Describing the System and Evaluating Its Performance

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- The objective of optimal control theory is *to determine the control signals* that will cause a process to satisfy the *physical constraints* and at the same time minimize (or maximize) some *performance criterion*.

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Problem Formulation – The Mathematical Model

If

$$x_1(t), x_2(t), \ldots, x_n(t)$$

are the state variables (or states) of the process at time t, and

$$u_1(t), u_2(t), \ldots, u_m(t)$$

are the control inputs to the process at time t, then the system may be described by n first-order differential equations

$$\dot{x}_1(t) = a_1(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t)$$
$$\dot{x}_2(t) = a_2(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t)$$
$$\vdots$$
$$\dot{x}_n(t) = a_n(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t)$$

(1.1-1)

We shall define

$$\mathbf{x}(t) \equiv \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_n(t) \end{bmatrix}^T$$

as the *state vector* of the system, and

$$\mathbf{u}(t) \equiv \begin{bmatrix} u_1(t) & u_2(t) & \dots & u_m(t) \end{bmatrix}^T$$

as the *control vector*. The *state equations* of the system can then be written

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) \qquad (1.1\text{-}1a)$$

Let the system be described by Eq. (1.1-1a) for

$t \in [t_0, t_f]$.

Definition 1-1

A history of control input values during the interval $[t_0, t_f]$ is denoted by $\mathbf{u}$ and is called a *control history*, or simply a *control*.

Definition 1-2

A history of state values in the interval $[t_0, t_f]$ is called a *state trajectory* and is denoted by $\mathbf{x}$.

Example 1.1-1. A car is to be driven in a straight line away from point O. The distance of the car from O at time t is denoted by $d(t)$. To simplify the model, let us approximate the car by a unit point mass that can be accelerated by using the throttle or decelerated by using the brake. The differential equation is

$$\ddot{d}(t) = \alpha(t) + \beta(t)$$

where the control α is throttle acceleration and β is braking deceleration.

Selecting position and velocity as state variables, that is, $x_1(t) = d(t)$ and $x_2(t) = \dot{d}(t)$,

and letting

$u_1(t) = \alpha(t)$ and $u_2(t) = \beta(t)$,

we find the state equations become

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u_1(t) + u_2(t). \end{aligned} \qquad (1.1\text{-}3)$$

Problem Formulation – Physical Constraints

Example 1.1-2. Consider the problem of driving the car in Example 1.1-1 between points O and e. Assume that the car starts from rest and stops upon reaching point e.

First let us define the *state constraints*. If t_0 is the time of leaving O, and t_f is he time of arrival at e, then, clearly,

$$x_1(t_0) = 0 \quad \text{and} \quad x_1(t_f) = e. \tag{1.1-4}$$

In addition, since the automobile starts from rest and stop at e,

$$x_2(t_0) = 0 \quad \text{and} \quad x_2(t_f) = 0. \tag{1.1-5}$$

In matrix notation these *boundary conditions* are

$$\mathbf{x}(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0} \quad \text{and} \quad \mathbf{x}(t_f) = \begin{bmatrix} e \\ 0 \end{bmatrix} \tag{1.1-6}$$

If we assume that the car does not back up, then the additional constraints

$$0 \le x_1(t) \le e, \quad \text{and} \quad 0 \le x_2(t) \tag{1.1-7}$$

are also imposed.

Since the acceleration is bounded by some upper limit which depends on the capability of the engine, and the maximum deceleration is limited by the braking system parameters.

If the maximum acceleration is $M_1 > 0$, and the maximum deceleration is $M_2 > 0$, then the *control constraints* can be expressed as

$$\begin{aligned} 0 \le u_1(t) \le M_1 \\ -M_2 \le u_2(t) \le 0. \end{aligned} \qquad (1.1\text{-}8)$$

In addition, if the car starts with G gallons of gas and these are no service stations on the way, another constraint is

$$\int_{t_0}^{t_f} [k_1 u_1(t) + k_2 x_2(t)]dt \le G \qquad (1.1\text{-}9)$$

which assumes that the rate of gas consumption is proportional to both acceleration and speed with constants of proportionality k_1 and k_2.

