CS 480	Computational Theory

Is Creativity Algorithmic?

How Mathematicians Think — By William Byers

The subtitle of this fascinating book is *Using Ambiguity, Contradiction, and Paradox to Create Mathematics* — and in light of the types of questions that I'm asking you to ponder this semester, I thought it good to share with you the author's concluding chapter.

Conclusion: Is Mathematics Algorithmic or Creative?

The objective of this concluding chapter is to summarize and draw out the implications of the view of mathematics I have been developing in the preceding chapters. I have been describing mathematics as an activity that is dynamic and creative, pulsating with the life of the mind. It is a "way of knowing" that is quite unique.

But mathematics is not off in some obscure corner of human activity; it is central to human experience and human culture. Thus it is not surprising that many of the great questions of the day have a certain reflection within mathematics. The question that I have chosen to explore in this chapter is the relationship between computing and mathematics, between mathematical thought and computer simulations of thought processes. This discussion will allow me to summarize much of what has been discussed in the book so far and demonstrate why its importance extends beyond mathematics itself. It will highlight the difference between the point of view I have been taking and the views one finds among many consumers of mathematics in the scientific and technological community as well as the public at large.

Normally using mathematics to investigate some subject means creating a mathematical model, using mathematical techniques to draw out the mathematical consequences of that model, and, finally, translating the conclusions back to the original situation. This is not the way I shall be using mathematics. The inferences I shall discuss come not from the content of some mathematical model or theory; they will come from the nature of mathematics itself.

Can Computers Do Mathematics?

The question that will be a springboard into my discussion will be the naive question of whether computers are capable of doing mathematics — today or conceivably in the future.

What do I mean by the question, "Can a computer do mathematics?" I am not asking whether computers can be helpful, even indispensable assistants to a human mathematician as, for example, in the case of the proof of the four-color theorem (the theorem that states that any map can be colored with four different colors in such a way that no two adjacent regions have the same color) when the computer checked the validity of the conjecture in a large but finite number of cases. After all, it was mathematicians who had shown that the problem could be reduced to the verification of these cases.

Nor am I asking whether a computer can be used to generate data or graphics that can give the human investigator a "feel" for some mathematical situation. Again, I am not asking if the computer can be used to attempt to ascertain the likelihood (formally or informally) that a certain conjecture is or is not the case.

I am asking whether it is conceivable that at sometime in the future computers could completely take over the show, whether a machine could be programmed to "do" mathematics from start to finish. This would involve (among other activities) examining mathematical situations or situations that potentially could be mathematized, producing data about these situations, generating conjectures, and demonstrating the validity or invalidity of these conjectures. Put in this way, the answer to the question of whether a computer could ever do mathematics is clearly "No!" (The discussion about whether a computer can do mathematics is usually restricted to the last of these activities, namely, demonstrating the validity of certain mathematical statements.)

William Thurston said, "In practice, mathematicians prove theorems in a social context. It [mathematics] is a socially conditioned body of knowledge and techniques." An increasing number of mathematicians would agree with the statement that mathematics is what (human) mathematicians do. If we take this view, then it is tautological that only mathematicians can do mathematics. Ironically, it is the very success of computers in profoundly infiltrating the day-to-day activities of many working mathematicians that has raised the question of the role of proof in mathematics. If mathematics is not defined merely as a proof-generating activity, then it is difficult to see it being developed by autonomous machine intelligence (if there ever is such a thing.) Thus the more extreme version of the question of computers and mathematics is not very profound.

A more interesting question is the following: What aspects of mathematical activity can computers duplicate and, on the other hand, what aspects of mathematics (if any) cannot be duplicated by machines? Success in isolating some of these characteristics would be important. It would have implications for other related questions like "Is intelligence algorithmic? What is creativity and what is its relationship to deductive, thinking? And even what is the nature of Mind?"

There is today a powerful point of view that the mind is a computer and that thinking and problem solving can now, or will in the future, be done by "thinking machines." This point of view is not restricted to the artificial intelligence community but is common in the entire field of cognitive science. More important, the idea that thinking is the kind of thing that computers do or simulate is a pervasive influence throughout our entire culture. Many people have the idea that computers have a kind of infinite potential — if there is some task that they cannot accomplish today, it seems inevitable that they will be able to do it tomorrow or the day after. This is almost an article of faith for many people.

