淡江大學

航空太空工程學系研究所

高等工程數學

作業 2

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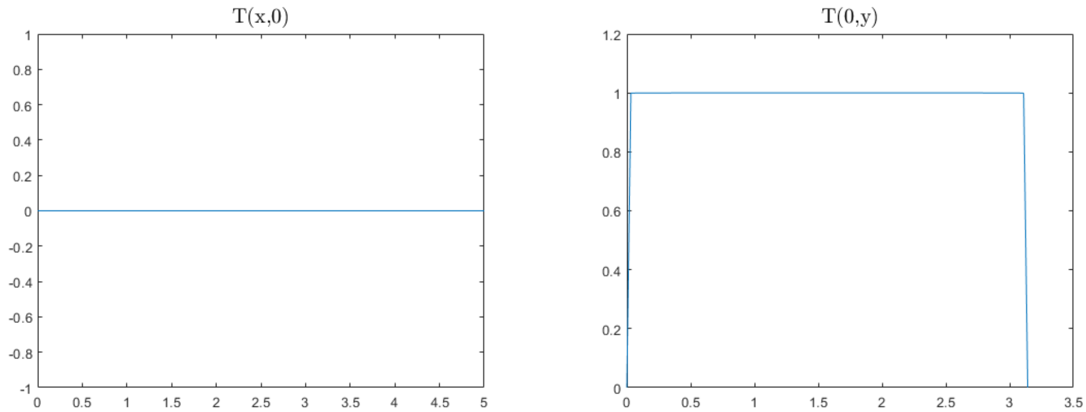
#1

```
clear;clc;close all
[x, y] = meshgrid(linspace(0,5,100),linspace(0,pi(),100));
len = size(x);
for i = 1:len(1)
    for j = 1:len(2)
        f = @(n) exp(-(2.*n-1).*x(i,j)).*sin((2.*n-1).*y(i,j))./(2.*n-1);
        T(i,j) = 4/pi()*limsum(f);
    end
end
figure()
surf(x,y,T)

f = @(n) 4/pi()*exp(-(2.*n-1).*1).*sin((2.*n-1).*(pi()/2))./(2.*n-1);
sum = 0;
n = 1;
while 1
    error = f(n);
    sum = sum + error;
    if abs(error) < 1e-6
        break
    end
    n = n+1;
end
fprintf("For T(1,pi/2): \n    Iteration times: %d \n    Value: %f \n    Error: %f\n\n", n, sum, er

f = @(n) 4/pi()*exp(-(2.*n-1).*0.0369).*sin((2.*n-1).*(0.01*pi()))./(2.*n-1);
sum = 0;
n = 1;
while 1
    error = f(n);
    sum = sum + error;
    if abs(error) < 1e-6
        break
    end
    n = n+1;
end
fprintf("For T(0.0369,0.01pi): \n    Iteration times: %d \n    Value: %f \n    Error: %f\n\n", n, 

figure()
plot(x(1,:), T(1,:))
title("T(x,0)",'FontSize',15,'interpreter','latex')

figure()
plot(y(:,1), T(:,1))
```

```
title("T(0,y)",'FontSize',15,'interpreter','latex')
```

....

```
For T(1,pi/2):
    Iteration times: 7
    Value: 0.448834
    Error: 0.000000

For T(0.0369,0.01pi):
    Iteration times: 98
    Value: 0.448859
    Error: -0.000001
```



#2 (a)

$$f(\rho) = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sin\theta}{1-\rho^2}\right) \quad , \quad \text{Let } x(\rho) = \frac{2\rho\sin\theta}{1-\rho^2}$$

$$\Rightarrow f(x) = \frac{1}{2}\tan^{-1}(x)$$

From Maclaurin series,

$$f(\rho) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\rho^n$$

for $n=0$,

$$f(0) = \frac{1}{2}\tan^{-1}(0) = 0$$

for $n=1$,

$$f'(\rho) = \frac{df}{d\rho} = \frac{df}{dx}\frac{dx}{d\rho} = \frac{1}{\cancel{2}(x^2+1)} \cdot \frac{\cancel{4}\sin\theta(1+\rho^2)}{(1-\rho^2)^2} = \frac{\sin\theta(\rho^2+1)}{1-2\rho^2+\rho^4+4\rho^2\sin^2\theta}$$

- $$\frac{df}{dx} = \frac{1}{2(x^2+1)}$$
- $$\frac{dx}{d\rho} = \frac{2\sin\theta}{1-\rho^2} + \frac{2\rho\sin\theta}{(1-\rho^2)^2}(-1)(-2\rho)$$

$$= \frac{2\sin\theta(1-\rho^2)+4\rho^2\sin\theta}{(1-\rho^2)^2} = \frac{2\sin\theta(1+\rho^2)}{(1-\rho^2)^2}$$

$$f'(0) = \frac{1}{2(x(0)^2+1)}(2\sin\theta) = \underline{\sin\theta}$$

for $n=2$,

$$f''(\rho) = \frac{2\rho\sin\theta}{1-2\rho^2+\rho^4+4\rho^2\sin^2\theta} - \frac{\sin\theta(\rho^2+1)}{(1-2\rho^2+\rho^4+4\rho^2\sin^2\theta)^2}(-4\rho+4\rho^3+8\rho\sin^2\theta)$$

