3.3 Optimization

The absolute maximum value of a function is the largest value of the function on its domain, the absolute minimum value of a function is the smallest value of the function on its domain.

How to find the absolute extrema of a continuous function $f(x)$ on a close interval $[a, b]$

1. Find all critical numbers of $f(x)$ in $[a, b]$
2. Evaluate $f(x)$ at the critical numbers and at the endpoints a and b

The largest and smallest values found in step 2 will be the absolute maximum and minimum values of $f(x)$ on $[a, b]$

example 1

Find the absolute extrema values of $f(x) = x^3 - 9x^2 + 15x$ on $[0, 3]$

$$f'(x) = 3x^2 - 18x + 15 = 3(x-1)(x-5) = 0 \Longrightarrow C.N. \begin{cases} x = 1 \\ x = 5(out\ of\ range) \end{cases}$$

$$\begin{cases} f(0) = 0 \\ f(1) = 7 \\ f(3) = -9 \end{cases}$$

⇨ $f(x)$ has a absolute max at $x = 1$, and $f(1) = 7$

⇨ $f(x)$ has a absolute min at $x = 3$, and $f(3) = -9$

example 2

$$V(t) = 96t^{\frac{1}{2}} - 6t \qquad t > 0$$

$$V'(t) = 48t^{-\frac{1}{2}} - 6 = 0 \Rightarrow t = 64$$

$$V''(t) = -24t^{-\frac{3}{2}},\ V''(64) = -24(64)^{-\frac{3}{2}} < 0$$

⇨ $V(t)$ has absolute maximum at $t = 64$, and $V(64) = 384$

Application : maximizing profit

$p(x) = Price\ function$

$C(x) = Cost\ function$

$R(x) = Revenue\ function$

$P(x) = Profit\ function$

Revenue $=$ (unit price) $\cdot$ $(quantity)$

Profit $=$ Revenue $-$ $Lost$

example 3

$p(x) = 22000 - 70x$

$R(x) = p \cdot x = (22000 - 70x) \cdot x = 22000x - 70x^2$

$C(x) = 8000 - 20000$

$P(x) = R(x) - C(x) = -70x^2 + 14000x - 20000$

(a) $P'(x) = -140x + 14000 = 0 \Rightarrow x = 100$

$P''(x) = -140,\ P''(100) = -140 < 0$

⇨ $P(x)$ has maximum at $x = 100$

(b) Selling price

$p(100) = 22000 - 70(100) = 15000$

(c) Max profit

$P(100) = -70(100)^2 + 14000(100) - 20000 = 680000$

To maximize the profit function $P(x) = R(x) - C(x)$

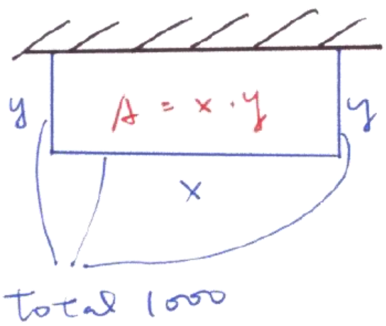
If $P(x)$ has a maximum at $x = a$, then $P'(a) = 0$

⇨ $P'(a) = R'(a) - C'(a) = 0$

⇨ $R'(a) = C'(a)$

Thus, profit is maximized at a production level for which marginal revenue equals marginal lost.

example 4



$Max\quad A = xy$

$Subject\ to\quad x + 2y = 1000$

$x = 1000 - 2y$

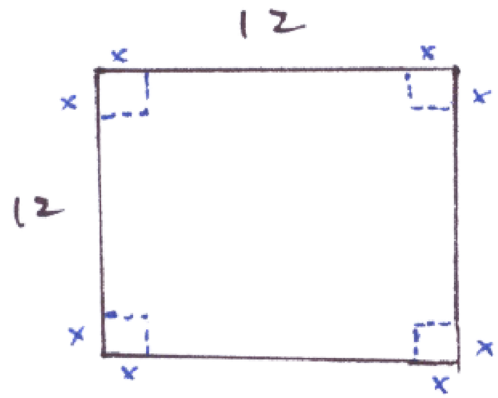
$A = xy \Rightarrow (1000 - 2y) \cdot y = 1000y - 2y^2$

$A' = 1000 - 4y = 0 \Rightarrow y = 250$

$A'' = -4,\ A''(250) = -4 < 0$

⇨ The area is indeed maximized when $x = 500,\ y = 250$ and area is 125000

example 5



The volume is

$V(x) = (12-2x)(12-2x)\cdot x$

$\quad\quad = 4x^3 - 48x^2 + 144x \quad 0 < x < 6$

$Max \quad V(x)$

$V'(x) = 12x^2 - 96x + 144$

$\quad\quad = (x-2)(x-6) = 0$

$$\Rightarrow C.N.\begin{cases} x = 2 \\ x = 6(out\ of\ range) \end{cases}$$

$V''(x) = 24x - 96,\ V'(2) = -48 < 0$

⇨ The volume is maximized at $x = 2$,

and maximum volume is 128.