MATH 0350 PSET #7

Section 5.3

1.

(a) <=> (iii)

(b) <=> (iv)

(c) <=> (ii)

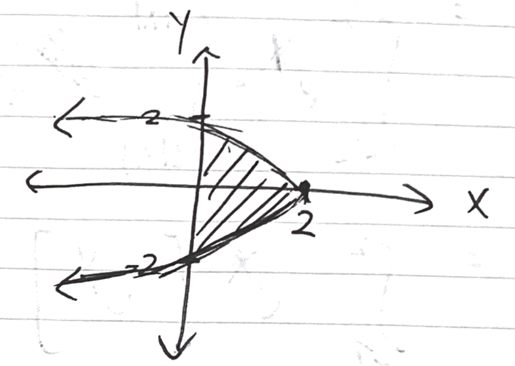
(d) <=> (i)

2. (a) $\int_{-2}^{2}\int_{0}^{4-y^2}(4-x)\,dx\,dy$

<=> $0 \le x \le 4-y^2$

<=> $-2 \le y \le 2$

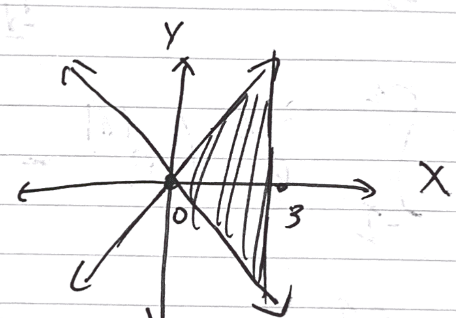
<=>



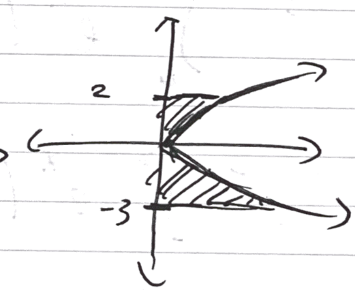
(b) $\int_{0}^{3}\int_{-x}^{x}(6+y-2x)\,dy\,dx$

<=> $-x \le y \le +x$

<=> $0 \le x \le 3$



4. (a) $\int_{-3}^{2}\int_{0}^{y^2}(x^2+y)\,dx\,dy$ <=> $0 \le x \le y^2$, $-3 \le y \le 2$ <=>



$= \int_{-3}^{2}\left[\frac{x^3}{3}+yx\right]\Big|_{0}^{y^2} dy$

$= \int_{-3}^{2}\frac{y^6}{3}+y^3\,dy = \left[\frac{y^7}{21}+\frac{y^4}{4}\right]\Big|_{-3}^{2}$

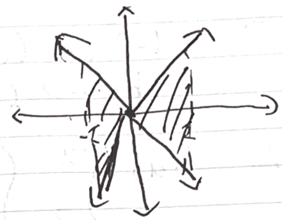
$= \frac{128}{21}+\frac{16}{4}-\left(-\frac{2187}{21}+\frac{81}{4}\right)$

$= \frac{2315}{21} - \frac{65}{4}$

4(b) $\displaystyle\int_{-1}^{1}\int_{-2|x|}^{|x|} e^{x+y}\,dy\,dx$ $\quad -2|x| \le y \le |x|$

$\Leftrightarrow -1 \le x \le 1$

$\Leftrightarrow$



$= \displaystyle\int_{-1}^{1} e^{x} \cdot \int_{-2|x|}^{|x|} e^{y}\,dy\,dx$

$= \displaystyle\int_{-1}^{1} e^{x} \left[e^{y} \right] \Big|_{-2|x|}^{|x|} dx$

$= \displaystyle\int_{-1}^{1} e^{x} \left(e^{|x|} - e^{-2|x|} \right) dx$

$= \displaystyle\int_{0}^{1} e^{x} \left(e^{|x|} - e^{-2|x|} \right) dx + \int_{-1}^{0} e^{x} \left(e^{|x|} - e^{-2|x|} \right) dx$

$= \displaystyle\int_{0}^{1} e^{x} \left(e^{x} - e^{-2x} \right) dx + \int_{-1}^{0} e^{x} \left(e^{-x} - e^{+2x} \right) dx$

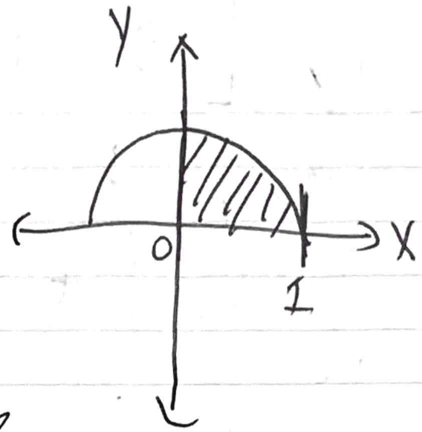
$= \displaystyle\int_{0}^{1} e^{2x} - e^{-x}\,dx + \int_{-1}^{0} 1 - e^{3x}\,dx$

$= \left[\frac{e^{2x}}{2} + e^{-x} \right] \Big|_{0}^{1} + \left[x - \frac{e^{3x}}{3} \right] \Big|_{-1}^{0}$

$= \left(\frac{e^{2}}{2} + \frac{1}{e} \right) - \left(\frac{1}{2} + 1 \right) + \left(0 - \frac{1}{3} \right) - \left(-1 - \frac{e^{-3}}{3} \right)$

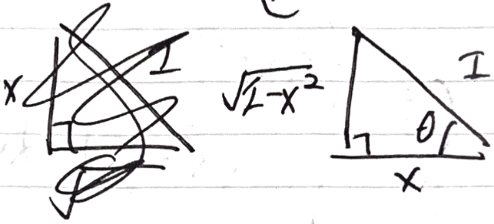
$$= \frac{-5}{6} + \frac{1}{3e^3} + \frac{1}{e} + \frac{e^2}{2}$$

4(c) $\displaystyle\int_0^1 \int_0^{(1-x^2)^{\frac{1}{2}}} dy\, dx$ <>



$$= \int_0^1 [y]\Big|_0^{(1-x^2)^{\frac{1}{2}}} dy\ dy$$

$$= \int_0^1 \sqrt{1-x^2}\, dx$$ <>



$$= \int_{\frac{\pi}{2}}^{0} -\sin^2\theta\, d\theta$$

$\frac{\sqrt{1-x^2}}{1} = \sin\theta$

$\frac{x}{1} = \cos\theta$

$dx = -\sin\theta\, d\theta$

$\theta = \cos^{-1}(x)$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1-\cos(2\theta)}{2}\, d\theta$$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

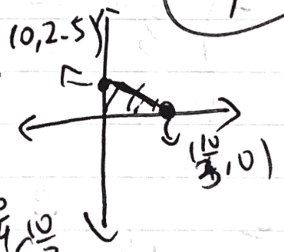
$\frac{\cos(2\theta)-1}{-2} = \sin^2\theta$

$$= \left[\frac{x}{2} - \frac{\sin(2x)}{4}\right]\Big|_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{4} - 0\right) - (0-0) = \frac{\pi}{4}$$

8 $+x, +y, 3x+4y=10$ <>

$y = \frac{-3x}{4} + \frac{10}{4}$



$$\iint_D (x^2+y^2)\, dA = \int_0^{-\frac{3}{4}x+\frac{10}{4}} \int_0^{\frac{10}{3}} (x^2+y^2)\, dx\ dy$$

$$= \int_0^{-\frac{3}{4}x+\frac{10}{4}} \left[\frac{x^3}{3} + y^2 x \right] \Bigg|_0^{\frac{10}{3}} dy$$

$$= \int_0^{-\frac{3}{4}x+\frac{10}{4}} \frac{1}{3}\left(\frac{10}{3}\right)^3 + y^2 \cdot \frac{10}{3} - (0+0) \; dy$$

$$= \int_0^{\frac{10}{3}} \int_0^{-\frac{3}{4}x+\frac{10}{4}} (x^2+y^2) \, dy \; dx$$

$$= \int_0^{\frac{10}{3}} \left[\frac{y^3}{3} + x^2 y \right] \Bigg|_0^{-\frac{3}{4}x+\frac{10}{4}} dx$$

$$= \int_0^{\frac{10}{3}} \frac{\left(-\frac{3}{4}x+\frac{10}{4}\right)^3}{3} + x^2 \cdot \left(-\frac{3}{4}x+\frac{10}{4}\right) \; dx$$

~~let~~ $$= \int_0^{\frac{10}{3}} \frac{\left(-\frac{3}{4}x+\frac{10}{4}\right)^3}{3} dx + \int_0^{\frac{10}{3}} x^2\left(-\frac{3}{4}x+\frac{10}{4}\right) dx$$

let $-\frac{3}{4}x + \frac{10}{4} = u$

$-\frac{3}{4}dx = du$

$dx = -\frac{4}{3}du$

$$= -\int_{\frac{10}{4}}^{0} \frac{4}{3} \cdot \frac{u^3}{3} \, du + \int_0^{\frac{10}{3}} (x^2)\left(-\frac{3}{4}x + \frac{10}{4}\right) dx$$

$$= +\frac{4}{9} \int_0^{\frac{10}{4}} u^3 \, du + \int_0^{\frac{10}{3}} (x)^2 \left(-\frac{3}{4}x + \frac{10}{4}\right) dx$$

$=$ $\cancel{\frac{4}{9}}$ $\cancel{(10)^4}$ $\cancel{4^4}$

$\cancel{\frac{4}{4}}\frac{1}{4}\frac{(2.5)^4}{9} + \frac{3}{4}\int_0^{\frac{10}{3}} x^3\,dx + \frac{10}{4}\int_0^{\frac{10}{3}} x^2\,dx$

D | I

$-\frac{3}{4}$ $x+2$

$= \frac{(2.5)^4}{9} - \frac{3}{4^2}\left(\frac{10}{3}\right)^4 + \frac{10}{4}\cdot\frac{1}{3}\left(\frac{10}{3}\right)^3$

$= \frac{(2.5)^4}{9} - \frac{\cancel{3}}{\cancel{16}}\cdot\frac{\cancel{100}\cdot\cancel{100}\,25}{\cancel{9}\,3^{\cancel{4}3}} + \frac{10}{4}\cdot\frac{1}{3}\cdot\frac{10^3}{3^3}$

$= \frac{2.5^4}{9} - \frac{25^2}{27} + \frac{10^4}{4\cdot 3^4}$

$= \frac{1}{9}\left(\left(\frac{5}{2}\right)^4 - \frac{25^2}{3} + \frac{10^4}{4\cdot 3^2}\right)$

$= \frac{1}{9}\cdot\frac{15625}{144} = \frac{15625}{1296}$

10. $\int_0^1\int_0^{x^2}(x^2+xy-y^2)\,dy\,dx$

$\hookrightarrow D = \{(x,y)\in\mathbb{R}^2 : 0\le x\le 1,\ 0\le y\le x^2\}$

$= \int_0^1 \left[x^2y + \frac{1}{2}xy^2 - \frac{1}{3}y^3\right]\Big|_0^{x^2}\,dx$

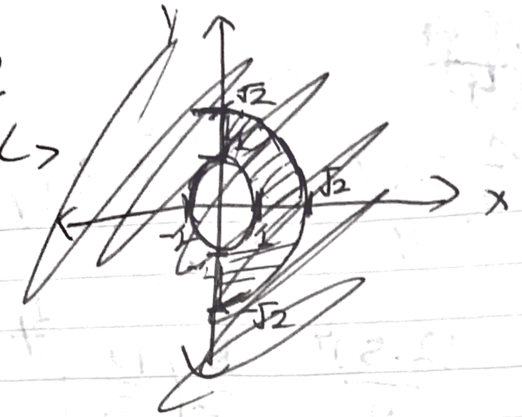
$= \int_0^1 \left[x^4 + \frac{1}{2}x^5 - \frac{1}{3}x^6\right]dx$

$= \frac{1}{5}x^5 + \frac{1}{12}x^6 - \frac{1}{21}x^7 \Big|_0^1$

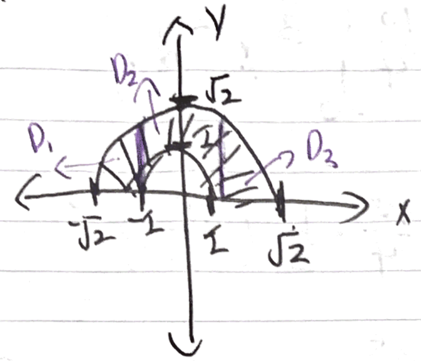
$\frac{17\cdot 21 - 60}{60\cdot 21} = \frac{297}{1260}$

$= \frac{1}{5} + \frac{1}{12} - \frac{1}{21} = \frac{17}{60} - \frac{1}{21} = \frac{33}{140}$

11. $D = \{1 \le x^2+y^2 \le 2,\ y \ge 0\} \Leftrightarrow$



$\Leftrightarrow$



$\Leftrightarrow$ $D = D_1 \cup D_2 \cup D_3$, where

$D_1 = -\sqrt{2} \le x \le -1,\quad 0 \le y \le \sqrt{2-y^2}$

$D_2 = -1 \le x \le 1,\quad \sqrt{1-x^2} \le y \le \sqrt{2-x^2}$

$D_3 : 1 \le x \le \sqrt{2},\quad 0 \le y \le \sqrt{2-x^2}$

So D is an y-simple region

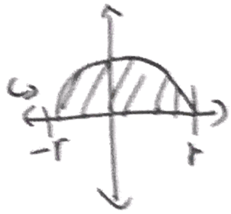
$$= \iint_D f(x,y)\,dA = \iint_{R_2} f(x,y)\,dA - \iint_{R_1} f(x,y)\,dA,$$

where R_2 is the ~~unit circle~~ upper semicircle of radius 2, R_1 is the unit upper semicircle of radius 1;

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} (1+xy)\,dy\,dx - \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (1+xy)\,dy\,dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left[y + \frac{xy^2}{2}\right]\Bigg|_0^{\sqrt{2-x^2}} dx - \int_{-1}^{1} \left[y + \frac{xy^2}{2}\right]\Bigg|_0^{\sqrt{1-x^2}} dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\sqrt{2-x^2} + \frac{x(\sqrt{2-x^2})^2}{2}\right) dx \int_{-1}^{1} \left(\sqrt{1-x^2} + \frac{x(1-x^2)}{2}\right) dx$$



$\Rightarrow \int_{-r}^{r} \sqrt{r^2-x^2}\,dx = \frac{\pi r^2}{2}$, so

$$= \frac{\pi(\sqrt{2})^2}{2} + \int_{-\sqrt{2}}^{\sqrt{2}} \frac{x\cdot(\sqrt{2-x^2})^2}{2} - \left(\frac{\pi(1)^2}{2}\right) - \int_{-1}^{1} \frac{x\cdot(\sqrt{1-x^2})^2}{2}$$

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$$= \pi + \cancel{0} - \frac{\pi}{2} - \cancel{0} = \frac{\pi}{2}$$

12. $\iint_D \cos y\,dx\,dy = \iint_D \cos y\,dy\,dx$
$\Rightarrow$

$$= \int_{\pi}^{2\pi}\int_{x}^{2x} \cos(y)\,dy\,dx$$

$$= \int_{\pi}^{2\pi} [\sin(y)]\Big|_{x}^{2x}\,dx$$

$$= \int_{\pi}^{2\pi} \sin(2x) - \sin(x)\,dx$$

$$= \left[-\frac{1}{2}\cos(2x) + \cos(x)\right]\Big|_{\pi}^{2\pi}$$

$$= -\frac{1}{2}(\cancel{1}) + 1 - \cancel{(-\frac{1}{2}} - 1)$$

$$= 2$$

Section 5.4

1(a)

$$\iint_D dx\,dy \Leftrightarrow \quad 0 \le y \le 8, \quad \frac{y}{2} \le x \le 4$$

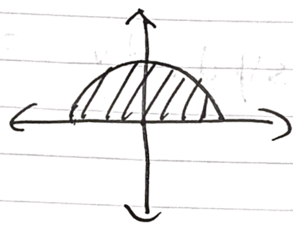
$\Leftrightarrow$ since D is simple, we can also describe the region as

$0 \le x \le 4, \quad 0 \le y \le 2x$, so

$$\int_0^8 \int_{y/2}^4 dx\,dy = \int_0^4 \int_0^{2x} dy\,dx$$

1(c)

$$\int_0^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} dx\,dy \Leftrightarrow$$



$$= \int_{-4}^{4} \left(\int_0^{\sqrt{16-x^2}} dy \right) dx$$

$-\sqrt{16-y^2} \le x \le \sqrt{16-y^2}$

$0 \le y \le 4$

$x = \sqrt{16-y^2}$,

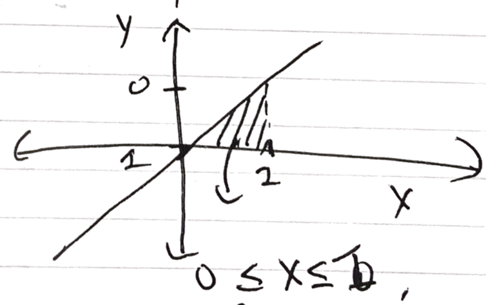
$y = \sqrt{16-x^2}$, $\Rightarrow$ $0 \le y \le \sqrt{16-x^2}$

$-4 \le x \le +4$

2.

$$\int_0^1 \int_y^1 \sin(x^2)\,dx\,dy \Leftrightarrow \quad y \le x \le 1$$

$0 \le y \le 1$



$$= \int_0^1 \left(\int_0^x \sin(x^2)\,dy \right) dx$$

$$= \int_0^1 x \cdot \sin(x^2)\,dx$$

let $u = x^2$

$\frac{1}{2}du = x\,dx$ $= \int_0^1 \frac{1}{2} \sin(u)\,du$

$0 \le x \le 1$,

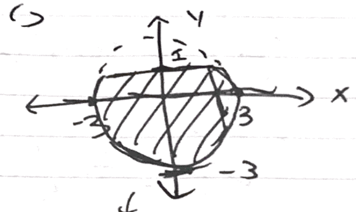
$0 \le y \le x$

$$= -\frac{1}{2}\cos(u)\Big|_0^1$$

$$= -\frac{1}{2}(\cos(1) - 1) = \frac{1-\cos(1)}{2}$$

4(b) $$\int_{-3}^{1}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2 \, dx \, dy$$

$-\sqrt{9-y^2} \le x \le \sqrt{9-y^2}$
$-3 \le y \le 1$



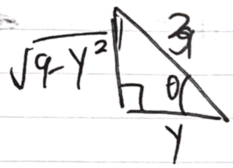
$$= \frac{1}{3}\int_{-3}^{1} 2\left(\sqrt{9-y^2}\right)^3 dy$$

$$= \frac{2}{3}\int_{-3}^{1} \cancel{\left(\sqrt{9-y^2}\right)} (9-y^2)^{\frac{3}{2}} dy$$

~~$3 \le x \le 3$~~

~~$1 \le y \le \sqrt{9-x^2} + 1$~~

$$= \frac{2}{3}\int_{-3}^{1} (9-y^2)^{\frac{3}{2}} dy$$



$$\frac{\sqrt{9-y^2}}{3} = \sin\theta$$

$$\frac{y}{3} = \cos\theta$$

$$y = 3\cos\theta$$

$$dy = -3\sin\theta \, d\theta$$

$$= \frac{2}{3}\int_{-3}^{1} (3\sin\theta)^3 \cdot (-3)(\sin\theta)\, d\theta$$

$$= 3^4 \cdot \frac{2}{3}\int_{\arccos(\frac{1}{3})}^{\pi} (\sin\theta)^4 \, d\theta$$

$\sqrt{9-y^2} = 3\sin\theta$ $\qquad \cos^{-1}\left(\frac{y}{3}\right) = \theta$

$(9-y^2)^{\frac{3}{2}} = (3\sin\theta)^3$

~~$\cos^{-1}\left(\frac{1}{9}y\right) = \theta$~~

$-3 \to \pi$
$1 \to \arccos\left(\frac{1}{3}\right)$

$$= 54 \cdot \int_{\arccos(\frac{1}{3})}^{\pi} \sin^4\theta \, d\theta$$

$$\int \sin^4\theta \, d\theta - \int \sin^4\theta \, d\theta$$

$$= \int \sin^2\theta\,(1-\sin^2\theta)\, d\theta +$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$= \int \sin^2\theta \cdot \cos^2\theta \, d\theta$$

$$= \frac{1}{4}\int 1 - \cos^2(2\theta)\, d\theta$$

$$= \frac{1}{4}\theta - \frac{1}{4}\int \frac{1+\cos(4\theta)}{2}\, d\theta$$

$$= \frac{1}{4}\theta - \frac{1}{8}\theta - \frac{1}{8}\int \cos(4\theta)\, d\theta$$

$$= \frac{1}{4}\theta - \frac{1}{8}\theta - \frac{1}{32}\sin(4\theta)$$

we know that

$$\int \sin^2\theta \, d\theta = \int \frac{1-\cos(2\theta)}{2} = \frac{1}{2}\theta - \frac{1}{4}\cdot\sin(2\theta),$$

So

$$\int \sin^4\theta \, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) - \left(\frac{1}{4}\theta - \frac{1}{8}\theta - \frac{1}{32}\sin(4\theta)\right)$$

$$= \frac{1}{32}\sin(4\theta) - \frac{1}{4}\sin(2\theta) + \frac{3}{8}\theta$$

$$= \frac{\sin(4\theta) - 8\sin(2\theta) + 12\theta}{32} + C$$

So

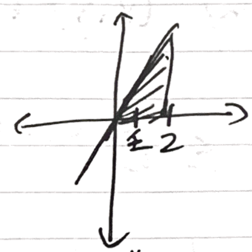
$$54\int_{\arccos(\frac{1}{3})}^{\pi} \sin^4\theta \, d\theta$$

$$= 54\left(\frac{0 - 0 + 12\pi}{32} - \left(\frac{\sin(4\arccos(\frac{1}{3})) - 8\sin(2\arccos(\frac{1}{3})) + 12\arccos(\frac{1}{3})}{32} \right) \right)$$

$$= \frac{81}{4}\pi - \frac{27}{16}\left(\sin(4\arccos(\tfrac{1}{3})) - 8\sin(2\arccos(\tfrac{1}{3})) + 12\arccos(\tfrac{1}{3})\right)$$

$$\approx 48.825$$

4(c) $\displaystyle\int_0^1 \int_{y/2}^{2} e^{x^2} \, dx \, dy$ $\rightarrow$



$$= \int_0^2 \int_0^{2x} e^{x^2} \, dy \, dx$$

$\frac{y}{2} \le x \le 2$

$0 \le y \le 1$

$\downarrow$

$0 \le y \le 2x$

$0 \le x \le 2$

$$= \int_0^2 e^{x^2} \int_0^{2x} dy \, dx$$

$$= \int_0^2 e^{x^2} \, 2x \, dx$$

let $u = x^2$

$du = 2x\,dx$

$$= \int_0^4 e^u \, du$$

$$= e^4 - 1$$

7.

$$f(x,y) = e^{\sin(x+y)}, \quad D = [-\pi, \pi] \times [-\pi, \pi]$$

Show that,

$$\frac{1}{e} \le \frac{1}{4\pi^2} \iint_D f(x,y)\, dA \le e,$$

~~By Fubini's Thm since f(x,y) is continuous over rectangle D, we have~~;

$$\frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{\sin(x+y)}\, dx\, dy,$$ notice that

$-1 \le \sin(x+y) \le 1$, for all $(x,y) \in D$, so

and $f(z) = e^z$ is an increasing function on $\mathbb{R}$, so,

for any $(x,y) \in D$;

$e^{-1} \le e^{\sin(x+y)} \le e^1$, therefore, we have

$$\frac{1}{4\pi^2} \cdot A \cdot \frac{1}{e} \le \frac{1}{4\pi^2} \iint_D f(x,y)\, dA \le \frac{A}{4\pi^2} \cdot e,$$

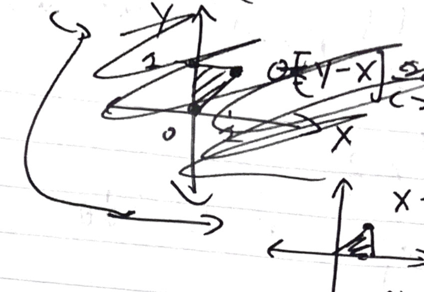
since A is $2\pi \times 2\pi = 4\pi^2$, we have

$$\underline{\frac{1}{e} \le \frac{1}{4\pi^2} \iint_D f(x,y)\, dA \le e} \text{ by}$$

<u>Mean Value Inequality</u>

10. Show

$$\frac{1}{6} \le \iint_D \frac{dA}{y-x+3} \le \frac{1}{4}$$ where D is the triangle



$0 \le [y-x]$ so ↦ 1 to 0

$x-y \mapsto 1$ to 0

$y-x \mapsto -1$ to 0

So

~~max f(x) on D ≤ ... max f~~

By Mean-Value Inequality, since

$f(z) = \frac{1}{z}$ is a decreasing function on $\mathbb{R}_{>0}$, and

for all $(x,y) \in D$, ~~we have~~ $-1 \le y-x \le 0$, or

$2 \le y-x+3 \le 3$,

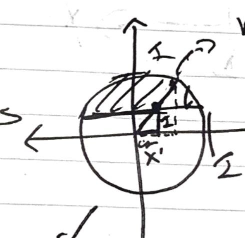
and the area of D is $\frac{1 \cdot 1}{2} = \frac{1}{2}$, so we have

$$A \cdot \frac{1}{3} \le \iint_D \frac{dA}{y-x+3} \le A \cdot \frac{1}{2}$$

, or

$$\frac{1}{6} \le \iint_D \frac{dA}{y-x+3} \le \frac{1}{4}$$

15. $$\iint_D y^3(x^2+y^2)^{-3/2}\, dx\, dy$$ ↦ D is



r

$r = 3 \sqrt{1^2 + x}$

$x' = \tan\theta$

$\frac{y}{x} = \tan\theta$

$$= \int_{+\frac{1}{2}}^{1} y^3 \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2+y^2)^{-3/2}\, dx \right) dy$$

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$

$\frac{1}{2} \le y \le 1$

$$\int_{+\frac{1}{2}}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^3(x^2+y^2)^{-3/2}\, dx\, dy$$

$x^2+y^2 = r^2$

$x = r\cos\theta$

$y = r\sin\theta$,

$dx \cdot dy = r \cdot dr \cdot d\theta$

$$= \iint r \cdot (\sin\theta)^3 \frac{1}{r^3}\, dr\, d\theta$$

$$= \iint r\sin^3\theta \, dr \, d\theta$$

$\sqrt{1-y^2}$

$\downarrow$

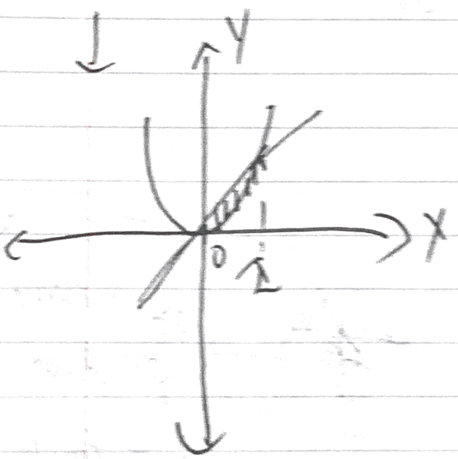
$r: [3, \text{ to } \sqrt{\cancel{\quad}}$

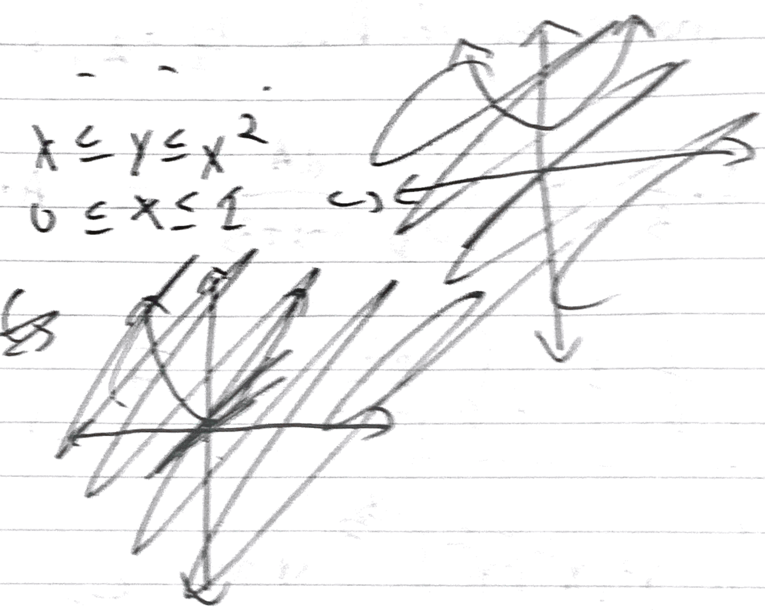
$\theta = [0, \pi]^{3 - \sqrt{1^2 + \frac{1}{\tan^2\theta}}}$

$$= \int_0^{\pi} \int_{3\sqrt{1^2+\frac{1}{\tan^2\theta}}}^{3} r \, dr \, d\theta$$

$$= \int_0^{\pi} \frac{1}{6} \cdot \left(9 - 1 + \frac{1}{\tan^2\theta}\right) dr \, d\theta$$

16. $D = \cancel{[x, x^2]}$. $x \le y \le x^2$
$0 \le x \le 1$ $\Leftrightarrow$





$\Leftrightarrow$ $0 \le y \le 1$, $y \le x \le \sqrt{y}$, so

$$\iint_D f(x,y) \, dx \, dy = \int_0^1 \left[\int_y^{\sqrt{y}} f(x,y) \, dx\right] dy$$

Section 5.5

2.

$$\iiint_W \sin(x)\,dx\,dy\,dz$$

$$= \int_0^x \int_0^1 \int_0^\pi \sin(x)\,dx\,dy\,dz \quad , \text{ since } 0 \le x \le \pi, \text{ and } 0 \le z \le x, \text{ we have } 0 \le z \le \pi, \text{ so}$$

$$= \int_0^1 \int_0^\pi \sin(x) \int_0^x dz\,dx\,dy$$

$$= \int_0^1 \int_0^\pi \sin(x)\,x\,dx\,dy$$

	D	I
+	x	$\sin(x)$
	1 −	$-\cos(x)$

$$x\cos(x) - 1 \Big|_0^\pi$$

$$\Rightarrow \pi$$

$$= \int_0^1 \pi\,dy$$

$$= \pi$$

4.

$$\int_0^1\int_0^1\int_0^1 e^{-xy}\,y\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left[\frac{-1}{\cancel{y}} \cdot e^{-xy} \cdot \cancel{y}\right]\Big|_0^1 dy\,dz$$

$$= \int_0^1\int_0^1 (-e^{-y} - 1)\,dy\,dz$$

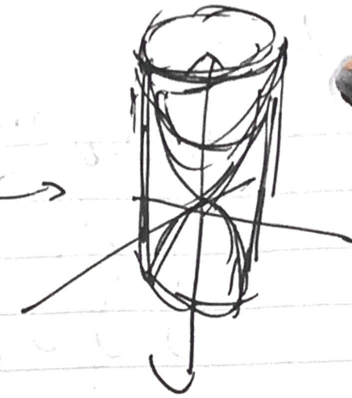
$$= \int_0^1 +e^{-1} \cancel{-1} - (1)\,dz$$

$$= \frac{1}{e}$$

7. $z = \sqrt{x^2+y^2}$, $z = x^2+y^2$,

Elementary Region

$$x^2+y^2 \le z \le \sqrt{x^2+y^2},$$
$$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2},$$
$$-1 \le y \le 1$$

$\hookrightarrow$ 

11.

$\hookrightarrow$ $$x^2+y^2 \le z \le 10-x^2-2y^2$$
$$-\sqrt{(10-2x^2)/3} \le y \le \sqrt{(10-2x^2)/3},$$
$$-\sqrt{5} \le x \le \sqrt{5}, \text{ so}$$

$\hookrightarrow$
$$\iiint_W dx\,dy\,dz = \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{(10-2x^2)/3}}^{\sqrt{\frac{(10-2x^2)}{3}}} \int_{x^2+y^2}^{10-x^2-2y^2} dz\,dy\,dx$$

$$= \int_{-\sqrt{5}}^{\sqrt{5}} \left(\int_{-\sqrt{\frac{(10-2x^2)}{3}}}^{\sqrt{\frac{10-2x^2}{3}}} (10-2x^2-3y^2)\,dy \right) dx$$

$$= \int_{-\sqrt{5}}^{\sqrt{5}} 2\left(\frac{(10-2x^2)^{3/2}}{\sqrt{3}} - \cancel{1}\,\frac{(10-2x^2)^{3/2}}{3^{3/2}} \right) dx$$

$$= \frac{2}{\sqrt{3}} \int_{-\sqrt{5}}^{\sqrt{5}} (10-2x^2)^{3/2} - \frac{1}{3}(10-2x^2)^{3/2}\,\cancel{dx}$$

$$= \frac{4}{3\sqrt{3}} \cdot \int_{-\sqrt{5}}^{\sqrt{5}} (10-2x^2)^{3/2}\, dx$$, let

$\sqrt{5}\sin(\theta) = x$

$$= \frac{4}{3\sqrt{3}} \int_{-\pi}^{\pi} (10-10\sin^2\theta)^{3/2} \cdot \sqrt{5} \cdot \cos\theta \, d\theta$$

$$= \frac{4}{3\sqrt{3}} \cdot 10^{3/2} \cdot \sqrt{5} \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(\theta)\, d\theta$$

$$= \frac{4}{3\sqrt{3}} \cdot 10^{3/2} \cdot \sqrt{5} \cdot 2 \int_{0}^{\frac{\pi}{2}} \cos^4(\theta)\, d\theta ,$$

~~Recall~~ $\cancel{\int \cos^4(\theta)\, d\theta =}$

14. $\Leftrightarrow -a \le x \le a, \quad -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2},$

$-\sqrt{a^2-y^2} \le z \le \sqrt{a^2-x^2}$

$$\Leftrightarrow \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\, dy\, d\cancel{x}$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2 \cdot \sqrt{a^2-x^2}\, dy\, dx$$

$$= \int_{-a}^{a} \left(2 \cdot \sqrt{a^2-x^2}\right)^2 dx$$

$$= 4\int_{-a}^{a} \cancel{f} (a^2-x^2)\, dx$$

$$= 4 \cdot \left(a^2 x - \frac{x^3}{3}\right)\Big|_{-a}^{a} = 4 \cdot \left(\frac{4a^3}{3}\right)$$

$$= \frac{16a^3}{3}$$

16.
$$\int_0^1\int_0^x\int_0^y (y+xz)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^x yz + x\left(\tfrac{1}{2}z^2\right)\Big|_0^y \,dy\,dx$$

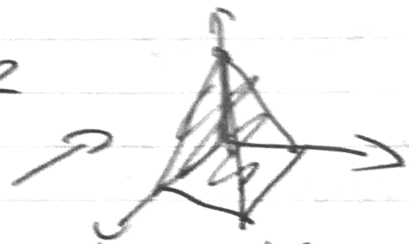
$$= \int_0^1\int_0^x y^2 + x\frac{y^2}{2}\,dy\,dx$$

$$= \int_0^1 \left(\frac{y^3}{3} + x\cdot\frac{y^3}{6}\right)\Big|_0^x \,dx$$

$$= \int_0^1 \frac{x^3}{3} + \frac{x^4}{6}\,dx$$

$$= \underline{\frac{1}{12} + \frac{1}{30}} = \underline{\frac{7}{60}}$$

22. $\iiint (x^2+y^2)\,dx\,dy\,dz$



For the Pyramid, we have

$\Rightarrow$ $0 \le z \le 1$, $0 \le y \le 1-z$, $0 \le x \le 1-z$,

$$= \int_0^1\int_0^{1-z}\int_0^{1-z} (x^2+y^2)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^{1-z} \left[\frac{x^3}{3} + y^2x\right]\Big|_0^{1-z} \,dy\,dz$$

$$= \int_0^1\int_0^{1-z} \left(\frac{(1-z)^3}{3} + (1-z)y^2\right)dy\,dz$$

$$= \int_0^1 \frac{(1-z)^4}{3} + \frac{(1-z)^4}{3}\, dz$$

$$= \frac{2}{3}\int_0^1 (1-z)^4\, dz$$

$$= \frac{2}{3}\cdot\frac{1}{5}\left|(1-z)^5\right|_0^1$$

$$= \frac{2}{3}\cdot\frac{1}{5}\cdot(1) = \frac{2}{15}$$

30.

~~ex~~. $x=0$, $y=0$, $z=0$, $x+y=1$, $z=x+y$

$y=1-x$

(a) $$\int_0^1\int_0^{1-x}\int_0^{x+y} dz\, dy\, dx$$

$$= \int_0^1\int_0^{1-x} (x+y)\, dy\, dx$$

$$= \int_0^1 \left(\frac{(1-x)^2}{2} + x(1-x)\right) dx$$

$$= \frac{1}{3} \quad ?$$

(b) $\iiint_0 x\, dx\, dy\, dz$ = (c) $\iiint_W y\, dr\, dy\, dz$