Optimal Control

Final exam

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I did not offer assistances to nor receive assistances from others in this exam.

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1

$$\min -\int_0^1 y\,dt$$

s.t.

$$\begin{cases} \dot{x} = V\cos u & x(0)=0 \\ \dot{y} = V\sin u & y(0)=0 \end{cases}$$

$$\psi(x(t_f)) = \begin{bmatrix} x(1) \\ y(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$H = -y + \lambda_1 (V\cos u) + \lambda_2 (V\sin u)$$

$$\begin{cases} \dot{x} = V\cos u \\ \dot{y} = V\sin u \end{cases} \quad - ①$$

$$\begin{cases} \dot{\lambda}_1 = -\frac{\partial H}{\partial x} = 0 \\ \dot{\lambda}_2 = -\frac{\partial H}{\partial y} = 1 \end{cases} \quad - ②$$

$$H_u = -\lambda_1 V\sin u + \lambda_2 V\cos u = 0 \quad - ③$$

$$\begin{bmatrix} \lambda_1(1) \\ \lambda_2(1) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \gamma \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad - ④$$

$$\begin{bmatrix} x(1) \\ y(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad - ⑤$$

From ②,

$$\begin{cases} \lambda_1 = \pi_1 \\ \lambda_2 = t + \pi_2 \end{cases}$$

From ③,

$$-\lambda_1 V\sin u + \lambda_2 V\cos u = 0$$

$$\Rightarrow \frac{\sin u}{\cos u} = \tan u = \frac{\lambda_2}{\lambda_1} = \frac{t+\pi_2}{\pi_1} = \frac{1}{\pi_1} t + \frac{\pi_2}{\pi_1} = \underline{C_1 + C_2 t} \; \#$$

#2

$$\min \; x_2(1)$$

s.t.

$$\begin{cases} \dot{x}_1 = x_2 \;, & x_1(0) = 0 \\ \dot{x}_2 = u \;, & x_2(0) = 0 \end{cases}$$

$$x_1(1) = 0$$

$$|u| \le 1$$

$$H = \lambda_1 (x_2) + \lambda_2 (u)$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \qquad -①$$

$$\begin{cases} \dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \\ \dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \end{cases} \qquad -②$$

$$\min_{u \in \{-1,1\}} H = \lambda_1 (x_2) + \lambda_2 (u)$$

$$\Rightarrow \begin{cases} \text{if } \lambda_2 \ge 0 \,, & u = -1 \\ \text{if } \lambda_2 < 0 \,, & u = 1 \end{cases} \qquad -③$$

$$\begin{bmatrix} \lambda_1(1) \\ \lambda_2(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \nu \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad -④$$

$$x_1(1) = 0 \qquad -⑤$$

From ②,

$$\lambda_1 = \pi_1$$

$$\lambda_2 = -\pi_1 t + \pi_2$$

$$\lambda_2(1) = -\pi_1 t + \pi_2 = 1$$

Let switch time is t_s

for $0 \le t \le t_s$,

$$\begin{cases} \dot{x}_1 = x_2 \,, & x_1(0) = 0 \\ \dot{x}_2 = 1 \,, & x_2(0) = 0 \end{cases}$$

$$\Rightarrow x_2(t) = \int_0^t 1 \, d\tau = \tau \Big|_0^t = t$$

$$x_1(t) = \int_0^t \tau \, d\tau = \frac{1}{2}\tau^2 \Big|_0^t = \frac{1}{2} t^2$$

for $t_s < t \le t_f$,

$$\begin{cases} \dot{x}_1 = x_2, & x_1(t_s) = t_s \\ \dot{x}_2 = -1, & x_2(t_s) = \frac{1}{2}t_s^2 \end{cases}$$

$$\Rightarrow x_2(t) = \int_{t_s}^{t} -1 \, d\tau = -\tau \Big|_{t_s}^{t} = -t + t_s$$

$$x_1(t) = \int_{t_s}^{t} -\tau + t_s \, d\tau = -\frac{1}{2}\tau^2 + t_s\tau \Big|_{t_s}^{t} = -\frac{1}{2}t^2 + t_s t + \frac{1}{2}t_s^2 - t_s^2$$

$$= -\frac{1}{2}t^2 + t_s t - \frac{1}{2}t_s^2$$

$$x_1(1) = 0 = t_s - \frac{1}{2} + t_s - \frac{1}{2}t_s^2$$

$$\Rightarrow \frac{1}{2}t_s^2 - 2t_s + \frac{1}{2} = 0$$

$$\Rightarrow t_s = 2 \pm \sqrt{4-1} = 0.2679, \underbrace{\cancel{3.7321}}_{> t_f}$$

$$\Rightarrow u = \begin{cases} 1, & \text{if } 0 \le t \le 0.2679 \\ -1, & \text{if } 0.2679 < t \le 1 \end{cases} \quad \#$$

\# 3.

$$\min \ \frac{1}{2}\int_0^{t_f} \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt$$

s.t.

$$\dot{x} = Ax + Bu$$

$$\psi(x(t_f)) = M_f \, x(t_f)$$

(a)

$$H = \frac{1}{2}\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \lambda^T(Ax + Bu)$$

$$= \frac{1}{2}x^T Q x + x^T N u + \frac{1}{2}u^T R u + \lambda^T A x + \lambda^T B u$$

① $\dot{x} = Ax + Bu$

② $\dot{\lambda} = -\frac{\partial H}{\partial x} = -Qx - Nu - A^T\lambda$

③ $H_u = 0 = N^T x + Ru + B^T\lambda$

④ $\lambda(t_f) = \cancel{\phi_x(x(t_f))}^{\,0} + \nu^T \psi_x(x(t_f))$

$$= \nu^T M_f$$

⑤ $\psi(x(t_f)) = M_f \, x(t_f)$ \#

(b)

From ③

$$\underline{u = R^{-1}(-N^T x - B^T \lambda)} \ \#$$

(c)

$$\dot{x} = Ax + Bu$$

$$= Ax + B(-R^{-1}N^T x - R^{-1}B^T \lambda)$$

$$= (A - BR^{-1}N^T)x - BR^{-1}B^T \lambda$$

$$\dot{\lambda} = -Qx - Nu - A^T \lambda$$

$$= -Qx - N(-R^{-1}N^T x - R^{-1}B^T \lambda) - A^T \lambda$$

$$= -Qx + NR^{-1}N^T x - NR^{-1}B^T \lambda - A^T \lambda$$

$$= (-Q + NR^{-1}N^T)x - (NR^{-1}B^T + A^T)\lambda$$

$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \underline{\begin{bmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -NR^{-1}B^T + A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}} \ \#$$

(d)

#4.

$$\min J = \frac{1}{2} e_f^T Q_f e_f + \frac{1}{2}\int_0^{t_f} (y-y_d)^T Q_d (y-y_d) + u^T R_d u \} dt$$

s.t.

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$e_f = M_f x(t_f) - \psi(x(t_f))$$

Parameters :

$$C = [0 \ 0 \ 1 \ 0]$$

$$D = [0 \ 0]$$

$$\psi(x(t_f)) = x(t_f) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$y_d = 1$$

To be determine parameter

$$Q_f \in \mathbb{R}^{4\times4},\ Q_d \in \mathbb{R}^{1\times1},\ R_d \in \mathbb{R}^{2\times2}$$

Def: $Q = C^T Q_d C$, $N = C^T Q_d D$, $R = R_d + D^T Q_d D$

$$\begin{cases} \dot{S} = -S(A - BR^{-1}N^T) - (A - BR^{-1}N^T)^T S - Q + NR^{-1}N^T + SBR^{-1}B^T S \\ \dot{g} = -[(A - BR^{-1}N^T)^T + (BR^{-1}S)^T] g + C^T Q_d y_d \end{cases}$$ #

$$\begin{cases} S(t_f) = M_f^T Q_f M_f \\ g(t_f) = -M_f^T Q_d \psi(x(t_f)) \end{cases}$$ #

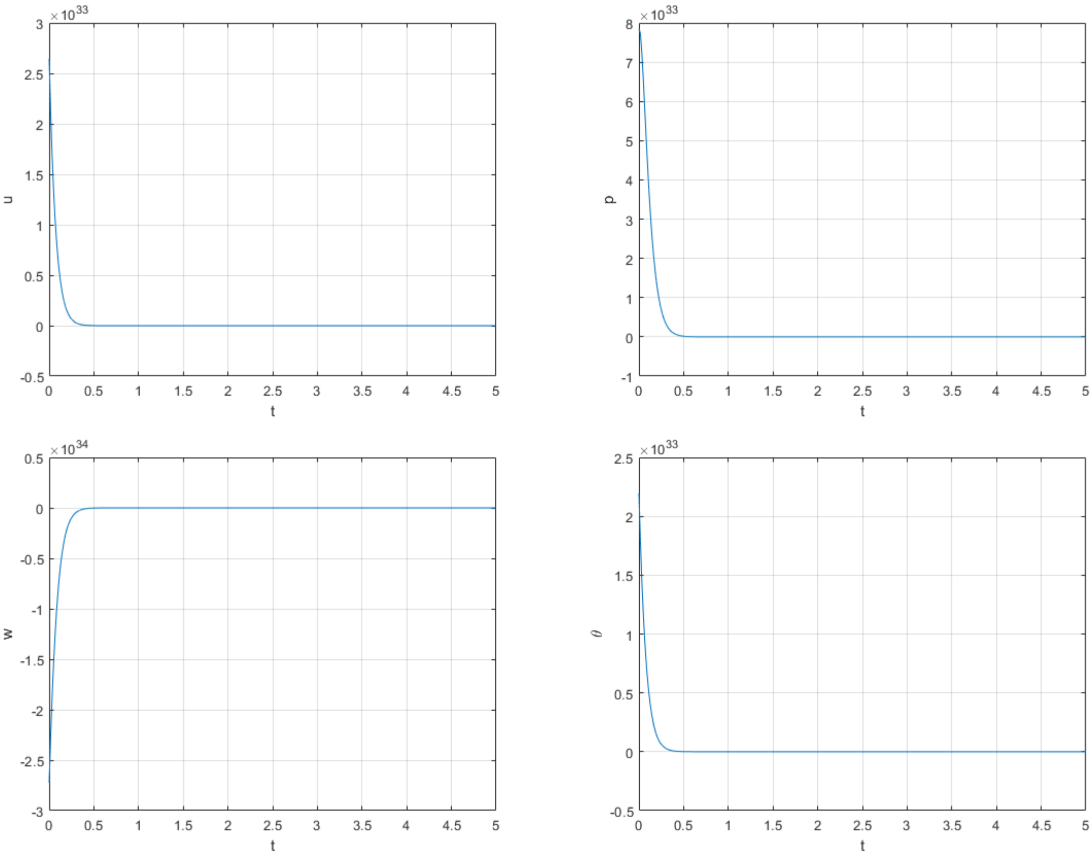
#4(b)

```
clear;clc;close all
% System parameter
A = [-0.2640   0.7203   0       -9.8063
     -1.4406  -8.0534  10.5360  0
      0.7936 -10.4977 -20.2975  0
      0        1        0       0      ];
B = [ 0       2.8510
     -0.1392  0
     -2.4104  0
      0       0     ];
C = [0 0 1 0];
D = zeros(1,2);

% Constrain parameter
yd = 1;
psi = [0 0 1 0]';
Mf = diag([1 1 1 1]).*factor;

% Tune parameter
factor = 1;
Qd = 100.*factor;                % weight of tracking
Rd = diag([100 100]).*factor;    % weight of control
Qf = diag([1 1 1 1]).*factor;    % weight of terminal error

Q = C'*Qd*C;
N = C'*Qd*D;
R = Rd + D'*Qd*D;

% Terminal condition
x_tf = psi;
s_tf = Mf'*Qf*Mf;
s_tf = reshape(s_tf, [16,1]);
g_tf = -Mf'*Qd*psi;
state_tf = [x_tf; s_tf; g_tf];

LQT = @(t, state) NC(t, state, A, B, C, D, Q, R, N, Qd, yd);
[t, state] = ode45(LQT, [5 0], state_tf);
```

```
% Plot of result
figure()
plot(t, state(:,1))
xlabel("t"); ylabel("u")
grid on

figure()
plot(t, state(:,2))
xlabel("t"); ylabel("w")
grid on

figure()
plot(t, state(:,3))
xlabel("t"); ylabel("p")
grid on

figure()
plot(t, state(:,4))
xlabel("t"); ylabel("\theta")
grid on
```

#4(c)

No, it can't be solved by this method. It may figure out an answer but the solution is out of sense because the initial state we figure out approaches to the infinity. And it is difficult to solve this problem by tuning the parameters.


×10^33
u
t
×10^33
p
t
×10^34
w
t
×10^33
θ
t

function of necessary condition

```
function dstate = NC(t, state, A, B, C, D, Q, R, N, Qd, yd)
    % state define: [x(1:4), s(5:20), g(21:24)]'

    x = state(1:4);
    s = reshape(state(5:20), [4,4]);
    g = state(21:24);
    lambda = s*x + g;
    u = inv(R)*(-N'*x-B'*lambda);

    inv_R = inv(R);

    dx = A*x + B*u;
    ds = -s*(A-B*inv_R*N') - (A-B*inv_R*N')'* - Q ...
         + N*inv_R*N' + s*B*inv_R*B'*s;
    dg = (-(A-B*inv_R*N')' + s*B*inv_R*B')*g + C'*Qd*yd;

    dstate = zeros(24,1);
    dstate(1:4) = dx;
    dstate(5:20) = reshape(ds, [16,1]);
    dstate(21:24) = dg;

end
```