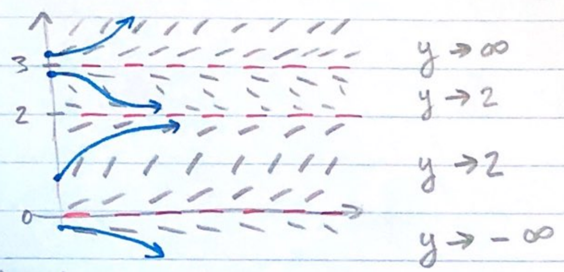
Exam 1 Study Guide

Direction Fields

Identifying Equilibria:

$y' = 0$

Ex) $y' = y(y-2)(y-3)$ Eq.: $y = 0, 2, 3$



Analyze behavior based on initial conditions:

$y_0 > 3, \quad y' = (+)(+)(+) = (+)$

$2 < y_0 < 3, \quad y' = (+)(+)(-) = (-)$

$0 < y_0 < 2, \quad y' = (+)(-)(-) = (+)$

$y_0 < 0, \quad y' = (-)(-)(-) = (-)$

Integrating Factor

☆ First Order ODE. $y' + p(t)y = g(t)$

$$\mu = e^{\int p(t)dt} \longrightarrow \frac{d}{dt}(\mu \cdot y) = \mu g$$

$$\mu \cdot y = \int \mu g \, dt$$

Tank Modeling

In: $r_{in} = \frac{V_{in}}{t}$ $c_{in} = \frac{Q_{out}}{V_{out}}$

Out: $r_{out} = \frac{V_{out}}{t}$ $c_{out} = \frac{Q(t)}{V(t)}$

$Q'(t) = r_{in}c_{in} - r_{out}c_{out}$ $V'(t) = r_{in} - r_{out}$

Overflow problem? Find t, $V_{max} = V(t)$

Euler's Method

$$y_n = y_{n-1} + f(t_{n-1}, y_{n-1}) \cdot h$$
$$t_n = t_{n-1} + h$$

Ex) $y' = t^2 - y^2 \Rightarrow f(t,y) = t^2 - y^2$ $\quad h = 0.1$

$y(0) = 1$

t	$y_n = y_{n-1} + hy$	$y' = t^2 - y^2$	hy'
0	1	-1	$(0.1)(-1) = -0.1$
0.1	$1 + (-0.1) = 0.9$	$(0.1)^2 - (0.9)^2 = -0.8$	$(0.1)(-0.8) = -0.08$ ...

Exam 2 Study Guide

All Equations are Linear

Homogeneous w/ Constant Coeffs

$$ay'' + by' + cy = 0$$

1. Assume $y = e^{rt}$
2. Solve $ar^2 + br + c = 0$
3. Three Cases:

① $r_1 \neq r_2$ and $r_1, r_2 \in \mathbb{R}$

$\Rightarrow$ $\underline{y_1 = e^{r_1 t}}$ $\underline{y_2 = e^{r_2 t}}$

② $r_1 = r_2$ and $r_1, r_2 \in \mathbb{R}$

$\Rightarrow$ Let $r = r_1 = r_2$

$\underline{y_1 = e^{rt}}$ $\underline{y_2 = t \cdot e^{rt}}$

③ $r_{1,2} = \lambda \pm \mu i$ $r_1, r_2 \in \mathbb{C}$

$\Rightarrow$ $\underline{y = e^{\lambda t}(c_1 \cdot \sin(\mu t) + c_2 \cdot \cos(\mu t))}$

Existence and Uniqueness Theorem

$$y'' + p(t)y' + q(t)y = g(t)$$

$$y(t_0) = y_0 \qquad y'(t_0) = y_0'$$

There exists one interval I containing t_0.