$$= \frac{2\rho\sin\theta[-\rho^4-2\rho^2+2\cos(2\theta)+1]}{1-2\rho^2+\rho^4+4\rho^2\sin^2\theta}$$

$$\Rightarrow f''(0) = 0 - 0 = \underline{0}$$

for $n=3$,

$$f^{(3)}(\rho) = \frac{2\sin\theta\left[-5\rho^4-6\rho^2+2\cos(2\theta)+1\right]}{1-2\rho^2+\rho^4+2\rho^2\sin^2\theta} - \frac{2\sin\theta\left[-\rho^5-2\rho^3+2\rho\cos(2\theta)+\rho\right]}{\left(1-2\rho^2+\rho^4+2\rho^2\sin^2\theta\right)^2} \times \left(-4\rho+4\rho^3+4\rho\sin^2\theta\right)$$

$$f^{(3)}(0) = \frac{2\sin\theta\left[2\cos(2\theta)+1\right]}{1} - 0$$

$$= 2\left[2\sin\theta\cos(2\theta)+\sin\theta\right] = 2\left[2\sin(\theta)\cos(2\theta)+\sin(2\theta-\theta)\right]$$

$$= 2\left[\cancel{2}\sin\theta\cos(2\theta)+\sin(2\theta)\cos\theta-\cancel{\cos\theta\sin(2\theta)}\right]$$

$$= 2\sin(\theta+2\theta) = \underline{2\sin(3\theta)}$$

$$f(\rho) = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sin\theta}{1-\rho^2}\right) = \cancelto{0}{\frac{0}{0!}}\cdot 1 + \frac{\sin\theta}{1!}\rho + \cancelto{0}{\frac{0}{2!}}\rho^2 + \frac{2\sin(3\theta)}{3!}\rho^3 + \cdots$$

$$= \underline{\rho\sin\theta + \frac{1}{3}\rho\sin(3\theta) + \cdots \quad \#}$$

(b)

$$\sum_{n=0}^{\infty} \frac{e^{-(2n+1)x}}{2n+1}\sin(2n+1)\frac{\pi y}{H} = \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}e^{-nx}\sin\frac{n\pi}{H}y$$

Let $\rho = e^{-x}$, $\theta = \frac{\pi y}{H}$, then

$$\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\rho^n\sin n\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sin\theta}{1-\rho^2}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2e^{-x}\sin\frac{\pi}{H}y}{1-e^{-2x}}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{\sin\frac{\pi}{H}y}{\frac{e^{x}-e^{-x}}{2}}\right)$$

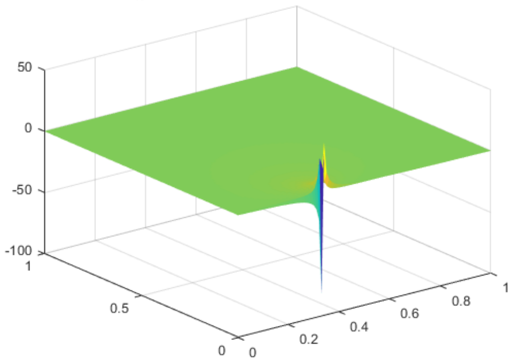
$$= \frac{1}{2}\tan^{-1}\left(\frac{\sin\frac{\pi}{H}y}{\sinh x}\right)$$

#3

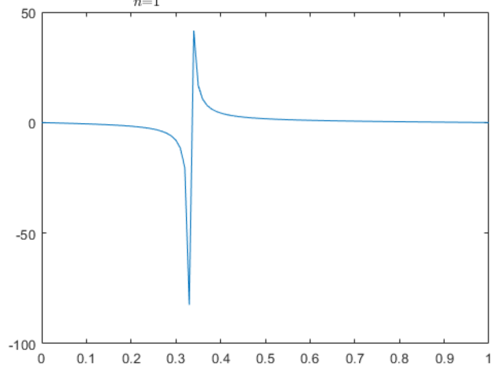
```
clear;clc;close all
[x, y] = meshgrid(0:0.01:1,0:0.01:1);
len = size(x);
for i = 1:len(1)
    for j = 1:len(2)
        f = @(n) 2*exp(-n*pi().*y(i,j)).*sin(n*pi()/3).*sin(n*pi()*x(i,j));
        T(i,j) = limsum(f);
    end
end

figure()
surf(x,y,T, 'edgecolor', 'none')
title("$2\sum\limits^{\infty}_{n=1} e^{-n \pi y}\sin{\frac{n\pi}{3}}\sin{n \pi x}$ 3D Plot",'FontS

figure()
plot(x(1,:), T(1,:))
% contour(x,y,T,'ShowText','on')
title("$2\sum\limits^{\infty}_{n=1} e^{-n \pi y}\sin{\frac{n\pi}{3}}\sin{n \pi x}$ 2D Plot",'FontS

x = 0;
ended = 1;
step = 0.0001;
sum = 0;
while x<=ended
    f = @(n) 2*sin(n*pi()/3).*sin(n*pi()*x);
    sum = sum + limsum(f)*step;
    x = x+step;
end
fprintf("The answer of integral of T(x,0) from 0 to 1 is %.4f. \n", sum)
```

```
The answer of integral of T(x,0) from 0 to 1 is 1.0000.
```

