Gödel's Incompleteness Theorem

CS 480
Computational Theory

Machines will never have *all* the answers...

There will always be room for a creative person who can think of a better way of doing things.

Try to catch the universe in a finite net of axioms and the universe will fight back. Reality is, on the deepest level, essentially infinite. No finitely programmed machine can ever exhaust the richness of the mental and physical world we inhabit.

The proof of Gödel's Incompleteness Theorem is so simple, and so sneaky, that it is almost embarassing to relate. His basic procedure is as follows:

Gödel Meets the Universal Truth Machine

1. Someone introduces Gödel to UTM, a machine that is supposed to be a Universal Truth Machine, capable of correctly answering any question at all.
2. Gödel asks for the program and circuit diagrams of the UTM. The program may be complicated, but it can only be finitely long. Call the program $P(UTM)$ for Program of the Universal Truth Machine.
3. Smiling a little, Gödel writes out the following sentence: "The machine constructed on the basis of the program $P(UTM)$ will never say that this sentence is true." Call this sentence G for Gödel. *Note that G is equivalent to "UTM will never say G is true."*
4. Now Gödel laughs and asks UTM whether G is true or not.

The Liar Paradox **Exponentiated**

5. If UTM says G is true, then "UTM will never say G is true" is false. If "UTM will never say G is true" is false, then G is false (since $G =$ "UTM will never say G is true.") So if UTM says that G is true, then G is in fact false, and UTM has made a false statement. So UTM will never say that G is true, since UTM makes only true statements.
6. We have established that UTM will never say G is true. So "UTM will never say G is true" is in fact a true sentence. So G is true (since $G =$ "UTM will never say G is true.").
7. "I know a truth that UTM can never utter," Gödel says. "I know that G is true. UTM is not truly universal."

Think about it — it grows on you.

Gimmick?

The gimmick in Gödel's proof is very similar to the gimmick in the famous Liar paradox of Epimenides: "I am lying," says Epimenides. Is he? [...] Define B to be the sentence "B is not true." Is B true? The problem is that B is true if and only if B is not true. So B is somehow outside the scope of the applicability of the concepts "true" and "not true." There is something viciously meaningless about the sentence B, and one is inclined just to try to forget about it.

But Gödel's G sentence cannot be so lightly dismissed. With his great mathematical and logical genius, Gödel was able to find a way (for any given $P(UTM)$) actually to write down a complicated polynomial equation that has a solution if and only if G is true. So G is not at all some vague or non-mathematical sentence. *G is a specific mathematical problem that we know the answer to, even though UTM does not!* So UTM does not, and cannot, embody a best and final theory of mathematics.

— Rudy Rucker, *Infinity and the Mind*

Gödel Sums It Up

"The human mind is incapable of formulating (or mechanizing) all its mathematical intuitions, i.e., if it has succeeded in formulating some of them, this very fact yields new intuitive knowledge, e.g., the consistency of this formalism. This fact may be called the 'incompletability' of mathematics. On the other hand, on the basis of what has been proved so far, it remains possible that there may exist (and even be empirically discoverable) a theorem-proving machine which in fact is equivalent to mathematical intuition, but cannot be proved to be so, nor even be proved to yield only **correct** *theorems of finitary number theory."*

— Kurt Gödel