#1 (a) max $P = 4(x+y)$ s.t. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 = f$

$\Rightarrow L = -4(x+y)$

Def. $H = P + \lambda f$

$$= -4(x+y) + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

$$= \frac{\lambda}{a^2}x^2 + \frac{\lambda}{b^2}y^2 - 4x - 4y - 1$$

$$\frac{\partial H}{\partial x} = \frac{2\lambda}{a^2}x - 4 = 0 \quad — ①$$

$$\frac{\partial H}{\partial y} = \frac{2\lambda}{b^2}y - 4 = 0 \quad — ②$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \quad — ③$$

From eq. ① & ②

$$x = 4 \cdot \frac{a^2}{2\lambda} = \frac{2a^2}{\lambda}$$

$$y = 4 \cdot \frac{b^2}{2\lambda} = \frac{2b^2}{\lambda}$$

Substitude x, y into eq ③,

$$\frac{\left(\frac{2a^2}{\lambda}\right)^2}{a^2} + \frac{\left(\frac{2b^2}{\lambda}\right)^2}{b^2} - 1 = 0$$

$$\Rightarrow \frac{4a^4}{a^2\lambda^2} + \frac{4b^4}{b^2\lambda^2} - 1 = 0 \Rightarrow \frac{4a^2}{\lambda^2} + \frac{4b^2}{\lambda^2} = 1$$

$$\Rightarrow \lambda^2 = 4a^2 + 4b^2 \Rightarrow \lambda = \sqrt{4a^2+4b^2}$$

$$\frac{\partial^2 H}{\partial \begin{bmatrix} x \\ y \end{bmatrix}^2} = \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial y} \\ \frac{\partial^2 H}{\partial y \partial x} & \frac{\partial^2 H}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \frac{2\lambda}{a^2} & 0 \\ 0 & \frac{2\lambda}{b^2} \end{bmatrix} \geq 0$$

Then the max P is at

$$\begin{cases} x = \frac{-2a^2}{\sqrt{4a^2+4b^2}} \\ y = \frac{-2b^2}{\sqrt{4a^2+4b^2}} \end{cases} \#$$

(b) From the previous solution, the max P will happen at

$$x = \frac{-2a^2}{\sqrt{4a^2+4b^2}}$$

$$y = \frac{-2b^2}{\sqrt{4a^2+4b^2}}$$

For $a=1$, $b=2$,

$$\begin{cases} x = \dfrac{-2}{\sqrt{20}} = -\dfrac{1}{\sqrt{5}} \\ y = \dfrac{-8}{\sqrt{20}} = -\dfrac{4}{\sqrt{5}} \end{cases} \#$$

#2. (a) Assume $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \end{bmatrix}_{j\times n}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_j \end{bmatrix}_{j\times 1}$, $\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_j \end{bmatrix}_{j\times 1}$

$$\min\ L = \|x\|_2^2 \quad \text{s.t.} \quad Ax - b = 0.$$

Def. $H = L + \lambda^T(Ax-b)$

$$\frac{\partial H}{\partial x} = \frac{\partial L}{\partial x} + \frac{\partial}{\partial x}\left[\lambda^T(Ax-b)\right]$$

$$= \frac{\partial L}{\partial x} + \cancel{\frac{\partial \lambda^T}{\partial x}}^{\,0}(Ax-b) + \lambda^T\frac{\partial}{\partial x}(Ax+b)$$

$$= \frac{\partial L}{\partial x} + \lambda^T A$$

$$= 2x^T + \lambda^T A = 0 \quad \text{— } ①$$

$$Ax - b = 0 \quad \text{— } ②$$

Aside:

$$L = \|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$\frac{\partial L}{\partial x} = \left[\frac{\partial L}{\partial x_1}\ \frac{\partial L}{\partial x_2}\ \cdots\ \frac{\partial L}{\partial x_n}\right]$$

$$= \left[2x_1\ 2x_2\ \cdots\ 2x_n\right]$$

$$= 2\left[x_1\ x_2\ \cdots\ x_n\right)$$

$$= 2x^T$$

From eq ①,

$$2x^T + \lambda^T A = 0 \Rightarrow x^T = \frac{1}{2}(-\lambda^T A) = \frac{-1}{2}\lambda^T A$$

$$\overset{\{\cdot\}^T}{\Rightarrow} x = \left(\frac{-1}{2}\lambda^T A\right)^T = \frac{-1}{2}(A^T\lambda)$$

Substitude x into eq ②

$$A\left[\frac{-1}{2}A^T\lambda\right]-b=0$$

$$\Rightarrow \frac{-1}{2}AA^T\lambda=b$$

$$\Rightarrow \lambda=-2(AA^T)^{-1}b$$

$$=-2(AA^T)^{-1}b$$

$$\frac{\partial^2 H}{\partial x^2}=\begin{bmatrix}2 & & \\ & 2 & \\ & & \ddots & \\ & & & 2\end{bmatrix}>0$$

Then $x=\frac{-1}{2}A^T\left[-2(AA^T)^{-1}b\right]$

$$=A^T(AA^T)^{-1}b \quad \#$$

(b) From the previous soluation, min L will happen at

$$x=A^T(AA^T)^{-1}b$$

For $A=\begin{bmatrix}1 & 2 & 3\\ 1 & -1 & 2\end{bmatrix}$, $b=\begin{bmatrix}10\\10\end{bmatrix}$

$$x=\begin{bmatrix}1 & 1\\ 2 & -1\\ 3 & 2\end{bmatrix}\left(\begin{bmatrix}1 & 2 & 3\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & -1\\ 3 & 2\end{bmatrix}\right)^{-1}\begin{bmatrix}10\\10\end{bmatrix}$$

$$=\begin{bmatrix}\frac{100}{59}\\ \frac{-70}{59}\\ \frac{210}{59}\end{bmatrix} \quad \#$$

