Navigation Equations

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- Mechanization equations (Geodetic frame)

Kinematic equations
- position equations

- ECEF position – if navigation is being performed in the ECEF coordinate frame

$$\mathbf{R}^e(t) = \int_0^t \mathbf{v}_e^e(\tau)d\tau + \mathbf{R}^e(0)$$

where $\mathbf{v}_e^e$ is the earth's relative velocity expressed in ECEF frame.

- Geodetic position

$$\begin{bmatrix} \dot{\phi} \\ \dot{\lambda} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{R_M + h} & 0 & 0 \\ 0 & \dfrac{1}{(R_P + h)\cos\phi} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_N \\ v_E \\ v_D \end{bmatrix}$$

Note that $\mathbf{v}_e^n = \begin{bmatrix} v_N & v_E & v_D \end{bmatrix}$ is the vehicle earth's relative velocity expressed in *local level coordinate frame*.

In the above,

$$R_M = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{1.5}}$$

$$R_P = \frac{a}{(1-e^2\sin^2\phi)^{0.5}}$$

where R_M is the meridian radius of curvature, R_P is the prime radius of curvature, a is the equatorial radius, and e is the eccentricity.

Kinematic equations
- velocity equations

- Inertial frame – In an inertial frame of reference, the differential equation for the position vector is

$$\frac{d^2}{dt^2}\mathbf{R}^i = \mathbf{f}^i + \mathbf{G}^i(\mathbf{R})$$

where $\mathbf{G}^i(\mathbf{R})$ is the gravitational acceleration, and the specific force vector $\mathbf{f}^i$ in inertial frame would be calculated based on the body-frame inertial measurements:

$$\mathbf{f}^i = \mathbf{R}_{b2i}\mathbf{f}^b$$

Therefore, the direction-cosine matrix differential equation,

$$\dot{\mathbf{R}}_{b2i} = \mathbf{R}_{b2i}\varpi_{ib}^{b}$$

is also of interest, where

$$\varpi_{ib}^{b} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

Note that the vector $\boldsymbol{\omega}_{ib}^{b} = \begin{bmatrix} p & q & r \end{bmatrix}^{T}$ is the inertial angular rate of the body frame expressed in body frame, which would be measured by body-mounted gyros.

- ECEF – In the following the differential equations are derived for earth's relative velocity as expressed in ECEF coordinates, which is useful for determining the ECEF position vector.

The earth's relative velocity vectors as represented in ECEF and inertial frame are related by

$$\mathbf{v}_e^e = \mathbf{R}_{i2e}\mathbf{v}_e^i$$

Take the derivative

$$\frac{d}{dt}\mathbf{v}_e^e = \frac{d}{dt}(\mathbf{R}_{i2e}\mathbf{v}_e^i) = (\frac{d}{dt}\mathbf{R}_{i2e})\mathbf{v}_e^i + \mathbf{R}_{i2e}\frac{d}{dt}\mathbf{v}_e^i$$

$$= -\mathbf{R}_{i2e}\varpi_{ie}^e\mathbf{v}_e^i + \mathbf{R}_{i2e}\frac{d}{dt}\mathbf{v}_e^i$$

$$= \mathbf{R}_{i2e}(\frac{d}{dt}\mathbf{v}_e^i - \varpi_{ie}^i\mathbf{v}_e^i)$$

$$= \mathbf{R}_{i2e}(\mathbf{f}^i + \mathbf{g}^i - \varpi_{ie}^i\mathbf{v}_e^i - \varpi_{ie}^i\mathbf{v}_e^i)$$

$$= (\mathbf{f}^e + \mathbf{g}^e - 2\varpi_{ie}^e\mathbf{v}_e^e)$$

The specific force in ECEF is calculated as

$$\mathbf{f}^e = \mathbf{R}_{b2e}\mathbf{f}^b$$

The rotation matrix $\mathbf{R}_{b2e}$ is determined as the solution to the differential equation

$$\dot{\mathbf{R}}_{b2e} = \mathbf{R}_{b2e}\varpi_{eb}^b$$

where

$$\boldsymbol{\omega}_{eb}^b = \boldsymbol{\omega}_{ib}^b - \mathbf{R}_{e2b}\boldsymbol{\omega}_{ie}^e$$

- Geodetic – In the following the differential equations are derived for geodetic velocity.

The geodetic velocity vectors can be related to inertial frame by

$$\mathbf{v}_e^n = \mathbf{R}_{i2n}\mathbf{v}_e^i$$

Take the derivative

$$
\begin{aligned}
\frac{d}{dt}\mathbf{v}_e^n &= \frac{d}{dt}(\mathbf{R}_{i2n}\mathbf{v}_e^i) = (\frac{d}{dt}\mathbf{R}_{i2n})\mathbf{v}_e^i + \mathbf{R}_{i2n}\frac{d}{dt}\mathbf{v}_e^i \\
&= -\mathbf{R}_{i2n}\varpi_{in}^i\mathbf{v}_e^i + \mathbf{R}_{i2n}\frac{d}{dt}\mathbf{v}_e^i = \mathbf{R}_{i2n}(\frac{d}{dt}\mathbf{v}_e^i - \varpi_{in}^i\mathbf{v}_e^i) \\
&= \mathbf{R}_{i2n}(\mathbf{f}^i + \mathbf{g}^i - \varpi_{ie}^i\mathbf{v}_e^i - \varpi_{in}^i\mathbf{v}_e^i) \\
&= \mathbf{R}_{i2n}(\mathbf{f}^i + \mathbf{g}^i - \varpi_{ie}^i\mathbf{v}_e^i - (\varpi_{ie}^i + \varpi_{en}^i)\mathbf{v}_e^i) \\
&= (\mathbf{f}^n + \mathbf{g}^n - (\varpi_{en}^n + 2\varpi_{ie}^n)\mathbf{v}_e^n)
\end{aligned}
$$

The specific force in geodetic is calculated as

$$\mathbf{f}^n = \mathbf{R}_{b2n}\mathbf{f}^b$$

The rotation matrix $\mathbf{R}_{b2\mathrm{n}}$ is determined as the solution to the differential equation

$$\dot{\mathbf{R}}_{b2n} = \mathbf{R}_{b2n}\varpi_{nb}^{b}$$

where

$$\boldsymbol{\omega}_{nb}^{b} = \boldsymbol{\omega}_{ib}^{b} - \mathbf{R}_{n2b}\boldsymbol{\omega}_{in}^{n} = \boldsymbol{\omega}_{ib}^{b} - \mathbf{R}_{n2b}(\boldsymbol{\omega}_{ie}^{n} + \boldsymbol{\omega}_{en}^{n})$$

and

$$\boldsymbol{\omega}_{en}^{n} = \begin{bmatrix} \dot{\lambda}\cos\phi \\ -\dot{\phi} \\ -\dot{\lambda}\sin\phi \end{bmatrix} = \begin{bmatrix} \dfrac{v_E}{R_M + h} & -\dfrac{v_N}{R_P + h} & -\dfrac{v_E \tan\phi}{R_P + h} \end{bmatrix}^T$$

