Geometric Applications of Calculus

concavity: The shape of a curve as it bends; it can be concave up or concave down.

global maximum or minimum: The absolute highest or lowest value of a function over a given domain.

horizontal point of inflection: A stationary point where the concavity of the curve changes.

local maximum or minimum: a relatively high or low value of a function shown graphically as a turning point.

maximum point: A stationary point where the curve reaches a peak.

minimum point: A stationary point where the curve reaches a trough.

monotonic increasing or **decreasing:** A function that is always increasing or decreasing.

point of inflection: A point at which the curve is neither concave upwards nor downwards, but where the concavity changes.

stationary point: A point on the graph of $y = f(x)$ where the tangent is horizontal and its gradient $f'(x) = 0$. It could be a maximum point, minimum point or a horizontal point of inflection.

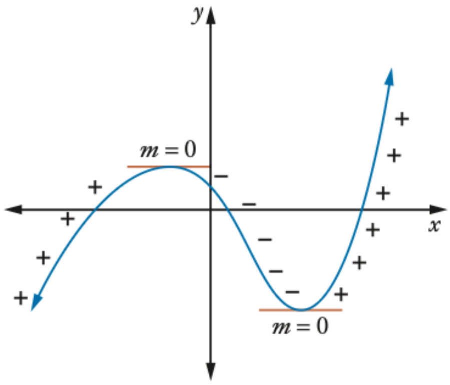
WRITING NOTES

Increasing and Decreasing Curves

A curve can be considered to be increasing or decreasing at certain points along the curve.

Increasing refers to the y-value increasing as it moves from left to right along the curve, in essence, a positive gradient.

Decreasing refers to the y-value decreasing as it moves from left to right along the curve, in essence, a negative gradient.



Sign of the first derivative

If $f'(x) > 0$, the graph of $y = f(x)$ is increasing.

If $f'(x) < 0$, the graph of $y = f(x)$ is decreasing.

If $f'(x) = 0$, the graph of $y = f(x)$ has a **stationary point**.

Sometimes a curve is **monotonic increasing** or **decreasing** (*always* increasing or decreasing).

Monotonic increasing or decreasing functions

A curve is monotonic increasing if $f'(x) > 0$ for all x.

A curve is monotonic decreasing if $f'(x) < 0$ for all x.

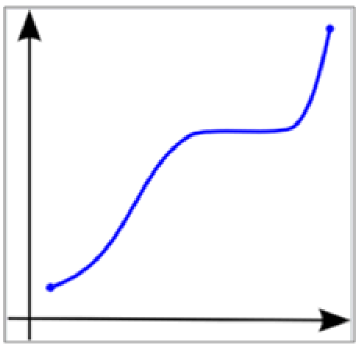


Figure 1 - A monotonically increasing function

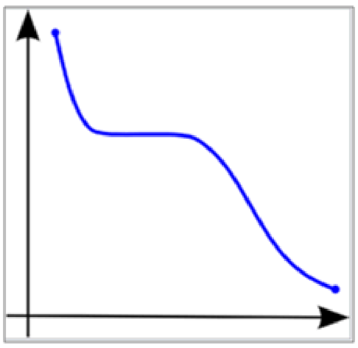


Figure 2 - A monotonically decreasing function

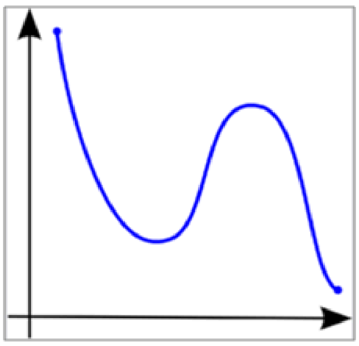


Figure 3 - A function that is not monotonic

For what x values is the function $f(x) = -2x^2 + 8x - 1$ increasing?

Sketch a curve with $\frac{dy}{dx} < 0$ for $x < 4$, $\frac{dy}{dx} = 0$ when $x = 4$ and $\frac{dy}{dx} > 0$ for $x > 4$.

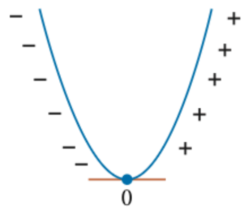
Stationary Points

A **stationary point** refers to a point where the function is near increasing or decreasing, in essence, the gradient is zero.

There are three main types:

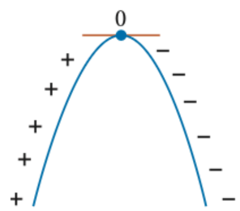
At a local **minimum point**, the curve is decreasing on the LHS and increasing on the RHS.

x	LHS	Minimum	RHS
$f'(x)$	< 0	0	> 0

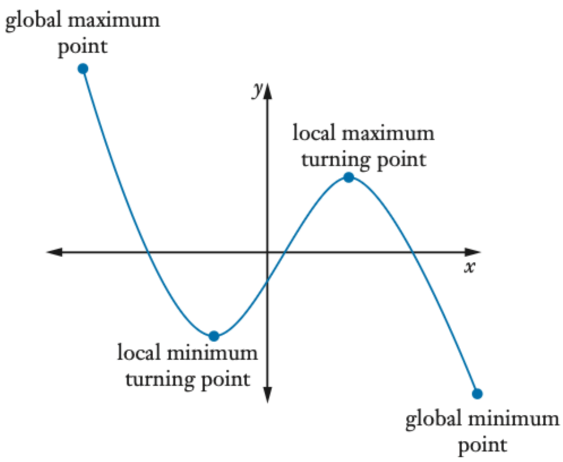


At a local **maximum point**, the curve is increasing on the LHS and decreasing on the RHS.

x	LHS	Minimum	RHS
$f'(x)$	> 0	0	< 0



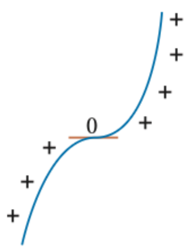
These stationary points are called **local maximum or minimum** points because they are not necessarily the **global maximum or minimum** points on the curve.

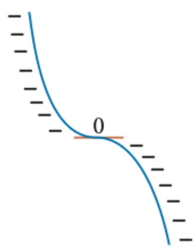


Horizontal point of inflection

These curves are increasing or decreasing on **both** sides of the horizontal **point of inflection**.
It is not a turning point since the curve does not turn around at this point.

We will learn more about points of inflection in the next section.





Find the turning point on the curve $y = 3x^2 + 6x + 1$ and determine its nature.

Find any stationary points on the curve $S = 2\pi r + \frac{120}{r}$ correct to 2 decimal places, and determine their nature.

Concavity and Points of Inflection

Concavity

Relationship between 1st and 2nd derivatives

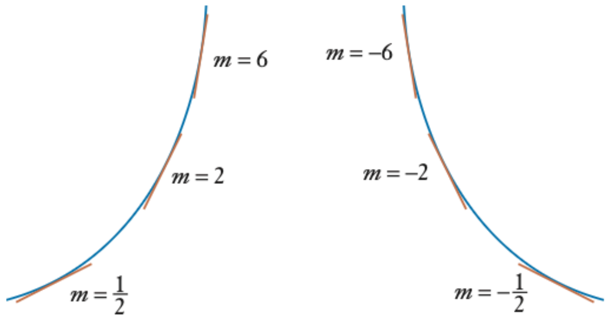
If $f''(x) > 0$ then $f'(x)$ is increasing.

If $f''(x) < 0$ then $f'(x)$ is decreasing.

If $f''(x) = 0$ then $f'(x)$ is stationary.

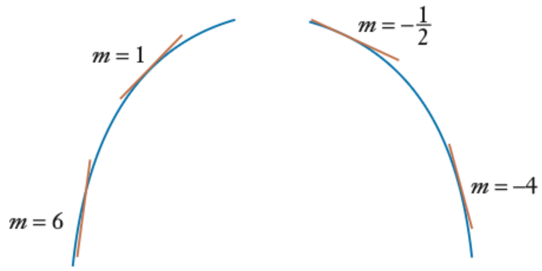
The sign of the second derivative shows the shape of the graph.

If $f''(x) > 0$ then $f'(x)$ is increasing. This means that the gradient of the tangent is increasing.



Notice the upward shape of these curves. The curve lies above the tangents. We say that the curve is **concave upwards**.

If $f''(x) < 0$ then $f'(x)$ is decreasing. This means that the gradient of the tangent is decreasing.



Notice the downward shape of these curves. The curve lies below the tangents. We say that the curve is **concave downwards**.

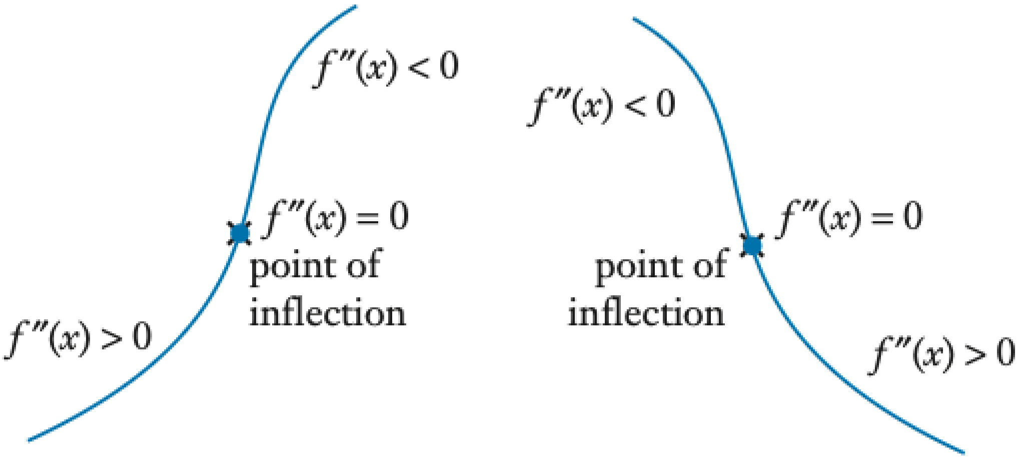
Sign of 2nd derivative

If $f''(x) > 0$, the curve is concave upwards.

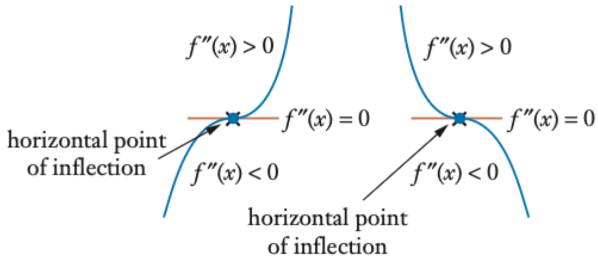
If $f''(x) < 0$, the curve is concave downwards.

Point of inflection

Refers to the point where concavity changes, a curve goes from increasing to decreasing vice versa.



The diagrams on the right show a **horizontal point of inflection** that occurs at a stationary point. Notice that the tangent is horizontal at the point of inflection.



Points of inflection

If $f''(x) = 0$, and concavity changes, it is a **point of inflection**.

If $f'(x) = 0$ also, it is a **horizontal point of inflection.**

For what values of x is the curve $y = x^3 + x^2 - 2x - 1$ concave upwards?

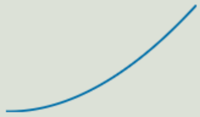
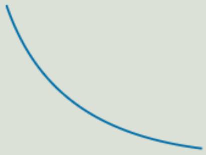
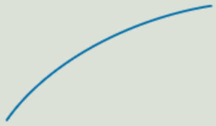
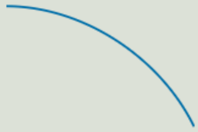
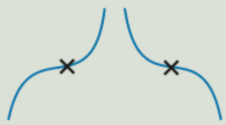
For the function $f(x) = 3x^5 - 10x^3 + 7$:

a Find any points of inflection.

b Find which of these points are horizontal points of inflection (stationary points).

Stationary Points and the Second Derivative

Shape of a curve and the derivatives

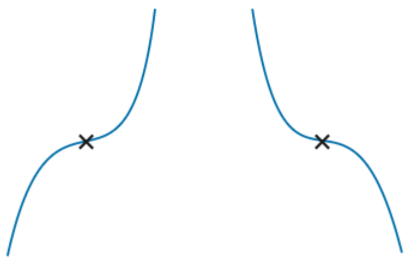
	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$
$\frac{d^2y}{dx^2} > 0$			
$\frac{d^2y}{dx^2} < 0$			
$\frac{d^2y}{dx^2} = 0$			

We can use the table to find the requirements for stationary points.

If $f'(x) = 0$ and $f''(x) > 0$, there is a minimum turning point (concave upwards).

If $f'(x) = 0$ and $f''(x) < 0$, there is a maximum turning point (concave downwards).

If $f'(x) = 0$ and $f''(x) = 0$ and concavity changes, then there is a horizontal point of inflection.



Stationary points and the derivatives

Minimum turning point: $f'(x) = 0$ and $f''(x) > 0$

Maximum turning point: $f'(x) = 0$ and $f''(x) < 0$

Horizontal point of inflection: $f'(x) = 0, f''(x) = 0$ and concavity changes

Find the stationary point on the curve $y = x^2 - 2x + 1$ and determine its nature.

The curve $y = x^3 - mx^2 + 5x - 7$ has a stationary point where $x = -1$. Find the value of m.

Curve Sketching

CHECKLIST:

Have I found:

1. ***Stationary points*** through f'(x)=0 and solving for X
2. ***Points of inflection*** through f''(x)=0, and if concavity changes.
3. Any ***intercepts***
4. ***Domain and range***
5. ***Asymptotes***

Find any stationary points and points of inflection on the curve $y = f(x) + g(x)$ where $f(x) = x^3 - 7x^2 - 1$ and $g(x) = x^2 + 4$ and hence sketch the curve.

Global Maxima and Minima

The same as curve sketching, with the inclusion of adding in max and min points at the endpoints of a domain.

Sketch the graph of $f(x) = 2x^3 + 3x^2 - 36x + 5$ for $-3 \leq x \leq 3$, showing any stationary points. Find the global maximum and minimum values of the function.

Optimisation

This combines theoretical maths into practical applications. Examples can involve:

- finding the volume of a box built from a rectangular sheet.
- Minimise the fuel used in a flight
- Maximise profit from manufacturing

Generally speaking, there will always be something to:

- Create an equation for
- Rearrange an equation
- Substitute into an equation

You will need to have a recall of every formula essentially, as questions can ask about measurement (pythagoras, DST, midpoint formula) or space (area and volume).

In the diagram to the right, $PQRS$ is a rectangle with sides $PQ = 6$ cm and $QR = 4$ cm. The side SP is extended to T, and the side SR is extended to U, so that T, Q and U are collinear. Let $PT = x$ cm and $RU = y$ cm.

a Show that $xy = 24$.

b Show that the area of $\triangle TSU$ is given by $A = 24 + 3x + \frac{48}{x}$.

c Hence find the minimum possible area of $\triangle TSU$.