The Navigation Basics

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Geometry of the Earth

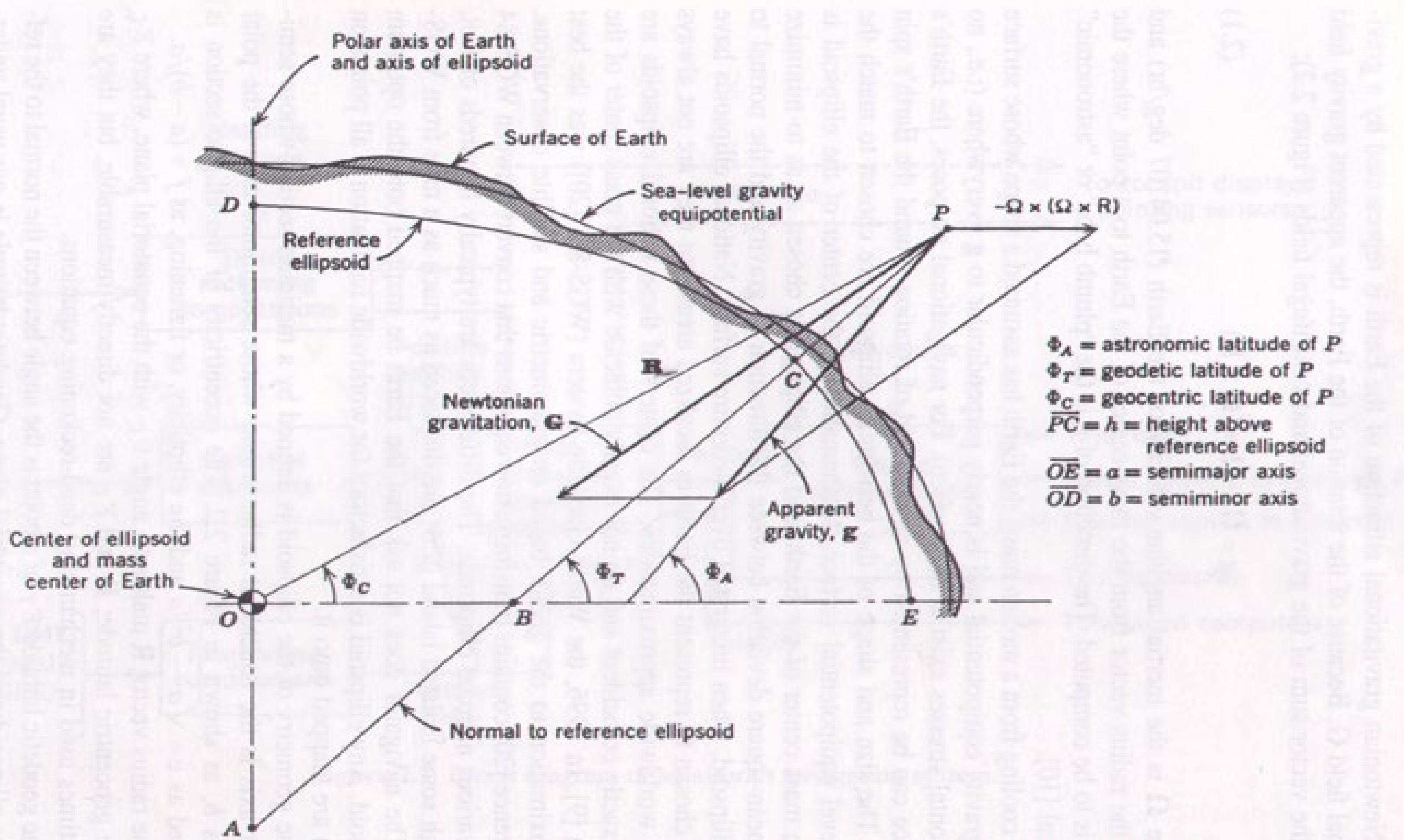

Figure 2.2 Meridian section of the Earth, showing the reference ellipsoid and gravity field.

- The *meridian radius of curvature*, R_M, is the radius of the best-fitting circle to a meridian section of the ellipsoid. The *prime radius of curvature*, R_P, is the radius of the best-fitting circle to a vertical east-west section of the ellipsoid. The *Gaussian radius of curvature*, R_G, is the radius of the best-fitting sphere to the ellipsoid at any point. They are respectively:

$$R_M = \frac{a(1-e^2)}{(1-e^2\sin^2\phi_T)^{3/2}} \approx a[1+e^2(\frac{3}{2}\sin^2\phi_T - 1)]$$

$$R_P = \frac{a}{(1-e^2\sin^2\phi_T)^{1/2}} \approx a[1+\frac{e^2}{2}\sin^2\phi_T]$$

$$R_G = \sqrt{R_M R_P} \approx a[1-\frac{e^2}{2}\cos 2\phi_T]$$

The aircraft's position vector expressed in ECEF is given as

$$x = (R_p + h)\cos\phi_T \cos\lambda$$

$$y = (R_p + h)\cos\phi_T \sin\lambda$$

$$z = (R_p(1-e^2) + h)\sin\phi_T$$

and

$$\sin^2\phi_T = \frac{z^2}{(R_p(1-e^2)+h)^2}$$

$$\cos^2\phi_T = \frac{x^2+y^2}{(R_p+h)^2}$$

From the definition of R_p, we have

$$\sin^2 \phi_T = \frac{R_p^2 - a^2}{e^2 R_p^2}$$

Combine the above equations, we have

$$\frac{z^2}{(R_p(1-e^2)+h)^2} = \frac{R_p^2 - a^2}{e^2 R_p^2}$$

$$\frac{x^2+y^2}{(R_p+h)^2} + \frac{R_p^2 - a^2}{e^2 R_p^2} = 1$$

Let

$$v = \frac{R_p^2}{a^2}$$

the altitude is given by

$$h = R_p \left\{ \sqrt{\frac{(x^2 + y^2)e^2}{a^2[(e^2 - 1)v + 1]}} - 1 \right\}$$

and the above equations can be expressed as

$$f(v) = c_0 v^4 + c_1 v^3 + c_2 v^2 + c_3 v + c_4 = 0$$

where

$$c_0 = e^4(e^2-1)^2, \quad c_1 = 2e^2(e^2-1)(d_2-2d_1)$$

$$c_2 = d_2^2 - 2e^2(e^2-1)d_3 - 4e^2(3-2e^2)d_1$$

$$c_3 = -2d_2d_3 - 4d_1e^2(e^2-3)$$

$$c_4 = d_3^2 - 4d_1e^2$$

$$d_1 = \frac{x^2+y^2}{a^2}, \quad d_2 = \frac{z^2+x^2+(1-e^2)y^2}{a^2} + e^2(2-e^2)$$

$$d_3 = \frac{x^2+y^2+z^2}{a^2} + e^2$$

The above nonlinear equation can be solved for the variable ν by the *Newton-Raphson* method, then the geodetic latitude can be obtained

$$\phi_T = \pm \tan^{-1} \sqrt{\frac{\nu - 1}{(e^2 - 1)\nu + 1}}$$

- The numerical data for the Earth (WGS-84):

$$a = 6378.137 \quad \text{Km}$$

$$e^2 = f(2-f), \quad f = 1/298.257.$$

$$g_0(\phi_T) = 9.780327(1 + 0.0053024\sin^2\phi_T$$
$$-0.0000058\sin^2(2\phi_T)) \quad \text{m/sec}^2$$

$$g(\phi_T, h) = g_0(\phi_T) - [3.0877\times10^{-6} - 0.0044\times10^{-6}\sin^2\phi_T]h$$
$$+0.072\times10^{-12}h^2 \quad \text{m/sec}^2$$

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Coordinate Frames and Transformations

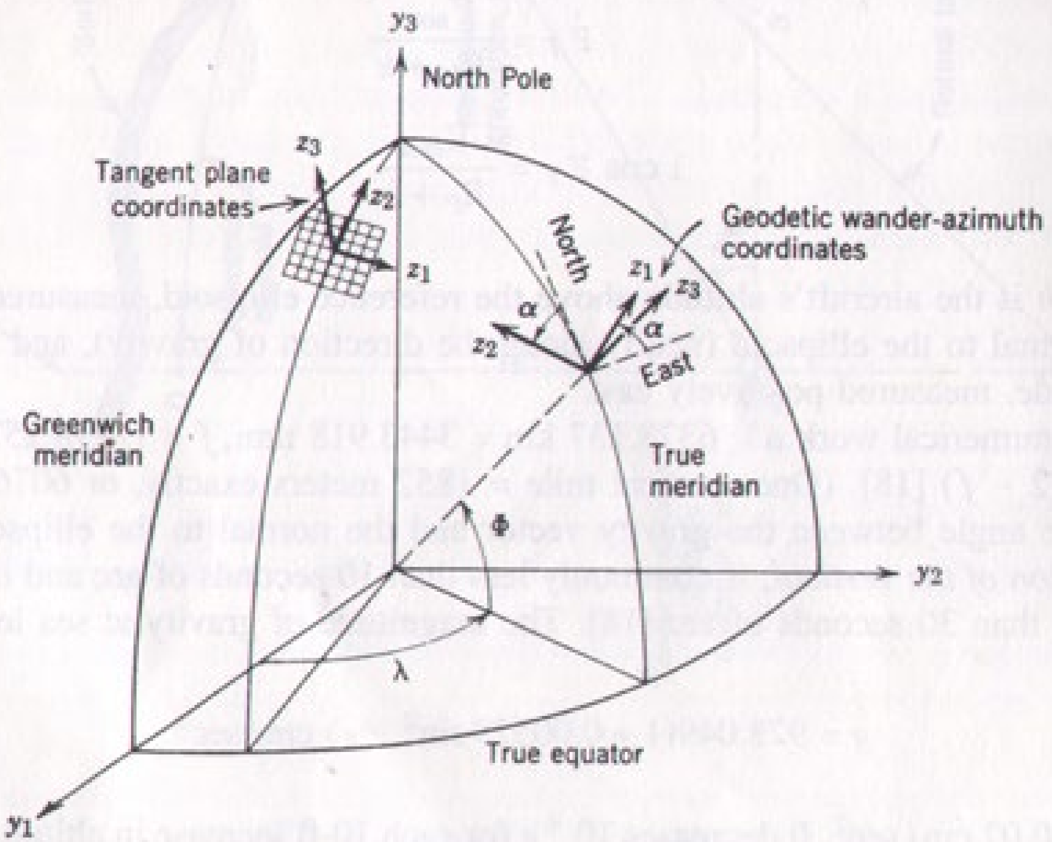

Figure 2.3 Navigation coordinate frames.

Coordinate Frames and Transformations

- Earth-centered, Earth-fixed (ECEF).
- Earth-centered inertial (ECI).
- Geodetic spherical coordinates.
  - These are the spherical coordinates of the normal to the reference ellipsoid.
- Geodetic wander azimuth.
  - These coordinates are always parallel to the locally level to the reference ellipsoid.
- Geocentric spherical coordinates.
- Tangent plane coordinates.
  - These coordinates are always parallel to the locally level axes.

○地心慣性座標(ECI)至地球座標(ECEF)

$$\mathbf{R}_{I2e}=\begin{bmatrix}\cos\omega_E t & \sin\omega_E t & 0\\ -\sin\omega_E t & \cos\omega_E t & 0\\ 0 & 0 & 1\end{bmatrix}$$

其中地球自轉角速度 $\omega_E = 7.292115062\times10^{-5}\ \text{rad/sec}$

- ECEF to Geodetic transformation

$$\mathbf{R}_{e2d} = \begin{bmatrix} \sin\phi_T & 0 & \cos\phi_T \\ 0 & 1 & 0 \\ \cos\phi_T & 0 & -\sin\phi_T \end{bmatrix} \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sin\phi_T \cos\lambda & \sin\phi_T \sin\lambda & \cos\phi_T \\ -\sin\lambda & \cos\lambda & 0 \\ \cos\phi_T \cos\lambda & \cos\phi_T \sin\lambda & -\sin\phi_T \end{bmatrix}$$

- Geodetic to Body axes transformation

$$\mathbf{R}_{d2b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & -\sin\theta \\ \begin{matrix} -\sin\psi\cos\phi \\ +\cos\psi\sin\theta\sin\phi \end{matrix} & \begin{matrix} \cos\psi\cos\phi \\ +\sin\psi\sin\theta\sin\phi \end{matrix} & \cos\theta\sin\phi \\ \begin{matrix} \sin\psi\sin\phi \\ +\cos\psi\sin\theta\cos\phi \end{matrix} & \begin{matrix} -\cos\psi\sin\phi \\ +\sin\psi\sin\theta\cos\phi \end{matrix} & \cos\theta\cos\phi \end{bmatrix}$$