$$2\sum_{n=1}^{\infty} e^{-n\pi y} \sin \tfrac{n\pi}{3} \sin n\pi x \text{ 3D Plot}$$



$$2\sum_{n=1}^{\infty} e^{-n\pi y} \sin \tfrac{n\pi}{3} \sin n\pi x \text{ 2D Plot}$$



#4 (a)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Let $T(x,y) = X(x)\,Y(y)$

$$\Rightarrow X''Y + XY'' = 0$$

$$\Rightarrow \frac{Y''}{Y} = -\frac{X''}{X} = \mu = -\lambda^2$$

$$\begin{cases} Y'' + \lambda^2 Y = 0 & -① \\ X'' + \lambda^2 X = 0 & -① \end{cases}$$

For eq. ①

$$Y'' + \lambda^2 Y = 0 \Rightarrow Y(y) = c_1 \cos\lambda y + c_2 \sin\lambda y$$

from B.C. ,

$$T(x,0) = C_1 \cdot 1 = 0 \Rightarrow C_1 = 0$$

$$T(x,\pi) = X(x)\, C_2 \sin\lambda\pi = 0 \quad \Rightarrow \lambda = n = 1, 2, 3, \cdots$$

For eq. ③

$$X'' + \lambda^2 X = 0 \Rightarrow X(x) = C_3 \sinh\lambda x + C_4 \cosh\lambda x$$

from B.C.

$$T(\infty, y) = C_3 \sinh\lambda\infty + C_4 \cosh\lambda\infty = 0$$

$$\Rightarrow C_3 = -C_4 \frac{\cosh\lambda\infty}{\sinh\lambda\infty} = C_4 \frac{\sinh\lambda\infty \cosh\lambda x - \sinh\lambda x \cosh\lambda\infty}{\sinh\lambda\infty}$$

$$= \frac{C_4}{\sinh\lambda\infty} \sinh(\infty - \lambda) = C_4^{*} \sinh(\infty - \lambda)$$

#4(b)

```
clear;clc;close all

f = @(n) (4/pi).*(1./(2.*n-1)).*exp(-(2.*n-1).*0.88).*sin((2.*n-1).*pi/2);
sum = 0;
n = 1;
while 1
    error = f(n);
    sum = sum + error;
    if abs(error) < 1e-6
        break
    end
    n = n+1;
end
fprintf("For T0=1, then T(0.88,pi/2): \n    Iteration times: %d \n    Value: %f \n    Error: %f\n\
```

```
For T0=1, then T(0.88,pi/2):
    Iteration times: 8
    Value: 0.500619
    Error: -0.000000
```

#4(c)

```
clear;clc;close all
T0 = 1;
x = 0.88;
y = pi()/2;
T = 2*T0/pi()*atan2(sin(y),sinh(x));
fprintf("T(0.88,pi/2) = %.4f\n", T)
```

```
T(0.88,pi/2) = 0.5006
```

#5.

$$y_{A_1}(x) = x(1-x) = -x^2+x$$

$$\tilde{\lambda}_1 = \frac{\int_0^1 (1+x^2)\, y_{A_1}'^{\,2}(x)\,dx}{\int_0^1 2x\, y_{A_1}^2(x)\,dx} = \frac{\frac{7}{15}}{\frac{1}{30}} = \frac{7\times\cancel{30}^{\,2}}{\cancel{15}} = \underline{14 \;\#}$$

Aside:

$$(1+x^2)(4x^2-4x+1)$$

$$= 4x^2-4x+1+4x^4-4x^3+x^2$$

$$= 4x^4-4x^3+5x^2-4x+1$$

Num:

$$y_{A_1}' = -2x+1$$

$$\int_0^1 (1+x^2)(-2x+1)^2\,dx = \int_0^1 (1+x^2)(4x^2-4x+1)\,dx$$

$$= \int_0^1 4x^4-4x^3+5x^2-4x+1\,dx = \frac{4}{5}x^5 - x^4 + \frac{5}{3}x^3 - 2x^2 + x \Big|_0^1$$

$$= \frac{4}{5} - 1 + \frac{5}{3} - 2 + 1 = \frac{7}{15}$$

Dem:

$$\int_0^1 2x(-x^2+x)^2\,dx = \int_0^1 2x(x^4-2x^3+x^2)\,dx = \int_0^1 2x^5-4x^4+2x^3\,dx$$

$$= \frac{1}{3}x^6 - \frac{4}{5}x^5 + \frac{1}{2}x^4 \Big|_0^1 = \frac{1}{3} - \frac{4}{5} + \frac{1}{2} = \frac{1}{30}$$