Generating Functions

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Prepare For Some Stretching

Consider and reconsider this very advanced counting technique called generating functions by recalling the basic fact of exponentiation that underlies their usefulness:

$\mathrm{x}^{\mathrm{a}}\,\mathrm{x}^{\mathrm{b}} = \mathrm{x}^{\mathrm{a} + \mathrm{b}}$

Hold On For The Ride

Now look at these facts:

If $\mathrm{G(x)} = \sum_{\mathrm{k}\,=\,0}^{\infty} \mathrm{a_k}\,\mathrm{x}^{\mathrm{k}}$

then $\mathrm{xG(x)} = \sum_{\mathrm{k}\,=\,0}^{\infty} \mathrm{a_k}\,\mathrm{x}^{\mathrm{k}\,+\,1} = \sum_{\mathrm{k}\,=\,1}^{\infty} \mathrm{a_{k\,-\,1}}\,\mathrm{x}^{\mathrm{k}}$

and $\mathrm{x^2G(x)} = \sum_{\mathrm{k}\,=\,0}^{\infty} \mathrm{a_k}\,\mathrm{x}^{\mathrm{k}\,+\,2} = \sum_{\mathrm{k}\,=\,2}^{\infty} \mathrm{a_{k\,-\,2}}\,\mathrm{x}^{\mathrm{k}}$

etc.

Consider the sequence $\mathrm{H_n} = \{0, 1, 3, 7, 15, \ldots\}$, coming from the Tower of Hanoi puzzle. The number $\mathrm{H_n}$ is the minimal number of moves needed to transfer a stack of n disks to an empty peg. As can be concretized from a look at some code, the recurrence relation is

$\mathrm{H_n} = 2\mathrm{H_{n\text{-}1}} + 1.$

The initial condition $\mathrm{H_1} = 1$ is typically given, but in fact we can define $\mathrm{H_0} = 0$ just as easily (it takes 0 moves to move 0 disks)! Then the recurrence holds for $n \geq 1$. At first this may be a problem (especially if you haven't taken DM3 yet) but with practice it becomes a straightforward iterative exercise to use substitution to solve this recurrence relation:

$$
\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&\vdots \\
&= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \\
&= 2^n - 1
\end{aligned}
$$

This approach repeatedly uses the recurrence relation to express $\mathrm{H_n}$ in terms of previous terms of the sequence. In the next to last equality, the initial condition $\mathrm{H_1} = 1$ has been used. (And the last equality is based on the formula for the sum of a geometric series, which should be familiar to every DM student.)

Let H be the generating function for the sequence $\mathrm{H_n}$. Then $\mathrm{H} = \mathrm{H_0} + \mathrm{H_1\ x} + \mathrm{H_2\ x^2} + \mathrm{H_3\ x^3} + \cdots$.

Then, since $\mathrm{H_0} = 0$ and $\mathrm{H_n} = 2\mathrm{H_{n\ -\ 1}} + 1$ for $n \geq 1$:

$$
\begin{aligned}
H &= 0 + (2H_0 + 1)x + (2H_1 + 1)x^2 + (2H_2 + 1)x^3 + \cdots \\
&= 2H_0x + 1x + 2H_2x^2 + 1x^2 + 2H_2x^3 + 1x^3 + \cdots \\
&= 2x(H_0 + H_1x + H_2x^2 + H_3x^3 + \cdots) + x + x^2 + x^3 + \cdots \\
&= 2xH + x(1 + x + x^2 + x^3 + \cdots)
\end{aligned}
$$

Then $\mathrm{H - 2xH} = \mathrm{x}(1 + \mathrm{x} + \mathrm{x^2} + \mathrm{x^3} + \cdots)$,
or $\mathrm{H(1 - 2x)} = \mathrm{x}(1 + \mathrm{x} + \mathrm{x^2} + \mathrm{x^3} + \cdots) = \mathrm{x}\frac{}{(1-x),}$
where the last equation follows from easily-verifiable fact that
$\sum_{\mathrm{i}=0}^{\infty} \mathrm{x^i} = 1/(1 - \mathrm{x})$. Thus, by dividing both sides by (1 - 2x):
$\mathrm{H} = \mathrm{x}\frac{}{(1-2x)(1-x).}$

We want to get a formula for the coefficients of H, but there's no easy way to do that without breaking the denominator into a sum!

So, recalling (!) how in calculus we integrated rational functions using partial fraction decomposition, we can rewrite the fraction on the right in the form $\mathrm{A}\frac{}{1-2x+\frac{B}{1-x},where A and B are constants. That means}$

$$
\frac{x}{(1-2x)(1-x)} = \frac{A}{1-2x} + \frac{B}{1-x} = \frac{A(1-x) + B(1-2x)}{(1-2x)(1-x)} = \frac{(A+B) + (-A-2B)x}{(1-2x)(1-x)},
$$

and therefore, by equating coefficients in the numerators, we get A + B = 0 and -A -2B = 1. These equations are easily seen to have the solution A = 1 and B = -1. Thus

$$H = \frac{x}{(1-2x)(1-x)} = \frac{1}{1-2x} - \frac{1}{1-x}.$$

But then the above easily-verifiable fact combined with another fairly-easily-verifiable (and similar) fact yield:

$$\begin{aligned} H &= 1/(1-2x) - 1/(1-x) \\ &= (1 + 2x + 4x^2 + 8x^3 + \cdots) - (1 + x + x^2 + x^3 + \cdots) \\ &= (1-1) + (2-1)x + (4-1)x^2 + (8-1)x^3 + \cdots . \end{aligned}$$

Thus we see that H_n, the coefficient of x^n, is 2^n - 1, which agrees with what we found by the iterative approach.

Generalize

Here's the general method for solving recurrences using generating functions:

1. Multiply both sides of the recurrence equation by x^k.
2. Sum both sides over all k for which the equation is valid.
3. Choose a generating function G(x). (Usually $G(x) = \sum_{k=0}^{\infty} a_k x^k$.)
4. Rewrite the equation in terms of the generating function G(x).
5. Solve for G(x).
6. The coefficient of x^k in G(x) is a_k.

Recur

Here is one author's blurb about the myth of the Tower of Hanoi:

> A myth created to accompany the puzzle tells of a tower in Hanoi where monks are transferring 64 gold disks from one [diamond] peg to another, according to the rules of the puzzle. The myth says that the world will end when they finish the puzzle. How long after the monks started will the world end if the monks take one second to move a disk?
>
> From the explicit formula, the monks require

2^{64} - 1 = 18,446,744,073,709,551,615

moves to transfer the disks. Making one move per second, it will take them more than 500 billion years to complete the transfer, so the world should survive a while longer than it already has.

– Kenneth H. Rosen, *Discrete Mathematics and Its Applications*

If even 500 billion years seems inadequate to fully comprehend the use of generating functions in solving recurrence relations, it should be encouraging to contemplate the following "grook" by Piet Hein:

T. T. T.

Put up in a place
where it's easy to see
the cryptic admonishment
T. T. T.

When you feel how depressingly
slowly you climb,
it's well to remember that
Things Take Time.

Here is another one — also a favorite!

The road to wisdom? –
Well, it's plain
and simple to express:

Err
and err
and err again

but less
and less
and less.