飛機體座標至地心慣性座標

$$\mathbf{R}_{b2I} = \mathbf{R}_{e2I}\mathbf{R}_{d2e}\mathbf{R}_{b2d}$$

Quaternions

- The quaternion is a four-element vector $\mathbf{q} = [q_1, q_2, q_3, q_4]^T$ that can be partitioned as

$$\mathbf{q} = \begin{bmatrix} \mathbf{e}\sin(\zeta/2) \\ \cos(\zeta/2) \end{bmatrix}$$

where $\mathbf{e}$ is a unit vector and ζ is a positive rotation about $\mathbf{e}$. If the quaternion $\mathbf{q}$ represents the rotational transformation from reference frame a to reference frame b, then frame a is aligned with frame b when frame a is rotated by ζ radians about $\mathbf{e}$. Note that $\mathbf{q}$ has The normality property that $\|\mathbf{q}\|=1$.

- The rotation matrix from a frame to b frame, in terms of quaternion is

$$\mathbf{R}_{a2b} = \begin{bmatrix} q_1^2 + q_4^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_3 q_4) & 2(q_1 q_3 + q_2 q_4) \\ 2(q_1 q_2 + q_3 q_4) & q_2^2 + q_4^2 - q_1^2 - q_3^2 & 2(q_2 q_3 - q_1 q_4) \\ 2(q_1 q_3 - q_2 q_4) & 2(q_2 q_3 + q_1 q_4) & q_3^2 + q_4^2 - q_1^2 - q_2^2 \end{bmatrix}$$

- Initialization of quaternions from a known direction cosine matrix is

$$\mathbf{q} = \begin{bmatrix} \dfrac{\mathbf{R}(3,2) - \mathbf{R}(2,3)}{4q_4} \\ \dfrac{\mathbf{R}(1,3) - \mathbf{R}(3,1)}{4q_4} \\ \dfrac{\mathbf{R}(2,1) - \mathbf{R}(1,2)}{4q_4} \\ \dfrac{1}{2}\sqrt{1 + \mathbf{R}(1,1) + \mathbf{R}(2,2) + \mathbf{R}(3,3)} \end{bmatrix}$$

- The Euler angles can be obtained from the of quaternion

$$\theta = \sin^{-1}(-2(q_2 q_4 + q_1 q_3))$$

$$\phi = \arctan 2[2(q_2 q_3 - q_1 q_4), 1 - 2(q_1^2 + q_2^2)]$$

$$\psi = \arctan 2[2(q_1 q_2 - q_3 q_4), 1 - 2(q_2^2 + q_3^2)]$$

- Quaternion derivatives

$$\dot{\mathbf{q}} = \frac{1}{2}\begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

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Range and Bearing Calculation

