#1 (a)

$$\text{Basis}: \left\{ \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$$

$$a\begin{bmatrix}1\\-1\\0\end{bmatrix} + b\begin{bmatrix}1\\0\\-1\end{bmatrix} + c\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\4\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a+b\\-a+c\\-b\end{bmatrix} = \begin{bmatrix}1\\4\\0\end{bmatrix} \Rightarrow \begin{cases} a=1\\ b=0\\ c=5\end{cases}$$

$(1, 0, 5)$ with respect to $\left\{ \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$ #

(b)

$$a\begin{bmatrix}1\\-1\\0\end{bmatrix} + b\begin{bmatrix}1\\0\\-1\end{bmatrix} + c\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}s+2\\ \frac{1}{s}\\ -2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a+b\\-a+c\\-b\end{bmatrix} = \begin{bmatrix}s+2\\ \frac{1}{s}\\ -2\end{bmatrix} \Rightarrow \begin{cases} a=s\\ b=2\\ c=\frac{1}{s}+s=\frac{s^2+1}{s}\end{cases}$$

$\left(s, 2, \frac{s^2+1}{s}\right)$ w.r.t. $\left\{ \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$ #

#2. $V = \left\{ \begin{bmatrix}x\\ \alpha x\end{bmatrix} \middle| x\in\mathbb{R}, \alpha = \text{const} \right\}$ over $\mathbb{R}$

Let $X = \left\{ \begin{bmatrix}1\\ \alpha\end{bmatrix} \right\}$

$\alpha_1 \begin{bmatrix}1\\ \alpha\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \alpha_1 = 0 \Rightarrow$ linearly indep.

$sp(x) = a\begin{bmatrix}1\\ \alpha\end{bmatrix} = \begin{bmatrix}x\\ \alpha x\end{bmatrix} \Rightarrow a = x$

$\Rightarrow X$ span V

$\Rightarrow \left\{ \begin{bmatrix}1\\ \alpha\end{bmatrix} \right\}$ is a basis of V. #

#3 (a) $S = \left\{ \begin{bmatrix} s^2 \\ s \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} \right\}$ over rational function

$$\alpha \begin{bmatrix} s^2 \\ s \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \alpha s^2 + \beta = 0 \\ \alpha s + \frac{\beta}{s} = 0 \end{cases} \Rightarrow \begin{cases} \alpha s^2 + \beta = 0 \\ \alpha s^2 + \beta = 0 \end{cases}$$

$\therefore$ $\alpha = \frac{1}{s^2}$, $\beta = -1$ is a solution.

$\therefore$ S over rationl function is NOT linear independent. #

(b) $S = \left\{ \begin{bmatrix} s^2 \\ s \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} \right\}$ over $\mathbb{R}$

$$\alpha \begin{bmatrix} s^2 \\ s \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \alpha s^2 + \beta = 0 \\ \alpha s + \frac{\beta}{s} = 0 \end{cases} \Rightarrow \begin{cases} \alpha s^2 + \beta = 0 \\ \alpha s^2 + \beta = 0 \end{cases}$$

$\because$ $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only solution.

$\therefore$ S over $\mathbb{R}$ is linear independent #

#4 $P(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$

$P'(0) = 4a_4 s^3 + 3a_3 s^2 + 2a_2 s + a_1 \Big|_{s=0} = a_1$

$P(0) = a_0$

$\Rightarrow P'(0) + P(0) = a_1 + a_0 = 0$

$P(1) = a_4 + a_3 + a_2 + a_1 + a_0 = 0$

$\Rightarrow a_4 + a_3 + a_2 = 0$

$V = \{ (a_4, a_3, a_2, a_1, a_0) \mid a_1 + a_0 = 0,\ a_4 + a_3 + a_2 = 0 \}$

(a) Let $x = x_4 s^4 + x_3 s^3 + x_2 s^2 + x_1 s + x_0 = (x_4, x_3, x_2, x_1, x_0) \in V$

$y = y_4 s^4 + y_3 s^3 + y_2 s^2 + y_1 s + y_0 = (y_4, y_3, y_2, y_1, y_0) \in V$

$\alpha, \beta \in \mathbb{R}$

(A) (a1) $x + y = (x_4 + y_4) s^4 + (x_3 + y_3) s^3 + (x_2 + y_2) s^2 + (x_1 + y_1) s + (x_0 + y_0)$

$y + x = (y_4 + x_4) s^4 + (y_3 + x_3) s^3 + (y_2 + x_2) s^2 + (y_1 + x_1) s + (y_0 + x_0)$

$= x + y$

(a2) $(x+y) + z = \left[(x_4 + y_4) s^4 + (x_3 + y_3) s^3 + (x_2 + y_2) s^2 + (x_1 + y_1) s + (x_0 + y_0) \right]$

$+ z_4 s^4 + z_3 s^3 + z_2 s^2 + z_1 s + z_0$

$= (x_4 + y_4 + z_4) s^4 + (x_3 + y_3 + z_3) s^3 + (x_2 + y_2 + z_2) s^2$

$+ (x_1 + y_1 + z_1) s + (x_0 + y_0 + z_0)$

$x + (y + z) = x_4 s^4 + x_3 s^3 + x_2 s^2 + x_1 s + x_0$

$+ (y_4 + z_4) s^3 + (y_3 + z_3) s^3 + (y_2 + z_2) s^2 + (y_1 + z_1) s + (y_0 + z_0)$

