Particle Motion in the Equatorial Plane of a Dipole Magnetic Field

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Abstract. An exact relation is derived which describes bound particle orbits in the equatorial plane of a dipole magnetic field. An exact expression is then obtained for the average angular velocity of the particle about the dipole axis. The corresponding drift velocity is compared with the usual first-order expression based on a constant local field gradient. It is shown that the first-order expression for the drift velocity can be considerably in error when the particle loops are not small compared with the mean distance from the dipole axis.

Introduction. The bound motion of charged particles in the magnetic field of a dipole has been of considerable interest since the discovery of charged particles trapped in the geomagnetic field. Most calculations of such motions, for example, those of *Hamlin, Karplus, Vik, and Watson* [1961] published earlier in this journal, are based on the justifiable assumption of an average field and field gradient over the individual loops made by the particle. However, when conditions are such that the size of these loops is not negligibly small compared with the distance from the dipole axis, errors are introduced by not accounting for the changes in the field and field gradient over the path of an individual gyration. The extent of these errors in a special but typical case is the subject of this paper. We treat exactly the nonrelativistic bound motion of a charged particle in the equatorial plane of the dipole magnetic field and compare the exact drift velocity with the approximate expression. See equations 26 and 24, respectively. These equations indicate the parameter values for which the approximate expression for the drift velocity becomes inadequate.

We consider a magnetic field $\mathbf{B}$ of magnitude $B = M/r^3$ (M constant) directed everywhere perpendicularly to a given plane. The radius vector $\mathbf{r}$ locates a particle moving with velocity v in this plane; the angles θ and ϕ are related to $\mathbf{r}$ and $\mathbf{v}$ as indicated in Figure 1. (The velocity magnitude v of course remains constant throughout the motion.)

An expression for $r(\phi)$. A description of the trajectory in the form $r = r(\phi)$ is easily derived, and it forms a convenient basis for the subsequent evaluations. We start with the following relations:

$$dr/dt = v \cos \phi \tag{1}$$

$$r(d\theta/dt) = v \sin \phi \tag{2}$$

$$\frac{d}{dt}(\theta + \phi) = \frac{eB}{mc} = \frac{eM}{mc}\frac{1}{r^3} \tag{3}$$

Equations 1 and 2 are identities which follow from the resolution of $\mathbf{v}$ and $d\mathbf{r}/dt$ into components parallel and perpendicular to $\mathbf{r}$.

Equation 3 is verified as follows. When $v \equiv |\mathbf{v}|$ is constant, the equation of motion $d\mathbf{v}/dt = (e/mc)\mathbf{v} \times \mathbf{B}$ reduces to $d\hat{n}/dt = (e/mc)\hat{n} \times \mathbf{B}$, where $\hat{n}$ is a unit vector along $\mathbf{v}$. For motion in the plane perpendicular to $\mathbf{B}$, we may write $\hat{n} \times \mathbf{B} = B\hat{l}$, where $\hat{l}$ is a unit vector perpendicular to both $\hat{n}$ and $\mathbf{B}$. The derivative of $\hat{n}$ with respect to the angle $(\theta + \phi)$ between $\hat{n}$ and a fixed direction is also equal to $\hat{l}$. Thus it follows that $d\hat{n}/dt = \hat{l}d(\theta + \phi)/dt$. The equation $d\hat{n}/dt = (e/mc)\hat{n} \times \mathbf{B}$ then becomes $\hat{l}[d(\theta + \phi)/dt - (eB)/(mc)] = 0$ from which equation 3 follows.

Equations 2 and 3 may be combined to give

$$\frac{d\phi}{dt} = \frac{eM}{mc}\frac{1}{r^3} - \frac{v \sin \phi}{r} \tag{4}$$

Writing (1) as $(dr/d\phi)(d\phi/dt) = v \cos \phi$ and substituting (4) leads to the relation

$$r \cos \phi (d\phi/dr) + \sin \phi = (eM)/(mcvr^2)$$

which is integrated directly to give

$$r \sin \phi + \frac{v_c r_0^2}{v}\frac{1}{r} = \frac{v_c r_0}{v} \tag{5}$$

With no loss of generality for the present purposes, the constant of integration has been

