1 π attenuator

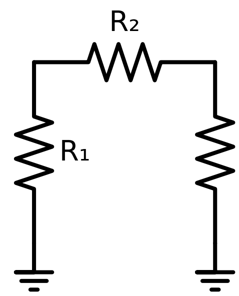

Figure 1: π-attenuator

The three constraints for the attenuator,

1. $R_1 = R_3$
2. $S_{11} = 0$, or $Z_{in} = R_0$
3. $S_{21} = A$

Starting with the second constraint, when looking into the left side of the network, and assuming the right side is terminated in $R_0 = 50\Omega$, Z_{in} is,

$$Z_{in} = (R_3||R_0 + R_2)||R_1$$

$$Z_{in} = \frac{\left(\frac{R_3 R_0}{R_3+R_0} + R_2\right)R_1}{\left(\frac{R_3 R_0}{R_3+R_0} + R_2\right) + R_1}$$

$$Z_{in} = \frac{\Big(R_3 R_0 + R_2(R_3 + R_0)\Big)R_1}{\Big(R_3 R_0 + R_2(R_3 + R_0)\Big) + R_1(R_3 + R_0)}$$

Using $R_1 = R_3$, and the constraint that $Z_{in} = R_0$,

$$R_0 = \frac{\Big(R_1 R_0 + R_2(R_1 + R_0)\Big)R_1}{\Big(R_1 R_0 + R_2(R_1 + R_0)\Big) + R_1(R_1 + R_0)}$$

Solve for R_2,

$$R_0\Big(\Big(R_1 R_0 + R_2(R_1 + R_0)\Big) + R_1(R_1 + R_0)\Big) = \Big(R_1 R_0 + R_2(R_1 + R_0)\Big)R_1$$

$$R_0 R_2(R_1 + R_0) - R_1 R_2(R_1 + R_0) = R_1^2 R_0 - R_1 R_0^2 - R_0 R_1(R_1 + R_0)$$

$$R_2\Big(R_0(R_1 + R_0) - R_1(R_1 + R_0)\Big) = R_1^2 R_0 - R_1 R_0^2 - R_0 R_1(R_1 + R_0)$$

$$R_2 = \frac{R_1^2 R_0 - R_1 R_0^2 - R_0 R_1 (R_1 + R_0)}{R_0(R_1 + R_0) - R_1(R_1 + R_0)}$$

$$R_2 = \frac{2R_1 R_0^2}{R_1^2 - R_0^2}$$

The third constraint requires the voltage transfer function to be equal to the desired voltage loss, A.

$$S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^+=0} = A$$

Since the input impedance is matched, $V_1^- = 0$, and because the s-parameter definition requires that $V_2^+ = 0$,

$$A = \frac{V_2}{V_1}$$

If a voltage V_1 is applied at port 1, the voltage at port 2 is,

$$V_2 = V_1 \frac{R_3 || R_0}{R_2 + R_3 || R_0}$$

$$\frac{V_2}{V_1} = \frac{R_3 || R_0}{R_2 + R_3 || R_0}$$

$$\frac{V_2}{V_1} = \frac{\frac{R_3 R_0}{R_3 + R_0}}{R_2 + \frac{R_3 R_0}{R_3 + R_0}}$$

$$\frac{V_2}{V_1} = \frac{R_3 R_0}{R_2(R_3 + R_0) + R_3 R_0}$$

Again using $R_1 = R_3$,

$$A = \frac{V_2}{V_1} = \frac{R_1 R_0}{R_2(R_1 + R_0) + R_1 R_0}$$

Solve for R_2,

$$A R_2 (R_1 + R_0) + A R_1 R_0 = R_1 R_0$$

$$R_2 = \frac{R_1 R_0 - A R_1 R_0}{A(R_1 + R_0)}$$

$$R_2 = \frac{R_1 R_0 (1 - A)}{A(R_1 + R_0)}$$

We have two unknowns, R_1 and R_2, with two equations,

$$R_2 = \frac{2R_1 R_0^2}{R_1^2 - R_0^2}$$

$$R_2 = \frac{R_1 R_0 (1-A)}{A(R_1 + R_0)}$$

Equating,

$$\frac{R_1 R_0 (1-A)}{A(R_1 + R_0)} = \frac{2R_1 R_0^2}{R_1^2 - R_0^2}$$

Solving for R_1,

$$R_1 R_0 (1-A)(R_1^2 - R_0^2) = 2R_1 R_0^2 A(R_1 + R_0)$$

$$(1-A)(R_1^2 - R_0^2) = 2R_0 A(R_1 + R_0)$$

$$(1-A)(R_1^2) - 2R_0 A(R_1) = 2R_0^2 A + R_0^2(1-A)$$

$$(1-A)(R_1^2) - 2R_0 A(R_1) - R_0^2(A+1) = 0$$

$$(1-A)(R_1^2) - 2R_0 A(R_1) + R_0^2(1-A) = 0$$

Using the quadratic formula to solve for R_1,

$$R_1 = \frac{2R_0 A \pm \sqrt{4A^2 R_0^2 - 4R_0^2(1-A)(A+1)}}{2(1-A)}$$

$$R_1 = \frac{R_0 A \pm R_0\sqrt{A^2 - (A^2+1)}}{(1-A)}$$

$$R_1 = R_0 \frac{A \pm 1}{1-A}$$

A is less than 1 (since attenuator is passive), so the denominator is always positive. For R_1 to be non-negative, we must take the $A+1$ solution,

$$R_1 = R_0 \frac{A+1}{1-A}$$

Using this in the equation for R_2,

$$R_2 = \frac{R_1 R_0 (1-A)}{A(R_1 + R_0)}$$

$$R_2 = \frac{R_0 \frac{A+1}{1-A} R_0 (1-A)}{A(R_0 \frac{A+1}{1-A} + R_0)}$$

$$R_2 = \frac{R_0 (A+1)}{A(\frac{A+1}{1-A} + 1)}$$

$$R_2 = R_0 \frac{(A+1)(1-A)}{A(A+1+(1-A))}$$

$$R_2 = R_0 \frac{1-A^2}{2A}$$