I did not offer assistances to nor receive assistances from others in this exam.

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1

$$\min -\int_0^1 y\,dt$$

s.t.

$$\begin{cases} \dot{x} = V\cos u \\ \dot{y} = V\sin u \end{cases} \qquad \begin{matrix} x(0) = 0 \\ y(0) = 0 \end{matrix}$$

$$\psi(x(t_f)) = \begin{bmatrix} x(1) \\ y(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$H = -y + \lambda_1 (V\cos u) + \lambda_2 (V\sin u)$$

$$\begin{cases} \dot{x} = V\cos u \\ \dot{y} = V\sin u \end{cases} \quad - ①$$

$$\begin{cases} \dot{\lambda}_1 = -\dfrac{\partial H}{\partial x} = 0 \\ \dot{\lambda}_2 = -\dfrac{\partial H}{\partial y} = 1 \end{cases} \quad - ②$$

$$H_u = -\lambda_1 V\sin u + \lambda_2 V\cos u = 0 \quad - ③$$

$$\begin{bmatrix} \lambda_1(1) \\ \lambda_2(1) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \gamma \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad - ④$$

$$\begin{bmatrix} x(1) \\ y(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad - ⑤$$

From ②,

$$\begin{cases} \lambda_1 = \pi_1 \\ \lambda_2 = t + \pi_2 \end{cases}$$

From ③,

$$-\lambda_1 V\sin u + \lambda_2 V\cos u = 0$$

$$\Rightarrow \frac{\sin u}{\cos u} = \tan u = \frac{\lambda_2}{\lambda_1} = \frac{t + \pi_2}{\pi_1} = \frac{1}{\pi_1} t + \frac{\pi_2}{\pi_1} = C_1 + C_2 t \;\#$$

2

$$\min \quad x_2(1)$$

s.t.

$$\begin{cases} \dot{x}_1 = x_2 , & x_1(0) = 0 \\ \dot{x}_2 = u , & x_2(0) = 0 \end{cases}$$

$$x_1(1) = 0$$

$$|u| \le 1$$

$$H = \lambda_1 (x_2) + \lambda_2 (u)$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \quad - ①$$

$$\begin{cases} \dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \\ \dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \end{cases} \quad - ②$$

$$\min_{u \in \{-1,1\}} H = \lambda_1 (x_2) + \lambda_2 (u)$$

$$\Rightarrow \begin{cases} \text{if } \lambda_2 \ge 0 , \; u = -1 \\ \text{if } \lambda_2 < 0 , \; u = 1 \end{cases} \quad - ③$$

$$\begin{bmatrix} \lambda_1(1) \\ \lambda_2(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \nu \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad - ④$$

$$x_1(1) = 0 \quad - ⑤$$

From ②,

$$\lambda_1 = \pi_1$$

$$\lambda_2 = -\pi_1 t + \pi_2$$

$$\lambda_2(1) = -\pi_1 t + \pi_2 = 1$$

Let switch time is t_s

for $0 \le t \le t_s$,

$$\begin{cases} \dot{x}_1 = x_2 , & x_1(0) = 0 \\ \dot{x}_2 = 1 , & x_2(0) = 0 \end{cases}$$

$$\Rightarrow x_2(t) = \int_0^t 1 \, d\tau = \tau \Big|_0^t = t$$

$$x_1(t) = \int_0^t \tau \, d\tau = \frac{1}{2}\tau^2 \Big|_0^t = \frac{1}{2} t^2$$

for $t_s < t \le t_f$,

$$\begin{cases} \dot{x}_1 = x_2, & x_1(t_s) = t_s \\ \dot{x}_2 = -1, & x_2(t_s) = \frac{1}{2}t_s^2 \end{cases}$$

$$\Rightarrow x_2(t) = \int_{t_s}^{t} -1 \, d\tau = -\tau \Big|_{t_s}^{t} = -t + t_s$$