What this book has emphasized is how our current expectations for computers and algorithmic thought arise out of such elements as formalism and the older "dream of reason." This is such a basic element of the cultural heritage handed down to us from the Greeks that it is understandable that as a society we are unwilling to relinquish this dream even in the face of all kinds of evidence to the contrary. Relinquishing this dream means also relinquishing the hope that all of our problems will be eliminated by computer technology. In a way the dream expressed by formalism has morphed into today's dreams of artificial intelligence. These dreams have in common the faith that it is possible to banish ambiguity and, therefore, a certain form of complexity from human life.

Roger Penrose has taken strong issue with the belief in the unlimited potential of algorithmic thought in a series of books that have become very popular. For Penrose there is something other than complicated calculation involved in mathematical activity — there is an aspect of mathematical truth that goes beyond anything that can be accessed by algorithmic thinking. This book has, in a sense, been a search for this noncomputational factor in mathematical thought and therefore, by extension, in all thought.

The idea that "the mind is a computer" is a dangerous one, and it is important that it be refuted. However, any refutation will not come easily precisely because we have all become part of the culture of information technology, and in that culture the notion that just about everything is programmable is a seductive and powerful dream — a modern myth. Why do I say that modeling thought on computer-generated activity is dangerous? It is because such ideas can easily become self-fulfilling prophesies. We may soon come to *define* thought as computer-like activity. Steven Pinker, for example, is a well-known cognitive scientist and author of the immensely popular book, *How the Mind Works*. In this book he gives two criteria for intelligence. The first is "to make decisions rationally by means of some set of rules" and the second is "the ability to attain goals in the face of obstacles based on rational truth-obeying rules." What is interesting is Pinker's repeated use of the terms "rational" and "rules." He is *defining* intelligence as a kind of algorithmic rationality such as may be found in a mathematical proof or a computer algorithm. If we accept this kind of definition then the question of whether a computer is intelligent is a circular one. In this way computer thought may well have become metaphor for human thought.

An all-pervasive metaphor becomes its own reality. In the sense that the metaphor "time is money" conditions the way in which we experience time, so "the brain is a computer" is conditioning the way we look at the human mind, even the manner in which we look at what it means to be human. The danger is that our infatuation with the computer as a model for ourselves will result in our forgetting about other ways of using the mind. The danger is that we shall lose touch with the deeper aspects of what it means to be human. This is the larger context of this discussion.

Computers do what they do according to the algorithms that are programed into them. Questions about the theoretical capabilities of artificial intelligence are really questions about the nature and limits, if any, of algorithmic thought. Now mathematics has a great deal to say about algorithmic thought. In fact, for my purposes the question "Can a computer think?" which is equivalent to the question "Is thought algorithmic?" could be replaced by “Can a computer do mathematics?” or “Is mathematics algorithmic?” Mathematics is the part of our culture that has investigated algorithmic thought in the most profound and systematic way. The results of that investigation should be available to inform the contemporary discussion of the nature of human thought. Unfortunately, many people who address such questions know little of the culture of mathematics. On the other hand, mathematicians, as a rule, are not inclined to stand back from their subject — from the day-to-day problems of their research and consider the larger metaquestions that are raised, at least implicitly, by the nature of their work. For this reason, there is scarcely any input from mathematics or mathematicians on these questions, even though mathematics provides much of the cultural context within which such questions arise.

Most claims that computers can do mathematics are really claims that computers can generate correct proofs. Thus it may be worthwhile to spend another moment discussing the relationship between proof and mathematics. What is the value of a proof? The irony is that, as was mentioned above, it is precisely the computational and graphical abilities of computers that have thrown into doubt the centrality of proof in the mathematical firmament. Nevertheless we would certainly not want to argue that the computer is incapable of spinning out lines of code that could be interpreted as formal proofs. Indeed, if one is a formalist, that is, if one believes that there is no deeper content to mathematics other than the formal set of axioms, definitions and proofs, then a computer *can* do mathematics. However Chapter 8 contained a discussion, of the severe limitations of that way of looking at mathematics.

The difference between a computer and a human mathematician can be demonstrated by asking the question, “What stands behind the formal mathematical proof?” For the computer the formal proof *is* mathematics. For the mathematician there is a whole universe of intuition and understanding that lies behind the formal proof. Proofs, or more precisely the ideas on which the proofs are based, come out of this domain of informal mathematics. But for the computer this domain does not exist.

Mathematics is about understanding! Proofs are important to the extent that they help develop an understanding of some mathematical situation. What a computer can simulate is the proof-generating aspect of mathematics but not the understanding. What computers mimic is a secondary mathematical activity, not the primary activity.

Suppose that a computer succeeded in generating a set of mathematically valid results. What criteria would enable the computer to decide which were important, what directions of inquiry were worth pursuing, and so forth? Mathematics is not a game like bridge or chess. It has no obvious beginning and no well-defined ending. Mathematics is an open-ended exploration. Unlike physics, no one will ever proclaim the “end of mathematics.” Mathematics is a human activity intimately connected with the human need to discern patterns in their environment. As such it is related to properties of the human mind and our need to draw conceptual maps that facilitate our understanding of the natural world and ourselves.

Is Mathematical Thought Algorithmic?

Is mathematical thought algorithmic? In keeping with the theme of basing my discussion of mathematics on what mathematicians actually do, I shall approach this question through a discussion of two terms that have great currency in the informal discussions of mathematicians — the terms "trivial" and "deep." The highest compliment a mathematician can give to a piece of mathematics is to say that it is "deep." Although most people would be hard pressed to make precise what is meant by "depth" in mathematics (or in anything else), nevertheless most good mathematicians would have no hesitation in classifying a particular result as "deep" as opposed to superficial or "trivial." In fact, an instinct for which problems and results are deep is often taken as a criterion of how good a mathematician a particular individual is. An excellent mathematician works on "deep" problems and produces "deep" results.

In fact this entire book has given us various hints, not of what to take as a definition of depth, but of conditions that may accompany depth or even produce it. Consider the definition of ambiguity and its essential ingredient, namely, a double perspective. This dualism may be seen to produce depth by analogy with the way visual depth is produced in binocular vision. Two visual perspectives each of which is consistent but which are not identical combine to produce a single, unified perspective that is richer than either of the originals. This is analogous to what happens when you go from a surface appreciation of a particular mathematical phenomenon to a deeper understanding. Understanding seems to carry with it a sense of depth, a multidimensionality or sense of multiple perspectives. Thus the entire discussion of ambiguity is connected to the notion of depth.

Furthermore, the depth of a particular idea seems to sometimes be related to that other aspect of ambiguity — the degree of incompatibility between the original frameworks. Concepts that are difficult in this way — zero, irrational numbers, infinity — turn out to be important, turn out to have a great deal of mathematical content. The resolution of paradoxical situations depends on the development of significant mathematical ideas. This leads to the conclusion that has been mentioned a number of times now: depth resides in ambiguity but only when the situation is resolved by an act of creativity. One might say that the depth of a mathematical situation is a measure of the creativity that accompanies its birth.

"Trivial" results, on the other hand, are results that follow in a mechanical way from the premises; that are superficial, formal, and require no "idea" or act of understanding. I am suggesting that algorithmic thinking is trivial. Though users of mathematics are required to master a certain number of algorithms, the advantage of this mode of thinking is precisely that it can be applied mechanically.

The difference between the "deep" and the "trivial" is crucial to our conception of mathematics. The great French mathematician Henri Poincaré was well aware of what was at stake here. He said:

> the very possibility of mathematical science seems an insoluble contradiction. If this science is only deductive in appearance, from whence is derived that

> perfect rigour which is challenged by none? If, on the contrary, all the propositions which it enunciates may be derived in order by the rules of formal logic, how is it that mathematics is not reduced to a gigantic tautology? The syllogism can teach us nothing essentially new, and if everything should be capable of being reduced to the principle of identity, then everything should be capable of being reduced to that principle. Are we then to admit that the enunciations of all the theorems with which so many volumes are filled, are only indirect ways of saying that A is A?

Poincaré rejects the vision that mathematics is no more than mere tautology. He goes on to say, "it must be granted that mathematical reasoning has of itself a kind of creative virtue, and is therefore to be distinguished from the syllogism. *The difference must be profound* ." Poincaré compares the creative to the syllogistic, whereas I use the term algorithmic, but the point is the same — there is a creative depth to mathematics. This is the mystery of mathematics which cannot be understood through a kind of logical reductionism.

"Trivial" Mathematics: "I Follow It But I Don't Understand It!"

Mathematics is usually approached in one of two ways. The first approaches it *instrumentally* as a body of useful results, techniques, formulas, equations,and so on that are assumed to be valid and can be applied to solve various kinds of problems. This is how engineers use mathematics, for example, or psychologists use statistics. One doesn't delve too deeply into the derivation of the techniques or the question of why they work. One accepts that they work and one moves on.

The second approach to mathematics ostensibly addresses the question of *why* mathematics works, The question of "why" is generally answered in theoretic mathematics, the approach you will find in books written by mathematicians and in courses taught by them. The question of why mathematics is valid is answered by embedding that bit of mathematics into a deductive system.

The development of an area within pure mathematics is characterized by a certain approach — the deductive, axiomatic approach. As in Euclidean geometry, this involves starting with axioms and definitions and building toward results that follow rigorously from these assumptions. One begins with an abstract characterization of the system to be considered. Consider the following example.

[Example 1 elided — see Brother Neff if you're curious!]

That is the end of the argument. It this argument frustrates you, if your reaction is that the argument is much ado about nothing or if the argument is too abstract for you, you have a point. In fact I am taking a chance in using this example because I may just succeed in "turning off" my reader just as so many students are turned off in theoretical courses in pure mathematics.

It is difficult to understand what is going on in this proof precisely because this kind of argument is all at the formal, logical level. The point of the example is that one can follow the logic of the definitions and argument without knowing anything about the subject, with no experience with anything connected to the subject, and therefore certainly no intuition or "feel" for the subject. In response to this argument one of my students made the observation, "I follow it but I don't understand it." This is precisely the point that I am making! With a little care and effort one can verify that the argument is logically correct, but, because one has no feeling for the context or ideas that are involved, one feels that one does not understand what is going on. Verification is one thing, understanding quite another. Verification is superficial, restricted to the surface of things. This is what I mean by "trivial."

Whereas an argument that is structured around some mathematical idea (see Chapter 5) can be understood by grasping that idea, a formal argument like the one above cannot really be understood because there is, in a sense, nothing to understand. All that can be done with an argument like this is to verify it, and a computer could do this verification. This is what it means for an argument to be "trivial"!

Almost all of the mathematics that I have discussed in the earlier chapters has been non-trivial to a certain extent at least. However I will now repeat a very simple result just to contrast it with the previous one.

Example 2: Adding Integers

Consider the formula for the sum of the first n integers,

$$1+2+\cdots+n=\frac{n(n+1)}{2}$$

Here is a simple argument for this result: Let

$$S=1+2+\cdots+(n-1)+n;$$

Reversing the order for addition gives

$$S=n+(n-1)+\cdots+2+1;$$

Adding we see that $2S$ is given by adding up the n columns each of which sums to $n+1$. That is,

$$2S=n(n+1)$$

Thus $S=n(n+1)/2$ as required.

This is a very simple argument, but it is not completely trivial because it contains an idea, namely reversing the sum and adding by columns. This idea shows us not only that the formula is valid but also *why* it is true. We understand this result in a way that we do not understand the previous formal argument.

One of the main goals of this work has been to differentiate between the algorithmic and the profound in mathematics. How, that is, can we distinguish between "trivial" thought and "complex" thought? In the above examples a distinction was tentatively drawn between the "trivial" and the "simple" — the first argument was called "trivial," the second is "simple." Mathematical thought can be simple and it can also be complex but mostly it is nontrivial. Computer thought, on the other hand, even though it may be very lengthy and complicated, is essentially trivial.

Is the distinction between the "trivial" and the "deep" of any value? Is mathematics trivial or nontrivial? Is science? Is human thought in general nontrivial? Does the algorithmic subsume all other thought, that is, is all thought trivial? In the case of mathematics, the famous attempt of Russell and Whitehead (1925) to show that mathematics could be reduced to logic was therefore an attempt to *prove* that mathematics was trivial. Gödel showed that arithmetic was nontrivial and therefore the attempt of Russell and Whitehead was a failure. This is why Penrose evokes the work of Gödel to support his claim. He knows in his gut that mathematics is nontrivial.