back

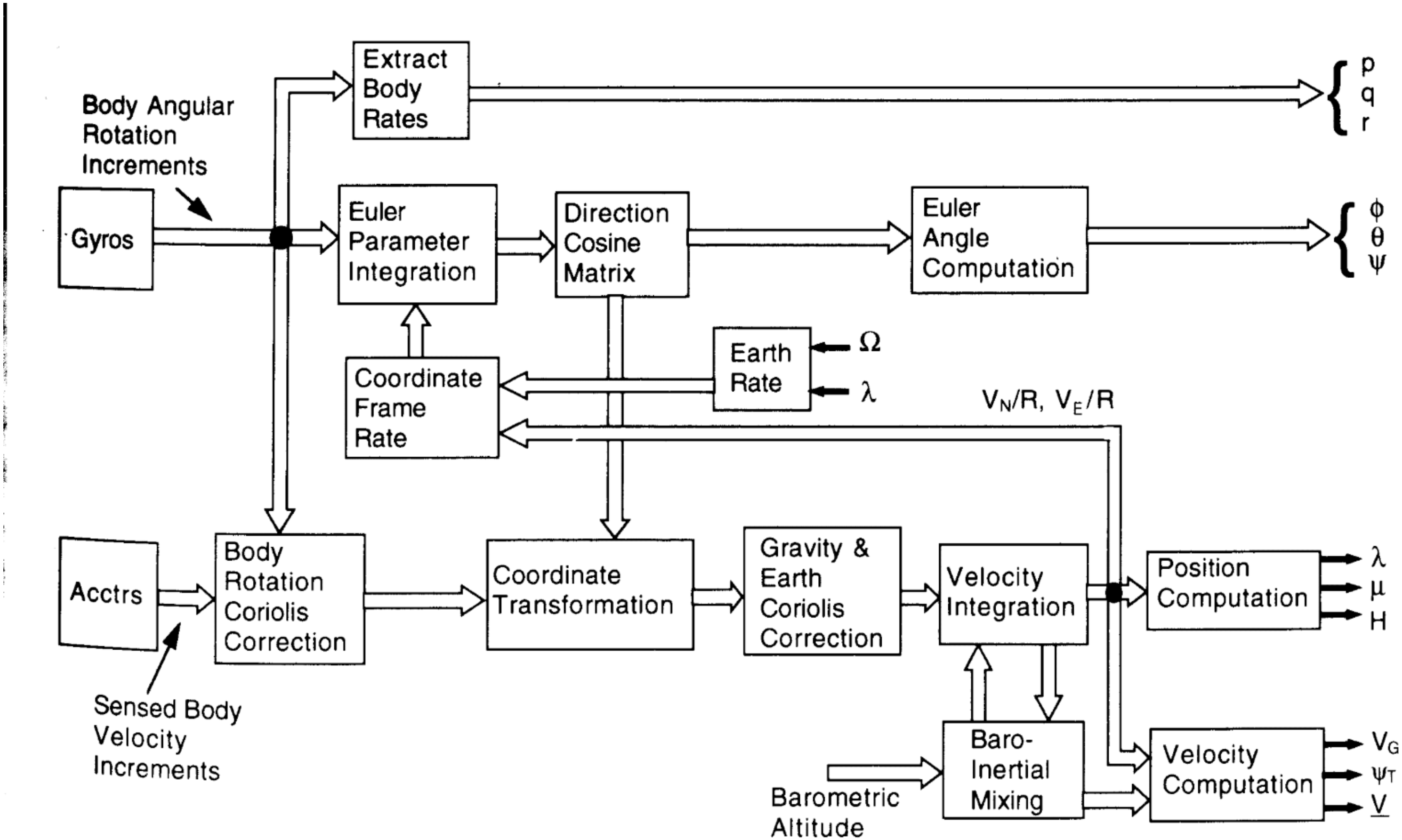
Mechanization equations (Geodetic frame)

$$\begin{bmatrix} \dot{\phi} \\ \dot{\lambda} \\ \dot{h} \\ \dot{v}_N \\ \dot{v}_E \\ \dot{v}_D \end{bmatrix} = \begin{bmatrix} \frac{1}{R_M + h} & 0 & 0 \\ 0 & \frac{1}{(R_P + h)\cos\phi} & 0 \\ 0 & 0 & -1 \\ 0 & -(\dot{\lambda} + 2\omega_{ie})\sin\phi & \dot{\phi} \\ (\dot{\lambda} + 2\omega_{ie})\sin\phi & 0 & (\dot{\lambda} + 2\omega_{ie})\cos\phi \\ -\dot{\phi} & (\dot{\lambda} + 2\omega_{ie})\cos\phi & 0 \end{bmatrix} \begin{bmatrix} v_N \\ v_E \\ v_D \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_N \\ f_E \\ f_D \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ g \end{bmatrix}$$

$$\begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\Phi\tan\Theta & \cos\Phi\tan\Theta \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \dfrac{\sin\Phi}{\cos\Theta} & \dfrac{\cos\Phi}{\cos\Theta} \end{bmatrix} \begin{bmatrix} \hat{p} \\ \hat{q} \\ \hat{r} \end{bmatrix}$$

$$\begin{bmatrix} \hat{p} \\ \hat{q} \\ \hat{r} \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} - \mathbf{R}_{n2b}(\omega_{ie} \begin{bmatrix} \cos\phi \\ 0 \\ -\sin\phi \end{bmatrix} + \begin{bmatrix} \dfrac{v_E}{R_M + h} \\ -\dfrac{v_N}{R_P + h} \\ -\dfrac{v_E \tan\phi}{R_P + h} \end{bmatrix})$$


Body Angular Rotation Increments
Gyros
Extract Body Rates
p q r
Euler Parameter Integration
Direction Cosine Matrix
Euler Angle Computation
ϕ θ ψ
Coordinate Frame Rate
Earth Rate
Ω
λ
V_N/R, V_E/R
Acctrs
Body Rotation Coriolis Correction
Coordinate Transformation
Gravity & Earth Coriolis Correction
Velocity Integration
Position Computation
λ μ H
Sensed Body Velocity Increments
Baro-Inertial Mixing
Barometric Altitude
Velocity Computation
V_G ψ_T V