$$x_1(t) = \int_{t_s}^{t} -\tau + t_s \, d\tau = -\frac{1}{2}\tau^2 + t_s\tau \Big|_{t_s}^{t} = -\frac{1}{2}t^2 + t_s t + \frac{1}{2}t_s^2 - t_s^2$$

$$= -\frac{1}{2}t^2 + t_s t - \frac{1}{2}t_s^2$$

$$x_1(1) = 0 = t_s - \frac{1}{2} + t_s - \frac{1}{2}t_s^2$$

$$\Rightarrow \frac{1}{2}t_s^2 - 2t_s + \frac{1}{2} = 0$$

$$\Rightarrow t_s = 2 \pm \sqrt{4-1} = 0.2679, \cancel{3.7321}_{> t_f}$$

$$\Rightarrow u = \begin{cases} 1, & \text{if } 0 \le t \le 0.2679 \\ -1, & \text{if } 0.2679 < t \le 1 \end{cases} \quad \#$$

\# 3.

$$\min \ \frac{1}{2}\int_0^{t_f} \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt$$

s.t.

$$\dot{x} = Ax + Bu$$

$$\psi(x(t_f)) = M_f \, x(t_f)$$

(a)

$$H = \frac{1}{2}\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \lambda^T(Ax + Bu)$$

$$= \frac{1}{2}x^T Q x + x^T N u + \frac{1}{2}u^T R u + \lambda^T A x + \lambda^T B u$$

① $\dot{x} = Ax + Bu$

② $\dot{\lambda} = -\frac{\partial H}{\partial x} = -Qx - Nu - A^T\lambda$

③ $H_u = 0 = N^T x + Ru + B^T\lambda$

④ $\lambda(t_f) = \underbrace{\cancel{\phi_x(x(t_f))}}_{0} + \gamma^T \psi_x(x(t_f))$

$$= \gamma^T M_f$$

⑤ $\psi(x(t_f)) = M_f \, x(t_f)$ #

(b)

From ③

$$u = R^{-1}(-N^T x - B^T \lambda) \quad \#$$

(c)

$$\dot{x} = Ax + Bu$$

$$= Ax + B(-R^{-1}N^T x - R^{-1}B^T \lambda)$$

$$= (A - BR^{-1}N^T)x - BR^{-1}B^T \lambda$$

$$\dot{\lambda} = -Qx - Nu - A^T \lambda$$

$$= -Qx - N(-R^{-1}N^T x - R^{-1}B^T \lambda) - A^T \lambda$$

$$= -Qx + NR^{-1}N^T x - NR^{-1}B^T \lambda - A^T \lambda$$

$$= (-Q + NR^{-1}N^T)x - (NR^{-1}B^T + A^T)\lambda$$

$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -NR^{-1}B^T + A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \quad \#$$

(d)

#4.

$$\min J = \frac{1}{2} e_f^T Q_f e_f + \frac{1}{2}\int_0^{t_f} (y-y_d)^T Q_d (y-y_d) + u^T R_d u \} dt$$

s.t.

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$e_f = M_f x(t_f) - \psi(x(t_f))$$

Parameters :

$$C = [0 \ 0 \ 1 \ 0]$$

$$D = [0 \ 0]$$

$$\psi(x(t_f)) = x(t_f) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$y_d = 1$$

To be determine parameter

$$Q_f \in \mathbb{R}^{4\times4}, Q_d \in \mathbb{R}^{1\times1}, R_d \in \mathbb{R}^{2\times2}$$

Def: $Q = C^T Q_d C$, $N = C^T Q_d D$, $R = R_d + D^T Q_d D$

$$\begin{cases} \dot{S} = -S(A - BR^{-1}N^T) - (A - BR^{-1}N^T)^T S - Q + NR^{-1}N^T + SBR^{-1}B^T S \\ \dot{g} = -[(A - BR^{-1}N^T)^T + (BR^{-1}S)^T] g + C^T Q_d y_d \end{cases} \#$$

$$\begin{cases} S(t_f) = M_f^T Q_f M_f \\ g(t_f) = -M_f^T Q_d \psi(x(t_f)) \end{cases} \#$$