#1.

$$e^{(A_1+A_2)t} = \sum_{n=0}^{\infty} \frac{t^n (A_1+A_2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{t^n}{n!} C_m^n A_1^m A_2^{n-m}, \quad \text{if } A_1 A_2 = A_2 A_1$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\cancel{n!}\, A_1^m A_2^{n-m}}{(n-m)!\, m!\, \cancel{n!}} t^n$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{A_1^m}{m!} t^m\right)\left(\frac{A_2^{n-m}}{(n-m)!} t^{n-m}\right)$$

$$= e^{A_1 t} e^{A_2 t} \quad \#$$

#2

$$\phi_1(s) = \det(sI - A_1) = \det \begin{bmatrix} s-1 & -2 & 0 \\ 0 & s & -2 \\ 0 & 0 & s-1 \end{bmatrix} = (s-1)s(s-1) \Rightarrow s = 0, 1, 1$$

Let $p(s) = \alpha_0 + \alpha_1 s + \alpha_2 s^2$, $f_1(s) = s^{100}$, $f_2(s) = e^{st}$

for $s_1 = 0$,

$$\alpha_0 = 0^{100} = 0$$

for $s_{2,3} = 1$,

$$\alpha_1 + \alpha_2 = 1^{100} = 1 \quad -①$$

$$p'(s) = \alpha_1 + 2\alpha_2 s$$

$$f_1'(s) = 100 s^{99}$$

$$p'(1) = \alpha_1 + 2\alpha_2 = 100 \times 1^{99} = 100 \quad -②$$

from ① & ②

$$\begin{cases} \alpha_1 + \alpha_2 = 1 \\ \alpha_1 + 2\alpha_2 = 100 \end{cases} \Rightarrow \begin{cases} \alpha_1 = -98 \\ \alpha_2 = 99 \end{cases}$$

$$s^{100} = -98s + 99s^2$$

$$\Rightarrow A_1^{100} = -98A_1 + 99A_1^2 = \begin{bmatrix} -98 & -196 & 0 \\ 0 & 0 & -196 \\ 0 & 0 & -98 \end{bmatrix} + \begin{bmatrix} 99 & 198 & 396 \\ 0 & 0 & 198 \\ 0 & 0 & 99 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 396 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \#$$

for $s=0$,

$$\alpha_0 = e^0 = 1$$

for $s=1$

$$1+\alpha_1+\alpha_2 = e^t \quad -① $$

$$P'(s) = \alpha_1 + 2\alpha_2 s$$

$$f_2'(s) = te^{\lambda t}$$

$$P'(1) = \alpha_1 + 2\alpha_2 = te^t \quad -②$$

from ① & ②

$$\begin{cases} \alpha_1+\alpha_2 = e^t - 1 \\ \alpha_1 + 2\alpha_2 = te^t \end{cases} \Rightarrow \begin{cases} \alpha_1 = (-t+2)e^t - 2 \\ \alpha_2 = (t-1)e^t + 1 \end{cases}$$

$$e^s = 1 + \left[(-t+2)e^t - 2\right]s + \left[(t-1)e^t + 1\right]s^2$$

$$\Rightarrow e^{A_1 t} = 1 + \left[(-t+2)e^t - 2\right]A_1 + \left[(t-1)e^t + 1\right]A_1^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \left[(-t+2)e^t - 2\right]\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} + \left[(t-1)e^t + 1\right]\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 2(e^t-1) & 4\left[(t-1)e^t+1\right] \\ 0 & 1 & 2(e^t-1) \\ 0 & 0 & e^t \end{bmatrix} \#$$

$$\phi_2(s) = \det(sI - A_2) = \det\begin{bmatrix} s & 1 & 0 & 0 \\ 0 & s & 1 & 0 \\ 0 & 0 & s & 1 \\ 0 & 0 & 0 & s \end{bmatrix} = s^4 \Rightarrow s=0$$

Let $P(s) = \alpha_0 + \alpha_1 s + \alpha_2 s^2 + \alpha_3 s^3$, $f_1(s) = s^{100}$, $f_2(s) = e^{st}$

$P'(s) = \alpha_1 + 2\alpha_2 s + 3\alpha_3 s^2$, $f_1'(s) = 100 s^{99}$, $f_2'(s) = te^{st}$

$P''(s) = 2\alpha_2 + 6\alpha_3 s$, $f_1''(s) = 9900 s^{98}$, $f_2''(s) = t^2 e^{st}$

