#1 For $\{\mathcal{A}v_1, \mathcal{A}v_2, \cdots, \mathcal{A}v_n\}$ is linearly independent,

$$\alpha_1 \mathcal{A}(v_1) + \alpha_2 \mathcal{A}(v_2) + \cdots + \alpha_n \mathcal{A}(v_n) = 0_W, \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

$\because$ $\mathcal{A}$ is a linear transformation.

$$\therefore \mathcal{A}(\alpha_1 v_1) + \mathcal{A}(\alpha_2 v_2) + \cdots + \mathcal{A}(\alpha_n v_n) = \mathcal{A}(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = 0_W$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0_V, \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

$\Rightarrow \{v_1, v_2, \cdots, v_n\}$ is linearly independent. #

Let $V, W \in \mathbb{R}^2$. $\mathcal{A}: V \to W$

$$V = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{A}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 3x_1 - 2x_2 \\ 3x_1 - 2x_2 \end{bmatrix} = W$$

$$w_1 = \mathcal{A}(v_1) = \begin{bmatrix} 3\cdot 1 - 2\cdot 0 \\ 3\cdot 1 - 2\cdot 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$w_2 = \mathcal{A}(v_2) = \begin{bmatrix} 3\cdot 0 - 2\cdot 1 \\ 3\cdot 0 - 2\cdot 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$\Rightarrow W = \left\{ \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right\}$$

$$\alpha_1 \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \alpha_1 = \frac{1}{3}, \quad \alpha_2 = \frac{1}{2}$$

$\because$ $\alpha_1, \alpha_2 \neq 0$

$\therefore$ W is NOT linear independent #

#2. Let $x \in N(\mathcal{A})$, $v \in V$, but $x \in V$, $v \notin N(\mathcal{A})$

$$\mathcal{A}(x+v) = \mathcal{A}(x) + \mathcal{A}(v)$$
$$= \mathcal{A}(v)$$

If $\mathcal{A}$ is one-to-one mapping,

$$\mathcal{A}(x+v) = \mathcal{A}(v) \Rightarrow x+v = v$$

then $x = 0_V$

$\Rightarrow \mathcal{N}(\mathcal{A}) = 0_V$ #

#3.

#4. Let $W = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, $V = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

$P(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = (a_4, a_3, a_2, a_1, a_0)$ w.r.t. W

$$\mathcal{A}(P(s)) = \int_0^s a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \, ds$$

$$= \frac{a_4}{5} s^5 + \frac{a_3}{4} s^4 + \frac{a_2}{3} s^3 + \frac{a_1}{2} s^2 + a_0 s \Rightarrow A_W = \begin{bmatrix} \frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_W$$

(a) Let P_1, P_2 w.r.t. W

$$\mathcal{A}(P_1(s)+P_2(s)) = A(P_1+P_2) = AP_1+AP_2$$
$$= \mathcal{A}(P_1(s)) + \mathcal{A}(P_2(s))$$
$$\mathcal{A}(\alpha P_1(s)) = A(\alpha P_1) = \alpha(AP_1) = \alpha \mathcal{A}(P_1)$$

$\therefore$ $\mathcal{A}$ is a linear transformation #

(b) Let (b_1, b_2, b_3) w.r.t. V

$$\begin{cases} v_1 = 1\cdot w_1 + (-1)\cdot w_2 + \\ v_2 = 0\cdot w_1 + 1\cdot w_2 + (-1)\, w_3 \\ v_3 = \qquad\qquad\qquad\qquad + 1\cdot w_4 + (-1)\, w_5 \end{cases}$$

$$T_{W2V} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A_V = A_W T_{W2V} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R(\mathcal{A}) = A_V \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} b_1 \\ -\frac{1}{4} b_1 + \frac{1}{4} b_2 \\ -\frac{1}{3} b_2 \\ \frac{1}{2} b_3 \\ -b_3 \\ 0 \end{bmatrix} = b_1 \begin{bmatrix} \frac{1}{5} \\ -\frac{1}{4} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ \frac{1}{4} \\ -\frac{1}{3} \\ 0 \\ 0 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix}$$

$$\mathcal{R}(\mathcal{A}) = sp\left(\begin{bmatrix} 1/5 \\ -1/4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/4 \\ -1/3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \\ -1 \\ 0 \end{bmatrix}\right) \text{ w.r.t. } \{s^6, s^5, \dots, s, 1\}$$

(c)

$$A = \begin{bmatrix} 1/5 & 0 & 0 \\ -1/4 & 1/4 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = A_V \quad \#$$

(d) Let $x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ w.r.t V

$$A_V x = 0$$

$$\Rightarrow \begin{bmatrix} 1/5 & 0 & 0 \\ -1/4 & 1/4 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \frac{1}{5}a_1 = 0 \\ \frac{-1}{4}a_1 + \frac{1}{4}a_2 = 0 \\ -\frac{1}{3}a_2 = 0 \\ \frac{1}{2}a_3 = 0 \\ -a_3 = 0 \\ 0 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases} \Rightarrow \mathcal{N}(\mathcal{A}) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\} \#$$

(e)

$$\dim(\mathcal{R}(\mathcal{A})) + \dim(\mathcal{N}(\mathcal{A})) = 3 + 0 = 3 = \dim(V) \#$$