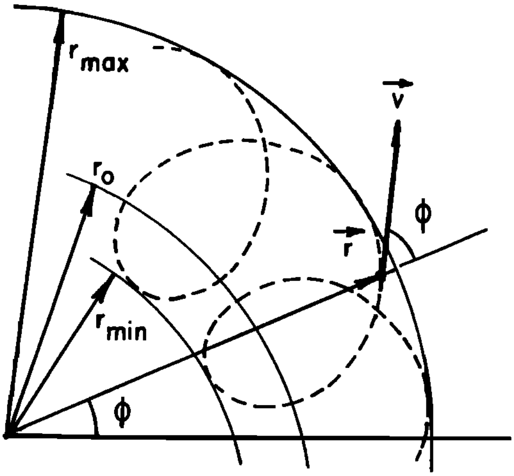

Fig. 1.

chosen so that $r = r_0$ when $\phi = 0$. Thus r_0 is the radius of the circle that the orbit crosses at right angles (see Fig. 1). The parameter $v_c = (eM)/(mcr_0^2)$ is the velocity with which a particle would move along the circle of radius r_0.

The solution of (5) corresponding to bound orbits is

$$r = \frac{2r_0}{1 + \sqrt{1 - 4(v/v_c)\sin\phi}} \tag{6}$$

where we require that $v < (v_c/4)$. It is clear that

$$r_{\max} = \frac{2r_0}{1 + \sqrt{1 - 4(v/v_c)}} \tag{7a}$$

and

$$r_{\min} = \frac{2r_0}{1 + \sqrt{1 + 4(v/v_c)}} \tag{7b}$$

The relation 6 constitutes an implicit description of the orbit.

An expression for $\overline{d\theta/dt}$. An important parameter which we now compute is the time average of $d\theta/dt$ over the period during which ϕ increases by 2π. $d\theta/dt$ is the angular velocity of the particle about the dipole axis.

Using (5) to express (1) and (2) in terms of r gives

$$\frac{d\theta}{dt} = \frac{v_c}{r_0}\left[\left(\frac{r_0}{r}\right)^2 - \left(\frac{r_0}{r}\right)^3\right] \tag{8}$$

and

$$\frac{dr}{dt} = v\left\{1 - \left(\frac{v_c}{v}\right)^2\left[\left(\frac{r_0}{r}\right) - \left(\frac{r_0}{r}\right)^2\right]^2\right\}^{1/2} \tag{9}$$

From (9), the time T required for r to execute one cycle is

$$T = \frac{2}{v}\int_{r\min}^{r\max} \frac{dr}{\sqrt{1 - \left(\frac{v_c}{v}\right)^2\left[\left(\frac{r_0}{r}\right) - \left(\frac{r_0}{r}\right)^2\right]^2}} \tag{10}$$

where $r_{\min}$ and $r_{\max}$ are given by (7). Utilizing (8), (9), and (10), the required relation

$$\overline{\frac{d\theta}{dt}} \equiv \frac{2}{T}\int_{r\min}^{r\max} \left(\frac{d\theta}{dt}\right)\Big/\left(\frac{dr}{dt}\right) dr$$

may be written

$$\overline{\frac{d\theta}{dt}} = \frac{v_c}{r_0}\left[I_1/I_2\right] \tag{11}$$

where

$$I_1 = \int_{1/2(1+\sqrt{1-4/V})}^{1/2(1+\sqrt{1+4/V})} \frac{(1 - R)\,dR}{\sqrt{1 - V^2R^2(1 - R)^2}} \tag{12a}$$

and

$$I_2 = \int_{1/2(1+\sqrt{1-4/V})}^{1/2(1+\sqrt{1+4/V})} \frac{dR}{R^2\sqrt{1 - V^2R^2(1 - R)^2}} \tag{12b}$$

We have made the change of variable $r_0/r = R$ and have defined $V \equiv v_c/v$. V must be greater than 4.

It is possible to factor the radicand in the above integrals into the form

$$\begin{aligned} 1 - V^2R^2(1 - R)^2 &\equiv [A_1(R - \alpha)^2 - B_1(R - \beta)^2] \\ &\quad \cdot [A_2(R - \alpha)^2 - B_2(R - \beta)^2] \end{aligned} \tag{13}$$

after which we follow the method of reducing a general elliptic integral to standard tabulated forms, obtaining as the final result

$$I_1 = K_1F(k) + K_2\Pi(k, k) \tag{14a}$$

$$I_2 = K_3F(k) + K_4E(k) \tag{14b}$$

F, E, and Π are complete elliptic integrals of the first, second, and third kinds, respectively, in the form

$$F(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 z)^{-1/2}\, dz \quad (15a)$$

$$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 z)^{1/2}\, dz \quad (15b)$$

$$\Pi(k, k) = \int_0^{\pi/2} (1 + k \sin^2 z)^{-1} \cdot (1 - k^2 \sin^2 z)^{-1/2}\, dz \quad (15c)$$

The parameters k, K_1, K_2, K_3, K_4 (as well as α, β, A_1, A_2, B_1, B_2 in equation 13) are given in terms of V through the set of relations listed in Appendix 1. Equation 11 with I_1 and I_2 given by (14) constitutes the desired analytical expression for $\overline{d\theta/dt}$.

When V is sufficiently large, the parameters in (14) may be evaluated much more easily from the initial terms of their series expansions in $1/V$. These expansions, determined from the relations in Appendix 1, are

$$k = -\frac{1}{V^2}\left(1 + \frac{10}{V^2} + \frac{117}{V^4} + \frac{1478}{V^6} + \cdots\right) \quad (16a)$$

$$K_1 = \frac{2}{V}\left(1 + \frac{1}{V^2} + \frac{8}{V^4} + \frac{83}{V^6} + \cdots\right) \quad (16b)$$

$$K_2 = -\frac{2}{V}\left(1 - \frac{1}{V^2} - \frac{10}{V^4} - \frac{117}{V^6} - \cdots\right) \quad (16c)$$

$$K_3 = 2V\left(1 - \frac{2}{V^2} - \frac{10}{V^4} - \frac{84}{V^6} - \cdots\right) \quad (16d)$$

$$K_4 = -2V\left(1 - \frac{3}{V^2} - \frac{17}{V^4} - \frac{154}{V^6} - \cdots\right) \quad (16e)$$

The initial terms in the expansions of I_1 and I_2 are then found to be

$$I_1 = \frac{\pi}{2}\frac{3}{V^3}\left(1 + \frac{35}{4V^2} + \frac{385}{4V^4} + \cdots\right) \quad (17a)$$

and

$$I_2 = \frac{\pi}{2}\frac{2}{V}\left(1 + \frac{15}{2V^2} + \frac{315}{4V^4} + \cdots\right) \quad (17b)$$

Then from (11),

$$\overline{\frac{d\theta}{dt}} = \frac{v_c}{r_0}\frac{3}{2}\frac{1}{V^2}\left(1 + \frac{5}{4V^2} + \frac{65}{8V^4} + \cdots\right) \quad (18)$$

as a more tractable form of the desired result.

Again, $V = v_c/v$, where $v_c = (eM)/(mcr_0^2)$. For a given charged particle and a given dipole field, v_c is a function only of r_0, the radius of that circle which the orbit crosses normally (see Fig. 1). v_c has the physical significance of the velocity needed for a particle to move in a circle with center at $r = 0$ and radius r_0.

Other quantities of interest, expressed as expansions, are r_{max}, r_{min}, $(\overline{r_{max} - r_{min}})$ (see equations 7 and 10), and $\theta_T \equiv T\, \overline{d\theta/dt}$. These are given by

$$\frac{r_{max}}{r_0} = \left(1 + \frac{1}{V} + \frac{2}{V^2} + \frac{5}{V^3} + \frac{14}{V^4} + \cdots\right) \quad (19a)$$

$$\frac{r_{min}}{r_0} = \left(1 - \frac{1}{V} + \frac{2}{V^2} - \frac{5}{V^2} + \frac{14}{V^4} + \cdots\right) \quad (19b)$$

$$\frac{r_{max} - r_{min}}{r_0} = \frac{2}{V}\left(1 + \frac{5}{V^2} + \frac{42}{V^4} + \cdots\right) \quad (19c)$$

$$T = \frac{2\pi r_0}{v_c}\left(1 + \frac{15}{2V^2} + \frac{315}{4V^4} + \cdots\right) \quad (19d)$$

and

$$\theta_T = 3\pi\frac{1}{V^2}\left(1 + \frac{35}{4V^2} + \frac{385}{4V^4} + \cdots\right) \quad (19e)$$

Comparison with the drift due to a constant gradient field. The first-order expression that is ordinarily used to calculate the drift velocity of a particle in a magnetic field having a gradient normal to the field direction is given by

$$v_{D1} = \frac{c}{e}\left(\tfrac{1}{2}mv^2\right)\frac{1}{B_0^{\,2}}\,|\nabla B|_0 \quad (20)$$

v_{D1} is the average velocity of the particle in the plane perpendicular to the magnetic field, the direction of this velocity being perpendicular to the field gradient. $|\nabla B|_0$ is assumed constant over the orbit and small. See, for example, *Spitzer* [1956].

Applying this to the dipole case for

$$(r_{max} - r_{min}) \ll r_0$$

we put

$$B_0 = M/r_0^{\,3} \quad \text{and} \quad |\nabla B|_0 = 3M/r_0^{\,4}$$

and obtain

$$v_{D1} = \frac{3}{2}\frac{v^2}{v_c} \quad (21)$$

The derivation of an exact expression corresponding to this case is given in Appendix 2. The result is

$$v_{D2} = v\left[-C_0 + (1 + C_0)\frac{E\left(\sqrt{\dfrac{2}{1 + C_0}}\right)}{F\left(\sqrt{\dfrac{2}{1 + C_0}}\right)}\right] \quad (22)$$

where $C_0 = (eB_0^2)(2mc\, v\, |\nabla B|_0)$ and F and E are complete elliptic integrals of the first and second kinds, respectively. Putting $B_0 = M/r_0^3$ and $|\nabla B|_0 = 3M/r_0^4$ shows that $C_0 = V/6$. Equation 22 may then be expanded in powers of $1/V$, giving

$$v_{D2} = \frac{3}{2}\frac{v}{V}\left(1 + \frac{81}{8}\frac{1}{V^2} + \cdots\right) \quad (23)$$

Relations (21) and (23) along with (18) are slightly rewritten and given below for comparison.

$$\frac{v_{D1}}{v} = \frac{3}{2}\frac{1}{V} \quad (24)$$

$$\frac{v_{D2}}{v} = \frac{3}{2}\frac{1}{V}\left(1 + \frac{81}{8}\frac{1}{V^2} + \cdots\right) \quad (25)$$

$$\frac{v_{D3}}{v} = \frac{3}{2}\frac{1}{V}\left(1 + \frac{5}{4}\frac{1}{V^2} + \frac{65}{8}\frac{1}{V^4} + \cdots\right) \quad (26)$$

Above, v_{D1} is the first-order linear drift velocity based on a constant field and field gradient over the orbit; v_{D2} is the series expression for the exact linear drift velocity, which follows from equation 22 (v_{D1} and v_{D2} both neglect the curvature of the lines of constant B in a dipole field); $v_{D3} = \overline{r_0(d\theta/dt)}$ is the series expression for the exact drift velocity in a dipole field, which follows from (11) and (14) and Appendix 1. v is the particle velocity and $V = v_c/v$ ($V > 4$) where $v_c = (eM)/(mcr_0^2)$ is the velocity with which a particle would move in a circle of radius r_0 about the dipole axis.

A comparison of (24) and (26) shows the range of validity of the usual first-order expression. Equation 25 shows that the neglect of the curvature of the constant B lines essentially cancels any advantage of using a second- or higher-order linear expression. However, the derivation of (25) shows that when (24) is used, B_0 and $|\nabla B|_0$ should be evaluated at r_0, the radius of that circle crossed normally by the trajectory, and not, for example, the average of r_{min} and r_{max}. For values of V such that $\theta_T = \overline{T\ (d\theta/dt)}$ is a small fraction of 2π, (24) can differ from (26) by about 10 per cent at most. However, when θ_T is greater than, say, $\pi/4$, (24) becomes completely inadequate, and it is necessary to use (26), or (11) and (14) and Appendix 1, which constitute the exact form of (26).

Not for every combination of V and r_0 will v be nonrelativistic, as has been assumed throughout. The smaller values of V correspond to nonrelativistic particle velocities only for fairly large r_0.

Appendix 1

The parameters k, K_1, K_2, K_3, K_4 in equations 14 are given in terms of V by the following sequence of relations:

$$K_1 = D_1 D_2 \qquad K_2 = -D_1$$

$$K_3 = D_1(D_3 + D_4)$$

$$K_4 = -D_1 D_3 \qquad k = -A_1 B_1^{-1}$$

$$D_1 = 2(B_1 A_2)^{-1/2} \qquad D_2 = (1 - \alpha)(\beta - \alpha)^{-1}$$

$$D_3 = (\beta - \alpha) B_1 A_2$$

$$D_4 = \alpha^2\beta^{-2}(\alpha - \beta) A_1 A_2 + \beta^{-2}(\beta - \alpha)^{-1}$$

$$A_1 = -A_2 L_2 V^2 \quad A_2 = (1 - L_1)(L_2 - L_1)^{-1}$$

$$B_1 = -B_2 L_1 V^2 \quad B_2 = (1 - L_2)(L_2 - L_1)^{-1}$$

$$\alpha = (1 - \tfrac{1}{2}a)(b^2 V^2 - L_1)(1 - L_1)^{-1}$$

$$\beta = (1 - \tfrac{1}{2}a)(b^2 V^2 - L_2)(1 - L_2)^{-1}$$

$$L_1 = \lambda_1[1 + (1 - \lambda_2)^{1/2}]$$

$$L_2 = \lambda_1[1 - (1 - \lambda_2)^{1/2}]$$

$$\lambda_1 = b^2 V^2(a^2 + 2b - 2b^3 V^2)(a^2 + 4b^3 V^2)^{-1}$$

$$\lambda_2 = (a^2 - 4b)(a^2 + 4b^3 V^2) \cdot (a^2 + 2b - 2b^3 V^2)^{-2}$$

$$b = V^{-1} a^{1/2}(2 - a)^{-1/2}$$

$$a = 1 + 2^{-1/2}[1 + (1 - 16V^{-2})^{1/2}]^{1/2}$$

The following relations between the above coefficients have been used to greatly simplify final results:

$$Q_2 = Q_1 = (-k)^{-1/2}$$

where

$$Q_2 = [1 - 2\alpha + (1 - 4V^{-1})^{1/2}] \cdot [1 - 2\beta + (1 - 4V^{-1})^{1/2}]^{-1}$$

$$Q_1 = [1 - 2\alpha + (1 + 4V^{-1})^{1/2}] \cdot [1 - 2\beta + (1 + 4V^{-1})^{1/2}]^{-1}$$

which helps to show that the elliptic integrals are complete and that certain odd terms vanish. Also,

$$(3\alpha - \beta)C_1 = (\alpha - \beta)C_2$$

where

$$C_1 = \alpha^4\beta^{-4}A_1A_2 - \alpha^2\beta^{-2}(A_1B_2 + B_1A_2) + B_1B_2$$

$$C_2 = 3\alpha^4\beta^{-4}A_1A_2 - 2\alpha^2\beta^{-2}(A_1B_2 + B_1A_2) + B_1B_2$$

which helps to show that the coefficient of an elliptic integral of the third kind in equation 14*b* vanishes.

Finally, $A_1A_2 = B_1B_2$, which shows that the two arguments of Π in (14*a*) are the same.

Appendix 2

The exact relation for the drift due to a constant gradient field is derived as follows. Consider a magnetic field $\mathbf{B} = B(y)\hat{k} = (-sy + B_0)\hat{k}$, ($s$, B_0 const.) directed perpendicularly to the xy plane. The vector extending from the position of the particle to the instantaneous center of curvature of its trajectory is given by $(mc/eB)(\mathbf{v} \times \hat{k})$, $B(y)$ being evaluated at the particle position. With $\mathbf{P}$ and $\mathbf{R}$ position vectors of the particle and the center of curvature, respectively, we then have

$$\mathbf{P} = \mathbf{R} + \frac{mcv}{eB}\left(\frac{\hat{k} \times \mathbf{v}}{v}\right)$$

differentiation shows that

$$\dot{\mathbf{R}} = \frac{mcv}{e}\frac{\dot{B}}{B^2}\left(\frac{\hat{k} \times \mathbf{v}}{v}\right) \qquad \text{(A-1)}$$

Introducing $B = -sy + B_0$ and resolving (A-1) into components gives

$$\dot{X} = -\frac{mcv}{es}\frac{\dot{y}\sin\varphi}{(y - B_0/s)^2} \qquad \text{(A-2}a\text{)}$$

$$\dot{Y} = \frac{mcv}{es}\frac{\dot{y}\cos\varphi}{(y - B_0/s)^2} \qquad \text{(A-2}b\text{)}$$

where φ is the angle between $\mathbf{v}$ and the x axis, and X and Y are the components of $\mathbf{R}$.

The identity $\dot{y} = v \sin\varphi$ can easily be integrated, using the fact that $\dot{\varphi} = (e/mc)(-sy + B_0)$. giving

$$\frac{es}{mcv}\left(y - \frac{B_0}{s}\right)^2 = 2\left(\cos\varphi + \frac{eB_0^2}{2mcvs}\right) \qquad \text{(A-3)}$$

where the condition $y(\varphi = \pi/2) = 0$ has been imposed. Thus (A-2*a*) and (A-2*b*) become

$$\dot{X} = -\frac{v}{2}\frac{\sin^2\varphi}{(\cos\varphi + C_0)} \qquad \text{(A-4}a\text{)}$$

$$\dot{Y} = \frac{v}{2}\frac{\sin\varphi\cos\varphi}{(\cos\varphi + C_0)} \qquad \text{(A-4}b\text{)}$$

where $C_0 = eB_0^2/2mcvs$. We will consider only $C_0 > 1$, which is the (periodic) case of interest here.

(A-3) combined with $\dot{\varphi} = (e/mc)(-sy + B_0)$ shows that $dt = A_0(\cos\varphi + C_0)^{-1/2}\, d\varphi$, where $A_0 = (mc/2evs)^{1/2}$. Thus the time T required for φ to increase by 2π is given by

$$T = A_0 \int_0^{2\pi} (\cos\varphi + C_0)^{-1/2}\, d\varphi \qquad \text{(A-5)}$$

The expression $\bar{\dot{Y}} = T^{-1}\int_0^T \dot{Y}\, dt$ represents the time average of the y velocity component of the center of curvature and is, in this case,

$$\bar{\dot{Y}} = \frac{v}{2}\frac{\int_0^{2\pi} \sin\varphi\cos\varphi\,(\cos\varphi + C_0)^{-3/2}\, d\varphi}{\int_0^{2\pi} (\cos\varphi + C_0)^{-1/2}\, d\varphi} \qquad \text{(A-6)}$$

which vanishes. This implies that the average particle velocity has no component in the direction of the field gradient.

In the same way, the average x velocity component of the center of curvature, and thus the average x component of velocity of the particle, is given by

$$\bar{\dot{X}} = -\frac{v}{2}\frac{\int_0^{\pi} \sin^2\varphi\,(\cos\varphi + C_0)^{-3/2}\, d\varphi}{\int_0^{\pi} (-\cos\varphi + C_0)^{-1/2}\, d\varphi} \qquad \text{(A-7)}$$

It may be shown without great difficulty that this is equivalent to

$$\bar{\dot{X}} = v\left[-C_0 + (1 + C_0)\,\frac{E\left(\sqrt{\dfrac{2}{1 + C_0}}\right)}{F\left(\sqrt{\dfrac{2}{1 + C_0}}\right)}\right] \qquad \text{(A-8)}$$

as the final expression for the drift velocity due to a constant gradient. E and F are complete elliptic integrals of the same form as equations 14.

Acknowledgments. This work has been supported by the Air Force Cambridge Research Laboratories, Geophysics Research Directorate, through contract AF19(604)-4545 with Harvard University. I am indebted to Professor Max Krook and to Dr. C. A. Whitney for helpful advice.

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(Manuscript received September 1, 1961; revised October 16, 1961.)