Mathematics is a wonderful vantage point from which to take up the question of "triviality" versus profundity. Return for a moment to the origins of systematic mathematical thought. In Chapter 2 I discussed the "dream of reason" that emerged in the civilization of ancient Greece, the discovery of a new way of thinking that seemed to have the potential of allowing human beings access to a permanent and objective truth. This vision is still a powerful cultural force in our society. In the terminology of Chapter 7, it is a "great idea." In that chapter I discussed formalism as an example of a great idea, but really formalism is just a particular development of a much older idea — the idea of reason itself. This is a core idea for our civilization which has been at work changing human beings and their cultures, their sense of themselves and of the natural world for thousands of years.

Today, with the dawning of the age of information technology we stand at a new and crucial stage in the development of this core idea. What has changed is our technology. The computer is the tool through which this great idea is being implemented. All technology takes a human characteristic, makes it autonomous, and extends it. Behind every technological advance there is an idea — the machine is really this idea given a physical form. The telescope extends the act of seeing; the telephone the act of speaking. The computer extends a certain way of using the mind — it extends rationality, logic, and, in particular, what I have called algorithmic thinking. Now the idea of the computer has all the characteristics of a great idea. It is an extraordinary development and we are all, to a certain extent at least, caught up in the rush of enthusiasm that accompanies the birth of this great idea. However, remember that a great idea is always wrong in that it always overreaches itself when it claims universality. A technology can only objectify a certain aspect of the human potential. It inevitably approaches human beings from a certain perspective and inevitably ignores other perspectives. Human beings are not only logical animals; we are also creative animals. Thus the dream of the human being as computer, of human thought as computer thought, is false even if the computers in question are supercomputers whose computational powers dwarf those of present day machines. Nevertheless, the "dream of reason" in its most modern incarnation is refashioning modern culture in its own image.

Yet, in a way, this modern version of the "dream of reason" is the dream that thought is "trivial" or can be made "trivial."

I claim that there is something that is going on in mathematical thought that is nontrivial. What is this other factor? Mathematics is deep; mathematics is basically a creative activity; mathematics is one of the most profound ways of using the mind. Much of this book has been an attempt to get a handle on what it means for a piece of mathematics to be "deep." In a sense the entire book is a discussion of the distinction between the "trivial" and the "nontrivial" within mathematical thought. The question of distinguishing between "depth" and "triviality" is vital! Is profundity a matter of quality or quantity? The proponents of artificial intelligence might argue that it is one of quantity and that the advent of bigger and faster machines will make all of human thought accessible to machine replication. I maintain that the difference is qualitative.

One of the great mysteries of mathematics is contained in this question about the nature of "depth." It is similar to the question of what makes a work of art great. The best practitioners agree that "depth" refers to something that is real, but what is it exactly? This has become an important question ever since the advent of formalism in mathematics and even more so with the advent of computers. Could a computer be programmed to distinguish between the trivial and the deep? Or is depth merely subjective, completely in the mind of the beholder; and thus not really there at all? This is a very important question, a deep question, I am tempted to say. The introduction of the idea of "ambiguity" was an attempt to get a handle on this question.

In his introduction to Davis and Hersh; Gian-Cario Rota says, "The mystery of mathematics is that . . . conclusions originating in the play of the mind do find striking physical applications." It is indeed this surprising correspondence that pushes us to take a more careful look at what is going on in mathematics. In his famous paper I referred to earlier on the "Unreasonable Effectiveness of Mathematics in the Natural Sciences," Wigner claims that the power of mathematics lies in the ingenious definitions that mathematicians have developed. These definitions, he feels, capture some very deep aspects of reality and are therefore the secret of why mathematics works as well as it does. This is fine as far as it goes, but it cries out for additional clarification. For one thing, what is the nature of these ingenious definitions? What is it, exactly, about mathematical concepts that make them so fruitful?

In my attempt to get a handle on "depth," I found it necessary to reconsider the very things that mathematical reasoning seems to be delivering us from; to reconsider the myth of reason itself. I did this by introducing the notion of "ambiguity" in all of its many guises as another aspect of mathematics that complements the logical structure. Ambiguity and paradox are aspects of mathematical thought that differentiate the "trivial" from the "deep." The "trivial" arises from the elimination of the ambiguous. The "deep" involves a complex multidimensionality such as those evoked by the successful resolution of situations of ambiguity and paradox. Even the word "resolution" is misleading in this context because it usually implies the reduction of the ambiguous to the logical and linear. What really happens is that the ambiguity gives birth to a larger context, a unified framework that contains the various potentialities that were inherent in the original situation.