<u>Definition 1-3</u>

A control history which satisfies the control constraints during the entire time interval $[t_0, t_f]$ is called an *admissible control*.

We denote the set of admissible controls by $\mathbf{U}$, and the notation $\mathbf{u} \in \mathbf{U}$ means that the control history $\mathbf{u}$ is admissible.

Definition 1-4

A state trajectory which satisfies the state variable constraints during the entire time interval $[t_0, t_f]$ is called an *admissible trajectory*.

The set of admissible trajectories by $\mathbf{X}$, and the notation $\mathbf{x} \in \mathbf{X}$ means that the trajectory $\mathbf{x}$ is admissible.

Problem Formulation – The Performance Measure

Assumed that the performance of a system is evaluated by a measure of the form

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\,dt, \qquad (1.1\text{-}10)$$

where h and g are scalar functions. The performance measure assigns a unique real number to each trajectory of the system.

Problem Formulation – The Optimal Control Problem Statement

Find an *admissible control* $\mathbf{u}^*$ which causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)$$

to follow an *admissible trajectory* $\mathbf{x}^*$ that minimizes the performance measure

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)dt,$$

$\mathbf{u}^*$is called an *optimal control* and $\mathbf{x}^*$ an *optimal trajectory.*

Problem Formulation – Comments on Optimal Control Problem

- The optimal control may not exists.
  - Since existence theorems are in rather short supply, we shall, in most cases, attempt to find an optimal control rather than try to prove that one exists.
- Even if an optimal control exists, it may not be unique.
- Thus, we are seeking the absolute or *global minimum* of J not merely *local minima*.
- If the objective is to maximize the performance measure, the theory we shall develop still applies because this is the same as minimizing the negative of this measure.

The Performance Measure – Minimum-Time Problems

Problem: To transfer a system from an arbitrary initial state to a specified target set S in minimum time.

The performance measure to be minimized is

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt \qquad (2.1\text{-}1)$$

The Performance Measure – Terminal Control Problems

Problem: To minimize the deviation of the final state of a system from its desired value $\mathbf{r}(t_f)$.

A possible performance measure is

$$J = \sum_{i=1}^{n} \left[x_i(t_f) - r_i(t_f) \right]^2 \qquad (2.1\text{-}2)$$

To allow greater generality, we can insert a real symmetric positive semi-definite n x n weighting matrix $\mathbf{H}$ to obtain

$$J = \left[\mathbf{x}(t_f) - \mathbf{r}(t_f) \right]^T \mathbf{H} \left[\mathbf{x}(t_f) - \mathbf{r}(t_f) \right] \qquad (2.1\text{-}3)$$

The Performance Measure – Minimum-Control-Effort Problems

Problem: To transfer a system from an arbitrary initial state to a specified target set S, with a minimum expenditure of control effort.

The performance measure to be minimized is

$$J = \int_{t_0}^{t_f} \left| u(t) \right| dt \qquad (2.1\text{-}4)$$

The Performance Measure – Tracking Problems

Problem: To maintain the system state $\mathbf{x}(\mathrm{t})$ as close as possible to the desired state $\mathbf{r}(\mathrm{t})$ in the interval $[t_0,t_\mathrm{f}]$.

The performance measure to be minimized is

$$J = \int_{t_0}^{t_f} \left[\mathbf{x}(t) - \mathbf{r}(t)\right]^T \mathbf{Q}(t)\left[\mathbf{x}(t) - \mathbf{r}(t)\right]dt \qquad (2.1\text{-}5)$$

where $\mathbf{Q}(t)$ is a real symmetric n x n matrix that is positive semi-definite for all $t \in [t_0, t_f]$

The Performance Measure – Regulator Problems

A regulator problem is the special case of a tracking problem which results when the desired state values are zero.
($\mathbf{r}(t) = 0$ for all $t \in [t_0, t_f]$)

References

- Donald E Kirk, *Optimal Control Theory An Introduction*, Prentice-Hall, 1970.
- Frank L. Lewis and Vassilis L. Syrmos, *Optimal Control*, 2nd edition, John Wiely & Sons, 1995. (高立圖書代理)