#3 (a) Def $H=L+\lambda f$

$$=\frac{y_1\sec\theta_1}{v_1}+\frac{(y_2-y_1)\sec\theta_2}{v_2}+\lambda\left[x_2-y_1\tan\theta_1-(y_2-y_1)\tan\theta_2\right]$$

$$\frac{\partial H}{\partial \theta_1} = \frac{y_1 \sec\theta_1 \tan\theta_1}{v_1} + \lambda\left[-y_1 \sec^2\theta_1\right] = 0 \quad \text{— ①}$$

$$\frac{\partial H}{\partial \theta_2} = \frac{(y_2-y_1)\sec\theta_2\tan\theta_2}{v_2} + \lambda\left[-(y_2-y_1)\sec^2\theta_2\right] = 0 \quad \text{— ②}$$

$$x_2 - y_1\tan\theta_1 - (y_2-y_1)\tan\theta_2 = 0 \quad \text{— ③}$$

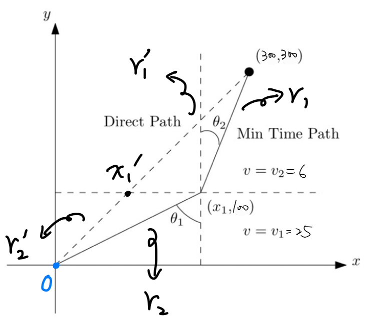
From eq ① & ②.

$$\tan\theta_1 = \frac{(\lambda \cancel{y_1} \sec^{\cancel{2}}\theta_1) v_1}{\cancel{y_1 \sec\theta_1}} = \lambda v_1 \sec\theta_1 \Rightarrow \sin\theta_1 = \lambda v_1$$

$$\tan\theta_2 = \frac{\lambda \cancel{(y_2-y_1)} \sec^{\cancel{2}}\theta_2 v_2}{\cancel{(y_2-y_1)\sec\theta_2}} = \lambda v_2 \sec\theta_2 \Rightarrow \sin\theta_2 = \lambda v_2$$

$$\Rightarrow \lambda = \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \quad \#$$

(b) Figure out x_1.



$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

$$\Rightarrow \frac{x_1/r_2}{v_1} = \frac{(300-x_1)/r_1}{v_2}$$

$$\Rightarrow \frac{x_1}{25\sqrt{x_1^2+100^2}} = \frac{(300-x_1)}{6\sqrt{(300-x_1)^2+200^2}}$$

$$\Rightarrow x_1 = 254.1752$$

$$t = t_1 + t_2 = \frac{r_2}{v_1} + \frac{r_1}{v_2}$$

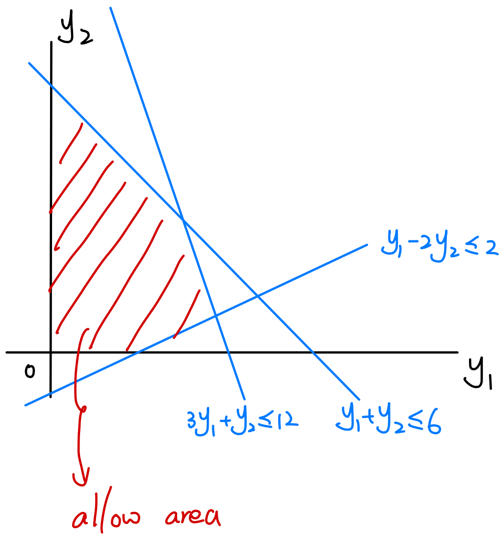
$$= \frac{\left[(300-x_1)^2+200^2\right]^{0.5}}{6} + \frac{\left[x_1^2+100^2\right]^{0.5}}{25} = 45.12 \text{ sec} \quad \#$$

$$x_1' = 100$$

$$t' = t_1' + t_2' = \frac{r_2'}{v_1} + \frac{r_1'}{v_2}$$

$$= \frac{\left[(300-x_1')^2+200^2\right]^{0.5}}{6} + \frac{x_1'^2+100^2}{25} = 52.79 \text{ sec} \quad \#$$

#5 From figure we guess $y^* = \left(\frac{26}{7}, \frac{6}{7}\right)$



$$L_y = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$g_{1y} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad g_1(y^*) = -\frac{26}{7} \Rightarrow \mu_1 = 0$$

$$g_{2y} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad g_2(y^*) = -\frac{6}{7} \Rightarrow \mu_2 = 0$$

$$g_{3y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad g_3(y^*) = \frac{-10}{7} \Rightarrow \mu_3 = 0$$

$$g_{4y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad g_4(y^*) = 0 \Rightarrow \mu_4 \ge 0$$

$$g_{5y} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad g_5(y^*) = 0 \Rightarrow \mu_5 \ge 0$$

$$H = L + \mu_1 g_{1y} + \mu_2 g_{2y} + \mu_3 g_{3y} + \mu_4 g_{4y} + \mu_5 g_{5y} = 0$$

$$\Rightarrow \begin{bmatrix} -5 \\ 1 \end{bmatrix} + \cancelto{0}{\mu_1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \cancelto{0}{\mu_2} \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \cancelto{0}{\mu_3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \mu_4 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \mu_5 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3\mu_4 + \mu_5 \\ \mu_4 - 2\mu_5 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} 1.2857 \\ 1.1429 \end{bmatrix}$$

Minimum L will happen at $(y_1, y_2) = \left(\frac{26}{7}, \frac{6}{7}\right)$ and $L = \frac{-136}{7}$ #