#1.

$$\frac{df}{dx} = 3x^2+8x-3=0$$

$$\Rightarrow x = \frac{1}{3}, -3$$

$$\frac{d^2f}{dx^2} = 6x+8$$

(a) For $x=\frac{1}{3} \in \mathbb{R}$

$$\left.\frac{d^2f}{dx^2}\right|_{x=\frac{1}{3}} = 10$$

$$\because \left.\frac{d^2f}{dx^2}\right|_{x=\frac{1}{3}} > 0$$

$\therefore$ $x=\frac{1}{3}$ is a local minimum, and $f(\frac{1}{3}) = \frac{121}{27}$ #

For $x=-3 \in \mathbb{R}$

$$\left.\frac{d^2f}{dx^2}\right|_{x=-3} = -10$$

$$\because \left.\frac{d^2f}{dx^2}\right|_{x=-3} < 0$$

$\therefore$ $x=-3$ is a local maximum, and $f(-3) = 23$ #

(b) Critical pts: $x = 0, 2, \frac{1}{3}, -3$

$f(0) = 5$

$f(2) = 23$

$f(\frac{1}{3}) = \frac{121}{27}$

$f(-3) = 23$

Local minimum is at $x=\frac{1}{3}$, the value is $\frac{121}{27}$. #

Local maximum is at $x=2,-3$, the value is 23. #

(C) $$\frac{f(1+\varepsilon)-f(1)}{\varepsilon} = \frac{[(1+\varepsilon)^3+4(1+\varepsilon)^2-3(1+\varepsilon)+5]-[1+4-3+5]}{\varepsilon}$$

$$= \frac{[\varepsilon^3+7\varepsilon^2+8\varepsilon+7]-7}{\varepsilon} = \varepsilon^2+7\varepsilon+8$$

$$\lim_{\varepsilon\to 0^+} \varepsilon^2+7\varepsilon+8 = 8$$

$$\lim_{\varepsilon\to 0^-} \varepsilon^2+7\varepsilon+8 = 8$$

$$\because \lim_{\varepsilon\to 0^+} \frac{f(1+\varepsilon)-f(1)}{\varepsilon} = \lim_{\varepsilon\to 0^-} \frac{f(1+\varepsilon)-f(1)}{\varepsilon}$$

$\therefore$ The function at $x=1$ is differentiable, the value is 8. #

#2 (a) $$x^T A x = x_1(2x_1+x_2)+x_2(x_1+2x_2)$$

$$= 2x_1^2+x_1x_2+x_1x_2+2x_2^2$$

$$= 2x_1^2+2x_1x_2+2x_2^2$$

$$= 2(x_1+x_2)^2-2x_1x_2 \geq 0$$

A is a positive semi-definite matrix #

(b) $$L(x) = [x_1 \quad x_2]\begin{bmatrix} 2x_1+x_2 \\ x_1+2x_2 \end{bmatrix} = 2x_1^2+2x_1x_2+2x_2^2$$

$$\nabla L(x) = \left[\frac{\partial L}{\partial x_1} \quad \frac{\partial L}{\partial x_2}\right] = [4x_1+2x_2 \quad 2x_1+4x_2] = 0$$

$$\Rightarrow \begin{cases} 4x_1+2x_2=0 \\ 2x_1+4x_2=0 \end{cases} \Rightarrow \begin{cases} x_1=0 \\ x_2=0 \end{cases}$$

$$H = \frac{\partial^2 H}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$\because$ H is a positive semi-definite matrix.

$\therefore$ $x=(0,0)$ is a minimum. #