The Fundamental Concepts

Prof. Der-Ming Ma, Ph.D.
Department of Aerospace Engineering
Tamkang University

- Definitions
  - *Function*
  - *Functional*
  - *Linearity of Functions*
  - *Linearity of Functionals*
  - *Closeness of Functions*
  - *The Increment of Functions*
  - *The Increment of Functionals*
  - *The Differential of a Function*
  - *The Variation of a Functional*
  - *Maxima and Minima of Functions*
  - *Maxima and Minima of Functionals*
- The Fundamental Theorem of Calculus of variations

Definitions – *Function*

Definition 3-1

A *function* f is a rule of correspondence that assigns to each element $\mathbf{q}$ in a certain set $\mathcal{D}$ a unique element in a set $\mathcal{R}$. $\mathcal{D}$ is called the *domain* of f and $\mathcal{R}$ is the *range*.

Example 3.1-1 Suppose $q_1, q_2, \ldots, q_n$ are the coordinates of a point in n-dimensional Euclidean space and

$$f(\mathbf{q}) = \sqrt{q_1^2 + q_2^2 + \cdots + q_n^2} \qquad (3.1\text{-}1)$$

the real number assigned by f is the distance of the point $\mathbf{q}$ from the origin.

Definitions – *Functional*

<u>*Definition 3-2*</u>

A *functional* J is a rule of correspondence that assigns to each function $\mathbf{x}$ in a certain class Ω a unique real number. Ω is called the *domain* of functional, and the set of real numbers associated with the functions in Ω is called the *range* of the functional.

Example 3.1-2 Suppose x is a continuous function of t defined in the interval $[t_0, t_\mathrm{f}]$ and

$$J(x) = \int_{t_0}^{t_f} x(t)dt \qquad (3.1\text{-}2)$$

the real number assigned by J is the area under the $x(t)$ curve.

Definitions – *Linearity of Functions*

Definition 3-3

f is a *linear function* of $\mathbf{q}$ if and only if it satisfies the *principle of homogeneity*

$$f(\alpha\mathbf{q}) = \alpha f(\mathbf{q}) \tag{3.1-3}$$

for all $\mathbf{q} \in \mathcal{D}$ and for all real numbers α such that $\alpha\mathbf{q} \in \mathcal{D}$, and the *principle of additivity*

$$f(\mathbf{q}^{(1)} + \mathbf{q}^{(2)}) = f(\mathbf{q}^{(1)}) + f(\mathbf{q}^{(2)}) \tag{3.1-4}$$

for all $\mathbf{q}^{(1)}$, $\mathbf{q}^{(2)}$, and $\mathbf{q}^{(1)} + \mathbf{q}^{(2)}$ in $\mathcal{D}$.

Example 3.1-3. Check the linearity of the functions $f(t) = 5t$ for all t, and $g(t) = 2/t$ for $t > 0$.

$$f(2t) = 5(2t) = 10t = 2 \cdot f(t)$$

$$f(t+2t) = 5(t+2t) = 5t + 10t = f(t) + f(2t)$$

$$g(2t) = \frac{2}{2t} = \frac{1}{t} = 2 \cdot g(t)$$

$$g(t+2t) = \frac{2}{t+2t} \neq g(t) + g(2t)$$

Definitions – *Linearity of Functionals*

Definition 3-4

J is a linear functional of $\mathbf{x}$ if and only if it satisfies the *principle of homogeneity*

$$J(\alpha\mathbf{x}) = \alpha J(\mathbf{x}) \qquad (3.1\text{-}5)$$

for all $\mathbf{x} \, \varepsilon \, \Omega$ and for all real numbers α such that $\alpha\mathbf{x} \in \Omega$, and the *principle of additivity*

$$J(\mathbf{x}^{(1)} + \mathbf{x}^{(2)}) = J(\mathbf{x}^{(1)}) + J(\mathbf{x}^{(2)}) \qquad (3.1\text{-}6)$$

for all $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, and $\mathbf{x}^{(1)} + \mathbf{x}^{(2)}$ in Ω.

Example 3.1-4. Check the linearity of the functionals

$$J(x) = \int_{t_0}^{t_f} x(t)dt \quad \text{and} \quad F(x) = \int_{t_0}^{t_f} x^2(t)dt$$

$$J(2x) = \int_{t_0}^{t_f} 2x(t)dt = 2\int_{t_0}^{t_f} x(t)dt = 2J(x)$$

$$J(x+2x) = \int_{t_0}^{t_f} [x(t)+2x(t)]dt = \int_{t_0}^{t_f} x(t)dt + \int_{t_0}^{t_f} 2x(t)dt$$

$$F(2x) = \int_{t_0}^{t_f} [2x(t)]^2 dt = 4\int_{t_0}^{t_f} x(t)dt \neq 2F(x)$$

$$F(x+2x) = \int_{t_0}^{t_f} [x(t)+2x(t)]^2 dt \neq F(x) + F(2x)$$

Definitions – *Closeness of Functions*

Definition 3-5

The *norm* in n-dimensional Euclidean space is a rule of correspondence that assigns to each point $\mathbf{q}$ a real number. The *norm* of $\mathbf{q}$ denote by $\|\mathbf{q}\|$, satisfies the following properties:

1. $\|\mathbf{q}\| \geq 0$ and $\|\mathbf{q}\| = 0$ if and only if $\mathbf{q} = 0$.
2. $\|\alpha \mathbf{q}\| = |\alpha| \|\mathbf{q}\|$ for all real number α.
3. $\left\|\mathbf{q}^{(1)} + \mathbf{q}^{(2)}\right\| \leq \left\|\mathbf{q}^{(1)}\right\| + \left\|\mathbf{q}^{(2)}\right\|$.

When we say two points $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ are close together, we mean that $\|\mathbf{q}^{(1)} - \mathbf{q}^{(2)}\|$ is small.

Definition 3-6

The *norm of a function* is a rule of correspondence that assigns to each function $\mathbf{x} \in \Omega$, defined for $t \in [t_0, t_\mathrm{f}]$, a real number. The *norm* of $\mathbf{x}$ denote by $\|\mathbf{x}\|$, satisfies the following properties:

1. $\|\mathbf{x}\| \geq 0$ and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = 0$ for all $t \in [t_0, t_f]$.
2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all real number α.
3. $\|\mathbf{x}^{(1)} + \mathbf{x}^{(2)}\| \leq \|\mathbf{x}^{(1)}\| + \|\mathbf{x}^{(2)}\|$.

Considering the closeness of two functions $\mathbf{y}$ and $\mathbf{z}$, let $\mathbf{x}(t) = \mathbf{y}(t) - \mathbf{z}(t)$, the difference of two functions. $\|\mathbf{x}(t)\|$ is small if the functions are "close".

Definitions – *The Increment of Functions*

Definition 3-7

If $\mathbf{q}$ and $\mathbf{q} + \Delta\mathbf{q}$ are elements for which the function f is defined, then *the increment of* f, denoted by Δf, is

$$\Delta f \triangleq f(\mathbf{q} + \Delta\mathbf{q}) - f(\mathbf{q}) \qquad (3.1\text{-}7)$$

Notice that Δf depends on both $\mathbf{q}$ and $\Delta\mathbf{q}$, in general, so to be more explicit we would write $\Delta f(\mathbf{q}, \Delta\mathbf{q})$

Example 3.1-5. Consider the function

$$f(\mathbf{q}) = q_1^2 + 2q_1q_2 \text{ for all real } q_1, \quad q_2$$

The increment of f is

$$\begin{aligned}\Delta f &= f(\mathbf{q} + \Delta\mathbf{q}) - f(\mathbf{q}) \\ &= [q_1 + \Delta q_1]^2 + 2[q_1 + \Delta q_1][q_2 + \Delta q_2] - [q_1^2 + 2q_1q_2] \\ &= 2q_1\Delta q_1 + [\Delta q_1]^2 + 2\Delta q_1 q_2 + 2\Delta q_2 q_1 + 2\Delta q_1 \Delta q_2\end{aligned}$$

Definitions – *The Increment of Functionals*

Definition 3-8

If $\mathbf{x}$ and $\mathbf{x} + \delta\mathbf{x}$ are functions for which the functional J is defined, then *the increment of* J, denoted by ΔJ, is

$$\Delta J \triangleq J(\mathbf{x} + \delta\mathbf{x}) - J(\mathbf{x}) \qquad (3.1\text{-}8)$$

Again, notice that ΔJ depends on both $\mathbf{x}$ and $\delta\mathbf{x}$, in general, so to be more explicit we would write $\Delta J(\mathbf{x}, \delta\mathbf{x})$

Example 3.1-6. Find the increment of the functional

$$J(x) = \int_{t_0}^{t_f} x^2 dt,$$

where x is a continuous function of t.

The increment of J is

$$\begin{aligned} \Delta J &= J(x+\delta x) - J(x) \\ &= \int_{t_0}^{t_f} [x(t)+\delta x(t)]^2 dt - \int_{t_0}^{t_f} x^2(t) dt \\ &= \int_{t_0}^{t_f} [2x(t)\delta x(t) + [\delta x(t)]^2] dt \end{aligned}$$

Definitions – *The Differential of a Function*

Definition 3-9

The increment of a function of n variables can be written as

$$\Delta f(\mathbf{q}+\Delta\mathbf{q}) = df(\mathbf{q},\Delta\mathbf{q}) + g(\mathbf{q},\Delta\mathbf{q})\bullet\left\|\Delta\mathbf{q}\right\| \qquad (3.1\text{-}9)$$

where df is a linear function of $\Delta\mathbf{q}$. If

$$\lim_{\left\|\Delta\mathbf{q}\right\|\to 0}\{g(\mathbf{q},\Delta\mathbf{q})\} = 0,$$

then f is said to be *differentiable* at $\mathbf{q}$, and df is the *differential* of f at the point $\mathbf{q}$.

Notice:

If f is a differentiable function of *one* variable t, then the differential can be written

$$df(t, \Delta t) = f'(t)\Delta t$$

$f'(t)$ is called the *derivative* of f at t.

Example 3.1-7. Find the differential of the function

$$f(\mathbf{q}) = q_1^2 + 2q_1q_2 \text{ for all real } q_1,\ \ q_2$$

The increment of f is

$$\begin{aligned}
&\Delta f(\mathbf{q} + \Delta\mathbf{q}) \\
&= f(\mathbf{q} + \Delta\mathbf{q}) - f(\mathbf{q}) \\
&= [q_1 + \Delta q_1]^2 + 2[q_1 + \Delta q_1][q_2 + \Delta q_2] - [q_1^2 + 2q_1q_2] \\
&= \underbrace{[(2q_1 + 2q_2)\Delta q_1 + 2q_1\Delta q_2]}_{df(\mathbf{q},\Delta\mathbf{q})} + \underbrace{\frac{[\Delta q_1]^2 + 2\Delta q_1\Delta q_2}{\sqrt{[\Delta q_1]^2 + [\Delta q_2]^2}}}_{g(\mathrm{q},\Delta\mathbf{q})} \sqrt{[\Delta q_1]^2 + [\Delta q_2]^2}
\end{aligned}$$

Definitions – *The Variation of a Functional*

<u>Definition 3-10</u>

The increment of a functional can be written as

$$\Delta J(\mathbf{x}+\delta\mathbf{x}) = \delta J(\mathbf{x},\delta\mathbf{x}) + g(\mathbf{x},\delta\mathbf{x})\bullet\left\|\delta\mathbf{x}\right\| \qquad (3.1\text{-}10)$$

where δJ is linear in $\delta\mathbf{x}$. If

$$\lim_{\|\delta\mathbf{x}\|\to 0}\{g(\mathbf{x},\delta\mathbf{x})\} = 0,$$

then J is said to be *differentiable* on $\mathbf{x}$, and δJ is the *variation* of J evaluated for the function $\mathbf{x}$.