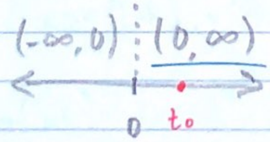
Ex) $y'' + \frac{3}{t}y = 1$, $y(1) = 1$ $y'(1) = 2$

$t_0 = 1$

$p = 0$

$q = \frac{3}{t}$; $t \neq 0$

$g = 1$



$\underline{I = (0, \infty)}$

Reduction of Order

$$P(t)\,y'' + Q(t)\,y' + R(t)\,y = 0$$

Given: y_1

1. **Assume:** $y_{gen} = v(t)\cdot y_1(t)$

$$y_g = v\,y$$
$$y_g' = v'y + y'v$$
$$y_g'' = v''y + y''v + 2y'v'$$

2. **Plug in and Factor v's**

$$(\quad)v'' + (\quad)v' + \underbrace{(\quad)}_{\text{must } = 0}v = 0$$

$$\Rightarrow (\quad)v'' + (\quad)v' = 0$$

3. **Let** $w = v'$

$$\Rightarrow \underbrace{(\quad)w' + (\quad)w = 0}_{\text{First-Order Linear Eq.}}$$

4. Integrating Factor $\mu = e^{\int p(t)\,dt}$

5. Solve for w and back-substitute for v

Non-Homogeneous Eq's

$$y'' + p(t)y' + q(t)y = g(t)$$

Method of Undetermined Coefficients

Only works if:

$g(t) =$ Exponential, Polynomial, sin, cos — Addition, Subtraction, and Multiplication of any or all Functions is permitted

Find y_1 and y_2 (Homogeneous Solutions)

Solve

Find $y_p = Y(t)$ from $g(t)$ using table.

$g = e^{at} \rightarrow y_p = Ae^{at} \cdot t^m$

$g = t^n \longrightarrow y_p = (A_n t^n + A_{n-1} t^{n-1} + \dots + A_0) \cdot t^m$

$g = \sin(bt)$ or $g = \cos(bt)$
$\rightarrow y_p = [A\cos(bt) + B\sin(bt)] \cdot t^m$

$g = t^n \cdot e^{at}$
$\rightarrow y_p = e^{at}(A_n t^n + A_{n-1} t^{n-1} + \dots + A_0) \cdot t^m$

$g = e^{at} \cdot \sin(bt)$ or $e^{at} \cdot \cos(bt)$
$\rightarrow y_p = e^{at}(A\cos(bt) + B\sin(bt)) \cdot t^m$

$g = t^n \cdot e^{at} \cdot \sin(bt)$ or $t^n \cdot e^{at} \cdot \cos(bt)$

$$y_p = e^{at} \cdot [\sin(bt)(A_n t^n + A_{n-1} t^{n-1} + \dots + A_0) + \cos(bt)(B_n t^n + B_{n-1} t^{n-1} + \dots + B_0)] \cdot t^m$$

t^m is the extra factors that might be needed for y_p due to repeats of y_1 or y_2 found in y_p.

Variation of Parameters

Used for when $g(t)$ is not exp, poly, sin, or cos.

$$y'' + p(t)y' + q(t)y = g(t)$$

<u>Assume</u>: $y_g = u_1 y_1 + u_2 y_2$

$$W = W(y_1, y_2)$$

$$u_1' = \frac{-y_2 \cdot g}{W} \qquad u_2' = \frac{y_1 \cdot g}{W}$$

$$y_g = u_1 y_1 + u_2 y_2 = y_c + y_p$$

Electrical and Mechanical Vibrations

$$mu'' + \gamma u' + ku = F(t)$$

$$u(0) = u_0 \qquad u'(0) = u_0'$$

<u>Formulas</u>: $w = mg$, $w = kL$, $F_d = \gamma v$

<u>Undamped</u> <u>Free</u> Vibrations

$\gamma = 0$ $\quad$ $F(t) = 0$

$\Rightarrow mu'' + ku = 0$. Assume $u = e^{rt}$

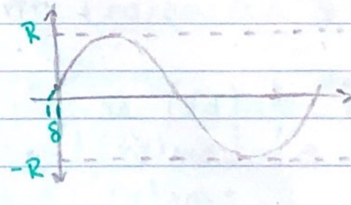
$r = \pm \omega_0 i$ $\quad$ ω_0: Natural Circular Frequency $= \sqrt{\frac{k}{m}}$

$$u_g = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

T: Natural Period

R: Amplitude

δ: Phase Shift



Damped Free Vibrations

$\gamma \neq 0 \qquad F(t) = 0$

$\Longrightarrow \quad mu'' + \gamma u' + ku = 0$, Assume $u = e^{rt}$

Three Cases:

$r_1 \quad r_2$ and $r_1, r_2 \in \mathbb{R}^-$

$\Longrightarrow \quad \underline{u_1 = e^{r_1 t}} \qquad \underline{u_2 = e^{r_2 t}}$ Over Damping

$r_1 \quad r_2$ and $r_1, r_2 \in \mathbb{R}^-$

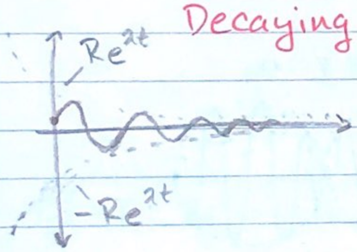
$\Longrightarrow$ Let $r = r_1 = r_2$

$\underline{u_1 = e^{rt}} \qquad \underline{u_2 = t \cdot e^{rt}}$ Critical Damping

$r_{1,2} = \lambda \pm \mu i \qquad \lambda < 0$

$\Longrightarrow \quad \underline{u = e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t))}$

Decaying Oscillation



μ: Quasi Frequency

T: Quasi Period

R: Amplitude $= \sqrt{A^2 + B^2}$

δ: Phase Shift $= \tan^{-1}\left(\frac{B}{A}\right)$

B: sin
A: cos

$$u(t) = R \cdot e^{\lambda t} \cdot \cos(\mu t - \delta)$$

To obtain this equation...

$u = c_1 \cos(\mu t) + c_2 \sin(\mu t)$

$u = A \cos(\mu t) + B \sin(\mu t)$

$\longrightarrow R = \sqrt{A^2 + B^2}$

$u = R \cos(\mu t - \delta)$ Angle Subtraction Identity

$= R(\cos(\mu t) \cdot \cos(\delta) + \sin(\mu t)\sin(\delta))$

$= R\cos(\delta)\cos(\mu t) + R\sin(\delta)\sin(\mu t)$

$$\begin{array}{c|l} \cos(\mu t) & A = R\cos(\delta) = x \longrightarrow A^2 = R^2\cos^2(\delta) \\ \sin(\mu t) & B = R\sin(\delta) = y \longrightarrow B^2 = R^2\sin^2(\delta) \end{array}$$

$\tan(\delta) = \frac{B}{A} \quad \Longrightarrow \quad \boxed{\delta = \tan^{-1}\left(\frac{B}{A}\right)}$ Add π if needed based on Quadrant

Electrical Vibrations

I = Current $\qquad$ L = Inductor

R = Resistance $\qquad$ E = Voltage

C = Capacitor $\qquad$ Q = Charge

$$I = Q'$$

$$L I' + R I + \frac{1}{C} Q = E(t)$$

or $$L Q'' + R Q' + \frac{1}{C} Q = E(t)$$

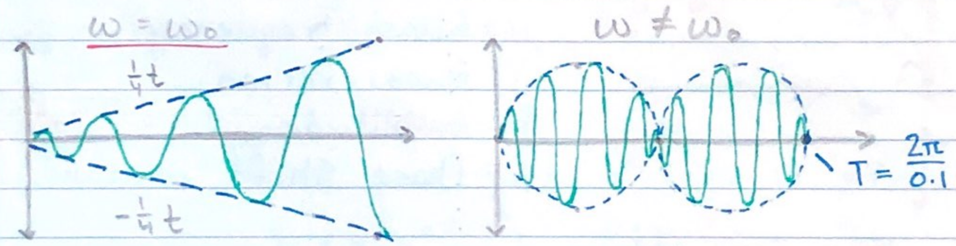
Forced Vibrations

$F(t) \neq 0$

$$mu'' + \gamma u' + ku = F(t) \leftarrow \omega_0 \text{ lives here}$$

$$\rightarrow u_c = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \leftarrow \omega_0 \text{ lives with } u_c$$

Solve for nonhom. Two Cases:



Resonance

$u = \frac{1}{4}t \cdot \sin(t)$

Beat

$u = \alpha \sin(0.1t) \sin(0.9t)$

slower freq.

Trigonometric Identity

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\cos(t) - \cos(0.8t) = -2\sin(0.9t)\sin(0.1t)$$

Ch 5 Series Solutions

Power Series

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

$$y'' = \sum_{n=2}^{\infty} n \cdot (n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + \ldots$$

Reindexing

$$\sum_{n=0}^{\infty} a_n x^{n+4} \xrightarrow[+4]{-4} \sum_{n=4}^{\infty} a_{n-4} x^n$$

Inverse Operations!

Series Solutions near an Ordinary Point

x_0 is an Ordinary Point if $P(x_0) \neq 0$ for

$$P(x) y'' + Q(x) y' + R(x) y = 0$$ ← Homogeneous Equation

Ex) $2y'' + xy' + 3y = 0$

$$2\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} + 3\sum_{n=0}^{\infty} a_n x^n = 0$$

Reindex until each term is x^n

$$2\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \boxed{x^n} + \sum_{n=1}^{\infty} n a_n \boxed{x^n} + 3\sum_{n=0}^{\infty} a_n \boxed{x^n} = 0$$

Plug in: $n = 0 \rightarrow 2(2)(1)a_2 + 3a_0 = 0$

$$a_2 = -\frac{3}{4} a_0$$ Solve for bigger index

$$4a_2 + 3a_0 + \sum_{n=1}^{\infty} \left(2(n+2)(n+1) a_{n+2} + n a_n + 3a_n \right) x^n = 0$$

$$\begin{cases} 4a_2 + 3a_0 = 0 & \rightarrow \boxed{a_2 = -\frac{3}{4} a_0} \\ 2(n+2)(n+1) a_{n+2} + (n+3) a_n = 0 & \rightarrow \boxed{a_{n+2} = \dfrac{-(n+3) a_n}{2(n+2)(n+1)}} \end{cases}$$

Solve for coefficients, and factor out y_1 and y_2

Systems of First-Order Linear Equations

Eigenvalues and Eigenvectors

- Satisfy $A\vec{x} = \lambda\vec{x}$

Eigenvalues

Eigenvectors can be determined by solving $A\vec{x} = \lambda\vec{x}$ for some eigenvalue λ.

Shortcut: $A - \lambda I = \begin{pmatrix} -2 & 6 \\ -1 & 3 \end{pmatrix}$ $\quad \vec{x}^{(1)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ Right Col, Simplified / Left Col, Simplified

Negate one but not both.

Homogeneous Linear Systems

$$\vec{x}' = A\vec{x}$$

Assume $\vec{x} = \vec{\xi}_1 e^{rt}$

r = Eigenvalue

$\vec{\xi}_1$ = Eigenvector

$r \in \mathbb{R}$

$$\vec{x}_{gen}(t) = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)}$$
$$= c_1 \vec{\xi}_1^{(1)} e^{r_1 t} + c_2 \vec{\xi}_1^{(2)} e^{r_2 t}$$

$r \in \mathbb{C}$

$$r_{1,2} = \lambda \pm \mu i \implies \vec{x} = \vec{\xi}_1 e^{(\lambda + \mu i)t} = \vec{\xi}_1 e^{\lambda t}(\cos \mu t + i \cdot \sin \mu t)$$

$$\vec{x}_{gen}(t) = c_1 \operatorname{Re}(\vec{x}^{(1)}) + c_2 \operatorname{Im}(\vec{x}^{(2)})$$

PDEs and Fourier Series

Two-Point Boundary Value Problems

$$ay'' + by' + cy = 0$$

$$y(t_0) = y_0 \qquad y(t_1) = y_1$$

1. Solve for $y_{gen}(t)$
2. Plug in Boundary Conditions
3. Solve for c_1 and c_2
   → End up with one sol, inf sol, or zero sol.

