I did not offer assistances to nor receive assistances from others in this exam.

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#1. (a)

$$sp\left(\left\{\begin{bmatrix}1\\-1\\3\end{bmatrix},\begin{bmatrix}2\\2\\2\end{bmatrix}\right\}\right) = \alpha_1\begin{bmatrix}1\\-1\\3\end{bmatrix}+\alpha_2\begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}\alpha_1+2\alpha_2\\-\alpha_1+2\alpha_2\\3\alpha_1+2\alpha_2\end{bmatrix}$$

If $u_3 \in sp\left(\left\{\begin{bmatrix}1\\-1\\3\end{bmatrix},\begin{bmatrix}2\\2\\2\end{bmatrix}\right\}\right)$ then the u_1, u_2, u_3 can only span $\mathbb{R}^2$

$$\begin{bmatrix}\alpha_1+2\alpha_2\\-\alpha_1+2\alpha_2\\3\alpha_1+2\alpha_2\end{bmatrix} = \begin{bmatrix}z_1\\-4\\z_2\end{bmatrix} \Rightarrow -\alpha_1+2\alpha_2=-4 \Rightarrow \alpha_1=2\alpha_2+4$$

$$\Rightarrow \begin{bmatrix}z_1\\-4\\z_2\end{bmatrix} = \begin{bmatrix}(2\alpha_2+4)+2\alpha_2\\-(2\alpha_2+4)+2\alpha_2\\3(\alpha_2+4)+2\alpha_2\end{bmatrix} = \begin{bmatrix}4\alpha_2+4\\-4\\5\alpha_2+12\end{bmatrix}$$

If $\begin{bmatrix}z_1\\z_2\end{bmatrix} = \begin{bmatrix}4\alpha_2+4\\5\alpha_2+12\end{bmatrix}$, $\alpha_2 \in \mathbb{R}$, then u_1, u_2, u_3 can't span $\mathbb{R}^3$ #

(b)

$$sp\left(\left\{\begin{bmatrix}1\\-1\\3\end{bmatrix},\begin{bmatrix}2\\2\\2\end{bmatrix},\begin{bmatrix}4\alpha_2+4\\-4\\5\alpha_2+12\end{bmatrix}\right\}\right) = \alpha_1\begin{bmatrix}1\\-1\\3\end{bmatrix}+\alpha_2\begin{bmatrix}2\\2\\2\end{bmatrix}+\alpha_3\begin{bmatrix}4\alpha_2+4\\-4\\5\alpha_2+12\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\alpha_1+2\alpha_2+4\alpha_2\alpha_3+4\alpha_3\\-\alpha_1+2\alpha_2 \quad -4\alpha_3\\3\alpha_1+2\alpha_2+5\alpha_2\alpha_3 \quad +12\alpha_3\end{bmatrix} = \alpha_1\begin{bmatrix}1\\-1\\3\end{bmatrix}+\alpha_2\begin{bmatrix}2\\2\\2\end{bmatrix}+\alpha_3\begin{bmatrix}4\\-4\\12\end{bmatrix}+\alpha_2\alpha_3\begin{bmatrix}4\\0\\5\end{bmatrix}$$

$$= (\alpha_1+4\alpha_3)\begin{bmatrix}1\\-1\\3\end{bmatrix}+\alpha_2\begin{bmatrix}2\\2\\2\end{bmatrix}+\alpha_2\alpha_3\begin{bmatrix}4\\0\\5\end{bmatrix}$$

$$\left\{\begin{bmatrix}1\\-1\\3\end{bmatrix},\begin{bmatrix}2\\2\\2\end{bmatrix},\begin{bmatrix}4\\0\\5\end{bmatrix}\right\}$$ #

#2 Define $V=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$, $W=\{x^2, x, 1\}$

$T: V \to W$

The matrix representation of T in basis V to W is

$$\begin{bmatrix}a\\b\\c\end{bmatrix}_W = A\begin{bmatrix}a\\b\\c\end{bmatrix}_V \Rightarrow A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

