#1 (a)

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, where $a_{ij} \in \mathbb{R}$

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, where $x_i, y_i \in \mathbb{R}$

1. $f(cx) = A\cdot(cx)$

$$= A\cdot\begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$$

$$= \begin{bmatrix} ca_{11}x_1 & ca_{12}x_2 & \cdots & ca_{1n}x_n \\ ca_{21}x_1 & ca_{22}x_2 & \cdots & ca_{2n}x_n \\ \vdots & \vdots & \ddots & \vdots \\ ca_{n1}x_1 & ca_{n2}x_2 & \cdots & ca_{nn}x_n \end{bmatrix}$$

$$c\cdot f(x) = c\cdot Ax = c\cdot\begin{bmatrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & \cdots & a_{2n}x_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}x_1 & a_{n2}x_2 & \cdots & a_{nn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} ca_{11}x_1 & ca_{12}x_2 & \cdots & ca_{1n}x_n \\ ca_{21}x_1 & ca_{22}x_2 & \cdots & ca_{2n}x_n \\ \vdots & \vdots & \ddots & \vdots \\ ca_{n1}x_1 & ca_{n2}x_2 & \cdots & ca_{nn}x_n \end{bmatrix}$$

$\Rightarrow f(cx) = c\cdot f(x)$

2. $f(x+y) = A(x+y) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$

$$= \begin{bmatrix} a_{11}(x_1+y_1) & a_{12}(x_2+y_2) & \cdots & a_{1n}(x_n+y_n) \\ a_{21}(x_1+y_1) & a_{22}(x_2+y_2) & \cdots & a_{2n}(x_n+y_n) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x_n+y_n) & a_{n2}(x_n+y_n) & \cdots & a_{nn}(x_n+y_n) \end{bmatrix}$$

$$f(x)+f(y) = Ax+Ay = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & \cdots & a_{2n}x_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}x_1 & a_{n2}x_2 & \cdots & a_{nn}x_n \end{bmatrix} + \begin{bmatrix} a_{11}y_1 & a_{12}y_2 & \cdots & a_{1n}y_n \\ a_{21}y_1 & a_{22}y_2 & \cdots & a_{1n}y_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}y_1 & a_{n2}y_2 & \cdots & a_{nn}y_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1+a_{11}y_1 & a_{12}x_2+a_{12}y_2 & \cdots & a_{1n}x_n+a_{1n}y_n \\ a_{21}x_1+a_{21}y_1 & a_{22}x_2+a_{22}y_2 & \cdots & a_{2n}x_n+a_{2n}y_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}x_1+a_{21}y_1 & a_{n2}x_2+a_{n2}y_2 & \cdots & a_{nn}x_n+a_{nn}y_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(x_1+y_1) & a_{12}(x_2+y_2) & \cdots & a_{1n}(x_n+y_n) \\ a_{21}(x_1+y_1) & a_{22}(x_2+y_2) & \cdots & a_{2n}(x_n+y_n) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x_1+y_1) & a_{n2}(x_2+y_2) & \cdots & a_{nn}(x_n+y_n) \end{bmatrix}$$

$\Rightarrow f(x+y) = f(x)+f(y)$

$\Rightarrow$ From 1 and 2, we know

$$f(cx) = c \cdot f(x)$$
$$f(x+y) = f(x)+f(y)$$

so that $f(x)$ is linear. #

(b) $f(x) = \sin x$

$\because f(cx) = \sin(cx)$

$c \cdot f(x) = c \cdot \sin(x) \neq f(cx)$

$\therefore f(x)$ is NOT linear. #

#2. (a) $\mathcal{F} = \{0, 1\}$, with standard operation.

A. $\begin{cases} 0+0=0 \\ 0+1=1 \\ 1+0=0 \\ 1+1=2 \end{cases}$

$\because 2 \notin \mathcal{F}$

$\therefore \mathcal{F}$ is NOT a field. #

(b) $\mathcal{F} = \left\{ x, y \in \mathbb{R} \;\middle|\; \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \right\}$

Let $\alpha = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}$, $\beta = \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix}$, $\gamma = \begin{bmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{bmatrix}$, $\alpha, \beta, \gamma \in \mathcal{F}$

A. $\alpha + \beta = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} + \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix}$

$= \begin{bmatrix} x_1+x_2 & y_1+y_2 \\ -y_1-y_2 & x_1+x_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 & y_1+y_2 \\ -(y_1+y_2) & x_1+x_2 \end{bmatrix}$

$\alpha + \beta \in \mathcal{F}$, and is unique

(a1) $\beta + \alpha = \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} + \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}$

$= \begin{bmatrix} x_2+x_1 & y_2+y_1 \\ -y_2-y_1 & x_2+x_1 \end{bmatrix} = \begin{bmatrix} x_1+x_2 & y_1+y_2 \\ -y_1-y_2 & x_1+x_2 \end{bmatrix} = \alpha + \beta$

(a2) $(\alpha+\beta)+\gamma = \left(\begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} + \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} \right) + \begin{bmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{bmatrix}$

$= \begin{bmatrix} x_1+x_2 & y_1+y_2 \\ -y_1-y_2 & x_1+x_2 \end{bmatrix} + \begin{bmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{bmatrix}$

$= \begin{bmatrix} x_1+x_2+x_3 & y_1+y_2+y_3 \\ -y_1-y_2-y_3 & x_1+x_2+x_3 \end{bmatrix}$

$$\alpha+(\beta+\gamma) = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} + \left(\begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} + \begin{bmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} + \begin{bmatrix} x_2+x_3 & y_2+y_3 \\ -y_2-y_3 & x_2+x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1+x_2+x_3 & y_1+y_2+y_3 \\ -y_1-y_2-y_3 & x_1+x_2+x_3 \end{bmatrix} = (\alpha+\beta)+\gamma$$

(a3) Let $0 = \begin{bmatrix} x_0 & y_0 \\ -y_0 & x_0 \end{bmatrix}$

$$\alpha + 0 = \alpha$$

$$\Rightarrow \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} + \begin{bmatrix} x_0 & y_0 \\ -y_0 & x_0 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_0 & y_0 \\ -y_0 & x_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in F$$

(a4) Let $-\alpha = \begin{bmatrix} x' & y' \\ -y' & x' \end{bmatrix}$

$$\alpha + (-\alpha) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} + \begin{bmatrix} x' & y' \\ -y' & x' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' & y' \\ -y' & x' \end{bmatrix} = \begin{bmatrix} -x_1 & -y_1 \\ y_1 & -x_1 \end{bmatrix} = \begin{bmatrix} -x_1 & (-y_1) \\ -(-y_1) & -x_1 \end{bmatrix} \in F$$

M. $\alpha \cdot \beta = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix}$

$$= \begin{bmatrix} x_1x_2 - y_1y_2 & x_1y_2 + x_2y_1 \\ -x_2y_1 - x_1y_2 & -y_1y_2 + x_1x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1x_2 - y_1y_2 & x_1y_2 + x_2y_1 \\ -(x_1y_2 + x_2y_1) & x_1x_2 - y_1y_2 \end{bmatrix} \in F$$

(m1) $\beta\cdot\alpha = \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}$

$$= \begin{bmatrix} x_2x_1 - y_2y_1 & x_2y_1 + y_2x_1 \\ -y_2x_1 - x_2y_1 & -y_2y_1 + x_2x_1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1x_2 - y_1y_2 & x_1y_2 + x_2y_1 \\ -(x_1y_2 + x_2y_1) & x_1x_2 - y_1y_2 \end{bmatrix} = \alpha\cdot\beta$$

(m2) $(\alpha\cdot\beta)\cdot\gamma = \left(\begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}\begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix}\right)\begin{bmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{bmatrix}$

$$= \begin{bmatrix} x_1x_2 - y_1y_2 & x_1y_2 + x_2y_1 \\ -(x_1y_2 + x_2y_1) & x_1x_2 - y_1y_2 \end{bmatrix}\begin{bmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_3(x_1x_2 - y_1y_2) - y_3(x_1y_2 + x_2y_1) & y_3(x_1x_2 - y_1y_2) + x_3(x_1y_2 + x_2y_1) \\ -x_3(x_1y_2 + x_2y_1) - y_3(x_1x_2 - y_1y_2) & -y_3(x_1y_2 + x_2y_1) + x_3(x_1x_2 - y_1y_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1x_2x_3 - x_1y_2y_3 - x_2y_1y_3 - x_3y_1y_2 & x_1x_2y_3 + x_1x_3y_2 + x_2x_3y_1 - y_1y_2y_3 \\ -x_1x_2y_3 - x_1x_3y_2 - x_2x_3y_1 + y_1y_2y_3 & x_1x_2x_3 - x_1y_2y_3 - x_2y_1y_3 - x_3y_1y_2 \end{bmatrix}$$

$\alpha\cdot(\beta\cdot\gamma) = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}\left(\begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix}\begin{bmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{bmatrix}\right)$

$$= \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}\begin{bmatrix} x_2x_3 - y_2y_3 & x_2y_3 + x_3y_2 \\ -(x_2y_3 + x_3y_2) & x_2x_3 - y_2y_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1(x_2x_3 - y_2y_3) - y_1(x_2y_3 + x_3y_2) & x_1(x_2y_3 + x_3y_2) + y_1(x_2x_3 - y_2y_3) \\ -y_1(x_2x_3 - y_2y_3) - x_1(x_2y_3 + x_3y_2) & -y_1(x_2y_3 + x_3y_2) + x_1(x_2x_3 - y_2y_3) \end{bmatrix}$$

$$= \begin{bmatrix} x_1x_2x_3 - x_1y_2y_3 - x_2y_1y_3 - x_3y_1y_2 & x_1x_2y_3 + x_1x_3y_2 + x_2x_3y_1 - y_1y_2y_3 \\ -x_1x_2y_3 - x_1x_3y_2 - x_2x_3y_1 + y_1y_2y_3 & x_1x_2x_3 - x_1y_2y_3 - x_2y_1y_3 - x_3y_1y_2 \end{bmatrix}$$

$$= (\alpha\cdot\beta)\cdot\gamma$$

(m3) Let $1 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

$$\alpha \cdot 1 = \alpha$$

$$\Rightarrow \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} ax_1 - by_1 & bx_1 + ay_1 \\ -ay_1 - bx_1 & -by_1 + ax_1 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}$$

$$\begin{cases} ax_1 - by_1 = x_1 \\ bx_1 + ay_1 = y_1 \\ -ay_1 - bx_1 = -y_1 \\ -by_1 + ax_1 = x_1 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 0 \end{cases}$$

$$\Rightarrow 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{F}$$

(m4) Let $\alpha^{-1} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

$$\alpha \cdot \alpha^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \frac{1}{x_1^2 - y_1^2} \begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix} \in \mathcal{F}$$

$$\alpha \cdot (\beta + \gamma) = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} \cdot \left(\begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} + \begin{bmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_2 + x_3 & y_2 + y_3 \\ -(y_2 + y_3) & x_2 + x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1(x_2 + x_3) - y_1(y_2 + y_3) & x_1(y_2 + y_3) + y_1(x_2 + x_3) \\ -y_1(x_2 + x_3) - x_1(y_2 + y_3) & -y_1(y_2 + y_3) + x_1(x_2 + x_3) \end{bmatrix}$$

$$= \begin{bmatrix} x_1x_2 + x_1x_3 - y_1y_2 - y_1y_3 & x_1y_2 + x_1y_3 + x_2y_1 + x_3y_1 \\ -x_1y_2 - x_1y_3 - x_2y_1 - x_3y_1 & x_1x_2 + x_1x_3 - y_1y_2 - y_1y_3 \end{bmatrix}$$

$$\alpha\beta+\alpha\gamma=\begin{bmatrix}x_1 & y_1\\ -y_1 & x_1\end{bmatrix}\begin{bmatrix}x_2 & y_2\\ -y_2 & x_2\end{bmatrix}+\begin{bmatrix}x_1 & y_1\\ -y_1 & x_1\end{bmatrix}\begin{bmatrix}x_3 & y_3\\ -y_3 & x_3\end{bmatrix}$$

$$=\begin{bmatrix}x_1x_2-y_1y_2 & x_1y_2+x_2y_1\\ -x_2y_1-x_1y_2 & x_1x_2-y_1y_2\end{bmatrix}+\begin{bmatrix}x_1x_3-y_1y_3 & x_1y_3+x_3y_1\\ -x_3y_1-x_1y_3 & x_1x_3-y_1y_3\end{bmatrix}$$

$$=\begin{bmatrix}x_1x_2+x_1x_3-y_1y_2-y_1y_3 & x_1y_2+x_1y_3+x_2y_1+x_3y_1\\ -x_1y_2-x_1y_3-x_2y_1-x_3y_1 & x_1x_2+x_1x_3-y_1y_2-y_1y_3\end{bmatrix}$$

$$=\alpha\cdot(\beta+\gamma)$$

$F=\left\{x,y\in\mathbb{R}\;\middle|\;\begin{bmatrix}x & y\\ -y & x\end{bmatrix}\right\}$ is a field #

#3 (a) $V=\left\{x=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}\;\middle|\; a_{ij}\in\mathbb{R},\ \det(x)\neq 0\right\}$ over $\mathbb{R}$

Let $x=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$, $y=\begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix}$,

$\because$ $\det(x)=1\times 1-(-1)\times 1=1-(-1)=2$

$\det(y)=(-1)\times(-1)-1\times(-1)=1-(-1)=2$

$\therefore$ $x,y\in V$

$$x+y=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}+\begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$\because$ $\det(x+y)=0$

$\therefore$ $x+y\notin V$

$\therefore$ V is NOT a vector space #

(b) $V = \left\{ X = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \middle| \; a_{ij} \in \mathbb{R}, \; a_{ij} = a_{ji} \right\}$ over $\mathbb{R}$

Let $X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix}$, $Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{12} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1n} & y_{2n} & \cdots & y_{nn} \end{bmatrix}$, $Z = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{12} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{1n} & z_{2n} & \cdots & z_{nn} \end{bmatrix}$

for $X, Y, Z \in V$ and $\alpha, \beta \in \mathbb{R}$

A. $X + Y = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix} + \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{12} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1n} & y_{2n} & \cdots & y_{nn} \end{bmatrix}$

$$= \begin{bmatrix} x_{11}+y_{11} & x_{12}+y_{12} & \cdots & x_{1n}+y_{1n} \\ x_{12}+y_{12} & x_{22}+y_{22} & \cdots & x_{2n}+y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n}+y_{1n} & x_{2n}+y_{2n} & \cdots & x_{nn}+y_{nn} \end{bmatrix} \in V$$

(a1) $Y + X = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{12} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1n} & y_{2n} & \cdots & y_{nn} \end{bmatrix} + \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix}$

$$= \begin{bmatrix} y_{11}+x_{11} & y_{12}+x_{12} & \cdots & y_{1n}+x_{1n} \\ y_{12}+x_{12} & y_{22}+x_{22} & \cdots & y_{2n}+x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1n}+x_{1n} & y_{2n}+x_{2n} & \cdots & y_{nn}+x_{nn} \end{bmatrix} = X + Y$$

(a2) $(X+Y) + Z = \left(\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix} + \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{12} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1n} & y_{2n} & \cdots & y_{nn} \end{bmatrix} \right) + \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{12} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{1n} & z_{2n} & \cdots & z_{nn} \end{bmatrix}$

$$= \begin{bmatrix} x_{11}+y_{11} & x_{12}+y_{12} & \cdots & x_{1n}+y_{1n} \\ x_{12}+y_{12} & x_{22}+y_{22} & \cdots & x_{2n}+y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n}+y_{1n} & x_{2n}+y_{2n} & \cdots & x_{nn}+y_{nn} \end{bmatrix} + \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{12} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{1n} & z_{2n} & \cdots & z_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11}+y_{11}+z_{11} & x_{12}+y_{12}+z_{12} & \cdots & x_{1n}+y_{1n}+z_{1n} \\ x_{12}+y_{12}+z_{12} & x_{22}+y_{22}+z_{22} & \cdots & x_{2n}+y_{2n}+z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n}+y_{1n}+z_{1n} & x_{2n}+y_{2n}+z_{2n} & \cdots & x_{nn}+y_{nn}+z_{nn} \end{bmatrix}$$

$$x+(y+z) = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix} + \left(\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{12} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1n} & y_{2n} & \cdots & y_{nn} \end{bmatrix} + \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{12} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{1n} & z_{2n} & \cdots & z_{nn} \end{bmatrix} \right)$$

$$= \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix} + \begin{bmatrix} y_{11}+z_{11} & y_{12}+z_{12} & \cdots & y_{1n}+z_{1n} \\ y_{12}+z_{12} & y_{22}+z_{22} & \cdots & y_{2n}+z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1n}+z_{1n} & y_{2n}+z_{2n} & \cdots & y_{nn}+z_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11}+y_{11}+z_{11} & x_{12}+y_{12}+z_{12} & \cdots & x_{1n}+y_{1n}+z_{1n} \\ x_{12}+y_{12}+z_{12} & x_{22}+y_{22}+z_{22} & \cdots & x_{2n}+y_{2n}+z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n}+y_{1n}+z_{1n} & x_{2n}+y_{2n}+z_{2n} & \cdots & x_{nn}+y_{nn}+z_{nn} \end{bmatrix} = (x+y)+z$$

(a3) Let $\theta = \begin{bmatrix} \theta_{11} & \theta_{12} & \cdots & \theta_{1n} \\ \theta_{21} & \theta_{22} & \cdots & \theta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{n1} & \theta_{n2} & \cdots & \theta_{nn} \end{bmatrix}$

$$x + \theta = x$$

$$\Rightarrow \theta = x - x = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix} - \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in V$$

(a4) Let $(-x) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

$$x + (-x) = \theta$$

$$\Rightarrow (-x) = \theta - x$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} -x_{11} & -x_{12} & \cdots & -x_{1n} \\ -x_{12} & -x_{22} & \cdots & -x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -x_{1n} & -x_{2n} & \cdots & -x_{nn} \end{bmatrix} \in V$$

M. $\alpha \cdot x = \alpha \cdot \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix} = \begin{bmatrix} \alpha x_{11} & \alpha x_{12} & \cdots & \alpha x_{1n} \\ \alpha x_{12} & \alpha x_{22} & \cdots & \alpha x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha x_{1n} & \alpha x_{2n} & \cdots & \alpha x_{nn} \end{bmatrix} \in V$

(m1) $(\alpha \cdot \beta) \cdot x = (\alpha\beta) \cdot \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix}$

$$= \begin{bmatrix} \alpha\beta x_{11} & \alpha\beta x_{12} & \cdots & \alpha\beta x_{1n} \\ \alpha\beta x_{12} & \alpha\beta x_{22} & \cdots & \alpha\beta x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha\beta x_{1n} & \alpha\beta x_{2n} & \cdots & \alpha\beta x_{nn} \end{bmatrix}$$

$\alpha \cdot (\beta \cdot x) = \alpha \cdot \left(\beta \cdot \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix} \right)$

$$= \alpha \cdot \begin{bmatrix} \beta x_{11} & \beta x_{12} & \cdots & \beta x_{1n} \\ \beta x_{12} & \beta x_{22} & \cdots & \beta x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta x_{1n} & \beta x_{2n} & \cdots & \beta x_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha\beta x_{11} & \alpha\beta x_{12} & \cdots & \alpha\beta x_{1n} \\ \alpha\beta x_{12} & \alpha\beta x_{22} & \cdots & \alpha\beta x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha\beta x_{1n} & \alpha\beta x_{2n} & \cdots & \alpha\beta x_{nn} \end{bmatrix} = (\alpha \cdot \beta) \cdot x$$

(m2) $\alpha \cdot (x+y) =$

$$\alpha \cdot \left(\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix} + \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{12} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1n} & y_{2n} & \cdots & y_{nn} \end{bmatrix} \right)$$

$$= \alpha \cdot \begin{bmatrix} x_{11}+y_{11} & x_{12}+y_{12} & \cdots & x_{1n}+y_{1n} \\ x_{12}+y_{12} & x_{22}+y_{22} & \cdots & x_{2n}+y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n}+y_{1n} & x_{2n}+y_{2n} & \cdots & x_{nn}+y_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(x_{11}+y_{11}) & \alpha(x_{12}+y_{12}) & \cdots & \alpha(x_{1n}+y_{1n}) \\ \alpha(x_{12}+y_{12}) & \alpha(x_{22}+y_{22}) & \cdots & \alpha(x_{2n}+y_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(x_{1n}+y_{1n}) & \alpha(x_{2n}+y_{2n}) & \cdots & \alpha(x_{nn}+y_{nn}) \end{bmatrix}$$

$\alpha \cdot x + \alpha \cdot y =$

$$\alpha \cdot \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix} + \alpha \cdot \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{12} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1n} & y_{2n} & \cdots & y_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_{11} & \alpha x_{12} & \cdots & \alpha x_{1n} \\ \alpha x_{12} & \alpha x_{22} & \cdots & \alpha x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha x_{1n} & \alpha x_{2n} & \cdots & \alpha x_{nn} \end{bmatrix} + \begin{bmatrix} \alpha y_{11} & \alpha y_{12} & \cdots & \alpha y_{1n} \\ \alpha y_{12} & \alpha y_{22} & \cdots & \alpha y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha y_{1n} & \alpha y_{2n} & \cdots & \alpha y_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_{11}+\alpha y_{11} & \alpha x_{12}+\alpha y_{12} & \cdots & \alpha x_{1n}+\alpha y_{1n} \\ \alpha x_{12}+\alpha y_{12} & \alpha x_{22}+\alpha y_{22} & \cdots & \alpha x_{2n}+\alpha y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha x_{1n}+\alpha y_{1n} & \alpha x_{2n}+\alpha y_{2n} & \cdots & \alpha x_{nn}+\alpha y_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(x_{11}+y_{11}) & \alpha(x_{12}+y_{12}) & \cdots & \alpha(x_{1n}+y_{1n}) \\ \alpha(x_{12}+y_{12}) & \alpha(x_{22}+y_{22}) & \cdots & \alpha(x_{2n}+y_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(x_{1n}+y_{1n}) & \alpha(x_{2n}+y_{2n}) & \cdots & \alpha(x_{nn}+y_{nn}) \end{bmatrix} = \alpha(x+y)$$

(m3) $(\alpha+\beta) \cdot x =$

$$(\alpha+\beta) \cdot \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha+\beta)x_{11} & (\alpha+\beta)x_{12} & \cdots & (\alpha+\beta)x_{1n} \\ (\alpha+\beta)x_{12} & (\alpha+\beta)x_{22} & \cdots & (\alpha+\beta)x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha+\beta)x_{1n} & (\alpha+\beta)x_{2n} & \cdots & (\alpha+\beta)x_{nn} \end{bmatrix}$$