Mathematics, as I have been describing it, is an art form. The words ambiguity and metaphor are much more acceptable in the arts than they are in the sciences. But ambiguity and metaphor are the mechanisms through which that ultimate ambiguity, the one that divides the objective from the subjective, the natural world from the mind, is bridged. It is the same mechanism that operates at all levels — from a simple mathematical concept like “variable” to the mathematics of “string theory.” The same mechanism even applies to a discussion of mathematics as a whole.

Mathematics As Complexity

One of the truly remarkable things about mathematics is the manner in which it has provided human beings with models and metaphors that have been used to make sense of the world. Euclidean geometry is important, not just as geometry but as the model of reason it provided for other areas of mathematics and for human thought in general. It operates in a double way — as scientific theory and as metaphor. Differential equations provided the model of the machine universe in which the past and the future are absolutely determined from knowledge of the present state of affairs. The list goes on and on.

In Chapter 7, I briefly discussed the modem theories of chaos and complexity. These theories have created a great stir in recent years — not because they have succeeded in making original and verifiable predictions — but because they provide a new metaphor, a new framework for thinking about the world, including the world of science. Viewed in this way, “chaos and complexity theory” can be thought of as self-referential — it describes aspects of the natural world and it provides a way of thinking about mathematics of which it forms a part. In a sense this book is a part of this new way of thinking. It is an attempt to develop a new way of thinking about mathematics — an attempt to think out what it would mean to apply the metaphoric complexity to mathematics itself.

What would a description of “mathematics as complexity” be like? The following paragraphs are a brief indication of some of the characteristics that such a description might include.

First of all, one must draw a distinction between the terms “complex” and “complicated.” As the terms are used in this book, “complicated” refers to the quantitative,”complex” to the qualitative. As I have noted, it is possible for logical mathematical reasoning to be complicated without being “deep” — without containing very interesting mathematical ideas. “Complexity” refers primarily to the world of mathematical insights. It refers to mathematics both as a body of knowledge and as a practice — the extraordinary mixture of subtle reasoning and profound intuition that characterizes both mathematics and the manner in which the mind is used in mathematics. Mathematics is both complicated and complex, but one’s choice of which property to emphasize as essential will reveal one’s orientation toward mathematics. Is it toward the algorithmic or the creative?

The theory of chaos arises from the study of *nonlinearity*. Complexity is fundamentally nonlinear. If mathematics is nonlinear, then its essence cannot be captured by algorithmic procedures or by the linear strings of reasoning that characterize both mathematical proofs and deductive systems. Mathematics is a world of dynamic change — an extraordinarily complex world with its own ecological structure. In this world new concepts are continually coming into being while others are sinking into irrelevance. Knowledge is continually being reorganized and reevaluated in the light of new interests and new ways of thinking. One metaphor for nonlinear mathematics comes from the theory of turbulence, like that describing the flow of water in a river. In a situation of turbulence there are features, such as whirlpools, that are stable but can disappear if the rate of flow of the water changes. A turbulent system is constantly reorganizing itself — depending on the rate of flow of the fluid involved. In the same way, mathematics can be thought of, not as permanent and absolute, but as in a continual state of dynamic rearrangement. There are whole areas of mathematics that were once of great interest but that today are not pursued and sometimes not even remembered.

Another aspect of complex systems is that they are *open*. For example, a biological system is in continuous interaction with other systems, including the ambient environment. Thus a biological organism, like a human being, for example, can only be considered in isolation as a temporary expedient, not as an absolute reality. Even though it may sound strange, mathematical theories also cannot be considered in isolation from other parts of mathematics, from the sciences and computer science, and from human culture in general. Mathematics is a human endeavor with roots in the natural world and human biology. In particular, mathematics changes and evolves. However it is often described as though it were absolute and timeless.