The null space of A is

$$A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow \ker(T)=\{0\}$

$\Rightarrow T$ is one-to-one #

#3. (a) $B_1=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$, $B_2=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}4\\-4\\1\end{bmatrix}\right\}$

$$\begin{bmatrix}1\\0\\0\end{bmatrix}=1\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}+0\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}+0\cdot\begin{bmatrix}4\\-4\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix}=-1\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}+1\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}+0\cdot\begin{bmatrix}4\\-4\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix}=-8\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}+4\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}+1\cdot\begin{bmatrix}4\\-4\\1\end{bmatrix}$$

$$T=\begin{bmatrix}1&-1&-8\\0&1&4\\0&0&1\end{bmatrix}$$ #

(b)

Let $P(x) = ax^2+bx+c$, then

$$
\begin{aligned}
P(2x+1) &= a(2x+1)^2+b(2x+1)+c \\
&= a(4x^2+4x+1)+b(2x+1)+c \\
&= 4ax^2+(4a+2b)x+(a+b+c) \\
&= \begin{bmatrix} a+b+c \\ 4a+2b \\ 4a \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} c \\ b \\ a \end{bmatrix}
\end{aligned}
$$

$$A_{B_1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$
\begin{tikzcd}
B_1 \arrow[r, "A_{B_1}"] \arrow[d, "T"'] & B_1 \arrow[d, "T"] \\
B_2 \arrow[r, "A_{B_2}"] & B_2
\end{tikzcd}
$$

$$
\begin{aligned}
A_{B_2} &= T A_{B_1} T^{-1} \\
&= \begin{bmatrix} 1 & 0 & -27 \\ 0 & 2 & 12 \\ 0 & 0 & 4 \end{bmatrix} \quad \#
\end{aligned}
$$

#4. Def $\|x\| = \langle x, x \rangle$

$$
\begin{aligned}
\|u-v\| &= \langle u-v, u-v \rangle \\
&= \langle u-v, u \rangle + \langle u-v, -v \rangle \\
&= \langle u, u \rangle + \langle -v, u \rangle + \langle u, -v \rangle + \langle -v, -v \rangle \\
&= \langle T(u), T(u) \rangle + \langle T(-v), T(u) \rangle + \langle T(u), T(-v) \rangle + \langle T(-v), T(-v) \rangle \\
&= \langle T(u)+T(-v), T(u) \rangle + \langle T(u)+T(-v), T(-v) \rangle \\
&= \langle T(u)+T(-v), T(u)+T(-v) \rangle
\end{aligned}
$$

$\because$ T is a linear transformation

$$= \langle T(u)-T(v), T(u)-T(v) \rangle = \|T(u)-T(v)\| \quad \#$$

$$\text{Angle between } u \,\&\, v = \frac{\langle u, v\rangle}{\|u\|\|v\|}$$

$$= \frac{\langle u, v\rangle}{\langle u, u\rangle \cdot \langle v, v\rangle}$$

$$= \frac{\langle T(u), T(v)\rangle}{\langle T(u), T(u)\rangle \langle T(v), T(v)\rangle}$$

$$= \frac{\langle T(u), T(v)\rangle}{\|T(u)\| \cdot \|T(v)\|} = \text{Angle between } T(u) \,\&\, T(v)$$

#

#5

Define $A(v), B(v), C(v) \in \mathcal{L}(V, U)$, $\alpha, \beta, \gamma \in F$

(A1) $(A+B)(v) = A(v) + B(v)$

$= B(v) + A(v) = (B+A)(v)$

(A2) $[(A+B)+C](v) = (A+B)(v) + C(v)$

$= A(v) + B(v) + C(v)$

$= A(v) + (B+C)(v) = [A + (B+C)](v)$

(A3) Def. $\Theta \in \mathcal{L}(V, U)$

$A(v) + \Theta(v) = A(v)$

$\Rightarrow \Theta(v) = 0$

(A4) Def. $(-x)(v) \in \mathcal{L}(V, U)$

$A(v) + (-x)(v) = \Theta(v) = 0$

$\Rightarrow (-x)(v) = -A(v)$

(SM1) $(\alpha\beta)A(v) = \alpha\beta A(v)$

$$= \alpha(\beta A(v)) = \alpha(\beta A(v))$$

(SM2) $\alpha(A+B)(v) = \alpha(A(v)+B(v))$

$$= \alpha A(v) + \alpha B(v)$$

(SM3) $(\alpha+\beta)A(v) = \alpha A(v) + \beta A(v)$

(SM4) Def $1 \in F$

$$(1)A(v) = A(v)$$

$$\Rightarrow 1 = 1$$

$\Rightarrow L(V, U)$ over F is a vector space #