

Linear Systems

HW3

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#1 (a)

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$a \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+b \\ -a+c \\ -b \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} a=1 \\ b=0 \\ c=5 \end{cases}$$

$$(1, 0, 5) \text{ with respect to } \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \#$$

(b)

$$a \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} s+2 \\ \frac{1}{s} \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+b \\ -a+c \\ -b \end{bmatrix} = \begin{bmatrix} s+2 \\ \frac{1}{s} \\ -2 \end{bmatrix} \Rightarrow \begin{cases} a=s \\ b=2 \\ c=\frac{1}{s}+s = \frac{s^2+1}{s} \end{cases}$$

$$(s, 2, \frac{s^2+1}{s}) \text{ w.r.t. } \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \#$$

#2. $V = \left\{ \begin{bmatrix} x \\ \alpha x \end{bmatrix} \mid x \in \mathbb{R}, \alpha = \text{const} \right\}$ over \mathbb{R}

$$\text{Let } X = \left\{ \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \right\}$$

$$\alpha_1 \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha_1 = 0 \Rightarrow \text{linearly indep.}$$

$$\text{Sp}(x) = a \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \begin{bmatrix} x \\ \alpha x \end{bmatrix} \Rightarrow a = x$$

$$\Rightarrow x \text{ span } V$$

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \right\} \text{ is a basis of } V. \#$$

#3 (a) $S = \left\{ \begin{bmatrix} s^2 \\ s \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} \right\}$ over rational function

$$\alpha \begin{bmatrix} s^2 \\ s \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \alpha s^2 + \beta = 0 \\ \alpha s + \frac{\beta}{s} = 0 \end{cases} \Rightarrow \begin{cases} \alpha s^2 + \beta = 0 \\ \alpha s^2 + \beta = 0 \end{cases}$$

$\therefore \alpha = \frac{1}{s^2}, \beta = -1$ is a solution.

$\therefore S$ over rational function is NOT linear independent. #

(b) $S = \left\{ \begin{bmatrix} s^2 \\ s \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} \right\}$ over \mathbb{R}

$$\alpha \begin{bmatrix} s^2 \\ s \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \alpha s^2 + \beta = 0 \\ \alpha s + \frac{\beta}{s} = 0 \end{cases} \Rightarrow \begin{cases} \alpha s^2 + \beta = 0 \\ \alpha s^2 + \beta = 0 \end{cases}$$

$\therefore \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only solution.

$\therefore S$ over \mathbb{R} is linear independent #

$$\#4 \quad P(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

$$P'(s) = 4a_4 s^3 + 3a_3 s^2 + 2a_2 s + a_1 \Big|_{s=0} = a_1$$

$$P(0) = a_0$$

$$\Rightarrow P'(0) + P(0) = a_1 + a_0 = 0$$

$$P(1) = a_4 + a_3 + a_2 + a_1 + a_0 = 0$$

$$\Rightarrow a_4 + a_3 + a_2 = 0$$

$$V = \{ (a_4, a_3, a_2, a_1, a_0) \mid a_1 + a_0 = 0, a_4 + a_3 + a_2 = 0 \}$$

$$(a) \quad \text{Let } x = x_4 s^4 + x_3 s^3 + x_2 s^2 + x_1 s + x_0 = (x_4, x_3, x_2, x_1, x_0) \in V$$

$$y = y_4 s^4 + y_3 s^3 + y_2 s^2 + y_1 s + y_0 = (y_4, y_3, y_2, y_1, y_0) \in V$$

$$\alpha, \beta \in \mathbb{R}$$

$$(A) \quad (a1) \quad x+y = (x_4+y_4)s^4 + (x_3+y_3)s^3 + (x_2+y_2)s^2 + (x_1+y_1)s + (x_0+y_0)$$

$$y+x = (y_4+x_4)s^4 + (y_3+x_3)s^3 + (y_2+x_2)s^2 + (y_1+x_1)s + (y_0+x_0)$$

$$= x+y$$

$$(a2) \quad (x+y)+z = \left[(x_4+y_4)s^4 + (x_3+y_3)s^3 + (x_2+y_2)s^2 + (x_1+y_1)s + (x_0+y_0) \right]$$

$$+ z_4 s^4 + z_3 s^3 + z_2 s^2 + z_1 s + z_0$$

$$= (x_4+y_4+z_4)s^4 + (x_3+y_3+z_3)s^3 + (x_2+y_2+z_2)s^2$$

$$+ (x_1+y_1+z_1)s + (x_0+y_0+z_0)$$

$$x+(y+z) = x_4 s^4 + x_3 s^3 + x_2 s^2 + x_1 s + x_0$$

$$+ (y_4+z_4)s^4 + (y_3+z_3)s^3 + (y_2+z_2)s^2 + (y_1+z_1)s + (y_0+z_0)$$

$$= (x+y)+z$$

$$(a3) \text{ Let } O_v = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

$$\chi + O_v = \chi$$

$$\Rightarrow a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

$$\Rightarrow O_v = 0 = (0, 0, 0, 0, 0)$$

$$(a4) \text{ Let } -\chi = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

$$\chi + (-\chi) = 0$$

$$(-\chi) = -\chi = -a_4 s^4 - a_3 s^3 - a_2 s^2 - a_1 s - a_0$$

$$= (-a_4, -a_3, -a_2, -a_1, -a_0)$$

$$(M) (m1) (\alpha \cdot \beta) \cdot \chi = (\alpha \beta) \cdot \chi = \alpha \beta \chi_4 s^4 + \alpha \beta \chi_3 s^3 + \alpha \beta \chi_2 s^2 + \alpha \beta \chi_1 s + \alpha \beta \chi_0$$

$$\alpha \cdot (\beta \cdot \chi) = \alpha (\beta \chi_4 s^4 + \beta \chi_3 s^3 + \beta \chi_2 s^2 + \beta \chi_1 s + \beta \chi_0)$$

$$= (\alpha \cdot \beta) \cdot \chi$$

$$(m2) \alpha \cdot (\chi + y) = \alpha \cdot [(\chi_4 + y_4) s^4 + (\chi_3 + y_3) s^3 + (\chi_2 + y_2) s^2 + (\chi_1 + y_1) s + (\chi_0 + y_0)]$$

$$= \alpha (\chi_4 + y_4) s^4 + \alpha (\chi_3 + y_3) s^3 + \alpha (\chi_2 + y_2) s^2 + \alpha (\chi_1 + y_1) s + \alpha (\chi_0 + y_0)$$

$$\alpha \chi + \alpha y = \alpha \chi_4 s^4 + \alpha \chi_3 s^3 + \alpha \chi_2 s^2 + \alpha \chi_1 s + \alpha \chi_0$$

$$+ \alpha y_4 s^4 + \alpha y_3 s^3 + \alpha y_2 s^2 + \alpha y_1 s + \alpha y_0$$

$$= (\alpha \chi_4 + \alpha y_4) s^4 + (\alpha \chi_3 + \alpha y_3) s^3 + (\alpha \chi_2 + \alpha y_2) s^2$$

$$+ (\alpha \chi_1 + \alpha y_1) s + (\alpha \chi_0 + \alpha y_0)$$

$$= \alpha (\chi + y)$$

$$(m3) \quad (\alpha + \beta) \cdot \chi = (\alpha + \beta) \chi_4 s^4 + (\alpha + \beta) \chi_3 s^3 + (\alpha + \beta) \chi_2 s^2 + (\alpha + \beta) \chi_1 s + (\alpha + \beta) \chi_0$$

$$\begin{aligned} \alpha \chi + \beta \chi &= \alpha \chi_4 s^4 + \alpha \chi_3 s^3 + \alpha \chi_2 s^2 + \alpha \chi_1 s + \alpha \chi_0 \\ &\quad + \beta \chi_4 s^4 + \beta \chi_3 s^3 + \beta \chi_2 s^2 + \beta \chi_1 s + \beta \chi_0 \\ &= (\alpha \chi_4 + \beta \chi_4) s^4 + (\alpha \chi_3 + \beta \chi_3) s^3 + (\alpha \chi_2 + \beta \chi_2) s^2 \\ &\quad + (\alpha \chi_1 + \beta \chi_1) s + \alpha \chi_0 + \beta \chi_0 \\ &= (\alpha + \beta) \chi \end{aligned}$$

$$(m4) \quad \text{Let } 1_V = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

$$1_V \cdot \chi = \chi$$

$$\Rightarrow 1_V = 1 = (0, 0, 0, 0, 1)$$

$\therefore V$ is a vector space. #

$$(b) \quad \text{Let } \chi = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\} \quad \chi \subseteq V$$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\Rightarrow \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ -\alpha_1 + \alpha_2 \\ -\alpha_2 \\ \alpha_3 \\ -\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \\ \alpha_3 = 0 \end{cases}$$

$$\therefore \alpha_1 = \alpha_2 = \alpha_3 = 0$$

$\therefore X$ is linearly independent

$$\begin{aligned} \text{sp}(X) &= \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \\ &= \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ -\alpha_1 + \alpha_2 \\ -\alpha_2 \\ \alpha_3 \\ -\alpha_3 \end{bmatrix} \\ &= \alpha_1 s^4 + (-\alpha_1 + \alpha_2) s^3 + (-\alpha_2) s^2 + (\alpha_3) s + (-\alpha_3) \end{aligned}$$

check conditions:

$$p(s) = \alpha_1 s^4 + (-\alpha_1 + \alpha_2) s^3 + (-\alpha_2) s^2 + (\alpha_3) s + (-\alpha_3)$$

$$p'(s) = 4\alpha_1 s^3 + 3(-\alpha_1 + \alpha_2) s^2 + 2(-\alpha_2) s + \alpha_3$$

$$p'(0) + p(0) = \alpha_3 + (-\alpha_3) = 0$$

$$\begin{aligned} p(1) &= \cancel{\alpha_1} + (-\cancel{\alpha_1} + \cancel{\alpha_2}) + (-\cancel{\alpha_2}) + \cancel{\alpha_3} + (-\cancel{\alpha_3}) \\ &= 0 \end{aligned}$$

X can span V .

$\Rightarrow X$ is a basis for V . #

(c) $\dim(V) = 3$. #

(d) $p(s) = s^4 + s^3 - 2s^2 + 3s - 3$

$$\Rightarrow (1, 2, 3) \text{ w.r.t. } \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\} \quad \#$$