Complex systems inevitably involve an element of *contingency*, an element of uncertainty. "Randomness" is discussed in Chapter 7 as basic to two of the most far-reaching scientific theories of our time — quantum mechanics and evolution. Human life as we experience it involves a healthy dose of the accidental and the unpredictable. Where is the uncertain in our description of mathematics? Conventionally mathematics is thought of as the antithesis of the uncertain. It is the epitome of the certain and the absolute. In my approach, the uncertain exists as a key element of mathematics yet in a manner that is unique and quite characteristic of the unique way in which mathematics approaches the world. The French philosopher Edgar Morin says, "Logical positivism could not avoid taking the role of an epistemological policeman forbidding us to look precisely where we must look today, toward the uncertain, the ambiguous, the contradictory." In mathematics the role of logical positivism is taken by formalism, which renders invisible the properties of uncertainty and ambiguity that form an indispensable aspect of mathematics.

If mathematics cannot be accurately represented as isolated from its environment, it follows that an adequate description of mathematics will have to include the related processes of creating and understanding mathematics. The normal view is that there is an objective body of mathematical theory on the one side and, on the other, the mathematician who creates the theory or the consumer who uses it. The idea that there could be a point of view that contains both elements, that is, the introduction of this human element, changes everything. Mathematics is no longer a strictly "objective" theory. "Objectivity" is merely a description of one dimension of mathematics; that is, objectivity is merely an approximation to what is going on. It is one point of view — a useful one — but not the definitive one.

If mathematics includes the mathematician, then it is reasonable to see intelligence as an essential ingredient of mathematics. Mathematics is a form of intelligence in action; that is, it is not only the objective result of an act of intelligence but rather a demonstration of the nature of intelligence itself. It is a major way in which intelligence functions. When we study mathematics we are not so much absorbing some predetermined set of facts as we are studying the manner in which the mind works — the manner in which it produces mathematics. (In saying this I am aware that I am using "mathematics" in an ambiguous way, both as content and as process.)

One of the essential properties of intelligence is flexibility. What does it mean to be an expert at something? What makes someone an expert is her ability to respond to a completely novel situation — to solve original problems, for example. Expertise does not only involve having command of a huge amount of factual knowledge — it does not mean being a human data bank. It involves the capacity for flexible response. It is a form of creativity. This description applies equally to the notion of "understanding." Understanding is a kind of expertise. A true measure of intelligence is this capacity for flexible and original response. How does this flexibility arise in mathematics? As was mentioned above, the essence of algorithmic thought is the elimination of flexibility by providing predetermined responses to any given eventuality. Biological systems that are successful in an evolutionary sense have, many believe, the flexibility to adapt to a changing environment. Complex thought in mathematics also contains a kind of flexibility that is part of any reasonable definition of intelligence.

There is a profound relationship between complexity and simplicity. For example, the well-known mathematician and popularizer of "catastrophe theory" E. C. Zeeman said that "Technical skill is the mastery of complexity, while creativity is mastery of simplicity." What then is simplicity? What is it that makes a piece of mathematics simple? It is a commonly observed phenomenon that a piece of mathematics is simple if you understand it and difficult if you do not. This is not a joke or a definition of the word "simple," but it reflects the way the word is often used. This usage of the word implies that the mathematical situation is simplified, if you will, by an act of intelligence, that is, of understanding. Thus a picture of mathematics emerges, a picture that was discussed in the chapters on "mathematical ideas," of mathematics as a hierarchy of the simple. Data look complicated and "hard" until the emergence of a mathematical idea structures the data and even makes them "simple." The ideas that make things simple are not objective in any absolute sense. It makes more sense to call them optimizations of the potential contained in the original situation. These ideas often solidify into new mathematical objects that form the data that may be organized by new, higher order ideas. This is the way in which "complex" situations develop. Both the process of mathematical thought and its product are now unified in a complex system that on one hand is an intricate structure and on the other is flexible and capable of dynamic change. In one dimension there is the order and permanence of the formal, deductive system. The other dimension is open to ambiguity and contradiction, open to insight and creativity, open to change.

Are Human Beings Trivial?

The question for the proponents of "algorithmic intelligence" is the following. Do the creative uses of the mind that I have discussed throughout this book arise from subtle algorithmic processes, or is the reverse the case, namely, that algorithmic processes arise from another and more basic way of using the mind? This book has been an argument for the existence of other intelligent ways of thinking, ways of thinking that do not reduce to Pinker's "rational truth-obeying rules." What happens in a situation of ambiguity? You may have one set of rational rules, one context, that pushes you in one direction and a second that pushes you in a different direction. It is precisely at the level of rational rules that things break down. How does one operate in a situation of conflict and incompatibility? Is intelligence inoperative in such situations? Of course not. A situation of conflict is precisely where we most need the intervention of creative intelligence.

Thus the larger question concerning the centrality of algorithmic thought comes down to whether or not you believe that at the most basic level things are ordered, rational, and algorithmic — what I think of as a kind of Platonic vision of the human mind. Now there is much to be said for this vision of the power and importance of the algorithmic — it is, in fact, a great idea. We are in the inevitable period of inflation that comes with any great idea — the time when it is claimed that this particular insight will explain everything. Thus this book is merely saying, "No, this is not the only way of using the mind. There is another way." In this other way, from this different perspective, disorder and conflict are never definitively eliminated. In fact the very attempt to describe human activity in an algorithmic, rule-based way leads to the problematic situations that I have called ambiguous, contradictory, and paradoxical. Working with these situations, attempting to understand them in the face of their problematic aspects, gives rise to acts of intelligence and creativity that produce the order that we observe.

The implications of this discussion are considerable. What is at stake is nothing less than our conception of what it means to be a human being. To claim that thought is "trivial" is to claim that human beings are "trivial" In my opinion to hold that human beings are trivial is to miss something vital, something that all previous human cultures would have seen as self-evident. *Human beings are not machines! Human beings are not trivial!* What could be more important than this question? It is perhaps ironic that mathematics, the area out of which computer science developed, and the discipline with which it has the greatest affinity, should be the domain in which the distinction between algorithmic and non-algorithmic should be the most accessible.

In other cultures this kind of question about the nature of the human mind would have religious implications. It would be answered with reference to "God" or "spirit" or the "soul." In our secular culture these questions are addressed by computer scientists, cognitive psychologists, and philosophers of science. Nevertheless the stakes remain high and involve everyone, for what is at issue is human self-definition — the nature of the beings we tell ourselves that we are.

Mathematics In The Light Of Ambiguity — A Great Idea

Maybe this is a good place to come back to the beginning and summarize the point of view that is developed in the preceding pages. I have attempted to lay the foundations for a great idea, an idea of human nature as fundamentally creative. This creative process that we call mathematics, for example, has no end, no ultimate objective, and therefore will never be completed. It follows that the problematic situations such as ambiguity and contradiction, are never completely eliminated. In fact the creative process thrives on such situations. Now in order for such a point of view to be considered seriously it has been necessary to put limitations upon a competing great idea — the idea of the human mind as a logical machine. Mathematics in its largest sense is the arena in which this particular battle has been fought, but its implications apply to the worlds of science and technology and to the culture that these have spawned.

In a sense the questions that are raised in this book involve matters of life and death. Not life and death in a literal sense but in the metaphorical sense of living systems versus mechanical ones. In one sense biology is *about* living systems, whereas physics is *about* nonliving ones. However in another sense, physics, biology, and mathematics can be alive or dead. At the research level these subjects are all alive, all growing and changing. The practice of a scientific discipline has many of the characteristics of a living organism. On the other hand, a discipline may stop developing, may become moribund because it is thought of as a completed, theoretical "truth" to which nothing will ever be added or taken away. This attitude signals the end of mathematics and science. If a description of mathematics is to have any value it must describe a discipline that is living and growing, not a subject that is frozen.

One of the aspects of a living system is that it is creative. Life is continually being confronted with problems and the necessity to resolve these problems. Problems, in life, in art, and in science, are inevitable. Not only can they not be avoided but they are the very things that spur development, that spur evolution. The solutions that these problematic situations bring forth are unpredictable, a priori. The solution to such problems often involves an element that is entirely unexpected — the creative element. A creative solution is not mechanical — it does not involve juggling a number of predetermined elements according to predetermined rules. It involves the emergence of a novel way of looking at the original situation. This new way of seeing is often generated by very incompatible tendencies within the original situation that made it problematic in the first place. This is the essence of a living system as it is the essence of mathematics. This living essence is intimately connected to what I have been calling "the light of ambiguity." The essence of mathematics is that it is nontrivial, creative, and alive.