1

(a) $\det\left(sI-\begin{bmatrix}1&0&-4\\0&3&0\\-2&0&-1\end{bmatrix}\right)=\det\left(\begin{bmatrix}s-1&0&4\\0&s-3&0\\2&0&s+1\end{bmatrix}\right)=s^3-3s^2-9s+27$

$\Rightarrow s=-3,3,3.$ #

For $s=-3$,

$$-3I-\begin{bmatrix}1&0&-4\\0&3&0\\-2&0&-1\end{bmatrix}=\begin{bmatrix}-4&0&4\\0&-6&0\\2&0&-2\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}-4&0&4&0\\0&-6&0&0\\2&0&-2&0\end{array}\right]\Rightarrow\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&0&0\\0&0&0&0\end{array}\right]\Rightarrow\begin{cases}x_1=x_3\\x_2=0\\x_3=x_3\end{cases}$$

eigenvector of $s=-3$ is $\begin{bmatrix}1\\0\\1\end{bmatrix}$ #

For $s=3$

$$3I-\begin{bmatrix}1&0&-4\\0&3&0\\-2&0&-1\end{bmatrix}=\begin{bmatrix}2&0&4\\0&0&0\\2&0&4\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}2&0&4&0\\0&0&0&0\\2&0&4&0\end{array}\right]\Rightarrow\left[\begin{array}{ccc|c}1&0&2&0\\0&0&0&0\\0&0&0&0\end{array}\right]\Rightarrow\begin{cases}x_1=-2x_3\\x_2=x_2\\x_3=x_3\end{cases}$$

eigenvector of $s=3$ are $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix}\right\}$ #

$$J=\begin{bmatrix}1&0&-2\\0&1&0\\1&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&-4\\0&3&0\\-2&0&-1\end{bmatrix}\begin{bmatrix}1&0&-2\\0&1&0\\1&0&1\end{bmatrix}=\begin{bmatrix}-3&0&0\\0&3&0\\0&0&3\end{bmatrix}$$ #

(b) $\det\left(sI-\begin{bmatrix}2&1&1\\0&3&1\\0&-1&1\end{bmatrix}\right)=s^3-6s^2+12s-8$

$\Rightarrow s=2,2,2.$ #

For $s=2$,

$$2I - \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = x_1 \\ x_2 = -x_3 \\ x_3 = x_3 \end{cases}$$

Select $v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = x_1 \\ x_2 = -1-x_3 \\ x_3 = x_3 \end{cases}$$

eigenvector of $s=2$ are $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ #

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \#$$

(c) $\det\left(sI - \begin{bmatrix} 7 & -11 & -30 \\ -14 & 13 & -43 \\ -8 & 10 & 29 \end{bmatrix}\right) = s^3 - 49s^2 + 707s + 1521$

$\Rightarrow s = -1.8933,\ 25.4467 \pm 12.4830i$ #

eigenvector of $s=-1.8933$ is $\begin{bmatrix} 0.7534 \\ 0.6573 \\ -0.0177 \end{bmatrix}$ #

eigenvector of $s = 25.4467 \pm 12.4830i$ is $\begin{bmatrix} 0.2928 \pm 0.3453i \\ 0.7637 \\ -0.3164 \mp 0.3344i \end{bmatrix}$ #

$$J = \begin{bmatrix} -1.8933 & 0 & 0 \\ 0 & 25.4467+12.483i & 0 \\ 0 & 0 & 25.4467-12.483i \end{bmatrix} \#$$

(d) $\det\left(sI-\begin{bmatrix}\alpha & w\\ -w & \alpha\end{bmatrix}\right)=\begin{vmatrix}s-\alpha & -w\\ w & s-\alpha\end{vmatrix}=(s-\alpha)^2+w^2=0$

