PHYSICS 5300 Final Project

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Physical Problem: Double Spherical Pendulum

Describing system using Cartesian coordinates:

$$\begin{aligned}
x_1 &= R_1 \sin\theta_1 \cos\phi_1 \\
y_1 &= R_1 \sin\theta_1 \sin\phi_1 \\
z_1 &= R_1 \cos\theta_1 \\
x_2 &= R_1 \sin\theta_1 \cos\phi_1 + R_2 \sin\theta_2 \cos\phi_2 \\
y_2 &= R_1 \sin\theta_1 \sin\phi_1 + R_2 \sin\theta_2 \sin\phi_2 \\
z_2 &= R_1 \cos\theta_1 + R_2 \cos\theta_2
\end{aligned}$$

Velocities:

$$\begin{aligned}
\dot{x}_1 &= R_1(\dot{\theta}_1 \cos\theta_1 \cos\phi_1 - \dot{\phi}_1 \sin\theta_1 \sin\phi_1) \\
\dot{y}_1 &= R_1(\dot{\theta}_1 \cos\theta_1 \sin\phi_1 + \dot{\phi}_1 \sin\theta_1 \cos\phi_1) \\
\dot{z}_1 &= -R_1\dot{\theta}_1 \sin\theta_1 \\
\dot{x}_2 &= R_1(\dot{\theta}_1 \cos\theta_1 \cos\phi_1 - \dot{\phi}_1 \sin\theta_1 \sin\phi_1) + R_2(\dot{\theta}_2 \cos\theta_2 \cos\phi_2 - \dot{\phi}_2 \sin\theta_2 \sin\phi_2) \\
\dot{y}_2 &= R_1(\dot{\theta}_1 \cos\theta_1 \sin\phi_1 + \dot{\phi}_1 \sin\theta_1 \cos\phi_1) + R_2(\dot{\theta}_2 \cos\theta_2 \sin\phi_2 + \dot{\phi}_2 \sin\theta_2 \cos\phi_2) \\
\dot{z}_2 &= -R_1\dot{\theta}_1 \sin\theta_1 - R_2\dot{\theta}_2 \sin\theta_2
\end{aligned}$$

Velocities squared:

$$\begin{aligned}
\dot{x}_1^2 &= R_1^2(\dot{\theta}_1^2 \cos^2\theta_1 \cos^2\phi_1 - 2\dot{\theta}_1\dot{\phi}_1 \cos\theta_1 \cos\phi_1 \sin\theta_1 \sin\phi_1 + \dot{\phi}_1^2 \sin^2\theta_1 \sin^2\phi_1) \\
\dot{y}_1^2 &= R_1^2(\dot{\theta}_1^2 \cos^2\theta_1 \sin^2\phi_1 + 2\dot{\theta}_1\dot{\phi}_1 \cos\theta_1 \sin\phi_1 \sin\theta_1 \cos\phi_1 + \dot{\phi}_1^2 \sin^2\theta_1 \cos^2\phi_1) \\
\dot{z}_1^2 &= R_1^2\dot{\theta}_1^2 \sin^2\theta_1 \\
\dot{x}_2^2 &= R_1^2(\dot{\theta}_1^2 \cos^2\theta_1 \cos^2\phi_1 - 2\dot{\theta}_1\dot{\phi}_1 \cos\theta_1 \cos\phi_1 \sin\theta_1 \sin\phi_1 + \dot{\phi}_1^2 \sin^2\theta_1 \sin^2\phi_1) \\
&\quad + 2R_1R_2(\dot{\theta}_1\dot{\theta}_2 \cos\theta_1 \cos\theta_2 \cos\phi_1 \cos\phi_2 - \dot{\theta}_1\dot{\phi}_2 \cos\theta_1 \cos\phi_1 \sin\theta_2 \sin\phi_2 \\
&\quad - \dot{\phi}_1\dot{\theta}_2 \sin\theta_1 \sin\phi_1 \cos\theta_2 \cos\phi_2 + \dot{\phi}_1\dot{\phi}_2 \sin\theta_1 \sin\phi_1 \sin\theta_2 \sin\phi_2) \\
&\quad + R_2^2(\dot{\theta}_2^2 \cos^2\theta_2 \cos^2\phi_2 - 2\dot{\theta}_2\dot{\phi}_2 \cos\theta_2 \cos\phi_2 \sin\theta_2 \sin\phi_2 + \dot{\phi}_2^2 \sin^2\theta_2 \sin^2\phi_2) \\
\dot{y}_2^2 &= R_1^2(\dot{\theta}_1^2 \cos^2\theta_1 \sin^2\phi_1 + 2\dot{\theta}_1\dot{\phi}_1 \cos\theta_1 \sin\phi_1 \sin\theta_1 \cos\phi_1 + \dot{\phi}_1^2 \sin^2\theta_1 \cos^2\phi_1) \\
&\quad + 2R_1R_2(\dot{\theta}_1\dot{\theta}_2 \cos\theta_1 \sin\phi_1 \cos\theta_2 \sin\phi_2 + \dot{\theta}_1\dot{\phi}_2 \cos\theta_1 \sin\phi_1 \sin\theta_2 \cos\phi_2 \\
&\quad + \dot{\phi}_1\dot{\theta}_2 \sin\theta_1 \cos\phi_1 \cos\theta_2 \sin\phi_2 + \dot{\phi}_1\dot{\phi}_2 \sin\theta_1 \cos\phi_1 \sin\theta_2 \cos\phi_2) \\
&\quad + R_2^2(\dot{\theta}_2^2 \cos^2\theta_2 \sin^2\phi_2 + 2\dot{\theta}_2\dot{\phi}_2 \cos\theta_2 \sin\phi_2 \sin\theta_2 \cos\phi_2 + \dot{\phi}_2^2 \sin^2\theta_2 \cos^2\phi_2) \\
\dot{z}_2^2 &= R_1^2\dot{\theta}_1^2 \sin^2\theta_1 + 2R_1R_2\dot{\theta}_1\dot{\theta}_2 \sin\theta_1 \sin\theta_2 + R_2^2\dot{\theta}_2^2 \sin^2\theta_2
\end{aligned}$$

Lagrangian:

$$\mathcal{L} = \frac{1}{2}m_1\Big[R_1^2(\dot\theta_1^2\cos^2\theta_1\cos^2\phi_1 - 2\dot\theta_1\dot\phi_1\cos\theta_1\cos\phi_1\sin\theta_1\sin\phi_1 + \dot\phi_1^2\sin^2\theta_1\sin^2\phi_1)$$
$$+ R_1^2(\dot\theta_1^2\cos^2\theta_1\sin^2\phi_1 + 2\dot\theta_1\dot\phi_1\cos\theta_1\sin\phi_1\sin\theta_1\cos\phi_1 + \dot\phi_1^2\sin^2\theta_1\cos^2\phi_1) + R_1^2\dot\theta_1^2\sin^2\theta_1\Big]$$
$$+ \frac{1}{2}m_2\Big[R_1^2(\dot\theta_1^2\cos^2\theta_1\cos^2\phi_1 - 2\dot\theta_1\dot\phi_1\cos\theta_1\cos\phi_1\sin\theta_1\sin\phi_1 + \dot\phi_1^2\sin^2\theta_1\sin^2\phi_1)$$
$$+ 2R_1R_2(\dot\theta_1\dot\theta_2\cos\theta_1\cos\theta_2\cos\phi_1\cos\phi_2 - \dot\theta_1\dot\phi_2\cos\theta_1\cos\phi_1\sin\theta_2\sin\phi_2$$
$$- \dot\phi_1\dot\theta_2\sin\theta_1\sin\phi_1\cos\theta_2\cos\phi_2 + \dot\phi_1\dot\phi_2\sin\theta_1\sin\phi_1\sin\theta_2\sin\phi_2)$$
$$+ R_2^2(\dot\theta_2^2\cos^2\theta_2\cos^2\phi_2 - 2\dot\theta_2\dot\phi_2\cos\theta_2\cos\phi_2\sin\theta_2\sin\phi_2 + \dot\phi_2^2\sin^2\theta_2\sin^2\phi_2)$$
$$+ R_1^2(\dot\theta_1^2\cos^2\theta_1\sin^2\phi_1 + 2\dot\theta_1\dot\phi_1\cos\theta_1\sin\phi_1\sin\theta_1\cos\phi_1 + \dot\phi_1^2\sin^2\theta_1\cos^2\phi_1)$$
$$+ 2R_1R_2(\dot\theta_1\dot\theta_2\cos\theta_1\sin\phi_1\cos\theta_2\sin\phi_2 + \dot\theta_1\dot\phi_2\cos\theta_1\sin\phi_1\sin\theta_2\cos\phi_2$$
$$+ \dot\phi_1\dot\theta_2\sin\theta_1\cos\phi_1\cos\theta_2\sin\phi_2 + \dot\phi_1\dot\phi_2\sin\theta_1\cos\phi_1\sin\theta_2\cos\phi_2)$$
$$+ R_2^2(\dot\theta_2^2\cos^2\theta_2\sin^2\phi_2 + 2\dot\theta_2\dot\phi_2\cos\theta_2\sin\phi_2\sin\theta_2\cos\phi_2 + \dot\phi_2^2\sin^2\theta_2\cos^2\phi_2)$$
$$+ R_1^2\dot\theta_1^2\sin^2\theta_1 + 2R_1R_2\dot\theta_1\dot\theta_2\sin\theta_1\sin\theta_2 + R_2^2\dot\theta_2^2\sin^2\theta_2\Big] + m_1gR_1\cos\theta_1 + m_2g(R_1\cos\theta_1 + R_2\cos\theta_2)$$

$$\mathcal{L} = \frac{1}{2}m_1\Big[R_1^2(\dot\theta_1^2\cos^2\theta_1 + \dot\phi_1^2\sin^2\theta_1) + R_1^2\dot\theta_1^2\sin^2\theta_1\Big] + \frac{1}{2}m_2\Big[R_1^2(\dot\theta_1^2\cos^2\theta_1 + \dot\phi_1^2\sin^2\theta)$$
$$+ 2R_1R_2(\dot\theta_1\dot\theta_2\cos\theta_1\cos\theta_2\cos\phi_1\cos\phi_2 - \dot\theta_1\dot\phi_2\cos\theta_1\cos\phi_1\sin\theta_2\sin\phi_2$$
$$- \dot\phi_1\dot\theta_2\sin\theta_1\sin\phi_1\cos\theta_2\cos\phi_2 + \dot\phi_1\dot\phi_2\sin\theta_1\sin\phi_1\sin\theta_2\sin\phi_2) + R_2^2(\dot\theta_2^2\cos^2\theta_2 + \dot\phi_2^2\sin^2\theta_2)$$
$$+ 2R_1R_2(\dot\theta_1\dot\theta_2\cos\theta_1\sin\phi_1\cos\theta_2\sin\phi_2 + \dot\theta_1\dot\phi_2\cos\theta_1\sin\phi_1\sin\theta_2\cos\phi_2$$
$$+ \dot\phi_1\dot\theta_2\sin\theta_1\cos\phi_1\cos\theta_2\sin\phi_2 + \dot\phi_1\dot\phi_2\sin\theta_1\cos\phi_1\sin\theta_2\cos\phi_2) + R_1^2\dot\theta_1^2\sin^2\theta_1$$
$$+ 2R_1R_2\dot\theta_1\dot\theta_2\sin\theta_1\sin\theta_2 + R_2^2\dot\theta_2^2\sin^2\theta_2\Big] + (m_1+m_2)gR_1\cos\theta_1 + m_2gR_2\cos\theta_2$$

$$\mathcal{L} = \frac{1}{2}m_1\Big[R_1^2(\dot\theta_1^2 + \dot\phi_1^2\sin^2\theta_1)\Big] + \frac{1}{2}m_2\Big[R_1^2(\dot\theta_1^2 + \dot\phi_1^2\sin^2\theta) + 2R_1R_2\Big(\dot\theta_1\dot\theta_2\cos\theta_1\cos\theta_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2)$$
$$+ \dot\theta_1\dot\phi_2\cos\theta_1\sin\theta_2(\sin\phi_1\cos\phi_2 - \cos\phi_1\sin\phi_2) + \dot\phi_1\dot\theta_2\sin\theta_1\cos\theta_2(\cos\phi_1\sin\phi_2 - \sin\phi_1\cos\phi_2)$$
$$+ \dot\phi_1\dot\phi_2\sin\theta_1\sin\theta_2(\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2)\Big) + R_2^2(\dot\theta_2^2 + \dot\phi_2^2\sin^2\theta_2) + 2R_1R_2\dot\theta_1\dot\theta_2\sin\theta_1\sin\theta_2\Big]$$
$$+ (m_1+m_2)gR_1\cos\theta_1 + m_2gR_2\cos\theta_2$$

$$\mathcal{L} = \frac{1}{2}(m_1+m_2)\Big[R_1^2(\dot\theta_1^2 + \dot\phi_1^2\sin^2\theta_1)\Big] + \frac{1}{2}m_2\Big\{2R_1R_2\Big[\dot\theta_1\dot\theta_2\big(\cos\theta_1\cos\theta_2\cos(\phi_1-\phi_2) + \sin\theta_1\sin\theta_2\big)$$
$$+ \dot\theta_1\dot\phi_2\cos\theta_1\sin\theta_2\sin(\phi_1-\phi_2) + \dot\phi_1\dot\theta_2\sin\theta_1\cos\theta_2\sin(\phi_2-\phi_1) + \dot\phi_1\dot\phi_2\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\Big]$$
$$+ R_2^2(\dot\theta_2^2 + \dot\phi_2^2\sin^2\theta_2)\Big\} + (m_1+m_2)gR_1\cos\theta_1 + m_2gR_2\cos\theta_2$$

Equation of motion for θ_1:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta_1}\right) - \frac{\partial\mathcal{L}}{\partial\theta_1} = 0$$

$$\frac{d}{dt}\left[(m_1+m_2)R_1^2\dot\theta_1 + m_2R_1R_2\{\dot\theta_2[\cos\theta_1\cos\theta_2\cos(\phi_1-\phi_2) + \sin\theta_1\sin\theta_2] + \dot\phi_2\cos\theta_1\sin\theta_2\sin(\phi_1-\phi_2)\}\right]$$
$$-(m_1+m_2)R_1^2\dot\phi_1^2\sin\theta_1\cos\theta_1 - m_2R_1R_2\Big[\dot\theta_1\dot\theta_2\left(-\sin\theta_1\cos\theta_2\cos(\phi_1-\phi_2) + \cos\theta_1\sin\theta_2\right)$$
$$-\dot\theta_1\dot\phi_2\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2) + \dot\phi_1\dot\theta_2\cos\theta_1\cos\theta_2\sin(\phi_2-\phi_1) + \dot\phi_1\dot\phi_2\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\Big] + (m_1+m_2)gR_1\sin\theta_1 = 0$$

$$
\begin{aligned}
&(m_1+m_2)R_1^2\ddot{\theta}_1 + m_2R_1R_2\Big\{\ddot{\theta}_2\big[\cos\theta_1\cos\theta_2\cos(\phi_1-\phi_2)+\sin\theta_1\sin\theta_2\big] + \dot{\theta}_2\big[-\dot{\theta}_1\sin\theta_1\cos\theta_2\cos(\phi_1-\phi_2) \\
&-\dot{\theta}_2\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2) - \cos\theta_1\cos\theta_2\sin(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2) + \dot{\theta}_1\cos\theta_1\sin\theta_2 + \dot{\theta}_2\sin\theta_1\cos\theta_2\big] \\
&+\ddot{\phi}_2\cos\theta_1\sin\theta_2\sin(\phi_1-\phi_2) - \dot{\theta}_1\dot{\phi}_2\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2) + \dot{\theta}_2\dot{\phi}_2\cos\theta_1\cos\theta_2\sin(\phi_1-\phi_2) \\
&+\dot{\phi}_2\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2)\Big\} - (m_1+m_2)R_1^2\dot{\phi}_1^2\sin\theta_1\cos\theta_1 - m_2R_1R_2\Big[\dot{\theta}_1\dot{\theta}_2\big(-\sin\theta_1\cos\theta_2\cos(\phi_1-\phi_2) \\
&+\cos\theta_1\sin\theta_2\big) - \dot{\theta}_1\dot{\phi}_2\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2) + \dot{\phi}_1\dot{\theta}_2\cos\theta_1\cos\theta_2\sin(\phi_2-\phi_1) + \dot{\phi}_1\dot{\phi}_2\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\Big] \\
&+(m_1+m_2)gR_1\sin\theta_1 = 0
\end{aligned}
$$

$$
\begin{aligned}
&(m_1+m_2)R_1(R_1\ddot{\theta}_1 - R_1\dot{\phi}_2\sin\theta_1\cos\theta_1 + g\sin\theta_1) + m_2R_1R_2\big\{\ddot{\theta}_2\big[\cos\theta_1\cos\theta_2\cos(\phi_1-\phi_2)+\sin\theta_1\sin\theta_2\big] \\
&+\dot{\theta}_2\big[-\dot{\theta}_2\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2) - \cos\theta_1\cos\theta_2\sin(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2) + \dot{\theta}_2\sin\theta_1\cos\theta_2\big] \\
&+\ddot{\phi}_2\cos\theta_1\sin\theta_2\sin(\phi_1-\phi_2) + \dot{\theta}_2\dot{\phi}_2\cos\theta_1\cos\theta_2\sin(\phi_1-\phi_2) + \dot{\phi}_2\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2) \\
&-\dot{\phi}_1\dot{\theta}_2\cos\theta_1\cos\theta_2\sin(\phi_2-\phi_1) + \dot{\phi}_1\dot{\phi}_2\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\big\} = 0
\end{aligned}
$$

Equation of motion for θ_2:

$$
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\right) - \frac{\partial\mathcal{L}}{\partial\theta_2} = 0
$$

$$
\begin{aligned}
&\frac{d}{dt}\Big[m_2\Big\{R_1R_2\Big[\dot{\theta}_1\big(\cos\theta_1\cos\theta_2\cos(\phi_1-\phi_2)+\sin\theta_1\sin\theta_2\big) + \dot{\phi}_1\sin\theta_1\cos\theta_2\sin(\phi_2-\phi_1)\Big] + R_2^2\dot{\theta}_2\Big\}\Big] \\
&-\frac{1}{2}m_2\Big\{2R_1R_2\big[\dot{\theta}_1\dot{\theta}_2\big(-\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2)+\sin\theta_1\cos\theta_2\big) + \dot{\theta}_1\dot{\phi}_2\cos\theta_1\cos\theta_2\sin(\phi_1-\phi_2) \\
&-\dot{\phi}_1\dot{\theta}_2\sin\theta_1\sin\theta_2\sin(\phi_2-\phi_1) + \dot{\phi}_1\dot{\phi}_2\sin\theta_1\cos\theta_2\cos(\phi_1-\phi_2)\big] + 2R_2^2\dot{\phi}_2^2\sin\theta_2\cos\theta_2\Big\} + m_2gR_2\sin\theta_2 = 0
\end{aligned}
$$

$$
\begin{aligned}
&m_2\Big\{R_1R_2\big[\ddot{\theta}_1\left(\cos\theta_1\cos\theta_2\cos(\phi_1-\phi_2)+\sin\theta_1\sin\theta_2\right) + \dot{\theta}_1\big(-\dot{\theta}_1\sin\theta_1\cos\theta_2\cos(\phi_1-\phi_2) \\
&-\dot{\theta}_2\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2) - \cos\theta_1\cos\theta_2\sin(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2) + \dot{\theta}_1\cos\theta_1\sin\theta_2 + \dot{\theta}_2\sin\theta_1\cos\theta_2\big) \\
&\ddot{\phi}_1\sin\theta_1\cos\theta_2\sin(\phi_2-\phi_1) + \dot{\phi}_1\dot{\theta}_1\cos\theta_1\cos\theta_2\sin(\phi_2-\phi_1) - \dot{\phi}_1\dot{\theta}_2\sin\theta_1\sin\theta_2\sin(\phi_2-\phi_1) \\
&+\dot{\phi}_1\sin\theta_1\cos\theta_2\cos(\phi_2-\phi_1)(\dot{\phi}_2-\dot{\phi}_1)\big] + R_2^2\ddot{\theta}_2\Big\} - \frac{1}{2}m_2\Big\{2R_1R_2\big[\dot{\theta}_1\dot{\theta}_2\big(-\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2)+\sin\theta_1\cos\theta_2\big) \\
&+\dot{\theta}_1\dot{\phi}_2\cos\theta_1\cos\theta_2\sin(\phi_1-\phi_2) - \dot{\phi}_1\dot{\theta}_2\sin\theta_1\sin\theta_2\sin(\phi_2-\phi_1) + \dot{\phi}_1\dot{\phi}_2\sin\theta_1\cos\theta_2\cos(\phi_1-\phi_2)\big] \\
&+2R_2^2\dot{\phi}_2^2\sin\theta_2\cos\theta_2\Big\} + m_2gR_2\sin\theta_2 = 0
\end{aligned}
$$

$$
\begin{aligned}
&m_2R_2\Big\{R_1\Big[\ddot{\theta}_1\big(\cos\theta_1\cos\theta_2\cos(\phi_1-\phi_2)+\sin\theta_1\sin\theta_2\big) + \dot{\theta}_1\big(-\dot{\theta}_1\sin\theta_1\cos\theta_2\cos(\phi_1-\phi_2) \\
&-\cos\theta_1\cos\theta_2\sin(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2) + \dot{\theta}_1\cos\theta_1\sin\theta_2\big) + \ddot{\phi}_1\sin\theta_1\cos\theta_2\sin(\phi_2-\phi_1) \\
&+\dot{\phi}_1\dot{\theta}_1\cos\theta_1\cos\theta_2\sin(\phi_2-\phi_1) - \dot{\phi}_1^2\sin\theta_1\cos\theta_2\cos(\phi_2-\phi_1) - \dot{\theta}_1\dot{\phi}_2\cos\theta_1\cos\theta_2\sin(\phi_1-\phi_2)\Big] \\
&+R_2\big(\ddot{\theta}_2 - \dot{\phi}_2^2\sin\theta_2\cos\theta_2\big) + g\sin\theta_2\Big\} = 0
\end{aligned}
$$

Equation of motion for ϕ_1:

$$
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}_1}\right) - \frac{\partial\mathcal{L}}{\partial\theta_1} = 0
$$

$$
\begin{aligned}
&\frac{d}{dt}\Big[(m_1+m_2)R_1^2\dot{\phi}_1\sin^2\theta_1 + m_2R_1R_2\big(\dot{\theta}_2\sin\theta_1\cos\theta_2\sin(\phi_2-\phi_1) + \dot{\phi}_2\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\big)\Big] \\
&-m_2R_1R_2\Big[-\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2\sin(\phi_1-\phi_2) + \dot{\theta}_1\dot{\phi}_2\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2) - \dot{\phi}_1\dot{\theta}_2\sin\theta_1\cos\theta_2\cos(\phi_2-\phi_1) \\
&-\dot{\phi}_1\dot{\phi}_2\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2)\Big] = 0
\end{aligned}
$$

$$(m_1+m_2)R_1^2(\ddot{\phi}_1\sin^2\theta_1+2\dot{\phi}_1\dot{\theta}_1\sin\theta_1\cos\theta_1)+m_2R_1R_2\Big[\ddot{\theta}_2\sin\theta_1\cos\theta_2\sin(\phi_2-\phi_1)+\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2\sin(\phi_2-\phi_1)$$
$$-\dot{\theta}_2^2\sin\theta_1\sin\theta_2\sin(\phi_2-\phi_1)+\dot{\theta}_2\sin\theta_1\cos\theta_2\cos(\phi_2-\phi_1)(\dot{\phi}_2-\dot{\phi}_1)+\ddot{\phi}_2\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)$$
$$+\dot{\phi}_2\dot{\theta}_1\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2)+\dot{\phi}_2\dot{\theta}_2\sin\theta_1\cos\theta_2\cos(\phi_1-\phi_2)-\dot{\phi}_2\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2)\Big]$$
$$-m_2R_1R_2\Big[-\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2\sin(\phi_1-\phi_2)+\dot{\theta}_1\dot{\phi}_2\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2)-\dot{\phi}_1\dot{\theta}_2\sin\theta_1\cos\theta_2\cos(\phi_2-\phi_1)$$
$$-\dot{\phi}_1\dot{\phi}_2\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2)\Big]=0$$

$$(m_1+m_2)R_1^2(\ddot{\phi}_1\sin^2\theta_1+2\dot{\phi}_1\dot{\theta}_1\sin\theta_1\cos\theta_1)+m_2R_1R_2\Big[\ddot{\theta}_2\sin\theta_1\cos\theta_2\sin(\phi_2-\phi_1)-\dot{\theta}_2^2\sin\theta_1\sin\theta_2\sin(\phi_2-\phi_1)$$
$$+2\dot{\theta}_2\dot{\phi}_2\sin\theta_1\cos\theta_2\cos(\phi_2-\phi_1)+\ddot{\phi}_2\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)+\dot{\phi}_2^2\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2)\Big]=0$$

Equation of motion for ϕ_2:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}_2}\right)-\frac{\partial\mathcal{L}}{\partial\phi_2}=0$$

$$\frac{d}{dt}\Big[m_2R_2\big\{R_1\big[\dot{\theta}_1\cos\theta_1\sin\theta_2\sin(\phi_1-\phi_2)+\dot{\phi}_1\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\big]+R_2\dot{\phi}_2\sin^2\theta_2\big\}\Big]$$
$$-m_2R_1R_2\big[\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2\sin(\phi_1-\phi_2)-\dot{\theta}_1\dot{\phi}_2\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2)$$
$$+\dot{\theta}_2\dot{\phi}_1\sin\theta_1\cos\theta_2\cos(\phi_2-\phi_1)+\dot{\phi}_1\dot{\phi}_2\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2)\big]=0$$

$$m_2R_2\Big\{R_1\big[\ddot{\theta}_1\cos\theta_1\sin\theta_2\sin(\phi_1-\phi_2)-\dot{\theta}_1^2\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2)+\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2\sin(\phi_1-\phi_2)$$
$$+\dot{\theta}_1\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2)+\ddot{\phi}_1\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)+\dot{\theta}_1\dot{\phi}_1\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2)$$
$$+\dot{\theta}_2\dot{\phi}_1\sin\theta_1\cos\theta_2\cos(\phi_1-\phi_2)-\dot{\phi}_1\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2)\big]+R_2\big(\ddot{\phi}_2\sin^2\theta_2+2\dot{\theta}_2\dot{\phi}_2\sin\theta_2\cos\theta_2\big)\Big\}$$
$$-m_2R_1R_2\big[\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2\sin(\phi_1-\phi_2)-\dot{\theta}_1\dot{\phi}_2\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2)+\dot{\theta}_2\dot{\phi}_1\sin\theta_1\cos\theta_2\cos(\phi_2-\phi_1)$$
$$+\dot{\phi}_1\dot{\phi}_2\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2)\big]=0$$

$$m_2R_2\Big\{R_1\big[\ddot{\theta}_1\cos\theta_1\sin\theta_2\sin(\phi_1-\phi_2)-\dot{\theta}_1^2\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2)+2\dot{\theta}_1\dot{\phi}_1\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2)$$
$$+\ddot{\phi}_1\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)-\dot{\phi}_1^2\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2)\big]+R_2\big(\ddot{\phi}_2\sin^2\theta_2+2\dot{\theta}_2\dot{\phi}_2\sin\theta_2\cos\theta_2\big)\Big\}=0$$