Proof

- Verification of propositions by a chain of logical deductions from a set of Axioms

$$\forall n \in \mathbb{N},\ \underbrace{n^2+n+41 = \text{a prime number}}_{\text{predicate}}$$

$\forall$ for all natural number

$\mathbb{N}$: a set of natural numbers
$(0, 1, 2, 3, \ldots)$

predicate: a kind of proposition whose truth value depends on one or more variables

note: $n^2+n+41 = \text{prime} \Rightarrow$ False statement

when $n = 40$, $40^2 + 40 + 41$

$$= 40(40+1) + 40 + 1$$

$$= 41(40+1) = \boxed{41 \cdot 41} \not\text{prime}.$$

$\exists\, a, b, c, d \in \mathbb{N}^+ \quad a^4 + b^4 + c^4 = d^4$

Note $\forall$ & $\exists$ (for all, there exists): quantifier

$P \to Q$, implication

⇓

P implies Q or if P, then Q.

Implication Truth table.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

e.g. $\underbrace{\text{if pigs fly}}_{}$, then you'll be awesome.

$F = \boxed{P} \Rightarrow Q$

⇓

From the truth table above when $P = F$,

then $P \to Q$ always T

$\forall n \in \mathbb{Z} \quad (n \geq 2) \Longleftrightarrow (n^2 \geq 4)$

$P \Leftrightarrow Q$, P if and only if Q. (iff).

P	Q	$P \to Q$	$Q \to P$	$P \leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note: for $P \leftrightarrow Q$ is true, provided P & Q are both true or both false.

e.g. $(n \geq 2) \leftrightarrow (n^2 \geq 4)$

T, when $n = 1$ F $\leftrightarrow$ F

T, when $n = 3$ T $\leftrightarrow$ T

F, when $n = -3$ F $\leftrightarrow$ T

as a whole: F

Axioms

a proposition, assumed to be true because it's somehow reasonable

e.g. if $a = b$, and $b = c$, then $a = c$

axiom
- Consistent: $\emptyset$ proposition can be proved both true and false
- Complete: can be proved or disapproved

Logical deduction (inference rules)

- combine axioms and true propositions in order to form more true propositions.

One fundamental inference rule: modus ponens.

, if P is true and $P \to Q$ is true, then Q is also true

$$\begin{array}{c} P \\ P \Rightarrow Q \\ \hline Q \end{array} \quad \begin{array}{l} \big\} T \\ \\ \} T \end{array}$$

$(P \land (P \Rightarrow Q)) \Rightarrow Q$

if P and $P \to Q$ is true, then Q is true

$$(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$$

$$\frac{\begin{matrix} P \Rightarrow Q \\ Q \Rightarrow R \end{matrix}}{P \Rightarrow R}$$

$$(P \Rightarrow Q) \wedge \neg Q \Rightarrow \neg P$$

$$\frac{\begin{matrix} P \Rightarrow Q \\ \neg Q \end{matrix}}{\neg P}$$

A tautology:

$$(X \Rightarrow Y) \Leftrightarrow \underbrace{(\neg Y \Rightarrow \neg X)}_{\text{contrapositive of } X \Rightarrow Y}.$$

- an implication is true if and only if its contrapositive is true.

e.g "if you are wise, then you attend recitation"

==

"if you don't attend recitation, then you're $\not\!$ wise"

Proof $(X \Rightarrow Y) \Leftrightarrow (\neg Y \Rightarrow \neg X)$

X	Y	$X \Rightarrow Y$	$\neg Y \Rightarrow \neg X$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

, showing: the left side is logically equivalent to the right side for every setting of the variables X & Y.

thus,

$(X \Rightarrow Y) \Leftrightarrow (\neg Y \Rightarrow \neg X)$ is true in every case (tautology)

Inference rules

$$\frac{P \to Q}{\neg Q \to \neg P}$$

$$\frac{\neg Q \to \neg P}{P \to Q}.$$

A Proof by contradiction. (indirect proof).

idea: to assume that the desired conclusion is false and then show that that assumption leads to an absurdity or contradiction.

, meaning: the assumption = wrong, so the desired conclusion is actually true

$$\frac{\neg p \Rightarrow \text{False.}}{p}$$

if $\neg p$ implies some falsehood, p must actually be true.

p	$(\neg p \Rightarrow \text{False}) \Rightarrow p$
T	T
F	T

$\sqrt{2}$ is an irrational number.

assume $\sqrt{2}$ is rational.

$\sqrt{2} = a/b.$, a & b integers & $b \neq 0$.

$2 = \frac{a^2}{b^2}$ ⇒ the fraction is in lowest form

$2b^2 = a^2$. : a is even, a is a multiple of 2.

⇓

a^2 is a multiple of 4.

because of the equality.

$2b^2 = a^2$, $2b^2$ must be a multiple of 4.

⇓

b^2 is even

& a & b are both even

, so the fraction $a/b \neq$ in lowest terms

so the assumption $\sqrt{2}$ is rational: F (False)