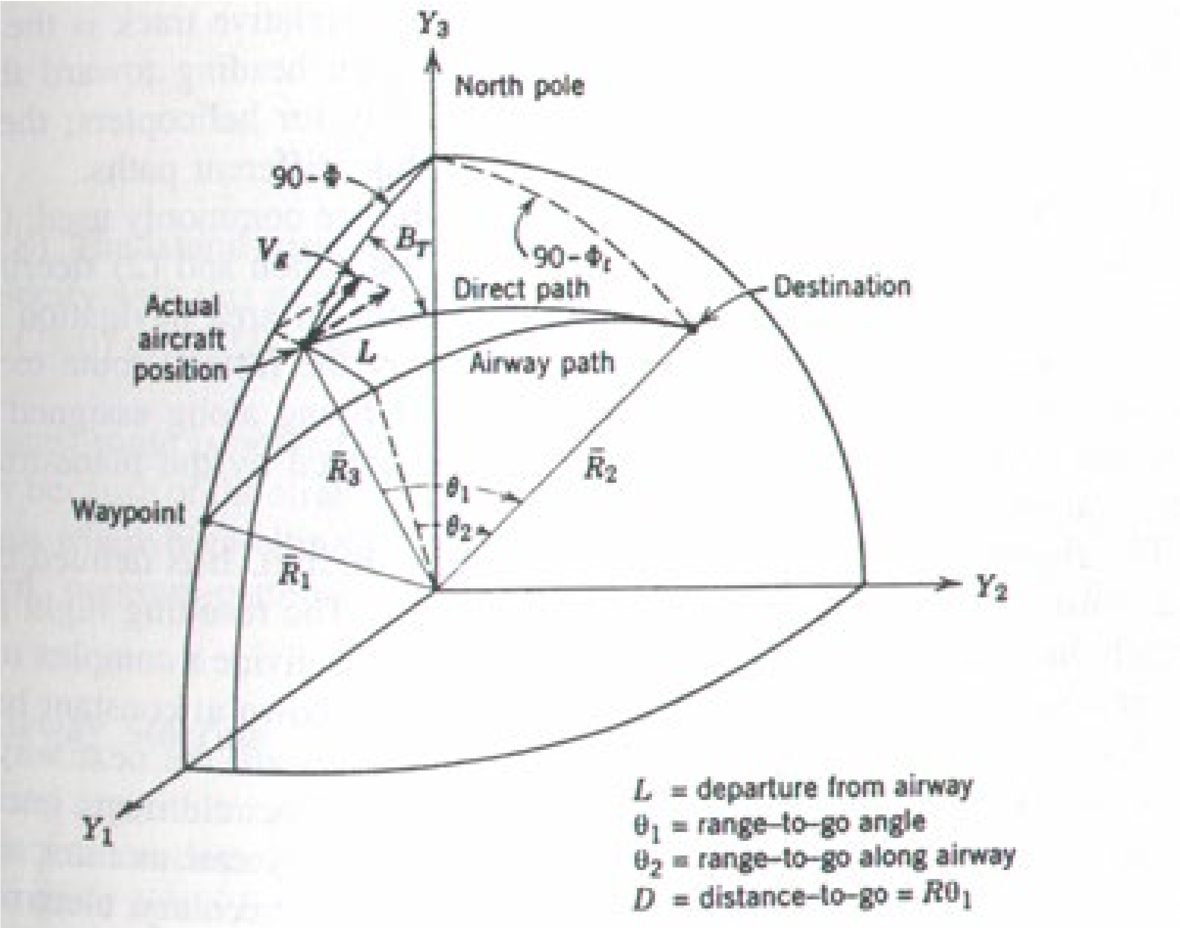

Figure 2.8 Course-line calculations.

$$\cos\frac{D}{R_G} = \sin\phi\sin\phi_t + \cos\phi\cos\phi_t\cos(\lambda-\lambda_t)$$

$$\sin B_T = \frac{\cos\phi_t}{\sin(D/R_G)}\sin(\lambda-\lambda_t)$$

where R_G is the Gaussian radius of curvature. Within 100 nm of the aircraft, the Earth can be assumed flat within an error of 0.3 nm.

Direct Steering

- The steering computer calculates the ground speed V_1 along the direction to the destination and V_2 normal to the *line of sight* to the destination.
- The command bank angle is made proportional to V_2 in order that the aircraft's heading rate be driven to zero when flying along the desired great circle.
- If V_a is airspeed, the heading rate and the command bank angle are

$$\dot{H} = (g / V_a) \tan \phi$$

$$\phi_c = K_1 V_2 + K_2 \dot{V}_2$$

Direct Steering

- The normal to the great circle plane connecting present position $\mathbf{R}_3$ to the destination $\mathbf{R}_2$ is defined by the unit vector:

$$\hat{\mathbf{u}} = \frac{\mathbf{R}_2 \times \mathbf{R}_3}{\left|\mathbf{R}_2 \times \mathbf{R}_3\right|}$$

- The lateral speed V_2 is the magnitude of the dot product of the aircraft's velocity with this unit vector.

Airway Steering

- The steering algorithm calculates a great circle from the takeoff point (or from a waypoint) to the destination (or to another waypoint).
- The aircraft is steered along this great circle by calculating the lateral deviation L from the desired great circle and commanding a bank angle:

$$\phi_c = K_1 L + K_2 \dot{L} + K_3 \int L dt$$

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