#1 (a) $f(x) = x^2$

$$f(x_1+x_2) = (x_1+x_2)^2$$

$$= x_1^2+2x_1x_2+x_2^2$$

$$f(x_1)+f(x_2) = x_1^2+x_2^2$$

$\because$ $f(x_1+x_2) \neq f(x_1)+f(x_2)$

$\therefore$ $f(x)$ is non-linear #

(b)

$$f(1) = 1^2 = 1$$

$$f(-1) = (-1)^2 = 1$$

$\because$ Both $x=1$ and $x=-1$ will equal to 1.

$\therefore$ When $x \in \mathbb{R}$ the function is not 1-1. #

(c)

Let $f(x) = a$, $a \in [0, \infty)$

$$\Rightarrow x^2 = a$$

$$\Rightarrow x = \pm\sqrt{a}$$

if $x \geq 0$, then

$$x = \sqrt{a}$$

$\because$ for one a will only have one x.

$\therefore$ when $x \geq 0$ the function is 1-1. #

#2

$x_1 = (1, 2, 3, 4, 5)$

$$\|x_1\|_1 = 1+2+3+4+5 = 15$$

$$\|x_1\|_2 = \sqrt{1^2+2^2+3^2+4^2+5^2} = \sqrt{55}$$

$$\|x_1\|_5 = (1^5+2^5+3^5+4^5+5^5)^{\frac{1}{5}} = 5.3602$$

$$\|x_1\|_\infty = max(1, 2, 3, 4, 5) = 5$$

$x_2 = (1, -2, 3, -4, -6)$

$$\|x_2\|_1 = 1+|-2|+3+|-4|+|-6| = 16$$

$$\|x_2\|_2 = \sqrt{1^2+(-2)^2+3^2+(-4)^2+(-6)^2} = \sqrt{66}$$

$$\|x_2\|_5 = (1^5+|-2|^5+3^5+|-4|^5+|-6|^5)^{\frac{1}{5}} = 6.1884$$

$$\|x_2\|_\infty = max(1, 2, 3, 4, 6) = 6$$

	x_1		x_2
1-norm	15	<	16
2-norm	$\sqrt{55}$	<	$\sqrt{66}$
5-norm	5.3602	<	6.1884
∞-norm	5	<	6

#

#3

(a)

$$V(x_1+\Delta x_1,\ x_2+\Delta x_2) = (x_1+\Delta x_1)^2+(x_2+\Delta x_2)^2$$
$$= x_1^2+2x_1\Delta x_1+\Delta x_1^2+x_2^2+2x_2\Delta x_2+\Delta x_2^2$$
$$= x_1^2+x_2^2+2x_1\Delta x_1+2x_2\Delta x_2+\Delta x_1^2+\Delta x_2^2$$

$$\Delta V(x_1, x_2, \Delta x_1, \Delta x_2) = V(x_1+\Delta x_1, x_2+\Delta x_2) - V(x_1, x_2)$$
$$= \cancel{(x_1^2+x_2^2)}+(2x_1\Delta x_1+2x_2\Delta x_2)+(\Delta x_1^2+\Delta x_2^2)-\cancel{(x_1^2+x_2^2)}$$
$$= \underline{2x_1\Delta x_1+2x_2\Delta x_2+\Delta x_1^2+\Delta x_2^2} \quad \#$$

(b)

Suppose $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$, $\Delta X = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$, $\|\Delta X\|_2 = 0.01$ Aside =

$$\|\Delta X\|_2 = \sqrt{\Delta x_1^2+\Delta x_2^2} = 0.01$$
$$\Rightarrow \Delta x_1^2+\Delta x_2^2 = 10^{-4} \quad -① $$
$$\Rightarrow \Delta x_2 = \sqrt{10^{-4}-\Delta x_1^2} \quad -② $$

Subsitude ①, ② into ΔV

$$\Delta V = 2\sqrt{2}\,\Delta x_1 + 2\sqrt{2}\sqrt{10^{-4}-\Delta x_1^2} + 10^{-4}$$
$$= 2\sqrt{2}\left(\Delta x_1+\sqrt{10^{-4}-\Delta x_1^2}\right)+10^{-4}$$

$$\frac{d\Delta V}{d\Delta x_1} = 2\sqrt{2}\left[1+\frac{1}{2}(10^{-4}-\Delta x_1^2)^{-\frac{1}{2}}(-2\Delta x_1)\right] = 0$$
$$\Rightarrow 1-\Delta x_1(10^{-4}-\Delta x_1^2)^{-\frac{1}{2}} = 0$$
$$\Rightarrow \Delta x_1(10^{-4}-\Delta x_1^2)^{-\frac{1}{2}} = 1$$
$$\Rightarrow \Delta x_1^2 = 10^{-4}-\Delta x_1^2$$
$$\Rightarrow \Delta x_1^2 = 0.5\times10^{-4} \Rightarrow \Delta x_1 = \sqrt{0.5}\times10^{-2}$$

The greatest ΔV is at $\Delta X_1 = \Delta X_2 = \sqrt{0.5} \times 10^{-2}$

$$\Delta V = 2X_1 \Delta X_1 + 2X_2 \Delta X_2 + \Delta X_1^2 + \Delta X_2^2$$

$$= 2\sqrt{2}\left(\sqrt{0.5} \times 10^{-2} + \sqrt{0.5} \times 10^{-2}\right) + 10^{-4}$$

$$= 4 \times 10^{-2} + 10^{-4} = \underline{0.0401} \ \#$$