

for $n=3$,

$$f^{(3)}(y) = \frac{2\sin\theta [-5y^4 - 6y^2 + 2\cos(2\theta) + 1]}{1 - 2y^2 + y^4 + 2y^2\sin^2\theta} - \frac{2\sin\theta [-y^5 - 2y^3 + 2y\cos(2\theta) + y]}{(1 - 2y^2 + y^4 + 2y^2\sin^2\theta)^2} \times (-4y + 4y^3 + 4y\sin^2\theta)$$

$$\begin{aligned} f^{(3)}(0) &= \frac{2\sin\theta [2\cos(2\theta) + 1]}{1} - 0 \\ &= 2[2\sin\theta\cos(2\theta) + \sin\theta] = 2[2\sin(\theta)\cos(2\theta) + \sin(2\theta - \theta)] \\ &= 2[\cancel{\sin\theta\cos(2\theta)} + \sin(2\theta)\cos\theta - \cancel{\cos\theta\sin(2\theta)}] \\ &= 2\sin(\theta + 2\theta) = \underline{2\sin(3\theta)} \end{aligned}$$

$$\begin{aligned} f(y) &= \frac{1}{2} \tan^{-1}\left(\frac{2y\sin\theta}{1-y^2}\right) = \cancel{\frac{0}{1}} + \frac{\sin\theta}{1} y + \cancel{\frac{0}{2}} y^3 + \frac{2\sin(3\theta)}{3} y^3 + \dots \\ &= \underline{y\sin\theta + \frac{1}{3}y\sin(3\theta) + \dots} \quad \# \end{aligned}$$

(b)

$$\sum_{n=0}^{\infty} \frac{e^{-(2n+1)x}}{2n+1} \sin(2n+1)\frac{\pi y}{H} = \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} e^{-nx} \sin \frac{n\pi y}{H}$$

Let $y = e^{-x}$, $\theta = \frac{\pi y}{H}$, then

$$\begin{aligned} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} y^n \sin n\theta &= \frac{1}{2} \tan^{-1}\left(\frac{2y\sin\theta}{1-y^2}\right) \\ &= \frac{1}{2} \tan^{-1}\left(\frac{2e^{-x}\sin\frac{\pi y}{H}}{1-e^{-2x}}\right) \\ &= \frac{1}{2} \tan^{-1}\left(\frac{\sin\frac{\pi y}{H}}{\frac{e^x - e^{-x}}{2}}\right) \\ &= \frac{1}{2} \tan^{-1}\left(\frac{\sin\frac{\pi y}{H}}{\sinh x}\right) \end{aligned}$$