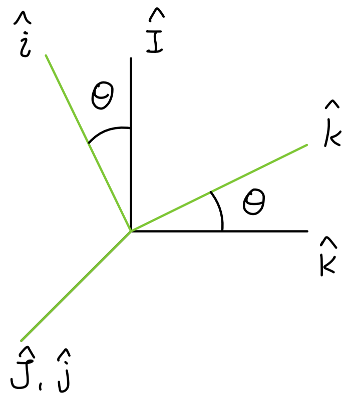
#1.

$$\hat{i} = \cos\theta\,\hat{I} + 0\,\hat{J} + \sin\theta\,\hat{K}$$

$$\hat{j} = 0\,\hat{I} + 1\,\hat{J} + 0\,\hat{K}$$

$$\hat{k} = -\sin\theta\,\hat{I} + 0\,\hat{J} + \cos\theta\,\hat{K}$$



$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{bmatrix} \quad \#$$

#2. Let

$$x = \begin{bmatrix} a_1 + \bar{j} b_1 \\ a_2 + \bar{j} b_2 \\ \vdots \\ a_n + \bar{j} b_n \end{bmatrix}, \quad y = \begin{bmatrix} c_1 + \bar{j} d_1 \\ c_2 + \bar{j} d_2 \\ \vdots \\ c_n + \bar{j} d_n \end{bmatrix}, \quad z = \begin{bmatrix} e_1 + \bar{j} f_1 \\ e_2 + \bar{j} f_2 \\ \vdots \\ e_n + \bar{j} f_n \end{bmatrix},$$

$x, y, z \in \mathbb{C}^n$ over $\mathbb{R}$.

$$\langle x, y \rangle = \mathrm{Re}(x) \cdot \mathrm{Re}(y)$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = a_1 c_1 + a_2 c_2 + \cdots + a_n c_n$$

$$= \sum_{i=1}^{n} a_i c_i \in \mathbb{R}$$

(1) $$\langle x, y+z \rangle = \mathrm{Re}(x) \cdot \mathrm{Re}(y+z)$$

$$= \mathrm{Re}(x) \cdot \mathrm{Re}\left(\begin{bmatrix} (c_1+e_1) + \bar{j}(d_1+f_1) \\ (c_2+e_2) + \bar{j}(d_2+f_2) \\ \vdots \\ (c_n+e_n) + \bar{j}(d_n+f_n) \end{bmatrix} \right)$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} c_1+e_1 \\ c_2+e_2 \\ \vdots \\ c_n+e_n \end{bmatrix} = a_1(c_1+e_1) + a_2(c_2+e_2) + \cdots + a_n(c_n+e_n)$$

$$= \sum_{i=1}^{n} a_i (c_i + e_i)$$

$$\langle x, y\rangle + \langle x, z\rangle = Re(x)\cdot Re(y) + Re(x) + Re(z)$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

$$= a_1c_1 + a_2c_2 + \cdots + a_nc_n + a_1e_1 + a_2e_2 + \cdots + a_ne_n$$

$$= a_1(c_1+e_1) + a_2(c_2+e_2) + \cdots + a_n(c_n+e_n)$$

$$= \sum_{i=1}^{n} a_i(c_i+e_i) = \langle x, y+z\rangle$$

(2) $\langle x, \alpha y\rangle = Re(x)\cdot Re(\alpha y)$

$$= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} \alpha(c_1+jd_1) \\ \alpha(c_2+jd_2) \\ \vdots \\ \alpha(c_n+jd_n) \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} \alpha c_1 \\ \alpha c_2 \\ \vdots \\ \alpha c_n \end{bmatrix} = \alpha a_1c_1 + \alpha a_2c_2 + \cdots + \alpha a_nc_n$$

$$= \alpha(a_1c_1 + a_2c_2 + \cdots + a_nc_n)$$

$$= \alpha\sum_{i=1}^{n} a_ic_i = \alpha\langle x, y\rangle$$

(3) $\langle y, x\rangle^* = [Re(y)\cdot Re(x)]^*$

$$= \left(\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\right)^* = (c_1a_1 + c_2a_2 + \cdots + c_na_n)^*$$

$$= a_1c_1 + a_2c_2 + \cdots + a_nc_n$$

$$= \sum_{i=1}^{n} a_ic_i = \langle x, y\rangle$$

(4) $\langle x, x\rangle = Re(x)\cdot Re(x)$

$$= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1^2 + a_2^2 + \cdots + a_n^2$$

when $\langle x, x\rangle = 0$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 = 0 \Rightarrow a_1, a_2, \cdots, a_n = 0$$

$\because$ $b_1 \sim b_n$ can be any number.

$\therefore \langle x, x \rangle = 0 \Leftrightarrow x = 0$ is NOT satisfied.

$\therefore$ This define was NOT inner product. #

#3 (a) $\langle x+y, z \rangle = \langle z, x+y \rangle^*$

$$= \left(\langle z, x \rangle + \langle z, y \rangle\right)^*$$

$$= \langle z, x \rangle^* + \langle z, y \rangle^*$$

$$= \langle x, z \rangle + \langle y, z \rangle \quad \#$$

(b) $\langle \alpha x, y \rangle = \langle y, \alpha x \rangle^*$

$$= \left(\alpha \langle y, x \rangle\right)^*$$

$$= \alpha^* \langle x, y \rangle \quad \#$$

#4. $\langle \sin nt, \cos mt \rangle = \int_{-\pi}^{\pi} \sin nt \cdot \cos mt \, dt$

Aside: $u = \sin nt$
$\Rightarrow du = n \cos nt \, dt$
$dv = \cos mt \, dt$
$\Rightarrow v = \frac{1}{m} \sin mt$

$u = \cos nt$
$\Rightarrow du = -n \sin nt \, dt$
$dv = \sin mt \, dt$
$\Rightarrow v = \frac{-1}{m} \cos mt$

$$= \frac{1}{m} \sin nt \sin mt \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{n}{m} \cos nt \sin mt \, dt$$

$$= \frac{1}{m} \sin nt \sin mt \Big|_{-\pi}^{\pi}$$

$$- \left(\frac{-1}{m} \cos nt \cos mt \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{n}{m} \sin nt \cos mt \, dt \right)$$

$$\left(1 + \frac{n}{m}\right) \int_{-\pi}^{\pi} \sin nt \cos mt \, dt = \frac{1}{m} \left[\cancelto{0}{\sin n\pi \sin m\pi} - \cancelto{0}{\sin(-n\pi) \sin(-m\pi)} \right]$$

$$+ \frac{1}{m} \left[\cancelto{1}{\cos n\pi \cos m\pi} - \cancelto{1}{\cos(-n\pi) \cos(-m\pi)} \right]$$

$$= \frac{1}{m}(1 - 0) = 0$$

$$\Rightarrow \int_{-\pi}^{\pi} \sin nt \cos nt \, dt = 0$$

$\because \langle \sin nt, \cos mt \rangle = 0$

$\therefore$ $\sin nt$ and $\cos mt$ are orthogonal. #

#5. (a) Let $V = C([t_0, t_1], \mathbb{R})$, $F = \mathbb{R}$, $W = \{ f \in V \mid f(t_0) = 0 \}$

Let $g(t), h(t) \in W$, $\alpha \in F$

For $g(t) + h(t) \in W$?

$\because g(t_0) + h(t_0) = 0 + 0 = 0$

$\therefore g(t) + h(t) \in W$

For $\alpha g(t) \in W$?

$\because \alpha g(t_0) = \alpha \cdot 0 = 0$

$\therefore \alpha g(t) \in W$

Exist a zero vector ?

$g(t) + O = g(t)$

$\Rightarrow O = 0 \in W$

$\Rightarrow$ The subset is a subspace #

(b) Let $V = C([t_0, t_1], \mathbb{R})$, $F = \mathbb{R}$, $W = \{ f \in V \mid f(t_0) = 5 \}$

Let $g(t), h(t) \in W$, $\alpha \in F$

For $g(t) + h(t) \triangleq y(t) \in W$?

$\because y(t_0) = g(t_0) + h(t_0)$

$= 5 + 5 = 10 \neq 5$

$\therefore y(t) \notin W \Rightarrow$ The subset is NOT subspace. #