#4.

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+\lambda\sigma(x)y(x)=0$$

$$\Rightarrow P(x)\frac{d^2y}{dx^2}+\frac{dP(x)}{dx}\frac{dy}{dx}+\lambda\sigma(x)y(x)=0$$

$$\Rightarrow x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{\lambda}{x}y(x)=0$$

$$\Rightarrow P(x)=x,\ \frac{dP}{dx}=1,\ \sigma(x)=\frac{1}{x}$$

$$\lambda=\frac{\int_1^2 x\left(\frac{dy}{dx}\right)^2dx}{\int_1^2\frac{1}{x}y^2(x)dx}$$

$$y_A(x)=(x-1)(2-x)=-x^2+3x-2$$

$$\frac{dy_A}{dx}=-2x+3$$

$$\tilde{\lambda}_1=\frac{\int_1^2 x(-2x+3)^2dx}{\int_1^2\frac{1}{x}(-x^2+3x-2)^2dx}$$

$$=\frac{1/2}{\ln 16-\frac{11}{4}}=\underline{22.1349}\ \#$$

Aside:

$$\int_1^2 x(-2x+3)^2dx=\int_1^2 4x^3-12x^2+9x\,dx$$

$$=x^4-4x^3+\frac{9}{2}x^2\Big|_1^2$$

$$=(16-32+18)-\left(1-4+\frac{9}{2}\right)=\frac{1}{2}$$

$$\int_1^2\frac{1}{x}(-x^2+3x-2)^2dx$$

$$=\int_1^2 x^3-6x^2+13x-12+\frac{4}{x}\,dx$$

$$=\frac{1}{4}x^4-2x^3+\frac{13}{2}x^2-12x+4\ln x\Big|_1^2$$

$$=(4-16+26-24+\ln 16)-\left(\frac{1}{4}-2+\frac{13}{2}-12+0\right)$$

$$=\frac{-11}{4}+\ln 16$$

Let $z=\ln x$, $\frac{dz}{dx}=\frac{1}{x}$

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{\lambda}{x}y=0$$

$$\Rightarrow x\left(\frac{1}{x^2}\frac{d^2y}{dz^2}-\frac{1}{x^2}\frac{dy}{dz}\right)+\frac{1}{x}\frac{dy}{dz}+\frac{\lambda}{x}y=0$$

$$\Rightarrow \frac{1}{x}\frac{d^2y}{dz^2}-\cancel{\frac{1}{x}\frac{dy}{dz}}+\cancel{\frac{1}{x}\frac{dy}{dz}}+\frac{\lambda}{x}y=0$$

$$\Rightarrow \frac{1}{x}\frac{d^2y}{dz^2}+\frac{\lambda}{x}y=0$$

$x\cdot$
$$\Rightarrow \frac{d^2y}{dz^2}+\lambda y=0$$

$$y(z)=c_1\cos\sqrt{\lambda}z+c_2\sin\sqrt{\lambda}z$$

$$\Rightarrow y(x)=c_1\cos(\sqrt{\lambda}\ln x)+c_2\sin(\sqrt{\lambda}\ln x)$$

Aside:

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$=\frac{-1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$=\frac{1}{x^2}\frac{d^2y}{dz^2}-\frac{1}{x^2}\frac{dy}{dz}$$

from B.C.

$$y(1) = C_1 \cos(\sqrt{\lambda}\cdot 0) + \cancelto{0}{C_2 \sin(\sqrt{\lambda}\cdot 0)} = 0 \quad \Rightarrow C_1 = 0$$

$$y(2) = C_2 \sin(\sqrt{\lambda}\,\ln 2) = 0 \quad \Rightarrow \quad \sin(\sqrt{\lambda}\,\ln 2) = 0$$

$$\Rightarrow \sqrt{\lambda}\,\ln 2 = n\pi$$

$$\Rightarrow \lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2, \quad n = 1, 2, 3, \cdots$$

$$\lambda_1 = \left(\frac{\pi}{\ln 2}\right)^2 = \underline{20.5423} \ \#$$