$$\alpha\cdot X+\beta\cdot X = \alpha\cdot\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn}\end{bmatrix} + \beta\cdot\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn}\end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_{11} & \alpha x_{12} & \cdots & \alpha x_{1n} \\ \alpha x_{12} & \alpha x_{22} & \cdots & \alpha x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha x_{1n} & \alpha x_{2n} & \cdots & \alpha x_{nn}\end{bmatrix} + \begin{bmatrix} \beta x_{11} & \beta x_{12} & \cdots & \beta x_{1n} \\ \beta x_{12} & \beta x_{22} & \cdots & \beta x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta x_{1n} & \beta x_{2n} & \cdots & \beta x_{nn}\end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_{11}+\beta x_{11} & \alpha x_{12}+\beta x_{12} & \cdots & \alpha x_{1n}+\beta x_{1n} \\ \alpha x_{12}+\beta x_{12} & \alpha x_{22}+\beta x_{22} & \cdots & \alpha x_{2n}+\beta x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha x_{1n}+\beta x_{1n} & \alpha x_{2n}+\beta x_{2n} & \cdots & \alpha x_{nn}+\beta x_{nn}\end{bmatrix}$$

$$= \begin{bmatrix} (\alpha+\beta) x_{11} & (\alpha+\beta) x_{12} & \cdots & (\alpha+\beta) x_{1n} \\ (\alpha+\beta) x_{12} & (\alpha+\beta) x_{22} & \cdots & (\alpha+\beta) x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha+\beta) x_{1n} & (\alpha+\beta) x_{2n} & \cdots & (\alpha+\beta) x_{nn}\end{bmatrix} = (\alpha+\beta)\cdot X$$

(m4) Let $1 = a$, $a \in \mathbb{R}$

$$1\cdot X = X$$

$$\Rightarrow a\cdot\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn}\end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn}\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a x_{11} & a x_{12} & \cdots & a x_{1n} \\ a x_{12} & a x_{22} & \cdots & a x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a x_{1n} & a x_{2n} & \cdots & a x_{nn}\end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn}\end{bmatrix}$$

$$\Rightarrow a = 1 \in \mathbb{R}$$

The symmetric matrices are vector space. #

(c) Let $x, y, z \in V$, $\alpha, \beta \in \mathbb{R}$

A. $x + y = xy$

$\because$ both x, y are bigger than 0.

$\therefore xy > 0 \Rightarrow xy \in V$

(a1) $x+y=xy$

$y+x=yx=xy=x+y$

(a2) $(x+y)+z=(xy)+z$

$=xyz$

$x+(y+z)=x+(yz)$

$=xyz=(x+y)+z$

(a3) Let $\theta=a$

$x+\theta=x$

$\Rightarrow xa=x$

$\Rightarrow a=1\in V$

(a4) Let $(-x)=a$

$x+(-x)=1$

$\Rightarrow xa=1$

$\Rightarrow a=\frac{1}{x}\in V$

M. $\alpha\cdot x=x^{\alpha}\in V$

(m1) $(\alpha\cdot\beta)\cdot x=\alpha\beta\cdot x$

$=x^{\alpha\beta}$

$\alpha\cdot(\beta\cdot x)=\alpha\cdot(x^{\beta})$

$=(x^{\beta})^{\alpha}$

$=x^{\alpha\beta}=(\alpha\cdot\beta)\cdot x$

(m2) $\alpha\cdot(x+y)=\alpha\cdot(xy)$

$=(xy)^{\alpha}$

$\alpha\cdot x+\alpha\cdot y=x^{\alpha}+y^{\alpha}=x^{\alpha}y^{\alpha}=(xy)^{\alpha}$

$=\alpha\cdot(x+y)$

(m3) $(\alpha+\beta)\cdot x = x^{\alpha+\beta}$

$\alpha\cdot x+\beta x = x^{\alpha}+x^{\beta} = x^{\alpha+\beta} = (\alpha+\beta)\cdot x$

(m4) Let $1 = a$

$1\cdot x = ax = x$

$\Rightarrow a=1 \in \mathbb{R}$

It is a vector space. #