1 Introduction

In this project you will write a computer program in python to solve a system of first order differential equations, used to model a spherical symmetric explosion. Throughout the text there are **problems** labeled in bold, make sure you do all of them. Most of the text serve as a review, so if your comfortable with the material you can go straight to the problems. Each student is expected to hand in their own solutions.

2 Physics

2.1 Rankine-Hugoniot Conditions

In class we derived the Euler equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{1}$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \nabla)\vec{v} = -\nabla p \tag{2}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\,\vec{v}] = 0 \tag{3}$$

where $E = \frac{1}{2}\rho v^2 + \rho u$ is the total energy per volume, the sum of kinetic energy and internal energy per volume. Coupled with an equation of state $p = p\,(\rho, T)$ our system of equations is closed.

In astrophysical systems, the propagation of shock waves occur frequently, for example in a supernova explosion. Shock waves are abrupt, nearly discontinuous, changes in the flow. Our current form of the Euler equations cannot handle discontinuities. Therefore, we have to supplement our equations with the appropriate conditions at the shock front.

Let us idealize the surface of the discontinuity as a plane moving to the right, see Figure 1. We want to apply the Euler equations across the shock front. The problem is one dimensional so we write the Euler equations with our axis perpendicular to the shock. Further, to simplify our analysis we move from our laboratory frame to the frame where the shock is stationary. So in our new frame, the shock is at the origin separating two states of the fluid, left the post-shock gas and right the pre-shock gas.

We now have to relate our variables in the laboratory frame to the shock frame. Let us label our laboratory pre-shock values as $\hat{v_1}$, $\hat{\rho_1}$, $\hat{T_1}$ and post-shock values as $\hat{v_2}$, $\hat{\rho_2}$, $\hat{T_2}$. To transform to the laboratory frame we must subtract the shock speed v_s from both velocities, call them v_1 and v_2. Now the density $\hat{\rho}$ and temperature $\hat{T}$ are scalar quantities and are unaffected in the frame transformation (i.e. $\hat{\rho} = \rho$ and $\hat{T} = T$).

Now to relate the pre-shock and post-shock quantities we start with the

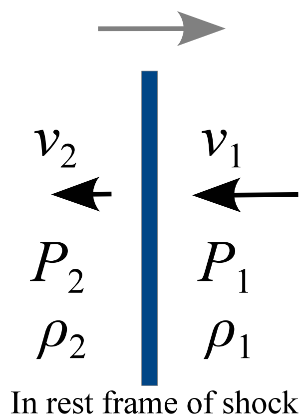

Figure 1: Diagram of a plane shock in a frame where the shock speed is zero. In the laboratory frame the shock is traveling in the right direction

continuity equation (1) and integrate it over an infinitesimal length dx,

$$\int_{-dx/2}^{dx/2} \left(\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho v) \right) dx = 0 \tag{4}$$

$$\frac{\partial}{\partial t} \int_{-dx/2}^{dx/2} \rho dx + \rho v \Big|_{-dx/2}^{dx/2} = 0. \tag{5}$$

Here we assumed the shock to be steady, non changing in the rest frame the shock, therefore all $\partial/\partial t$ are zeros and we are left with $\rho_1 v_1 = \rho_2 v_2$. Similarly, we can repeat the analysis on the momentum and energy equation. Our results are known as the Rankine-Hugoniot conditions:

$$\rho_1 v_1 = \rho_2 v_2 \tag{6}$$

$$\rho v_1^2 + p_1 = \rho v_2^2 + p_2 \tag{7}$$

$$\frac{1}{2} v_1^2 + u_1 + \frac{p_1}{\rho_1} = \frac{1}{2} v_2^2 + u_2 + \frac{p_2}{\rho_2} \tag{8}$$

These expression merely express the conservation laws. The first, the conservation of mass, the second the conversion of ram pressure to thermal pressure, and the third the conversion of kinetic energy to enthalpy.

Now that we have the Rankine-Hugoniot conditions we would like to put them in a form that would be more useful for our analysis. Specifically, we would like to have them in a form where the ratio of post-shock to pre-shock depends only on the Mach number M, ratio of the flow speed to sound speed, and the ratio of specific heats, γ. After a bit of algebra we have the following result,

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M^2}{(\gamma-1) M^2 + 2} \tag{9}$$

$$\frac{p_2}{p_1} = \frac{1+\gamma\left(2M^2-1\right)}{\gamma+1} \tag{10}$$

$$\frac{T_2}{T_1} = \frac{\left((\gamma-1) M^2+2\right)\left(1+\gamma\left(2M^2-1\right)\right)}{(\gamma+1)^2 M^2}. \tag{11}$$

Notice the behavior of equation (9) as M varies. If $M > 1$ then ρ_2/ρ_1 is greater than 1 and increases with increasing M. Thus the gas behind the shock is always compressed and becomes further compressed for a faster moving shock. When $M = 1$ then $\rho_1 = \rho_2$ and therefore the shock disappears. Further, the gas has a maximum compression value when $M \to \infty$,

$$\frac{\rho_2}{\rho_1} \to \frac{\gamma+1}{\gamma-1}. \tag{12}$$

For $\gamma = 5/3$ the maximum compression is 4, thus if a disturbance compresses the gas by a factor of 4 a shock develops moving with an infinite Mach speed.

2.2 Blast Wave

A sudden release of large amount of energy E creates a strong explosion, characterized by a strong shock wave, which progress in an ambient medium of density ρ_0. This is a natural model for supernova explosions or atomic bombs. We would like to know how fast will the shock propagate and how quantities like density or velocity vary with radius and time. We can find the behavior of the shock radius $R(t)$, if we assume that the blast wave evolves in a self-similar fashion, we will justify this in the next section. That is, the evolution of some initial configuration expands uniformly. So later times must appear as enlargement of the previous configuration. Let λ be a scale parameter characterizing the size of the blast wave at time t after the explosion. The dimensions of the problem are:

$$[E] = \frac{\mathrm{ML}^2}{\mathrm{T}^2}$$
$$[\rho_0] = \frac{\mathrm{M}}{\mathrm{L}^3}$$
$$[t] = \mathrm{T}.$$

To produce a dimension of length, the only possible combination is

$$\lambda = \left(Et^2/\rho_0\right)^{1/5}. \tag{13}$$

Since our solution is self-similar, any shell of gas with radius r has to evolve in the same way as λ. Thus we introduce a dimensionless distant parameter

$$\xi = \frac{r}{\lambda} = r\left(\frac{Et^2}{\rho_0}\right)^{1/5}. \tag{14}$$

Therefore each shell can be labeled by a particular ξ. Let ξ_s designate the shock front so that radius of the blast wave is given by

$$R(t) = \xi_s \left(\frac{Et^2}{\rho_0} \right)^{1/5}$$

The velocity of the shock front follows by time differentiation

$$v_s(t) = \frac{dR_s}{dt} = \frac{2}{5}\frac{R_s}{t} = \frac{2}{5}\xi_s \left(\frac{E}{\rho t^3} \right)^{1/5} .$$

So we have found that the shock front increases as $t^{2/5}$ and the velocity decreases as $t^{-3/5}$. Figure 2 shows confirmation of the $t^{5/2}$ dependence for the first atomic explosion in New Mexico in 1945.

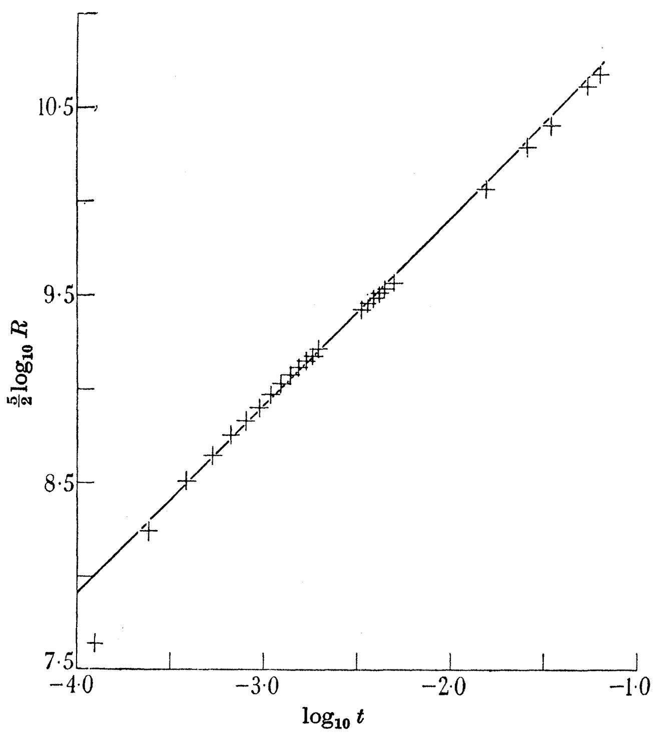

Figure 2: Plot showing the $R_s(t) \propto t^{5/2}$ relation for the first atomic explosion.

3 Mathematics

3.1 Differential Equations for the Blast Wave

We are now left to solve the Euler equations. Since the problem has spherical symmetry we will use the Euler equations in spherical coordinates dropping the polar and azimuthal terms, which are

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho v\right) = 0 \tag{15}$$

$$\rho\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right) v + \frac{\partial P}{\partial r} = 0 \tag{16}$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)\frac{P}{\rho^\gamma} = 0. \tag{17}$$

The last equation is the conservation of entropy which we replaced the energy equation.

As usual we should scale out the dimensions of the problem by defining new dimensionless functions. You should convince yourself that the following new functions are dimensionless:

$$f(x) = \rho/\rho_0 \tag{18}$$

$$g(x) = \frac{v}{(R_s/t)} \tag{19}$$

$$h(x) = \frac{P}{\rho_0 (R_s/t)^2} \tag{20}$$

$$\tag{21}$$

Our new functions $f(x)$, $g(x)$, and $h(x)$ represent the dimensionless density, velocity, and pressure respectively and are functions of the dimensionless scale length $x = r/R_s$.

Problem 1: Substitute expressions into to derive the Euler equations in term of our new functions $f(x)$, $g(x)$, and $h(x)$. Hint: you might find it easier to first work out all the derivatives first by using the chain rule and then plugging them. For example to calculate $\partial\rho/\partial t$, you would first calculate $\partial x/\partial t$,

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}\xi_0^{-1}\left(\frac{\rho_0}{E}\right)^{1/5} t^{-2/5} r = -\frac{2}{5}\frac{x}{t}, \tag{22}$$

and then use the chain rule

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial\left(\rho_0 f(x)\right)}{\partial x} = -\frac{2}{5}\frac{\rho_0 x}{t}\frac{\partial f(x)}{\partial x}. \tag{23}$$

Your final answer after should be

$$\boxed{\begin{aligned} 10fg + 5x(f'g + fg') - 2x^2 f' &= 0 \quad (24)\\ (5gg' - 2xg' - 3g)f + 5h' &= 0 \quad (25)\\ (fh' - \gamma h f')(5g - 2x) - 6hf &= 0, \quad (26)\end{aligned}}$$

where I suppressed the x dependence and the prime symbol means differentiation respect to x. Notice that in our new equations r and t have canceled out, leaving x as the only independent variable. This confirms our assumption that the solution is self-similar.

3.2 Initial Conditions

Now we just need the initial conditions to start the integration to solve for $f(x)$, $g(x)$, and $h(x)$. Lets put our coordinate system at the center of the blast wave. The blast wave expands from the origin with the shock wave as a spherical shell. Lets looks at the shock front along the positive x-axis. From this point of view the shock has a positive velocity since its flowing along the positive x-axis. If we zoom in infinitesimally, the circular shock can be treated as a plane shock, see Figure 3. We can now use the Rankine-Hugoniot conditions

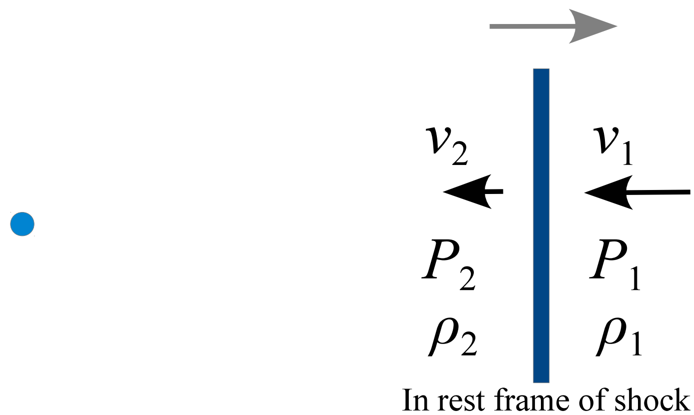

Figure 3: Diagram of a plane shock in a frame where the shock speed is zero. In the laboratory frame the shock is traveling in the right direction

on the shock. Exterior to the shock is our background values, the state of the system before the explosion. Our background values are the density $\rho_1 = \rho_0$, velocity $v_1 = 0$, and pressure P_1. Interior to the shock is our unknown values ρ_2, $v_2 = v - v_s$, and P_2. Remember these values are in the frame of the shock, specifically only the velocities are affected in the transformation. Now we make the following assumption on the structure of the explosion. We suppose it's a strong explosion, meaning we use equation (12). Now we can solve for the initial condition for $f(x)$ at the shock

$$\rho(R_s) = \rho_2 = \rho_1 f(1) = \rho_1 \frac{\gamma+1}{\gamma-1} \tag{27}$$

$$\boxed{\Rightarrow f(1) = \frac{\gamma+1}{\gamma-1}}. \tag{28}$$

You can do the same in order to find the initial conditions for $g(x)$ and $h(x)$.

Problem 2: Find the initial conditions for $g(x)$ and $h(x)$. For $g(x)$ you should use equation (6) and (9). Remember $v_1 = -v_s$, since this the initial velocity of the gas is assumed to be stationary and the velocity just behind the shock is $v_2 = v - v_s$. For $h(x)$ use equation (7) and ignoring $P_1 \approx 0$ since we are assuming a strong shock. Your answers should be

$$\boxed{\begin{aligned} g(1) &= \frac{4}{5}\frac{1}{\gamma+1} \qquad (29)\\ h(1) &= \frac{8}{25}\frac{1}{\gamma+1}. \qquad (30)\end{aligned}}$$

You may have noticed we derived our differential equations (24-26) but we have completely ignored the constant ξ_s. You can see from equation (2.2) we need ξ_s to evaluate the radius of the shock. We can solve for ξ_s by conservation of energy in the blast wave,

$$\begin{aligned} E &= \int_0^{Rs} \left(\frac{1}{2}\rho v^2 + \frac{P}{\gamma-1}\right) 4\pi r^r dr \\ &= \int_0^1 \left(\frac{1}{2}\rho_0 f(x)\left(\frac{g(x)R_s}{t}\right)^2 + \frac{h(x)\rho_0}{\gamma-1}\left(\frac{R_s}{t}\right)^2\right) 4\pi\left(xR_s\right)^2 R_s dx \\ &= \rho_0 \frac{R_s^5}{t^2}\int_0^1 \left(\frac{1}{2}f(x)g(x)^2 + \frac{h(x)}{\gamma-1}\right) 4\pi x^2 dx \\ &= \frac{\rho_0 \xi_s^5}{t^2}\left(\frac{Et^2}{\rho_0}\right)\int_0^1 \left(\frac{1}{2}f(x)g(x)^2 + \frac{h(x)}{\gamma-1}\right) 4\pi x^2 dx, \end{aligned}$$

solving for ξ_s we have

$$\boxed{\Rightarrow \xi_s = \left[4\pi\int_0^1 \left(\frac{1}{2}f(x)g(x)^2 + \frac{h(x)}{\gamma-1}\right) x^2 dx\right]^{-1/5}.} \qquad (31)$$

4 Numerical

4.1 Solving System of First Order Differential Equations

You are now ready to solve equations (24-26) with initial conditions (28-30) by the Runge-Kutta method that we have learned. Let's write our system of differential equations in vector form

$$\mathbf{r} = (f, g, h) \qquad (32)$$

$$\mathbf{f} = \left(f'(x,f,g,h), g'(x,f,g,h), h'(x,f,g,h)\right), \qquad (33)$$

so that our equations become

$$\frac{d\mathbf{r}}{dt} = \mathbf{f}. \qquad (34)$$

However our equations (24-26) are coupled, we need to solve for f', g', h' in terms of x, f, g, and h. To save you time, I have solved it for you and the result is

$$\boxed{\begin{aligned} f' &= \frac{-5f\left(fg(5g-2x)(10g-x)-30hx\right)}{x\left(5g-2x\right)\left(f(5g-2x)^2-25\gamma h\right)} && (35)\\ g' &= \frac{-30hx+3fg(5g-2x)x+50\gamma gh}{x\left(f(5g-2x)^2-25\gamma h\right)} && (36)\\ h' &= \frac{fh\left(6(5g-2x)x+5g(x-10g)\gamma\right)}{x\left(f(5g-2x)^2-25\gamma h\right)} && (37)\end{aligned}}$$

Problem 3: Integrate equations (35-37) with initial conditions (28-30) using the Runge-Kutta scheme. Remember you are integrating from the shock front to the origin. Notice that the differential equations blows up at the origin. To get around this stop the integration one step away from the origin. You should make a plot your solutions in one graph. You can do this by normalizing your solution, meaning dividing by the largest value. Your plot should look like Figure 4.

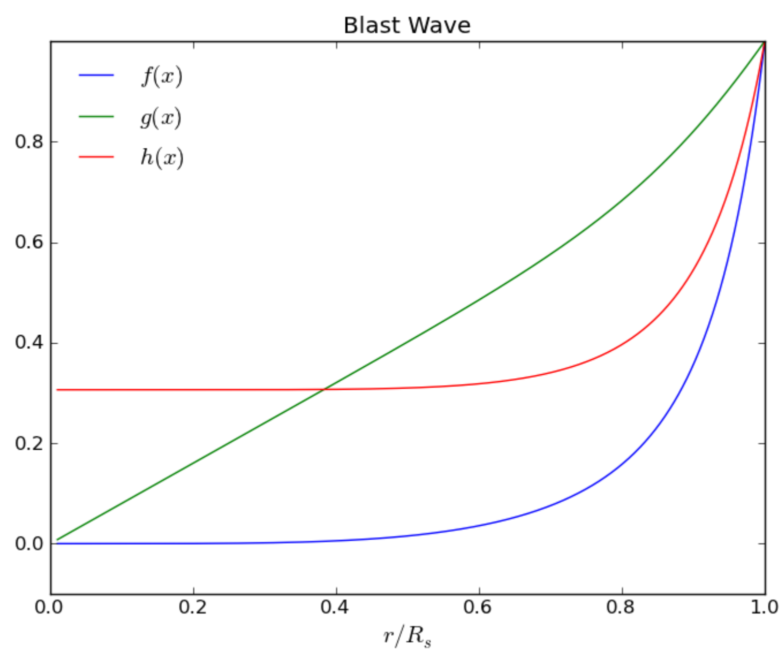

Figure 4: Solution for $f(x)$, $g(x)$, and $h(x)$

4.2 Numerical Integration

Once you solved for f, g, and h you can use them to solve for ξ_s by (31). You will have to do a numerical integration. Let's review Simpson rule of integration that we went over in class. In the Simpson rule, we approximate the area under the curve of $y = f(x)$ by a series of piecewise quadratic curves. For example,

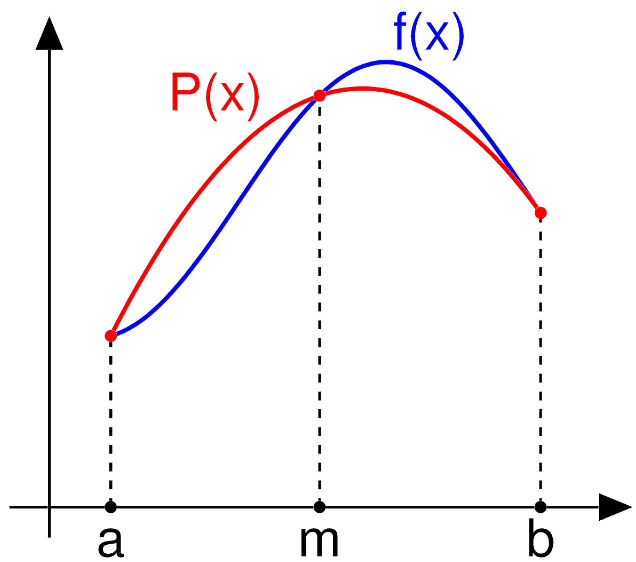

Figure 5: Plot of $f(x)$ and its quadratic fit $P(x)$

in Figure 5 we would like to calculate the integral of $f(x)$ but we have only the values of f at the locations a, m, and b equally spaced apart. The general equation of a quadratic polynomial has the form

$$A(x - x_0)^2 + B(x - x_0) + C, \tag{38}$$

where A, B, and C are constants and x_0 corresponds to shift from the origin. We set $x_0 = a$ and evaluate (38) at a, m, and b

$$\begin{aligned} f(a) &= A(a - a)^2 + B(a - a) + C \\ f(m) &= A(m - a)^2 + B(m - a) + C \\ f(b) &= A(b - a)^2 + B(b - a) + C \\ &. \end{aligned}$$

Let the spacing be called δ such that $m - a = \delta$ and $b - a = 2\delta$. Then we have

$$\begin{aligned} f(a) &= C \\ f(m) &= A\delta^2 + B\delta + C \\ f(b) &= 4A\delta^2 + 2B\delta + C \end{aligned}$$

Solving for then unknown coefficients we have

$$A = \frac{f(a) - 2f(m) + f(b)}{2\delta^2}, \; B = -\frac{f(b) - 4f(m) + 3f(a)}{2\delta}, \; C = f(a).$$

We now have a parabola that approximates $f(x)$, so now we can integrate it

$$\begin{aligned} I &= \int_a^b \left(A(x-a)^2 + B(x-a) + C \right) dx \\ &= \frac{\delta}{3} \left(f(a) + 4f(m) + f(b) \right). \end{aligned}$$

To achieve a better approximation we simply divide the domain to into many slices, repeat Simpson's rule on successive pairs and then sum up the estimates of all pairs. So the final answer is

$$\int_a^b f(x)dx \approx \frac{\delta}{3} \left[f(x_0 = a) + f(x_n = b) + 2 \sum_{j=1}^{n/2-1} f(x_{2j}) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) \right], \tag{39}$$

where n is the number of subintervals. Notice that to fit the parabola we needed two slices, thus we need an even number n of slices to perform the integration. This also implies we need an odd number of data values of $f(x)$. On a final note, equation (39) can be easily implemented in python by using slices. For example the term

$$\sum_{j=1}^{n/2} f(x_{2j-1}) = f(x_1) + f(x_3) + \cdots + f(x_{n-1}) \tag{40}$$

can be written in python as `np.sum(f[1:-1:2])`, where `f` is an numpy array holding values of f and `np.sum()` is a function that sums all the elements of an array. You can do something similarly for the last term in equation (39).

Problem 4: Use Simpson's rule to integrate equation (31) for $\gamma = 5/3$. Before doing this integral you should test your Simpson's implementation by doing integrals that you know the answer. Your answer should be $\xi_s \approx 1.15$.

5 Remaining Problems

Problem 5: In this problem you will make a series of 2d slices of the density from the blast wave that later you can turn into a movie, see Figure 6. To create a 2d density slice of your solution you will first need to specify a Cartesian grid. Pick a grid such that $x, y \in [-0.5, 0.5] \times [-0.5, 0.5]$. For simplicity set ρ_0 and E equal to one and then use equation (2.2) to find the time elapsed for the shock to reach edge of the box (i.e $R(t) = 0.5 = \xi_s(Et^2/\rho_0)^{1/5}$). Then divide

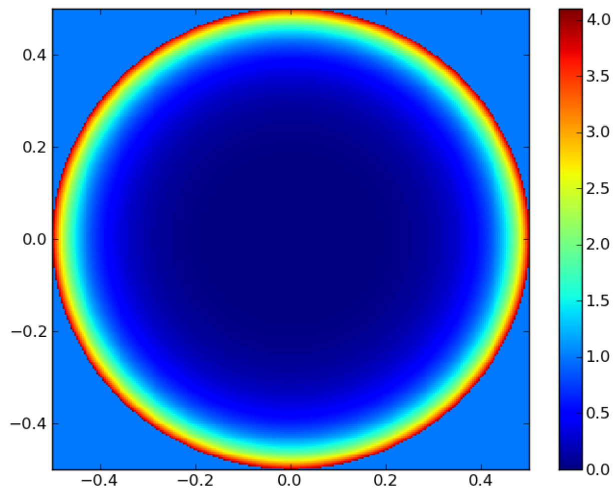

Figure 6: 2d slice of blast wave

that time into a series of time snapshots. Use $R(t)$ to scale your solution to the appropriate time slice and then map your solution to the grid of density values, $\rho(x, y)$. The mapping corresponds to going through each value of x and y and calculating its radius r from the origin and retrieve its corresponding value from $\rho(r)$ and store in a $\rho(x, y)$. You will have to handle two cases if the radius falls out of the integration regions. The first case is if r is smaller then the initial radius of integration. You can simply assign it the smallest value of $\rho(r)$. The second case is when r is larger than the final radius of integration. This case corresponds to values outside the shock which is simply ρ_0. To plot your solution you can use `pcolor` or `imshow` in pyplot. You may or may not find the following numpy functions helpful.

```
linspace(start, stop): Return evenly spaced numbers
  over a specified interval
meshgrid(x,y): Return coordinate matrices from two
  or more coordinate vectors
ndarray.flatten(): Return a copy of the array collapsed
  into one dimension
digitize(x,bins): Return indices of the bins to which
  each value in input array belongs
```

Check the online documentation for examples and further details.