$P^{(3)}(s) = 6\alpha_3$, $f_1^{(3)}(s) = 970200 s^{97}$, $f_2^{(3)}(s) = t^3 e^{st}$

$$P(0) = \alpha_0 = 0^{100}$$

$$P'(0) = \alpha_1 = 100 \cdot 0^{99} = 0$$

$$P''(0) = 2\alpha_2 = 9900 \cdot 0^{98} = 0$$

$$P^{(3)}(0) = 6\alpha_3 = 970200 \cdot 0^{97} = 0$$

$$s^{100} = 0 \Rightarrow A^{100} = \underline{0_{4\times4}} \ \#$$

$$P(0) = \alpha_0 = e^0 = 1$$

$$P'(0) = \alpha_1 = te^0 = t$$

$$P''(0) = 2\alpha_2 = t^2e^0 = t^2 \Rightarrow \alpha_2 = \frac{1}{2}t^2$$

$$P^{(3)}(0) = 6\alpha_3 = t^3e^0 = t^3 \Rightarrow \alpha_3 = \frac{1}{6}t^3$$

$$e^{st} = 1 + ts + \frac{1}{2}t^2s^2 + \frac{1}{6}t^3s^3$$

$$e^{At} = I + tA + \frac{1}{2}t^2A^2 + \frac{1}{6}t^3A^3$$

$$= \begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \frac{1}{6}t^3 \\ 0 & 1 & t & \frac{1}{2}t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \#$$

\# 3.

$$\det(sI - A) = \det\begin{bmatrix} s-a & b \\ -b & s-a \end{bmatrix} = (s-a)^2 + b^2 = 0$$

$$\Rightarrow (s-a)^2 = -b^2 \Rightarrow s - a = \pm bi \Rightarrow s = a \pm bi$$

Let $P(s) = \alpha_0 + \alpha_1 s$, $f(s) = e^{st}$

for $s = a + bi$,

$$P(a+bi) = \alpha_0 + \alpha_1(a+bi) = e^{(a+bi)t} = e^{at}(\cos bt + i\sin bt) \quad \text{—①}$$

for $s = a - bi$

$$P(a-bi) = \alpha_0 + \alpha_1(a-bi) = e^{(a-bi)t} = e^{at}(\cos bt - i\sin bt) \quad \text{—②}$$

from ①−②,

$$\cancel{2}\alpha_1 b \cancel{i} = e^{at}(\cancel{i}\,\cancel{2}\sin bt)$$

$$\Rightarrow \alpha_1 = \frac{e^{at}}{b}\sin bt$$

from ①+②,

$$\cancel{2}\alpha_0 + \cancel{2}a\alpha_1 = e^{at}(\cancel{2}\cos bt)$$

$$\Rightarrow \alpha_0 = e^{at}\cos bt - a\alpha_1$$

$$= e^{at}\cos bt - \frac{a}{b}e^{at}\sin bt$$

$$e^{At} = e^{at}\left(\cos bt - \frac{a}{b}\sin bt\right) + \left(\frac{e^{at}}{b}\sin bt\right)\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$= \begin{bmatrix} e^{at}\cos bt - \cancel{\frac{a}{b}\sin bt} & 0 \\ 0 & e^{at}\cos bt - \cancel{\frac{a}{b}\sin bt} \end{bmatrix} + \begin{bmatrix} \cancel{\frac{a}{b}e^{at}\sin bt} & e^{at}\sin bt \\ -e^{at}\sin bt & \cancel{\frac{a}{b}e^{at}\sin bt} \end{bmatrix}$$

$$= \begin{bmatrix} e^{at}\cos bt & e^{at}\sin bt \\ -e^{at}\sin bt & e^{at}\cos bt \end{bmatrix} = \underline{e^{at}\begin{bmatrix} \cos bt & \sin bt \\ \sin bt & \cos bt \end{bmatrix}} \quad \#$$

#4

(a)

$$x = [1 \;\; 0 \;\; 1]^T \Rightarrow e_x = \left[\frac{1}{\sqrt{2}} \;\; 0 \;\; \frac{1}{\sqrt{2}}\right]^T$$

$$R(x, 45^\circ) = \cos 45^\circ\, I_3 + \sin 45^\circ\, [x]_\times + (1 - \cos 45^\circ)\, x x^T$$

$$= \frac{\sqrt{2}}{2} I_3 + \frac{\sqrt{2}}{2}[x]_\times + \left(1 - \frac{\sqrt{2}}{2}\right) x x^T$$

$$= \underline{\begin{bmatrix} \frac{2+\sqrt{2}}{4} & \frac{-1}{2} & \frac{2-\sqrt{2}}{4} \\ \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{-1}{2} \\ \frac{2-\sqrt{2}}{4} & \frac{1}{2} & \frac{2+\sqrt{2}}{4} \end{bmatrix}} \quad \#$$

Aside:

$$[x]_\times = \begin{bmatrix} 0 & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$x x^T = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(b),(c)

$$\phi(s) = \det(sI - R) = (s-1)\left(s - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\left(s - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$$

$$s = 1,\ \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i$$

for $s = 1$,

$$\begin{bmatrix} \frac{2-\sqrt{2}}{4} & \frac{1}{2} & \frac{-2+\sqrt{2}}{4} \\ \frac{-1}{2} & \frac{2-\sqrt{2}}{2} & \frac{1}{2} \\ \frac{-2+\sqrt{2}}{4} & \frac{-1}{2} & \frac{2-\sqrt{2}}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2+\sqrt{2} & -1 & 0 \\ -1 & 2-\sqrt{2} & 1 & 0 \\ -2+\sqrt{2} & -2 & 2-\sqrt{2} & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2+\sqrt{2} & -1 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 12 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvector of $s = 1$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ #