#4 (a)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Let $T(x,y) = X(x)\,Y(y)$

$$\Rightarrow X''Y + XY'' = 0$$

$$\Rightarrow \frac{Y''}{Y} = -\frac{X''}{X} = \mu = -\lambda^2$$

$$\begin{cases} Y'' + \lambda^2 Y = 0 & -① \\ X'' + \lambda^2 X = 0 & -② \end{cases}$$

For eq. ①

$$Y'' + \lambda^2 Y = 0 \Rightarrow Y(y) = c_1 \cos\lambda y + c_2 \sin\lambda y$$

from B.C. ,

$$T(x,0) = C_1 \cdot 1 = 0 \Rightarrow C_1 = 0$$

$$T(x,\pi) = X(x)\,C_2 \sin\lambda\pi = 0 \quad \Rightarrow \lambda = n = 1, 2, 3, \cdots$$

For eq. ③

$$X'' + \lambda^2 X = 0 \Rightarrow X(x) = C_3 \sinh\lambda x + C_4 \cosh\lambda x$$

from B.C.

$$T(\infty, y) = C_3 \sinh\lambda\infty + C_4 \cosh\lambda\infty = 0$$

$$\Rightarrow C_3 = -C_4 \frac{\cosh\lambda\infty}{\sinh\lambda\infty} = C_4 \frac{\sinh\lambda\infty \cosh\lambda x - \sinh\lambda x \cosh\lambda\infty}{\sinh\lambda\infty}$$

$$= \frac{C_4}{\sinh\lambda\infty} \sinh(\infty - \lambda) = C_4^* \sinh(\infty - \lambda)$$

$$T(x,y) = C_2 C_4^* \sinh(\infty - x) \sin ny = A_n \sinh \lambda_n (\infty - x) \sin ny \ , \ n = 1,2,3,\cdots$$

$$= \sum_{n=1}^{\infty} A_n \sin n(\infty - x) \sin ny$$

from B.C.

$$T(0,y) = \sum_{n=1}^{\infty} \underbrace{(A_n \sinh n\infty)}_{\triangleq a_n} \sin ny = \sum_{n=1}^{\infty} a_n \sin ny = f(y)$$

$$\Rightarrow a_n = \frac{\int_0^{\pi} f(y) \sin ny \, dy}{\int_0^{\pi} \sin^2 ny \, dy} = \frac{2}{\pi} \int_0^{\pi} f(y) \sin ny \, dy = A_n \sinh n\infty$$

$$\Rightarrow A_n = \frac{2}{\pi \sinh n\infty} \int_0^{\pi} f(y_0) \sin ny_0 \, dy_0$$

$$\Rightarrow T(x,y) = \sum_{n=1}^{\infty} \frac{2}{\pi} \int_0^{\pi} f(y_0) \sin ny_0 \, dy_0 \ \frac{\sinh n(\infty - x)}{\sinh n\infty} \sin ny$$

$$= \frac{2}{\pi} \int_0^{\pi} f(y_0) \sum_{n=1}^{\infty} \frac{\sinh n(\infty - x)}{\sinh n\infty} \sin ny \sin ny_0 \, dy_0$$

$$= \int_0^{\pi} f(y_0) K(x,y,y_0) \, dy_0$$

$$K(x,y,y_0) = \sum_{n=1}^{\infty} \frac{\sinh n(\infty - x)}{\sinh n\infty} \sin ny \sin ny_0 \, dy_0$$

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#4 (c) $$T(x,y) = \frac{4T_0}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n} e^{-nx} \sin ny$$

from #2(b), $$\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n} e^{-nx} \sin ny = \frac{1}{2} \tan^{-1}\left(\frac{\sin y}{\sinh x}\right)$$

$$= \frac{2T_0}{\pi} \tan^{-1}\left(\frac{\sin y}{\sinh x}\right)$$

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