Eigenfunctions

$$y'' + \lambda y = 0$$

1. Solve for $\lambda < 0$, $\lambda = 0$, $\lambda > 0$.
2. Find conditions for c_1, c_2 to be nonzero.
3. Solve for all λ_n if it allows $y \neq 0$.

Fourier Series

Given a periodic function f $f(x+T) = f(x)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

if $f(x)$ is even:

$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$b_n = 0$

if $f(x)$ is odd:

$a_n = 0$

$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Even and Odd Extensions

Given f on an interval $[0, L]$

Even:

$$g(x) = \begin{cases} f(x), & \underline{0 \le x \le L} \\ f(-x), & \underline{-L < x < 0} \end{cases}$$

close $f(x)$

open extension,

flip and negate boundaries

Odd:

$$g(x) = \begin{cases} f(x) & 0 < x < L \\ 0 & \underline{x = 0, L} \\ -f(-x) & \underline{-L < x < 0} \end{cases}$$

open $f(x)$

close $0, L$

open extension

flip and negate boundaries

Ch 6 The Laplace Transform

Definition

Given $f(t)$, $t \geq 0$, $\mathcal{L}\{f(t)\}$ is:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \quad , \quad s > 0$$

Inverse

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Ex) $\mathcal{L}^{-1}\{\frac{3}{s}\} = 3 \cdot \mathcal{L}\{\frac{1}{s}\} = 3 \cdot 1 = \underline{3}$

Ex) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s+1)}\right\}$

Partial Fraction Decomposition

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{1}{s+1} + \frac{-1}{s+2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-1}{s+2}\right\}$$

$$= e^{-t} - e^{-2t}$$

Methods:

- Partial Fractions
- Completing the Square
- Unit Function Shifts

Second-Order Linear Equation

$\mathcal{L}\{y(t)\} \longrightarrow Y(s)$

$\mathcal{L}\{y'(t)\} \longrightarrow sY(s) - y(0)$

$\mathcal{L}\{y''(t)\} \longrightarrow s^2Y(s) - sy(0) - y'(0)$

build up the derivatives

Unit Step Function

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

- If you're using several $u(t)$'s, cancel them out along the number line.

$\mathcal{L}\{u_c f(t-c)\} = e^{-cs} F(s)$

$\mathcal{L}\{u_c\} = \frac{1}{s} e^{-cs}$

Ex) $\mathcal{L}\{5 \cdot \underbrace{u_{27}(t)}_{c=27} \cdot \underbrace{(t-27)^3}_{f(t)=t^3}\} = 5 \cdot e^{-27s} \frac{3!}{s^4}$

Ex) $\mathcal{L}\{t + u_2(t)(t-2) + u_5(t) \cdot (5-t) + u_7(t)(t-7)\}$
$= \frac{1}{s^2} + e^{-2s} \cdot \frac{1}{s^2} - e^{-5s} \cdot \frac{1}{s^2} + e^{-7s} \cdot \frac{1}{s^2}$

Adjusting Functions

Ex) $g(t) = 7u_6(t)\cdot(t+4) = 7u_6(t)\cdot\left[t-6+4+6\right]$

$= 7u_6(t)\left[t-6\right] + 70u_6(t)$

$G(s) = 7e^{-6s}\cdot\frac{1}{s^2} + 70\, e^{-6s}\cdot\frac{1}{s}$

Ex) $G(s) = \frac{12\, e^{-4s}}{(s-7)^{11}} = 12\cdot e^{-4s}\cdot\frac{1}{(s-7)^{11}} = \frac{12\cdot e^{-4s}}{10!}\cdot\frac{10!}{(s-7)^{10+1}}$

① $\underline{F(s)} = \frac{12}{10!}\cdot\frac{10!}{(s-7)^{10+1}} \rightarrow$ ② $\underline{f(t)} = \frac{12}{10!}\cdot t^{10}\, e^{7t}$

③ $g(t) = u_c(t)\cdot \underline{f(t-c)}$

$= u_4(t)\cdot\frac{12}{10!}\cdot(t-4)^{10}\cdot e^{7(t-4)}$