)$
(and any convex combination of these)
$\frac{1}{2} < c < 1$
$x^* = (2,4)$
$c = 1$
$x^* = (2,4)$ and $(6,0)$
(and convex combinations)



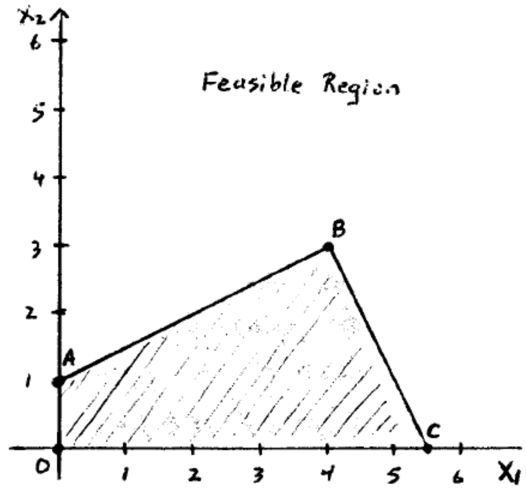
3.1-13.

First note that $(2,3)$ satisfies the three constraints, i.e., $(2,3)$ is always feasible for any value of k. Moreover, the third constraint is always binding at $(2,3)$, $kx_1 + x_2 = 2k + 3$. To check if $(2,3)$ is optimal, observe that changing k simply rotates the line that always passes through $(2,3)$. Rewriting this equation as $x_2 = -kx_1 + (2k+3)$, we see that the slope of the line is $-k$, and therefore, the slope ranges from 0 to $-\infty$.



As we can see, $(2,3)$ is optimal as long as the slope of the third constraint is less than the slope of the objective line, which is $-\frac{1}{2}$. If $k < \frac{1}{2}$, then we can increase the objective by traveling along the third constraint to the point $(2 + \frac{3}{k}, 0)$, which has an objective value of $2 + \frac{3}{k} > 8$ when $k < \frac{1}{2}$. For $k \geq \frac{1}{2}$, $(2, 3)$ is optimal.

3.1-14.



Case 1: $c_2 = 0$ (vertical objective line)

If $c_1 > 0$, the objective value increases as x_1 increases, so $x^* = (\frac{11}{2}, 0)$, point C.

If $c_1 < 0$, the opposite is true so that all the points on the line from $(0,0)$ to $(0,1)$, line $\overline{OA}$, are optimal.

If $c_1 = 0$, the objective function is $0x_1 + 0x_2 = 0$ and every feasible point is optimal.

Case 2: $c_2 > 0$ (objective line with slope $-\frac{c_1}{c_2}$)

If $-\frac{c_1}{c_2} > \frac{1}{2}$, $x^* = (0, 1)$, point A.

If $-\frac{c_1}{c_2} < -2$, $x^* = (\frac{11}{2}, 0)$, point C.

If $\frac{1}{2} > -\frac{c_1}{c_2} > -2$, $x^* = (4, 3)$, point B.

If $-\frac{c_1}{c_2} = \frac{1}{2}$, any point on the line $\overline{AB}$ is optimal. Similarly, if $-\frac{c_1}{c_2} = -2$, any point on the line $\overline{BC}$ is optimal.

Case 3: $c_2 < 0$ (objective line with slope $-\frac{c_1}{c_2}$, objective value increases as the line is shifted down)

If $-\frac{c_1}{c_2} > 0$, i.e., $c_1 > 0$, $x^* = (\frac{11}{2}, 0)$, point C.

If $-\frac{c_1}{c_2} < 0$, i.e., $c_1 < 0$, $x^* = (0, 0)$, point O.

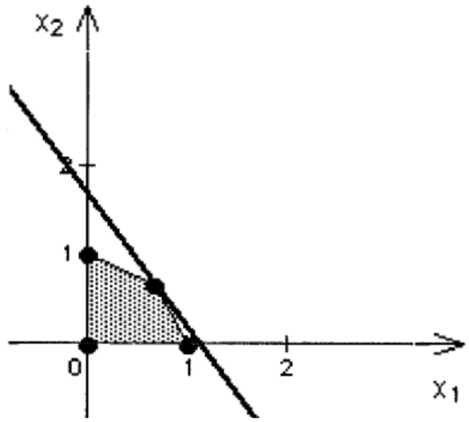
If $-\frac{c_1}{c_2} = 0$, i.e., $c_1 = 0$, x^* is any point on the line $\overline{OC}$.

3.2-1.

(a) maximize $P = 3A + 2B$

$$\begin{aligned} \text{subject to} \quad 2A + \ \ B &\leq 2 \\ A + 2B &\leq 2 \\ 3A + 3B &\leq 4 \\ A, B &\geq 0 \end{aligned}$$

(b) Optimal Solution: $(A, B) = (x_1^*, x_2^*) = (2/3, 2/3)$ and $P^* = 3.33$



(c) We have to solve $2A + B = 2$ and $A + 2B = 2$. By subtracting the second equation from the first one, we obtain $A - B = 0$, so $A = B$. Plugging this in the first equation, we get $2 = 2A + B = 3A$, hence $A = B = 2/3$.

3.2-2.

(a) TRUE (e.g., maximize $z = -x_1 + 4x_2$)

(b) TRUE (e.g., maximize $z = -x_1 + 3x_2$)

(c) FALSE (e.g., maximize $z = -x_1 - x_2$)

3.2-3.

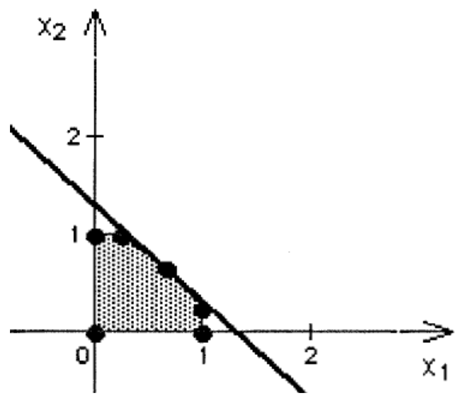
(a) As in the Wyndor Glass Co. problem, we want to find the optimal levels of two activities that compete for limited resources. Let x_1 and x_2 be the fraction purchased of the partnership in the first and second friends venture respectively.

	Resource Usage per Unit of Activity		
Resource	**1**	**2**	**Available Amount**
Fraction of partnership in 1st	1	0	1
Fraction of partnership in 2nd	0	1	1
Money	$\$10,000$	\$8000	$\$12,000$
Summer work hours	400	500	600
Unit Profit	\$9000	\$9000	

(b) maximize $P = 9000x_1 + 9000x_2$

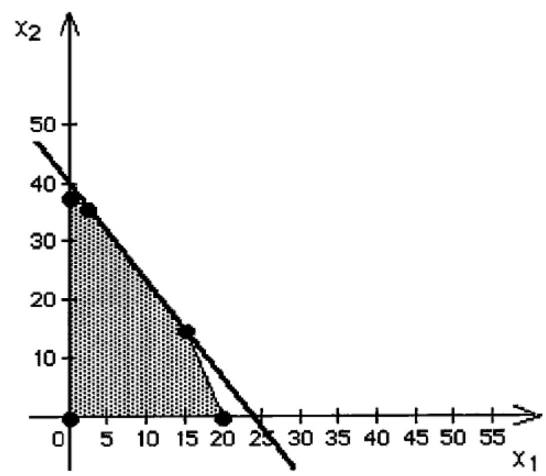
$$
\begin{aligned}
\text{subject to} \quad x_1 \qquad\qquad\qquad & \leq 1 \\
x_2 & \leq 1 \\
10,000x_1 + 8000x_2 & \leq 12,000 \\
400x_1 + \ \ 500x_2 & \leq 600 \\
x_1, x_2 & \geq 0
\end{aligned}
$$

(c) Optimal Solution: $(x_1^*, x_2^*) = (2/3, 2/3)$ and $P^* = 12,000$

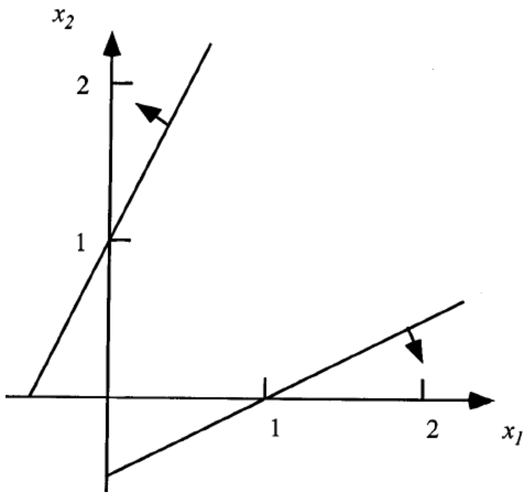


3.2-4.

Optimal Solutions: $(x_1^*, x_2^*) = (15, 15)$, $(2.5, 35.833)$ and all points lying on the line connecting these two points, $Z^* = 12,000$

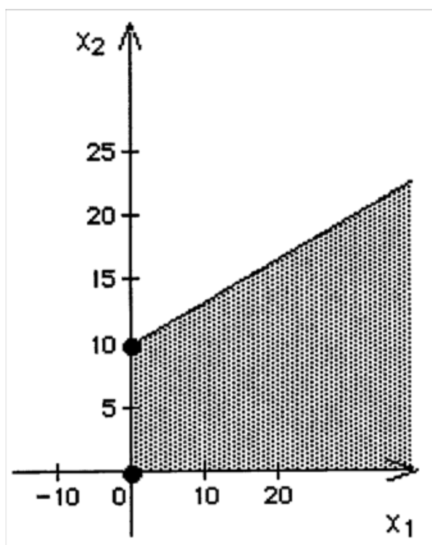


3.2-5.

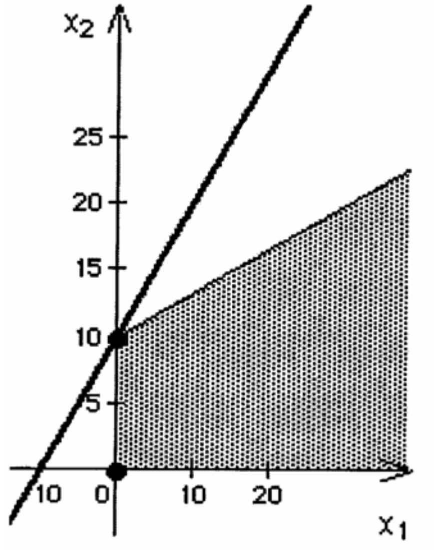


3.2-6.

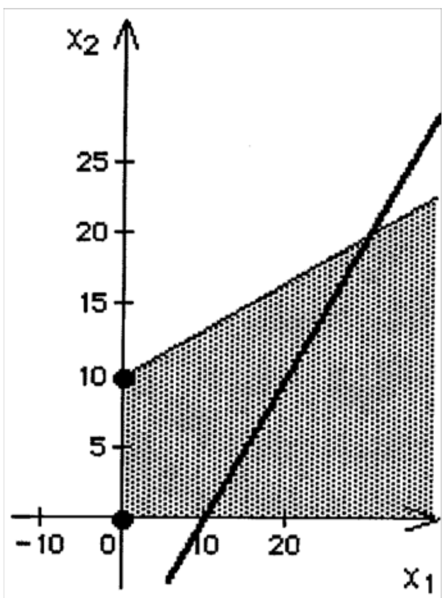
(a)



(b) Yes. Optimal solution: $(x_1^*, x_2^*) = (0, 10)$ and $Z^* = 10$



(c) No. The objective function value rises as the objective line is slid to the right and since this can be done forever, so there is no optimal solution.



(d) No, if there is no optimal solution even though there are feasible solutions, it means that the objective value can be made arbitrarily large. Such a case may arise if the data of the problem are not accurately determined. The objective coefficients may be chosen incorrectly or one or more constraints might have been ignored.

3.3-1.

Proportionality: It is fair to assume that the amount of work and money spent and the profit earned are directly proportional to the fraction of partnership purchased in either venture.

Additivity: The profit as well as time and money requirements for one venture should not affect neither the profit nor time and money requirements of the other venture. This assumption is reasonably satisfied.

Divisibility: Because both friends will allow purchase of any fraction of a full partnership, divisibility is a reasonable assumption.

Certainty: Because we do not know how accurate the profit estimates are, this is a more doubtful assumption. Sensitivity analysis should be done to take this into account.

3.3-2.

Proportionality: If either variable is fixed, the objective value grows proportionally to the increase in the other variable, so proportionality is reasonable.

Additivity: It is not a reasonable assumption, since the activities interact with each other. For example, the objective value at $(1,1)$ is not equal to the sum of the objective values at $(0,1)$ and $(1,0)$.

Divisibility: It is not justified, since activity levels are not allowed to be fractional.

Certainty: It is reasonable, since the data provided is accurate.

3.4-1.

In this study, linear programming is used to improve prostate cancer treatments. The treatment planning problem is formulated as an MIP problem. The variables consist of binary variables that represent whether seeds were placed in a location or not and the continuous variables that denote the deviation of received dose from desired dose. The constraints involve the bounds on the dose to each anatomical structure and various physical constraints. Two models were studied. The first model aims at finding the maximum feasible subsystem with the binary variables while the second one minimizes a weighted sum of the dose deviations with the continuous variables.

With the new system, hundreds of millions of dollars are saved and treatment outcomes have been more reliable. The side effects of the treatment are considerably reduced and as a result of this, postoperation costs decreased. Since planning can now be done just before the operation, pretreatment costs decreased as well. The number of seeds required is reduced, so is the cost of procuring them. Both the quality of care and the quality of life after the operation are improved. The automated computerized system significantly eliminates the variability in quality. Moreover, the speed of the system allows the clinicians to efficiently handle disruptions.

3.4-2.

(a) Proportionality: OK, since beam effects on tissue types are proportional to beam strength.

Additivity: OK, since effects from multiple beams are additive.

Divisibility: OK, since beam strength can be fractional.

Certainty: Due to the complicated analysis required to estimate the data about radiation absorption in different tissue types, sensitivity analysis should be employed.

(b) Proportionality: OK, provided there is no setup cost associated with planting a crop.