Linear Systems

HW6

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#(a)

$$Ax_1 = y_1 \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$

$$Ax_2 = y_2 \Rightarrow \begin{bmatrix} a_{11} & a_{12} & -2 \\ a_{21} & a_{22} & 4 \\ a_{31} & a_{32} & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -a_{12}-4 \\ -a_{22}+8 \\ -a_{32}+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \\ 2 \end{bmatrix}$$

$$Ax_3 = y_3 \Rightarrow \begin{bmatrix} a_{11} & -5 & -2 \\ a_{21} & 7 & 4 \\ a_{31} & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11}+5-4 \\ a_{21}-7+8 \\ a_{31}-2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -1 & -5 & -2 \\ 2 & 7 & 4 \\ -1 & 2 & 1 \end{bmatrix} \#$$

#(b)

$$\{x_1, x_2, x_3\} = \left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right\}, \{e_1, e_2, e_3\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$\begin{cases} e_1 = (0)\cdot x_3 + (-1)\cdot x_2 + (1)\cdot x_3 \\ e_2 = (2)\cdot x_1 + (-1)\cdot x_2 + (0)\cdot x_3 \\ e_3 = (1)\cdot x_1 + (0)\cdot x_2 + (0)\cdot x_3 \end{cases} \Rightarrow P = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \#$$

#(c)

$$\tilde{A} x_1 = y_1 \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \\ -2 \end{bmatrix}$$

$$\tilde{A} x_2 = y_2 \Rightarrow \begin{bmatrix} 9 & a_{12} & a_{13} \\ -2 & a_{22} & a_{23} \\ -2 & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\tilde{A} x_3 = y_3 \Rightarrow \begin{bmatrix} 9 & 2 & a_{13} \\ -2 & -2 & a_{23} \\ -2 & 1 & a_{33} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} 9 & 2 & 5 \\ -2 & -2 & -3 \\ -2 & 1 & 0 \end{bmatrix} \#$$

#(d)

$$\begin{tikzcd} \{e\} \arrow[r, "A"] & \{e\} \arrow[d, "P"] \\ \{x\} \arrow[u, "P^{-1}"] & \{x\} \end{tikzcd}$$

$$\tilde{A} = PAP^{-1} = \begin{bmatrix} 9 & 2 & 5 \\ -2 & -2 & -3 \\ -2 & 1 & 0 \end{bmatrix} \#$$

#(e)

$$\begin{tikzcd} \{e\} \arrow[r, "A"] & \{e\} \arrow[d, "Q^{-1}"] \\ \{x\} \arrow[u, "P^{-1}"] & \{y\} \end{tikzcd}$$

$$\begin{aligned} y_1 &= (-2)e_1 + (4)e_2 + (1)e_3 \\ y_2 &= (1)e_1 + (1)e_2 + (0)e_3 \\ y_3 &= (0)e_1 + (3)e_2 + (-1)e_3 \end{aligned} \Rightarrow Q = \begin{bmatrix} -2 & 1 & 0 \\ 4 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\bar{A} = Q^{-1}AP^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \#$$

#(f)

$$\bar{A}x_1 = y_1 \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{A}x_2 = y_2 \Rightarrow \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\bar{A}x_3 = y_3 \Rightarrow \begin{bmatrix} 1 & 0 & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \#$$

#(9)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}, \quad y = \begin{bmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{bmatrix}_{n\times 1}$$

$$Ay = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{bmatrix} = \begin{bmatrix} \frac{1}{n}(a_{11}+a_{12}+\cdots+a_{1n}) \\ \frac{1}{n}(a_{21}+a_{22}+\cdots+a_{2n}) \\ \vdots \\ \frac{1}{n}(a_{m1}+a_{m2}+\cdots+a_{mn}) \end{bmatrix}$$

$$\begin{aligned} \|Ay\|_1 &= \frac{1}{n}\left(|a_{11}+a_{12}+\cdots+a_{1n}| + |a_{21}+a_{22}+\cdots+a_{2n}| + \cdots + |a_{m1}+a_{m2}+\cdots+a_{mn}| \right) \\ &= \frac{1}{n}\left(\sum_{j=1}^{m} \|\{a_j\}\|_1 \right) \end{aligned}$$

$$\begin{aligned} \|A\|_1 &= \sup \|Ay\|_1 \\ &= \sup\left(\frac{1}{n} \sum_{j=1}^{m} \right. \end{aligned}$$

$$A = [\, a_1 \;\; a_2 \;\; \cdots \;\; a_n \,] \text{ for } a_i \in \mathbb{R}^{m\times 1}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ for } y_i \in \mathbb{R}, \quad \|y\|_1 = 1$$

$$Ay = a_1 y_1 + a_2 y_2 + \cdots + a_n y_n$$

$$\begin{aligned} \|A\|_1 &= \|Ay\|_1 \\ &= \|a_1 y_1 + a_2 y_2 + \cdots + a_n y_n\|_1 \\ &\le \|a_1 y_1\|_1 + \|a_2 y_2\|_1 + \cdots + \|a_n y_n\|_1 \\ &= |y_1| \cdot \|a_1\|_1 + |y_2| \cdot \|a_2\|_1 + \cdots + |y_n| \cdot \|a_n\|_1 \\ &\le (|y_1| + |y_2| + \cdots + |y_n|) \left(\max_{i=1,2,\cdots,n} \|a_i\|_1 \right) \\ &= \cancelto{1}{\|y\|_1} \max_{i=1,2,\cdots,n} \|a_i\|_1 \\ &= \max_{i=1,2,\cdots,n} \|a_i\|_1 \end{aligned}$$

$$\|A\| \le \max_{i=1,2,\cdots,n} \|a_i\|_1 \quad \#$$

#(h) $\left\| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\|_1 = 1$ #