Example 3.1-6. Find the variation of the functional

$$J(x) = \int_0^1 x^2 dt,$$

where x is a continuous function of t.

The increment of J is

$$\Delta J = J(x+\delta x) - J(x) = \int_0^1 [x(t)+\delta x(t)]^2 dt - \int_0^1 x^2(t) dt$$

$$= \underbrace{\int_0^1 [2x(t)\delta x(t)] dt}_{\delta J(\mathbf{x},\delta \mathbf{x})} + \underbrace{\int_0^1 \frac{[\delta x(t)]^2}{\|\delta x\|} dt}_{g(\mathbf{x},\delta \mathbf{x})} \bullet \|\delta x\|$$

Notice:

It is very important to keep in mind that δJ is ***the linear approximation*** to the difference in the functional J caused by two comparison curves. If the comparison curves are close, then the variation should be a good approximation to the increment; however, δJ may be a poor approximation to ΔJ if the comparison curves are far apart.

Definitions – *Maxima and Minima of Functions*

<u>Definition 3-11</u>

A *function* f with domain $\mathcal{D}$ has a relative extremum at the point $\mathbf{q}^*$ if there is an $\varepsilon > 0$ such that for all points $\mathbf{q}$ in $\mathcal{D}$ that satisfy $\|\mathbf{q} - \mathbf{q}^*\| < \varepsilon$ the increment of f has the same sign. If

$$\Delta f = f(\mathbf{q}) - f(\mathbf{q}^*) \geq 0 \qquad (3.1\text{-}11)$$

$f(\mathbf{q}^*)$ is a relative minimum; if

$$\Delta f = f(\mathbf{q}) - f(\mathbf{q}^*) \leq 0 \qquad (3.1\text{-}12)$$

$f(\mathbf{q}^*)$ is a relative maximum.

Notice

If (3.1-11) is satisfied for arbitrarily large ε, then $f(\mathbf{q}^*)$ is a *global*, or *absolute, minimum*. Similarly, if (3.1-12) holds for arbitrarily large ε, then $f(\mathbf{q}^*)$ is a *global*, or *absolute, maximum*.

Definitions – *Maxima and Minima of Functionals*

Definition 3-12

A *functional* J with domain Ω has a relative extremum at the point $\mathbf{x}^*$ if there is an $\varepsilon > 0$ such that for all functions $\mathbf{x}$ in Ω which satisfy $\|\mathbf{x} - \mathbf{x}^*\| < \varepsilon$, the increment of J has the same sign. If

$$\Delta J = J(\mathbf{x}) - J(\mathbf{x}^*) \geq 0 \tag{3.1-13}$$

$J(\mathbf{x}^*)$ is a relative minimum; if

$$\Delta J = J(\mathbf{x}) - J(\mathbf{x}^*) \leq 0 \tag{3.1-14}$$

$J(\mathbf{x}^*)$ is a relative maximum.

Notice

If (3.1-13) is satisfied for arbitrarily large ε, then $J(\mathbf{x}^*)$ is a *global*, or *absolute, minimum*. Similarly, if (3.1-14) holds for arbitrarily large ε, then $J(\mathbf{x}^*)$ is a *global*, or *absolute, maximum*. $\mathbf{x}^*$ is called an ***extremal***, and $J(\mathbf{x}^*)$ is referred to as an ***extremum***.