$\Rightarrow s=\alpha\pm wi$ #

For $s=\alpha+wi$

$$\begin{bmatrix}\alpha+wi & 0\\ 0 & \alpha+wi\end{bmatrix}-\begin{bmatrix}\alpha & w\\ -w & \alpha\end{bmatrix}=\begin{bmatrix}wi & -w\\ w & wi\end{bmatrix}$$

$$\left[\begin{array}{cc|c}wi & -w & 0\\ w & wi & 0\end{array}\right]\Rightarrow\left[\begin{array}{cc|c}-1 & -i & 0\\ 1 & i & 0\end{array}\right]\Rightarrow\left[\begin{array}{cc|c}1 & i & 0\\ 0 & 0 & 0\end{array}\right]\Rightarrow\begin{cases}x_1=-ix_2\\ x_2=x_2\end{cases}$$

eigenvector of $s=\alpha+wi$ is $\begin{bmatrix}-i\\1\end{bmatrix}$ #

For $s=\alpha-wi$

$$\begin{bmatrix}\alpha-wi & 0\\ 0 & \alpha-wi\end{bmatrix}-\begin{bmatrix}\alpha & w\\ -w & \alpha\end{bmatrix}=\begin{bmatrix}-wi & -w\\ w & -wi\end{bmatrix}$$

$$\left[\begin{array}{cc|c}-wi & -w & 0\\ w & -wi & 0\end{array}\right]\Rightarrow\left[\begin{array}{cc|c}1 & -i & 0\\ 1 & -i & 0\end{array}\right]\Rightarrow\left[\begin{array}{cc|c}1 & -i & 0\\ 0 & 0 & 0\end{array}\right]\Rightarrow\begin{cases}x_1=ix_2\\ x_2=x_2\end{cases}$$

eigenvector of $s=\alpha-wi$ is $\begin{bmatrix}i\\1\end{bmatrix}$ #

$$J=\begin{bmatrix}-i & i\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}\alpha & w\\ -w & \alpha\end{bmatrix}\begin{bmatrix}-i & i\\ 1 & 1\end{bmatrix}=\begin{bmatrix}a+bi & 0\\ 0 & a-bi\end{bmatrix}$$ #

#2.

(a) $\int_0^{2\pi}\sin(t-x)f(x)\,dx=\int_0^{2\pi}[\sin(t)\cos(-x)+\cos(t)\sin(-x)]f(x)\,dx$

$=\sin t\int_0^{2\pi}\cos x\,f(x)\,dx-\cos t\int_0^{2\pi}\sin x\,f(x)\,dx$

$$T=\begin{bmatrix}-\int_0^{2\pi}\sin x\,f(x)dx & 0\\ 0 & \int_0^{2\pi}\cos x\,f(x)\,dx\end{bmatrix}$$ #

(b)

$$\det(sI-T) = \det\left(\begin{bmatrix} s+\int_0^{2\pi}\sin x\, f(x)dx & 0 \\ 0 & s-\int_0^{2\pi}\cos x\, f(x)dx \end{bmatrix}\right)$$

$$= \left(s+\int_0^{2\pi}\sin x\, f(x)\,dx\right)\left(s-\int_0^{2\pi}\cos x\, f(x)\,dx\right)$$

$$\Rightarrow s = -\int_0^{2\pi}\sin x\, f(x)\,dx,\ \int_0^{2\pi}\cos x\, f(x)\,dx \quad \#$$

For $s = -\int_0^{2\pi}\sin x\, f(x)\,dx$

$$(sI-T)\,v = 0$$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -\int_0^{2\pi}\sin x\, f(x)\,dx - \int_0^{2\pi}\cos x\, f(x)\,dx & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = x_1 \\ x_2 = 0 \end{cases}$$

eigenvector of $s = -\int_0^{2\pi}\sin x\, f(x)\,dx$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ #

For $s = \int_0^{2\pi}\cos x\, f(x)\,dx$

$$\left[\begin{array}{cc|c} \int_0^{2\pi}\sin x\, f(x)\,dx + \int_0^{2\pi}\cos x\, f(x)\,dx & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = x_2 \end{cases}$$

eigenvector of $s = \int_0^{2\pi}\cos x\, f(x)\,dx$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ #