I did not offer assistances to nor receive assistances from others in this exam.

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(a) $\underbrace{(A-\lambda_2 I)\cdots(A-\lambda_{n-1}I)(A-\lambda_n I)}_{=\tilde{A}}(C_1 v_1 + C_2 v_2 + \cdots + C_{n-1}v_{n-1} + C_n v_n) = 0$

$\Rightarrow \tilde{A}C_1 v_1 + \cancelto{0}{\tilde{A}C_2 v_2} + \cdots + \cancelto{0}{\tilde{A}C_{n-1}v_{n-1}} + \cancelto{0}{\tilde{A}C_n v_n} = 0$

$\because \tilde{A} \neq 0,\ v_1 \neq 0$

$\therefore C_1 = 0$

$\underbrace{(A-\lambda_1 I)(A-\lambda_3 I)\cdots(A-\lambda_{n-1}I)(A-\lambda_n I)}_{=\tilde{A}}(C_1 v_1 + C_2 v_2 + \cdots + C_{n-1}v_{n-1} + C_n v_n) = 0$

$\Rightarrow \cancelto{0}{\tilde{A}C_1 v_1} + \tilde{A}C_2 v_2 + \cancelto{0}{\tilde{A}C_3 v_3} + \cdots + \cancelto{0}{\tilde{A}C_{n-1}v_{n-1}} + \cancelto{0}{\tilde{A}C_n v_n} = 0$

$\because \tilde{A} \neq 0,\ v_2 \neq 0$

$\therefore C_2 = 0$

$\vdots$

$\Rightarrow C_1 = C_2 = \cdots = C_n = 0$

$\Rightarrow v_{k,k+1},\ k = 1,3,5,\cdots,n-1$ are linear independent

$C_1(\alpha_1 + i\beta_1) + C_2(\alpha_1 - i\beta_1) + \cdots + C_{n-1}(\alpha_{n-1} + i\beta_{n-1}) + C_n(\alpha_{n-1} - i\beta_{n-1})$

$= \left[(C_1+C_2)\alpha_1 + \cdots + (C_{n-1}+C_n)\alpha_n\right] + i\left[(C_1+C_2)\beta_1 + \cdots + (C_{n-1}+C_n)\beta_n\right] \in \mathbb{C}^{n\times n}$

$\therefore v_{k,k+1},\ k = 1,3,5,\cdots,n-1$ is a basis of $\mathbb{C}^{n\times n}$ #

(b) $\Lambda = V^{-1}AV$ #

$$\begin{tikzcd}
\text{Standard basis} \arrow[r, "A"] & \text{Standard basis} \\
\text{eigenspace} \arrow[u, "V"] \arrow[r, "\Lambda"] & \text{eigenspace} \arrow[u, "V"']
\end{tikzcd}$$

#2

$$\phi(t,t_0) = e^{At} = e^{\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}t}$$

$$\det(A-sI) = (s-\lambda)^2 = 0 \Rightarrow s = \lambda, \lambda$$

$$P(s) = a_0 + a_1 s, \quad f(s) = e^{st}$$

$$P(\lambda) = a_0 + a_1\lambda = e^{\lambda t}$$

$$P'(\lambda) = a_1 = te^{\lambda t}$$

$$a_0 = (1-\lambda t)e^{\lambda t}$$

$$\phi(t,t_0) = (1-\lambda t)e^{\lambda t} I_2 + te^{\lambda t}\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix} = \begin{bmatrix}e^{\lambda t} & te^{\lambda t}\\ 0 & e^{\lambda t}\end{bmatrix}$$

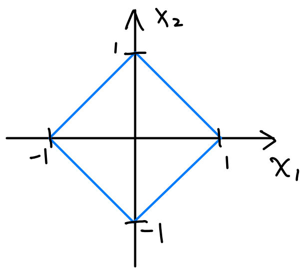
$$x(t) = \phi(t,t_0)\begin{bmatrix}1\\2\end{bmatrix}$$

$$= \begin{bmatrix}(1+2t)e^{\lambda t}\\ 2e^{\lambda t}\end{bmatrix} \quad \#$$

#3.

(a) Let $X = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

$$\|x\|_1 = |x_1| + |x_2| = 1$$



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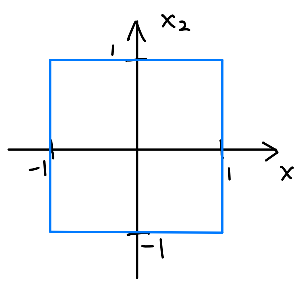
$$\|x\|_\infty = \max(x_1, x_2) = 1$$



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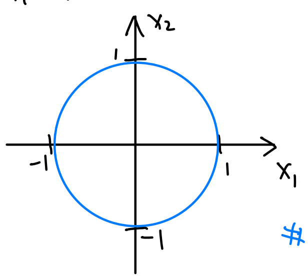
$$\|x\|_2 = \sqrt{x_1^2 + x_2^2} = 1$$



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(b) $y = Ax = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta + 2\sin\theta \\ -2\cos\theta + 4\sin\theta \end{bmatrix}$

(i)

$$\|y\|_1 = |\cos\theta + 2\sin\theta| + |-2\cos\theta + 4\sin\theta|$$

$$\|A\|_1 = \sup_{\theta\in[0,2\pi]} (\|y\|_1) = \underline{6 \ \#}$$

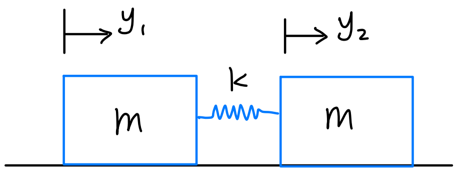
$$\|y\|_2 = \sqrt{(\cos\theta + 2\sin\theta)^2 + (-2\cos\theta + 4\sin\theta)^2}$$

$$\|A\|_2 = \sup_{\theta\in[0,2\pi]} (\|y\|_2) = \underline{4.7013 \ \#}$$

$$\|y\|_\infty = \max(\cos\theta + 2\sin\theta, -2\cos\theta + 4\sin\theta)$$

$$\|A\|_\infty = \sup_{\theta\in[0,2\pi]} (\|y\|_\infty) = \underline{4.4721 \ \#}$$

#4.



(a) $m\ddot{y}_1 = k(y_2 - y_1)$

$m\ddot{y}_2 = -k(y_2 - y_1)$

$$\dot{X} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \frac{k}{m}(y_2 - y_1) \\ -\frac{k}{m}(y_2 - y_1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m} & \frac{k}{m} & 0 & 0 \\ \frac{k}{m} & -\frac{k}{m} & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} \ \#$$

(b) $\det(A - sI) = \det \begin{bmatrix} -s & 0 & 1 & 0 \\ 0 & -s & 0 & 1 \\ -\frac{k}{m} & \frac{k}{m} & -s & 0 \\ \frac{k}{m} & -\frac{k}{m} & 0 & -s \end{bmatrix} = s^4 + 2\frac{k}{m}s^2$

$\Rightarrow s = 0, 0, \pm\sqrt{\frac{2k}{m}}\, i$

For $s = 0$

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{k}{m} & \frac{k}{m} & 0 & 0 & 0 \\ \frac{k}{m} & -\frac{k}{m} & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = x_2 \\ x_2 = x_2 \\ x_3 = 0 \\ x_4 = 0 \end{cases}$$

eigenvector of $s=0$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ #

For $s = \pm\sqrt{\frac{2k}{m}}\,i$,

$$\left[\begin{array}{cccc|c} \pm\sqrt{\frac{2k}{m}}\,i & 0 & 1 & 0 & 0 \\ 0 & \pm\sqrt{\frac{2k}{m}}\,i & 0 & 1 & 0 \\ -\frac{k}{m} & \frac{k}{m} & \pm\sqrt{\frac{2k}{m}}\,i & 0 & 0 \\ \frac{k}{m} & -\frac{k}{m} & 0 & \pm\sqrt{\frac{2k}{m}}\,i & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & \pm\sqrt{\frac{m}{2k}}\,i & 0 \\ 0 & 1 & 0 & \mp\sqrt{\frac{m}{2k}}\,i & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 = \mp\sqrt{\frac{m}{2k}}\,i \\ x_2 = \pm\sqrt{\frac{m}{2k}}\,i \\ x_3 = -x_4 \\ x_4 = x_4 \end{cases}$$

eigenvector of $s = \pm\sqrt{\frac{2k}{m}}\,i$ is $\left\{ \begin{bmatrix} -\sqrt{\frac{m}{2k}}\,i \\ \sqrt{\frac{m}{2k}}\,i \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\frac{m}{2k}}\,i \\ -\sqrt{\frac{m}{2k}}\,i \\ -1 \\ 1 \end{bmatrix} \right\}$ #