The Fundamental Theorem of Calculus of variations

Let $\mathbf{x}$ be a vector function of t in the class Ω, and $J(\mathbf{x})$ be a differentiable functional of $\mathbf{x}$. Assume that the functions in Ω are not constrained by any boundaries. If $\mathbf{x}^*$ is an extremal, the variation of J must vanish on $\mathbf{x}^*$; that is

$$\delta J\,(\mathbf{x}^*, \delta\mathbf{x}) = 0 \text{ for all admissible } \delta\mathbf{x}. \qquad (3.1\text{-}15)$$

Proof by contradiction:

Assume $\mathbf{x}^*$ is an extremal, and that $\delta J(\mathbf{x}^*, \delta\mathbf{x}) \neq 0$. We will show that these assumption imply that the increment ΔJ can be made to change sign in an arbitrarily small neighborhood of $\mathbf{x}^*$. (It contradicts to *Definition 3-12*, that is, ΔJ has the same sign.)

Proof :

The increment is

$$\Delta J(\mathbf{x}^*, \delta\mathbf{x}) = J(\mathbf{x}^* + \delta\mathbf{x}) - J(\mathbf{x}^*)$$

$$= \delta J(\mathbf{x}^*, \delta\mathbf{x}) + g(\mathbf{x}^*, \delta\mathbf{x}) \bullet \|\delta\mathbf{x}\| \qquad \text{(3.1-16)}$$

where $g(\mathbf{x}^*, \delta\mathbf{x}) \to 0$ as $\|\delta\mathrm{x}\| \to 0$; thus, there is a neighborhood, $\|\delta\mathbf{x}\| < \varepsilon$, where $g(\mathbf{x}^*, \delta\mathbf{x}) \bullet \|\delta\mathbf{x}\|$ is small enough so that δJ dominate the expression for ΔJ.

Now let us select the variation $\delta\mathbf{x} = \alpha\delta\mathbf{x}^{(1)}$, where $\alpha > 0$ and $\|\alpha\delta\mathbf{x}^{(1)}\| < \varepsilon$. Suppose that

$$\delta J(\mathbf{x}^*, \alpha\delta\mathbf{x}^{(1)}) < 0.$$

Since δJ is a linear functional of $\delta\mathbf{x}$, the principle of homogeneity gives

$$\delta J(\mathbf{x}^*, \alpha\delta\mathbf{x}^{(1)}) = \alpha\delta J(\mathbf{x}^*, \delta\mathbf{x}^{(1)}) < 0.$$

The sign of ΔJ and δJ are the same for $\|\delta\mathbf{x}\| < \varepsilon$; thus

$$\Delta J(\mathbf{x}^*, \alpha\delta\mathbf{x}^{(1)}) < 0.$$

Now let us select the variation $\delta \mathbf{x} = -\alpha\delta\mathbf{x}^{(1)}$. Clearly, $\|\alpha\delta\mathbf{x}^{(1)}\| < \varepsilon$ implies that $\|-\alpha\delta\mathbf{x}^{(1)}\| < \varepsilon$; therefore, the sign of $\Delta J(\mathbf{x}^*, -\alpha\delta\mathbf{x}^{(1)})$ is the same as the sign of $\delta J(\mathbf{x}^*, -\alpha\delta\mathbf{x}^{(1)})$. Again, using the principle of homogeneity, we obtain

$$\delta J(\mathbf{x}^*, -\alpha\delta\mathbf{x}^{(1)}) = -\alpha\delta J(\mathbf{x}^*, \delta\mathbf{x}^{(1)}) ;$$

therefore, since $\delta J(\mathbf{x}^*, \alpha\delta\mathbf{x}^{(1)}) < 0$, $\delta J(\mathbf{x}^*, -\alpha\delta\mathbf{x}^{(1)}) > 0$, and this implies

$$\Delta J(\mathbf{x}^*, -\alpha\delta\mathbf{x}^{(1)}) > 0.$$

To recapitulate, we have shown that if $\delta J(\mathbf{x}^*, \delta\mathbf{x}) \neq 0$, then in an arbitrary small neighborhood of $\mathbf{x}^*$

$$\Delta J(\mathbf{x}^*, \alpha\delta\mathbf{x}^{(1)}) < 0$$

and

$$\Delta J(\mathbf{x}^*, -\alpha\delta\mathbf{x}^{(1)}) > 0,$$

Thus contradicting the assumption that $\mathbf{x}^*$ is an extremal.