$= (x + y) + z$

(a3) Let $0_v = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$

$x + 0_v = x$

$\Rightarrow a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$

$\Rightarrow 0_v = 0 = (0, 0, 0, 0, 0)$

(a4) Let $-x = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$

$x + (-x) = 0$

$(-x) = -x = -a_4 s^4 - a_3 s^3 - a_2 s^2 - a_1 s - a_0$

$= (-a_4, -a_3, -a_2, -a_1, -a_0)$

(M) (m1) $(\alpha \cdot \beta) \cdot x = (\alpha\beta) \cdot x = \alpha\beta x_4 s^4 + \alpha\beta x_3 s^3 + \alpha\beta x_2 s^2 + \alpha\beta x_1 s + \alpha\beta x_0$

$\alpha \cdot (\beta \cdot x) = \alpha(\beta x_4 s^4 + \beta x_3 s^3 + \beta x_2 s^2 + \beta x_1 s + \beta x_0)$

$= (\alpha \cdot \beta) \cdot x$

(m2) $\alpha \cdot (x+y) = \alpha \cdot [(x_4+y_4)s^4 + (x_3+y_3)s^3 + (x_2+y_2)s^2 + (x_1+y_1)s$
$+ (x_0+y_0)]$

$= \alpha(x_4+y_4)s^4 + \alpha(x_3+y_3)s^3 + \alpha(x_2+y_2)s^2$
$+ \alpha(x_1+y_1)s + (x_0+y_0)]$

$\alpha x + \alpha y = \alpha x_4 s^4 + \alpha x_3 s^3 + \alpha x_2 s^2 + \alpha x_1 s + \alpha x_0$
$+ \alpha y_4 s^4 + \alpha y_3 s^3 + \alpha y_2 s^2 + \alpha y_1 s + \alpha x_0$

$= (\alpha x_4 + \alpha y_4)s^4 + (\alpha x_3 + \alpha y_3)s^3 + (\alpha x_2 + \alpha y_2)s^2$
$+ (\alpha x_1 + \alpha y_1)s + (\alpha x_0 + \alpha y_0)$

$= \alpha(x+y)$

(m3) $(\alpha+\beta)\cdot x = (\alpha+\beta)x_4 s^4 + (\alpha+\beta)x_3 s^3 + (\alpha+\beta)x_2 s^2 + (\alpha+\beta)x_1 s$
$+ (\alpha+\beta)x_0$

$$\alpha x + \beta x = \alpha x_4 s^4 + \alpha x_3 s^3 + \alpha x_2 s^2 + \alpha x_1 s + \alpha x_0$$
$$+ \beta x_4 s^4 + \beta x_3 s^3 + \beta x_2 s^2 + \beta x_1 s + \beta x_0$$
$$= (\alpha x_4 + \beta x_4)s^4 + (\alpha x_3 + \beta x_3)s^3 + (\alpha x_2 + \beta x_2)s^2$$
$$+ (\alpha x_1 + \beta x_1)s + \alpha x_0 + \beta x_0$$
$$= (\alpha+\beta)x$$

(m4) Let $1_V = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$

$$1_V \cdot x = x$$
$$\Rightarrow 1_V = 1 = (0, 0, 0, 0, 1)$$

$\therefore$ V is a vector space. #

(b) Let $X = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ $\quad X \subseteq V$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\Rightarrow \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ -\alpha_1 + \alpha_2 \\ -\alpha_2 \\ \alpha_3 \\ -\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \\ \alpha_3 = 0 \end{cases}$$

$\because \; \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\therefore \; \mathcal{X}$ is linearly independent

$$sp(\mathcal{X}) = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$= \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ -\alpha_1 + \alpha_2 \\ -\alpha_2 \\ \alpha_3 \\ -\alpha_3 \end{bmatrix}$$

$$= \alpha_1 s^4 + (-\alpha_1 + \alpha_2) s^3 + (-\alpha_2) s^2 + (\alpha_3) s + (-\alpha_3)$$

check conditions :

$$P(s) = \alpha_1 s^4 + (-\alpha_1 + \alpha_2) s^3 + (-\alpha_2) s^2 + (\alpha_3) s + (-\alpha_3)$$

$$P'(s) = 4\alpha_1 s^3 + 3(-\alpha_1 + \alpha_2) s^2 + 2(-\alpha_2) s + \alpha_3$$

$$P'(0) + P(0) = \alpha_3 + (-\alpha_3) = 0$$

$$P(1) = \cancel{\alpha_1} + (-\cancel{\alpha_1} + \cancel{\alpha_2}) + (-\cancel{\alpha_2}) + \cancel{\alpha_3} + (-\cancel{\alpha_3})$$

$$= 0$$

$\mathcal{X}$ can span $\mathbb{V}$.

$\Rightarrow \mathcal{X}$ is a basis for $\mathbb{V}$. #

(c) $\dim(\mathbb{V}) = 3$. #

(d) $P(s) = s^4 + s^3 - 2s^2 + 3s - 3$

$\Rightarrow (1, 2, 3)$ w.r.t. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ #