Importance Sampling with Monte Carlo VaR

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Why importance sampling

Typically, there are three major ways to measure value at risk, including parameteric, variance-covariance and monte carlo approaches. In monte carlo apporach, it simluate the loss of portfolio with changes in portfolio risks.

However, the computation of simulation usually cost enormous because the VaR often happens in rare events, causing waste of time and computation. Under this circumstance, importance sampling is used to deal with the issue.

Monte Carlo VaR

notation:

- S: vector of risk factors
- ΔS: change in risk factor during the holding period
- Δt: VaR horizon(e.g. one day or two weeks)
- L: Loss in portfolio value resulting from change ΔS over Δt

goal: find $x := P(L > x_p) = p$

Basic approach: repeat following steps with multiple x

- Generate N simulations $\Delta S^{(1)}, ..., \Delta S^{(N)}$
- Calculate Losses $L^{(1)}, ..., L^{(N)}$ with simulation ΔS
- $\alpha = \frac{1}{N} \sum I(L^{(i)} > x)$

Delta-Gamma approach

In order to apply importance sampling, we have to approximate portfolio loss first, and then find corresponding distribution.

$$L \approx a_0 - \delta' \Delta S - \frac{1}{2} \Delta S' \Gamma \Delta S$$

where $a_0 = -\Theta \Delta t$
Through change in variable and some matrix algebra, we can rewrite the approximation form:

$$L \approx a_0 + b'Z + Z'\Lambda Z = a_0 + Q$$

where Z is independent normal random variables, Λ is a diagonal matrix and $b' = -\delta' C$

Importance Sampling: Introduction

As mentioned before, we have to draw VaR from rare events, so we change the distribution from which underlying variables are generated in order to generate more samples from 'important' region.

$$
\begin{aligned}
P(L > x) &= E(I(L > x)) \\
&= \int I(L > x) f(z) dz \\
&= \int I(L > x) \frac{f(z)}{g(z)} g(z) dz \\
&= \tilde{E}(I(L > x)\ell(z))
\end{aligned}
$$

Question: How to find reasonable $g(z)$?

Second moment:

$$\tilde{E}(I(L > x)^2 \ell(z)^2) = \int I(L > x)^2 \frac{f(z)}{g(z)} f(z) dz$$
$$= E(I(L > x)^2 \frac{f(z)}{g(z)})$$

choose $g(z)$ carefully to make sure variance reduction
A transform from F to F_θ is called exponential change of measure. The density of F_θ is given by $f_\theta(x) = exp(\theta x - \psi(\theta)) f(x)$. As a result, $\ell(z) = exp(-\theta Q + \psi(\theta))$
Question: How to find reasonable θ

Numerical

1. Compute C satisfy $CC' = \Sigma$ and $-0.5C'\Gamma C = \lambda$
2. find θ
3. cal $\Sigma(\theta) = (1 - 2\theta\lambda)^{-1}$ and $\mu(\theta) = \theta\Gamma(\theta)b$
4. Simulate:
    1. Generate $Z^1, ..., Z^N$ from $N(\mu(\theta), \Sigma(\theta))$
    2. cal $\Delta S^i = CZ^i$
    3. cal L^i from ΔS^i and Q^i from Z^i
    4. return estimate:

$$\frac{1}{N}\sum exp(-\theta Q + \psi(\theta))I(L^i > x)$$