A crash course on Category Theory

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Introduction

This course will introduce category theory from an adjunction-driven point of view, which is somewhat unusual.

This course is a modest introduction to category theory, written as the first author discovered this topic while starting his PhD.

Note that this document is obviously not exhaustive. This non-exhaustivity comes from the context of the PhD; the first author did not need to learn about n-categories or categorical logic, so there is nothing about it here.

This document is also intended to be the starting point of a book. However, it is currently in an early stage (see Section 14 to convince yourself that this document needs more work to be done).

Also note that this Crash Course has nothing to do with Bartosz Milewski's *Crash Course in Category Theory*.

1. Basic notions

This section will introduce some basic notions about category theory: categories, functors, opposite categories, natural transformations.

Definition 1.1 (Category [1])**.** A *category* $\mathscr{C}$ consists of the following data:

- A collection of *objects*, denoted $\mathrm{Ob}_{\mathscr{C}}$
- A collection of *morphisms*, or *arrows*, denoted $\mathrm{Mor}_{\mathscr{C}}$
- A map $\mathrm{dom} : \mathrm{Mor}_{\mathscr{C}} \to \mathrm{Ob}_{\mathscr{C}}$; for each morphism f, $\mathrm{dom}(f)$ is called the *domain* of f
- A map $\mathrm{cod} : \mathrm{Mor}_{\mathscr{C}} \to \mathrm{Ob}_{\mathscr{C}}$; for each morphism f, $\mathrm{cod}(f)$ is the called *codomain* of f
- For each morphism $f \in \mathrm{Mor}_{\mathscr{C}}$, we write $f : A \to B$ if $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$
- A *composition law* $\circ$ such that, for all $f : A \to B$ and $g : B \to C$, there is a chosen morphism $g \circ f : A \to C$
- For each object $A \in \mathrm{Ob}_{\mathscr{C}}$, there is a chosen morphism $1_A : A \to A$ called *identity morphism of A*

The composition law is required to be associative: $\forall A, B, C, D \in \mathrm{Ob}_{\mathscr{C}}, \forall f : A \to B$ and $g : B \to C$ and $h : C \to D$, $(h \circ g) \circ f = h \circ (g \circ f)$. Identity morphisms are required to act like identities: $\forall A, B \in \mathrm{Ob}_{\mathscr{C}}$, $\forall f : A \to B$, $f \circ 1_A = 1_B \circ f = f$.

In the rest of the course, a category $\mathscr{C}$ will be described according to the following presentation:

Objects: An object in $\mathscr{C}$ is...

Morphisms: A morphism in $\mathscr{C}$ is...

Identities: An identity morphism is...

Composition: The composition law for morphisms is...

Usually, the description of morphisms suffices to implicitly define dom and cod, as in the following examples.

Example 1.2 (Category of **Sets**). One of the easiest categories is the category in of sets. We define the category **Sets** as the following:

Objects: An object in **Sets** is any set

Morphisms: A morphism in **Sets** is any function $f : A \to B$

Identities: An identity morphism is an identity function $\mathrm{id}_A : A \to A$

Composition: The composition law for morphisms is the usual composition of functions

Example 1.3 (Preorder category). Another different but useful example of category is the category based on a preorder. If $(P, \leqslant)$ is a preordered set (we will refer to this as a *proset*), then we can define the following category:

Objects: The objects are the elements of the set P

Morphisms: There is an arrow $p \to q$ if and only if $p \leqslant q$

Identities: An identity morphism is an arrow $p \to p$ representing the trivial equality $p = p$

Composition: The composition law for morphisms is the transitivity of the preorder $\leqslant$: if $p_0 \rightarrow p_1$ and $p_1 \rightarrow p_2$ then the transitivity of $\leqslant$ implies that $p_0 \rightarrow p_2$

Note that here, the arrows have a very different meaning to the ones in **Sets**. Arrows are not at all similar to functions, but rather the representation of the preorder. Note that there is at most one arrow between two objects in the proset.

This example will be useful not to base our intuition only on the category of **Sets**; **Sets** is a very nice category with lots of properties and examples, however, it does not represent the "generic" category. There are categories that behave differently and we need examples of them.

Definition 1.4 (Hom-set)**.** Let $\mathscr{C}$ be a category, and let A and B be two objects of $\mathscr{C}$. We denote by $\mathrm{Hom}_{\mathscr{C}}(A, B)$ the collection of arrows $A \rightarrow B$ in the category $\mathscr{C}$.

Example 1.5 (Hom-sets in **Sets**). In the category **Sets**, A and B are two sets, and $\mathrm{Hom}_{\mathbf{Sets}}(A, B)$ is the set of functions $f : A \rightarrow B$.

Example 1.6 (Hom-sets in a proset). In a proset $(P, \leqslant)$, we have $\mathrm{Hom}_P(p, q) = \{(p, q)\} \Leftrightarrow p \leqslant q$; otherwise, $\mathrm{Hom}_P(p, q) = \emptyset$.

Let us study some properties of the arrows of a category. We start by considering isomorphisms and will then study weaker properties (the categorical equivalents of surjective and injective functions).

Definition 1.7 (Isomorphism [1])**.** Let $\mathscr{C}$ be a category. A morphism $f : A \rightarrow B \in \mathrm{Mor}_{\mathscr{C}}$ is an *isomorphism* when there exists $g : B \rightarrow A \in \mathrm{Mor}_{\mathscr{C}}$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Such a g is denoted f^{-1}.

Example 1.8 (Isomorphisms in **Sets**). An isomorphism in **Sets** is a function that is invertible. Thus, an isomorphism in **Sets** is a bijection.

Example 1.9 (Isomorphisms in a proset). In a proset category, there is at most one arrow $p \rightarrow q$. Thus, an arrow is an isomorphism whenever we have two arrows $p \rightarrow q \rightarrow p$.

Remark 1.10 (Isomorphisms in other categories). **Sets** and preorders are the canonical examples of categories. There are lots of other categories. Some of them are refered to *categories of structured sets*:

1. **Lin**$_{\mathbb{F}}$: the category of vector spaces over a field $\mathbb{F}$, with linear mappings
2. **Groups** the category of groups, with group homomorphisms
3. **Rings** the category of rings, with ring homomorphisms
4. **Fields** the category of fields, with ring homomorphisms (this one has interesting properties)

In most *structured sets categories*, a bijective morphism is an isomorphism, just like in **Sets**. However, there exist bijective morphisms that are not isomorphisms (in **Top**, the category of topological spaces), and in more complicated categories, there exist isomorphisms that are not bijective (see the homotopy category of CW complexes). This is because bijectivity is *not* a property of morphisms that makes sense in terms of categories.

We have just introduced the notion of an isomorphism, and we saw that in **Sets**, they were exactly the bijections (Example 1.8). Thus, isomorphisms generalise the concept of bijection to other categories. Now, one could ask: how to generalise the concept of injections and surjections?

Definition 1.11 (Epimorphisms and monomorphisms [1])**.** Let $\mathscr{C}$ be a category and let $c : C \rightarrow C'$ be an arrow in $\mathscr{C}$.

The arrow c is a *monomorphism*, or is *monic*, if, for all $f, g : A \rightarrow C$, $c \circ f = c \circ g \Rightarrow f = g$:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \arrow[r, "c"] & C' \end{tikzcd}$$

The arrow c is an *epimorphism*, or is *epic*, if, for all $f, g : C' \to B$, $f \circ c = g \circ c \Rightarrow f = g$:

$$C \xrightarrow{c} C' \underset{g}{\overset{f}{\rightrightarrows}} B$$

Example 1.12 (Epis and monos in **Sets**). In **Sets**, suppose $c : C \to C'$ is monic. Let $x, y \in C$ such that $c(x) = c(y)$. Let f_x and f_y be the functions:

$$f_x : \begin{cases} 1 & \longrightarrow & C \\ i & \longmapsto & x \end{cases} \qquad \text{and} \qquad f_y : \begin{cases} 1 & \longrightarrow & C \\ i & \longmapsto & y \end{cases}$$

As c is monic, we have $c \circ f_x = c \circ f_y \Rightarrow f_x = f_y \Rightarrow x = y$.

Conversely, if c is injective, then for all $f, g : X \to C$, if $c \circ f = c \circ g$, then for all $x \in X$, $c \circ f(x) = c \circ g(x)$ which by injectivity means $f(x) = g(x)$ and then $f = g$.

Now, if $c : C \to C'$ is epic, let $\chi_{c(C)} : C' \to 2$ be the characteristic function of $c(C)$ (the image of c), and let $\mathrm{cste}_1 : x \to 1$ be the constant function. We have $\chi_{c(C)} \circ c = \mathrm{cste}_1 \circ c$, which by epicity gives $\chi_{c(C)} = \mathrm{cste}_1$, and thus $C' = c(C)$, from which we deduce the surjectivity.

If $c : C \to C'$ is surjective, let $f, g : C' \to B$ such that $f \circ c = g \circ c$. For all $y \in C'$, there exists an x such that $y = c(x)$ and $f \circ c(x) = g \circ c(x) = f(y) = g(y)$, which gives $f = g$, and c is epic.

In summary, in **Sets**, monomorphisms are exactly injective functions, and epimorphisms are exactly surjective functions.

Example 1.13 (Epis and monos in a proset). In a proset category $(P, \leqslant)$, every arrow is monic and epic. This is due to the unicity of the arrow between two objects. Note that, here, the arrows that are both monic and epic, are not necessarily isomorphisms.

Remark 1.14 (Epis and monos in other categories). In most "structured sets" categories, for example, in **Monoids**, in **Groups**, in $\mathbf{Lin}_{\mathbb{F}}$, the monomorphisms are exactly the injective morphisms. However, the epimorphisms are not exactly the surjective morphisms. For more information, see [1, Section 2.1, pp30-31].

Proposition 1.15. *Let $f : A \to B$ and $g : B \to A$ such that $g \circ f = \mathrm{id}_A$. Then f is monic while g is epic.*

Proof. Let $a, a' : A' \to A$ such that $f \circ a = f \circ a'$, then $g \circ f \circ a = g \circ f \circ a' \Rightarrow a = a'$, so f is monic.

Let $b, b' : A \to A'$ such that $b \circ g = b' \circ g$, then $b \circ g \circ f = b' \circ g \circ f \Rightarrow b = b'$, so g is epic. □

From Remark 1.12, we deduce that a function in **Sets** is an isomorphism if and only if it is both monic and epic. However, the "if and only if" does not hold for most categories (see Example 1.13 or [1, Section 2.1.1, pp32-33] for an example). What does hold is the following:

Corollary 1.16. *If $c : C \to C'$ is an isomorphism, then c is both a monomorphism and an epimorphism.*

We now go back to studying a bit more about categories. We consider here the size of categories, which might be a concern of a reader with set-theoretic background.

Nothing in Definition 1.1 implies that $\mathrm{Ob}_{\mathscr{C}}$ or $\mathrm{Mor}_{\mathscr{C}}$ should be sets (nor should be $\mathrm{Hom}_{\mathscr{C}}(A, B)$). In fact, $\mathrm{Ob}_{\mathbf{Sets}}$ is not a set. In that sense, categories can be as *big* as possible. However, in the scope of this course, we will only use *somewhat small* categories, in the following sense.

Definition 1.17 (Small, locally small and large categories [2])**.** A category $\mathscr{C}$ is *small* if both $\mathrm{Ob}_{\mathscr{C}}$ and $\mathrm{Mor}_{\mathscr{C}}$ are sets; otherwise, it is *large*.

A category $\mathscr{C}$ is *locally small* if, for all objects $A, B \in \mathscr{C}$, the Hom-set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a set.

Example 1.18. **Sets** is large but locally small.

Example 1.19. If $(P, \leqslant)$ is a proset, then it is a small (thus locally small) category.

Example 1.20. The following example is inspired from set-theory. If V_α is the α-th set of the von Neumann hierarchy [3, Definition 2.1, p. 95], and if λ is a limit ordinal, then we define the category V_λ by:

Objects: An object in V_λ is any set $A \in V_\lambda$

Morphisms: A morphism in V_λ is any function $f : A \to B$ for $A, B \in V_\lambda$

Identities: An identity morphism is an identity function $\mathrm{id}_A : A \to A$

Composition: The composition law for morphisms is the usual composition of functions

We can see V_λ as a truncated **Sets** category. The category V_λ is a small category.

Example 1.21. For an example of a large, non-locally small category, see [4].

Remark 1.22. Small categories are locally small (because "*sets contain sets*").

In this course, we will consider locally-small categories, for a reason explained later. For now, we continue with a few more basic notions.

We also define mappings somewhat similar to functions, or homomorphisms, between categories.

Definition 1.23 (Functor [1])**.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories.

A *functor* $F : \mathscr{C} \to \mathscr{D}$ is a mapping from $\mathscr{C}$ to $\mathscr{D}$ such that:

- $\forall C \in \mathrm{Ob}_{\mathscr{C}},\ F(C) \in \mathrm{Ob}_{\mathscr{D}}$
- $\forall f : A \to B \in \mathrm{Mor}_{\mathscr{C}},\ F(f) : F(A) \to F(B) \in \mathrm{Mor}_{\mathscr{D}}$
- $\forall A \in \mathrm{Ob}_{\mathscr{C}},\ F(1_A) = 1_{F(A)}$
- $\forall f : A \to B, g : B \to C \in \mathrm{Mor}_{\mathscr{C}},\ F(g \circ f) = F(g) \circ F(f)$

In other words, a functor $F : \mathscr{C} \to \mathscr{D}$ sends the objects (resp. morphisms) in $\mathscr{C}$ to objects (resp. morphisms) in $\mathscr{D}$, preserving domains and codomains of morphisms, as well as identities and composition.

Example 1.24 (Functors between prosets). If $(P_1, \leqslant_1)$ and $(P_2, \leqslant_2)$ are prosets, then a functor between those two categories is a monotone function such that $p \leqslant_1 q \Rightarrow F(p) \leqslant_2 F(q)$.

Example 1.25 (Forgetful functors). Every category of structured sets $\mathscr{C}$, for example $\mathscr{C} = \mathbf{Lin}_{\mathbb{F}}$ or $\mathscr{C} = \mathbf{Fields}$, comes with a functor $F : \mathscr{C} \to \mathbf{Sets}$ that "takes away the structure". For example, if $\mathscr{C} = \mathbf{Lin}_{\mathbb{F}}$, then it sends a vector space to its underlying set. Such a functor generally has interesting properties as well (but we will have to wait until Section 5.7).

One can interpret a functor $\mathscr{C} \to \mathscr{D}$ as a way to have the *picture* of the category $\mathscr{C}$ into the category $\mathscr{D}$ ([1]). It is the idea behind diagrams as we will see in Section 7.

Remark 1.26. It is important to note here that the image of a category by a functor is not necessarily a category. Consider the following functor:

$$\begin{tikzcd}
A \arrow[d, "f"] & & F(A) \arrow[dd, "F(f)"] \\
B & & \\
& \mapsto & F(B) = F(C) \arrow[dd, "F(g)"] \\
C \arrow[d, "g"] & & \\
D & & F(D)
\end{tikzcd}$$

In the domain category, there is no composite $g \circ f$ because the domain of g is not the codomain of f. However, in the image of the functor, we have an arrow $F(g)$ whose domain coincides with the codomain of $F(f)$. If it were a category, it would need a composite arrow $F(?) = F(g) \circ F(f)$, which doesn't exist in the first category.

Of course, we can complete the image of a functor and make it a category.

Sometimes, we come across some functors that behave strangely. Namely, sometimes a functor $F : \mathscr{C} \to \mathscr{D}$ may send $c : C \to C'$ to $F(c) : F(C') \to F(C)$ (note the inversion). We will give an example of such a functor. What is happening, is that F is actually not a functor $\mathscr{C} \to \mathscr{D}$ but somehow defined on a similar, but "reversed" category of $\mathscr{C}$.

Definition 1.27 (Opposite category [1])**.** Let $\mathscr{C}$ be any category. We call *opposite, or dual category of* $\mathscr{C}$, denoted by $\mathscr{C}^{\mathrm{op}}$, the following category:

Objects: An object in $\mathscr{C}^{\mathrm{op}}$ is an object in $\mathscr{C}$

Morphisms: An arrow $f : B \to A$ in $\mathscr{C}^{\mathrm{op}}$ is an arrow $f : A \to B$ in $\mathscr{C}$

Identities: An identity in $\mathscr{C}^{\mathrm{op}}$ is an identity in $\mathscr{C}$

Composition: The composition law in $\mathscr{C}^{\mathrm{op}}$ is the same as in $\mathscr{C}$

Basically, the opposite category $\mathscr{C}^{\mathrm{op}}$ is the same category as $\mathscr{C}$, with inverted arrows.

If a functor $F : \mathscr{C} \to \mathscr{D}$ sends $c : C \to C'$ to $F(c) : F(C') \to F(C)$, then F is not actually defined on $\mathscr{C}$ but rather on $\mathscr{C}^{\mathrm{op}}$: $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$. However, it is often simpler to consider only functors on $\mathscr{C}$, hence the following notions:

Definition 1.28 (Covariant and contravariant functor)**.** A functor $F : \mathscr{C} \to \mathscr{D}$ is called *covariant* if it sends $f : A \to B$ to $F(f) : F(A) \to F(B)$.

A functor $G : \mathscr{C} \to \mathscr{D}$ is called *contravariant* if it sends $f : A \to B$ to $G(f) : G(B) \to G(A)$, or equivalently, if $G : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ is a covariant functor.

Two examples of such functors are the following:

Definition 1.29 (Covariant Hom-set functor [5])**.** Let $\mathscr{C}$ be a (locally small) category, and let $A \in \mathscr{C}$ be an object.

The mapping $\mathrm{Hom}_{\mathscr{C}}(A) - : \left\{ \begin{array}{ccc} \mathscr{C} & \longrightarrow & \mathbf{Sets} \\ B & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(A, B) \end{array} \right.$ defines the *covariant Hom-set functor*. It sends an object $B \in \mathscr{C}$ to the set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of arrows from A to B, and an arrow $b : B \to B'$ to the arrow $\mathrm{Hom}_{\mathscr{C}}(A, b) : \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{C}}(A, B')$ in **Sets**.

Definition 1.30 (Contravariant Hom-set functor [5])**.** Let $\mathscr{C}$ be a (locally small) category, and let $B \in \mathscr{C}$ be an object.

The mapping $\mathrm{Hom}_{\mathscr{C}}(-, B) : \left\{ \begin{array}{ccc} \mathscr{C}^{\mathrm{op}} & \longrightarrow & \mathbf{Sets} \\ A & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(A, B) \end{array} \right.$ defines the *contravariant Hom-set functor*. It sends an object $A \in \mathscr{C}^{\mathrm{op}}$ to the set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of arrows from A to B, and an arrow $a : A \to A'$ to the arrow $\mathrm{Hom}_{\mathscr{C}}(a, B) : \mathrm{Hom}_{\mathscr{C}}(A', B) \to \mathrm{Hom}_{\mathscr{C}}(A, B)$ in **Sets**.

Remark 1.31. Their names are not stolen: $B \to \mathrm{Hom}_{\mathscr{C}}(A)\, B$ is a covariant functor and $A \to \mathrm{Hom}_{\mathscr{C}}(A)\, B$ is a contravariant functor.

Note that both Hom-set functors imply $\mathscr{C}$ to be locally small. As stated a few paragraphs before, all the categories we will encounter in this course are locally small, unless stated otherwise, because we will often need this functor to be defined.

Also note that along this course, we will encounter lots of examples of contravariant functors. This notion may look confusing. With a bit of practice, it is no more a problem.

We continue and end this section with a final basic notion of category theory, namely, natural transformations, which are a kind of mappings between functors.

Definition 1.32 (Natural transformation [1]). Let $\mathscr{C}$ and $\mathscr{D}$ be two categories, and let $F, G : \mathscr{C} \to \mathscr{D}$ be functors. A *natural transformation* $\theta : F \to G$ consists of a collection of morphisms in $\mathscr{D}$ $(\theta_C : F(C) \to G(C))_{C \in \mathrm{Ob}_{\mathscr{C}}}$ such that, for all $C, D \in \mathscr{C}$, and for all $h : C \to D$, the following square commutes:

$$\begin{tikzcd} C \arrow[d, "h"'] & & F(C) \arrow[r, "\theta_C"] \arrow[d, "F(h)"'] \arrow[dr, phantom, "\checkmark"] & G(C) \arrow[d, "G(h)"] \\ D & \rightsquigarrow & F(D) \arrow[r, "\theta_D"'] & G(D) \end{tikzcd} \tag{1}$$

For each object $C \in \mathscr{C}$, the morphism θ_C is called the C-component of θ.
The natural transformation $\theta : F \to G$ can be written in the following diagram:

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F", ""{name=U, below}] \arrow[r, bend right, "G"', ""{name=D, above}] & \mathscr{D} \arrow[Rightarrow, from=U, to=D, "\theta"] \end{tikzcd}$$

We denote by $\mathrm{Nat}\,(F, G)$ the collection of all natural transformations between F and G.

Depending on the context, and for the sake of readability, the C-component of a natural transformation θ can be written θ_C as above (C as an index) or $\theta(C)$ (C as a parameter).

Natural transformations can be seen as a way to extract the parameters C, D and h from $F(C)$, $F(D)$ and $F(h)$ and input them into G, while preserving arrows. It's a variable substitution.

Remark 1.33. Consider two functors $F, G : \mathscr{C} \to \mathscr{X}$, and their respective (categorified) images $\mathrm{Im}\,(F)$ and $\mathrm{Im}\,(G)$. A natural transformation $\theta : F \to G$ may be seen as a functor $\hat{\theta} : \mathrm{Im}\,(F) \to \mathrm{Im}\,(G)$ such that:

1. for all object $C \in \mathscr{C}$, $\hat{\theta}(F(C)) = G(C)$ ($\hat{\theta}$ preserves the objects)
2. for all arrow $c : C \to C' \in \mathscr{C}$, $\hat{\theta}(F(c)) = G(c)$ with $F(c) : F(C) \to F(C')$ and $G(c) : G(C) \to G(C')$ ($\hat{\theta}$ preserves the arrows)
3. $\hat{\theta}$ makes the natural transformation diagram (Diagram 1) commute

Note that this view of natural transformations is not standard, but it may help some readers to grasp this notion.

Before introducing the notion of natural isomorphism, we need to make something clear on the nature of natural transformations.

Definition 1.34 (Composition of natural transformations). Let $\mathscr{C}$, $\mathscr{D}$ be categories, and let F, G and H be functors $\mathscr{C} \to \mathscr{D}$.

If $\theta : F \to G$ is the natural transformation $\theta = \left(F(A) \xrightarrow{\theta_A} G(A)\right)_{A \in \mathscr{C}}$ and $\eta : G \to H$ is the natural transformation $\eta = \left(G(A) \xrightarrow{\eta_A} H(A)\right)_{A \in \mathscr{C}}$ then the composition of θ by η is $\eta \circ \theta : F \to H$, defined by $\eta \circ \theta = \left(F(A) \xrightarrow{\eta_A \circ \theta_A} H(A)\right)_{A \in \mathscr{C}}$.

Definition 1.35 (Functor category [1])**.** Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. The *functor category*, denoted by $\mathbf{Func}(\mathscr{C}, \mathscr{D})$, or by $\mathscr{D}^{\mathscr{C}}$, is the following category:

Objects: The objects are the functors $F : \mathscr{C} \to \mathscr{D}$

Morphisms: A morphism between two functors F and G is a natural transformation $\theta : F \to G = \left(F(A) \xrightarrow{\theta_A} G(A)\right)_{A \in \mathscr{C}}$

Identities: An identity on a functor F is the identity natural transformation $\mathrm{id}_F = \left(F(A) \xrightarrow{\mathrm{id}_{F(A)}} F(A)\right)_{A \in \mathscr{C}}$

Composition: The composition law in $\mathbf{Func}(\mathscr{C}, \mathscr{D})$ is defined in Definition 1.34.

Natural transformations are morphisms between functors. Besides, if $F, G : \mathscr{C} \to \mathscr{D}$ are two functors, then the notation $\mathrm{Nat}(F, G)$ actually stands for $\mathrm{Hom}_{\mathbf{Func}(\mathscr{C},\mathscr{D})}(F, G)$; however $\mathrm{Nat}(F, G)$ is usually more convenient.

Using Definition 1.35 (functor category), and Definition 1.7 (isomorphism), we deduce the definition of a natural isomorphism:

Definition 1.36 (Natural isomorphism [1])**.** Let $F, G : \mathscr{C} \to \mathscr{D}$ be functors. A *natural isomorphism* $\theta : F \to G$ is a natural transformation that is an isomorphism in the functor category $\mathbf{Func}(\mathscr{C}, \mathscr{D})$.

It is easy to see that:

Lemma 1.37. *A natural transformation $\theta : F \to G$ is a natural isomorphism whenever the C-components $\theta_C : F(C) \to G(C)$ are isomorphisms.*

This lemma gives a useful description of what a natural isomorphism is. It makes it easier to look for an inverse. We will use this lemma in the following section.

This lemma does not exactly hold for monic or epic natural transformations. In fact, we have only one implication.

Proposition 1.38. *Let $\mathscr{X}$ and $\mathscr{C}$ any categories. Let $F, G : \mathscr{X} \to \mathscr{C}$ be two functors and let $\alpha : F \to G$ be a natural transformation between those two functors.*

If for all $X \in \mathscr{X}$, $\alpha_X : F(X) \to G(X)$ is monic (resp. epic), then so is $\alpha : F \to G$.

Proof. Suppose that each X-component is monic. The proof is similar when we are considering epic components.

Consider $\beta, \beta' : H \to F$ such that $\alpha \circ \beta = \alpha \circ \beta'$.

$$H \underset{\beta'}{\overset{\beta}{\rightrightarrows}} F \xrightarrow{\alpha} G \qquad \Leftrightarrow \qquad H(X) \underset{\beta'_X}{\overset{\beta_X}{\rightrightarrows}} F(X) \xrightarrow{\alpha_X} G(X)$$

In terms of components, this means that for all $X \in \mathscr{X}$, we have $\alpha_X \circ \beta_X = \alpha_X \circ \beta'_X$. As every component is monic, this gives $\beta_X = \beta'_X$, and then $\beta = \beta'$. Thus, α is monic. $\square$

Surprisingly, the converse does not hold in general. In fact, it needs some more properties about the codomain category, but this is far beyond the scope of this crash course.

We have now introduced the basic notions of category theory: categories, hom-sets, isomorphisms, monomorphisms, epimorphisms, opposite categories, (covariant or contravariant) functors, natural transformations. We can now move on to the next section, in which we introduce the very first important result about category theory.

2. Yoneda lemma

Given a functor $F : \mathscr{C} \longrightarrow$ **Sets**, can we transform it into a Hom-set functor? The answer is provided by the Yoneda lemma. The Yoneda lemma is based on a natural transformation, as illustrated by the following series of figures.

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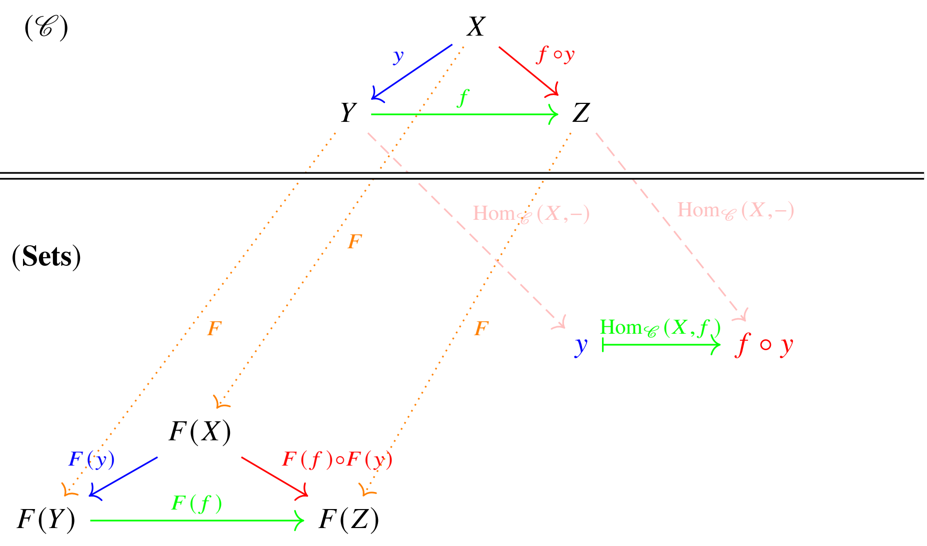

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The Yoneda lemma is surprisingly treated as a full theorem. However, the Yoneda lemma requires lemmas.

Lemma 2.1 (The simplest representation lemma)**.** *Let $\mathscr{X}, \mathscr{C}$ be categories, and let $U : \mathscr{X} \longrightarrow \mathscr{C}$ be a functor.*

1. $\forall C \in \mathrm{Ob}_{\mathscr{C}}, \forall x : X \longrightarrow X' \in \mathrm{Mor}_{\mathscr{X}}, \forall c \in Hom_{\mathscr{C}}(C, U(X)), Hom_{\mathscr{C}}(C, U(x))(c) = U(x) \circ c$
2. $\forall X \in \mathrm{Ob}_{\mathscr{X}}, \forall x : X' \longrightarrow X'' \in \mathrm{Mor}_{\mathscr{X}}, \forall y \in Hom_{\mathscr{X}}(X, X'), x \circ y = Hom_{\mathscr{X}}(X, x)(y)$
3. $\forall X \in \mathrm{Ob}_{\mathscr{X}}, \forall x : X \longrightarrow X' \in \mathrm{Mor}_{\mathscr{X}}, x = Hom_{\mathscr{X}}(X, x)(\mathrm{id}_X)$

Proof. 1. By direct application of the definitions of a functor (Definition 1.23) and of the covariant Hom-set functor (Definition 1.29):

$$\mathrm{Hom}_{\mathscr{C}}(C, U(x)) : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(C, U(X)) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(C, U(X')) \\ a & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(C, U(x))(a) \end{array} \right.$$

where $\mathrm{Hom}_{\mathscr{C}}(C, U(x))(a)$ is the morphism such that:

$$\begin{tikzcd} & C \arrow[dl, "a"'] \arrow[dr, "\mathrm{Hom}_{\mathscr{C},C}(U(x))(a)"] \arrow[d, phantom, "\checkmark"] & \\ U(X) \arrow[rr, "U(x)"'] & & U(X') \end{tikzcd}$$

which gives $\mathrm{Hom}_{\mathscr{C}}(C, U(x))(a) = U(x) \circ a$, hence the result.
2. Consequence of first part of the lemma with $\mathscr{C} = \mathscr{X}$, $U = \mathrm{Id}_{\mathscr{C}}$, $c = x$ and $X = C$.
3. Consequence of second part of the lemma with $X = X'$ and $y = \mathrm{id}_X$. □

Lemma 2.2. *Let $\mathscr{X}, \mathscr{C}$ be two categories and let $H : \mathscr{X} \to$ **Sets** be a functor.*

Given any $X \in \mathrm{Ob}_{\mathscr{X}}$ and any natural transformation $\varphi = \left(Hom_{\mathscr{X}}(X,Y) \xrightarrow{\varphi_Y} H(Y)\right)_{Y \in \mathrm{Ob}_{\mathscr{X}}}$ such that:

$$\begin{tikzcd} \mathscr{X} \arrow[r, bend left, "Hom_{\mathscr{X}}(X,Y)"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\varphi"] & \textbf{Sets} \end{tikzcd}$$

then $\varphi_X(\mathrm{id}_X)$ is the unique element $e \in H(X)$ such that:

$$\forall Y \in \mathrm{Ob}_{\mathscr{X}}, \forall y \in Hom_{\mathscr{X}}(X,Y),\ \varphi_Y(y) = H(y)(e)$$

Proof. Let $Y \in \mathrm{Ob}_{\mathscr{X}}$, and let $y \in \mathrm{Hom}_{\mathscr{X}}(X,Y)$.

By simplest representation lemma (Lemma 2.1, item 3), we have:

$$y = \mathrm{Hom}_{\mathscr{X}}(X,y)(\mathrm{id}_X)$$

Thus:

$$\begin{aligned} \varphi_Y(y) &= \varphi_Y(\mathrm{Hom}_{\mathscr{X}}(X,y)(\mathrm{id}_X)) \\ &= (\varphi_Y \circ \mathrm{Hom}_{\mathscr{X}}(X,y))(\mathrm{id}_X) \end{aligned}$$

Besides, $\varphi : \mathrm{Hom}_{\mathscr{X}}(X,-) \to H$ is a natural transformation; thus using Definition 1.32, diagram 1 with $F = \mathrm{Hom}_{\mathscr{X}}(X,-)$ and $G = H$, we have:

$$\varphi_Y \circ \mathrm{Hom}_{\mathscr{X}}(X,y) = H(y) \circ \varphi_X$$

which yields:

$$\begin{aligned} \varphi_Y(y) &= (H(y) \circ \varphi_X)(\mathrm{id}_X) \\ &= H(y)(\varphi_X(\mathrm{id}_X)) \end{aligned}$$

Hence the result. Now we have to prove that $e = \varphi_X(\mathrm{id}_X)$ is unique with that property. Let $e' \in H(X)$ such that $\forall Y \in \mathrm{Ob}_{\mathscr{X}}, \forall y \in \mathrm{Hom}_{\mathscr{X}}(X,Y),\ \varphi_Y(y) = H(y)(e') = H(y)(e)$. Using $X = Y$ and $y = \mathrm{id}_X$ yields:

$$\begin{aligned} H(\mathrm{id}_X)(e') &= H(\mathrm{id}_X)(e) \\ \mathrm{id}_{H(X)}(e') &= \mathrm{id}_{H(X)}(e) \\ e' &= e \end{aligned}$$

□

Lemma 2.3. *Let $\mathscr{X}, \mathscr{C}$ be two categories and let $H : \mathscr{X} \to$ **Sets** be a functor. Let $X \in \mathrm{Ob}_{\mathscr{X}}$ and $e \in H(X)$.*

The mapping $\varphi = \left(Hom_{\mathscr{X}}(X,Y) \xrightarrow{\varphi_Y} H(Y)\right)_{Y \in \mathrm{Ob}_{\mathscr{X}}}$ defined by:

$$\varphi_Y : \left\{ \begin{array}{ccc} Hom_{\mathscr{X}}(X,Y) & \longrightarrow & H(Y) \\ y & \longmapsto & H(y)(e) \end{array} \right.$$

is a natural transformation such that $\varphi_X(\mathrm{id}_X) = e$.

Proof. We need to prove that, for any $y : Y \longrightarrow Y' \in \mathrm{Mor}_{\mathscr{X}}$, the following square commutes:

$$
\begin{tikzcd}
Y \arrow[d, "y"] & & \mathrm{Hom}_{\mathscr{X}}(X,Y) \arrow[r, "\varphi_Y"] \arrow[d, "\mathrm{Hom}_{\mathscr{X}}(X,y)"'] \arrow[dr, phantom, "\checkmark"] & H(Y) \arrow[d, "H(y)"] \\
Y' & \rightsquigarrow & \mathrm{Hom}_{\mathscr{X}}(X,Y') \arrow[r, "\varphi_{Y'}"'] & H(Y')
\end{tikzcd}
$$

that is, we want:

$$\forall y : Y \longrightarrow Y',\ \varphi_{Y'} \circ \mathrm{Hom}_{\mathscr{X}}(X, y) = H(y) \circ \varphi_Y$$

Let $y : Y \to Y'$ be an arrow in $\mathscr{X}$. For all $x \in \mathrm{Hom}_{\mathscr{X}}(X, Y) = \mathrm{dom}\left(\mathrm{Hom}_{\mathscr{X}}(X, y)\right)$:

$$
\begin{aligned}
\varphi_{Y'} \circ \mathrm{Hom}_{\mathscr{X}}(X, y)(x) &= \varphi_{Y'}(y \circ x) && (2)\\
&= H(y \circ x)(e) && (3)\\
&= H(y) \circ H(x)(e) && (4)\\
&= H(y) \circ \varphi_Y(x) && (5)
\end{aligned}
$$

Equation 2 is due to the simplest representation lemma (Lemma 2.1-2); Equations 3 and 5 are due to the definition of φ and Equation 4 comes from the definition of a functor (Definition 1.23).

Besides:

$$
\begin{aligned}
\varphi_X(\mathrm{id}_X) &= H(\mathrm{id}_X)(e)\\
&= \mathrm{id}_{H(X)}(e)\\
&= e
\end{aligned}
$$

□

Definition 2.4 (The ξ natural isomorphism)**.** Let $\mathscr{X}$ be a category, let $H : \mathscr{X} \to \mathbf{Sets}$ be a functor and let X be an object in $\mathrm{Ob}_{\mathscr{X}}$.

We define:

$$\xi_{H,X} : \begin{cases} \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(X, -), H\right) & \longrightarrow & H(X) \\ \varphi & \longmapsto & \varphi_X(\mathrm{id}_X) \end{cases}$$

The ξ natural isomorphism is the mapping $\xi : H, X \to \xi_{H,X}$.

Definition 2.5 (The θ natural isomorphism)**.** Let $\mathscr{X}$ be a category, let $H : \mathscr{X} \to \mathbf{Sets}$ be a functor and let X be an object in $\mathrm{Ob}_{\mathscr{X}}$.

We define:

$$\theta_{H,X} : \begin{cases} H(X) & \longrightarrow & \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(X, -), H\right) \\ e & \longmapsto & \varphi_e^{H,X} = \left(\varphi_{e,Y}^{H,X} : \begin{cases} \mathrm{Hom}_{\mathscr{X}}(X, Y) & \longrightarrow & H(Y) \\ y & \longmapsto & H(y)(e) \end{cases}\right)_{Y \in \mathrm{Ob}_{\mathscr{X}}} \end{cases}$$

The θ natural isomorphism is the mapping $\theta : H, X \to \theta_{H,X}$.

Please note that those two natural isomorphisms are standard in the demonstrations of the Yoneda lemma; however their notation isn't. We highlight those two ismorphisms because they will have several occurrences in the current course.

Proposition 2.6. *The mappings ξ and θ are both actual natural isomorphisms, covariant in H and contravariant in X, and they are inverse of each other.*

Corollary 2.7 (Yoneda lemma [2])**.** *Let $\mathscr{X}$ be a category, let $H : \mathscr{X} \longrightarrow$ **Sets** be a functor and let X be an object in $\mathrm{Ob}_{\mathscr{X}}$.*

Then, $Nat\left(Hom_{\mathscr{X}}(X,-),H\right) \cong H(X)$.

Proof. Let $H : \mathscr{X} \longrightarrow$ **Sets** and let $X \in \mathrm{Ob}_{\mathscr{X}}$.

[Inverse]

We first prove that θ is the inverse of ξ. Then, we will prove that both are natural transformations.

Let $e \in H(X)$.

$$\begin{aligned}\xi_{H,X} \circ \theta_{H,X}(e) &= \xi_{H,X}\left(\varphi_e^{H,X}\right)\\ &= \varphi_{e,X}^{H,X}\left(\mathrm{id}_X\right) &\quad (6)\\ &= e &\quad (7)\end{aligned}$$

The transition $6 \Rightarrow 7$ comes from Lemma 2.3.

Similarly, let $\varphi \in \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(X,-),H\right)$. Note that, according to Lemma 2.2, φ is:

$$\varphi = \left(\varphi_Y : \left\{\begin{array}{ccc}\mathrm{Hom}_{\mathscr{X}}(X,Y) & \longrightarrow & H(Y)\\ y & \longmapsto & H(y)\left(\varphi_X\left(\mathrm{id}_X\right)\right)\end{array}\right.\right)_{Y\in\mathrm{Ob}_{\mathscr{X}}}$$

Thus:

$$\begin{aligned}\theta_{H,X} \circ \xi_{H,X}(\varphi) &= \theta_{H,X}\left(\varphi_X\left(\mathrm{id}_X\right)\right)\\ &= \left(\varphi^{H,X}_{\varphi_X(\mathrm{id}_X),Y} : \left\{\begin{array}{ccc}\mathrm{Hom}_{\mathscr{X}}(X,Y) & \longrightarrow & H(Y)\\ y & \longmapsto & H(y)\left(\varphi_X\left(\mathrm{id}_X\right)\right)\end{array}\right.\right)_{Y\in\mathrm{Ob}_{\mathscr{X}}}\\ &= \varphi\end{aligned}$$

Consequently, θ and ξ are mutually inverses.

We only have to check their naturalities.

[ξ is a natural transformation in H]

Let $\alpha : H \longrightarrow H'$ be a natural transformation. We want to check if the following diagram commutes:

$$\begin{tikzcd} H \arrow[d, "\alpha"'] & \rightsquigarrow & \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(X,-),H\right) \arrow[r, "\xi_{H,X}"] \arrow[d, "\mathrm{Nat}(\mathrm{Hom}_{\mathscr{X}}(X,-),\alpha)"'] & H(X) \arrow[d, "\alpha_X"] \\ H' & & \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(X,-),H'\right) \arrow[r, "\xi_{H',X}"'] & H'(X) \end{tikzcd}$$

Let $\varphi \in \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(X,-),H\right)$.

$$\begin{aligned}\xi_{H,X} \circ \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(X,-),\alpha\right)(\varphi) &= \xi_{H,X}(\alpha \circ \varphi)\\ &= \left(\alpha_X \circ \varphi_X\right)\left(\mathrm{id}_X\right)\\ &= \alpha_X\left(\varphi_X\left(\mathrm{id}_X\right)\right)\\ &= \alpha_X \circ \xi_{H,X}(\varphi)\end{aligned}$$

which gives the expected result.

[ξ is a natural transformation in X]
Recall that ξ is contravariant in X. Let $x : X' \to X \in \mathrm{Mor}_{\mathscr{X}}$. We want to check if the following diagram commutes:

$$\begin{tikzcd} X' \arrow[d, "x"] & \rightsquigarrow & \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(X,-), H\right) \arrow[r, "\xi_{H,X}"] \arrow[d, "\mathrm{Nat}(\mathrm{Hom}_{\mathscr{X}}(x,-),H)"'] & H(X) \arrow[d, "H(x)"] \\ X & & \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(X',-), H\right) \arrow[r, "\xi_{H,X'}"'] & H(X') \end{tikzcd}$$

The arrows $\mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(x,-), H\right)$ and $H(x)$ are inverted because ξ is supposed to be contravariant in X.
Let $\varphi \in \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(x,-), H\right)$.
On the one hand:

$$\begin{aligned} \xi_{H,X'} \circ \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(x,-), H\right)(\varphi) &= \xi_{H,X'}\left(\varphi \circ \mathrm{Hom}_{\mathscr{X}}(x,-)\right) \\ &= \left(\varphi \circ \mathrm{Hom}_{\mathscr{X}}(x,-)\right)_{X'}(\mathrm{id}_{X'}) \\ &= \varphi_{X'} \circ \mathrm{Hom}_{\mathscr{X}}(x, X')(\mathrm{id}_{X'}) \\ &= \varphi_{X'}(\mathrm{id}_{X'} \circ x) \\ &= \varphi_{X'}(x) \end{aligned}$$

On the other hand, note that φ is also a natural transformation. Thus, for $x : X' \to X$ in $\mathscr{X}^{\mathrm{op}}$ (note that we are in the opposite category), the following diagram does commute:

$$\begin{tikzcd} X' \arrow[d, "x"] & \rightsquigarrow & \mathrm{Hom}_{\mathscr{X}}(X,X) \arrow[r, "\varphi_X"] \arrow[d, "\mathrm{Hom}_{\mathscr{X}}(X,x)"'] \arrow[dr, phantom, "\checkmark"] & H(X) \arrow[d, "H(x)"] \\ X & & \mathrm{Hom}_{\mathscr{X}}(X,X') \arrow[r, "\varphi_{X'}"'] & H(X') \end{tikzcd}$$

In particular: $H(x) \circ \varphi_X = \varphi_{X'} \circ \mathrm{Hom}_{\mathscr{X}}(X, x)$, which gives:

$$\begin{aligned} H(x) \circ \xi_{H,X}(\varphi) &= H(x)\left(\varphi_X(\mathrm{id}_X)\right) \\ &= \varphi_{X'} \circ \mathrm{Hom}_{\mathscr{X}}(X,x)(\mathrm{id}_X) \\ &= \varphi_{X'}(\mathrm{id}_X \circ x) \\ &= \varphi_{X'}(x) \\ &= \xi_{H,X'} \circ \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(x,-), H\right)(\varphi) \end{aligned}$$

Consequently, ξ is natural in both its parameters X and H.

[θ is a natural transformation in H]
The idea is similar to ξ. Let $\alpha : H \to H'$ be a natural transformation. We want to prove that $\mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(X,-), \alpha\right) \circ \theta_{H,X} = \theta_{H,X} \circ \alpha_X$.
Let $e \in H(X)$.

$$\theta_{H,X} \circ \alpha_X(e) = \varphi^{H',X}_{\alpha_X(e)}$$

and:

$$\begin{aligned}
\mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}\left(X,-\right),\alpha\right)\circ\theta_{H,X}(e) &= \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}\left(X,-\right),\alpha\right)\left(\varphi_e^{H,X}\right)\\
&= \alpha\circ\varphi_e^{H,X}
\end{aligned}$$

where $\alpha\circ\varphi_e^{H}$ is the natural transformation:

$$\alpha\circ\varphi_e^{H} = \left(\alpha_Y\circ\varphi_{e,Y}^{H,X} : \left\{\begin{array}{ccc} \mathrm{Hom}_{\mathscr{X}}\left(X,Y\right) & \longrightarrow & H'(Y)\\ y & \longmapsto & \alpha_Y\left(H(y)\left(e\right)\right)\end{array}\right.\right)_{Y\in\mathrm{Ob}\,\mathscr{X}} \tag{8}$$

$$= \left(\alpha_Y\circ\varphi_{e,Y}^{H,X} : \left\{\begin{array}{ccc} \mathrm{Hom}_{\mathscr{X}}\left(X,Y\right) & \longrightarrow & H'(Y)\\ y & \longmapsto & H'(y)\circ\alpha_X\left(e\right)\end{array}\right.\right)_{Y\in\mathrm{Ob}\,\mathscr{X}} \tag{9}$$

$$\begin{aligned}
&= \varphi_{\alpha_X(e)}^{H',X}\\
&= \theta_{H,X}\circ\alpha_X(e)
\end{aligned}$$

The transition $8\Rightarrow 9$ is due to the naturality of α.

[θ is a natural transformation in X]

Let $x : X'\to X$ be a morphism in $\mathscr{X}$. We want $\mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}\left(x,-\right),H\right)\circ\theta_{H,X} = \theta_{H,X'}\circ H(x)$. Let $e\in H(X)$:

$$\begin{aligned}
\mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}\left(x,-\right),H\right)\circ\theta_{H,X}(e) &= \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}\left(x,-\right),H\right)\left(\varphi_e^{H,X}\right)\\
&= \varphi_e^{H,X}\circ\mathrm{Hom}_{\mathscr{X}}\left(x,-\right)
\end{aligned}$$

where $\varphi_e^{H,X}\circ\mathrm{Hom}_{\mathscr{X}}\left(x,-\right)$ is the following natural transformation:

$$\varphi_e^{H,X}\circ\mathrm{Hom}_{\mathscr{X}}\left(x,-\right) = \left(\varphi_{e,Y}^{H,X}\circ\mathrm{Hom}_{\mathscr{X}}\left(x,Y\right) : \left\{\begin{array}{ccc} \mathrm{Hom}_{\mathscr{X}}\left(X',Y\right) & \longrightarrow & H(Y)\\ y & \longmapsto & \varphi_{e,Y}^{H,X}\left(y\circ x\right)\end{array}\right.\right)_{Y\in\mathrm{Ob}\,\mathscr{X}}$$

However:

$$\begin{aligned}
\varphi_{e,Y}^{H,X}\left(y\circ x\right) &= H(y\circ x)(e)\\
&= H(y)\left(H(x)(e)\right)
\end{aligned}$$

which yields:

$$\begin{aligned}
\varphi_e^{H,X}\circ\mathrm{Hom}_{\mathscr{X}}\left(x,-\right) &= \varphi_{H(x)(e)}^{H,X'}\\
&= \theta_{H,X'}\circ H(x)(e)
\end{aligned}$$

Consequently, θ is natural in both its parameters X and H.

[Conclusion]

Both ξ and θ are natural transformations in H and X, and they are mutually inverses. As a consequence, ξ and θ are natural isomorphisms between $\mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}\left(X,-\right),H\right)$ and $H(X)$.

□

Remark 2.8. As stated in Definition 1.35, $\mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(X,-), H\right)$ corresponds to the Hom-set: $\mathrm{Hom}_{\mathbf{Func}(\mathscr{X},\mathbf{Sets})}\left(\mathrm{Hom}_{\mathscr{X}^{\mathrm{op}}}\right.$ $(X$
Note that it's $\mathscr{X}^{\mathrm{op}}$ and not $\mathscr{X}$, because the natural isomorphism is contravariant in X.

Remark 2.9. The Yoneda lemma has a central role due to its various meanings and consequences.

1. First, depending on the "size" of $\mathscr{X}$, we have different interpretations. If $\mathscr{X}$ is small, then $\mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(X,-), H\right)$ is a set because $\mathbf{Sets}^{\mathscr{X}}$ becomes locally small. If $\mathscr{X}$ is locally small, then it says nothing on $\mathbf{Sets}^{\mathscr{X}}$. However, the Yoneda lemma states that $\mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{X}}(X,-), H\right)$ is always a set. If $\mathscr{X}$ is non-locally small, then the functor $\mathrm{Hom}_{\mathscr{X}}(X,-)$ doesn't exist and the Yoneda lemma doesn't hold there.

2. Secondly, from a set-theoretic point of view, the Yoneda lemma states that there are not *that many natural transformations*: there are exactly $\mathrm{card}\,(H(X))$ natural transformations $\mathrm{Hom}_{\mathscr{X}}(X,-) \to H$, as each of these natural transformations is entirely determined by one element in $H(X)$.

3. Thirdly, according to the Yoneda lemma, if $H = \mathrm{Hom}_{\mathscr{X}}(Y,-)$:

$$\mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{C}}(X,-), \mathrm{Hom}_{\mathscr{C}}(Y,-)\right) \cong \mathrm{Hom}_{\mathscr{C}}(Y,X)$$

(Note the inversion) As stated in the previous paragraph, each element in $\mathrm{Hom}_{\mathscr{C}}(Y,X)$ characterises one natural transformation in $\mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{C}}(X,-), \mathrm{Hom}_{\mathscr{C}}(Y,-)\right)$. Consequently, any natural transformation $\mathrm{Hom}_{\mathscr{C}}(X,-) \to \mathrm{Hom}_{\mathscr{C}}(Y,-)$ is determined by an arrow $Y \to X$ using the application θ seen in the proof of the Yoneda lemma. Consequently, the only arrows $\mathrm{Hom}_{\mathscr{C}}(X,A) \to \mathrm{Hom}_{\mathscr{C}}(Y,A)$ are of the form $\mathrm{Hom}_{\mathscr{C}}(f,A)$ for some $f : Y \to X$.

The dual version of the Yoneda lemma is as follows:

Lemma 2.10 (Contravariant Yoneda lemma)**.** *Let $\mathscr{X}$ be a category, let $G : \mathscr{X}^{op} \to$ **Sets** be a contravariant functor and let X be an object in* $\mathrm{Ob}_{\mathscr{X}}$.
Then, $Nat\left(Hom_{\mathscr{X}}(-,X), G\right) \cong G(X)$.

The functor $X \to \mathrm{Hom}_{\mathscr{X}}(X,-)$ has good properties. Let's spend some time studying them.

Definition 2.11 (Yoneda embedding [1])**.** Let $\mathscr{C}$ be a category.
The *Yoneda embedding* is the functor:

$$y : \begin{cases} \mathscr{C}^{\mathrm{op}} & \longrightarrow & \mathbf{Func}\,(\mathscr{C}, \mathbf{Sets}) \\ C & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(C,-) \\ f : D \to C & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(f,-) : \mathrm{Hom}_{\mathscr{C}}(D,-) \to \mathrm{Hom}_{\mathscr{C}}(C,-) \end{cases}$$

Remark 2.12. Note that the Yoneda embedding is defined $\mathscr{C}^{\mathrm{op}}$. Thus, $f : C \to D$ in $\mathscr{C}$ becomes $f : D \to C$ in $\mathscr{C}^{\mathrm{op}}$, and $y(f)$ has the direction of $f \in \mathscr{C}^{\mathrm{op}}$.

Definition 2.13 (Injective, surjective, full, faithful, embedding [1])**.** Let $F : \mathscr{C} \to \mathscr{D}$ be a functor.

1. (a) The functor F is said *injective (resp. surjective) on objects* if $\mathrm{Ob}_F : \mathrm{Ob}_{\mathscr{C}} \to \mathrm{Ob}_{\mathscr{D}}$ is injective (resp. surjective).
   (b) The functor F is said *injective (resp. surjective) on arrows* if $\mathrm{Mor}_F : \mathrm{Mor}_{\mathscr{C}} \to \mathrm{Mor}_{\mathscr{D}}$ is injective (resp. surjective).

2. For all $A, B \in \mathrm{Ob}_{\mathscr{C}}$, define the mapping:

$$F_{A,B} : \begin{cases} \mathrm{Hom}_{\mathscr{C}}(A,B) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(F(A),F(B)) \\ f & \longmapsto & F(f) \end{cases}$$

(a) The functor F is said *faithful* if $\forall A, B \in \mathrm{Ob}_{\mathscr{C}}$, $F_{A,B}$ is injective.

(b) The functor F is said *full* if $\forall A, B \in \mathrm{Ob}_{\mathscr{C}}$, $F_{A,B}$ is surjective.

3. The functor F is called an *embedding* if it is injective on objects, full and faithful.

Difference between injective on arrows and faithful?

Proposition 2.14. *The Yoneda embedding is an actual embedding.*

Proof. The injectivity on objects is easy. Suppose $y(C) = y(D)$; then:

$$\begin{aligned} y(C) &= y(D) \\ \mathrm{Hom}_{\mathscr{C}}(C, -) &= \mathrm{Hom}_{\mathscr{C}}(D, -) \\ \Rightarrow \mathrm{Hom}_{\mathscr{C}}(C, C) &= \mathrm{Hom}_{\mathscr{C}}(D, C) \end{aligned}$$

Those two sets are equal. Thus, $\mathrm{id}_C \in \mathrm{Hom}_{\mathscr{C}}(C, C) \Rightarrow \mathrm{id}_C \in \mathrm{Hom}_{\mathscr{C}}(D, C) \Rightarrow C = D$. Thus, y is injective on objects.

As noted in Remark 2.9-3, the Yoneda lemma implies that:

$$\begin{aligned} \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{C}}(C, -), \mathrm{Hom}_{\mathscr{C}}(D, -)\right) &\cong \mathrm{Hom}_{\mathscr{C}}(D, C) \\ \mathrm{Hom}_{\mathbf{Sets}^{\mathscr{C}^{\mathrm{op}}}}(y(C), y(D)) &\cong \mathrm{Hom}_{\mathscr{C}}(D, C) \end{aligned}$$

Proposition 2.6 also states that the following natural transformation is an isomorphism:

$$\theta_{y(C),D} : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(C, D) & \longrightarrow & \mathrm{Hom}_{\mathbf{Sets}^{\mathscr{C}^{\mathrm{op}}}}(y(D), y(C)) \\ f & \longmapsto & \varphi_f^{y(C),D} \end{array} \right.$$

where φ_f is:

$$\begin{aligned} \varphi_f^{y(C),D} &= \left(\varphi_{f,X}^{y(C),D} : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(D, X) & \longrightarrow & y(C)(X) \\ g & \longmapsto & y(C)(g)(f) \end{array} \right. \right)_{X \in \mathrm{Ob}_{\mathscr{C}}} \\ &= \left(\varphi_{f,X}^{y(C),D} : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(D, X) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(C, X) \\ g & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(C, g)(f) \end{array} \right. \right)_{X \in \mathrm{Ob}_{\mathscr{C}}} \\ &= \left(\varphi_{f,X}^{y(C),D} : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(D, X) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(C, X) \\ g & \longmapsto & g \circ f \end{array} \right. \right)_{X \in \mathrm{Ob}_{\mathscr{C}}} \end{aligned}$$

We compare with what $y(f)$ is:

$$\begin{aligned} y(f) &= \left(y_X(f) : \left\{ \begin{array}{ccc} y(D)(X) & \longrightarrow & y(C)(X) \\ g & \longmapsto & y_X(f)(g) \end{array} \right. \right)_{X \in \mathrm{Ob}_{\mathscr{C}}} \\ &= \left(y_X(f) : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(D, X) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(C, X) \\ g & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(f, X)(g) \end{array} \right. \right)_{X \in \mathrm{Ob}_{\mathscr{C}}} \\ &= \left(y_X(f) : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(D, X) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(C, X) \\ g & \longmapsto & g \circ f \end{array} \right. \right)_{X \in \mathrm{Ob}_{\mathscr{C}}} \\ &= \varphi_f^{y(C),D} \end{aligned}$$

Consequently:

$$\theta_{y(C),D} = \begin{cases} \mathrm{Hom}_{\mathscr{C}}(C, D) & \longrightarrow & \mathrm{Hom}_{\mathbf{Sets}^{\mathscr{C}^{\mathrm{op}}}}(y(D), y(C)) \\ f & \longmapsto & y(f) \end{cases}$$

which yields that y is full and faithful. □

Using the Yoneda lemma (both covariant and contravariant) and the fact that the Yoneda embedding is an embedding, one can show the following corollaries:

Corollary 2.15. *Let $\mathscr{C}$ be a locally small category. Then, $\forall C, D \in \mathrm{Ob}_{\mathscr{C}}$, $f : C \to D$ is an isomorphism $\Leftrightarrow$ $Hom_{\mathscr{C}}(-, f) : Hom_{\mathscr{C}}(-, C) \to Hom_{\mathscr{C}}(-, D)$ is an isomorphism.*

Corollary 2.16. *Let $\mathscr{C}$ be a locally small category. Then, $\forall C, D \in \mathrm{Ob}_{\mathscr{C}}$, $f : C \to D$ is an isomorphism $\Leftrightarrow$ $Hom_{\mathscr{C}}(f, -) : Hom_{\mathscr{C}}(D, -) \to Hom_{\mathscr{C}}(C, -)$ is an isomorphism.*

Corollary 2.17. *Let $\mathscr{C}$ be a locally small category. Then, $\forall C, D \in \mathrm{Ob}_{\mathscr{C}}$, $Hom_{\mathscr{C}}(C, -) \cong Hom_{\mathscr{C}}(D, -) \Rightarrow C \cong D$.*

Corollary 2.18. *Let $\mathscr{C}$ be a locally small category. Then, $\forall C, D \in \mathrm{Ob}_{\mathscr{C}}$, $Hom_{\mathscr{C}}(-, C) \cong Hom_{\mathscr{C}}(-, D) \Rightarrow C \cong D$.*

3. Universal elements, universal arrows, representations

Introduction lol.

Definition 3.1 (Universal element [2])**.** Let $\mathscr{C}$ be a category, and $F : \mathscr{C} \to \mathbf{Sets}$ a functor.

The pair $(X^*, e^*) \in \mathrm{Ob}_{\mathscr{C}} \times F(X^*)$ is a universal element for F if the natural transformation $\theta_{F,X^*}(e^*) : \mathrm{Hom}_{\mathscr{C}}(X^*, -) \to F$ is an isomorphism.

Remark 3.2. When one sees Definition 3.1, the two natural questions should be:

- Is this universal element unique?
- Does the Yoneda embedding have a universal element?

The answer to the second question is easy: no, the Yoneda embedding doesn't have a universal element, because it is a functor $\mathscr{C} \to \mathbf{Func}\left(\mathscr{C}^{\mathrm{op}}, \mathbf{Sets}\right)$. However, for $C \in \mathscr{C}$, its C-component $y(C) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Sets}$ could have one.

We are looking for a pair (X^*, e^*) such that:

$$\begin{aligned}
\theta_{y(C),X^*}(e^*) &= \left(\varphi_X : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(X^*, X) & \longrightarrow & y(C)(X) \\ x & \longmapsto & y(C)(x)(e^*) \end{array} \right. \right)_{X \in \mathrm{Ob}_{\mathscr{C}}} \\
&= \left(\varphi_X : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(X^*, X) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(C, X) \\ x & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(C, x)(e^*) \end{array} \right. \right)_{X \in \mathrm{Ob}_{\mathscr{C}}} \\
&= \left(\varphi_X : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(X^*, X) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(C, X) \\ x & \longmapsto & x \circ e^* \end{array} \right. \right)_{X \in \mathrm{Ob}_{\mathscr{C}}}
\end{aligned}$$

What could the pair (X^*, e^*) be for $\theta_{y(C),X^*}(e^*)$ to be an isomorphism? There is one obvious answer: take (C, id_C).

But is this answer unique? Probably not. But is it unique up to isomorphism? The answer to this question lies in Proposition 3.5. Before, we have to show an intermediate proposition.

Remark 3.3. If the functor $F : \mathscr{C} \to \mathbf{Sets}$ is contravariant, then the universal element is a pair (X^*, e^*) such that $\theta^{\mathrm{op}}_{F,X^*}(e^*) : \mathrm{Hom}_{\mathscr{C}}(-, X^*) \to F$, where $\theta^{\mathrm{op}}_{F,X} : F(X) \to \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{C}}(-, X), F\right)$ is the dual of $\theta_{F,X}$.

Proposition 3.4 (Universal mapping property [5])**.** *Let $\mathscr{C}$ be a category and $F : \mathscr{C} \to \mathbf{Sets}$ a functor.*

The pair (C^*, e^*) is a universal element for F if and only if $\forall X \in \mathrm{Ob}_{\mathscr{C}}, \forall e \in F(X), \exists! x \in Hom_{\mathscr{C}}(C^*, X) \;\; e = F(x)(e^*)$.

Proof. Using Definition 3.1:

$$\begin{aligned}
&(C^*, e^*) \text{ is a universal element for } F \\
\Leftrightarrow\; &\theta_{F,X^*}(e^*) = \left(\varphi^{F,X^*}_{e^*,X} : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{X}}(X^*, X) & \longrightarrow & F(X) \\ x & \longmapsto & F(x)(e^*) \end{array} \right. \right)_{X \in \mathrm{Ob}_{\mathscr{X}}} \text{ is an isomorphism} \\
\Leftrightarrow\; &\forall X \in \mathrm{Ob}_{\mathscr{C}},\ \varphi^{F,X^*}_{e^*,X} : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{X}}(X^*, X) & \longrightarrow & F(X) \\ x & \longmapsto & F(x)(e^*) \end{array} \right. \text{ is an isomorphism} \\
\Leftrightarrow\; &\forall X \in \mathrm{Ob}_{\mathscr{C}}, \forall e \in F(X), \exists! x \in \mathrm{Hom}_{\mathscr{C}}(C^*, X) \;\; e = F(x)(e^*)
\end{aligned}$$

□

Proposition 3.5. *Let $\mathscr{C}$ be a category, and $F : \mathscr{C} \to \mathbf{Sets}$ a functor.*

If (X_0, e_0) and (X_1, e_1) are universal elements for F, then there exists a unique isomorphism $\varphi : X_0 \to X_1$ such that $F(\varphi)(e_0) = e_1$.

Proof. If (X_0, e_0) and (X_1, e_1) are universal elements for F, then by Universal Mapping Property (Proposition 3.4):

1. there is a unique $\varphi_0 \in \mathrm{Hom}_{\mathscr{C}}(X_0, X_1)$ such that $F(\varphi_0)(e_0) = e_1$.
2. there is a unique $\varphi_1 \in \mathrm{Hom}_{\mathscr{C}}(X_1, X_0)$ such that $F(\varphi_1)(e_1) = e_0$.
3. there exists a unique $\psi_0 \in \mathrm{Hom}_{\mathscr{C}}(X_0, X_0)$ such that $F(\psi_0)(e_0) = e_0$. However, id_{X_0} also has this property, so $\psi_0 = \mathrm{id}_{X_0}$.
4. there exists a unique $\psi_1 \in \mathrm{Hom}_{\mathscr{C}}(X_1, X_1)$ such that $F(\psi_1)(e_1) = e_1$. However, id_{X_1} also has this property, so $\psi_1 = \mathrm{id}_{X_1}$.

Now let us study $\varphi_0 \circ \varphi_1$ and $\varphi_1 \circ \varphi_0$. Combining items 1 and 2, we have:

$$\begin{aligned} F(\varphi_0)(F(\varphi_1)(e_1)) &= F(\varphi_0)(e_0) \\ F(\varphi_0) \circ F(\varphi_1)(e_1) &= e_1 \\ F(\varphi_0 \circ \varphi_1)(e_1) &= e_1 \end{aligned} \tag{10}$$

$$\begin{aligned} F(\varphi_1)(F(\varphi_0)(e_0)) &= F(\varphi_1)(e_1) \\ F(\varphi_1) \circ F(\varphi_0)(e_0) &= e_0 \\ F(\varphi_1 \circ \varphi_0)(e_0) &= e_0 \end{aligned} \tag{11}$$

As id_{X_0} (resp. id_{X_1}) is the unique arrow such that $F\left(\mathrm{id}_{X_0}\right)(e_0) = e_0$ (resp. $F\left(\mathrm{id}_{X_1}\right)(e_1) = e_1$), we deduce from Equation 10 (resp. from Equation 11) that $\varphi_1 \circ \varphi_0 = \mathrm{id}_{X_0}$ (resp. $\varphi_0 \circ \varphi_1 = \mathrm{id}_{X_1}$). Consequently, φ_0 is the isomorphism described in the proposition. □

Definition 3.6 (Representable functor)**.** Let $\mathscr{C}$ be a category, and $F : \mathscr{C} \to \mathbf{Sets}$ a functor.
A *representation of F* is a pair (X^*, ψ) where:

- $X^* \in \mathrm{Ob}_{\mathscr{C}}$ is called the *representing object of F*
- $\psi : \mathrm{Hom}_{\mathscr{C}}(X^*, -) \to F$ is a natural isomorphism.

The functor F is said *representable* if such a representation exists.

Remark 3.7. As in Remark 3.3, a representation of a contravariant functor F is a pair (X^*, ψ) such that $\psi : \mathrm{Hom}_{\mathscr{C}}(-, X^*) \to F$ is a natural isomorphism.

Lemma 3.8. *Let $\mathscr{C}$ be a category, and $F : \mathscr{C} \to$ **Sets** a functor.*
If (X^*, e^*) is a universal element for F then $\left(X^*, \theta_{F,X^*}(e^*)\right)$ is a representation of F.

Lemma 3.9. *Let $\mathscr{C}$ be a category, and $F : \mathscr{C} \to$ **Sets** a functor.*
If (X^*, ψ) is a representation of F, then $(X^*, \psi_{X^*}(\mathrm{id}_{X^*}))$ is a universal element for F.

Theorem 3.10. *Let $\mathscr{C}$ be a category, and $F : \mathscr{C} \to$ **Sets** a functor.*
There exists a universal element for $F \Leftrightarrow F$ is representable.

Theorem 3.10 is an immediate consequence of the two previous lemmas. Besides, the two lemmas give a way to convert a universal element into a representation.

Proof of Lemma 3.8. Let (X^*, e^*) be a universal element for F; it follows from Definition 3.1 that $\theta_{F,X^*}(e^*) : \mathrm{Hom}_{\mathscr{C}}(X^*, -) \to F$ is a natural isomorphism. Thence, $\left(X^*, \theta_{F,X^*}(e^*)\right)$ is a representation of F. □

Proof of Lemma 3.9. Let (X^*, ψ) be a representation of F. By Proposition 2.6, we have $\psi = \theta_{F,X^*}(\psi_{X^*}(\mathrm{id}_{X^*}))$. Besides, by definition of a representation, $\psi = \theta_{F,X^*}(\psi_{X^*}(X^*))$ is an isomorphism, which gives that $(X^*, \psi_{X^*}(\mathrm{id}_{X^*}))$ is a universal element for F. □

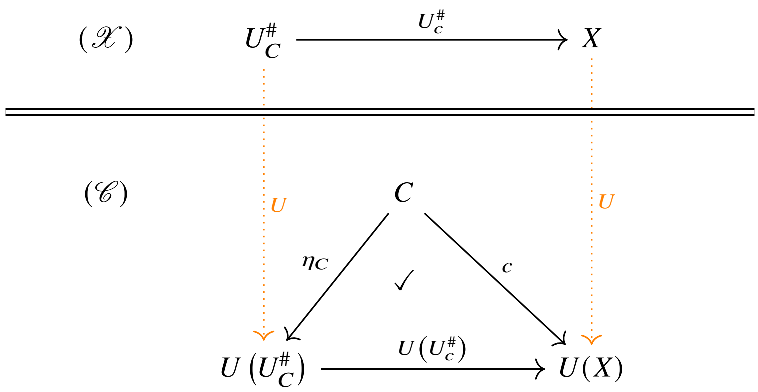
Corollary 3.11. *Representations of a functor F are unique up to isomorphism.*

Definition 3.12 (Universal arrows [2])**.** Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ be a functor and let $C \in \mathrm{Ob}_{\mathscr{C}}$.

A *universal arrow from C to U* is a pair $\left(U_C^\#, \eta_C\right)$, where $U_C^\# \in \mathrm{Ob}_{\mathscr{X}}$ and $\eta_C \in \mathrm{Hom}_{\mathscr{C}}\left(C, U\left(U_C^\#\right)\right)$, such that, for all $X \in \mathrm{Ob}_{\mathscr{X}}$, for all $c \in \mathrm{Hom}_{\mathscr{C}}(C, U(X))$, there exists a unique $x \in \mathrm{Hom}_{\mathscr{X}}\left(U_C^\#, X\right)$ such that $c = U(x) \circ \eta_C$.

In the following, this unique x will be denoted $U_c^\#$, such that: $c = U\left(U_c^\#\right) \circ \eta_C$.

Here is a diagram that sums up the idea behind universal arrows:



Lemma 3.13. *Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ be a functor and let $C \in \mathrm{Ob}_{\mathscr{C}}$.*

If $\left(U_C^\#, \eta_C\right)$ is a universal arrow from C to U, then $U_{\eta_C}^\# = \mathrm{id}_{U_C^\#}$.

Proof. The arrow $U_{\eta_C}^\#$ is the unique arrow that verifies: $\eta_C = U\left(U_{\eta_C}^\#\right) \circ \eta_C$. Of course, $\eta_C = U\left(\mathrm{id}_{U_C^\#}\right) \circ \eta_C$, so $\mathrm{id}_{U_C^\#} = U_{\eta_C}^\#$. □

Proposition 3.14. *Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ be a functor and let $C \in \mathrm{Ob}_{\mathscr{C}}$.*

1. *$\left(U_C^\#, \eta_C\right)$ is a universal arrow from C to U $\Leftrightarrow$ $\theta_{Hom_{\mathscr{C}}(C, U(-)), U_C^\#}(\eta_C)$ is a natural isomorphism.*

2. *If for all $C \in \mathrm{Ob}_{\mathscr{C}}$, there exists $X_C \in \mathrm{Ob}_{\mathscr{C}}$ and a natural isomorphism $\varphi : Hom_{\mathscr{X}}(X_C, -) \to Hom_{\mathscr{C}}(C, U(-))$, then $\left(X_C, \varphi_{X_C}\left(\mathrm{id}_{X_C}\right)\right)$ is a universal arrow from C to U.*

Proof. **[Item 1]**

By definition of a universal arrow $\left(U_C^\#, \eta_C\right)$, for all $X \in \mathrm{Ob}_{\mathscr{X}}$, for all $c \in \mathrm{Hom}_{\mathscr{C}}(C, U(X))$, there exists a unique $x \in \mathrm{Hom}_{\mathscr{X}}\left(U_C^\#, X\right)$ such that $c = U(x) \circ \eta_C$; equivalently, for all $X \in \mathrm{Ob}_{\mathscr{X}}$, the function:

$$\varphi_X : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{X}}\left(U_C^\#, X\right) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(C, U(X)) \\ x & \longmapsto & U(x) \circ \eta_C \end{array} \right.$$

is an isomorphism; that is, those φ_X are the components of $\theta_{\mathrm{Hom}_{\mathscr{C}}(C, U(-)), U_C^\#}(\eta_C)$, which is a natural isomorphism.

[Item 2]

Let $C \in \mathscr{C}$ and $X \in \mathscr{X}$.

We have $X_C \in \mathrm{Ob}_{\mathscr{C}}$ and a natural isomorphism $\varphi : \mathrm{Hom}_{\mathscr{X}}(X_C, -) \to \mathrm{Hom}_{\mathscr{C}}(C, U(-))$. We have $\varphi = \theta_{\mathrm{Hom}_{\mathscr{C}}(C, U(-)), X_C}\left(\varphi_{X_C}\left(\mathrm{id}_{X_C}\right)\right)$, which by Item 1 yields that $\left(X_C, \varphi_{X_C}\left(\mathrm{id}_{X_C}\right)\right)$ is a universal arrow from C to U. □

When seeing the definitions of universal elements and arrows, we wonder what could be the link between both. In fact, universal arrows and universal elements are two very close notions.

Proposition 3.15. *Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ be a functor and let $C \in \mathrm{Ob}_{\mathscr{C}}$.*
If $\left(U^{\#}_C, \eta_C\right)$ is a universal arrow from C to U, then $\left(U^{\#}_C, \eta_C\right)$ is also a universal element for $Hom_{\mathscr{C}}(C, U(-))$.

Proof. This proposition directly follows from Definitions 3.1 (universal element) and 3.12 (universal arrow). In fact, by definition of a universal element for $\mathrm{Hom}_{\mathscr{C}}(C, U(-))$, the following natural transformation should be an isomorphism:

$$\theta_{\mathrm{Hom}_{\mathscr{C}}(C,U(-)),U^{\#}_C}(\eta_C) = \left(\varphi_Y : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}\left(U^{\#}_C, Y\right) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(C, U(Y)) \\ x & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(C, U(x))(\eta_C) \end{array} \right.\right)_{Y \in \mathrm{Ob}_{\mathscr{C}}}$$

By simplest representation lemma (Lemma 2.1), we have:

$$\mathrm{Hom}_{\mathscr{C}}(C, U(x))(\eta_C) = U(x) \circ \eta_C$$

By definition of a universal arrow, $\forall Y \in \mathrm{Ob}_{\mathscr{X}}, \forall c \in \mathrm{Hom}_{\mathscr{C}}(C, U(Y)), \exists! c = U(x) \circ \eta_C = F_{\eta_C}(Y)(x)$. Thus, $\varphi_Y \in \mathrm{Mor}_{\mathbf{Sets}}$ is a bijection, thus an isomorphism; consequently, the natural transformation $\theta_{\mathrm{Hom}_{\mathscr{C}}(C,U(-)),X_C}(\eta_C)$ is also an isomorphism. □

Remember that universal elements are defined for a functor $\mathscr{C} \to \mathbf{Sets}$, and not just for a functor between any two categories. The converse proposition is a bit less general.

Proposition 3.16. *We denote by 1 the set $\mathcal{P}(\emptyset) = \{\emptyset\} = 1$ where $\emptyset$ is the empty set. For any set E, for any $e \in E$, we define:*

$$\delta^E_e : \left\{ \begin{array}{ccc} 1 & \longrightarrow & E \\ x & \longmapsto & e \end{array} \right.$$

*Let $\mathscr{X}$ be a category, let $U : \mathscr{X} \to$ **Sets** be a functor.*
If (X^*, e^*) is a universal element for U, then $\left(X^*, \delta^{U(X^*)}_{e^*}\right)$ is a universal arrow from 1 to U.

Proof. By Proposition 3.4, if (X^*, e^*) is a universal element for U, then $\forall X \in \mathscr{X}, \forall e \in U(X), \exists! x \in \mathrm{Hom}_{\mathscr{X}}(X^*, X)$ such that:

$$\begin{aligned} & e = U(x)(e^*) \\ \Leftrightarrow\ & \delta^{U(X)}_e(0) = U(x)\left(\delta^{U(X^*)}_{e^*}(0)\right) \\ \Leftrightarrow\ & \delta^{U(X)}_e = U(x) \circ \delta^{U(X^*)}_{e^*} \end{aligned}$$

Consequently, we have: $\forall X \in \mathscr{X}, \forall \delta^{U(X)}_e \in \mathrm{Hom}_{\mathbf{Sets}}(1, U(X)), \exists! x \in \mathrm{Hom}_{\mathscr{X}}(X^*, X)$, $\delta^{U(X)}_e = U(x) \circ \delta^{U(X^*)}_{e^*}$, which yields that $\left(X^*, \delta^{U(X^*)}_{e^*}\right)$ is a universal arrow from 1 to U. □

We sum up the results into this theorem:

Theorem 3.17. *Let $\mathscr{X}$ be a category, and let $U : \mathscr{X} \to$ **Sets** be a functor.*

1. *(X^*, e^*) is a universal element for $U \Leftrightarrow \left(X^*, \theta_{U,X^*}(e^*)\right)$ is a representation of U.*
2. *(X^*, e^*) is a universal element for $U \Leftrightarrow \left(X^*, \delta^{U(X^*)}_{e^*}\right)$ is a universal arrow from 1 of U.*
3. *(X^*, ψ) is a representation of $U \Leftrightarrow (X^*, \psi_X(\mathrm{id}_X))$ is a universal element for U.*
4. *$\left(U^{\#}_C, \eta_C\right)$ is a universal arrow from C to $U \Leftrightarrow \left(U^{\#}_C, \eta_C\right)$ is a universal element for $Hom_{\mathbf{Sets}}(C, U(-))$.*

4. Towards adjunctions

Definition 4.1 (Left adjoint - from universal arrows)**.** Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ be a functor. We suppose that for all $C \in \mathrm{Ob}_{\mathscr{C}}$, there exists a universal arrow $\left(U^\#_C, \eta_C\right)$ from C to U.

The *left adjoint of* U, denoted by U^*, is the mapping:

$$U^* : \begin{cases} \mathscr{C} & \longrightarrow & \mathscr{X} \\ C & \longmapsto & U^\#_C \\ c : C \to C' & \longmapsto & U^\#_{\eta_{C'} \circ c} \end{cases}$$

Let's study some properties of the left adjoint.

Lemma 4.2. *Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ be a functor, and let U^* be the left adjoint of U.*

For any $c \in Hom_{\mathscr{C}}(C, C')$, $U^*(c)$ is the unique solution in $x \in Hom_{\mathscr{X}}\left(U^\#_C, U^\#_{C'}\right)$ to the equation: $\eta_{C'} \circ c = U(x) \circ \eta_C$.

Proof. We have $\eta_{C'} \circ c \in \mathrm{Hom}_{\mathscr{C}}\left(C, U\left(U^\#_{C'}\right)\right)$. By definition of a universal arrow $\left(U^\#_C, \eta_C\right)$ from C to U, there exists a unique $U^\#_{\eta_{C'} \circ c} \in \mathrm{Hom}_C\left(C'\right)$ such that: $\eta_{C'} \circ c = U\left(U^\#_{\eta_{C'} \circ c}\right) \circ \eta_C = U\left(U^*(c)\right) \circ \eta_C$. □

Proposition 4.3. *Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ be a functor, and let U^* be the left adjoint of U.*

The left adjoint $U^* : \mathscr{C} \to \mathscr{X}$ is a functor.

Proof. The mapping U^* sends objects (resp. arrows) in $\mathscr{C}$ to objects (resp. arrows) in $\mathscr{X}$.

Using Lemma 3.13, we check the behaviour of U^* on identity arrows:

$$U^*\left(\mathrm{id}_C\right) = U^\#_{\eta_C \circ \mathrm{id}_C} = U^\#_{\eta_C} = \mathrm{id}_{U^\#_C}$$

As for the composition, let $c : C \to C'$ and $c' : C' \to C''$. By definition of U^*, we have:

$$\begin{aligned} \eta_{C'} \circ c &= U\left(U^*(c)\right) \circ \eta_C \\ \eta_{C''} \circ c' &= U\left(U^*\left(c'\right)\right) \circ \eta_{C'} \\ \eta_{C''} \circ c' \circ c &= U\left(U^*\left(c' \circ c\right)\right) \circ \eta_C \end{aligned} \tag{12}$$

But also:

$$\begin{aligned} \eta_{C''} \circ c' \circ c &= \left(U\left(U^*\left(c'\right)\right) \circ \eta_{C'}\right) \circ c \\ &= U\left(U^*\left(c'\right)\right) \circ \left(\eta_{C'} \circ c\right) \\ &= U\left(U^*\left(c'\right)\right) \circ U\left(U^*\left(c\right)\right) \\ &= U\left(U^*\left(c'\right) \circ U^*\left(c\right)\right) \end{aligned} \tag{13}$$

Equations 12 and 13, together with Lemma 4.2, yield:

$$U^*\left(c' \circ c\right) = U^*\left(c'\right) \circ U^*\left(c\right)$$

□

Proposition 4.4. *Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ be a functor, and let U^* be the left adjoint of U.*

The mapping $\eta = \left(\eta_C : C \to U \circ U^*(C)\right)$ is a natural transformation $\eta : \mathrm{Id}_{\mathscr{C}} \to U \circ U^*$.

Proof. We need to check if, for each $c : C \to C'$, the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "c"'] \arrow[dr, phantom, "\checkmark"] & U\left(U^*(C)\right) \arrow[d, "U(U^*(c))"] \\ C' \arrow[r, "\eta_{C'}"'] & U\left(U^*\left(C'\right)\right) \end{tikzcd}$$

That is, we need to check whether $U\left(U^*(c)\right) \circ \eta_C = \eta_{C'} \circ c$, which is the result of Lemma 4.2. □

Proposition 4.5. *Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ be a functor, and let U^* be the left adjoint of U.*

For all $X \in \mathscr{X}$, for all $C \in \mathscr{C}$, we define:

$$\beta_{C,X} : \begin{cases} Hom_{\mathscr{X}}\left(U^*(C), X\right) & \longrightarrow & Hom_{\mathscr{C}}\left(C, U(X)\right) \\ x & \longmapsto & U(x) \circ \eta_C \end{cases}$$

The mapping $\beta : C, X \mapsto \beta_{C,X}$ is a natural isomorphism $Hom_{\mathscr{X}}\left(U^*(-), -\right) \to Hom_{\mathscr{C}}\left(-, U(-)\right)$, contravariant in C and covariant in X.

Proof. For $C \in \mathscr{C}$, $\beta_{C,-} : \mathrm{Hom}_{\mathscr{X}}\left(U^*(C), -\right) \to \mathrm{Hom}_{\mathscr{C}}\left(C, U(-)\right)$ is the same function as $\beta_{C,-} = \theta_{\mathrm{Hom}_{\mathscr{C}}(C,U(-)),U^*(C)}\left(\eta_C\right)$ which we know is a natural isomorphism (cf. Proposition 3.14, item 1).

For $X \in \mathscr{X}$, we study $\beta_{-,X} : \mathrm{Hom}_{\mathscr{X}}\left(U^*\left(-\right), X\right) \to \mathrm{Hom}_{\mathscr{C}}\left(-, U(X)\right)$. Let $c : C \to C'$, we want the following diagram to commute:

$$\begin{tikzcd} C \arrow[d, "c"'] & & \mathrm{Hom}_{\mathscr{X}}\left(U^*\left(C'\right), X\right) \arrow[r, "\beta_{C',X}"] \arrow[d, "\mathrm{Hom}_{\mathscr{X}}(U^*(c),X)"'] \arrow[dr, phantom, "?"] & \mathrm{Hom}_{\mathscr{C}}\left(C', U(X)\right) \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(c,U(X))"] \\ C' & \rightsquigarrow & \mathrm{Hom}_{\mathscr{X}}\left(U^*\left(C\right), X\right) \arrow[r, "\beta_{C,X}"'] & \mathrm{Hom}_{\mathscr{C}}\left(C, U(X)\right) \end{tikzcd}$$

(Note that $\beta_{-,X}$ is supposed to be contravariant in C).

Let $f \in \mathrm{Hom}_{\mathscr{X}}\left(U^*\left(C'\right), X\right)$. On the one hand:

$$\begin{aligned} \beta_{C,X} \circ \mathrm{Hom}_{\mathscr{X}}\left(U^*\left(c\right), X\right)(f) &= \beta_{C,X}\left(f \circ U^*\left(c\right)\right) \\ &= U\left(f \circ U^*\left(c\right)\right) \circ \eta_C \end{aligned}$$

while on the other hand:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}\left(c, U(X)\right) \circ \beta_{C',X}(f) &= \mathrm{Hom}_{\mathscr{C}}\left(c, U(X)\right)\left(U\left(f\right) \circ \eta_{C'}\right) \\ &= U\left(f\right) \circ \eta_{C'} \circ c \\ &= U\left(f\right) \circ U\left(U^*\left(c\right)\right) \circ \eta_C \\ &= U\left(f \circ U^*\left(c\right)\right) \circ \eta_C \\ &= \beta_{C,X} \circ \mathrm{Hom}_{\mathscr{X}}\left(U^*\left(c\right), X\right)(f) \end{aligned}$$

Thus, $\beta_{-,X}$ is a natural transformation. Note that each component in C of $\beta_{-,X} = \left(\beta_{C,X}\right)_{C \in \mathrm{Ob}_{\mathscr{C}}}$ is an isomorphism (because $\theta_{\mathrm{Hom}_{\mathscr{C}}(C,U(-)),U^*(C)}\left(\eta_C\right)$, for any $C \in \mathrm{Ob}_{\mathscr{C}}$, is a natural isomorphism), thus so is $\beta_{-,X}$. □

Proposition 4.6. *Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ be a functor, and let U^* be the left adjoint of U.*

If $F : \mathscr{C} \to \mathscr{X}$ is a functor for which there exists a natural isomorphism $\gamma : Hom_{\mathscr{X}}(F(-), -) \to Hom_{\mathscr{C}}(-, U(-))$, then there exists a unique natural isomorphism $\alpha : F \to U^*$; in other words, the left adjoint is unique up to a unique isomorphism.

Proof. Let $F : \mathscr{C} \to \mathscr{X}$ such that $\gamma : \mathrm{Hom}_{\mathscr{X}}(F(-), -) \to \mathrm{Hom}_{\mathscr{C}}(-, U(-))$ is a natural isomorphism.

For all $C \in \mathscr{C}$, we have $\gamma_{C,-} = \theta_{\mathrm{Hom}_{\mathscr{C}}(C,U(-)),F(C)}(e_C)$ for some $e_C : C \to U(F(C)) = \gamma_{C,F(C)}\left(\mathrm{id}_{F(C)}\right)$. We deduce from Proposition 3.14-2 that F is a left adjoint for U. As U^* and F are left adjoints for U, then $(F(C), e_C)$ and $(U^*(C), \eta_C)$ are universal arrows from C to U (Definition 4.1), so $(F(C), e_C)$ and $(U^*(C), \eta_C)$ are also universal elements for $\mathrm{Hom}_{\mathscr{C}}(C, U(-))$ (Proposition 3.15). According to Proposition 3.5, there exists a unique isomorphism $a_C : U^*(C) \to F(C)$ such that $U(a_C)(e_C) = \eta_C$.

We now have to show that $\alpha = (\alpha_C)_{C \in \mathrm{Ob}_{\mathscr{C}}}$ is natural in C.

We have the following diagram:

$$\begin{tikzcd}
& C \arrow[dl, "\eta_C"'] \arrow[dr, "e_C"] \arrow[dd, "c"] & \\
U(F(C)) \arrow[rr, "U(\alpha_C)"] \arrow[dd, "U(F(c))"'] & & U(U^*(C)) \arrow[dd, "U(U^*(c))"] \\
& C' \arrow[dl, "\eta_{C'}"'] \arrow[dr, "e_{C'}"] & \\
U(F(C')) \arrow[rr, "U(\alpha_{C'})"] & & U(U^*(C'))
\end{tikzcd}$$

where the following subdiagrams commute:

$$\begin{tikzcd}
& C \arrow[dl, "\eta_C"'] \arrow[dr, "e_C"] \arrow[d, phantom, "\checkmark"] & \\
U(F(C)) \arrow[rr, "U(\alpha_C)"] & {} & U(U^*(C))
\end{tikzcd}
\qquad
\begin{tikzcd}
& C' \arrow[dl, "\eta_{C'}"'] \arrow[dr, "e_{C'}"] \arrow[d, phantom, "\checkmark"] & \\
U(F(C')) \arrow[rr, "U(\alpha_{C'})"] & {} & U(U^*(C'))
\end{tikzcd}$$

due to the construction of α_C and:

$$\begin{tikzcd}
U(F(C)) \arrow[d, "U(F(c))"'] \arrow[dr, phantom, "\checkmark"] & C \arrow[l, "\eta_C"'] \arrow[r, "e_C"] \arrow[d, "c"] \arrow[dr, phantom, "\checkmark"] & U(U^*(C)) \arrow[d, "U(U^*(c))"] \\
U(F(C')) & C' \arrow[l, "\eta_{C'}"] \arrow[r, "e_{C'}"'] & U(U^*(C'))
\end{tikzcd}$$

due to the naturality of $e : \mathrm{Id}_{\mathscr{C}} \to U \circ F$ and $\eta : \mathrm{Id}_{\mathscr{C}} \to U \circ U^*$.

By diagram chasing, we have:

$$U(\alpha_{C'}) \circ U(F(c)) \circ \eta_C = U(U^*(c)) \circ U(\alpha_C) \circ \eta_C$$
$$U(\alpha_{C'} \circ F(c)) \circ \eta_C = U(U^*(c) \circ \alpha_C) \circ \eta_C$$
$$\alpha_{C'} \circ F(c) = U^*(c) \circ \alpha_C$$

The last equation is due to $\beta_{C,X}$ being an isomorphism. This equation makes the following diagram commute:

$$\begin{tikzcd}
U\left(F(C)\right) \arrow[rr, "\alpha_C"] \arrow[d, "F(c)"'] & \checkmark & U^*(C) \arrow[d, "U^*(c)"] \\
F(C') \arrow[rr, "\alpha_{C'}"] & & U^*(C')
\end{tikzcd}$$

which makes $\alpha = (\alpha_C)_{C\in \mathrm{Ob}_{\mathscr{C}}}$ natural in C. □

Definition 4.7 (Adjunction - official)**.** Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ and $F : \mathscr{C} \to \mathscr{X}$ be two functors.

The 3-tuple (F, U, β) is called an *adjunction* whenever β is a natural isomorphism $\beta : \mathrm{Hom}_{\mathscr{X}}(F(-), -) \to \mathrm{Hom}_{\mathscr{C}}(-, U(-))$.

We also say that F is *left adjoint to* U and U is *right adjoint to* F. We will refer to β as the *adjunctor*[1]of F and B.

If (F, U, β) is an adjunction, we may write $F \dashv U$ or $\dfrac{F(C) \to X}{C \to U(X)}\ (\beta)$.

The following lemma proves that if (F, U, β) is an adjunction, then F is actually left adjoint to U as defined in Definition 4.1. In fact, both definitions are equivalent.

Lemma 4.8. *Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ and $F : \mathscr{C} \to \mathscr{X}$ be two functors.*

$F \dashv U \Leftrightarrow$ there exists a natural transformation $\eta : \mathrm{Id}_{\mathscr{C}} \to U \circ F$ such that $\forall C \in \mathrm{Ob}_{\mathscr{C}},\ (F(C), \eta_C)$ is a universal arrow from C to U.

Proof. **[Proof of ⇐]**

Suppose that we have a $\eta : \mathrm{Id}_{\mathscr{C}} \to U \circ F$ such that $\forall C \in \mathrm{Ob}_{\mathscr{C}},\ (F(C), \eta_C)$ is a universal arrow from C to U. According to the definition of a left adjoint, F corresponds to a left adjoint on objects. We have to check if, for all $X \in \mathscr{X}$, for all $c : C \to C'$, $F(c)$ is the unique solution in $x \in \mathrm{Hom}_{\mathscr{X}}(F(C), X)$ to the equation:

$$\eta'_C \circ c = U(x) \circ \eta_C \tag{14}$$

The natural transformation $\eta : \mathrm{Id}_{\mathscr{C}} \to U \circ F$ makes the following diagram commute:

$$\begin{tikzcd}
C \arrow[rr, "\eta_C"] \arrow[d, "c"'] & \checkmark & U\left(F(C)\right) \arrow[d, "U(F(c))"] \\
C' \arrow[rr, "\eta_{C'}"'] & & U\left(F(C')\right)
\end{tikzcd}$$

which proves that $F(c)$ is indeed a solution to Equation 14. The uniqueness of the solution comes from the definition of a universal arrow (Definition 3.12).

Consequently, F is a left adjoint to U. By Proposition 4.5, we can define a β from η that is a natural isomorphism $\mathrm{Hom}_{\mathscr{X}}(F(C), X) \to \mathrm{Hom}_{\mathscr{C}}(C, U(X))$. Finally, (F, U, β) is an adjunction.

[Proof of ⇒]

Suppose $F \dashv U$, and suppose (F, U, β) is an adjunction. Define η to be the natural transformation with components:

$$\eta_C = \beta_{C,F(C)}\left(\mathrm{id}_{F(C)}\right) \in \mathrm{Hom}_{\mathscr{C}}\left(C, U\left(F\left(C\right)\right)\right)$$

[1]The β natural isomorphism appears to be unamed in most references. However, in the rest of this course, it may be convenient to give it a name. Please note that nobody but the authors give this name to that isomorphism.

The naturality of η comes from the naturality of $\beta = \left(\beta_{C,X}\right)_{C\in\mathscr{C},X\in\mathscr{X}}$ in both its variables.
The naturality in X gives, when $c : C \to C'$, for $F(c) : F(C) \to F(C') \in \mathscr{X}$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{X}}\left(F(C),F(C)\right) \arrow[r, "\beta_{C,F(C)}"] \arrow[d, "\mathrm{Hom}_{\mathscr{X}}(F(C),F(c))"'] \arrow[dr, phantom, "\checkmark"] & \mathrm{Hom}_{\mathscr{X}}\left(C,U\circ F(C)\right) \arrow[d, "\mathrm{Hom}_{\mathscr{X}}(C,U\circ F(c))"] \\
\mathrm{Hom}_{\mathscr{X}}\left(F(C),F(C')\right) \arrow[r, "\beta_{C,F(C')}"'] & \mathrm{Hom}_{\mathscr{X}}\left(C,U\circ F(C')\right)
\end{tikzcd}$$

while the naturality in C gives, for $c : C \to C'$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{X}}\left(F(C'),F(C')\right) \arrow[r, "\beta_{C',F(C')}"] \arrow[d, "\mathrm{Hom}_{\mathscr{X}}(F(c),F(C'))"'] \arrow[dr, phantom, "\checkmark"] & \mathrm{Hom}_{\mathscr{X}}\left(C',U\circ F(C')\right) \arrow[d, "\mathrm{Hom}_{\mathscr{X}}(c,U\circ F(C'))"] \\
\mathrm{Hom}_{\mathscr{X}}\left(F(C),F(C')\right) \arrow[r, "\beta_{C,F(C')}"'] & \mathrm{Hom}_{\mathscr{X}}\left(C,U\circ F(C')\right)
\end{tikzcd}$$

From the first diagram, we obtain:

$$\begin{aligned}
\beta_{C,F(C')} \circ \mathrm{Hom}_{\mathscr{X}}\left(F(C),F(c)\right)\left(\mathrm{id}_{F(C)}\right) &= \mathrm{Hom}_{\mathscr{X}}\left(C,(U\circ F)(c)\right)\circ\beta_{C,F(C)}\left(\mathrm{id}_{F(C)}\right) \\
\beta_{C,F(C')} \circ F(c) &= (U\circ F)(c)\circ\beta_{C,F(C)}\left(\mathrm{id}_{F(C)}\right)
\end{aligned} \tag{15}$$

and from the second diagram:

$$\begin{aligned}
\beta_{C,F(C')} \circ \mathrm{Hom}_{\mathscr{X}}\left(F(c),F(C')\right)\left(\mathrm{id}_{F(C')}\right) &= \mathrm{Hom}_{\mathscr{X}}\left(c,U\circ F(C')\right)\circ\beta_{C',F(C')}\left(\mathrm{id}_{F(C')}\right) \\
\beta_{C,F(C')} \circ F(c) &= \beta_{C',F(C')}\left(\mathrm{id}_{F(C')}\right)\circ c
\end{aligned} \tag{16}$$

Combining Equations 15 and 16, we obtain:

$$\begin{aligned}
(U\circ F)(c)\circ\beta_{C,F(C)}\left(\mathrm{id}_{F(C)}\right) &= \beta_{C',F(C')}\left(\mathrm{id}_{F(C')}\right)\circ c \\
(U\circ F)(c)\circ\eta_C &= \eta_{C'}\circ c
\end{aligned}$$

which proves that η is a natural transformation.

We have to show that each $(F(C),\eta_C)$ is a universal arrow from C to U. We have a natural isomorphism $\beta_{C,-} : \mathrm{Hom}_{\mathscr{X}}\left(F(C),-\right) \to \mathrm{Hom}_{\mathscr{C}}\left(C,U(-)\right)$; so according to Proposition 3.14, $\left(F(C),\beta_{C,F(C)}\left(\mathrm{id}_{F(C)}\right)\right) = (F(C),\eta_C)$ is a universal arrow. Besides, F is a left adjoint to U. □

Definition 4.9 (Unit of an adjunction)**.** Let (F,U,β) be an adjunction.

The *unit of the adjunction* (F,U,β) is the natural transformation $\eta : \mathrm{Id}_{\mathscr{C}} \to U\circ F$ such that $\forall C \in \mathrm{Ob}_{\mathscr{C}},\ \eta_C = \beta_{C,F(C)}\left(\mathrm{id}_{F(C)}\right)$.

We will define the dual notion of a counit. However, we will not construct it explictly as we did the unit (that is, using universal arrows, then left adjoints), because it is not that interesting. We will first compute the inverse of the adjunctor β.

Note that Lemma 4.8 proves that Definitions 4.1 and 4.7 are not only equivalent, but also that we can construct the unit η from the adjunctor β and conversely. The same goes from the counit ε that we will define right after the following lemma.

Lemma 4.10. *Let (F, U, β) be an adjunction.*

For all $C \in \mathrm{Ob}_{\mathscr{C}}$, define $\eta_C = \beta_{C,F(C)}\left(\mathrm{id}_{F(C)}\right)$. Then, $\beta_{C,X}$ is:

$$\beta_{C,X} : \begin{cases} Hom_{\mathscr{X}}\left(F(C), X\right) & \longrightarrow & Hom_{\mathscr{C}}\left(C, U(X)\right) \\ x & \longmapsto & U(x) \circ \eta_C \end{cases}$$

For all $X \in \mathrm{Ob}_{\mathscr{X}}$, define $\varepsilon_X = \beta^{-1}_{U(X),X}\left(\mathrm{id}_{U(X)}\right)$. Then, $\beta^{-1}_{C,X}$ is:

$$\beta^{-1}_{C,X} : \begin{cases} Hom_{\mathscr{C}}\left(C, U(X)\right) & \longrightarrow & Hom_{\mathscr{X}}\left(F(C), X\right) \\ c & \longmapsto & \varepsilon_X \circ F(c) \end{cases}$$

Proof. By naturality of β and β^{-1}, and for $c : C \to U(X)$ and $x : F(C) \to X$, the following two diagrams commute:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{X}}\left(F\left(C\right), F\left(C\right)\right) \arrow[r, "\beta_{C,F(C)}"] \arrow[d, "\mathrm{Hom}_{\mathscr{X}}(F(C),x)"'] \arrow[dr, phantom, "\checkmark"] & \mathrm{Hom}_{\mathscr{C}}\left(C, U \circ F\left(C\right)\right) \arrow[d, "\mathrm{Hom}_{\mathscr{X}}(C,U(x))"] \\
\mathrm{Hom}_{\mathscr{X}}\left(F(C), X\right) \arrow[r, "\beta_{C,X}"'] & \mathrm{Hom}_{\mathscr{C}}\left(C, U(X)\right)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}\left(U\left(X\right), U\left(X\right)\right) \arrow[r, "\beta^{-1}_{U(X),X}"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(c,U(X))"'] \arrow[dr, phantom, "\checkmark"] & \mathrm{Hom}_{\mathscr{X}}\left(F \circ U\left(X\right), X\right) \arrow[d, "\mathrm{Hom}_{\mathscr{X}}(F(c),X)"] \\
\mathrm{Hom}_{\mathscr{C}}\left(C, U(X)\right) \arrow[r, "\beta^{-1}_{C,X}"'] & \mathrm{Hom}_{\mathscr{X}}\left(F(C), X\right)
\end{tikzcd}$$

Suppose we have $\beta_{C,X}(x) = c$ (or equivalently $\beta^{-1}_{C,X}(c) = x$). Those two diagrams combine into this one:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{X}}\left(F\left(C\right), F\left(C\right)\right) \arrow[d, "\beta_{C,F(C)}"'] \arrow[ddr, "\mathrm{Hom}_{\mathscr{X}}(F(C),x)"' near end] & \mathrm{Hom}_{\mathscr{C}}\left(U\left(X\right), U\left(X\right)\right) \arrow[d, "\beta^{-1}_{U(X),X}"] \arrow[ddl, "\mathrm{Hom}_{\mathscr{C}}(c,U(X))" near end] \\
\mathrm{Hom}_{\mathscr{C}}\left(C, U \circ F\left(C\right)\right) \arrow[d, bend right, "\mathrm{Hom}_{\mathscr{X}}(C,U(x))"'] & \mathrm{Hom}_{\mathscr{X}}\left(F \circ U\left(X\right), X\right) \arrow[d, bend left, "\mathrm{Hom}_{\mathscr{X}}(F(c),X)"] \\
\mathrm{Hom}_{\mathscr{C}}\left(C, U(X)\right) \arrow[r, bend left, "\beta^{-1}_{C,X}"] \arrow[r, phantom, "\checkmark"] & \mathrm{Hom}_{\mathscr{X}}\left(F(C), X\right) \arrow[l, bend left, "\beta_{C,X}"]
\end{tikzcd}$$

Firstly, with $\mathrm{id}_{F(C)} \in \mathrm{Hom}_{\mathscr{X}}\left(F(C), F(C)\right)$, we have:

$$\begin{aligned}
\beta^{-1}_{C,X} \circ \mathrm{Hom}_{\mathscr{C}}\left(C, U(x)\right) \circ \beta_{C,F(C)}\left(\mathrm{id}_{F(C)}\right) &= \mathrm{Hom}_{\mathscr{X}}\left(F(C), x\right)\left(\mathrm{id}_{F(C)}\right) \\
\beta^{-1}_{C,X}\left(U(x) \circ \beta_{C,F(C)}\left(\mathrm{id}_{F(C)}\right)\right) &= x \circ \mathrm{id}_{F(C)} \\
\beta^{-1}_{C,X}\left(U(x) \circ \eta_C\right) &= x
\end{aligned}$$

Secondly, with $\mathrm{id}_{U(X)} \in \mathrm{Hom}_{\mathscr{C}}\left(U(X), U(X)\right)$, we have:

$$\begin{aligned}
\beta_{C,X} \circ \mathrm{Hom}_{\mathscr{X}}\left(F(c), X\right) \circ \beta^{-1}_{U(X),X}\left(\mathrm{id}_{U(X)}\right) &= \mathrm{Hom}_{\mathscr{C}}\left(c, U(X)\right)\left(\mathrm{id}_{U(X)}\right) \\
\beta_{C,X} \circ \beta^{-1}_{U(X),X}\left(\mathrm{id}_{U(X)}\right) \circ F(c) &= \mathrm{id}_{U(X)} \circ c \\
\beta_{C,X}\left(\varepsilon_X \circ F(c)\right) &= c
\end{aligned}$$

The first calculation shows that $\beta_{C,X}(x) = U(x) \circ \eta_C$ and the second shows that $\beta^{-1}_{C,X}(c) = \varepsilon_X \circ F(c)$. $\square$

Lemma 4.11. *Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ and $F : \mathscr{C} \to \mathscr{X}$ be two functors.*
$F \dashv U \Leftrightarrow$ there exists a natural transformation $\varepsilon : F \circ U \to \mathrm{Id}_{\mathscr{X}}$ such that $\forall X \in \mathrm{Ob}_{\mathscr{X}}$, $\forall C \in \mathrm{Ob}_{\mathscr{C}}$ and $\forall x : F(C) \to X$, there exists a unique arrow $c : C \to U(X)$ such that: $\varepsilon_X \circ F(c) = x$.

Proof. **[Proof of $\Rightarrow$]**
If $F \dashv U$, then let (F, U, β) be the adjunction. We have $\beta^{-1}_{C,X} : \mathrm{Hom}_{\mathscr{X}}\left(-, U(-)\right) \to \mathrm{Hom}_{\mathscr{X}}\left(F(-), -\right)$.
Define $\varepsilon = (\varepsilon_X)_{X \in \mathrm{Ob}_{\mathscr{X}}}$ to be:

$$\varepsilon_X = \beta^{-1}_{U(X),X}\left(\mathrm{id}_{U(X)}\right)$$

A diagram chasing very similar to the one in the proof of Lemma 4.8 shows that ε is a natural transformation.

Let $X \in \mathrm{Ob}_{\mathscr{X}}$, let $C \in \mathrm{Ob}_{\mathscr{C}}$ and let $x : F(C) \to X$. The existence and unicity of the $c : C \to U(X)$ such that $\varepsilon_X \circ F(c) = x$ comes from the bijectivity of $\beta^{-1}_{C,X}$ as the equation is also: $\beta^{-1}_{C,X}(c) = x$. Of course, that c is $c = \beta_{C,X}(x) = U(x) \circ \eta_C$.

[Proof of $\Leftarrow$]
Define:

$$\gamma_{C,X} : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}\left(C, U(X)\right) & \longrightarrow & \mathrm{Hom}_{\mathscr{X}}\left(F(C), X\right) \\ c & \longmapsto & \varepsilon_X \circ F(x) \end{array} \right.$$

The definition of ε states that each $\gamma_{C,X}$ is an isomorphism. Now we have to prove that $\gamma = \left(\gamma_{C,X}\right)_{C \in \mathrm{Ob}_{\mathscr{C}}, X \in \mathrm{Ob}_{\mathscr{X}}}$ is natural (but contravariant) in C and (covariant) in X.

$$\begin{tikzcd}
C \arrow[d, "c"'] & & \mathrm{Hom}_{\mathscr{X}}\left(C', U(X)\right) \arrow[r, "\gamma_{C',X}"] \arrow[d, "\mathrm{Hom}_{\mathscr{X}}(c, U(X))"'] \arrow[dr, phantom, "?"] & \mathrm{Hom}_{\mathscr{X}}\left(F\left(C'\right), X\right) \arrow[d, "\mathrm{Hom}_{\mathscr{X}}(F(c), X)"] \\
C' & \rightsquigarrow & \mathrm{Hom}_{\mathscr{X}}\left(C, U(X)\right) \arrow[r, "\gamma_{C,X}"'] & \mathrm{Hom}_{\mathscr{X}}\left(F\left(C\right), X\right)
\end{tikzcd}$$

For $f \in \mathrm{Hom}_{\mathscr{X}}\left(C', U(X)\right)$, we have:

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{X}}\left(F\left(c\right), X\right) \circ \gamma_{C',X}(f) &= \varepsilon_X \circ F(f) \circ F(c) \\
\gamma_{C,X} \circ \mathrm{Hom}_{\mathscr{X}}\left(c, U(X)\right)(f) &= \varepsilon_X \circ F(f \circ c)
\end{aligned}$$

So γ is natural in C. As for the naturality in X:

$$
\begin{tikzcd}
X \arrow[d, "x"] & \rightsquigarrow & \mathrm{Hom}_{\mathscr{X}}\left(C, U(X)\right) \arrow[r, "\gamma_{C,X}"] \arrow[d, "\mathrm{Hom}_{\mathscr{X}}\left(C, U(x)\right)"'] \arrow[dr, phantom, "?"] & \mathrm{Hom}_{\mathscr{X}}\left(F\left(C\right), X\right) \arrow[d, "\mathrm{Hom}_{\mathscr{X}}\left(F\left(C\right), x\right)"] \\
X' & & \mathrm{Hom}_{\mathscr{X}}\left(C, U\left(X'\right)\right) \arrow[r, "\gamma_{C,X'}"'] & \mathrm{Hom}_{\mathscr{X}}\left(F\left(C\right), X'\right)
\end{tikzcd}
$$

Let $f \in \mathrm{Hom}_{\mathscr{X}}\left(C, U(X)\right)$:

$$
\begin{aligned}
\gamma_{C,X} \circ \mathrm{Hom}_{\mathscr{X}}\left(F\left(C\right), x\right)\left(f\right) &= x \circ \varepsilon_X \circ F(f) \\
\mathrm{Hom}_{\mathscr{X}}\left(C, U(x)\right) \circ \gamma_{C,X'}(f) &= \varepsilon_{X'} \circ F\left(U(x)\right) \circ F\left(f\right)
\end{aligned}
$$

Don't forget that ε is a natural transformation $F \circ U \to \mathrm{Id}_{\mathscr{X}}$. Thence, we have the following commutative diagram:

$$
\begin{tikzcd}
F \circ U(X) \arrow[r, "\varepsilon_X"] \arrow[d, "F \circ U(x)"'] \arrow[dr, phantom, "\checkmark"] & X \arrow[d, "x"] \\
F \circ U\left(X'\right) \arrow[r, "\varepsilon_{X'}"'] & X
\end{tikzcd}
$$

which gives:

$$
x \circ \varepsilon_X = \varepsilon_{X'} \circ F\left(U(x)\right)
$$

and finally:

$$
\begin{aligned}
x \circ \varepsilon_X \circ F(f) &= \varepsilon_{X'} \circ F\left(U(x)\right) \circ F\left(f\right) \\
\gamma_{C,X} \circ \mathrm{Hom}_{\mathscr{X}}\left(F\left(C\right), x\right)\left(f\right) &= \mathrm{Hom}_{\mathscr{X}}\left(C, U(x)\right) \circ \gamma_{C,X'}(f)
\end{aligned}
$$

Thus, γ is a natural transformation in both X and C; each component is an isomorphism, so γ is a natural isomorphism $\mathrm{Hom}_{\mathscr{C}}\left(-, U(-)\right) \to \mathrm{Hom}_{\mathscr{X}}\left(F(-), -\right)$; so γ^{-1} is a natural isomorphism $\mathrm{Hom}_{\mathscr{X}}\left(F(-), -\right) \to \mathrm{Hom}_{\mathscr{C}}\left(-, U(-)\right)$.

For all $C \in \mathrm{Ob}_{\mathscr{C}}$, define $\eta_C = \gamma^{-1}_{C,F(C)}\left(\mathrm{id}_{F(C)}\right)$. By Proposition 3.14, item 2, we know that $\left(F(C), \eta_C\right)$ is a universal arrow from C to U, which makes F the left adjoint of U by Lemma 4.8. □

Definition 4.12 (Counit of an adjunction)**.** Let (F, U, β) be an adjunction.

The *counit of the adjunction* (F, U, β) is the natural transformation $\varepsilon : F \circ U \to \mathrm{Id}_{\mathscr{X}}$ such that $\forall X \in \mathrm{Ob}_{\mathscr{X}},\ \varepsilon_X = \beta^{-1}_{U(X),X}\left(\mathrm{id}_{U(X)}\right)$.

The notion of adjunction appears everywhere in mathematics. As this notion is very important, we need to give many examples.

Example 4.13 (Identity). Let $\mathscr{C}$ be a category. Then the identity functor $\mathrm{Id}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ is both left and right adjoint of itself; and $\mathrm{Id}_{-,-} : \mathrm{Hom}_{\mathscr{C}}\left(\mathrm{Id}_{\mathscr{C}}(-), -\right) \to \mathrm{Hom}_{\mathscr{C}}\left(-, \mathrm{Id}_{\mathscr{C}}(-)\right)$ is the adjunctor. The unit and counit are: $\eta, \varepsilon : \mathrm{Id}_{\mathscr{C}} \to \mathrm{Id}_{\mathscr{C}}$.

Example 4.14 (Isomorphisms). Let $\mathscr{C}, \mathscr{X}$ be categories, and let $F : \mathscr{C} \to \mathscr{X}$ be an isomorphism between those categories. Then $F \dashv F^{-1} \dashv F$. In fact, $x : F(C) \to X \in \mathscr{X} \Leftrightarrow F^{-1}(x) : C \to F^{-1}(X)$ and $c : F^{-1}(X) \to C \Leftrightarrow F(c) : X \to F(C)$.

For the adjunction $\left(F, F^{-1}, \beta\right)$, the adjunctor β has components:

$$\beta_{C,X} : \begin{cases} \mathrm{Hom}_{\mathscr{X}}\left(F(C), X\right) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}\left(C, F^{-1}(X)\right) \\ x & \longmapsto & F^{-1}(x) \end{cases}$$

while for the second adjunction $\left(F^{-1}, F, \gamma\right)$, the adjunctor γ has components:

$$\gamma_{C,X} : \begin{cases} \mathrm{Hom}_{\mathscr{X}}\left(F^{-1}(X), C\right) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}\left(X, F(C)\right) \\ c & \longmapsto & F(c) \end{cases} = \beta^{-1}_{C,X}$$

The units and counits are the identity natural transformations.

Example 4.15 (Increasing linear function). Let $\mathscr{R} = (\mathbb{R}, \leqslant)$ be the category of the totally ordered set $\mathbb{R}$, equipped with the usual order on real numbers.

Objects: An object in $\mathscr{R}$ is a real number $x \in \mathbb{R}$

Morphisms: There is an arrow $x_0 \to x_1$ if and only if $x_0 \leqslant x_1$

Identities: An identity morphism is an arrow $x \to x$

Composition: If $x_0 \to x_1$ and $x_1 \to x_2$ are two arrows, then there is one arrow $x_0 \to x_2$

Note that there is only one arrow between two objects (real numbers) x_0, x_1; if $x_0 < x_1$ then $\mathrm{Hom}_{\mathbb{R}}\left(x_0, x_1\right)$ contains only one arrow, while $\mathrm{Hom}_{\mathbb{R}}\left(x_1, x_0\right)$ is empty. Similarly, there is only one arrow $x \to x$, and it is the identity. Finally, the composition law on arrows comes from the transitivity of the order relation $\leqslant$.

Let $F : \mathscr{R} \to \mathscr{R}$ be the functor: $F : x \to ax + b$ with $a > 0$. Let's check if F is actually a functor.

If $x_0 \to x_1$, then $x_0 \leqslant x_1$, which gives $ax_0 + b \leqslant ax_1 + b$, thus $F(x_0) \to F(x_1)$. If $x \to x$ then $F(x) \to F(x)$. Finally, if $x_0 \to x_1$ and $x_1 \to x_2$ then $F(x_0) \to F(x_1)$ and $F(x_1) \to F(x_2)$ and $F(x_0) \to F(x_2)$ (by transitivity of $\leqslant$).

Now suppose you have $x_0, x_1 \in \mathbb{R}$ such that:

$$\begin{aligned} F(x_0) \leqslant x_1 &\Leftrightarrow ax_0 + b \leqslant x_1 \\ &\Leftrightarrow x_0 \leqslant \frac{1}{a}x_1 - \frac{b}{a} \end{aligned}$$

Define $U : \begin{cases} \mathscr{R} & \longrightarrow & \mathscr{R} \\ x & \longmapsto & \dfrac{1}{a}x_1 - \dfrac{b}{a} \end{cases}$. Then F is left adjoint to U. The adjunctor β transforms arrows $F(x_0) \to x_1$ to arrows $x_0 \to U(x_1)$. The unit η and counit ε are the identity natural transformations $\mathrm{Id}_{\mathscr{R}} \to \mathrm{Id}_{\mathscr{R}}$.

Example 4.16 (Decreasing linear function). We can build a similar example of adjunctor using $\mathscr{R} = (\mathbb{R}, \leqslant)$ and its opposite category $\mathscr{R}^{\mathrm{op}} = (\mathbb{R}, \geqslant)$.

Let $\mathscr{R}^{\mathrm{op}} = (\mathbb{R}, \geqslant)$ be the category of the totally ordered set $\mathbb{R}$, equipped with the usual order on real numbers.

Objects: An object in $\mathscr{R}^{\mathrm{op}}$ is a real number $x \in \mathbb{R}$

Morphisms: There is an arrow $x_0 \to x_1$ if and only if $x_0 \geqslant x_1$

Identities: An identity morphism is an arrow $x \to x$

Composition: If $x_0 \to x_1$ and $x_1 \to x_2$ are two arrows, then there is one arrow $x_0 \to x_2$

Note that $(\mathbb{R}, \geqslant)$ is actually the opposite category of $(\mathbb{R}, \leqslant)$.
Let $F : \mathscr{R}^{\mathrm{op}} \to \mathscr{R}$ be the functor: $F : x \to ax + b$ with $a < 0$. Let $x_0, x_1 \in \mathbb{R}$ such that:

$$\begin{aligned} F(x_0) \geqslant x_1 &\Leftrightarrow ax_0 + b \geqslant x_1 \\ &\Leftrightarrow x_0 \leqslant \frac{1}{a} x_1 - \frac{b}{a} \\ &\Leftrightarrow x_0 \leqslant U(x_1) \end{aligned}$$

We can define the same U as in the previous example; and F is again left adjoint to U. The adjunctor β transforms arrows $F(x_0) \to x_1$ in $\mathscr{R}^{\mathrm{op}}$ to arrows $x_0 \to U(x_1)$ in $\mathscr{R}$. The unit and counit are the identity natural transformations as $U = F^{-1}$.

Example 4.17 (Image and inverse image of a function). Let $f : X \to Y$ be a function between two sets X and Y. The two categories will be the partially ordered sets $\mathscr{X} = (\mathcal{P}(X), \subseteq)$ and $\mathscr{Y} = (\mathcal{P}(Y), \subseteq)$ equipped with the usual inclusion of sets.

Define the three functors:

$$F : \begin{cases} \mathscr{X} & \longrightarrow & \mathscr{Y} \\ A & \longmapsto & f(A) = \{f(a) \mid a \in A\} \end{cases}$$

$$G : \begin{cases} \mathscr{Y} & \longrightarrow & \mathscr{X} \\ B & \longmapsto & f^{-1}(B) = \{b \in X \mid f(b) \in B\} \end{cases}$$

$$F^* : \begin{cases} \mathscr{X} & \longrightarrow & \mathscr{Y} \\ A & \longmapsto & \left\{y \in Y \mid f^{-1}(\{y\}) \subseteq A\right\} \end{cases}$$

The functor F gives the image of a subset of X, G gives the inverse image of a subset of Y and F^* gives the subset of inverse images of singletons of elements of Y. We let the reader check that those three functions are actually functors.

Suppose we have $F(A) \to B$. For all $a \in A$, $f(a) \in F(A) \subseteq B$, so for all $a \in A$, $a \in f^{-1}(B) = G(B)$, which gives $A \subseteq G(B)$. Conversely, suppose we have $A \to G(B)$. For all $a \in A$, $a \in G(B) = \{b \in X \mid f(b) \in B\}$, so for all $a \in A$, $f(a) \in B$, which gives $F(A) \subseteq B$.

We have $F \dashv G$:

$$F(A) \subseteq B \Leftrightarrow A \subseteq G(B)$$

The adjunctor β transforms arrows $F(A) \to B$ to arrows $A \to G(B)$. Note that $A \subset G \circ F(A)$ but there is in general no reason why A should be equal to $G \circ F(A)$ (except if f is injective). Consequently, the unit of the adjunction is:

$$\eta = \left(\eta_A : \begin{cases} A & \longrightarrow & G \circ F(A) \\ a & \longmapsto & a \end{cases}\right)_{A \subseteq X}$$

Similarly, note that $F \circ G(B) \subset B$ but there is in general no reason why $F \circ G(B)$ should be equal to B (except if f is surjective), so the counit is:

$$\varepsilon = \left(\varepsilon_B : \begin{cases} F \circ G(B) & \longrightarrow & B \\ b & \longmapsto & b \end{cases}\right)_{B \subseteq Y}$$

Besides, we also have $G \dashv F^*$. In fact, suppose we have $G(A) \subseteq B$. Then, $\forall x \in A$, $f^{-1}(\{x\}) \subset G(A) \subset B$, so $\forall x \in A, x \in F^*(B)$, hence $A \subseteq F^*(B)$. Conversely, if $A \subseteq F^*(B)$ then $\forall x \in G(A)$, we have:

$$\begin{aligned} x \in G(A) &\Rightarrow f(x) \in A \subset F^*(B) \\ &\Rightarrow f(x) \in \left\{y \in Y \mid f^{-1}(\{y\}) \subseteq B\right\} \\ &\Rightarrow f^{-1}(\{f(x)\}) \subseteq B \\ &\Rightarrow x \in B \end{aligned}$$

So we have $G(A) \subseteq B$.
The adjunctor β^* transforms arrows $G(A) \to B$ to arrows $A \to F^*(B)$.
Before computing the unit and counit, note that, for $B \subset Y$:

$$\begin{aligned} F^*(G(B)) &= F^*\left(f^{-1}(B)\right) \\ &= \left\{y \in Y \mid f^{-1}(\{y\}) \subseteq f^{-1}(B)\right\} \\ &= \left\{y \in Y \mid \forall x \in f^{-1}(\{y\}), f(x) \in B\right\} \\ &= \{y \in Y \mid y \in B \wedge \exists x \in X, y = f(x)\} \\ &= B \cap f(X) \subseteq B \end{aligned}$$

The interpretation is the following: $F^*(G(B))$ is the biggest subset of B that contains only images by f. Again, $F^*(G(B))$ has no reason to be equal to B, except if f is surjective.

Also, for $A \subset X$:

$$\begin{aligned} G\left(F^*(A)\right) &= f^{-1}\left(F^*(A)\right) \\ &= \{x \in X \mid f(x) \in F^*(A)\} \\ &= \left\{x \in X \mid f(x) \in \left\{y \in Y \mid f^{-1}(\{y\}) \subseteq A\right\}\right\} \\ &= \left\{x \in X \mid f^{-1}(\{f(x)\}) \subseteq A\right\} \\ &= \bigcup \{C \subseteq X \mid f(C) \subseteq f(A)\} \\ &\supseteq A \end{aligned}$$

The interpretation of $G\left(F^*(A)\right)$ is as follows: $G\left(F^*(A)\right)$ is the biggest subset of X that gives $f(A)$. Again, $G\left(F^*(A)\right)$ has no reason to be equal to A, except if f is injective.

In this case, the unit and counit are not easy to write. In fact, we will need to create an equivalence relation over X, for example $x = y \mod f \Leftrightarrow f(x) = f(y)$. Then we will need a section function that sends an equivalence class to its representative. Such a choice of section function should be chosen to be compatible with what we want from the unit and counit.

Example 4.18 (Galois connections). The previous three examples are special cases of monotone Galois connections. Every Galois connection between two posets is an adjunction.

Further examples of adjunctions will appear in the rest of the text, and we even propose a zoo of adjunctions in the next section.

Definition 4.19 (Whiskering)**.** Let $F, F' : \mathscr{C} \to \mathscr{C}'$ and $G, G : \mathscr{C}' \to \mathscr{D}$ be functors, and let $\alpha : F \to F'$ and $\beta : G \to G'$ be natural transformations.

1. The whiskering of G and α, denoted by $G \circ \alpha$, is the natural transformation: $G \circ \alpha : G \circ F \to G \circ F'$ with components $(G(\alpha_C) : G \circ F(C) \to G \circ F'(C))_{C \in \mathscr{C}}$.

2. The whiskering of β and F, denoted by $\beta \circ F$, is the natural transformation: $\beta \circ F : G \circ F \to G' \circ F$ with components $\left(\alpha_{G(C)} : G \circ F(C) \to G' \circ F(C)\right)_{C \in \mathscr{C}}$.

It is easy to check that:

Proposition 4.20. *Let $F, F' : \mathscr{C} \to \mathscr{C}'$ and $G, G : \mathscr{C}' \to \mathscr{D}$ be functors, and let $\alpha : F \to F'$ and $\beta : G \to G'$ be natural transformations.*

$$
\begin{aligned}
(H \circ G) \circ \beta &= H \circ (G \circ \beta) & \qquad \mathrm{Id}_{\mathscr{C}'} \circ \beta &= \beta \\
\alpha \circ (F \circ H) &= (\alpha \circ F) \circ H & \qquad \alpha \circ \mathrm{Id}_{\mathscr{C}} &= \alpha
\end{aligned}
$$

Remark 4.21. Proposition 4.20 simply states that whiskering (on the left or on the right) can be seen as a (left or right) monoid action of the monoid of functors (with composition) over the class of natural transformations. In other words, whiskerings and compositions are "associative" in a sense.

Proposition 4.22. *Let $F, F' : \mathscr{C} \to \mathscr{C}'$ and $G, G : \mathscr{C}' \to \mathscr{D}$ be functors, and let $\alpha : F \to F'$ and $\beta : G \to G'$ be natural transformations.*

Then the following diagram commutes:

$$
\begin{tikzcd}
G \circ F \arrow[r, "\beta \circ F"] \arrow[d, "G \circ \alpha"'] \arrow[dr, phantom, "\checkmark"] & G' \circ F \arrow[d, "G' \circ \alpha"] \\
G \circ F' \arrow[r, "\beta \circ F'"'] & G' \circ F'
\end{tikzcd}
$$

Proof. For $C \in \mathscr{C}$, consider the following "implemented" diagram:

$$
\begin{tikzcd}
G\left(F(C)\right) \arrow[r, "\beta_{F(C)}"] \arrow[d, "G(\alpha_C)"'] & G'\left(F(C)\right) \arrow[d, "G'(\alpha_C)"] \\
G\left(F'(C)\right) \arrow[r, "\beta_{F'(C)}"'] & G'\left(F'(C)\right)
\end{tikzcd}
$$

This diagram commutes because β is a natural transformation $G \to G'$ and α_C is an arrow $F(C) \to F'(C)$. □

Proposition 4.23 (Triangle identities)**.** *Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ and $F : \mathscr{C} \to \mathscr{X}$ be two functors. Let $\eta : \mathrm{Id}_{\mathscr{C}} \to U \circ F$ and $\varepsilon : F \circ U \to \mathrm{Id}_{\mathscr{X}}$ be natural transformations.*

The tuple $(F, U, \eta, \varepsilon)$ is an adjunction iff the following triangles commute:

$$
\begin{tikzcd}
F \arrow[r, "F \circ \eta"] \arrow[dr, "\mathrm{id}_F"'] & F \circ U \circ F \arrow[d, "\varepsilon \circ F"] \\
& F
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "\eta \circ U"] \arrow[dr, "\mathrm{id}_U"'] & U \circ F \circ U \arrow[d, "U \circ \varepsilon"] \\
& U
\end{tikzcd}
$$

Proof. **[Proof of ⇒]** Suppose $(F, U, \eta, \varepsilon)$ is an adjunction. According to Lemma 4.10, we can compute the adjunctor from the unit and counit:

$$
\beta_{C,X} : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{X}}\left(F(C), X\right) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}\left(C, U(X)\right) \\ x & \longmapsto & U(x) \circ \eta_C \end{array} \right.
$$

$$
\beta^{-1}_{C,X} : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}\left(C, U(X)\right) & \longrightarrow & \mathrm{Hom}_{\mathscr{X}}\left(F(C), X\right) \\ c & \longmapsto & \varepsilon_X \circ F(c) \end{array} \right.
$$

Also, from Definition 4.9 and 4.12, we deduce the triangle identities:

$$\begin{aligned}\eta_C = \beta_{C,F(C)}\left(\mathrm{id}_{F(C)}\right) &\Leftrightarrow \beta^{-1}_{C,F(C)}\left(\eta_C\right) = \mathrm{id}_{F(C)}\\ &\Leftrightarrow \varepsilon_{F(C)} \circ F\left(\eta_C\right) = \mathrm{id}_{F(C)}\\ &\Leftrightarrow (\varepsilon \circ F)_C \circ (F \circ \eta)_C = \mathrm{id}_{F(C)}\end{aligned}$$

$$\begin{aligned}\varepsilon_X = \beta^{-1}_{U(X),X}\left(\mathrm{id}_{U(X)}\right) &\Leftrightarrow \beta_{U(X),X}\left(\varepsilon_X\right) = \mathrm{id}_{U(X)}\\ &\Leftrightarrow U\left(\varepsilon_X\right) \circ \eta_{U(X)} = \mathrm{id}_{U(X)}\\ &\Leftrightarrow (U \circ \varepsilon)_X \circ (\eta \circ U)_X = \mathrm{id}_{U(X)}\end{aligned}$$

[Proof
(in place of β^{-1}) from ε, and we prove that γ is the inverse of β:

$$\beta_{C,X} : \begin{cases} \mathrm{Hom}_{\mathscr{X}}\left(F(C), X\right) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}\left(C, U(X)\right) \\ x & \longmapsto & U(x) \circ \eta_C \end{cases}$$

$$\gamma_{C,X} : \begin{cases} \mathrm{Hom}_{\mathscr{C}}\left(C, U(X)\right) & \longrightarrow & \mathrm{Hom}_{\mathscr{X}}\left(F(C), X\right) \\ c & \longmapsto & \varepsilon_X \circ F(c) \end{cases}$$

And then, for $x : F(C) \to X$, we have:

$$\gamma_{C,X} \circ \beta_{C,X}(x) = \varepsilon_X \circ F\left(U(x) \circ \eta_C\right)$$

This is equal to x due to the following diagram:

$$\begin{tikzcd} F(C) \arrow[r, "F(\eta_C)"] \arrow[dr, "\mathrm{id}_{F(C)}"'] & F \circ U \circ F(C) \arrow[r, "F \circ U(x)"] \arrow[d, "\varepsilon_{F(C)}"] & F \circ U(X) \arrow[d, "\varepsilon_X"] \\ & F(C) \arrow[r, "x"'] & X \end{tikzcd}$$

The left-hand triangle commutes because of the triangle identities; the right-hand square commutes because it represents the naturality of $\varepsilon : F \circ U \to \mathrm{Id}_{\mathscr{X}}$.

The converse equality is similarly proven:

$$\beta_{C,X} \circ \gamma_{C,X}(c) = U\left(\varepsilon_X \circ F(c)\right) \circ \eta_C$$

which is equal to c according to the following diagram:

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, "\eta_C"'] & U(X) \arrow[d, "\eta_{U(X)}"'] \arrow[dr, "\mathrm{id}_{U(X)}"] & \\ U \circ F(C) \arrow[r, "U \circ F(c)"'] & U \circ F \circ U(X) \arrow[r, "U(\varepsilon_X)"'] & U(X) \end{tikzcd}$$

The left-hand square commutes because η is a natural transformation, and the right-square commutes because of the triangle identities.

Thus, β and γ are both natural isomorphisms (the proof of naturality is not interesting and is left to the reader) and are inverses of each other. □

We finally give a third definition of adjunction:

Definition 4.24 (Adjoint - triangle identities)**.** Let $\mathscr{X}, \mathscr{C}$ be two categories. Let $U : \mathscr{X} \to \mathscr{C}$ and $F : \mathscr{C} \to \mathscr{X}$ be two functors. Let $\eta : \mathrm{Id}_{\mathscr{C}} \to U \circ F$ and $\varepsilon : F \circ U \to \mathrm{Id}_{\mathscr{X}}$ be natural transformations.

The tuple $(F, U, \eta, \varepsilon)$ is called an adjunction if the following triangles commute:

$$\begin{tikzcd} F \arrow[r, "F\circ\eta"] \arrow[dr, "\mathrm{id}_F"'] & F \circ U \circ F \arrow[d, "\varepsilon\circ F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "\eta\circ U"] \arrow[dr, "\mathrm{id}_U"'] & U \circ F \circ U \arrow[d, "U\circ\varepsilon"] \\ & U \end{tikzcd} \tag{17}$$

In the rest of this book, the left-hand diagram will be referred to as the "left-adjoint triangle identity" (because it mainly concerns F, the left adjoint) and the right-hand diagram will be referred to as the "right-adjoint triangle identity" (because it mainly concerns U, the right adjoint). Note however that the usage of this terminology is specific to this book. Other categorists will understand but might have come to other terms to refer to these triangles.

5. Zoo of adjunctions

Adjunctions are a huge part of category theory. In this section, we present a bunch of adjunctions and non-adjunctions. Most of them are examples or counterexamples of questions that the authors had at some point.

5.1. What is the difference between an adjunction and an equivalence of categories?

We start with a counterexample that we think is important to see. Rather, this is an ambiguity, and maybe a doubt, that is worth removing. In short: equivalences of categories are NOT adjunctions.

We need some definitions before showing this counterexample.

Definition 5.1 (Equivalence of categories). Let $\mathscr{C}$ and $\mathscr{X}$ be two categories, with functors $U : \mathscr{X} \to \mathscr{C}$ and $F : \mathscr{C} \to \mathscr{X}$.

The pair (F, U) is an *equivalence of categories* when there exist two natural isomorphisms $F \circ U \cong \mathrm{Id}_{\mathscr{X}}$ and $U \circ F \cong \mathrm{Id}_{\mathscr{C}}$.

It is easy to see that:

Proposition 5.2. *The equivalence of categories is an equivalence relation.*

The crucial thing to see here, is that an equivalence of categories is not the same as an isomorphism. In fact:

Proposition 5.3. *Let $F : \mathscr{C} \to \mathscr{X}$ be an isomorphic functor. Then $\left(F, F^{-1}\right)$ is an equivalence of categories.*

The converse is false; there are examples of equivalences of categories that are not isomorphisms. This is because equivalences of categories ("same worth") do not tell the same thing as isomorphisms ("same form"). Let us introduce a few notions as an example of the intuition that we will explain.

Definition 5.4 (Skeletal category). A category $\mathscr{C}$ is said *skeletal* when, for all objects $C, C' \in \mathscr{C}$, $C \cong C' \Rightarrow C = C'$.

Definition 5.5 (Skeleton of a category). Let $\mathscr{C}$ be a category.

A *skeleton of* $\mathscr{C}$, denoted by $S(\mathscr{C})$ is a full, skeletal subcategory of $\mathscr{C}$ such that the inclusion functor $i : (\mathscr{C}) \hookrightarrow \mathscr{C}$ verifies:

$$\forall C \in \mathscr{C},\ \exists X \in S(\mathscr{C}),\ C \cong X$$

Example 5.6. In **Sets**, the skeleton is the class of cardinals: two isomorphic sets in **Sets** are simply sets with the same cardinality.

Example 5.7. In a preorder category, the skeleton is the partial order on the equivalence classes of its elements.

In the previous examples, we refered to "the" skeleton of a category. This is due to the following proposition:

Proposition 5.8. *The skeleton of a category is unique up to isomorphism.*

Proof. Let $i : \mathscr{S} \hookrightarrow \mathscr{C}$ and $i' : \mathscr{S}' \hookrightarrow \mathscr{C}$ be two inclusion functors from two skeletons of $\mathscr{C}$ to $\mathscr{C}$.

Let $F : \mathscr{S} \to \mathscr{S}'$ be the following functor. For $S \in \mathscr{S}$, we have $S \in \mathscr{C}$, and there is a unique $S' \in \mathscr{S}'$ such that $S \cong S'$. We call $F(S)$ that $S' = F(S)$.

Also, choose (Axiom of Choice!) an isomorphism $i_S : S \to F(S) \in \mathscr{C}$ for each $S \in \mathscr{S}$. Then, for each $s : S \to T \in \mathscr{S} \subset \mathscr{C}$, define $F(s) = i_T \circ s \circ {i_S}^{-1}$. Then $F : \mathscr{S} \to \mathscr{S}'$ is an isomorphism. □

Some authors, like [6], consider this unicity up to isomorphism to be part of the definition of a skeleton.

The existence of a skeleton depends on the Axiom of Choice:

Proposition 5.9. *Assuming the Axiom of Choice, every category has a skeleton.*

Proof. For each equivalence class of objects (under the relation "is isomorphic to"), using a choice function, choose one representative, and keep the same morphisms between any two objects. Then the resulting category is a skeleton of its base category. □

However:

Proposition 5.10. *Every preorder category has a skeleton.*

Sketch of proof. Cf. Example 5.7. □

Here are two links between skeletons and equivalences of categories.

Proposition 5.11. *A category is equivalent to its skeleton.*

Proof. The pair consisting of the inclusion functor and a left inverse of it is an equivalence of categories. □

Lemma 5.12. *Two equivalent skeletal categories are isomorphic.*

Proof. Let $\mathscr{C}$ and $\mathscr{X}$ be two equivalent skeletal categories; let (F, U) be the pair of functors witnessing the equivalence. Then $F \circ U \cong \mathrm{Id}_{\mathscr{X}}$ and $U \circ F \cong \mathrm{Id}_{\mathscr{C}}$. However, in a skeletal category, for all $C \in \mathscr{C}$, we have: $U \circ F(C) \cong \mathrm{Id}_{\mathscr{C}}(C) = C \Rightarrow U \circ F(C) = C$. Similarly, for all $X \in \mathscr{X}$, $F \circ U(X) = X$. Of course, by naturality, this is also true for arrows in both categories:

$$\begin{tikzcd} U \circ F(C) \arrow[r, leftrightarrow, "="] \arrow[d, "U \circ F(c)"'] \arrow[dr, phantom, "\checkmark"] & C \arrow[d, "c"] \\ U \circ F(C') \arrow[r, leftrightarrow, "="'] & C' \end{tikzcd} \qquad \begin{tikzcd} F \circ U(X) \arrow[r, leftrightarrow, "="] \arrow[d, "F \circ U(x)"'] \arrow[dr, phantom, "\checkmark"] & X \arrow[d, "x"] \\ F \circ U(X') \arrow[r, leftrightarrow, "="'] & X' \end{tikzcd}$$

Then, we have $U \circ F = \mathrm{Id}_{\mathscr{C}}$ and $F \circ U = \mathrm{Id}_{\mathscr{X}}$, hence the result. □

Theorem 5.13. *Two categories are equivalent $\Leftrightarrow$ they have isomorphic skeletons.*

Proof of $\Rightarrow$. By transitivity of the equivalence of categories, if $\mathscr{C}$ and $\mathscr{X}$ are equivalent, then their skeletons are also equivalent. By Lemma 5.12, equivalent skeletons are isomorphic. □

Proof of $\Leftarrow$. Let $\mathscr{C}$ and $\mathscr{X}$ be two categories with isomorphic skeletons. Isomorphic skeletons are also equivalent, due to Proposition 5.3. According to Proposition 5.11, a category is equivalent to its skeleton; by transitivity of the relation of equivalence of categories, $\mathscr{C}$ and $\mathscr{X}$ are equivalent. □

Remark 5.14. An interpretation of this theorem is the following. In order to prevent confusion, we call "isomorphism class" an equivalence class of the relation "is isomorphic to". This is a relation on objects of a category. Two objects are in the same isomorphism class if they are isomorphic.

Isomorphic categories are categories that are "exactly the same": same number of objects, same number of arrows. Equivalent categories are categories that have the same number of arrows but not necessarily the same number of objects in each isomorphism class of objects.

The skeleton of a category has only one object. Given the skeleton $S(\mathscr{C})$ of a category $\mathscr{C}$, we can make a coproduct of that skeleton with itself, and then add one isomorphism between objects that are the same. Then, the resulting category $S(\mathscr{C}) \times 2$ would have two objects in each class of isomorphisms. This category $S(\mathscr{C}) \times 2$ is equivalent, but not isomorphic, to both $\mathscr{C}$ and $S(\mathscr{C})$. This construction can be made with any cardinal number, or even, different cardinal numbers per isomorphism class.

In pictures, the following categories are equivalent, but not isomorphic:

$$
\begin{tikzcd}
& * & \\
& * \arrow[u] & \\
* \arrow[ur] \arrow[rr, leftrightarrow] & & * \arrow[ul] \\
& * \arrow[ul] \arrow[ur] &
\end{tikzcd}
\quad eq. \quad
\begin{tikzcd}
* \\
* \arrow[u] \\
* \arrow[u] \\
* \arrow[u]
\end{tikzcd}
\quad eq. \quad
\begin{tikzcd}
* \arrow[r, leftrightarrow] & * \arrow[r, leftrightarrow] & * \arrow[r, leftrightarrow] & * \arrow[r, leftrightarrow] & * \\
* \arrow[u] \arrow[r, leftrightarrow] & * \arrow[u] \arrow[r, leftrightarrow] & * \arrow[u] \arrow[r, leftrightarrow] & * \arrow[u] \arrow[r, leftrightarrow] & * \arrow[u] \\
* \arrow[u] \arrow[r, leftrightarrow] & * \arrow[u] \arrow[r, leftrightarrow] & * \arrow[u] \arrow[r, leftrightarrow] & * \arrow[u] \arrow[r, leftrightarrow] & * \arrow[u] \\
* \arrow[u] \arrow[r, leftrightarrow] & * \arrow[u] \arrow[r, leftrightarrow] & * \arrow[u] \arrow[r, leftrightarrow] & * \arrow[u] \arrow[r, leftrightarrow] & * \arrow[u]
\end{tikzcd}
$$

This chapter is still about adjunctions, but we needed these notions in order to introduce the following:

Definition 5.15 (Adjoint equivalence)**.** Let $\mathscr{C}$ and $\mathscr{X}$ be two categories, with functors $U : \mathscr{X} \to \mathscr{C}$ and $F : \mathscr{C} \to \mathscr{X}$.

A pair of functors (F, U) is called an *adjoint equivalence* when $(F, U, \eta, \varepsilon)$ is an adjunction, and the unit $\eta : \mathrm{Id}_{\mathscr{C}} \to U \circ F$ and counit $\varepsilon : F \circ U \to \mathrm{Id}_{\mathscr{X}}$ are natural isomorphisms.

The following properties are easy to see:

Proposition 5.16. *Let $(F, U, \eta, \varepsilon)$ be an adjoint equivalence. Then:*

1. *$(F, U, \eta, \varepsilon)$ is an equivalence of categories*
2. *$\left(U, F, \varepsilon^{-1}, \eta^{-1}\right)$ is also an adjoint equivalence*
3. *$F \dashv U \dashv F$ (both F and U are left and right adjoints of the other)*

We can now procede to three counterexamples.

Theorem 5.17 (Informal)**.** *We can find examples and counterexamples of the following statements:*

1. *Adjunctions may or may not be equivalences*
2. *Equivalences may or may not be adjunctions, but any equivalence can be turned into an adjoint equivalence*
3. *If $F \dashv U \dashv F$ then (F, U) may or may not be an equivalence*

Proof. Of course, taking $F = U^{-1}$ yields examples of the positive statements in the theorem. The goal here is to show that this is not generally the case, by finding counterexamples.

[Proof of 1] The adjunction $- \times A \dashv (-)^A$ is obviously not an equivalence, as the counit $\mathrm{eval}_C : C^A \times A \to C$ is not invertible.

[Proof of 2] Basically, an equivalence doesn't need to satisfy the triangle identities. The fact that an equivalence can be turned into an adjoint equivalence will be proven in a next theorem.

[Proof of 3] This example comes from Tom Leinster's answer on MathOverflow. Now consider the terminal category $\mathscr{C}_1$ with one object 0 and one identity arrow id_0.

Consider a functor $U : \mathscr{C} \to \mathscr{C}_1$. Its left adjoint $F : \mathscr{C}_1 \to \mathscr{C}$ needs to verify:

$$\mathrm{Hom}_{\mathscr{C}}(F(0), C) \cong \mathrm{Hom}_{\mathscr{C}_1}(0, U(C)) \cong \mathrm{Hom}_{\mathscr{C}_1}(0, 0) = \{\mathrm{id}_0\} \cong 1$$

Necessarily, $F(0)$ needs to be the initial object of $\mathscr{C}$. Similarly, the right adjoint of U needs to map 0 to the terminal object of $\mathscr{C}$.

Note that **Vect**, the category of vector spaces and linear maps, has a zero object: the vector space of dimension 0 is both initial and terminal. So, the right and left adjoints of a $U : \mathbf{Vect} \to \mathscr{C}_1$ are equal. In **Vect**, we have $F \dashv U \dashv F$, and **Vect** is trivially non-equivalent to $\mathscr{C}_1$ (cf. Remark 5.14 for an intuition on why these two categories are non-equivalent). □

Lemma 5.18. *Let $(F, U, \eta, \varepsilon)$ be an equivalence of categories. If it verifies one triangle identity (Definition 4.24), then it verifies the other.*

Proof. We give the proof for one triangle identity. Suppose F and U verify the following triangle identity:

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "\mathrm{id}_F"'] & F \circ U \circ F \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \tag{18}$$

For $X \in \mathscr{X}$, we have the following natural transformation diagram:

$$\begin{tikzcd} U(X) \arrow[d, "\eta_{U(X)}"] & & U \circ F \circ U(X) \arrow[r, "U(\varepsilon_X)"] \arrow[d, "U \circ F(\eta_{U(X)})"'] \arrow[dr, phantom, "\checkmark"] & U(X) \arrow[d, "\eta_{U(X)}"] \\ U \circ F \circ U(X) & \rightsquigarrow & U \circ F \circ U \circ F \circ U(X) \arrow[r, "U(\varepsilon_{F \circ U(X)})"'] & U \circ F \circ U(X) \end{tikzcd} \tag{19}$$

The composite of the left and bottom arrows are actually the composition by U (adding one U on the left) and whiskering by U (adding one U on the right) of the assumed triangle identity:

$$\begin{tikzcd} U \circ F\,(U(X)) \arrow[r, "U(\varepsilon_{U(X)})"] \arrow[d, "U \circ F(\eta_{U(X)})"'] \arrow[dr, "\mathrm{id}_{U \circ F \circ U(X)}"] & U(X) \arrow[d, "\eta_{U(X)}"] \\ U \circ F \circ U \circ F\,(U(X)) \arrow[r, "U(\varepsilon_{F \circ U(X)})"'] & U \circ F \circ U(X) \end{tikzcd}$$

Which gives $\eta_{U(X)} \circ U(\varepsilon_X) = \mathrm{id}_{U \circ F \circ U(X)}$. Then, ε and η are natural isomorphisms, so are their respective whiskerings. We deduce the second triangle identity:

$$\begin{aligned} \eta_{U(X)} \circ U(\varepsilon_X) &= \mathrm{id}_{U \circ F \circ U(X)} \\ \eta_{U(X)} \circ U(\varepsilon_X) \circ U(\varepsilon_X)^{-1} &= \mathrm{id}_{U \circ F \circ U(X)} \circ U(\varepsilon_X)^{-1} \\ \eta_{U(X)} &= U(\varepsilon_X)^{-1} \\ U(\varepsilon_X) \circ \eta_{U(X)} &= \mathrm{id}_F \end{aligned}$$

The proof is similar for the other triangle identity, although it requires two more natural transformation diagrams and a more subtle, but not too subtle, argument. □

Remark 5.19. This lemma is from nLab. However, I do not understand the demonstration as I cannot read string diagrams. I prove the result differently.

Proposition 5.20. *If $(F, U, \eta, \varepsilon)$ is an equivalence, then there exists a unique ε_0 such that $(F, U, \eta, \varepsilon_0)$ is an adjoint equivalence.*

Proof. Define ε_0 to be:

$$\begin{tikzcd}
FUFUF \arrow[r, "F\eta^{-1}UF"] & FUF \arrow[d, "\varepsilon F"] \\
FUF \arrow[u, "FU\varepsilon^{-1}F"] \arrow[r, "\varepsilon_0 F"'] \arrow[ur, phantom, "\checkmark"] & F
\end{tikzcd}$$

And consider the natural transformation diagram:

$$\begin{tikzcd}
F \arrow[r, "F\eta"] \arrow[d, "\varepsilon^{-1}F"'] \arrow[dr, phantom, "\checkmark"] & FUF \arrow[d, "FU\varepsilon^{-1}F"] \\
FUF \arrow[r, "F\eta UF"'] & FUFUF
\end{tikzcd}$$

Gluing both diagrams together, it is easy to see that the following diagram commutes, and thus proves a triangle identity.

$$\begin{tikzcd}
F \arrow[rr, "\varepsilon^{-1}F"] \arrow[dr, "F\eta"] \arrow[dd, bend right, "\mathrm{id}_F"'] & & FUF \arrow[dr, "F\eta UF"] & \\
& FUF \arrow[rr, "FU\varepsilon^{-1}F"] \arrow[dl, "\varepsilon_0 F"] & & FUFUF \arrow[dl, "F\eta^{-1}UF"] \\
F & & FUF \arrow[ll, "\varepsilon F"] &
\end{tikzcd}$$

Which gives a triangle identity $\varepsilon_0 F \circ F\eta = \mathrm{id}_F$; by Lemma 5.18, we have an adjoint equivalence.

Finally, suppose (η, ε_1) also satisfies the triangle identities. Then, we have the following natural transformation diagram:

$$\begin{tikzcd}
FUFU \arrow[r, "\varepsilon_0 FU"'] \arrow[d, "FU\varepsilon_1"] \arrow[dr, phantom, "\checkmark"] & FU \arrow[d, "\varepsilon_1"] \\
FU \arrow[r, "\varepsilon_0"'] & \mathrm{Id}_{\mathscr{X}}
\end{tikzcd}$$

By pre-composing by $F\eta U$ and using both triangle identities, we obtain the following commutative diagram:

$$\begin{tikzcd}
FU \arrow[dr, "F\eta U"] \arrow[drr, "\mathrm{id}_{FU}"] \arrow[ddr, "\mathrm{id}_{FU}"'] & & \\
& FUFU \arrow[r, "\varepsilon_0 FU"'] \arrow[d, "FU\varepsilon_1"] \arrow[dr, phantom, "\checkmark"] & FU \arrow[d, "\varepsilon_1"] \\
& FU \arrow[r, "\varepsilon_0"'] & \mathrm{Id}_{\mathscr{X}}
\end{tikzcd}$$

which yields $\varepsilon_0 = \varepsilon_1$ and hence the unicity of ε_0. $\square$

This proof ends this subsection about the differences and links between adjunctions and equivalences.

5.2. An example of adjunction: inverse image of a function

Consider a function $f : A \to B$, A and B being sets. Define f^{-1} to be the following functor:

$$f^{-1} : \begin{cases} \mathcal{P}(B) & \longrightarrow & \mathcal{P}(A) \\ Y & \longmapsto & f^{-1}(Y) \end{cases}$$

where $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are seen as their partially-ordered counterparts.
This functor has both a left adjoint and a right adjoint.
The left adjoint L is pretty easy to see. We are looking for an $L : \mathcal{P}(A) \to \mathcal{P}(B)$ such that:

$$\mathrm{Hom}_{\mathcal{P}(B)}(L(X), Y) \cong \mathrm{Hom}_{\mathcal{P}(A)}\left(X, f^{-1}(Y)\right)$$

Let $X \in \mathcal{P}(A)$ and $Y \in \mathcal{P}(B)$. We are looking for an $L(X)$ such that $L(X) \subset Y$ if and only if $X \subset f^{-1}(Y)$. This happens when L is the direct image of f: $L : X \mapsto f(X)$.
What are the unit and counit of this adjunction?
The right adjoint is less commonly seen. It is a functor $R : \mathcal{P}(A) \to \mathcal{P}(B)$ such that $Y \subset R(X)$ if and only if $f^{-1}(Y) \subset X$. That is, for a $y \in Y \subset R(X)$, we have $\{y\}^{-1} \subset X$. In fact, $R(X)$ is defined as:

$$R(X) = \left\{y \in B \;\middle|\; f^{-1}(\{y\}) \subset X\right\}$$

If $X \subset A$, then $R(X)$ is the smallest set of y's whose fibers by f are in X. I don't know if this set has a name.

Here we have a chain of three adjoint functors. The reader might wonder if there are longer chains of adjoints. The following sections give answers to this question.

5.3. How long can a chain of adjoints be? Part 1: a chain of five adjoints

+ Units and counits?

5.4. How long can a chain of adjoints be? Part 2: a chain of adjoints for any odd integer

These strings of adjoints depend on a few notions that we will only introduce here without too many details.

Definition 5.21 ((Augmented) simplex category)**.** The *augmented simplex category*, denoted by Δ_a, is the full subcategory of **Cat** consisting of finite totally ordered sets together with monotonic maps between them.
The *simplex category*, denoted by Δ, is the full subcategory of Δ_a consisting of non-empty finite totally ordered sets together with monotonic maps between them.

Some authors use the skeleta of Δ and Δ_a. In that case, an object in (the skeleton of) Δ_a is a poset $(n, \leqslant)$, where n is the ordinal $\{0, 1, \ldots, n-1\}$, including the trivial (empty) order $(0, \leqslant)$. An object in (the skeleton of) Δ is also a poset $(n, \leqslant)$, without the trivial order $(0, \leqslant)$.
When we consider the skeletal versions of Δ, morphings of Δ can be decomposed into finite compositions of the following elementary maps:

Definition 5.22 (Face and degeneracy maps)**.** Let $n > 0$ and $i \in n$.
The *i-th face map for n* is the following functor:

$$\delta_i^n : \begin{cases} n-1 & \longrightarrow & n \\ j & \longmapsto & \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geqslant i \end{cases} \end{cases}$$

The *i-th degeneracy map for n* is the following functor:

$$\sigma_i^n : \left\{ \begin{array}{ccc} n+1 & \longrightarrow & n \\ j & \longmapsto & \begin{cases} j & \text{if } j < i \\ j-1 & \text{if } j \geqslant i \end{cases} \end{array} \right.$$

These maps verify the simplicial identities:

Proposition 5.23 (Simplicial identities)**.** *Let $n > 0$. Then the maps and degeneracy maps verify the following identities:*

$$\begin{array}{lll} \delta_j^{n+1} \circ \delta_i^n & = \delta_i^{n+1} \circ \delta_{j-1}^n & \textit{for } 0 \leqslant i < j \leqslant n \\ \sigma_j^n \circ \sigma_i^{n+1} & = \sigma_i^n \circ \sigma_{j+1}^{n+1} & \textit{for } 0 \leqslant i < j \leqslant n \end{array}$$

$$\sigma_j^n \circ \delta_i^{n+1} = \begin{cases} \delta_i^n \circ \sigma_{j-1}^{n-1} & \textit{for } 0 \leqslant i < j < n \\ \mathrm{Id}_n & \textit{for } j \in n \textit{ and } i \in \{j, j+1\} \\ \delta_{i-1}^n \circ \sigma_j^{n-1} & \textit{for } 0 \leqslant j < j+1 < i \leqslant n \end{cases}$$

Proof. This is basic arithmetic. □

The simplex category has uses that are way beyond the scope of this course (see quelque chose for more information). What matters to us is the following:

Proposition 5.24. *Let $n \in \Delta$, and $i \in n$. Then $\delta_{i+1}^{n+1} \dashv \sigma_i^n \dashv \delta_i^{n+1}$.*

Proof. This is a case-by-case analysis based on the definition of the functions. We prove $\sigma_i^n \dashv \delta_i^{n+1}$; the other proof is in the same vein.

Consider $\mathrm{Hom}_n\left(\sigma_i^n(a), b\right)$ and $\mathrm{Hom}_{n+1}\left(a, \delta_i^{n+1}(b)\right)$. We have four cases:

	$a \leqslant i$	$a > i$
$b < i$	Case 1	Case 2
$b \geqslant i$	Case 3	Case 4

1. If $a \leqslant i$ and $b < i$, then $\sigma_i^n(a) = a$ and $\delta_i^{n+1}(b) = b$. In that case, the equivalence $\mathrm{Hom}_n\left(\sigma_i^n(a), b\right) \cong \mathrm{Hom}_{n+1}\left(a, \delta_i^{n+1}(b)\right)$ is obvious.
2. If $a > i$ and $b < i$, then $\sigma_i^n(a) = a - 1$ and $\delta_i^{n+1}(b) = b$. In that case, we have $a > a - 1 \geqslant i > b$ hence the equivalence $\mathrm{Hom}_n(a-1, b) \cong \mathrm{Hom}_{n+1}(a, b)$.
3. If $a \leqslant i$ and $b \geqslant i$, then $\sigma_i^n(a) = a$ and $\delta_i^{n+1}(b) = b + 1$. We have $a \leqslant i \leqslant b < b + 1$, hence the equivalence: $\mathrm{Hom}_n(a, b) \cong \mathrm{Hom}_{n+1}(a, b+1)$.
4. If $a > i$ and $b \geqslant i$, then $\sigma_i^n(a) = a-1$ and $\delta_i^{n+1}(b) = b+1$. Then the equivalence $\mathrm{Hom}_n\left(\sigma_i^n(a), b\right) \cong \mathrm{Hom}_{n+1}\left(a, \delta_i^{n+1}(b)\right)$ is obvious.

□

+ Units and counits?

Theorem 5.25 (Informal)**.** *For each odd integer $m = 2k + 1$, there is a chain of m adjoints:*

$$\delta_k^{k+1} \dashv \sigma_{k-1}^k \dashv \delta_{k-1}^{k+1} \dashv \ldots \dashv \sigma_0^k \dashv \delta_0^{k+1}$$

We do not know of any similar result for even numbers.

Cf la première réponse à cette question.

5.5. How long can a chain of adjoints be? Part 3: an infinite chain of adjoints

There are other examples of infinite chains of adjoints (for example: this one) but we prefer this one, which requires not much side knowledge.

5.6. A logical adjunction

L'adjonction $\exists \dashv ?? \dashv \forall$

5.7. Forgetful and free functors

5.8. Other simple examples

Of course, this section could not be exhaustive. In the preface of the first edition of [5], Mac Lane wrote about Chapters III to V:

"The slogan is "Adjoint functors arise everywhere.""

So of course we cannot make a complete list of all things that happen to be adjoint functors, because there are loads of them. In this subsection, we introduce less epic examples of functors, without proof.

Initial and terminal objects

Consider the terminal category $\mathscr{C}_1$, consisting of only one object and only one identity morphism. Consider the functor $T : \mathscr{C} \to \mathscr{C}_1$ for some category $\mathscr{C}$. Then, T has a left (resp. right) adjoint $\Leftrightarrow$ $\mathscr{C}$ has an initial (resp. terminal) object, and that left (resp. right) adjoint is $\mathtt{init} : \mathscr{C}_1 \to \mathscr{C}$ (resp. $\mathtt{term} : \mathscr{C}_1 \to \mathscr{C}$) which sends the unique object of $\mathscr{C}_1$ to the initial (resp. terminal) object of $\mathscr{C}$. The unit $\eta : \mathrm{Id}_{\mathscr{C}} \to \mathtt{term} \circ T$ has components $\eta_C : C \to 1$ (the unique arrow from the UMP of the terminal object) and the counit $\varepsilon : T \circ \mathtt{term} \to \mathrm{Id}_{\mathscr{C}_1}$ consists in the only one arrow in $\mathscr{C}_1$.

Sets ***and*** **Rel*****: relations and powerset***

Consider the category **Rel** whose objects are sets and whose arrows are the relations (not just functions). We let the reader check that this actually defines a category. There is an obvious inclusion functor $\iota : \mathbf{Sets} \to \mathbf{Rel}$. We also have the powerset functor $\mathcal{P} : \mathbf{Rel} \to \mathbf{Sets}$ that sends a set X to its powerset $\mathcal{P}(X)$. Then, $i \dashv \mathcal{P}$.

This adjunction comes with a lot of structure. The unit of the adjunction $X \to \mathcal{P}(\iota(X))$ sends an element to its singleton $x \mapsto \{x\}$. The counit $\iota(\mathcal{P}(X)) \to X$ sends a subset $U \subset X$ to what? I don't get it.

Inclusion of preorders into **Cat**

Let **Pre** the category of preorders and monotone maps between them. Consider **Cat**, the category of small categories and functors between them. There is an obvious inclusion functor $I : \mathbf{Pre} \hookrightarrow \mathbf{Cat}$, because preorders are examples of small categories. This inclusion functor has a left adjoint $P : \mathbf{Cat} \to \mathbf{Pre}$. It is a nice exercise to find what that left adjoint is. We give the solution in the next paragraph.

The left adjoint P takes a small category $\mathscr{C}$, and turns it into a preorder with the following rule: for a category $\mathscr{C} \in \mathbf{Cat}$, $P(\mathscr{C})$ is the preorder defined by $C \leqslant C' \Leftrightarrow$ there is an arrow $C \to C'$ in $\mathscr{C}$.

The unit $\eta : \mathrm{Id}_{\mathbf{Cat}} \to I \circ P$ maps a category to its preorder and the counit $\varepsilon : P \circ I \to \mathrm{Id}_{\mathbf{Pre}}$ maps a preorder-ified preorder (thus a preorder) to itself; it is the identity.

5.9. A last word on adjunctions

In the following, these are purely non-mathematical thoughts and *opinions* on adjunctions.

When trying to understand adjunctions, I read and heard that they were fundamental not only in category theory, but also in mathematics, and why not, in life in general. However, many new-comers have a hard time figuring out what adjunctions say, while regular category-theorists just throw lots of (beautiful!) examples that prove that they *arise everywhere*.

Let me give an alternative opinion.

Adjunctions are thought as a weak form of equivalence of categories. As such, we would expect that adjunctions state something about the two categories that they make "*weakly equivalent*". They do not. Adjunctions do say something about categories, but only on arrows, and in a hard-to-parse, hard-to-think, hard-to-use way (think about the identity $\mathrm{Hom}_{\mathscr{C}}(F(C), X) \cong \mathrm{Hom}_{\mathscr{X}}(C, U(X))$). Otherwise, they don't say much about the two categories. The exception is, adjoint equivalences, which say something about categories, but more from being equivalences than adjunctions.

What is true though, is that they say something about the functors that they (ad)join. Adjunctions are a way to associate functors together in a unique way (left and right adjoints are unique up to isomorphism, thanks to the Yoneda lemma). We can also use this unicity in order to define a functor in terms of another. But that's it. The interpretation of this association is too context-dependent for one to extract the meaning of two functors being adjoints. They make good theorems, and it always fills one with wonder when they discover that two known functors, functions, mathematical things, are left or right adjoints of the other (this was the goal of this section). But that's it.

In fact, the fact that they arise everywhere (and they *do* arise everywhere) is also a hint about the other fact that they tell nothing. Otherwise, they would say much more than what currently appears in books.

The wise conclusion is that, adjunctions are beautiful flowers in the landscape of category theory (or mathematics, or life) but apart from their omnipresence, they say nothing about the soil they grew on (even flowers may say something about the acidity of the earth, the humidity, even wind or bugs).

Of course, this opinion is strictly personal, but this is how I would answer the questions "what is the use of adjunctions?" or "what do adjunctions say about category theory (or mathematics, or life)?". Adjunctions may be beautiful, but unless you really want to study them, or if you came across adjunctions while studying category theory, just do not spend too much time on it. Category theorists think they are important, because you encounter them everywhere, but they carry no information and almost no property other than the unique association of two functors.

6. Objects with some universality in them

A word about the UMP.

Definition 6.1 (Product [1])**.** Let $\mathscr{C}$ be a category and let A and B be objects in $\mathscr{C}$.

The *product of* A *and* B is 3-tuple $(A \times B, \pi_A, \pi_B)$ where $A \times B$ is an object in $\mathscr{C}$, and $\pi_A : A \times B \longrightarrow A$ and $\pi_B : A \times B \longrightarrow B$ are two arrows, such that, for all object P with two arrows $p_A : P \longrightarrow A$ and $p_B : P \longrightarrow B$, there exists a unique arrow $u : P \longrightarrow A \times B$ such that $\pi_A \circ u = p_A$ and $\pi_B \circ u = p_B$, that is, such that the following diagram commutes:

$$\begin{tikzcd}
 & P \arrow[ld, "p_A"'] \arrow[d, dashed, "u"'] \arrow[rd, "p_B"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}$$

We call π_A, π_B projections, and we denote u by $u = (p_A, p_B)$.

The definition of the product can be interpreted as follows. Given three objects A, B, C, the "shorter" path from C to A and B at the same time, always passes through $A \times B$. In a sense, $A \times B$ is an "optimised" link to A and B.

Example 6.2. In **Sets**, the product is the usual cartesian product of sets, and the projections are the usual projections $(a, b) \mapsto a$ and $(a, b) \mapsto b$.

Example 6.3. In a preorder category $(P, \leqslant)$, the product $p \times q$ of two elements p and q verifies $p \times q \leqslant p$ and $p \times q \leqslant q$, and for all $r \leqslant p$ $r \leqslant q$, we have $r \leqslant p \times q$. In fact, $p \times q = \inf(p, q)$.

Proposition 6.4. *Let $\mathscr{C}$ be a category and let A and B be objects in $\mathscr{C}$.*

The product $A \times B$ is unique up to isomorphism.

Proof. Let $(A \times B, \pi_A, \pi_B)$ and $\left(A \star B, \pi_A^\star, \pi_B^\star\right)$ be two products of A and B.

By definition of both products, there exists unique $u, u^\star$ such that the following diagram commutes:

$$\begin{tikzcd}
 & A \star B \arrow[ld, "\pi_A^\star"'] \arrow[rd, "\pi_B^\star"] \arrow[dd, dashed, bend right, "u^\star"'] & \\
A & & B \\
 & A \times B \arrow[lu, "\pi_A"] \arrow[ru, "\pi_B"'] \arrow[uu, dashed, bend right, "u"'] &
\end{tikzcd}$$

We then have: $\pi_A^\star \circ u \circ u^\star = \pi_A^\star = \pi_A^\star \circ \mathrm{id}_{A \star B}$ and $\pi_B^\star \circ u \circ u^\star = \pi_B^\star = \pi_B^\star \circ \mathrm{id}_{A \star B}$. The following diagram commutes:

$$\begin{tikzcd}
 & A \star B \arrow[ld, "\pi_A^\star"'] \arrow[rd, "\pi_B^\star"] \arrow[d, dashed, "u \circ u^\star"'] & \\
A & A \star B \arrow[l, "\pi_A^\star"] \arrow[r, "\pi_B^\star"'] & B \\
 & A \star B \arrow[lu, "\pi_A^\star"] \arrow[ru, "\pi_B^\star"'] \arrow[u, "\mathrm{id}_{A \star B}"'] &
\end{tikzcd}$$

By uniqueness condition, we have $u \circ u^\star = \mathrm{id}_{A \star B}$. A similar reasonning yields $u^\star \circ u = \mathrm{id}_{A \times B}$; then u and $u^\star$ are isomorphisms. $\square$

Consequently, it is natural to mention "the" product of two objects, instead of "a" product.

The definition of a product can be generalized from $n = 2$ to any $n \in \mathbb{N}$. When $n = 1$, the product of C_1 is just C_1 and the projection $\pi_1 \colon C_1 \to C_1$ is the identity. When $n = 0$, the empty product is an object $*$ such that for all objects P, there exists a unique arrow $u \colon P \to *$ (we will see later that this is the terminal object). Note that depending on the category $\mathscr{C}$, not all pairs (C_1, C_2) may have a product.

Definition 6.5 (Category with finite products)**.** The category $\mathscr{C}$ is said to have *finite products* if $\forall n \in \mathbb{N}, \forall (C_i)_{i\in n}$, the product $\prod_{i\in n} C_i$ exists.

The product of categories can also be defined. However, it is not always a product in the category of categories (if such a thing exists). It is still useful for further definitions.

Definition 6.6 (Product of categories)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. We define the *category of pairs*, or *the product category* $\mathscr{C} \times \mathscr{D}$ by:

Objects: An object in $\mathscr{C} \times \mathscr{D}$ is a pair (C, D) where $C \in \mathrm{Ob}_{\mathscr{C}}$ and $D \in \mathrm{Ob}_{\mathscr{D}}$

Morphisms: A morphism in $\mathscr{C} \times \mathscr{D}$ is a pair $(c, d) : (C, D) \to (C', D')$ where $c : C \to C' \in \mathrm{Mor}_{\mathscr{C}}$ and $d : D \to D' \in \mathrm{Mor}_{\mathscr{D}}$

Identities: An identity morphism is a pair $(\mathrm{id}_C, \mathrm{id}_D)$

Composition: The composition law for morphisms is pairwise: $(c, d) \circ (c', d') = (c \circ c', d \circ d')$ (using the composition laws of $\mathscr{C}$ and $\mathscr{D}$)

Remark 6.7. We can define the category **Cat** of small categories, where the morphisms are the functors between small categories. In that case, if $\mathscr{C}$ and $\mathscr{D}$ are small categories, then the product $\mathscr{C} \times \mathscr{D}$ is an actual product in this category.

Besides, any two categories give birth to a product category, however this product is not necessarily an actual product in the categorical sense.

Before checking on the dual notion of the product, let us have a look at the behaviour of the covariant and contravariant Hom-set functors in relation to the product.

Proposition 6.8. *Let $\mathscr{C}$ be a category with finite products. Then there is a natural isomorphism:*

$$Hom_{\mathscr{C}}(A, B \times C) \cong Hom_{\mathscr{C}}(A, B) \times Hom_{\mathscr{C}}(A, C)$$

in A, B and C.

Proof. As a product, there is a unique $u : \mathrm{Hom}_{\mathscr{C}}(A, B \times C) \to \mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(A, C)$ such that the following diagram commutes:

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathscr{C}}(A, B \times C) \arrow[dl, "\mathrm{Hom}_{\mathscr{C}}(A,\pi_B)"'] \arrow[d, "u"] \arrow[dr, "\mathrm{Hom}_{\mathscr{C}}(A,\pi_C)"] & \\
\mathrm{Hom}_{\mathscr{C}}(A, B) & \mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(A, C) \arrow[l, "\pi_{\mathrm{Hom}_{\mathscr{C}}(A,B)}"] \arrow[r, "\pi_{\mathrm{Hom}_{\mathscr{C}}(A,C)}"'] & \mathrm{Hom}_{\mathscr{C}}(A, C)
\end{tikzcd}$$

Conversely, given $(f_B, f_C) \in \mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(A, C)$, by definition of the product $B \times C$, there is a unique $v : A \to B \times C$ such that the following diagram commutes:

$$\begin{tikzcd}
& A \arrow[dl, "\pi_B \circ f"'] \arrow[d, "u"] \arrow[dr, "\pi_C \circ f"] & \\
B & B \times C \arrow[l, "\pi_B"] \arrow[r, "\pi_C"'] & C
\end{tikzcd}$$

Besides, $v = \langle f_B, f_C \rangle : A \to B \times C$, so we define:

$$\alpha_{A,B,C} : \begin{cases} \mathrm{Hom}_{\mathscr{C}}(A,B) \times \mathrm{Hom}_{\mathscr{C}}(A,C) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(A, B \times C) \\ (f_B, f_C) & \longmapsto & \langle f_B, f_C \rangle \end{cases}$$

By a reasonning similar to the one in the proof of Proposition 6.4, we have $\alpha_{A,B,C} = u^{-1}$, so $\alpha_{A,B,C}$ is a bijection.

The naturality is easy to check; let $a : A \to A'$:

$$\begin{tikzcd} A \arrow[d, "a"] & \rightsquigarrow & \mathrm{Hom}_{\mathscr{C}}(A',B) \times \mathrm{Hom}_{\mathscr{C}}(A',C) \arrow[r, "\alpha_{A',B,C}"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(a,B)\times\mathrm{Hom}_{\mathscr{C}}(a,C)"] & \mathrm{Hom}_{\mathscr{C}}(A', B \times C) \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(a,B\times C)"] \\ A' & & \mathrm{Hom}_{\mathscr{C}}(A,B) \times \mathrm{Hom}_{\mathscr{C}}(A,C) \arrow[r, "\alpha_{A,B,C}"'] & \mathrm{Hom}_{\mathscr{C}}(A, B \times C) \end{tikzcd}$$

We check that the diagram commutes:

$$\begin{aligned} \alpha_{A,B,C} \circ \mathrm{Hom}_{\mathscr{C}}(a,B) \times \mathrm{Hom}_{\mathscr{C}}(a,C)\,(f_B, f_C) &= \alpha_{A,B,C}\,(f_B \circ a, f_C \circ a) \\ &= \langle f_B \circ a, f_C \circ a \rangle \\ \mathrm{Hom}_{\mathscr{C}}(a,B \times C) \circ \alpha_{A',B,C}\,(f_B, f_C) &= \mathrm{Hom}_{\mathscr{C}}(a,B \times C)\,\langle f_B, f_C \rangle \\ &= \langle f_B, f_C \rangle \circ a \\ &= \langle f_B \circ a, f_C \circ a \rangle \end{aligned}$$

The other naturalities are as easy to check.

□

This property is not specific to the Hom-set functor.

Definition 6.9 (Preserving products)**.** The functor $F : \mathscr{C} \to \mathscr{D}$ is said to *preserve products* when, for all $A, B \in \mathrm{Ob}_{\mathscr{C}}$, if $A \times B$ exists, then $F(A \times B) \cong F(A) \times F(B)$.

Proposition 6.10. *The covariant Hom-set functor $\mathrm{Hom}_{\mathscr{C}}(A, -)$ preserves products.*

A similar question could be asked about the contravariant Hom-set functor: is there a natural isomorphism $\mathrm{Hom}_{\mathscr{C}}(A \times B, C) \to \mathrm{Hom}_{\mathscr{C}}(A, C) \times \mathrm{Hom}_{\mathscr{C}}(B, C)$? In fact, the answer is no. The right isomorphism is this one:

Proposition 6.11. *Let $\mathscr{C}$ be a category with finite products. Then there is a natural isomorphism:*

$$\mathrm{Hom}_{\mathscr{C}\times\mathscr{C}}((A, A), (B, C)) \cong \mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(A, C)$$

in A, B and C.

The proof is very similar to that of the covariant Hom-set functor.

Combining Proposition 6.8 and Proposition 6.11, we have:

Proposition 6.12. *Let $\mathscr{C}$ be a category with finite products. Then there is a natural isomorphism:*

$$\mathrm{Hom}_{\mathscr{C}}((A, A), (B, C)) \cong \mathrm{Hom}_{\mathscr{C}}(A, B \times C)$$

in A, B and C.

In other words, the diagonal functor:

$$\Delta_2 : \begin{cases} \mathscr{C} & \longrightarrow & \mathscr{C} \times \mathscr{C} \\ C & \longmapsto & (C, C) \\ c & \longmapsto & (c, c) \end{cases}$$

is right adjoint to the product functor $- \times - : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$.

Remark 6.13. The unit of the adjunction $\Delta_2 \dashv \times$ is:

$$\eta_C : C \to C \times C$$

and the counit is:

$$\varepsilon_{A,B} : (A \times B, A \times B) \to (A, B)$$

The dual notion of that of a product is the coproduct:

Definition 6.14 (Coproduct)**.** Let $\mathscr{C}$ be a category and let A and B be objects in $\mathscr{C}$.

The *coproduct of* A *and* B is 3-tuple $(A + B, i_A, i_B)$ where $A + B$ is an object in $\mathscr{C}$, and $i_A : A \to A + B$ and $i_B : B \to A + B$ are two arrows, such that, for all object X with two arrows $x_A : A \to X$ and $x_B : B \to X$, there exists a unique arrow $u : A + B \to X$ such that the following diagram commutes:

$$\begin{tikzcd} & X & \\ A \arrow[ur, "x_A"] \arrow[r, "i_A"'] & A + B \arrow[u, dashed, "u"] & B \arrow[ul, "x_B"'] \arrow[l, "i_B"] \end{tikzcd}$$

We call i_A, i_B *injections*, although they do not need to be injective.

Example 6.15. In **Sets**, the coproduct $A + B$ corresponds to the disjoint union of A and B, for example defined as

$$A + B = \{(a, 0) \mid a \in A\} \cup \{(b, 1) \mid b \in B\}$$

with injections being:

$$i_A : a \to (a, 0)$$
$$i_B : b \to (b, 1)$$

Example 6.16. In a preorder category $(P, \leqslant)$, the coproduct $p + q$ is the supremum: $p + q = \sup(p, q)$.

It is easy to see that:

Proposition 6.17. *Let* $\mathscr{C}$ *be a category and let A and B be objects in* $\mathscr{C}$*.*
$(A + B, i_A, i_B)$ *is a coproduct in* $\mathscr{C}$ *if and only if* $(A + B, i_A, i_B)$ *is a product in* $\mathscr{C}^{op}$.

Corollary 6.18. *The coproduct is unique up to isomorphism.*

The proof of the following is very similar to the proof of Proposition 6.8. We will just give the natural isomorphism to consider.

Proposition 6.19. *Let* $\mathscr{C}$ *be a category such that for all* A, B*, the coproduct* $A + B$ *exists. Then, there is a natural isomorphism:*

$$Hom_{\mathscr{C}}(A + B, C) \cong Hom_{\mathscr{C} \times \mathscr{C}}((A, B), (C, C))$$

In other words, the diagonal functor $\Delta_2 : C \mapsto (C, C)$ *is left adjoint to the coproduct functor* $- + - : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$.

Proof. If $f : A + B \to C$ then by definition of the coproduct, there are two arrows $i_A : A \to A + B$ and $i_B : B \to A + B$ such that $f = f \circ p_A + f \circ p_B$ ($f \circ p_A$ can be seen as the restriction of f to A). Consider the mapping:

$$\alpha_{A,B,C} : \begin{cases} \mathrm{Hom}_{\mathscr{C}}(A + B, C) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}\times\mathscr{C}}((A, B), (C, C)) \\ f & \longmapsto & (f \circ p_A, f \circ p_B) \end{cases}$$

Then it is not hard (but quite long) to prove that $\alpha_{A,B,C}$ defines a natural transformation $\mathrm{Hom}_{\mathscr{C}}(- + -, -) \to \mathrm{Hom}_{\mathscr{C}\times\mathscr{C}}((-, -), \Delta(-))$.

The unit is:

$$\eta_{A,B} : (A, B) \to (A + B, A + B)$$

and the counit is:

$$\varepsilon_C : C + C \to C$$

□

In summary:

Theorem 6.20. $+ \dashv \Delta_2 \dashv \times$.

Definition 6.21 (Exponential [1])**.** Let $\mathscr{C}$ be a category with finite products, and let B, C be objects of $\mathscr{C}$.

An *exponential of* B *and* C is a pair $\left(C^B, \varepsilon\right)$ where C^B is an object in $\mathscr{C}$ and $\varepsilon : C^B \times B \to C$, such that, for any arrow $f : A \times B \to C$, there exists a unique arrow $f^c : A \to C^B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[d, "f^c"'] \\ C^B \end{tikzcd} \qquad \begin{tikzcd} A \times B \arrow[d, "f^c \times \mathrm{id}_B"'] \arrow[dr, "f"] & \\ C^B \times B \arrow[r, "\varepsilon"'] & C \end{tikzcd}$$

The arrow ε is called *evaluation*; the arrow f^c is the *(exponential) transpose* of f.

Remark 6.22. Let's consider the category of sets $\mathscr{C} = \mathbf{Sets}$.

Let B, C be two sets; their exponential is $C^B = \mathrm{Hom}_{\mathbf{Sets}}(B, C)$ (note that this is specific to **Sets**). Let $f : A \times B \to C$ be a function. As a function in two variables, $f : a, b \mapsto f(a, b)$ can also be seen as a function $f^c : a \mapsto f(a, -) : b \mapsto f(a, b)$. The operation $f^c : a \mapsto f(a, -)$ is a function $A \to C^B$, it is called curryfication; however, the operation $g, b \mapsto g(b)$ is a function $C^B, B \to C$ called evaluation. The exponential of two sets B and C is the pair $\left(C^B, \varepsilon\right)$ where $C^B = \mathrm{Hom}_{\mathbf{Sets}}(B, C)$ and ε is the function:

$$\varepsilon : \begin{cases} C^B \times B & \longrightarrow & C \\ g, b & \longmapsto & g(b) \end{cases}$$

Thus, for all $f : A \times B \to C$, we have $\varepsilon \circ (f^c \times \mathrm{id}_B)(a, b) = \varepsilon(f(a, -), b) = f(a, b)$. The goal of the exponential is to generalise these notions of curryfication and evaluation to other categories.

Proposition 6.23. *Let $\mathscr{C}$ be a category with finite products. We also suppose that X^Y exists for all objects $X, Y \in \mathscr{C}$.*

Let A be an object of $\mathscr{C}$. Let P_A and E_A be the functors:

$$P_A : \begin{cases} \mathscr{C} & \longrightarrow & \mathscr{C} \\ X & \longmapsto & X \times A \\ x : X \to X' & \longmapsto & x \times \mathrm{id}_A : X \times A \to X' \times A \end{cases}$$

$$E_A : \left\{ \begin{array}{ccc} \mathscr{C} & \longrightarrow & \mathscr{C} \\ X & \longmapsto & X^A \\ x : X \to X' & \longmapsto & x^A : X^A \to X'^A \end{array} \right.$$

Then $P_A \dashv E_A$.

Proof. We let the reader check that P_A and E_A actually are functors.

We want to prove that there is a natural isomorphism γ with components:

$$\gamma_{X,Y} : \mathrm{Hom}_{\mathscr{C}}\left(X, Y^A\right) \to \mathrm{Hom}_{\mathscr{C}}\left(X \times A, Y\right)$$

For fixed Y and A, we consider the exponential $\left(Y^A, \varepsilon_Y\right)$.

Let $f : X \times A \to Y$; then by definition of the exponential Y^A, there exists a unique $f^c : X \to Y^A$ such that $f = \varepsilon_Y \circ f^c \times \mathrm{id}_A$. Consequently, there is a bijection:

$$\gamma_{X,Y} : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}\left(X, Y^A\right) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}\left(X \times A, Y\right) \\ g & \longmapsto & \varepsilon_Y \circ g \times \mathrm{id}_A \end{array} \right.$$

As for the naturality of γ, let $x : X \to X'$:

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}\left(X', Y^A\right) \arrow[rr, "\gamma_{X',Y}"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}}\left(x, Y^A\right)"'] & & \mathrm{Hom}_{\mathscr{C}}\left(X' \times A, Y\right) \arrow[d, "\mathrm{Hom}_{\mathscr{C}}\left(x \times A, Y\right)"] \\ \mathrm{Hom}_{\mathscr{C}}\left(X, Y^A\right) \arrow[rr, "\gamma_{X,Y}"'] \arrow[urr, phantom, "?"] & & \mathrm{Hom}_{\mathscr{C}}\left(X \times A, Y\right) \end{tikzcd}$$

For $f \in \mathrm{Hom}_{\mathscr{C}}\left(X', Y^A\right)$, we have:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}\left(x \times A, Y\right) \circ \gamma_{X',Y}\left(f\right) &= \mathrm{Hom}_{\mathscr{C}}\left(x \times A, Y\right) \circ \varepsilon_Y \circ \left(f \times \mathrm{id}_A\right) \\ &= \varepsilon_Y \circ \left(f \times \mathrm{id}_A\right) \circ \left(x \times \mathrm{id}_A\right) \\ &= \varepsilon_Y \circ \left(f \circ x \times \mathrm{id}_A\right) \\ \gamma_{X,Y} \circ \mathrm{Hom}_{\mathscr{C}}\left(x, Y^A\right)\left(f\right) &= \gamma_{X,Y}\left(f \circ x\right) \\ &= \varepsilon_Y \circ \left(f \circ x \times \mathrm{id}_A\right) \\ &= \mathrm{Hom}_{\mathscr{C}}\left(x \times A, Y\right) \circ \gamma_{X',Y}\left(f\right) \end{aligned}$$

Thus the diagram commutes.

As for the naturality in Y, let $y : Y \to Y'$:

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}\left(X, Y^A\right) \arrow[rr, "\gamma_{X,Y}"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}}\left(X, y^A\right)"'] & & \mathrm{Hom}_{\mathscr{C}}\left(X \times A, Y\right) \arrow[d, "\mathrm{Hom}_{\mathscr{C}}\left(X \times A, y\right)"] \\ \mathrm{Hom}_{\mathscr{C}}\left(X, Y'^A\right) \arrow[rr, "\gamma_{X,Y'}"'] \arrow[urr, phantom, "?"] & & \mathrm{Hom}_{\mathscr{C}}\left(X \times A, Y'\right) \end{tikzcd}$$

Let $\varepsilon_{Y'}$ be the evaluation that comes with Y'^A. For $f \in \mathrm{Hom}_{\mathscr{C}}\left(X, Y^A\right)$, we have:

$$
\begin{aligned}
\mathrm{Hom}_{\mathscr{C}}\left(X \times A, y\right) \circ \gamma_{X,Y}(f) &= y \circ \varepsilon_Y \circ (f \times \mathrm{id}_A) \\
\gamma_{X,Y'} \circ \mathrm{Hom}_{\mathscr{C}}\left(X, y^A\right)(f) &= \gamma_{X,Y'}\left(y^A \circ f\right) \\
&= \varepsilon_{Y'} \circ \left(y^A \circ f \times \mathrm{id}_A\right)
\end{aligned}
$$

Consider the following diagram:

$$
\begin{tikzcd}
& X \times A \arrow[dl, "f^c \times \mathrm{id}_A"'] \arrow[dr, "f"] \arrow[ddl, bend right=60, "y^A \circ f^c \times \mathrm{id}_A"'] \arrow[ddr, bend left=60, "y \circ f"] & \\
Y^A \times A \arrow[rr, "\varepsilon_Y"'] \arrow[d, "y^A \times \mathrm{id}_A"'] & ? & Y \arrow[d, "y"] \\
Y'^A \times A \arrow[rr, "\varepsilon_{Y'}"] & & Y'
\end{tikzcd}
$$

The following subdiagrams commute:

$$
\begin{tikzcd}
& X \times A \arrow[dl, "f^c \times \mathrm{id}_A"'] \arrow[d, "f"] \arrow[dr, "y \circ f"] & \\
Y^A \times A \arrow[r, "\varepsilon_Y"'] & Y \arrow[r, "y"'] & Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \times A \arrow[dl, "f^c \times \mathrm{id}_A"'] \arrow[d, "y^A \circ f^c \times \mathrm{id}_A"] \arrow[dr, bend left, "y \circ f"] & \\
Y^A \times A \arrow[r, "y^A \times \mathrm{id}_A"'] & Y'^A \times A \arrow[r, "\varepsilon_{Y'}"'] & Y'
\end{tikzcd}
$$

We deduce that the following diagram commutes too:

$$
\begin{tikzcd}
& X \times A \arrow[dl, "f^c \times \mathrm{id}_A"'] \arrow[dr, "f^c \times \mathrm{id}_A"] \arrow[ddd, "y \circ f"] & \\
Y^A \times A \arrow[d, "\varepsilon_Y"'] \arrow[r, phantom, "\checkmark"] & & Y^A \times A \arrow[d, "y^A \times \mathrm{id}_A"] \arrow[l, phantom, "\checkmark"] \\
Y \arrow[dr, "y"'] & & Y'^A \times A \arrow[dl, "\varepsilon_{Y'}"] \\
& Y' &
\end{tikzcd}
$$

which proves the equality:

$$
\begin{aligned}
&\varepsilon_{Y'} \circ \left(y^A \circ f \times \mathrm{id}_A\right) = y \circ \varepsilon_Y \circ (f \times \mathrm{id}_A) \\
\Leftrightarrow\ &\gamma_{X,Y'} \circ \mathrm{Hom}_{\mathscr{C}}\left(X, y^A\right)(f) = \mathrm{Hom}_{\mathscr{C}}\left(X \times A, y\right) \circ \gamma_{X,Y}(f)
\end{aligned}
$$

and thus the naturality of γ in Y.
The inverse natural isomorphism γ^{-1} is the adjunctor between P_A and E_A. □

Corollary 6.24. *The exponential is unique up to isomorphism.*

Proof. Consequence of the unicity of the right adjoint up to isomorphism. □

Remark 6.25. In **Sets**, the two functors P_A and E_A would be:

$$P_A : \begin{cases} \mathscr{C} & \longrightarrow & \mathscr{C} \\ X & \longmapsto & X \times A \\ x : X \to X' & \longmapsto & x \times \mathrm{id}_A : \begin{cases} X \times A & \longrightarrow & X' \times A \\ e, a & \longmapsto & (x(e), a) \end{cases} \end{cases}$$

$$E_A : \begin{cases} \mathscr{C} & \longrightarrow & \mathscr{C} \\ X & \longmapsto & X^A \\ x : X \to X' & \longmapsto & x^A : \begin{cases} X^A & \longrightarrow & X'^A \\ f & \longmapsto & x \circ f \end{cases} \end{cases}$$

Remark 6.26. If the category $\mathscr{C}$ has all exponentials $\left(Y^A, \varepsilon_Y\right)$ for all $A, Y \in \mathrm{Ob}_{\mathscr{C}}$, then $\varepsilon : (\varepsilon_Y)_{Y \in \mathrm{Ob}_{\mathscr{C}}}$ is the counit of the adjunction $P_A \dashv E_A$. In fact, using the γ seen in the proof of Proposition 6.23, we have $\gamma_{E_A(Y), Y}\left(\mathrm{id}_{E_A(Y)}\right) = \varepsilon_Y \circ \left(\mathrm{id}_{E_A(Y)}, \mathrm{id}_A\right) = \varepsilon_Y$.

The counit is less obvious. It is a natural transformation $\eta = \left(\eta_Y : Y \to (Y \times A)^A\right)$ with components η_Y such that the following diagram commutes:

$$\begin{tikzcd} Y \times A \arrow[d, "\eta_Y \times \mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_{Y \times A}"] & \\ (Y \times A)^A \times A \arrow[r, "\varepsilon_{Y \times A}"'] & Y \times A \end{tikzcd}$$

Definition 6.27 (Initial and terminal object [2])**.** Let $\mathscr{C}$ be a category, and let I, T be objects of $\mathscr{C}$.

The object I is called *initial* when, for every $C \in \mathrm{Ob}_{\mathscr{C}}$, there is only one arrow $I \to C$. The initial object is often denoted by 0.

The object T is called *terminal* when, for every $C \in \mathrm{Ob}_{\mathscr{C}}$, there is only one arrow $C \to T$. The terminal object is often denoted by 1.

Example 6.28. In **Sets**, any singleton $\{a\}$ is a terminal object, because there is only one function $A \to \{a\}$ for every set A (the constant function $x \mapsto a$). Besides, the empty set $\emptyset$ is the unique initial object; for set-theoretic reasons, there is only one function $\emptyset \to A$ (the empty function).

Example 6.29. If $(P, \leqslant)$ is a preorder, then the initial object is the minimal object $\min(P)$ (if it exists) and the terminal object is the maximum $\max(P)$ (if it exists).

Proposition 6.30.

1. *Let $\mathscr{C}$ be a category with initial object I. The initial object is unique up to unique isomorphism.*
2. *Let $\mathscr{C}$ be a category with terminal object T. The terminal object is unique up to unique isomorphism.*

Proof. **[Proof of 1]**

Let I and I' be two initial objects. Then there is only one arrow $I \to I$, $i : I \to I'$, $i' : I' \to I$ and $I' \to I'$. We have $i' \circ i : I \to I$, but the only arrow $I \to I$ is id_I so $i' \circ i = \mathrm{id}_I$. Similarly, we have $i \circ i' = \mathrm{id}_{I'}$, so i and i' are isomorphisms between I and I'.

[Proof of 2]

Same as with the initial objects. □

Lemma 6.31. *If $\mathscr{C}$ has finite products, then $\mathscr{C}$ has a terminal object.*

Proof. For any finite sequence of objects $(A_i)_{i \in n}$ there is a product $\prod_{i \in n} A_i$ together with projections $\pi_{A_i} : \prod_{i \in n} A_i \to A_i$.

If $n = 0$, we have an object 1 with no projections, such that for all $C \in \mathscr{C}$, there is a unique arrow $!_C : C \to 1$ such that no diagram commutes[2]. □

Proposition 6.32. *Let $\mathscr{C}$ be a category.*

If $\mathscr{C}$ has a terminal object 1, then $C \cong C^1 \cong C \times 1$.

Dually, if $\mathscr{C}$ has an initial object 0, then $C + 0 \cong C$.

Proof. For the equivalence $C \cong C \times 1$, it suffices to show that C is also a product of C and 1. For any $p : P \to C$, there is a unique arrow $!_P : P \to 1$. So, there is a unique arrow u such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, "p"'] \arrow[d, "u"] \arrow[dr, "!_P"] & \\ C & C \arrow[l, "\mathrm{id}_C"] \arrow[r, "!_C"'] & 1 \end{tikzcd}$$

and that u is $u = p$. So C and $C \times 1$ are both products of C and 1, so they are equivalent.

The same proof, with reverse arrows, yields that $C + 0 \cong C$.

As for the exponential, consider the adjunction $\mathrm{Hom}_{\mathscr{C}}\left(X, C^1\right) \cong \mathrm{Hom}_{\mathscr{C}}(X \times 1, C) \cong \mathrm{Hom}_{\mathscr{C}}(X, C)$. By Corollary 2.17, we have $C \cong C^1$.

(Proof without Yoneda? Only by diagam chase? Exponentials are not unique up to iso, apparently.) □

Proposition 6.33. *Let $\mathscr{C}$ be a category and let $F : \mathscr{C} \to$ **Sets**. Then there is a natural isomorphism $\mathrm{Hom}_{\mathbf{Sets}}(1, F(C)) \cong F(C)$, natural in both X and F.*

Proof. Recall that in **Sets**, $1 = \{0\}$.

Let $C \in \mathscr{C}$, we define the mapping:

$$\varphi_{F,C} : \begin{cases} \mathrm{Hom}_{\mathbf{Sets}}(1, F(C)) & \longrightarrow & F(C) \\ f & \longmapsto & f(0) \end{cases}$$

Of course, $\varphi_{F,C}$ is a bijection (isomorphism between sets): $y \in F(C)$ then there is exactly one function $f : 1 \to F(C)$ such that $f(0) = y$.

As for the naturality in C, if $c : C \to C'$ then we chekc if the following diagram commutes:

$$\begin{tikzcd} C \arrow[d, "c"] & & \mathrm{Hom}_{\mathbf{Sets}}(1, F(C)) \arrow[r, "\varphi_{F,C}"] \arrow[d, "\mathrm{Hom}_{\mathbf{Sets}}(1, F(c))"'] \arrow[dr, phantom, "?"] & F(C) \arrow[d, "F(c)"] \\ C' & \rightsquigarrow & \mathrm{Hom}_{\mathbf{Sets}}(1, F(C')) \arrow[r, "\varphi_{F,C'}"'] & F(C') \end{tikzcd}$$

For $f \in \mathrm{Hom}_{\mathbf{Sets}}(1, F(C))$:

[2] In fact I don't really understand this proof. It comes from [1, p. 47], and is supported by https://math.stackexchange.com/questions/1991522/terminal-objects-as-nullary-products. I don't find the proof convincing because I feel like we can define the product of n objects, for $n \geqslant 2$, or even $n = 1$. However, $n = 0$ seems like using the definition for a borderline case. As everyone seems to agree to this lemma (probably because it can be proven from elsewhere, using other tools), I mention it.

$$\begin{aligned}
F(c) \circ \varphi_{F,C}(f) &= F(c)(f(0)) \\
\varphi_{F,C'} \circ \mathrm{Hom}_{\mathscr{C}}(1, F(c))(f) &= \varphi_{F,C'}(F(c)(f)) \\
&= F(c)(f)(0) \\
&= F(c) \circ \varphi_{F,C}(f)
\end{aligned}$$

hence the naturality in C.

Then, for a fixed $C \in \mathscr{C}$, if $y : F \to F'$ is a natural transformation, we need to check if the following diagram commutes:

$$\begin{tikzcd}
F \arrow[d, "\alpha"'] & & \mathrm{Hom}_{\mathscr{C}}(1, F(C)) \arrow[r, "\varphi_{F,C}"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(1,\alpha_C)"'] \arrow[dr, phantom, "?"] & F(C) \arrow[d, "\alpha_C"] \\
F' & \rightsquigarrow & \mathrm{Hom}_{\mathscr{C}}(1, F'(C)) \arrow[r, "\varphi_{F',C}"'] & F'(C)
\end{tikzcd}$$

For $f \in \mathrm{Hom}_{\mathscr{C}}(1, F(C))$, it does:

$$\begin{aligned}
\alpha_C \circ \varphi_{F,C}(f) &= \alpha_C \circ f(0) \\
\varphi_{F',C} \circ \mathrm{Hom}_{\mathbf{Sets}}(1, \alpha_C)(f) &= \varphi_{F',C}(\alpha_C \circ f) \\
&= \alpha_C \circ f(0) \\
&= \alpha_C \circ \varphi_{F,C}(f)
\end{aligned}$$

□

Definition 6.34 (Cartesian closed category [2])**.** The category $\mathscr{C}$ is called *Cartesian closed* whenever the following three conditions hold:

1. There is a terminal object 1
2. $\mathscr{C}$ has finite products
3. For all objects $C, D \in \mathrm{Ob}_{\mathscr{C}}$, the exponential C^D exists

Example 6.35. The category **Sets** is Cartesian closed.

Recall from Section 1 the notions of epimorphisms and monomorphisms. The following notion of equaliser gives an example of monomorphism (and its dual notion is an example of epimorphism). In fact, it also gives a characterisation of isomorphisms.

Definition 6.36 (Equalisers [1])**.** Let $\mathscr{C}$ be a category, and let $f, g : A \to B$ be two arrows.

An *equaliser of f and g* is a pair (E, e) with $E \in \mathscr{C}$ and $e : E \to A$, such that $f \circ e = g \circ e$ and, for all $x : X \to A$ such that $f \circ x = g \circ x$, there exists a unique $u : X \to E$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[d, "u"'] \arrow[dr, "x"] & & \\
E \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

Example 6.37. (From [1]). In **Sets**, given two functions $f, g : A \to B$, their equaliser is (E, e) where $E = \{x \in A \mid f(x) = g(x)\}$ and $e : E \to A$ is the canonic inclusion.

Example 6.38. In a preorder category $(P, \leqslant)$, there is at most one arrow $p \to q$. Thus, the equaliser of $f, g : p \to q$, with $f = g$ is their domain together with its identity $\left(p, \mathrm{id}_p\right)$.

Proposition 6.39. *The equaliser is unique up to isomorphism.*

Proof. Let (E, e) and (E', e') be equalisers of f and g. There exist unique $u : E' \to E$ and $u' : E \to E'$ such that $e = u' \circ e'$ and $e' = u \circ e$, as in the following diagram:

$$\begin{tikzcd}
E \arrow[r, "e"] \arrow[d, bend left, "u'"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E' \arrow[u, bend left, "u"] \arrow[ur, "e'"'] & &
\end{tikzcd}$$

Thus, we have: $e = u' \circ e' = u' \circ u \circ e$, which gives the following diagram:

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E \arrow[u, bend left, "u' \circ u"] \arrow[u, bend right, "\mathrm{id}_E"'] \arrow[ur, "u' \circ u \circ e"'] & &
\end{tikzcd}$$

By unicity of the arrow $E \to E$ which makes the diagram commute, we have $u' \circ u = \mathrm{id}_E$. A similar reasoning yields $u \circ u' = \mathrm{id}_E$. ☐

Proposition 6.40. *Let (E, e) be an equaliser of $f, g : A \to B$.*
Then e is monic.

Proof. Let $c, c' : C \to E$ such that $e \circ c = e \circ c'$.

$$\begin{tikzcd}
C \arrow[d, shift right, "c'"'] \arrow[d, shift left, "c"] \arrow[dr, "e \circ c = e \circ c'"] & & \\
E \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

By definition of an equaliser, we have $f \circ e \circ c = g \circ e \circ c$, so there exists a unique $u : C \to E$ such that $e \circ u = e \circ c = e \circ c'$. By unicity of u, we have $u = c = c'$, hence e is monic. ☐

Proposition 6.41. *Let (E, e) be an equaliser of $f, g : A \to B$.*
If e is an epimorphism then e is an isomorphism.

Proof. Suppose e is epic. As an equaliser, we have the following diagram:

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

and as an epimorphism, we deduce that $f \circ e = g \circ e \Rightarrow f = g$.

Thus, the identity $\mathrm{id}_A : A \to A$ verifies $f \circ \mathrm{id}_A = g \circ \mathrm{id}_A$. Consequently, there exists a unique $u : A \to E$ such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
A \arrow[u, "u"] \arrow[ur, "\mathrm{id}_A"'] & &
\end{tikzcd}$$

from which we deduce $e \circ u = \mathrm{id}_A$.
The same occurs with $e : E \to A$:

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
A \arrow[u, "u"] \arrow[ur, "\mathrm{id}_A"'] & & \\
E \arrow[u, "e"] \arrow[uu, bend left, "\mathrm{id}_E"] \arrow[uur, bend right, "e=e\circ u\circ e"'] & &
\end{tikzcd}$$

We know that $e = e \circ \mathrm{id}_E = e \circ (u \circ e) = (e \circ u) \circ e = \mathrm{id}_A \circ e$. As an equaliser, e is monic, so $u \circ e = \mathrm{id}_E$; e is an isomorphism and $e^{-1} = u$. □

We deduce from this proposition what a *monic epimorphism* (or an *epic monomorphism*, or *monic/epic*) lacks to be an isomorphism:

Corollary 6.42. *Let $c : C \to C'$ be any arrow.*
The arrow c is an isomorphism $\Leftrightarrow$ c is an epic equaliser.

Take the arrows and reverse them; you get the definition of a coequaliser:

Definition 6.43 (Coequalisers [1])**.** Let $\mathscr{C}$ be a category, and let $f, g : A \to B$ be two arrows.
A *coequaliser of f and g* is a pair (Q, q) with $Q \in \mathscr{C}$ and $q : B \to Q$, such that $q \circ f = q \circ f$ and, for all $x : B \to X$ such that $x \circ f = x \circ g$, there exists a unique $u : E \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "x"'] & Q \arrow[d, "u"] \\
& & X
\end{tikzcd}$$

By duality, the following proposition holds:

Proposition 6.44. *The coequaliser is unique up to isomorphism.*

Proposition 6.45. *Let (Q, q) be a coequaliser of $f, g : A \to B$.*
Then q is epic.

Proposition 6.46. *Let (Q, q) be a coequaliser of $f, g : A \to B$.*
If q is a monomorphism then q is an isomorphism.

Corollary 6.47. *Let $c : C \to C'$ be any arrow.*
The arrow c is an isomorphism $\Leftrightarrow$ c is a monic coequaliser.

Example 6.48. In **Sets**, take $f, g : A \to B$. Let R be the relation such that $\forall a \in A, (f(a), g(a)) \in R$, and let $\bar{R}$ be the smallest equivalence relation containing R. Consider $\left(B_{/R}, b\right)$, where $B_{/R}$ is the quotient of B by the equivalence relation R, and b is the function that sends an element of B to its equivalence class. Then, $\left(B_{/R}, b\right)$ is the coequaliser of f and g.
For more details, see [2, Section 9.4.1, pp 278-279].

Example 6.49. Just as in Remark 6.38, as there is only one arrow between any two objects, the coequaliser of $f, g : p \to q$ is their codomain: $\left(q, \mathrm{id}_q\right)$.

We finish our presentation of the constructions with some universality in them, with pullbacks, and their dual, pushouts.

Definition 6.50 (Pullback [1]). Let $\mathscr{C}$ be a category. Let $f : A \to C$ and $g : B \to C$ be arrows with same codomain.

The *pullback of f and g* is a 3-tuple $(A \times_C B, p_A, p_B)$ such that the following diagram commutes:

$$\begin{tikzcd}
A \times_C B \arrow[r, "p_A"] \arrow[d, "p_B"'] \arrow[dr, phantom, "\checkmark"] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

and such that, for all (X, x_A, x_B) such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "x_A"] \arrow[d, "x_B"'] \arrow[dr, phantom, "\checkmark"] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

there is a unique arrow $u : X \to A \times_C B$ such that $x_A = p_A \circ u$ and $x_B = p_B \circ u$, that is, such that the triangles and squares commute:

$$\begin{tikzcd}
P \arrow[dr, "u"] \arrow[drr, "x_A"] \arrow[ddr, "x_B"'] & & \\
& A \times_C B \arrow[r, "p_A"'] \arrow[d, "p_B"] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

Example 6.51 (Pullbacks in **Sets**). In **Sets**, let $f : A \to C$ and $g : B \to C$ be two functions. Their pullback $(A \times_C B, \pi_A, \pi_B)$ is:

$$\begin{aligned}
A \times_C B &= \{z \in \mathcal{P}\left(\mathcal{P}\left(A \cup B\right)\right) \mid f \circ \pi_A(z) = g \circ \pi_B(z)\} \\
&\cong \{(x, y) \in A \times B \mid f(x) = g(y)\}
\end{aligned}$$

with projections $\pi_A : A \times_C B \to A$ and $\pi_B : A \times_C B \to B$.

Note that there is the idea of "equalising" two functions. As we will see in a following proposition, there is a link between equalisers and pullbacks, and the explicit construction is based on this idea.

Consider the special case where f and g are inclusion mappings (that is: functions of the form $f : \left\{\begin{array}{ccc} A & \longrightarrow & C \\ x & \longmapsto & x \end{array}\right.$ for $A \subset C$ and $g : \left\{\begin{array}{ccc} B & \longrightarrow & C \\ x & \longmapsto & x \end{array}\right.$ for $B \subset C$). The pullback of f and g is then:

$$\begin{aligned}
A \times_C B &= \{ (a, b) \in A \times B \mid a = b \} \\
&= \{(a, a) \in A \times B\} \\
&\cong \{a \in A\}\, a \in B \\
&= A \cap B
\end{aligned}$$

The intersection of sets consists in a pullback of inclusion mappings in **Sets**.

Example 6.52 (Pullbacks in a preorder). In a preorder category $(P, \leqslant)$, as there is at most one arrow between two objects, we don't need to check that any diagram commutes. In fact, the pullback is exactly the same as a product; that is, a pullback between $p \to q$ and $p' \to q$ is $p \times_q p' = p \times p' = \inf(p, p')$.

Proposition 6.53. *Pullbacks are unique up to isomorphism.*

Proof. This proof is similar to the ones for products, coproducts, equalisers, coequalisers. □

Let us study some more properties related to pullbacks. For example, pullbacks allow for a different characterisation of monomorphisms in a category.

Proposition 6.54. *Let $f : A \to B$ be an arrow. Then the following propositions are equivalent:*

1. *f is a monomorphism*
2. *The pullback of f with itself exists and is (P, p, p') with $p = p'$*
3. *The pullback of f with itself exists and is $(A, \mathrm{id}_A, \mathrm{id}_A)$*

Proof. **[(1) ⇒ (3)]** Suppose f is a monomorphism. Then for all $c, c' : C \to A$ such that $f \circ c = f \circ c'$, we have $c = c'$.

$$\begin{tikzcd} C \arrow[r, shift left, "c"] \arrow[r, shift right, "c'"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Then consider the triple $(A, \mathrm{id}_A, \mathrm{id}_A)$. Consider the following diagram:

$$\begin{tikzcd}
Q \arrow[dr, dashed, "u"] \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_2"'] & & \\
& A \arrow[r, "\mathrm{id}_A"'] \arrow[d, "\mathrm{id}_A"] \arrow[dr, phantom, "\checkmark"] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

such that $f \circ q_1 = f \circ q_2$. As f is monic, then $q_1 = q_2$ and the unique u is $u = q_1 = q_2$ and $(A, \mathrm{id}_A, \mathrm{id}_A)$ is the pullback of A with itself.

[(2) ⇒ (1)] Suppose (P, p, p) is the pullback of f with itself. Let Q be as in the following pullback diagram:

$$\begin{tikzcd}
Q \arrow[dr, dashed, "u"] \arrow[drr, "q_1"] \arrow[ddr, "q_2"'] & & \\
& P \arrow[r, "p"'] \arrow[d, "p"] \arrow[dr, phantom, "\checkmark"] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

such that $f \circ q_1 = f \circ q_2$. By definition of a pullback, the unique u verifies: $q_1 = p \circ u = q_2$, hence the monicity of f.

[(3) $\Rightarrow$ (2)] Obvious. □

Proposition 6.55. *Consider the following diagram:*

$$\begin{tikzcd}
A_1 \arrow[r, "a_1"] \arrow[d, "a"'] & B_1 \arrow[r, "b_1"] \arrow[d, "b"'] \arrow[dr, phantom, "\checkmark"] & C_1 \arrow[d, "c"] \\
A_2 \arrow[r, "a_2"'] & B_2 \arrow[r, "b_2"'] & C_2
\end{tikzcd}$$

Suppose that (B_1, b_1, b) is the pullback of c with b_2. Then (A_1, a_1, a) is the pullback of b with $a_2 \Leftrightarrow (A_1, b_1 \circ a_1, a)$ is the pullback of c with $b_2 \circ a_2$.

Suppose the right-hand square is a pullback; then the left-hand square is a pullback if and only if the whole rectangle is a pullback.

Proof. By diagram chase. □

This result proves sometimes to be useful, when some objects are defined in terms of pullbacks. More properties of pullbacks will come later. For now, let us just focus on the links between pullbacks, equalisers, products and terminal objects.

Lemma 6.56. *Let $\mathscr{C}$ be a category with products and equalisers. Let $f : A \longrightarrow C$ and $g : B \longrightarrow C$ be arrows.*

Let $(A \times B, \pi_A, \pi_B)$ be the product of A and B, $E \in \mathscr{C}$, $e : E \longrightarrow A \times B$, $e_A : E \longrightarrow A$ and $e_B : E \longrightarrow B$, as in the following diagram.

$$\begin{tikzcd}
E \arrow[dr, "e"] \arrow[drr, "e_A"] \arrow[ddr, "e_B"'] & & \\
& A \times B \arrow[r, "\pi_A"'] \arrow[d, "\pi_B"] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

We suppose that $e_A = \pi_A \circ e$ and $e_B = \pi_B \circ e$. The rest of the diagram is not supposed to commute otherwise.

(E, e) is an equaliser of $f \circ \pi_A$ and $g \circ \pi_B \Leftrightarrow (E, e_A, e_B)$ is a pullback of f and g.

Proof. Suppose (E, e) is an equaliser of $f \circ \pi_A$ and $g \circ \pi_B$. Let (P, p_A, p_B) such that $p_A : P \to A$, $p_B : P \to B$ and $f \circ p_A = g \circ p_B$. By definition of the product $A \times B$, there is a unique $u : P \to A \times B$ such that the two upper triangles commute:

$$\begin{tikzcd} & P \arrow[dl, "p_B"'] \arrow[d, "u"] \arrow[dr, "p_A"] & \\ B \arrow[dr, "g"'] & A \times B \arrow[l, "\pi_B"] \arrow[r, "\pi_A"'] & A \arrow[dl, "f"] \\ & C & \end{tikzcd}$$

Now, we have:

$$f \circ p_A = g \circ p_B$$
$$(f \circ \pi_A) \circ u = (g \circ \pi_B) \circ u$$

As (E, e) is an equaliser of $f \circ \pi_A$ and $g \circ \pi_B$, there exists a unique $v : P \to E$ such that $e \circ v = u$. Consequently, we have:

$$e \circ v = u$$
$$\pi_A \circ e \circ v = \pi_A \circ u$$
$$e_A \circ v = p_A$$

Similarly, for B, we have $e_B \circ v = p_B$. In summary, for any (P, p_A, p_B) such that $f \circ p_A = g \circ p_B$, there is a unique $v : P \to E$ such that $e_A \circ v = p_A$ and $e_B \circ v = p_B$; thence, (E, e_A, e_B) is a pullback of f and g.

Conversely, suppose (E, e_A, e_B) is a pullback of f and g; we have: $f \circ e_A = g \circ e_B$.

By definition of the product $A \times B$, there is a unique $u : E \to A \times B$ such that the two upper triangles commute:

$$\begin{tikzcd} & E \arrow[dl, "e_B"'] \arrow[d, "u"] \arrow[dr, "e_A"] & \\ B \arrow[dr, "g"'] & A \times B \arrow[l, "\pi_B"] \arrow[r, "\pi_A"'] & A \arrow[dl, "f"] \\ & C & \end{tikzcd}$$

We already have $e_A = \pi_A \circ e$ and $e_B = \pi_B \circ e$, so that unique u is e.

Let (P, p) be such that $p : P \to A \times B$ and $f \circ \pi_A \circ p = g \circ \pi_B \circ p$. Such a p induces arrows $p_A = \pi_A \circ p$ and $p_B = \pi_B \circ p$; we have:

$$f \circ p_A = g \circ p_B$$

As (E, e_A, e_B) is a pullback of f and g, there is a unique $v : P \to E$ such that $p_A = e_A \circ v$ and $p_B = e_B \circ v$.

We deduce:

$$p_A = e_A \circ v \Rightarrow \pi_A \circ p = \pi_A \circ e \circ v$$
$$p_B = e_B \circ v \Rightarrow \pi_B \circ p = \pi_B \circ e \circ v$$

which yields:

$$\begin{aligned}
(\pi_A \circ p, \pi_B \circ p) &= (\pi_A \circ e \circ v, \pi_B \circ e \circ v) \\
(\pi_A, \pi_B) \circ p &= (\pi_A, \pi_B) \circ e \circ v \\
\mathrm{id}_{A\times B} \circ p &= \mathrm{id}_{A\times B} \circ e \circ v \\
p &= e \circ v
\end{aligned}$$

To be an equaliser, there is one thing missing: $f \circ \pi_A \circ e = g \circ \pi_B \circ e$. This can be deduced from:

$$\begin{aligned}
f \circ e_A &= g \circ e_B \\
f \circ \pi_A \circ e &= g \circ \pi_B \circ e
\end{aligned}$$

□

Corollary 6.57. *If a category has finite products and equalisers, then it has pullbacks.*

Lemma 6.58. *Let $\mathscr{C}$ be a category.*
If $\mathscr{C}$ has pullbacks and a terminal object, then $\mathscr{C}$ has finite products.

Proof. Let $A, B \in \mathrm{Ob}_{\mathscr{C}}$. There are unique arrows $!_A : A \to 1$ and $!_B : B \to 1$. Let $(A \times_1 B, p_A, p_B)$ be the pullback of $!_A$ and $!_B$.

Let X be any object and let $x_A : X \to A$ and $x_B : X \to B$ be any arrows from X to A and B. By definition of a terminal object, there is a unique arrow $!_X : X \to 1$, so $!_X$ is:

$$!_X = !_A \circ x_A = !_B \circ x_B$$

Then X qualifies for the existence of a unique $u : X \to A \times_1 B$ such that the two triangles commute, as in the following diagram:

$$\begin{tikzcd}
X \arrow[dr, "u"] \arrow[drr, bend left, "x_A"] \arrow[ddr, bend right, "x_B"'] & & \\
& A \times_1 B \arrow[r, "p_A"] \arrow[d, "p_B"'] & A \arrow[d, "!_A"] \\
& B \arrow[r, "!_B"'] & 1
\end{tikzcd}$$

Thus, $A \times_1 B$ is a product of A and B. □

Lemma 6.59. *Let $\mathscr{C}$ be a category.*
If $\mathscr{C}$ has pullbacks and finite products, then $\mathscr{C}$ has equalisers.

Proof. The proof again consists in finding the right pullback that will be the equaliser. As $\mathscr{C}$ has products, we define $B \times B$. The pullback of $(\mathrm{id}_B, \mathrm{id}_B) : B \to B \times B$ and $(f, g) : A \to B \times B$ exists and is such that:

$$
\begin{aligned}
(\mathrm{id}_B, \mathrm{id}_B) \circ e' &= (f, g) \circ e \\
(e', e') &= (f \circ e, g \circ e) \\
&\Rightarrow f \circ e = g \circ e
\end{aligned}
$$

as in the diagram:

$$
\begin{tikzcd}
X \arrow[dr, "u"] \arrow[drr, bend left, "x_B"] \arrow[ddr, bend right, "x_A"'] & & \\
& E \arrow[r, "e"] \arrow[d, "e'"'] & A \arrow[d, "{(f,g)}"] \\
& B \arrow[r, "{(\mathrm{id}_B,\mathrm{id}_B)}"'] & B \times B
\end{tikzcd}
$$

The universality of the equaliser comes from that of the pullback. □

Finally, the dual notion of a pullback is a pushout:

Definition 6.60 (Pushout [5])**.** Let $\mathscr{C}$ be a category. Let $f : A \to B$ and $g : A \to C$ be arrows with same domain.

The *pushout of f and g* is a 3-tuple $(B +_A C, p_B, p_C)$ such that the following diagram commutes:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\checkmark"] & B \arrow[d, "p_B"] \\
C \arrow[r, "p_C"'] & B +_A C
\end{tikzcd}
$$

and such that, for all (X, x_A, x_B) such that the following diagram commutes:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\checkmark"] & B \arrow[d, "x_B"] \\
C \arrow[r, "x_C"'] & X
\end{tikzcd}
$$

there is a unique arrow $u : B +_A C \to X$ such that $x_B = p_B \circ u$ and $x_C = p_C \circ u$, that is, such that the triangles and squares commute:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "p_B"'] \arrow[ddr, "x_B"] & \\
C \arrow[r, "p_C"] \arrow[drr, "x_C"'] & B +_A C \arrow[dr, "u"] & \\
& & X
\end{tikzcd}
$$

The arrows $p_B : B \to B +_A C$ and $p_C : C \to B +_A C$ are often called the inclusion mappings, just like in the coproduct.

Example 6.61 (Pushout in **Sets**). In **Sets**, consider the functions $f : A \to B$ and $g : A \to C$. Then their pushout $B +_A C$ is identified with a subset of $B + C$; in fact, it is:

$$B +_A C = (B + C) /\equiv$$

where $\equiv$ is the smallest equivalence relation on $B + C$ such that for all $a \in A$, $f(a) \equiv g(a)$.

Another interesting special case is the following. In Example 6.51, we defined the intersection $A \cap B$ of two sets A and B. This intersection comes with trivial inclusion mappings $i_A : \begin{cases} A \cap B & \longrightarrow & A \\ x & \longmapsto & x \end{cases}$ and $i_B : \begin{cases} A \cap B & \longrightarrow & B \\ x & \longmapsto & x \end{cases}$, so we can compute its pushout.

$$\begin{tikzcd}
A \cap B \arrow[r, "i_A"] \arrow[d, "i_B"'] \arrow[dr, phantom, "\checkmark"] & A \arrow[d, "p_A"] \\
B \arrow[r, "p_B"'] & A +_{A \cap B} B
\end{tikzcd}$$

We have $A +_{A\cap B} B = (A + B) /\equiv$ where $\equiv$ is the smallest equivalence relation such that for all $a \in A$, $i_A(a) \equiv i_B(a)$. In our case, $i_A(a) = i_B(a) = a$, so $\equiv$ is simply the equality $=$. This means that, in the coproduct, which is a disjoint union in **Sets**, the pushout doesn't contain duplicates of the same element a if a is in both A and B. Thus, the pushout $A +_{A\cap B} B$ is simply the union $A \cup B$.

Example 6.62 (Pushout in a preorder). Just as pointed in Example 6.52 about pullbacks, in a preorder, the pushout is exactly the same as a coproduct.

The notions of equalisers and pullbacks will appear again in the rest of this course. The other two (coequalisers and pushouts) are introduced for the sake of completeness (and duality).

7. Your only colimit is yourself

Products/coproducts, initial/terminal objects, equalisers/coequalisers, pullbacks/pushouts are examples of the broader notion of limit. There are three ways to introduce limits, as illustrated in [2, Par. 9.2.6, p270]. We choose to introduce the limits using the characterisation with diagrams.

Definition 7.1 (Diagram [7])**.** Let $\mathscr{C}, \mathscr{I}$ be categories. A *diagram in $\mathscr{C}$ of shape $\mathscr{I}$* is a functor $\mathscr{I} \to \mathscr{C}$.

The category $\mathscr{I}$ is called the *index category* and it is usually (but not always!) small. If $\mathscr{I}$ is finite, then the diagram is said *finite*.

In the following, the objects of $\mathscr{I}$ will be denoted by $i, j, k, \ldots$ while the values of the functor $D : \mathscr{I} \to \mathscr{C}$ will be denoted by $D_i, D_j, D_k, \ldots$.

As explained in Section 1, a functor gives the picture of a category into another. A diagram $\mathscr{I} \to \mathscr{C}$ is no more than that: just a picture of the category $\mathscr{I}$ into the category $\mathscr{C}$, hence the name.

Definition 7.2 (Category of diagrams)**.** Let $\mathscr{C}, \mathscr{I}$ be categories. The *category of diagrams in $\mathscr{C}$ of shape $\mathscr{I}$* is the functor category $\mathbf{Func}\,(\mathscr{I}, \mathscr{C}) = \mathscr{C}^{\mathscr{I}}$.

Definition 7.3 (Diagonal functor)**.** Let $\mathscr{C}, \mathscr{I}$ be categories.

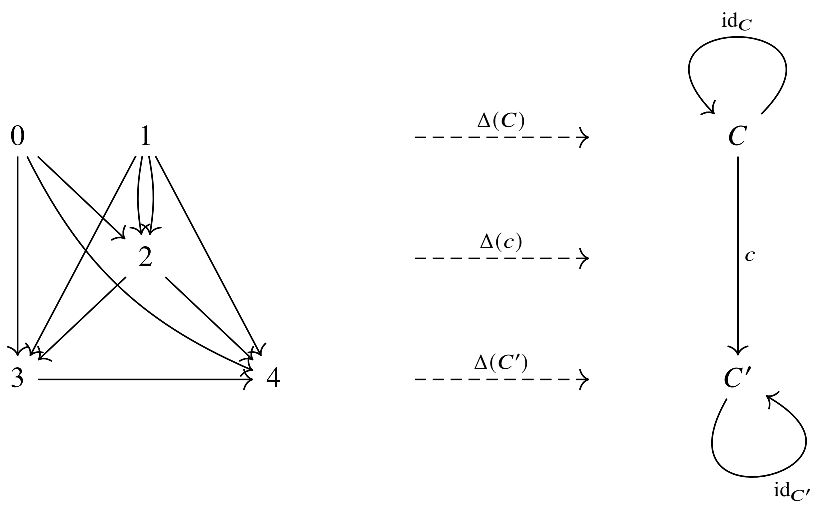
The *diagonal functor* Δ is the functor $\mathscr{C} \to \mathscr{C}^{\mathscr{I}}$ such that:

1. For all object $C \in \mathscr{C}$, $\Delta(C)$ is the diagram:

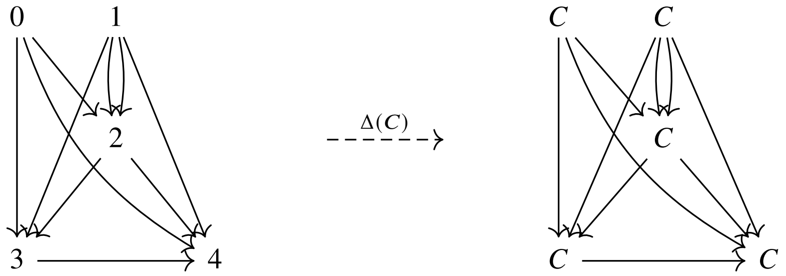
$$\Delta : \begin{cases} \mathscr{I} & \longrightarrow & \mathscr{C} \\ i & \longmapsto & C \\ i \to j & \longmapsto & \mathrm{id}_C \end{cases}$$

2. For all arrow $c : C \to C' \in \mathscr{C}$, $\Delta(c) : \Delta(C) \to \Delta(C')$ is the natural transformation $\Delta(c) = \left(C \xrightarrow{c} C'\right)_{i \in \mathscr{I}}$ (each component $\Delta(c)_i$ is a copy of c).

In summary, the functor $\Delta(C)$ "collapses" the category $\mathscr{I}$ into one element C. For example, if $\mathscr{I}$ is the following five-element category:



One can also see $\Delta(C)$ as a sequence of copies of C, indexed by the objects of $\mathscr{I}$. Here, the arrows of $\mathscr{I}$ don't matter, as they always become id_C. If $\mathscr{I}$ is a category with two objects, then $\Delta(C) = (C, C)$. A better view of the action of $\Delta(C)$ is the following diagram:



where all the arrows in the right diagram are identity arrows.

Definition 7.4 (Colimit)**.** Let $\mathscr{C}, \mathscr{I}$ be categories. Let $\Delta : \mathscr{C} \to \mathscr{C}^{\mathscr{I}}$ be the diagonal functor and let $D : \mathscr{I} \to \mathscr{C}$ be a diagram.

The pair $(\mathrm{Colim}\,(D), \eta_D)$ is the *colimit diagram for D* when $(\mathrm{Colim}\,(D)\,, \eta_D)$ is a universal arrow from D to Δ.

Remark 7.5. We have $\mathrm{Colim}\,(D) \in \mathrm{Ob}_{\mathscr{C}}$, $\eta_D : D \to \Delta\,(\mathrm{Colim}\,(D))$; that is, η_D is a natural transformation between the two diagrams $D : \mathscr{I} \to \mathscr{C}$ and $\Delta\,(\mathrm{Colim}\,(D)) : \mathscr{I} \to \mathscr{C}$.

Example 7.6. In **Sets**, a diagram $D : \mathscr{I} \to$ **Sets** is a functor that defines a small subcategory inside **Sets**. One might say it's a graph whose nodes are sets and whose arrows are functions such that the composite of two function is still an arrow in the graph. Note that, for all arrow $u : i \to j$ in $\mathscr{I}$, the arrow $D(u) : D_i \to D_j$ is a function between sets.

Define $X = \biguplus_{i \in \mathrm{Ob}_{\mathscr{I}}} D_i$ to be the coproduct in **Sets** of all D_i's. For the sake of clarity, let us explicitly define this coproduct as:

$$X = \biguplus_{i \in \mathrm{Ob}_{\mathscr{I}}} D_i = \{(i, x) \mid i \in \mathrm{Ob}_{\mathscr{I}}, x \in D_i\}$$

(You can check that this actually is a coproduct.)

Define the preorder $\asymp$ over X such that: $(i, x) \asymp (i', x')$ iff there exists some $u : i \to i'$ such that $D(u)(x) = x'$. Let $\sim$ be the equivalence relation generated by this preorder.

Then, the colimit of the diagram D is the quotient set:

$$\mathrm{Colim}\,(D) = X/{\sim} = \biguplus_{i \in \mathrm{Ob}_{\mathscr{I}}} D_i/{\sim}$$

The natural transformation η_D is composed of the inclusion maps $D_i \to \biguplus_{i \in \mathrm{Ob}_{\mathscr{I}}} D_i/{\sim}$.

Note that the coproduct of two sets corresponds to the special case where $\mathrm{card}\,(\mathrm{Ob}_{\mathscr{I}}) = 2$ and there is no arrow between the two objects, so that the equivalence relation $\sim$ is only the equality.

Example 7.7. In a preorder category $(P, \leqslant)$, the diagram $D : \mathscr{I} \to P$ defines a sub-order, and the colimit of that diagram, if it exists, is the sup of all D_i's: $\mathrm{Colim}\,(D) = \sup_{i \in \mathrm{Ob}_{\mathscr{I}}} D_i$. Note that we saw in Example 6.16 that the coproduct of a subset of a preorder was exactly its supremum. In fact, in a preorder, the arrows between two objects do not matter at all when computing colimits. This is because there is always at most one arrow between any two objects. Thus, in a preorder, the colimits are exactly the coproducts.

Remark 7.8. Let $(\mathrm{Colim}\,(D)\,, \eta_D)$ be a colimit. By definition, it is a universal arrow from D to Δ, so for all $C \in \mathrm{Ob}_{\mathscr{C}}$, for all $\alpha : D \to \Delta(C)$, there is a unique $x : \mathrm{Colim}\,(D) \to C$ such that:

$$\alpha = \Delta(x) \circ \eta_D \tag{20}$$

Note that $\alpha : D \to \Delta(C)$ is:

$$\alpha = (\alpha_i : D(i) \to \Delta(C)(i))_{i\in\mathscr{I}} = (\alpha_i : D_i \to C)_{i\in\mathscr{I}}$$

and η_D is:

$$\eta_D = (\eta_D(i) : D_i \to \Delta\left(\mathrm{Colim}\left(D\right)\right)(i))_{i\in\mathscr{I}} = (\eta_D(i) : D_i \to \mathrm{Colim}\left(D\right))_{i\in\mathscr{I}}$$

Finally, for $x : \mathrm{Colim}\left(D\right) \to C$, we have $\Delta(x) = (x : \mathrm{Colim}\left(D\right) \to C)_{i\in\mathscr{I}}$.
Thus, Equation 20 rewrites:

$$\begin{aligned}(\alpha_i : D_i \to C)_{i\in\mathscr{I}} &= (x : \mathrm{Colim}\left(D\right) \to C)_{i\in\mathscr{I}} \circ (\eta_D(i) : D_i \to \mathrm{Colim}\left(D\right))_{i\in\mathscr{I}} \\ &= \left(x \circ \left(D_i \overset{\eta_D(i)}{\longrightarrow} \mathrm{Colim}\left(D\right)\right)\right)_{i\in\mathscr{I}}\end{aligned}$$

Therefore, for all $i \in \mathscr{I}$:

$$\begin{tikzcd}
D_i \arrow[r, "\eta_D(i)"] \arrow[dr, "\alpha_i"'] & \mathrm{Colim}\left(D\right) \arrow[d, "x"] \\
& C
\end{tikzcd}$$

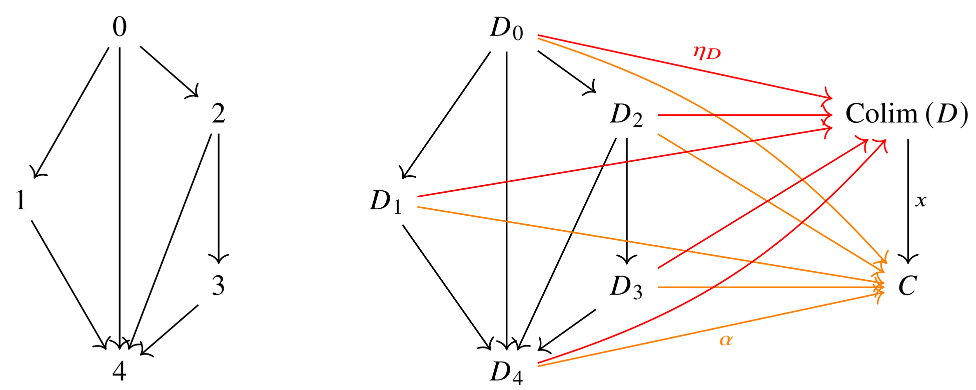
Besides, as $\alpha : D \to \Delta\left(C\right)$ and $\eta_D : D \to \Delta\left(\mathrm{Colim}\left(D\right)\right)$ are natural transformations, the following diagrams commute:

$$\begin{tikzcd}
i \arrow[dd, "u"] & & D_i \arrow[rr, "\eta_D(i)"] \arrow[drr, "\alpha_i"'] \arrow[ddl, dashed, "D(u)"] & & \mathrm{Colim}\left(D\right) \arrow[d, "x"] \arrow[ddl, dashed, "\Delta(\mathrm{Colim}(D))(u)"'] \\
& & & & C \arrow[ddl, dashed, "\Delta(C)(u)"] \\
j & D_j \arrow[rr, "\eta_D(j)"] \arrow[drr, "\alpha_j"'] & & \mathrm{Colim}\left(D\right) \arrow[d, "x"] & \\
& & & C &
\end{tikzcd}$$

which simplifies to:

$$\begin{tikzcd}
i \arrow[ddd, "u"'] & D_i \arrow[ddd, "D(u)"'] \arrow[drr, "\eta_D(i)"] \arrow[ddrr, "\alpha_i"'] & & \\
& & & \mathrm{Colim}\left(D\right) \arrow[d, "x"] \\
& & & C \\
j & D_j \arrow[uurr, "\eta_D(j)"] \arrow[urr, "\alpha_j"'] & &
\end{tikzcd}$$

Now, using a more complex starting graph:



where all triangles commute.

We see some cone-like figures in red and orange, the base of which is the diagram with the D_i's. W call these figures *cocones from D to C*. Definition 7.10 makes it more formal.

If follows from the previous remark that:

Proposition 7.9. *Let $\mathscr{C}, \mathscr{I}$ be categories. Let $\Delta : \mathscr{C} \to \mathscr{C}^{\mathscr{I}}$ be the diagonal functor and let $D : \mathscr{I} \to \mathscr{C}$ be a diagram. Let $C_D \in \mathrm{Ob}_{\mathscr{C}}$ and $\eta_D : D \to \Delta(C_D)$.*

The pair (C_D, η_D) is a colimit diagram for $D \Leftrightarrow \forall C \in \mathrm{Ob}_{\mathscr{C}}$, $\forall \alpha : D \to \Delta(C)$, $\exists! c \in Hom_{\mathscr{C}}(C_D, C)$ such that $\forall i \in \mathrm{Ob}_{\mathscr{I}}$, $\alpha_i = c \circ \eta_D$.

Proof. See Remark 7.8. Otherwise, it follows from the definition of a colimit. □

The notion of cocone was introduced in Remark 7.8. Here is the formal definition:

Definition 7.10 (Category of cocones)**.** Let $\mathscr{C}, \mathscr{I}$ be categories. Let $D : \mathscr{I} \to \mathscr{C}$ be a diagram.

The category **Cocones** (D) of cocones from D contains:

Objects: The objects are the natural transformations $\alpha : D \to \Delta(C) = \left(D_i \xrightarrow{\alpha_i} C\right)_{i \in \mathrm{Ob}_{\mathscr{I}}}$ for each $C \in \mathrm{Ob}_{\mathscr{C}}$, called *cocones from D to C*

Morphisms: Let $\alpha : \left(D_i \xrightarrow{\alpha_i} C\right)_{i \in \mathrm{Ob}_{\mathscr{I}}}$ and $\beta : \left(D_i \xrightarrow{\beta_i} C'\right)_{i \in \mathrm{Ob}_{\mathscr{I}}}$ be two cocones. An arrow $c : \alpha \to \beta$ is an arrow $c : C \to C'$ such that the following diagram commutes:

$$\begin{tikzcd} D_i \arrow[r, "\alpha_i"] \arrow[dr, "\beta_i"'] & C \arrow[d, "c"] \\ & C' \end{tikzcd}$$

Identities: An identity morphism is an arrow $\mathrm{id}_C : C \to C$

Composition: The composition law for morphisms is the composition law for morphisms in $\mathscr{C}$.

Example 7.11. In a preorder category, a cocone from D to C exists if and only if C is an upper bound of the D_i's. Note that the colimit is the least upper bound of the D_i's. This fact is made formal in the following proposition.

Proposition 7.12. *Let $\mathscr{C}, \mathscr{I}$ be categories. Let $D : \mathscr{I} \to \mathscr{C}$ be a diagram. Let* **Cocones** (D) *be the category of cocones from D. Let $C_D \in \mathscr{C}$ and $\eta_D : D \to \Delta(C_D)$.*

(C_D, η_D) is a colimit diagram for $D \Leftrightarrow \eta_D$ is an initial object in **Cocones** (D).

Proof. Using Proposition 7.9, the proof is easy:

$$
\begin{aligned}
&\eta_D \text{ is an initial object in } \mathbf{Cocones}\,(D)\\
\Leftrightarrow&\forall C \in \mathrm{Ob}_{\mathscr{C}}\, \forall \alpha : D \to \Delta(C),\ \exists! c : \eta_D \to \alpha\\
\Leftrightarrow&\forall C \in \mathrm{Ob}_{\mathscr{C}}\, \forall \alpha : D \to \Delta(C),\ \exists! c : C_D \to C \text{ such that } \forall i \in \mathrm{Ob}_{\mathscr{I}}\ \ \alpha_i = c \circ \eta_D(i)\\
\Leftrightarrow&\forall C \in \mathrm{Ob}_{\mathscr{C}}\, \forall \alpha : D \to \Delta(C),\ \exists! c : C_D \to C \text{ such that } \alpha = \Delta(c) \circ \eta_D\\
\Leftrightarrow&\ (C_D, \eta_D) \text{ is a universal arrow from } D \text{ to } \Delta\\
\Leftrightarrow&\ (C_D, \eta_D) \text{ is a colimit diagram for } D
\end{aligned}
$$

□

Corollary 7.13. *(C_D, η_D) is a colimit diagram for $D \Leftrightarrow \eta_D$ is a cocone which is universal: for any cone $\alpha : D \to \Delta(C)$, $\exists! c : C_D \to C$ such that $\alpha = \Delta(c) \circ \eta_D$.*

We gave a characterisation of a colimit (C_D, η_D) based on some property of η_D. There is also a characterisation of a colimit based on the object C_D.

Lemma 7.14. *If 0 is the initial object of $\mathscr{C}$, then the unique arrows $i_X : 0 \to X$ define the unique natural transformation $i : \Delta(0) \to \mathrm{Id}_{\mathscr{C}}$.*

If 1 is the terminal object of $\mathscr{C}$, then the unique arrows $!_X : X \to 1$ define the unique natural transformation $t : \mathrm{Id}_{\mathscr{C}} \to \Delta(1)$.

Proof. Let $f : X \to Y$. We need to check if the following diagram commutes:

$$
\begin{tikzcd}
\Delta(0)(X) \arrow[r, "i_X"] \arrow[d, "\Delta(0)(f)"'] & \mathrm{Id}_{\mathscr{C}}(X) \arrow[d, "\mathrm{Id}_{\mathscr{C}}(f)"] \\
\Delta(0)(Y) \arrow[r, "i_Y"'] & \mathrm{Id}_{\mathscr{C}}(X)
\end{tikzcd}
=
\begin{tikzcd}
0 \arrow[r, "i_X"] \arrow[d, "\mathrm{id}_0"'] & X \arrow[d, "f"] \\
0 \arrow[r, "i_Y"'] & Y
\end{tikzcd}
$$

We have $f \circ i_X : 0 \to Y$. By definition of an initial element, there is a unique arrow $i_Y : 0 \to Y$; thus, $i_Y = i_Y \circ \mathrm{id}_0 = f \circ i_X$. Besides, this natural transformation i is unique due to the uniqueness of the arrows i_X.

The statement with the terminal objects has a similar proof. □

Proposition 7.15. *Let $\mathscr{C}$ be a category and let $T \in \mathrm{Ob}_{\mathscr{C}}$.*

T is terminal in $\mathscr{C} \Leftrightarrow T$ is the colimit of $\mathrm{Id}_{\mathscr{C}}$.

Proof. **[Proof of ⇒]**

Suppose T is terminal in $\mathscr{C}$. By Lemma 7.14, there is a unique natural transformation $i : \mathrm{Id}_{\mathscr{C}} \to \Delta(T)$. This natural transformation is of course a cocone from T to $\mathrm{Id}_{\mathscr{C}}$.

Let $\alpha : \mathrm{Id}_{\mathscr{C}} \to \Delta(C)$ be a cocone from C to D for some object $C \in \mathscr{C}$. We are looking for an x so that the following diagram commutes:

$$
\begin{tikzcd}
X \arrow[r, "!_X"] \arrow[d, "\alpha_X"'] & T \arrow[dl, "x"] \\
A &
\end{tikzcd}
$$

If $X = T$, we have:

$$\begin{tikzcd} T \arrow[r, "!_T"] \arrow[d, "\alpha_T"'] & T \arrow[dl, "x"] \\ A & \end{tikzcd}$$

There is a unique arrow $!_T : T \to T$, and $!_T = \mathrm{id}_T$. Then, we have: $\alpha_T = x \circ !_T = x$. Consequently, α_T is a morphism of cocones; but if $m : A \to 1$ is another morphism of cocones, the following diagram commutes:

$$\begin{tikzcd} & T \arrow[dd, "\mathrm{id}_T"] \\ A \arrow[ur, "\alpha_T"] \arrow[dr, "m"'] & \\ & T \end{tikzcd}$$

which gives $m = \mathrm{id}_T \circ \alpha_T = \alpha_T = x$, hence the unicity of the x. Finally, (T, t) is the colimit of $\mathrm{Id}_{\mathscr{C}}$.

[Proof of $\Leftarrow$]

Let (T, η) be the colimit of $\mathrm{Id}_{\mathscr{C}}$.

The first step consists in proving that η_T is id_T.

For all $f : X \to Y$, the following diagram commutes:

$$\begin{tikzcd} X \arrow[dr, "\eta_T"] \arrow[dd, "f"'] & \\ & T \\ Y \arrow[ur, "\eta_Y"'] & \end{tikzcd}$$

In particular, if $f = \eta_X$:

$$\begin{tikzcd} X \arrow[dr, "\eta_T"] \arrow[dd, "\eta_T"'] & \\ & T \\ T \arrow[ur, "\eta_X"'] & \end{tikzcd}$$

As this is true for any X, we conclude that η_T is a morphism of cocones $\eta_T : \eta \to \eta$. By Proposition 7.12, η is initial in **Cocones** $(\mathrm{Id}_{\mathscr{C}})$, so the arrow η_T is $\eta_T = \mathrm{id}_T$.

The second step consists in showing the unicity of some arrow $X \to T$.

Let $f : X \to L$; the following diagram commutes:

$$\begin{tikzcd} \mathrm{Id}_{\mathscr{C}}(X) \arrow[r, "\eta_X"] \arrow[d, "\mathrm{Id}_{\mathscr{C}}(f)"'] \arrow[dr, phantom, "\checkmark"] & \Delta(T)(X) \arrow[d, "\Delta(T)(f)"] \\ \mathrm{Id}_{\mathscr{C}}(T) \arrow[r, "\eta_T"'] & \Delta(T)(T) \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\checkmark"] & T \arrow[d, "\mathrm{id}_T"] \\ T \arrow[r, "\mathrm{id}_T"'] & T \end{tikzcd}$$

which gives $\eta_X = f$, hence the unicity of η_X. Note that η defines one arrow $X \to T$ for each $X \in \mathrm{Ob}_{\mathscr{C}}$; so, for each $X \in \mathrm{Ob}_{\mathscr{C}}$, there is only one arrow $X \to T$, so T is terminal. $\square$

Proposition 7.16. *Let $\mathscr{C}$ be a category and let $A, B \in \mathrm{Ob}_{\mathscr{C}}$ such that the coproduct $(A+B, c_A, c_B)$ exists.*

Then $A+B$ is the colimit of the diagram $D : \begin{cases} 2 & \longrightarrow & \mathscr{C} \\ 0 & \longmapsto & A \\ 1 & \longmapsto & B \end{cases}$, where 2 is the category with two objects 0, 1 and no morphism between those two.

Proof. Note that for all $X \in \mathscr{C}$, the cocone $\alpha : D \to \Delta(X)$ has only two components $\alpha_A : A \to X$ and $\alpha_B : B \to X$. Besides, if $A+B$ exists, then there is a unique $u : A+B \to X$ such that $\alpha_A = u \circ c_A$ and $\alpha_B = u \circ c_B$; in other words, $(A+B, c)$, where $c : D \to \Delta(A+B)$ is the natural transformation with components c_A and c_B, is the colimit of D. □

We now introduce the dual notion of a colimit, namely that of a limit. We will need to introduce cones (the dual notion of cocones) as well.

Definition 7.17 (Limit)**.** Let $\mathscr{C}, \mathscr{I}$ be categories. Let $\Delta : \mathscr{C} \to \mathscr{C}^{\mathscr{I}}$ be the diagonal functor and let $D : \mathscr{I} \to \mathscr{C}$ be a diagram.

The pair $(\mathrm{Lim}\,(D), \varepsilon_D)$ is the *limit diagram for D* when for all $C \in \mathrm{Ob}_{\mathscr{C}}$, for all $\alpha : \Delta(C) \to D$, there is a unique $x : C \to \mathrm{Lim}\,(D)$ such that $\alpha = \varepsilon_D \circ \Delta(x)$.

We say that $\mathscr{C}$ *has finite limits* if every diagram $D : \mathscr{I} \to \mathscr{C}$ with finite index category $\mathscr{I}$ has a limit.

Remark 7.18. We have $\mathrm{Lim}\,(D) \in \mathrm{Ob}_{\mathscr{C}}$, $\varepsilon_D : \Delta\,(\mathrm{Lim}\,(D)) \to D$; that is, ε_D is a natural transformation between the two diagrams $\Delta\,(\mathrm{Lim}\,(D)) : \mathscr{I} \to \mathscr{C}$ and $D : \mathscr{I} \to \mathscr{C}$.

Example 7.19. As stated in Example 7.6, in **Sets**, a diagram $D : \mathscr{I} \to$ **Sets** is a functor that defines a small subcategory inside **Sets**.

The limit of D can be defined explicitly as:

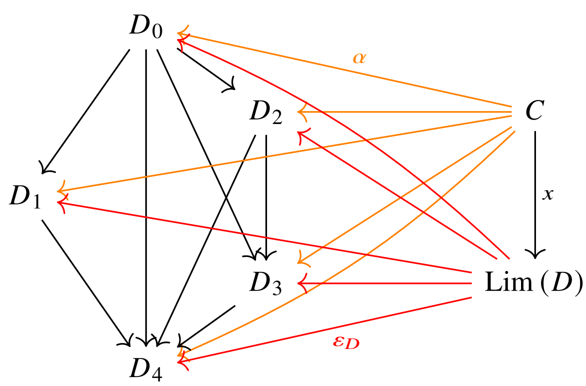
$$\mathrm{Lim}\,(D) = \left\{ (s_i)_{i \in \mathrm{Ob}_{\mathscr{I}}} \in \prod_{i \in \mathrm{Ob}_{\mathscr{I}}} D_i \;\middle|\; \forall u : i \to j,\ D(u)(s_i) = s_j \right\}$$

And the natural transformation ε_D is composed of each projection $\prod_{i \in \mathrm{Ob}_{\mathscr{I}}} D_i \to D_i$.

Again, the product is a special case of limit, when $\mathscr{I}$ is the category with only two objects and no arrow between them, so that the condition $\forall u : i \to j,\ D(u)(s_i) = s_j$ is vacuously true.

Example 7.20. In a preorder category $(P, \leqslant)$, the diagram $D : \mathscr{I} \to P$ defines a sub-order, and the limit of that diagram, if it exists, is the inf of all D_i's: $\mathrm{Lim}\,(D) = \inf_{i \in \mathrm{Ob}_{\mathscr{I}}} D_i$. Just as colimits (see Example 7.7), arrows between objects do not matter when computing limits. The limit of a diagram in a preorder is exactly the same as the product of its components.

Remark 7.21. Using the same diagram as in the last example in Remark 7.8, and using the duality, a limit illustrates this way:



Definition 7.22 (Cone)**.** Let $D : \mathscr{I} \to \mathscr{C}$ be a diagram.

We define the category **Cones** (D) of cones to D as the following category:

Objects: The objects are the natural transformations $\alpha : \Delta(C) \to D = \left(C \xrightarrow{\alpha_i} D_i\right)_{i \in \mathrm{Ob}_{\mathscr{I}}}$ for $C \in \mathrm{Ob}_{\mathscr{C}}$, called *cones from C to D*

Morphisms: Let $\alpha : \left(C \xrightarrow{\alpha_i} D_i\right)_{i \in \mathrm{Ob}_{\mathscr{I}}}$ and $\beta : \left(C' \xrightarrow{\beta_i} D_i\right)_{i \in \mathrm{Ob}_{\mathscr{I}}}$ be two cones. An arrow $c : \alpha \to \beta$ is an arrow $c : C \to C'$ such that the following diagram commutes:

$$\begin{tikzcd}
D_i & C \arrow[l, "\alpha_i"'] \arrow[d, "c"] \\
& C' \arrow[ul, "\beta_i"]
\end{tikzcd}$$

Identities: An identity morphism is an arrow $\mathrm{id}_C : C \to C$

Composition: The composition law for morphisms is the composition law for morphisms in $\mathscr{C}$.

Remark 7.23. Note that a cocone is *from* the diagram D *to* the object C, while a cone is *from* the object C *to* the diagram D.

Example 7.24. In a preorder category, a cone from C to D exists if and only if C is a lower bound of the D_i's. Just like in Example 7.11, note that the limit is the greatest lower bound of the D_i's. This fact is made formal in Proposition 7.25.

As the dual notion of colimit, we have the dual characterisations of limits:

Proposition 7.25. *Let $\mathscr{C}, \mathscr{I}$ be categories. Let $D : \mathscr{I} \to \mathscr{C}$ be a diagram. Let* **Cones** (D) *be the category of cones to D. Let $C_D \in \mathscr{C}$ and $\varepsilon_D : \Delta(C_D) \to D$.*

(C_D, ε_D) is a limit diagram for $D \Leftrightarrow \varepsilon_D$ is a terminal object in **Cones** (D).

Proof. Similar to the proof of Proposition 7.12. □

Proposition 7.26. *Let $\mathscr{C}$ be a category and let $I \in \mathrm{Ob}_{\mathscr{C}}$.*

I is initial in $\mathscr{C} \Leftrightarrow I$ is the limit of $\mathrm{Id}_{\mathscr{C}}$.

Proof. Similar to the proof of Proposition 7.15. □

Remark 7.27. Colimits are the initial objects of the category of cocones (Proposition 7.12), while the terminal object of a category is the colimit of the identity functor (Proposition 7.15). Dually, limits are the terminal objects of the category of cones (Proposition 7.25), while the initial object of a category is the limit of the identity functor (Proposition 7.26). Try not to confuse!

In the following, we mention the other limit diagrams; the proof is essentially the same as Proposition 7.16.

Proposition 7.28. *The following constructions are limits:*

1. *A terminal object in $\mathscr{C}$ is the limit of the empty diagram $D : \emptyset \to \mathscr{C}$, with $\emptyset$ as the empty category:*

$$\begin{tikzcd}
X \arrow[d, "u"] \\
1
\end{tikzcd}$$

2. *A product $A \times B$ in $\mathscr{C}$ is the limit of the diagram $D : \mathscr{C}_2 \to \mathscr{C}$, with $\mathscr{C}_2$ being the index category with two objects and no arrow between those two:*

$$\begin{tikzcd}
A & X \arrow[l] \arrow[dl] \arrow[d, "u"] \\
B & A \times B \arrow[l] \arrow[ul]
\end{tikzcd}$$

3. *A pullback of $f : A \to C$ and $g : B \to C$ is the limit of the diagram $D : \mathscr{C}_3 \to \mathscr{C}$, where $\mathscr{C}_3$ is the index category described below:*

$$\begin{tikzcd}
 & & & X \arrow[dlll] \arrow[dl] \arrow[ddll] \arrow[dd, "u"] \\
A \arrow[dr] & & B \arrow[dl] & \\
 & C & & A \times_C B \arrow[ulll] \arrow[ul] \arrow[ll]
\end{tikzcd}$$

4. *An equaliser of $f, g : A \to B$ is the limit of the diagram $D : \mathscr{C}_4 \to \mathscr{C}$ where $\mathscr{C}_4$ is the index category described below:*

$$\begin{tikzcd}
A \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & X \arrow[l] \arrow[dl] \arrow[d, "u"] \\
B & E \arrow[ul] \arrow[l]
\end{tikzcd}$$

The dual statement is also true:

Proposition 7.29. *The following constructions are colimits:*

1. *An initial object in $\mathscr{C}$ is the colimit of the empty diagram $D : \emptyset \to \mathscr{C}$, with $\emptyset$ as the empty category:*

$$\begin{tikzcd}
0 \arrow[d, "u"] \\
X
\end{tikzcd}$$

2. *A coproduct $A + B$ in $\mathscr{C}$ is the colimit of the diagram $D : \mathscr{C}_2 \to \mathscr{C}$, with $\mathscr{C}_2$ being the category with two objects and no arrow between those two:*

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & A + B \arrow[d, "u"] \\
B \arrow[ur] \arrow[r] & X
\end{tikzcd}$$

3. *A pushout of $f : A \to B$ and $g : A \to C$ is the colimit of the diagram $D : \mathscr{C}_3 \to \mathscr{C}$, where $\mathscr{C}_3$ is the category described below:*

$$\begin{tikzcd}
 & & & A +_C B \arrow[dd, "u"] \\
 & A \arrow[dl, "g"'] \arrow[dr, "f"] \arrow[urr] \arrow[drr] & & \\
C \arrow[uurrr] \arrow[rrr, bend right] & & B \arrow[uur] \arrow[r] & X
\end{tikzcd}$$

4. *A coequaliser of $f, g : A \to B$ is the colimit of the diagram $D : \mathscr{C}_4 \to \mathscr{C}$ where $\mathscr{C}_4$ is the category described below:*

$$\begin{tikzcd}
A \arrow[rr] \arrow[rrd] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & & Q \arrow[d, "u"] \\
B \arrow[rru] \arrow[rr] & & X
\end{tikzcd}$$

So in fact products, equalisers, terminal objects, pullbacks are special cases of limits (and their duals are special cases of colimits). Before exploring another link between those constructions, let us give two other instances of limits and colimits that may be useful in example-building.

Remark 7.30. Consider the category $\mathscr{D}$ with only one object (and the identity morphism).

A diagram $D : \mathscr{D} \to \mathscr{C}$ may be identified to the single object $D(0) = D_0$. It is easy to see that D_0 is its own limit and colimit.

Remark 7.31. Consider the category $\mathscr{D}$ consisting in two objects and one arrow:

$$\cdot \to \cdot$$

Consider a diagram $D : \mathscr{D} \to \mathscr{C}$; its image will be an arrow $D(d) : D_1 \to D_2$. What could be the limit and colimit of this diagram?

Let $(P, (p_1, p_2))$ be a cone to D, and suppose D has a limit $(\mathrm{Lim}\,(D), (l_1, l_2))$.

$$\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[dr, "p_2"] & D_1 \arrow[r, "q_1"] \arrow[d, "D(d)"] \arrow[ddr, "c_1"] & Q \\
& D_2 \arrow[ur, "q_2"] \arrow[dr, "c_2"] & \\
\mathrm{Lim}\,(D) \arrow[uur, "l_1"] \arrow[ur, "l_2"] & & \mathrm{Colim}\,(D)
\end{tikzcd}$$

The limit $\mathrm{Lim}\,(D)$ is such that there is a unique arrow $x : P \to \mathrm{Lim}\,(D)$ such that:

$$\begin{aligned}
p_i &= l_i \circ x \quad (i = 1, 2) \\
p_2 &= D(d) \circ p_1 \\
l_2 &= D(d) \circ l_1
\end{aligned}$$

A cone to D defines two arrows $p_1 : P \to D_1$ and $p_2 : P \to D_2$ such that $p_2 = D(d) \circ p_1$. So, given a cone $(P, (p_1, p_2))$ to D, there is a unique arrow $p_1 : P \to D_1$ such that the diagram commutes. In fact, $\big(D_1, (\mathrm{id}_{D_1}, D(d))\big)$ is the limit of D.

With the same reasonning, it is easy to see that $\big(D_2, (D(d), \mathrm{id}_{D_2})\big)$ is the colimit of D.

This is a better way to state this remark (better for memory): considering an arrow $A \to B$, A is its limit and B is its colimit. The limit is the domain, and the colimit is the codomain of the arrow.

Remark 7.32. Categories may or may not have all limits or all colimits, maybe for some diagrams and not others. However, diagrams from the one-object and the two-object categories always have a limit and a colimit in any category.

There is another, stronger link between products, equalisers, terminal objects and pullbacks. A similar link exists between their dual counterparts, see Theorem 7.35.

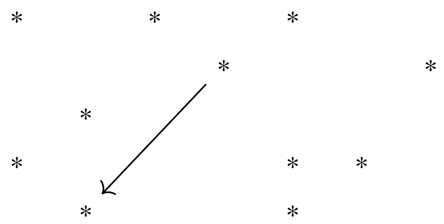
Lemma 7.33. *Let $\mathscr{C}$ be any category.*

If $\mathscr{C}$ has finite products and equalisers, then $\mathscr{C}$ has finite limits.

Proof. (The proof written here is a resolution of [2, Exercise 3, Section 2.13, Chapter 9])

We will start the proof with one special case of index category. We then give a hint for a second special case. Those two proofs generalise to any index category.

Suppose $\mathscr{I}$ is any finite category with only one non-identity arrow $a : j \to k$. It will then look like this category:



Now let $D : \mathscr{I} \to \mathscr{C}$ be any diagram.

As $\mathscr{C}$ has finite products, the product $\prod_{i \in n} D_i$ with arrows $\pi_i : \prod_{i \in \mathscr{I}} D_i \to D_i$ exists. As $\mathscr{C}$ has equalisers, consider the equaliser (E, e) of $D(a) \circ \pi_j$ and π_k.

$$E \xrightarrow{\quad e \quad} \prod_{i \in \mathscr{I}} D_i \overset{D(a)\circ\pi_j}{\underset{\pi_k}{\rightrightarrows}} D_k$$

Define $\varepsilon = (e_i = \pi_i \circ e)_{i \in \mathscr{I}}$. By definition of (E, e), we have:

$$D(a) \circ \pi_j \circ e = \pi_k \circ e$$

which proves that ε is a natural transformation $\Delta\,(\mathrm{Lim}\,(D)) \to D$ (there is only one arrow to check).

We now prove that (E, ε) is the limit of the diagram $D : \mathscr{I} \to \mathscr{C}$. Let $\alpha : \Delta(C) \to D$ be a cone to D; we have $D(a) \circ \alpha_j = \alpha_k$.

Consider the function $\Pi\alpha : C \to \prod_{i \in \mathscr{I}} D_i$ such that $\forall i \in \mathscr{I},\ \pi_i \circ \Pi\alpha = \alpha_i$. We have:

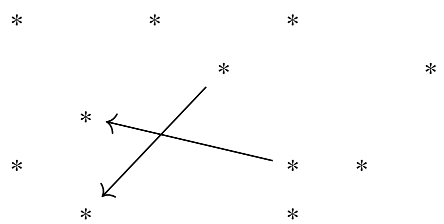
$$D(a) \circ \alpha_j = \alpha_k$$
$$D(a) \circ \pi_j \circ \Pi\alpha = \pi_j \circ \Pi\alpha$$

As (E, e) is an equaliser of $D(a) \circ \pi_j$ and π_j, there exists a unique $u : C \to E$ such that $e \circ u = \Pi\alpha$, from which we infer, for all $i \in \mathscr{I}$:

$$e \circ u = \Pi\alpha$$
$$\pi_i \circ e \circ u = \pi_i \circ \Pi\alpha$$
$$e_i \circ u = \alpha_i$$
$$\Rightarrow \ \varepsilon \circ \Delta(u) = \alpha$$

So (E, ε) is the limit of D.

Now suppose $\mathscr{I}$ is any finite category with only two non-identity arrow $a_0 : j_0 \to k_0$ and $a_1 : j_1 \to k_1$. It will then look like this category:



Note that no assumption is made about a_0 and a_1 being distinct; we only suppose that $k_0 \neq j_1$ and $j_0 \neq k_1$; otherwise they would compose and give birth to a third arrow.

For a diagram $D : \mathscr{I} \to \mathscr{C}$, we also build the product $\prod_{i \in \mathscr{I}} D_i$ with its projections $\pi_i : \prod_{i \in \mathscr{I}} D_i \to D_i$.

We also define the following arrows:

$$\begin{aligned} r_0 &= D(a_0) \circ \pi_{j_0} \\ r_1 &= D(a_1) \circ \pi_{j_1} \\ s_0 &= \pi_{k_0} \\ s_1 &= \pi_{k_1} \\ r &= (r_0, r_1) \\ s &= (s_0, s_1) \end{aligned}$$

As $\mathscr{C}$ has equalisers, consider the equaliser (E, e) of $r : \prod_{i \in \mathscr{I}} D_i \to D_{k_0} \times D_{k_1}$ and $s : \prod_{i \in \mathscr{I}} D_i \to D_{k_0} \times D_{k_1}$. The proof is very similar to the previous one. If $\alpha : \Delta(C) \to D$ is a cone to D, then we define $\Pi\alpha$ to be the concatenation of the components of α: $\forall i \in \mathscr{I}, \alpha_i = \pi_i \circ \Pi\alpha$. We check that $s \circ \Pi\alpha = r \circ \Pi\alpha$ using the fact that α is a natural transformation. As (E, e) is an equaliser, there exists a unique $u : C \to E$ such that $e \circ u = \Pi\alpha$, and we conclude that $\varepsilon \circ \Delta(u) = \Pi\alpha$, with $\varepsilon = (e_i = \pi_i \circ e)_{i \in \mathscr{I}}$ (which is a natural transformation $\Delta(E) \to D$). Finally, (E, ε) is the limit of D.

As the final case, let $\mathscr{I}$ be any finite category. Again, $\mathscr{C}$ has finite products, so we define $\prod_{i \in \mathscr{I}} D_i$ and its projections π_i. As the set of arrows in $\mathscr{I}$ is also finite, we can consider all arrows $a : j \to k \in \mathrm{Mor}_{\mathscr{I}}$ and define the product $\prod_{a:j \to k \in \mathrm{Mor}_{\mathscr{I}}} D_k$, that is, the product of all codomains of all arrows in $\mathscr{I}$. For $a_0 : j_0 \to k_0$, the projection of index a_0 will be denoted $\pi_{a_0} : \prod_{a:j \to k \in \mathrm{Mor}_{\mathscr{I}}} D_k \to D_{k_0}$.

We now define:

$$r, s : \prod_{i \in \mathscr{I}} D_i \to \prod_{a:j \to k \in \mathrm{Mor}_{\mathscr{I}}} D_k$$

such that, for all $a : j \to k \in \mathrm{Mor}_{\mathscr{I}}$, we have:

$$\begin{aligned} \pi_a \circ r &= D(a) \circ \pi_j = D(a) \circ \pi_{\mathrm{dom}\, a} \\ \pi_a \circ s &= \pi_k = \pi_{\mathrm{cod}\, a} \end{aligned}$$

Let (E, e) be an equaliser of r and s; the rest of the proof is very similar to the previous two ones. □

We can now bring together all the lemmas that we disseminated throughout the last two sections, and prove:

Theorem 7.34. *Let $\mathscr{C}$ be any category. The following propositions are equivalent:*

1. *$\mathscr{C}$ has finite products and equalisers*
2. *$\mathscr{C}$ has pullbacks and a terminal object*
3. *$\mathscr{C}$ has finite limits*

Proof. **(1 ⇒ 2)**

By Corollary 6.57, products and equalisers give pullbacks, while by Lemma 6.31, having all finite products gives a terminal object.

(2 ⇒ 1)

By Lemma 6.58, pullbacks and terminal object give products, while by Lemma 6.59, pullbacks and products give equalisers.

(1 ⇒ 3)

By Lemma 7.33, products and equalisers give limits.

(3 ⇒ 1)

By Proposition 7.28, products and equalisers are special cases of limits. □

Of course, the dual theorem is also true:

Theorem 7.35. *Let $\mathscr{C}$ be any category. The following propositions are equivalent:*

1. *$\mathscr{C}$ has finite coproducts and coequalisers*
2. *$\mathscr{C}$ has pushouts and an initial object*
3. *$\mathscr{C}$ has finite colimits*

Note that the theorems we mentionned with limits used any index category, be it small or large. Some results we proved only for finite limits, but, for example, Proposition 7.34 generalises to any cardinality (and thus, for any small category):

Theorem 7.36. *Let κ be a cardinal and let $\mathscr{C}$ be any category.*
$\mathscr{C}$ has all products of cardinality $\leqslant \kappa$ and equalisers $\Leftrightarrow$ $\mathscr{C}$ has all limits of cardinality $\leqslant \kappa$.
In other words, $\mathscr{C}$ has all small products and equalisers $\Leftrightarrow$ $\mathscr{C}$ has all small limits.

Remark 7.37 (Historical interlude). Among the many ways to introduce limits and colimits, we wanted to introduce the following version (for a source of that version, see [8, Exposé 1, section 2, page 9]).

Let $D : \mathscr{I} \to \mathscr{C}$ be a diagram. The limit of that diagram will be defined as the following functor:

$$\mathrm{Lim}\,(D) : \begin{cases} \mathscr{C}^{\mathrm{op}} & \longrightarrow & \mathbf{Sets} \\ C & \longmapsto & \mathrm{Hom}_{\mathscr{C}^{\mathscr{I}}}\,(\Delta(C), D) \end{cases}$$

and the colimit will be:

$$\mathrm{Colim}\,(D) : \begin{cases} \mathscr{C} & \longrightarrow & \mathbf{Sets} \\ C & \longmapsto & \mathrm{Hom}_{\mathscr{C}^{\mathscr{I}}}\,(D, \Delta(C)) \end{cases}$$

In other words, the limit (resp. colimit) of a diagram will be the functor that sends C to the set of cones from C to that diagram (resp. the set of cocones from that diagram to C). A diagram $\mathscr{I} \to \mathscr{C}$ has a (co)limit in the form of a functor $\mathscr{C}^{\mathrm{op}} \to \mathbf{Sets}$ or $\mathscr{C} \to \mathbf{Sets}$.

Now assume that $\mathrm{Lim}\,(D)$ the functor is represented by an object $R(D) \in \mathscr{C}$ (see Definition 3.6 for the definition of representable functor). Then for a fixed C, we have:

$$\mathrm{Hom}_{\mathscr{C}}\,(C, R(D)) \cong \mathrm{Hom}_{\mathscr{C}^{\mathscr{I}}}\,(\Delta(C), D)$$

That is, for a cone $\alpha : \Delta(C) \to D$, we have a unique arrow $C \to R(D)$ that makes the right diagrams commute... The representation $R(D)$ is exactly the limit $\mathrm{Lim}\,(D)$ of D in the sense of Definition 7.17! Similarly, the representative of $\mathrm{Colim}\,(D)$ will be the colimit $\mathrm{Colim}\,(D)$ of D.

8. Limits and adjunctions

We will now study some properties of adjunctions and their behaviour with regards to limits.

We start with a remark. We saw in Theorem 6.20 that $+ \dashv \Delta_2 \dashv \times$. There is a more general statement, that we mention but will only give a sketch of proof.

Let $\mathscr{C}$ be a category with finite limits, and let $\mathscr{I}$ be any finite category. The mappings:

$$\mathrm{Lim}\,(-) : \begin{cases} \mathscr{C}^{\mathscr{I}} & \longrightarrow & \mathscr{C} \\ D & \longmapsto & \mathrm{Lim}\,(D) \end{cases}$$

$$\mathrm{Colim}\,(-) : \begin{cases} \mathscr{C}^{\mathscr{I}} & \longrightarrow & \mathscr{C} \\ D & \longmapsto & \mathrm{Colim}\,(D) \end{cases}$$

are in fact functors. What is best, is that if $\Delta_{\mathscr{I}} : \mathscr{C} \to \mathscr{C}^{\mathscr{I}}$ is the diagonal functor of shape $\mathscr{I}$, then we have:

Theorem 8.1. $\mathrm{Colim}\,(-) \dashv \Delta_{\mathscr{I}} \dashv \mathrm{Lim}\,(-)$

Sketch of proof. This can be deduced from the definitions of a limit (resp. of a colimit), due to the existence and unicity of the arrow $C \to \mathrm{Lim}\,(D)$ (resp. $\mathrm{Colim}\,(D) \to C$) whenever we have a cone $\Delta_{\mathscr{I}}(C) \to D$ (resp. a cocone $D \to \Delta_{\mathscr{I}}(C)$). This gives the bijectivity between $\mathrm{Hom}_{\mathscr{C}^{\mathscr{I}}}(\Delta_{\mathscr{I}}(C), D)$ and $\mathrm{Hom}_{\mathscr{C}}(C, \mathrm{Lim}\,(D))$ (resp. $\mathrm{Hom}_{\mathscr{C}}(\mathrm{Colim}\,(D), C)$ and $\mathrm{Hom}_{\mathscr{C}^{\mathscr{I}}}(D, \Delta_{\mathscr{I}}(C))$). We then have to check that this defines a natural transformation. The (contravariant) naturality in C is easy, due to the definition of $\Delta_{\mathscr{I}}$, while the naturality in D requires a bit more attention not to confuse between cones and a natural transformation $\delta : D \to D'$. $\square$

The main question we will tackle in this section is the following. Suppose we have a functor $F : \mathscr{C} \to \mathscr{D}$. Does it have an adjoint? How to know if it does or not? And if it does, how to find it?

A first step may be to look at some properties of adjoints.

We deduce from Definition 4.1 and Theorem 3.17 that:

Proposition 8.2. *Let $U : \mathscr{C} \to$ **Sets** be functors.*

U has a left adjoint $\Leftrightarrow$ for all $C \in \mathscr{C}$, $\mathrm{Hom}_{\mathbf{Sets}}(C, U(-))$ has a universal element.

Another interesting property of adjoints is described right after the following definition. Just as some functors preserve products (see Definition 6.9), some functors preserve limits:

Definition 8.3 (Preserving limits and colimits)**.** Let $\mathscr{I}$ be an index category, and let $\mathscr{C}$, $\mathscr{D}$ be categories.

We say that the functor $F : \mathscr{C} \to \mathscr{D}$ *preserves all limits (resp. small limits; resp. finite limits)* when, for all index category (resp. small index category; resp. finite index category) $\mathscr{I}$, for all diagram $D : \mathscr{I} \to \mathscr{C}$, if the limit $(\mathrm{Lim}\,(D), \eta_D)$ exists, then $(F(\mathrm{Lim}\,(D)), F(\eta_D))$ is the limit of the diagram $F \circ D$.

Dually, we say that the functor $F : \mathscr{C} \to \mathscr{D}$ *preserves all colimits (resp. small colimits; resp. finite colimits)* when, for all index category (resp. small index category; resp. finite index category) $\mathscr{I}$, for all diagram $D : \mathscr{I} \to \mathscr{C}$, if the colimit $(\mathrm{Colim}\,(D), \varepsilon_D)$ exists, then $(F(\mathrm{Colim}\,(D)), F(\varepsilon_D))$ is the colimit of the diagram $F \circ D$.

Proposition 8.4. *Let $\mathscr{C}$ be a category with finite limits and let $C \in \mathrm{Ob}_{\mathscr{C}}$.*

The covariant Hom-set functor $\mathrm{Hom}_{\mathscr{C}}(C, -)$ preserves all finite limits.

Proof. By Theorem 7.34, it suffices to show that $\mathrm{Hom}_{\mathscr{C}}(C, -)$ preserves finite products and equalisers.

We already know from Proposition 6.10 that $\mathrm{Hom}_{\mathscr{C}}(C, -)$ preserves binary products. For it to preserve finite products, we need to show that it preserves the terminal object. If T is the terminal object in $\mathscr{C}$ then $\mathrm{Hom}_{\mathscr{C}}(C, T)$ contains only one arrow (the unique arrow $C \to T$ in $\mathscr{C}$). Consequently, $\mathrm{Hom}_{\mathscr{C}}(C, T) \cong 1$ (where 1 is the terminal object of **Sets**) and $\mathrm{Hom}_{\mathscr{C}}(C, -)$ preserves the terminal object.

Now, let (E, e) be the equaliser of $f, g : A \to B$. For all $z : Z \to A$ such that $f \circ z = g \circ z$, there is a unique $u : Z \to E$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, "u"] \arrow[ru, "z"'] & & \end{tikzcd}$$

The hom-set functor preserves the diagram (this is a property of functors). We need to check whether $(\mathrm{Hom}_{\mathscr{C}}(C, E), \mathrm{Hom}_{\mathscr{C}}(C, e))$ is an equaliser of $\mathrm{Hom}_{\mathscr{C}}(C, f)$ and $\mathrm{Hom}_{\mathscr{C}}(C, g)$. Let $h : X \to \mathrm{Hom}_{\mathscr{C}}(C, A)$ such that $\mathrm{Hom}_{\mathscr{C}}(C, f) \circ h = \mathrm{Hom}_{\mathscr{C}}(C, g) \circ h$ as in the diagram:

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(C, E) \arrow[r, "\mathrm{Hom}_{\mathscr{C}}(C,e)"] & \mathrm{Hom}_{\mathscr{C}}(C, A) \arrow[r, shift left, "\mathrm{Hom}_{\mathscr{C}}(C,f)"] \arrow[r, shift right, "\mathrm{Hom}_{\mathscr{C}}(C,g)"'] & \mathrm{Hom}_{\mathscr{C}}(C, B) \\ X \arrow[u, "v?"] \arrow[ru, "x"'] & & \end{tikzcd}$$

We need to find a $v : X \to \mathrm{Hom}_{\mathscr{C}}(C, E)$ such that $\mathrm{Hom}_{\mathscr{C}}(C, e) \circ v = h$.

Let $x \in X$. We have $h(x) : C \to A$ and:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(C, f) \circ h(x) &= \mathrm{Hom}_{\mathscr{C}}(C, g) \circ h(x) \\ f \circ (h(x)) &= g \circ (h(x)) \end{aligned}$$

So, the equaliser in $\mathscr{C}$ applies here: there is a unique $u(x) : C \to E$ such that $e \circ (u(x)) = h(x)$. Define u to be:

$$u : \begin{cases} X & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(C, E) \\ x & \longmapsto & u(x) \end{cases}$$

Then, by construction, for all $x \in X$, $h(x) = e \circ (u(x)) = (\mathrm{Hom}_{\mathscr{C}}(C, e) \circ u)(x)$ and u is unique. Consequently, $(\mathrm{Hom}_{\mathscr{C}}(C, E), \mathrm{Hom}_{\mathscr{C}}(C, e))$ is still an equaliser. □

Corollary 8.5. *Representable functors preserve all finite limits.*

The dual version of this theorem is the following:

Proposition 8.6. *The contravariant Hom-set functor $\mathrm{Hom}_{\mathscr{C}}(-, C)$ sends finite colimits to finite limits.*

Proof. We have to show that the contravariant sends the initial object to the terminal object, the coproduct to the product, and the coequalisers to equalisers.

We have $\mathrm{Hom}_{\mathscr{C}}(0, C) \cong 1$ because there is only one arrow $0 \to C$ (definition of an initial object).

Let $f : A \to C$ and $g : B \to C$ be two arrows; by definition of the coproduct $A + B$, there is a unique $u : A + B \to C$ such that the following diagram commutes:

$$\begin{tikzcd} & A + B \arrow[dd, "u"] & \\ A \arrow[ru, "i_A"] \arrow[rd, "f"'] & & B \arrow[lu, "i_B"'] \arrow[ld, "g"] \\ & C & \end{tikzcd}$$

This exactly says that there is a bijection (an isomorphism in **Sets**):

$$\alpha_{A,B} : \begin{cases} \mathrm{Hom}_{\mathscr{C}}(A+B,C) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(A,C) \times \mathrm{Hom}_{\mathscr{C}}(B,C) \\ u & \longmapsto & (u \circ i_A, u \circ i_B) \end{cases}$$

This isomorphism is natural in $A+B$:

$$\begin{tikzcd} A \arrow[d, "a"] & \rightsquigarrow & \mathrm{Hom}_{\mathscr{C}}(A'+B,C) \arrow[r, "\alpha_{A',B}"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(a+B,C)"'] \arrow[dr, phantom, "?"] & \mathrm{Hom}_{\mathscr{C}}(A',C) \times \mathrm{Hom}_{\mathscr{C}}(B,C) \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(a,C)\times\mathrm{Hom}_{\mathscr{C}}(B,C)"] \\ A' & & \mathrm{Hom}_{\mathscr{C}}(A+B,C) \arrow[r, "\alpha_{A,B}"'] & \mathrm{Hom}_{\mathscr{C}}(A,C) \times \mathrm{Hom}_{\mathscr{C}}(B,C) \end{tikzcd}$$

Let $f \in \mathrm{Hom}_{\mathscr{C}}(A'+B,C)$:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(a,C) \times \mathrm{Hom}_{\mathscr{C}}(B,C) \circ \alpha_{A',B}(f) &= \mathrm{Hom}_{\mathscr{C}}(a,C) \times \mathrm{Hom}_{\mathscr{C}}(B,C)\,(f \circ i_{A'}, f \circ i_B) \\ &= (f \circ i_{A'} \circ a, f \circ i_B) \\ \alpha_{A,B} \circ \mathrm{Hom}_{\mathscr{C}}(a+B,C)\,(f) &= \alpha_{A,B}\,(f \circ (a + \mathrm{id}_B)) \\ &= (f \circ (a+\mathrm{id}_B) \circ i_A, f \circ (a+\mathrm{id}_B) \circ i_B) \end{aligned}$$

Seeing the coproduct as a colimit, we deduce that the following diagram commutes:

$$\begin{tikzcd} & A+B \arrow[dd, "a+\mathrm{id}_B"] & \\ A \arrow[ur, "i_A"] \arrow[dd, "a"'] \arrow[r, phantom, "\checkmark"] & & B \arrow[ul, "i_B"'] \arrow[dd, "\mathrm{id}_B"] \arrow[l, phantom, "\checkmark"] \\ & A'+B & \\ A' \arrow[ur, "i_{A'}"'] & & B \arrow[ul, "i_B"] \end{tikzcd}$$

(Seeing $i = (i_A : A \longrightarrow A+B)_{A \in \mathscr{C}}$ as a natural transformation $\mathrm{Id}_{\mathscr{C}} \longrightarrow \mathrm{Id}_{\mathscr{C}} + \Delta(B)$)
We have:

$$\begin{aligned} (a+\mathrm{id}_B) \circ i_A &= i_{A'} \circ a \\ (a+\mathrm{id}_B) \circ i_B &= i_B \circ \mathrm{id}_B \\ \Rightarrow (f \circ (a+\mathrm{id}_B) \circ i_A, f \circ (a+\mathrm{id}_B) \circ i_B) &= (f \circ i_{A'} \circ a, f \circ i_B) \end{aligned}$$

The naturality in B is similar.

Finally, as for seeing that the contravariant Hom-set functor sends coequalisers to equalisers, the proof is very similar to showing that the covariant Hom-set functor preserves equalisers. □

In fact, these theorems are not only true for finite limits, but also for small limits. As there is something I don't understand here, because a product of any set of sets could be empty (without the Axiom of Choice) (but always exists?) , we will trust Awodey [1, Chapter 5, Proposition 5.25, p107] and admit the following proposition and corollaries:

Proposition 8.7. *Let $\mathscr{C}$ be a category with small limits and let $C \in \mathrm{Ob}_{\mathscr{C}}$.*
The covariant Hom-set functor $\mathrm{Hom}_{\mathscr{C}}(C,-)$ preserves all small limits.

Corollary 8.8. *Representable functors preserve all small limits.*

Proposition 8.9. *The contravariant Hom-set functor* $Hom_{\mathscr{C}}(-, C)$ *sends small colimits to small limits.*

Proposition 8.10 (Right Adjoints Preserve Limits [2], [1])**.** *Let* (F, U, β) *be an adjunction. Then* F *preserves colimits and* U *preserves limits.*

This proposition is commonly refered to as the the "RAPL" ("Right Adjoints Preserve Limits").

Proof. Suppose $F : \mathscr{C} \to \mathscr{X}$, $U : \mathscr{X} \to \mathscr{C}$ and let $D : \mathscr{I} \to \mathscr{X}$ be a diagram with a limit $\mathrm{Lim}\,(D)$. For $C \in \mathscr{C}$, we have:

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}}\,(C, U\,(\mathrm{Lim}\,(D))) &\cong \mathrm{Hom}_{\mathscr{C}}\,(F(C), \mathrm{Lim}\,(D)) \\
&\cong \mathrm{Lim}\,(\mathrm{Hom}_{\mathscr{C}}\,(F(C), D(-))) \\
&\cong \mathrm{Lim}\,(\mathrm{Hom}_{\mathscr{C}}\,(C, U \circ D(-))) \\
&\cong \mathrm{Hom}_{\mathscr{C}}\,(C, \mathrm{Lim}\,(U \circ D))
\end{aligned}$$

The first and third equations are due to the adjunction, while the second and fourth are due to the preservation of limits by the Hom-set functor (Proposition 8.4). As a consequence of Yoneda Lemma (Corollary 2.17), we deduce:

$$U\,(\mathrm{Lim}\,(D)) \cong \mathrm{Lim}\,(U \circ D)$$

Similarly, if $D : \mathscr{I} \to \mathscr{X}$ has a colimit $\mathrm{Colim}\,(D)$, and for $X \in \mathscr{X}$:

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}}\,(F\,(\mathrm{Colim}\,(D))\,, X) &\cong \mathrm{Hom}_{\mathscr{C}}\,(\mathrm{Colim}\,(D)\,, U(X)) \\
&\cong \mathrm{Lim}\,(\mathrm{Hom}_{\mathscr{C}}\,(D(-), U(X))) \\
&\cong \mathrm{Lim}\,(\mathrm{Hom}_{\mathscr{C}}\,(F \circ D(-), X)) \\
&\cong \mathrm{Hom}_{\mathscr{C}}\,(\mathrm{Colim}\,(F \circ D)\,, X)
\end{aligned}$$

Which also gives (by Corollary 2.18):

$$F\,(\mathrm{Colim}\,(D)) \cong \mathrm{Colim}\,(F \circ D) \tag{21}$$

□

Definition 8.11 (Complete category)**.** A category $\mathscr{C}$ is said *complete* (resp. *cocomplete*) when it has all small limits (resp. all small colimits).

Example 8.12. The category **Sets** is complete and cocomplete. Of course **Sets** has all finite limits (because it has products, equalisers, a terminal object and pullbacks) and colimits (because it has coproducts, coequalisers, an initial object and pushouts), but in the rest of this course, we will just admit that **Sets** has all small limits and colimits.

Definition 8.13 (Continuous functor)**.** A functor $F : \mathscr{C} \to \mathscr{D}$ is called *continuous* (resp. *cocontinuous*) if it preserves all small limits (resp. all small colimits).

Example 8.14. The covariant Hom-set functor is continuous but not cocontinuous.

We now move towards the next important theorem: the adjoint functor theorem. There are a few details to expand before.

Definition 8.15 (Weakly initial [7])**.** Let $\mathscr{C}$ be any category.

A set $C^* = \{C_i \in \mathrm{Ob}_{\mathscr{C}} \mid i \in I\}$ of objects in $\mathscr{C}$ is a *weakly initial set* when for all $C \in \mathscr{C}$, there is an $i \in I$ such that there is an arrow $C_i \to C$.

"There exists a set of objects that connects to any other object of the category" or "there is a (non-necessarily connected) subgraph that is connected to the rest of the category", or "there is some *weakly initial* subset of objects".

Lemma 8.16 ([7], [1])**.** *Let $\mathscr{C}$ be a locally small, complete category.*
$\mathscr{C}$ has an initial object $\Leftrightarrow$ There is a weakly initial set of objects in $\mathscr{C}$

Proof. If $\mathscr{C}$ has an initial object, then any set containing that initial object is weakly initial.
Suppose that $(C_i)_{i \in I}$ is a weakly initial set in $\mathscr{C}$.
Consider the category $\mathscr{C}_I$ defined by:

Objects: The objects of $\mathscr{C}_I$ are the C_i for $i \in I$

Morphisms: If $c \in \mathrm{Mor}_{\mathscr{C}}$ is an arrow $C_i \to C_j$ for $i, j \in I$, then $c \in \mathscr{C}_I$

Identities: An identity morphism of an object C_i is an identity morphism $\mathrm{id}_{C_i} \in \mathscr{C}$

Composition: The composition law for morphisms is the usual composition in $\mathscr{C}$

It is easy to see that $\mathscr{C}_I$ is a small subcategory of $\mathscr{C}$. Then, the inclusion mapping:

$$F : \begin{cases} \mathscr{C}_I & \longrightarrow & \mathscr{C} \\ C & \longmapsto & C \\ c & \longmapsto & c \end{cases}$$

is a functor; or rather, as $\mathscr{C}_I$ is small, F is a small diagram. As $\mathscr{C}$ is complete, it has a limit $\mathrm{Lim}\,(F)$.

We now show that $\mathrm{Lim}\,(F)$ is initial. Clearly, for all $C \in \mathscr{C}$, there is an arrow $c_i : C_i \to C$. As $\mathrm{Lim}\,(F)$ is the limit of F, there is also an arrow $\eta_i : \mathrm{Lim}\,(F) \to C_i$, so for all $C \in \mathscr{C}$, there is an arrow: $\mathrm{Lim}\,(F) \to C$, but this arrow is not necessarily unique.

Let $f, g : \mathrm{Lim}\,(F) \to C$ be two arrows, and let (E, e) be an equaliser of f and g. There is an $i \in I$ such that:

$$\begin{tikzcd} & E \arrow[r, "e"] & \mathrm{Lim}\,(F) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \\ C_i \arrow[ur, "c_i"] & & & \\ & \mathrm{Lim}\,(F) \arrow[ul, "\eta_i"] \arrow[uur, "\mathrm{id}_{\mathrm{Lim}(F)}"'] & & \end{tikzcd}$$

Besides, by unicity of the arrow $\mathrm{Lim}\,(F) \to \mathrm{Lim}\,(F)$, we deduce that:

$$e \circ c_i \circ \eta_i = \mathrm{id}_{\mathrm{Lim}(F)}$$

which gives:

$$\begin{aligned} f &= f \circ e \circ c_i \circ \eta_i \\ &= g \circ e \circ c_i \circ \eta_i \\ &= g \end{aligned}$$

using the fact that $f \circ e = g \circ e$.

Consequently, there is a unique arrow $\mathrm{Lim}\,(F) \to C$ for any $C \in \mathscr{C}$; so $\mathrm{Lim}\,(F)$ is an initial object. □

Definition 8.17 (Comma-category)**.** Let $F : \mathscr{C} \to \mathscr{X}$ and $G : \mathscr{D} \to \mathscr{X}$ be two functors.

The *comma-category* $(F \mid G)$ is the category described below:

Objects: The objects of $(F \mid G)$ are triples (C, f, D) such that $f : F(C) \to G(D)$, $C \in \mathscr{C}$ and $D \in \mathscr{D}$

Morphisms: A morphism $(C, f, D) \to (C', f', D')$ is a pair (c, d) such that $c : C \to C'$, $d : D \to D'$ and the following square commutes:

$$\begin{tikzcd} F(C) \arrow[r, "F(c)"] \arrow[d, "f"'] \arrow[dr, phantom, "\checkmark"] & F(C') \arrow[d, "f'"] \\ G(D) \arrow[r, "G(d)"'] & G(D') \end{tikzcd}$$

Identities: The identity morphism of an object (C, f, D) is the pair $(\mathrm{id}_C, \mathrm{id}_D)$

Composition: The composition law for morphisms is the usual composition $(c, d) \circ (c', d') = (c \circ c', d \circ d')$

If F is the diagonal functor $\Delta(C)$ for some $C \in \mathscr{C}$, then the comma-category is written $(C \mid G)$ and simplifies to:

Objects: The objects of $(C \mid G)$ are pairs (D, f) such that $f : C \to G(D)$ and $D \in \mathscr{D}$

Morphisms: A morphism $(D, f) \to (D', f')$ is an arrow $d : D \to D'$ such that the following square commutes:

$$\begin{tikzcd} & G(D) \arrow[dd, "G(d)"] \\ F(C) \arrow[ur, "h"] \arrow[dr, "h'"'] & \\ & G(D') \end{tikzcd}$$

Identities: The identity morphism of an object (D, f) is an identity arrow id_D

Composition: The composition law for morphisms is the usual composition in $\mathscr{D}$

Now, we consider the comma-category $(C \mid U)$ where $U : \mathscr{X} \to \mathscr{C}$ and $C \in \mathscr{C}$.

Lemma 8.18. *Let $(C \mid U)$ where $U : \mathscr{X} \to \mathscr{C}$ and $C \in \mathscr{C}$.*

If $\mathscr{X}$ is locally small, then the comma-category $(C \mid U)$ is also locally small.

Proof. Just note that, for any $X, X' \in \mathscr{X}$, $\mathrm{Hom}_{\mathscr{C}}(X, X')$ is a set. Also, note that $\mathrm{Hom}_{\mathscr{C}}((X, f), (X', f')) \subset \mathrm{Hom}_{\mathscr{C}}(X, X')$. □

Lemma 8.19. *Let $(C \mid U)$ where $U : \mathscr{X} \to \mathscr{C}$ and $C \in \mathscr{C}$.*

If $\mathscr{X}$ is complete and U preserves small limits, then $(C \mid U)$ is also complete.

Proof. It is easy to check that, due to the preservation of limits by U, the comma-category $(C \mid U)$ has products and equalisers.

J'ai la flemme □

Lemma 8.20. *Let $(C \mid U)$ where $U : \mathscr{X} \to \mathscr{C}$ and $C \in \mathscr{C}$.*
The universal arrows from C to U are the initial objects of $(C \mid U)$.

Proof. Let us recall the definition of a universal arrow from C to U: it is a pair $\left(U_C^\#, \eta_C\right)$ such that:

- $U_C^\# \in \mathscr{X}$ and $\eta_C : C \to U\left(U_C^\#\right)$
- for all $X \in \mathscr{X}$, for all $c : C \to U(X)$, there is a unique $x : U_C^\# \to X$ such that $c = U(x) \circ \eta_C$

An initial object in $(C \mid U)$ is a pair (I, i) where $I \in \mathscr{X}$ and $i : C \to U(I)$ such that for all object $(X, c) \in (C \mid U)$, there is a unique arrow x such that the following triangle commutes:

$$\begin{tikzcd} & U(I) \arrow[dd, "U(x)"] \\ C \arrow[ur, "i"] \arrow[dr, "c"'] & \\ & U(X) \end{tikzcd}$$

Both definitions are equivalent, hence the result. □

Lemma 8.21. *Let $U : \mathscr{X} \to \mathscr{C}$.*
U has a left adjoint $\Leftrightarrow$ for each $C \in \mathscr{C}$, the comma-category $(C \mid U)$ has an initial object.

Proof. Combine the previous lemma (Lemma 8.20) and Definition 4.1. □

We can finally prove the following version of the Adjoint Functor Theorem, as it appears in [1]:

Theorem 8.22 (Adjoint Functor Theorem - Awodey version [1])**.** *Let $\mathscr{X}$ be locally small and complete. Let $\mathscr{C}$ be any category and let $U : \mathscr{X} \to \mathscr{C}$ be a continuous functor.*
U has a left adjoint $\Leftrightarrow$ for each object $C \in \mathscr{C}$, the comma-category $(C \mid U)$ has a weakly initial set.

Proof. By Lemma 8.21, U has a left adjoint iff for each $C \in \mathscr{C}$, the comma-category $(C \mid U)$ has an initial object. As $\mathscr{X}$ is locally small and complete, by Lemma 8.16, for each $C \in \mathscr{C}$, the comma-category $(C \mid U)$ has an initial object iff for each object $C \in \mathscr{C}$, the comma-category $(C \mid U)$ has a weakly initial set. □

The following variant is also called Adjoint Functor Theorem:

Theorem 8.23 (Adjoint Functor Theorem - Leinster version [7])**.** *Let $\mathscr{X}$ be locally small and complete. Let $\mathscr{C}$ be any category and let $U : \mathscr{X} \to \mathscr{C}$ be a functor such that for each object $X \in \mathscr{X}$, the comma-category $(X \mid U)$ has a weakly initial set.*
U has a left adjoint $\Leftrightarrow$ U preserves limits.

Proof. **[Proof of $\Rightarrow$]**
Direct consequence of Proposition 8.10.
[Proof of $\Leftarrow$]
By Lemma 8.18, $\mathscr{X}$ is locally small, so is $(X \mid U)$. By Lemma 8.19, as U preserves limits and as $\mathscr{X}$ is complete, $(X \mid U)$ is complete. We can then use Lemma 8.16: for each object $X \in \mathscr{X}$, $(X \mid U)$ has a weakly initial set, so for all $X \in \mathrm{Ob}_{\mathscr{X}}$, $(X \mid U)$ has an initial object. By Lemma 8.21, U has a left adjoint. □

Note that if the category $\mathscr{X}$ is small (instead of only locally small) then the condition on the weakly initial set is useless. We then have the following corollary:

Corollary 8.24. *Let $\mathscr{X}$ be small (not only locally small) and complete. Let $U : \mathscr{X} \to$ **Sets** be a functor. The following propositions are equivalent:*

1. *U is continuous*
2. *U has a left adjoint*
3. *U is representable*

Proof. The equivalence $1 \Leftrightarrow 2$ is obvious.

The proof of $3 \Leftrightarrow 1$ is easy: if U is representable, then $U \cong \mathrm{Hom}_{\mathscr{C}}(C_U, -)$ for some $C_U \in \mathscr{C}$; and $\mathrm{Hom}_{\mathscr{C}}(C_U, -)$ is continuous (Proposition 8.4).

Je ne vois pas pour 1 ou 2 implique 3... □

Remark 8.25. A word on forgetful functors: See Awodey, p243-245 for an exaplanation, and Mac Lane, chapter V, for a proof.

The following remark is beyond the scope of this course, so we will not go into the technical details.

Chercher comment Cori et Lascar définissent les langages en théorie des modèles

An important application of the Adjoint functor theorem is the following:

If T is a finite theory, with $T - \mathbf{Models}$ being the category of the models of T and homomorphisms between them (in the model-theoretic sense), then the forgetful functor $T - \mathbf{Models} \longrightarrow \mathbf{Sets}$ has a left adjoint.

This is powerful because it means that we can, in a sense, add some structure to a Set in order to make it a group or a ring (not exactly because axiom of choice)

If $U : T - \mathbf{Models} \longrightarrow \mathbf{Sets}$ is the forgetful functor, then there is an adjoint $F : \mathbf{Sets} \longrightarrow T - \mathbf{Models}$ such that, for all $S \in \mathbf{Sets}$, for all $M \in T - \mathbf{Models}$, we have:

$$\mathrm{Hom}_{T-\mathbf{Models}}(F(S), M) \cong \mathrm{Hom}_{\mathbf{Sets}}(S, U(M))$$

The adjoint of U is called the free functor. In short, the free functor $F : \mathbf{Sets} \longrightarrow T - \mathbf{Models}$ is the functor that maps a set S to a structure "generated" by that set S (for example, if T is the theory of vector spaces, then the free functor F will consider that a given set S is a basis, and will build a vector space using this basis).

See MacLane Chapter IV, pp87-88 for a list of adjoints, some of them being between forgetful and free functors.

According to nLab: A general way to construct free functors is with a transfinite construction of free algebras (in set-theoretic foundations), or with an inductive type or higher inductive type (in type-theoretic foundations).

Remark 8.26. Consider two categories $\mathbf{Sets}^{\mathscr{C}}$ and $\mathscr{D}$, and a functor $L : \mathbf{Sets}^{\mathscr{C}} \longrightarrow \mathscr{D}$. Suppose we want to find $R : \mathscr{D} \longrightarrow \mathbf{Sets}^{\mathscr{C}}$ the right adjoint of L (we suppose that such an adjoint exists). We will study the behaviour of R on objects and arrows.

Let $D \in \mathscr{D}$; we have $R(D) \in \mathbf{Sets}^{\mathscr{C}}$: it is a contravariant functor $R(D) : \mathscr{C} \longrightarrow \mathbf{Sets}$. By Yoneda Lemma, for some $C \in \mathscr{C}$, we have:

$$\begin{aligned} R(D)(C) &\cong \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{C}}(C, -), R(D)\right) \\ &\cong \mathrm{Nat}\left(L\left(\mathrm{Hom}_{\mathscr{C}}(C, -)\right), D\right) \end{aligned}$$

where the second equation is the definition of an adjunction. The simplest choice of $R(D)$ should be:

$$R(D) = \mathrm{Nat}\left(L\left(\mathrm{Hom}_{\mathscr{C}}(-, -)\right), D\right) \tag{22}$$

So we have the behaviour of $R(D)$ on objects. On arrows $c : C \longrightarrow C'$, we suppose that $R(D)$ is a functor $\mathscr{C} \longrightarrow \mathbf{Sets}$; so, by Yoneda lemma, the following diagram should commute:

$$\begin{tikzcd} C \arrow[d, "c"] & \rightsquigarrow & R(D)(C) \arrow[r, "\theta_{R(D),C}"] \arrow[d, "R(D)(c)"'] \arrow[dr, phantom, "\checkmark"] & \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{C}}\left(C,-\right),R(D)\right) \arrow[d, "\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(c,-),R(D))"] \\ C' & & R(D)(C') \arrow[r, bend left, "\theta_{R(D),C'}"] & \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{C}}\left(C',-\right),R(D)\right) \arrow[l, bend left, "\xi_{R(D),C'}"] \end{tikzcd}$$

where $\xi_{R(D),C}$ is the Yoneda isomorphism $\mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{C}}\left(C,-\right),R(D)\right) \to R(D)(C)$ and $\theta_{R(D),C}$ is its inverse (cf. Definitions 2.4 and 2.5).

We deduce a (brutal) formula for $R(D)(c)$:

$$R(D)(c) = \xi_{R(D),C'} \circ \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{C}}\left(c,-\right),R(D)\right) \circ \theta_{R(D),C}$$

Finally, we want the behaviour of $R(d)$. For $C \in \mathscr{C}$, we have:

$$R(d)(D) = \mathrm{Nat}\left(L\left(\mathrm{Hom}_{\mathscr{C}}\left(C,-\right)\right),d\right)$$

So we have now described the functor $R : \mathscr{D} \to \mathbf{Sets}^{\mathscr{C}}$ in terms of functors and natural transformations whose expression we know.

In the same vein of the Adjoint Functor Theorem, the following proposition is sometimes useful when we have to prove that some functor has a right adjoint.

In the special case where we only have finite limits, we have the converse to Proposition 6.23 (exponential is right adjoint to product).

Proposition 8.27. *Let $\mathscr{C}$ be a category with all finite limits, and $C \in \mathscr{C}$.*

Let P_C be the functor:

$$P_C : \left\{ \begin{array}{ccc} \mathscr{C} & \longrightarrow & \mathscr{C} \\ X & \longmapsto & X \times C \\ x : X \to X' & \longmapsto & x \times \mathrm{id}_C : X \times C \to X' \times C \end{array} \right.$$

Then, there exists a right adjoint E_C to P_C $\Leftrightarrow$ for all A, the exponential A^C exists.

Proof. The proof falls beyond the scope of this course. See [2, Chapter 13, Section 13.3, Exercise 5, p359] for an exercise that will guide you into the proof of $\Rightarrow$.

Note that Proposition 6.23 is exactly $\Leftarrow$. □

This proposition will be useful later in order to prove that some functor in a category with finite limits, has a right adjoint.

9. Monads

Monads are yet another concept of category theory; basically, it is a functor from a category to itsef, together with two natural transformations that follow some rules. Monads have links with adjunctions and reciprocally; however, they are not equivalent.

Definition 9.1 (Monad)**.** A *monad on* $\mathscr{C}$ is a triple (M, η, μ) such that:

1. $M : \mathscr{C} \to \mathscr{C}$ is a functor
2. $\eta : \mathrm{Id}_{\mathscr{C}} \to M$ is a natural transformation, called the *unit*
3. $\mu : M \circ M \to M$ is a natural transformation, called the *multiplication*

and the following diagrams commute:

$$\begin{tikzcd} M \circ M \circ M \arrow[r, "\mu M"] \arrow[d, "M\mu"'] & M \circ M \arrow[d, "\mu"] \\ M \circ M \arrow[r, "\mu"'] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\eta M"] \arrow[dr, "="'] & M \circ M \arrow[d, "\mu"] & M \arrow[l, "M\eta"'] \arrow[dl, "="] \\ & M & \end{tikzcd}$$

The square diagram is sometimes referred to as the "associativity" diagram, and the bi-triangle one is sometimes called the "identity" diagram. Both names are not random. In fact, a monad is a generalisation of the notion of monoid in the form of a functor. M is an endofunctor (a functor from $\mathscr{C}$ to itself) that sets a framework for the monoid-looking structure given by the multiplication μ and the unit η, that behave as expected from them, according to the diagrams.

If there is no ambiguity, we refer to M as monad, instead of the whole triple (M, η, μ).

Just like adjunctions, examples of monads are legion. We are not going to make a whole section just for monads. We will just give a few examples. For now, the utility and context in which these monads appear will be intentionally left unexplained, but will be revealed as we progress through the section.

Example 9.2. The functor $M : \mathbf{Sets} \to \mathbf{Sets}$ defined by:

$$M : \left\{ \begin{array}{ccc} \mathbf{Sets} & \longrightarrow & \mathbf{Sets} \\ X & \longmapsto & X + 1 \end{array} \right.$$

where 1 is the terminal object and + denotes the coproduct, is a monad. Its unit $\eta = (\eta_X)_{X \in \mathbf{Sets}}$ and multiplication $\mu = (\mu_X)_{X \in \mathbf{Sets}}$ are:

$$\eta_X : \left\{ \begin{array}{ccc} X & \longrightarrow & X + 1 \\ x & \longmapsto & x \end{array} \right. \qquad \mu_X : \left\{ \begin{array}{ccc} X + 1 + 1 & \longrightarrow & X + 1 \\ x & \longmapsto & x \\ 0 & \longmapsto & 0 \end{array} \right.$$

where 0 is the unique element of 1 (it appears twice in $X + 1 + 1$ and only once in $X + 1$).

This example may be extended by replacing 1 with any given set $S \in \mathbf{Sets}$.

Example 9.3. If X is a set, denote by $X^{<\omega}$ the set of finite sequences over X, that is, $X^{<\omega} = \bigcup\limits_{n \in \omega} X^n$. The following functor is a monad:

$$M : \left\{ \begin{array}{ccc} \mathbf{Sets} & \longrightarrow & \mathbf{Sets} \\ X & \longmapsto & X^{<\omega} \end{array} \right.$$

Its unit $\eta = (\eta_X)_{X \in \mathbf{Sets}}$ is:

$$\eta_X : \begin{cases} X & \longrightarrow & X^{<\omega} \\ x & \longmapsto & (x) \end{cases}$$

and the components of its multiplication $\mu = \left(\mu_X : (X^{<\omega})^{<\omega} \to X^{<\omega}\right)_{X \in \mathbf{Sets}}$ send a nested tuple to its concatenation (we may also call this operation "flatten"):

$$\mu_X : \left(\left(x_0^0, \ldots, x_{n_0}^0\right), \ldots, \left(x_0^k, \ldots, x_{n_k}^k\right)\right) \mapsto \left(x_0^0, \ldots, x_{n_0}^0, \ldots, x_0^k, \ldots, x_{n_k}^k\right)$$

Example 9.4. Let $(E, e, *)$ be a monoid.

The functor $M : X \to X \times E$, equipped with the natural transformations with components:

$$\eta_X : \begin{cases} X & \longrightarrow & X \times E \\ x & \longmapsto & (x, e) \end{cases} \qquad \mu_X : \begin{cases} X \times E \times E & \longrightarrow & X \times E \\ (x, e_1, e_2) & \longmapsto & (x, e_1 * e_2) \end{cases}$$

is also a monad.

Example 9.5. Fix a set S. As seen in Proposition 6.23, the exponential $(-)^S$ is right adjoint to the product $- \times S$, and the evaluation $\varepsilon_X : X^S \times S \to X$ is in fact a component of the counit of that adjunction. As mentioned in Remark 6.26, the unit η is made of the arrows $\eta_X : X \to (X \times S)^S$ that are the curryfications of $\mathrm{id}_{X \times S}$. In short, $\eta_X : x \to (s \mapsto (x, s))$.

For a given set $S \in \mathbf{Sets}$, define the following natural transformation μ with components:

$$\mu_X : \begin{cases} \left((X \times S)^S \times S\right)^S & \longrightarrow & (X \times S)^S \\ f = (f_1, f_2) & \longmapsto & f_1(s)\,(f_2(s)) \end{cases}$$

where f is a function $f : s \mapsto (f_1(s), f_2(s))$, with, for all $s \in S$, $f_1(s) \in (X \times S)^S$ and $f_1(s) \in S$.

Then, the functor $X \mapsto (X \times S)^S$, equipped with the unit η of the product-exponential adjunction and the multiplication μ defined above, is a monad.

This last example is in fact an application of the following proposition:

Proposition 9.6. *Let $F \dashv U$ be an adjunction with unit $\eta : \mathrm{Id}_{\mathscr{C}} \to U \circ F$ and counit $\varepsilon : F \circ U \to \mathrm{Id}_{\mathscr{X}}$.*

Then the tuple $(U \circ F, \eta, U \circ \varepsilon \circ F)$ is a monad.

Proof. The identity diagram derives from the triangle identities of adjoints (cf. Definition 4.24). Composing the left-adjoint triangle with U on the left and composing the right-adjoint triangle with F on the right, we obtain:

$$\begin{tikzcd} U \circ F \arrow[r, "U \circ F \circ \eta"] \arrow[dr, "\mathrm{id}_{U \circ F}"'] & U \circ F \circ U \circ F \arrow[d, "U \circ \varepsilon \circ F"] \\ & U \circ F \end{tikzcd} \qquad \begin{tikzcd} U \circ F \arrow[r, "\eta \circ U \circ F"] \arrow[dr, "\mathrm{id}_{U \circ F}"'] & U \circ F \circ U \circ F \arrow[d, "U \circ \varepsilon \circ F"] \\ & U \circ F \end{tikzcd}$$

Gluing the two triangles on the common arrow $U \circ \varepsilon \circ F$ yields the identity diagram of a monad.

Then, consider an arrow $\varepsilon_X : F \circ U(X) \to X$; as ε is a natural transformation $\varepsilon : F \circ U \to \mathrm{Id}_{\mathscr{X}}$, the following square commutes:

$$\begin{tikzcd} F \circ U \circ F \circ U\,(X) \arrow[r, "\varepsilon_{F \circ U(X)}"] \arrow[d, "F \circ U(\varepsilon_X)"'] \arrow[dr, phantom, "\checkmark"] & F \circ U\,(X) \arrow[d, "\varepsilon_X"] \\ F \circ U\,(X) \arrow[r, "\varepsilon_X"'] & X \end{tikzcd}$$

which yields:

$$\begin{tikzcd}
F \circ U \circ F \circ U \arrow[r, "\varepsilon \circ F \circ U"] \arrow[d, "F \circ U \circ \varepsilon"'] \arrow[dr, phantom, "\checkmark"] & F \circ U \arrow[d, "\varepsilon"] \\
F \circ U \arrow[r, "\varepsilon"'] & \mathrm{Id}_{\mathscr{X}}
\end{tikzcd}$$

and by composing the previous diagram with U on the left, and F on the right, we obtain the associativity diagram of the monad. □

Of course, monads generate categories. We will see two of these categories:

Definition 9.7 (Eilenberg-Moore category)**.** Let (M, η, μ) be a monad on $\mathscr{C}$.
The Eilenberg-Moore category associated with (M, η, μ), denoted $\mathscr{C}^M$, is the following category:

Objects: An object is a pair (C, f) where $C \in \mathscr{C}$ and $f : M(C) \to C \in \mathscr{C}$ such that the following diagrams commute:

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[dr, "\mathrm{id}_C"'] & M(C) \arrow[d, "f"] \\
& C
\end{tikzcd}
\qquad
\begin{tikzcd}
M \circ M(C) \arrow[r, "\mu_C"] \arrow[d, "M(f)"'] & M(C) \arrow[d, "f"] \\
M(C) \arrow[r, "f"'] & C
\end{tikzcd}$$

Morphisms: An arrow $c : (C, f) \to (C', f')$ is an arrow $c : C \to C' \in \mathscr{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
M(C) \arrow[r, "M(c)"] \arrow[d, "f"'] & M(C') \arrow[d, "f'"] \\
C \arrow[r, "c"'] & C'
\end{tikzcd}$$

Identities: The identity of (C, f) is an identity id_C

Composition: The composition of arrows is the composition in $\mathscr{C}$

An object of the Eilenberg-Moore category $\mathscr{C}^M$ is often called an *algebra over M* or *M-algebra*. As stated before, monads generalise the idea of a monoid. Algebras over a monad generalise the notion of module over a ring.

Example 9.8. Recall from Example 9.3 the monad $M : X \mapsto X^{<\omega}$, with multiplication μ_X = "concatenation of tuples" and unit $\eta_X : x \mapsto (x)$. Let us study its Eilenberg-Moore category $\mathbf{Sets}^M$.

Every monoid $(E, *, e)$ induces a natural function $h : E^{<\omega} \to E$ that sends a tuple of elements $(e_1, \dots, e_n)$ to their multiplication $e_1 * \cdots * e_n$ (and the empty tuple () to the unit of the monoid e). It is easy to see that the pair (E, f) consisting of the underlying set of the monoid, with this natural function, is actually an object of $\mathbf{Sets}^M$ (making the right diagrams commute). It is also easy to see that monoid morphisms translate to an arrow in $\mathbf{Sets}^M$.

Now, each object $(X, f : X^{<\omega} \to X)$ of $\mathbf{Sets}^M$ can also be seen as a monoid. In fact, define $*$ as the restriction of f to X^2, and take $e = f(())$ (the image by f of the empty tuple); then, it is easy to see that $(X, *, e)$ follow the axioms of monoids. Then, a morphism in $\mathbf{Sets}^M$ is easily seen as a monoid morphism.

What this example claims is the following: $\mathbf{Sets}^M \cong \mathbf{Monoids}$, the isomorphism being the one we described above.

Definition 9.9 (Free and forgetful functor associated with a monad)**.** Let (M, η, μ) be a monad.

The *free functor associated with* M, denoted by Free^M, is the following functor:

$$\mathrm{Free}^M : \begin{cases} \mathscr{C} & \longrightarrow & \mathscr{C}^M \\ C & \longmapsto & (M(C), \mu_C) \\ c : C \to C' & \longmapsto & M(c) \end{cases}$$

The *forgetful functor associated with* M, denoted by Forget^M, is the following functor:

$$\mathrm{Forget}^M : \begin{cases} \mathscr{C}^M & \longrightarrow & \mathscr{C} \\ (C, f) & \longmapsto & C \\ u : (C, f) \to (C', f') & \longmapsto & u \end{cases}$$

In a sense, the free functor "creates", or "enforces", the structure of the monad, while the forgetful functor "forgets", or "nullifies", the structure of the monad.

The reader having read Sections 4 or 5 knows that examples of adjunctions include free/forgetful pairs. This is the reason why:

Proposition 9.10. *Let* (M, η, μ) *be a monad.*
Then, $\mathrm{Free}^M \dashv \mathrm{Forget}^M$.

Proof. We must compute the adjunctor:

$$\beta_{C,(C',f')} : \mathrm{Hom}_{\mathscr{C}}\left(\mathrm{Free}^M(C), (C', f')\right) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}\left(C, \mathrm{Forget}^M(C', f')\right)$$

However, first note that $\mathrm{Forget}^M \circ \mathrm{Free}^M = M$, so the unit of that adjunction will be the unit of the monad. By Lemma 4.10, the adjunctor is written $\beta_{C,(C',f')} : u \mapsto \mathrm{Forget}^M(u) \circ \eta_C$. We have obviously defined a natural transformation; we have to check that this defines a bijection.

Let us give the inverse of β. Let $v \in \mathrm{Hom}_{\mathscr{C}}\left(C, \mathrm{Forget}^M(C', f')\right)$; as $\mathrm{Forget}^M(C', f') = C'$, and $(C', f') \in \mathscr{C}^M$, the following diagram commutes:

$$\begin{tikzcd} M \circ M(C) \arrow[r, "M \circ M(v)"] \arrow[d, "\mu_C"'] \arrow[dr, phantom, "\checkmark"] & M \circ M(C') \arrow[r, "M(f')"] \arrow[d, "\mu_{C'}"'] \arrow[dr, phantom, "\checkmark"] & M(C') \arrow[d, "f'"] \\ M(C) \arrow[r, "M(v)"'] & M(C') \arrow[r, "f'"'] & C' \end{tikzcd} \tag{23}$$

This diagram shows that $f' \circ M(v)$ is an arrow $F(C) \to (C', f')$. Therefore, it suggests that for a given $v : C \to \mathrm{Forget}^M(C', f')$, the corresponding arrow in $\mathscr{C}^M$ should be $f' \circ M(v)$, which should be the inverse of $\beta_{C,(C',f')}$. Define $\gamma_{C,(C',f')} : v \mapsto f' \circ M(v)$.

Then let $u : F(C) \to (C', f')$; we have the following diagram:

$$\begin{tikzcd}
M(C) \arrow[r, "M(\eta_C)"] \arrow[dr, "\mathrm{id}_{M(C)}"'] & M \circ M(C) \arrow[r, "M(u)"] \arrow[d, "\mu_C"] \arrow[dr, phantom, "\checkmark"] & M(C') \arrow[d, "f'"] \\
 & M(C) \arrow[r, "u"'] & C'
\end{tikzcd}$$

The square commutes because u is an arrow $(M(C), \mu_C) \to (C', f')$ and the triangle commutes due to half of the identity diagram of a monad. We thus have:

$$\begin{aligned}
\gamma_{C,(C',f')} \circ \beta_{C,(C',f')}(u) &= \gamma_{C,(C',f')} \left(\mathrm{Forget}^M (u) \circ \eta_C\right) \\
&= f' \circ M \left(\mathrm{Forget}^M (u) \circ \eta_C\right) \\
&= f' \circ M \circ \mathrm{Forget}^M (u) \circ M(\eta_C) \\
&= f' \circ M(u) \circ M(\eta_C) \\
&= u
\end{aligned}$$

As for the converse, note that the image of Diagram 23 by Forget^M is:

$$M(C) \xrightarrow{M(v)} M(C') \xrightarrow{f'} C' \tag{24}$$

In the following diagram, the square commutes because $\eta : \mathrm{Id}_{\mathscr{C}} \to M$ is a natural transformation, and the triangle commutes because (C', f') is an object of an Eilenberg-Moore category:

$$\begin{tikzcd}
C \arrow[r, "v"] \arrow[dd, "\eta_C"'] & C' \arrow[dd, "\eta_{C'}"'] \arrow[dr, "\mathrm{id}_{C'}"] & \\
 & & C' \\
M(C) \arrow[r, "M(v)"'] & M(C') \arrow[ur, "f'"'] &
\end{tikzcd} \tag{25}$$

Using this diagram, we have:

$$\begin{aligned}
\beta_{C,(C',f')} \circ \gamma_{C,(C',f')}(v) &= \beta_{C,(C',f')} \left(f' \circ M(v)\right) \\
&= \mathrm{Forget}^M \left(f' \circ M(v)\right) \circ \eta_C \\
&= f' \circ M(v) \circ \eta_C && \text{(cf. Diagram 24)} \\
&= v && \text{(cf. Diagram 25)}
\end{aligned}$$

Thence, β is a natural isomorphism, and an adjunctor, and $\mathrm{Free}^M \dashv \mathrm{Forget}^M$. □

Definition 9.11 (Eilenberg-Moore adjunction)**.** Let (M, η, μ) be a monad.

The adjunction $\mathrm{Free}^M \dashv \mathrm{Forget}^M$ is called the *Eilenberg-Moore adjunction of M*. In order to set the notations, we denote by η^M and ε^M the unit and counit of that adjunction.

Remark 9.12. We can also deduce from the proof of Proposition 9.10 that the unit of the adjunction is $\eta^M = \eta$, that is, the unit of the monad, and the counit is defined by:

$$\varepsilon^M_{(C',f')} = f' \tag{26}$$

So, every time we have a monad, we have an adjunction, and every time we have an adjunction, we have a monad! However, there is no bijection between monads and adjunctions, as many adjunctions can give the same monad, and a monad gives rise to two adjunctions.

Definition 9.13 (Category of adjunctions)**.** Let (M, η, μ) be a monad on $\mathscr{C}$.

The category of adjunctions associated to M, denoted by $\mathrm{Adj}\,(M)$, is the following category:

Objects: Objects are the adjunctions $(F : \mathscr{C} \to \mathscr{X}, U, \eta, \varepsilon)$ such that

$$(M, \eta, \mu) = (U \circ F, \eta, U\varepsilon F)$$

Morphisms: An arrow $(F : \mathscr{C} \to \mathscr{X}, U, \eta, \varepsilon) \to (F' : \mathscr{C} \to \mathscr{X}', U', \eta', \varepsilon')$ is a functor $Z : \mathscr{X} \to \mathscr{X}'$ such that the following diagrams commute:

$$\begin{tikzcd} & \mathscr{X} \arrow[dd, "Z"] \\ \mathscr{C} \arrow[ur, "F"] \arrow[dr, "F'"'] & \\ & \mathscr{X}' \end{tikzcd} \qquad \begin{tikzcd} \mathscr{X} \arrow[dd, "Z"'] \arrow[dr, "U"] & \\ & \mathscr{C} \\ \mathscr{X}' \arrow[ur, "U'"'] & \end{tikzcd}$$

Identities: Identities are the identity functors

Composition: The composition is the usual composition of functors

Remark 9.14. In [5, section 7, chapter VI, page 99], there are two more conditions; in fact, that $Z\varepsilon = \varepsilon' Z$, or equivalently, that $\eta = \eta'$, which is necessarily the case here, because we are considering a category of adjunctions that generate the same given monad. In Definition 9.13, we are considering the special case where K from the book is Z here, and L from the book is $\mathrm{Id}_{\mathscr{C}}$ here.

Proposition 9.15. *Let $Z : (F, U, \eta, \varepsilon) \to (F', U', \eta', \varepsilon')$ be an arrow in $\mathrm{Adj}\,(M)$ for a given (M, η, μ). Denote by $\beta : Hom_{\mathscr{X}}(F(-), -) \to Hom_{\mathscr{C}}(-, U(-))$ and $\beta' : Hom_{\mathscr{X}'}(F'(-), -) \to Hom_{\mathscr{C}}(-, U'(-))$ the corresponding adjunctors.*

Then:

1. *For all $C \in \mathscr{C}$ and $X \in \mathscr{X}$, the following diagram commutes:*

$$\begin{tikzcd} Hom_{\mathscr{X}}(F(C), X) \arrow[r, "\beta_{C,X}"] \arrow[d, "Z"'] & Hom_{\mathscr{C}}(C, U(X)) \arrow[dd, "="] \\ Hom_{\mathscr{X}'}(Z \circ F(C), Z(X)) \arrow[d, "="'] & \\ Hom_{\mathscr{X}'}(F'(C), Z(X)) \arrow[r, "\beta'_{C,Z(X)}"'] & Hom_{\mathscr{C}}(C, U' \circ Z(X)) \end{tikzcd}$$

2. *For all $C \in \mathscr{C}$ and $X \in \mathscr{X}$, the following diagram commutes:*

$$\begin{tikzcd}
Hom_{\mathscr{C}}\left(C, U\left(X\right)\right) \arrow[r, "\beta^{-1}_{C,X}"] \arrow[dd, "="'] & Hom_{\mathscr{X}}\left(F\left(C\right), X\right) \arrow[d, "Z"] \\
& Hom_{\mathscr{X}'}\left(Z \circ F\left(C\right), Z\left(X\right)\right) \arrow[d, "="] \\
Hom_{\mathscr{C}}\left(C, U' \circ Z\left(X\right)\right) \arrow[r, "\beta'^{-1}_{C,Z(X)}"'] & Hom_{\mathscr{X}'}\left(F'\left(C\right), Z\left(X\right)\right)
\end{tikzcd}$$

3. $\eta = \eta'$
4. $Z\varepsilon = \varepsilon' Z$

Proof. **[Proof of Item 3]** Item 3 is obvious, as we are in a category of the form $\mathrm{Adj}\left(M\right)$, so:

$$\begin{aligned}
(M, \eta, \mu) &= (U \circ F, \eta, U\varepsilon F) \\
&= (U' \circ F', \eta', U'\varepsilon' F')
\end{aligned}$$

from which we deduce $\eta = \eta'$.

[Proof of Item 1] Let $x : F(C) \to X$ be an arrow. Then, $Z(x)$ is an arrow $Z \circ F(C) \to Z(X)$, and using Lemma 4.10, we have:

$$\beta'_{C,Z(X)}\left(Z(x)\right) = U'\left(Z(x)\right) \circ \eta'_C$$

As $\eta_C = \eta'_C$ (Item 3) and $U = U' \circ Z$, we have:

$$\beta_{C,X}\left(x\right) = U(x) \circ \eta_C = U' \circ Z(x) \circ \eta'_C = \beta'_{C,Z(X)}\left(Z(x)\right)$$

hence the result.

[Proof of Item 2] It is equivalent to Item 1.

[Proof of Item 4] In the diagram of Item 2, set $C = U(X)$, and choose $\mathrm{id}_{U(X)}$. We then have:

$$\begin{aligned}
\beta'^{-1}_{U(X),Z(X)} \circ Z\left(\mathrm{id}_{U(X)}\right) &= Z \circ \beta^{-1}_{U(X),X}\left(\mathrm{id}_{U(X)}\right) \\
\beta'^{-1}_{U(X),Z(X)}\left(\mathrm{id}_{Z \circ U(X)}\right) &= Z\left(\varepsilon_X \circ F\left(\mathrm{id}_{U(X)}\right)\right) \\
\varepsilon'_{Z(X)} \circ F\left(\mathrm{id}_{Z \circ U(X)}\right) &= Z\left(\varepsilon_X\right) \circ \left(Z \circ F\right)\left(\mathrm{id}_{U(X)}\right) \\
\varepsilon'_{Z(X)} &= Z\left(\varepsilon_X\right)
\end{aligned}$$

hence the result. □

The Eilenberg-Moore adjunction (Definition 9.11) is a specific object in that category:

Proposition 9.16. *Let M be a monad and* $\mathrm{Free}^M \dashv \mathrm{Forget}^M$ *its Eilenberg-Moore adjunction. Then* $\left(\mathrm{Free}^M, \mathrm{Forget}^M, \eta^M, \varepsilon^M\right)$ *is the terminal object of* $\mathrm{Adj}\left(M\right)$.

Proof. Let $(F, U, \eta, \varepsilon)$ be an adjunction $\mathscr{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathscr{X}$. We are looking for a unique arrow $Z : \mathscr{X} \to \mathscr{C}^M$ such that $\mathrm{Forget}^M \circ Z = U$. This gives the hint that $Z(X)$ should be of the form $(U(X), z)$ and for $x : X \to X'$, $Z(x)$ should be $U(x)$. As z should be an arrow $z : UFU(X) \to U(X)$ that verifies the diagrams of Definition 9.7, and in particular, the triangle identity, a good candidate for z seems to be $U(\varepsilon_X)$.

Define:

$$Z : \begin{cases} \mathscr{X} & \longrightarrow & \mathscr{C}^M \\ X & \longmapsto & (U(X), U(\varepsilon_X)) \\ x & \longmapsto & U(x) \end{cases}$$

We let the reader check that Z actually is a functor, that $\mathrm{Forget}^M \circ Z = U$ and that $Z \circ F = \mathrm{Free}^M$. We have the existence; we now have to check the unicity.

Let $Y : \mathscr{X} \to \mathscr{C}^M$ be another functor such that $\mathrm{Forget}^M \circ Y = U$ and $Y \circ \mathrm{Free}^M = U$. For $X \in \mathscr{X}$, we have $\mathrm{Forget}^M \circ Y(X) = U(X)$. As seen above, for $X \in \mathscr{X}$, $Y(X)$ will be of the form $(U(X), y)$ where $y : UFU(X) \to U(X)$, and for $x : X \to X'$, $Y(x)$ needs to be $U(x)$. The only potential difference between Z and Y lies in the comparison of the arrow in the pair $(U(X), y)$, in that y could be different to $U(\varepsilon_X)$.

Now, we have:

$$\begin{aligned} Y(\varepsilon_X) &= Z(\varepsilon_X) \\ \varepsilon'_{Y(X)} &= \varepsilon'_{Z(X)} && \text{(cf. Proposition 9.15, Item 4)} \\ \varepsilon'_{(U(X),h)} &= \varepsilon'_{(U(X),U(\varepsilon_X))} \\ h &= U(\varepsilon_X) && \text{(cf. Remark 9.12)} \end{aligned}$$

hence the unicity of Z. □

Definition 9.17 (Monadic adjunction)**.** Let $F \dashv U$ be an adjunction, with associated monad $M = U \circ F$.

The adjunction $F \dashv U$ is said monadic

Remark 9.18. Also note that all free/forgetful adjunctions are not necessarily monadic. First, note that if $M = \mathrm{Id}_{\mathscr{C}}$, then $\mathscr{C}^M \cong \mathscr{C}$. Then,

TODO:

1. Explications sur les noms des monades (cf la forme des EM-categories)
2. Kleisli category
3. Kleisli adjunction
4. Exampels

10. Sets-like categories

Besides adjoints, elementary topoi (plural of "topos" in Greek) are the second big part of this course. Before exploring this notion, we have to introduce some amount of notions around the following theme: introduction of set-like elements in categories.

We start with the categorical equivalent of a subset.

In **Sets**, when $X \subset Y$, if $x : X \to Y$ is the inclusion, then (X, x) is an equaliser, and x is a monomorphism.

In several categories based on sets (for example, the category of groups, the category of graphs, the category of rings; of "sets with structure"), when we have $X \subset Y$, if $x : X \to Y$ is a monic inclusion (that is, "an inclusion that respects the structure"), then $x(X)$ is a sub-"set with structure" (for example, a subgroup, a subgraph, a subring...) of Y.

Definition 10.1 (Category of subobjects)**.** Let $\mathscr{C}$ be a category and let C be an object of $\mathscr{C}$.

The *category of subobjects of* C, denoted by $\mathbf{SubObj}_{\mathscr{C}}(C)$ is the following category:

Objects: A subobject of C is a monomorphism $m : M \to C$

Morphisms: A morphism between subobjects $m : M \to C$ and $m' : M' \to C$ is an arrow $f : M \to M'$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[d, "f"'] \arrow[dr, "m"] & \\ M' \arrow[r, "m'"'] & C \end{tikzcd}$$

Identities: The identity morphism $m : M \to C$ is the identity morphism $\mathrm{id}_M : M \to M$

Composition: The composition law for morphisms is the usual composition of morphisms in the category $\mathscr{C}$

Example 10.2. Consider $\mathbb{R}$ the set of real numbers; let's study $\mathbf{SubObj}_{\mathbf{Sets}}(\mathbb{R})$.

The subobjects of $\mathbb{R}$ are any injections $x : X \to \mathbb{R}$. Consequently, the subobjects of $\mathbb{R}$ are not only the (inclusions of) subsets of $\mathbb{R}$ but also any injection from X to $\mathbb{R}$ where $\mathrm{card}(X) \leqslant \mathrm{card}(\mathbb{R})$.

Note that if $\mathrm{card}(X) \leqslant \mathrm{card}(\mathbb{R})$, then there are $\mathrm{card}\left(\mathbb{R}^X\right)$ injections from $X \to \mathbb{R}$, and each injection is a different subobject. As the collection of all sets with a certain cardinality is large (not a set) we deduce that $\mathbf{SubObj}_{\mathbf{Sets}}(\mathbb{R})$ is a large category (but locally small, according to the next proposition).

Proposition 10.3. *Let $\mathscr{C}$ be a category and let C be an object of $\mathscr{C}$. The category* $\mathbf{SubObj}_{\mathscr{C}}(C)$ *is a preorder.*

Proof. As any two subobjects $m : M \to C$ and $m' : M' \to C$ are monic, there is at most one arrow f such that $m' \circ f = m$. A category where there is at most one arrow is a preorder. □

Definition 10.4 (Inclusion and equivalence of subobjects)**.** Let $\mathbf{SubObj}_{\mathscr{C}}(C)$ be a category of subobjects.

For $m, m' \in \mathbf{SubObj}_{\mathscr{C}}(C)$, we define the inclusion relation as: $m \subset m' \Leftrightarrow$ there exists some arrow $f : m \to m$.

If $m \subset m'$ and $m' \subset m$, then we say that *m and m' are equivalent* and we write $m \equiv m'$.

Proposition 10.5. *For $m, m' \in \mathbf{SubObj}_{\mathscr{C}}(C)$, $m \equiv m' \Leftrightarrow m \cong m'$.*

In other words: two subobjects of C are equivalent iff they are isomorphic.

Proof. Suppose $m \equiv m'$; then there are two arrows $f : M \to M'$ and $f' : M' \to M$ such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[d, "f"'] \arrow[dr, "m"] & \\
M' \arrow[r, "m'"'] \arrow[d, "f'"'] & C \\
M \arrow[ur, "m"'] &
\end{tikzcd}$$

There are two arrows $f' \circ f, \mathrm{id}_M : M \to M$; as $\mathbf{SubObj}_{\mathscr{C}}(C)$ is a preorder, there is at most one arrow between two subobjects, so $f \circ f' = \mathrm{id}_M$. Similarly, $f' \circ f = \mathrm{id}_{M'}$, so $f' = f^{-1}$ and $m \cong m'$.

If $m \cong m'$, then let $f : m \to m'$ be an isomorphism; we deduce that $m \subset m'$. Also, f^{-1} is an isomorphism, so $m' \subset m$, and $m \equiv m'$. ☐

Corollary 10.6. *Equivalent subobjects have isomorphic domains.*

Remark 10.7. If (E, e) is the equaliser of $f, g : A \to B$, then E is a subobject of A (cf. Proposition 6.40).

Remark 10.8. In $\mathbf{SubObj}_{\mathbf{Sets}}(\mathbb{R})$, take $x : X \to \mathbb{R}$ and $x' : X' \to \mathbb{R}$. If $x \equiv x'$, then there is a bijection between both; equivalently, $\mathrm{card}(X) = \mathrm{card}(X')$. Consequently, the equivalence classes of the subobjects of $\mathbb{R}$ are the cardinals $\kappa \leqslant \mathrm{card}(\mathbb{R})$. If you consider a category **Sets** with the Continuum Hypothesis, then $\kappa \in \mathbb{N} \cup \left\{\aleph_0, 2^{\aleph_0}\right\}$.

If **Sets** respects the axiom of choice, then this is true for any set: the equivalence classes of the subobjects of a set X are the cardinals $\kappa \leqslant \mathrm{card}(X)$.

Proposition 10.9. *Let $\mathscr{C}$ be a category. Let $c : C \to C'$ be an arrow, and let $m' : M' \to C'$ be a subobject of C'.*

Suppose that the following diagram is a pullback:

$$\begin{tikzcd}
M \arrow[r, "k"] \arrow[d, "m"'] \arrow[dr, phantom, "\checkmark"] & M' \arrow[d, "m'"] \\
C \arrow[r, "c"'] & C'
\end{tikzcd}$$

Then $m : M \to C$ is also a subobject of C.

In other words: the pullback of a subobject is a subobject (or more generally: the pullback of a monomorphism is a monomorphism).

Proof. Suppose there are two arrows $z, z' : Z \to M$ such that $m \circ z = m \circ z'$.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "z"] \arrow[dr, shift right, "z'"'] \arrow[drr, "k \circ z"] \arrow[ddr, "m \circ z"'] & & \\
& M \arrow[r, "k"] \arrow[d, "m"'] \arrow[dr, phantom, "\checkmark"] & M' \arrow[d, "m'"] \\
& C \arrow[r, "c"'] & C'
\end{tikzcd}$$

As (M, m, k) is a pullback, we have $c \circ m = m' \circ k$, which yields:

$$\begin{aligned} c \circ m \circ z &= m' \circ k \circ z \\ &= m' \circ k \circ z' \\ &= c \circ m \circ z' \end{aligned}$$

By the universality of the pullback, the arrow $z : Z \to M$ is unique, so $z = z'$. Thus, m is monic. □

Proposition 10.10. *Let $\mathscr{C}$ be a category. Let $c : C \to C'$ be an arrow, let $m' : M' \to C'$ and $m'_0 : M'_0 \to C'$ be subobjects of C'.*

Suppose that the two following squares are pullbacks:

$$\begin{tikzcd} M \arrow[r, "k"] \arrow[d, "m"'] \arrow[dr, phantom, "\checkmark"] & M' \arrow[d, "m'"] \\ C \arrow[r, "c"'] & C' \end{tikzcd} \qquad \begin{tikzcd} M_0 \arrow[r, "k_0"] \arrow[d, "m_0"'] \arrow[dr, phantom, "\checkmark"] & M'_0 \arrow[d, "m'_0"] \\ C \arrow[r, "c"'] & C' \end{tikzcd} \tag{27}$$

If $m' \equiv m'_0$, then $m \equiv m_0$.

Proof. Let f'_0 be the isomorphism $f'_0 : m'_0 \to m'$. Consider the following diagram:

$$\begin{tikzcd} M_0 \arrow[r, "k_0"] \arrow[drr, "{f_0}' \circ k_0"'] \arrow[ddr, "m_0"'] & M'_0 \arrow[dr, "{f_0}'"] & \\ & M \arrow[r, "k"] \arrow[d, "m"'] \arrow[dr, phantom, "\checkmark"] & M' \arrow[d, "m'"] \\ & C \arrow[r, "c"'] & C' \end{tikzcd}$$

As $f'_0 : m'_0 \to m'$ is an isomorphism between subobjects, we have $m' \circ f'_0 = m'_0$. We then deduce from the diagrams 27 that:

$$c \circ m_0 = m'_0 \circ k_0 = m' \circ f'_0 \circ k_0$$

As M is the pullback of c and m', there is a unique arrow $u : M_0 \to M$ such that $k \circ u = f'_0 \circ k_0$ and $m \circ u = m_0$. With the same reasonning, we have a unique $u' : M \to M_0$ such that $k_0 \circ u = {f'}_0^{-1} \circ k$ and $m_0 \circ u = m'$. We then have to prove that $u' = u^{-1}$; this is because there is a unique arrow $\mathrm{id}_M : M \to M$ such that the diagram commutes; from which we deduce $u \circ u' = \mathrm{id}_M$ and $u' \circ u = \mathrm{id}_{M'}$. □

As stated in Remark 10.8, in **Sets**, subobjects can be grouped into equivalence classes, the representative of a given equivalence class being the cardinal of the subobjects. The collection of cardinals lower than a certain other cardinal is a set, while the collection of all subobjects generally is not. This is then easier, and more practical, not to refer to the collection of all subobjects $\mathbf{SubObj}_X$ (**Sets**), but rather, to the set of the equivalence classes of the subobjects:

Definition 10.11 (Set of subobjects)**.** Let $\mathscr{C}$ be a category.

For $C \in \mathrm{Ob}_{\mathscr{C}}$, we define $\mathrm{SubObj}_{\mathscr{C}}(C)$ to be the set of all equivalence classes[3]of subobjects of C; more explicitly:

$$\mathrm{SubObj}_{\mathscr{C}}(C) = \mathrm{Ob}_{\mathbf{SubObj}_C(\mathscr{C})}/\equiv$$

where $\equiv$ is the equivalence of subobjects (Definition 10.4).

In the rest of this course, we will refer to equivalence classes of subobjects, instead of bare subobjects. So, a set of subobjects is to be understood as the set of equivalence classes of subobjects.

Remark 10.12. As noticed in Proposition 10.10, the pullbacks of equivalent subobjects are equivalent. So, in a category with pullbacks, given an arrow $c : C \to C'$, for any subobject $m' : M' \to C'$ of C', there is a subobject $m : M \to C$ of C such that the following square is a pullback:

$$\begin{tikzcd} M \arrow[r, "k"] \arrow[d, "m"'] \arrow[dr, phantom, "\checkmark"] & M' \arrow[d, "m'"] \\ C \arrow[r, "c"'] & C' \end{tikzcd}$$

For an arrow $c : C \to C'$, there is some function that sends any subobject of C' to one subobject of C in a way that gives the above pullback. This function is denoted as $\mathrm{SubObj}_{\mathscr{C}}(c) : \mathrm{SubObj}_{\mathscr{C}}(C') \to \mathrm{SubObj}_{\mathscr{C}}(C)$ (beware of the inversion!).

Definition 10.13 (Subobject functor)**.** Let $\mathscr{C}$ be a category with pullbacks.

The *subobject functor* $\mathrm{SubObj}_{\mathscr{C}}$ (simply written SubObj when there is no doubt about the category) is the contravariant functor:

$$\mathrm{SubObj}_{\mathscr{C}} : \begin{cases} \mathscr{C} & \longrightarrow & \mathbf{Sets} \\ C & \longmapsto & \mathrm{SubObj}_{\mathscr{C}}(C) \\ c : C \to C' & \longmapsto & \mathrm{SubObj}_{\mathscr{C}}(c) \end{cases}$$

where $\mathrm{SubObj}_{\mathscr{C}}(C)$ is the set of subobjects (Definition 10.11) and $\mathrm{SubObj}_{\mathscr{C}}(c) : \mathrm{SubObj}_{\mathscr{C}}(C') \to \mathrm{SubObj}_{\mathscr{C}}(C)$ is the function introduced in Remark 10.12.

We now generalise the notion of characteristic function with the following definition.

Definition 10.14 (Subobject classifier)**.** Let $\mathscr{C}$ be a category with all finite limits.

A *subobject classifier in $\mathscr{C}$* is a pair (Ω, t) where $\Omega \in \mathrm{Ob}_{\mathscr{C}}$ and $t : 1 \to \Omega$ such that, for all $C \in \mathscr{C}$, and for any $m : M \to C$ subobject of C, there is a unique arrow $u : C \to \Omega$ such that the following diagram is a pullback:

$$\begin{tikzcd} M \arrow[r] \arrow[d, "m"'] \arrow[dr, phantom, "\checkmark"] & 1 \arrow[d, "t"] \\ C \arrow[r, "u"'] & \Omega \end{tikzcd}$$

The arrow $u : C \to \Omega$ is called the *classifying arrow for m* and is generally written χ_M.

[3]In fact, an equivalence class in this case, might not be a set. For example, in **Sets**, the collection of all subobjects of $\mathbb{R}$ of cardinality $\aleph_0$ is not a set. As a consequence, the collection of equivalence classes is not a set, as a set can only contain sets, at least in the set-theoretic sense (hereditarily: sets contain sets that contain sets and so on; there should not be any proper class in between). However, each representative is a cardinal, and there is only a set of cardinals below some cardinal. It is then more correct to refer to $\mathrm{SubObj}_{\mathscr{C}}(C)$ as the set of representatives.

In a sense, the subobject classifier is a "universal subobject".

Proposition 10.15. *Let $\mathscr{C}$ be a category with a subobject classifier (Ω, t).*
The subobject classifier is unique up to isomorphism.

Proof. Let (Ω, t) and (Ω', t') be two subobjects classifiers.

Note that an arrow $1 \to X$ is necessarily monic (since there is only one arrow to the terminal object). So, $t : 1 \to \Omega$ is a subobject of Ω. By definition of a subobject classifier, there exist unique $\chi_1 : \Omega \to \Omega'$ and $\chi_1' : \Omega' \to \Omega$ such that following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{id}_1"] \arrow[d, "t"] \arrow[dr, phantom, "\checkmark"] & 1 \arrow[r, "\mathrm{id}_1"] \arrow[d, "t'"] \arrow[dr, phantom, "\checkmark"] & 1 \arrow[d, "t"] \\
\Omega \arrow[r, "\chi_1"'] & \Omega' \arrow[r, "\chi_1'"'] & \Omega
\end{tikzcd}$$

As there is a unique arrow $\Omega \to \Omega$, which already is id_Ω (same for Ω'), we deduce that $\chi_1' = {\chi_1}^{-1}$. □

Example 10.16. What could the subobject classifier be in **Sets**? For now, let us justify the notation χ_M.

Suppose the simplest case. Consider a set X, a subset of Y (an actual subset, not only a subobject), and its canonical inclusion morphism $i : Y \to X$.

The terminal object in **Sets** is any one-element set; take the set-theoretic $1 = \{0\}$. Besides, there is a unique arrow $!_Y : Y \to 1$. We are looking for Ω and $t : 1 \to \Omega$. Let's be explicit: $!_Y$ and t are the functions:

$$!_Y : \begin{cases} Y & \longrightarrow & 1 \\ y & \longmapsto & 0 \end{cases} \qquad\qquad t : \begin{cases} 1 & \longrightarrow & \Omega \\ 0 & \longmapsto & \omega \end{cases}$$

where ω is some element in Ω.

As a subobject classifier, the following square is a pullback:

$$\begin{tikzcd}
Y \arrow[r, "!_Y"] \arrow[d, "i"'] \arrow[dr, phantom, "\checkmark"] & 1 \arrow[d, "t"] \\
X \arrow[r, "u"'] & \Omega
\end{tikzcd}$$

Let us observe the diagram in explicit terms. The function u is such that, for all $y \in Y$:

$$\begin{aligned} u \circ i(y) &= t \circ !_Y(y) \\ u(y) &= t(0) \\ u(y) &= \omega \end{aligned}$$

That is, for each $y \in Y$, the function u gives the same constant ω.

Now take another subset $Z \subset X$ with its inclusion mapping $i' : Z \to X$ and its terminal arrow $!_Z : Z \to 1$. We also suppose that $u \circ i' = t \circ !_Z$. As Y is a pullback, there is a unique $e : Z \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[dr, "e"] \arrow[drr, "!_Z"] \arrow[ddr, "i'"'] & & \\
& Y \arrow[r, "!_Y"] \arrow[d, "i"'] \arrow[dr, phantom, "\checkmark"] & 1 \arrow[d, "t"] \\
& X \arrow[r, "u"'] & \Omega
\end{tikzcd}$$

For all $z \in Z$, we have:

$$i'(z) = i \circ e(z)$$
$$z = e(z)$$

so e does not "alter" z; it is an inclusion mapping: $Z \subset Y$. So, for all $Z \subset X$ such that, for all $z \in Z$, we have $u(z) = \omega$, we deduce that $Z \subset Y$. Such a behaviour indicates that u should be the characteristic function χ_Y of Y, $\Omega = 2 = \{0, 1\}$ and $t : 0 \mapsto 1$ (the constant function that assigns 1 to its unique element 0).

In fact, we can check that $(2, 0 \mapsto 1)$ is the subobject classifier of **Sets**. In fact, the subobject classifier is designed to generate classifying arrows, which are the categorical generalisation of characteristic functions.

Proposition 10.17. *Let $\mathscr{C}$ be a category with all finite limits.*
$\mathscr{C}$ has a subobject classifier (Ω, t) $\Leftrightarrow$ the subobject functor $\mathrm{SubObj}_{\mathscr{C}}$ *is representable.*

Proof. By definition of a subobject classifier, for all C, for all $m : M \to C \in \mathrm{SubObj}_{\mathscr{C}}(C)$ subobject of C, there is a unique arrow $\chi_M : C \to \Omega$ such that the following diagram is a pullback:

$$\begin{tikzcd}
M \arrow[r, "!_M"] \arrow[d, "m"'] \arrow[dr, phantom, "\checkmark"] & 1 \arrow[d, "t"] \\
C \arrow[r, "\chi_M"'] & \Omega
\end{tikzcd}$$

By definition of the subobject functor, $m = \mathrm{SubObj}(\chi_M)(t)$. By Proposition 3.4, (Ω, t) is a universal element of SubObj, and by Theorem 3.17, $(\Omega, \theta_{\mathrm{SubObj},\Omega}(t))$ is a representation of SubObj. $\square$

Corollary 10.18. *If the subobject functor* $\mathrm{SubObj}_{\mathscr{C}}$ *is representable, then its universal element is (Ω, t).*

Corollary 10.19. $\mathrm{SubObj}_{\mathscr{C}}(-) \cong Hom_{\mathscr{C}}(-, \Omega)$.

We finish the section with the notion of a power object.

Definition 10.20 (Power object)**.** Let $\mathscr{C}$ be a category with finite limits. Let $C \in \mathscr{C}$.

The *power object of* C is a pair $(\mathcal{P}(C), p)$ where $\mathcal{P}(C)$ is an object of $\mathscr{C}$ and $p : \epsilon_C \to C \times \mathcal{P}(C)$ is a subobject of $C \times \mathcal{P}(C)$, such that, for all $D \in \mathscr{C}$, for all subobject $h : E \to C \times D$ of $C \times D$, there is a unique $u : D \to \mathcal{P}(C)$ such that the following diagram is a pullback:

$$\begin{tikzcd}
E \arrow[r, "e"] \arrow[d, "h"'] \arrow[dr, phantom, "\checkmark"] & \epsilon_C \arrow[d, "p"] \\
C \times D \arrow[r, "\mathrm{id}_C \times u"'] & C \times \mathcal{P}(C)
\end{tikzcd}$$

Remark 10.21. In terms of sets, the power object $\mathcal{P}(C)$ is the powerset of C (the set of all subsets of C). The object D is interpreted to be a family of subsets of C (a subset of $\mathcal{P}(C)$). The interpretation of $p : \epsilon_C \to C \times \mathcal{P}(C)$ will come later.

Proposition 10.22. *Let $\mathscr{C}$ be a category with finite limits. Let $C \in \mathscr{C}$.*
The power object of C is unique up to isomorphism.

Proof. Let $(\mathcal{P}(C), p)$ and (P', p') be two power objects of C.

$$\begin{tikzcd}
\epsilon_C \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\checkmark"] & \epsilon'_C \arrow[r] \arrow[d, "p'"] \arrow[dr, phantom, "\checkmark"] & \epsilon_C \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\checkmark"] & \epsilon'_C \arrow[d, "p'"] \\
C \times \mathcal{P}(C) \arrow[r, "\mathrm{id}_C \times u'"] \arrow[rr, bend right, "\mathrm{id}_C \times \mathrm{id}_{\mathcal{P}(C)}"'] & C \times P' \arrow[r, "\mathrm{id}_C \times u"] \arrow[rr, bend right, "\mathrm{id}_C \times \mathrm{id}_{P'}"'] & C \times \mathcal{P}(C) \arrow[r, "\mathrm{id}_C \times u'"] & C \times P'
\end{tikzcd}$$

Considering the unicity of u, u', $\mathrm{id}_{\mathcal{P}(C)}$ and $\mathrm{id}_{P'}$, and noticing that each adjacent two pullbacks are pullbacks, we have $u' = u^{-1}$. $\square$

Proposition 10.23. *Let $\mathscr{C}$ be a category with finite limits. Let $C \in \mathscr{C}$ with a power object $\mathcal{P}(C)$.*
For all $X \in \mathscr{C}$, $Hom_{\mathscr{C}}(X, \mathcal{P}(C)) \cong \mathrm{SubObj}_{\mathscr{C}}(X \times C)$ naturally in X.

Proof. By definition of a power object, there is a bijection $\varphi_X : \mathrm{Hom}_{\mathscr{C}}(X, \mathcal{P}(C)) \to \mathrm{SubObj}_{\mathscr{C}}(X \times C)$.

Let $x : X \to X'$ be an arrow; we check wether the following diagram commutes:

$$\begin{tikzcd}
X \arrow[d] & & \mathrm{Hom}_{\mathscr{C}}(X', \mathcal{P}(C)) \arrow[r, "\varphi_{X'}"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(x, \mathcal{P}(C))"'] \arrow[dr, phantom, "?"] & \mathrm{SubObj}_{\mathscr{C}}(X' \times C) \arrow[d, "\mathrm{SubObj}_{\mathscr{C}}(x \times C)"] \\
X' & \rightsquigarrow & \mathrm{Hom}_{\mathscr{C}}(X, \mathcal{P}(C)) \arrow[r, "\varphi_X"'] & \mathrm{SubObj}_{\mathscr{C}}(X \times C)
\end{tikzcd}$$

We want:

$$\mathrm{SubObj}_{\mathscr{C}}(x \times C) \circ \varphi_{X'} = \varphi_X \circ \mathrm{Hom}_{\mathscr{C}}(x, \mathcal{P}(C))$$

Let $u : X' \to \mathcal{P}(C)$; there is a unique subobject $h' : E' \to C \times X'$ such that the following is a pullback:

$$\begin{tikzcd}
E' \arrow[r] \arrow[d, "h'"'] \arrow[dr, phantom, "\checkmark"] & \epsilon_C \arrow[d, "p"] \\
C \times X' \arrow[r, "\mathrm{id}_C \times u"'] & C \times \mathcal{P}(C)
\end{tikzcd}$$

Besides, $\mathrm{SubObj}_{\mathscr{C}}(x \times C)(h')$ is defined as the unique h that makes the following diagram a pullback:

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "\checkmark"] & E' \arrow[d, "h'"] \\
C \times X \arrow[r, "\mathrm{id}_C \times x"'] & C \times X'
\end{tikzcd} \tag{28}$$

We deduce:

$$\begin{aligned}\mathrm{SubObj}_{\mathscr{C}}(x \times C) \circ \varphi_{X'}(u) &= \mathrm{SubObj}_{\mathscr{C}}(x \times C)(h') \\ &= h\end{aligned}$$

Then:

$$\varphi_X \circ \mathrm{Hom}_{\mathscr{C}}\left(x, \mathcal{P}\left(C\right)\right)(u) = \varphi_X(u \circ x) = h''$$

where h'' is the unique arrow such that the following diagram is a pullback:

$$\begin{tikzcd} E'' \arrow[r] \arrow[d, "h''"'] \arrow[dr, phantom, "\checkmark"] & \epsilon_C \arrow[d, "p"] \\ C \times X \arrow[r, "\mathrm{id}_C \times (u \circ x)"'] & C \times \mathcal{P}\left(C\right) \end{tikzcd} \tag{29}$$

Note that the above diagram decomposes into this:

$$\begin{tikzcd} E'' \arrow[r] \arrow[d, "h''"'] \arrow[dr, phantom, "\checkmark"] & E' \arrow[r] \arrow[d, "h'"'] \arrow[dr, phantom, "\checkmark"] & \epsilon_C \arrow[d, "p"] \\ C \times X \arrow[r, "\mathrm{id}_C \times x"'] & C \times X' \arrow[r, "\mathrm{id}_C \times u"'] & C \times \mathcal{P}\left(C\right) \end{tikzcd}$$

(The arrow $E'' \to E'$ can be obtained knowing that the right square is a pullback.)

By Proposition 6.55, if the right square $E'\epsilon_C(C \times \mathcal{P}\left(C\right))(C \times X')$ and the outer rectangle $E''\epsilon_C(C \times \mathcal{P}\left(C\right))(C \times X)$ are pullbacks, then the left square $E''E'(C \times X')(C \times X)$ is also a pullback; by unicity of h'' (from Diagram 29) and h (from Diagram 28), we deduce that $h = h''$, which let us conclude:

$$\begin{gathered} h = h'' \\ \mathrm{SubObj}_{\mathscr{C}}(x \times C) \circ \varphi_{X'}(u) = \varphi_X \circ \mathrm{Hom}_{\mathscr{C}}\left(x, \mathcal{P}\left(C\right)\right)(u) \end{gathered}$$

□

In **Sets**, we know that $\mathcal{P}\left(X\right) \cong 2^X$ (because each characteristic function $X \to 2$ defines a subset of X and conversely). There is a similar result with the categorical equivalents of a subobject classifier and the power object.

Proposition 10.24. *Let $\mathscr{C}$ be a category with finite limits and with a subobject classifier (Ω, t). Let $C \in \mathscr{C}$. Then, $\mathcal{P}\left(C\right) \cong \Omega^C$.*

Proof. By Proposition 10.23, we have:

$$\mathrm{Hom}_{\mathscr{C}}\left(-, \mathcal{P}\left(C\right)\right) \cong \mathrm{SubObj}_{\mathscr{C}}(- \times C)$$

By Corollary 10.18:

$$\mathrm{SubObj}_{\mathscr{C}}(-) \cong \mathrm{Hom}_{\mathscr{C}}\left(-, \Omega\right)$$

By Theorem 6.23:

$$\mathrm{Hom}_{\mathscr{C}}(- \times C, -) \cong \mathrm{Hom}_{\mathscr{C}}\left(-, (-)^{C}\right)$$

Combining those three results:

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(-, \mathcal{P}(C)) &\cong \mathrm{SubObj}_{\mathscr{C}}(- \times C) \\ &\cong \mathrm{Hom}_{\mathscr{C}}(- \times C, \Omega) \\ &\cong \mathrm{Hom}_{\mathscr{C}}\left(-, \Omega^{C}\right)\end{aligned}$$

By Corollary 2.18, we deduce that $\mathcal{P}(C) \cong \Omega^{C}$. □

Corollary 10.25. *If a category has all power objects, then $\mathcal{P}(1)$ is the subobject classifier.*

Proof. Just write the definition of the power object of 1. □

11. Elementary topoi

We can now introduce topoi. We will first see the general notion of elementary topoi (which we will simply call topoi) and we will then study the notion of a Grothendieck topos.

Definition 11.1 (Elementary topos (1) [1])**.** An *elementary topos* (or *topos* for short) is a category $\mathscr{E}$ that has finite limits, all exponentials, and a subobject classifier.

Definition 11.2 (Elementary topos (2) [2])**.** An *elementary topos* (or *topos* for short) is a category $\mathscr{E}$ that is cartesian closed, has finite limits, and a representable subobject functor.

Definition 11.3 (Elementary topos (3) [9])**.** An *elementary topos* (or *topos* for short) is a category $\mathscr{E}$ that has finite limits and such that every object has a power object.

Of course:

Proposition 11.4. *The three definitions of a topos are equivalent.*

In fact, it is easy to see that (2) $\Leftrightarrow$ (1) $\Rightarrow$ (3); however (3) $\Rightarrow$ (1) requires some more work. Let us just admit this proposition.

Example 11.5. As we have seen through this course, **Sets** has finite limits, all exponentials, a subobject classifier, every set has a powerset: **Sets** is an elementary topos. In fact, topoi are thought as categories that "roughly" behave like **Sets**.

As a first property of elementary topoi, we give the following theorem, but will not prove it as it is far from the scope of this course.

Proposition 11.6. *A topos has finite colimits.*

Proof. According to [2, Theorem 15.2.8, p389], this is very hard to prove, and the demonstration requires many notions that are beyond the scope of this course. We will just admit this theorem. □

Proposition 11.7. *Let $\mathscr{E}$ be a topos. A monomorphism in $\mathscr{E}$ is an equaliser.*

Proof. According to the definition of a subobject classifier (Ω, t), a monomorphism $m : M \longrightarrow C$ is indeed the equaliser of $\chi_M : C \longrightarrow \Omega$ and $!_C \circ t$. □

Proposition 11.8. *Let $\mathscr{E}$ be a topos. The isomorphisms in $\mathscr{E}$ are exactly the monic/epic.*

Proof. By Corollary 1.16, every isomorphism is both monic and epic; this is true in any category.

The converse comes from the topos-ness of $\mathscr{E}$. In a topos, all monomorphisms are equalisers, thus all monic/epic are epic equalisers, and by Proposition 6.42, all epic equalisers are isomorphisms. □

Definition 11.9 (Image of an arrow [10])**.** Let $f : A \longrightarrow C$ be an arrow.

We say that f *factors through* $m : B \longrightarrow C$ when there exists $e : A \longrightarrow B$ such that $f = m \circ e$.

The *image of* f is a monomorphism $m : B \longrightarrow C$ such that f factors through m, and for all monomorphism m', if f factors through m', then so does m.

Remark 11.10 (Explicit definition). The image of f is a monomorphism $m : B \longrightarrow C$ such that there exists $e : A \longrightarrow B$ such that $f = m \circ e$, and for all monomorphism $m' : B' \longrightarrow C$, if there exists $e' : A \longrightarrow B'$ such that $f = m' \circ e'$, then there exists $e'' : A \longrightarrow B'$ such that $m = m' \circ e''$.

In a sense, the image of f is the "smallest" subobject of C through which f can factor.

In **Sets**, the image of a function $f : X \longrightarrow Y$ is the inclusion mapping $f(X) \longrightarrow Y$.

Definition 11.11 (Epi-mono factorisation)**.** Let $f : A \longrightarrow C$ be an arrow in a category $\mathscr{C}$.

An *epi-mono factorisation of* f is a pair (m, e) where $m : B \longrightarrow C$ is a monomorphism and $e : A \longrightarrow B$ is an epimorphism such that $f = m \circ e$.

A category $\mathscr{C}$ is said to *have epi-mono factorisations* when every arrow f has an epi-mono factorisation.

In **Sets**, this property is very easy to see. Let $f : X \to Y$; then we can define $e : X \to f(X)$ (the restriction of f to its image) and then $m : f(X) \to Y$ as the inclusion function. Then e is epic (i.e. a surjection) and m is monic (i.e. an injection). As said before, topoi generalise in a sense the category of sets; and the following proposition holds:

Proposition 11.12. *Let $\mathscr{E}$ be a topos. Then $\mathscr{E}$ has epi-mono factorisations.*

Proof. We need the fact that a topos has colimits. Let $f : A \to B$ be a function; the pushout of f with f exists:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\checkmark"] & B \arrow[d, "x"] \\ B \arrow[r, "y"'] & Q \end{tikzcd}$$

and $x \circ f = y \circ f$.

Let (M, m) be the equaliser of x and y; there exists a unique e such that $f = m \circ e$:

$$\begin{tikzcd} M \arrow[r, "m"] & B \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & Q \\ A \arrow[u, "e"] \arrow[ur, "f"'] & & \end{tikzcd}$$

We now show that m is the image of f. Suppose $f = m' \circ e'$ for some monic $m' : M' \to B$. As we are in a topos, the monic m' is an equaliser of some $x', y' : B \to Q'$, as in the diagram:

$$\begin{tikzcd} M' \arrow[r, "m'"] & B \arrow[r, shift left, "x'"] \arrow[r, shift right, "y'"'] & Q' \\ A \arrow[u, "e'"] \arrow[ur, "f"'] & & \end{tikzcd}$$

As $x' \circ f = y' \circ f$, and using the fact that Q is a pushout of f with itself, there is a unique arrow $q : Q \to Q'$ such that $x' = q \circ x$ and $y' = q \circ y$, as in the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\checkmark"] & B \arrow[d, "x"] \arrow[ddr, "x'"] & \\ B \arrow[r, "y"] \arrow[drr, "y'"'] & Q \arrow[dr, "q"] & \\ & & Q' \end{tikzcd}$$

Now, we have:

$$x' \circ m = u \circ x \circ m = u \circ y \circ m = y' \circ m$$

so, m' being the equaliser of x' and y', there is a unique $e'' : M \to M'$ such that $m = m' \circ e''$; so m is indeed the image of f.

The only fact to prove now is the epicness of e. To do so, let us introduce and explain the following diagram:

$$\begin{tikzcd}
A \arrow[rrrr, "f"] \arrow[d, "e'"'] \arrow[dr, "e"] & & & & B \\
N \arrow[r, "n"'] & M \arrow[urrr, "m"] \arrow[dr, "e''"'] & & M \arrow[ur, "m"'] & \\
& & N \arrow[ur, "n"'] & &
\end{tikzcd}$$

We have proven that every arrow factors through some monomorphism thanks to some pushout; then there are two arrows $z, z' : M \to Q$ such that z and z' are the pushout of e with itself, and there exist $e' : A \to N$ and a monomorphism $n : N \to M$ such that $e = n \circ e'$, and n is the equaliser of z and z'.

First, let us focus on e' and n. We have $f = m \circ e = m \circ n \circ e'$; so f factors through $m \circ n$, which is monic (composition of monomorphisms is monic). As m is the image of f, m also does: there exists a $e'' : M \to N$ such that $m = m \circ n \circ e''$; as m is monic, we deduce:

$$\mathrm{id}_M = n \circ e'' \Rightarrow n = n \circ e'' \circ n \Rightarrow e'' \circ n = \mathrm{id}_N$$

as n is also monic; but as $e'' \circ n = \mathrm{id}_N$ and $n \circ e'' = \mathrm{id}_M$, we deduce that n is an isomorphism.

Now, consider z and z', the pushout of e with itself, with n being the equaliser of z and z'. As n is an isomorphism, we have $z = z'$. Now let $c, c' : M \to C$ be such that $c \circ e = c' \circ e$; as z and z' are a pushout of e with itself, there exists a unique u such that:

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "e"'] \arrow[dr, phantom, "\checkmark"] & M \arrow[d, "z"'] \arrow[ddr, "c"] & \\
M \arrow[r, "z'"] \arrow[drr, "c'"'] & Z \arrow[dr, "u"] & \\
& & C
\end{tikzcd}$$

and $c = u \circ z = u \circ z' = c'$. Thus, e is epic. □

For the following property of a topos, let us get back to slice categories. We did not introduce slice categories as such, but rather, comma-categories (Definition 8.17).

Definition 11.13 (Slice category)**.** Let $\mathscr{C}$ be a category, and let $C \in \mathscr{C}$ be an object.

The *slice category of $\mathscr{C}$ over C*, simply written $\mathscr{C}/C$, is the following category:

Objects: The objects are the arrows $x : X \to C \in \mathrm{Mor}_{\mathscr{C}}$

Morphisms: For $x : X \to C$ and $x' : X' \to C$, a morphism $f : x \to x'$ is defined as an arrow $f : X \to X' \in \mathrm{Mor}_{\mathscr{C}}$ such that the following triangle commutes:

$$\begin{tikzcd}
X \arrow[dr, "x"] \arrow[dd, "f"'] & \\
& C \\
X' \arrow[ur, "x'"'] &
\end{tikzcd}$$

Identities: The identity morphism of $x : X \to C$ is simply id_X

Composition: The composition law for morphisms is the usual composition law in $\mathscr{C}$

Confronting Definition 8.17 with Definition 11.13, it is easy to see that:

Proposition 11.14. *Let $\mathscr{C}$ be a category and $C \in \mathscr{C}$ be an object.*
The slice category $\mathscr{C}/C$ is the comma-category $(\mathrm{Id}_{\mathscr{C}} \mid \Delta(C))$.

Proposition 11.15. *Let $\mathscr{C}$ be a category with a terminal object* 1*. Then $\mathscr{C}/1 \cong \mathscr{C}$.*

Proof. The proof is quite easy. Define the functors:

$$F : \begin{cases} \mathscr{C} & \longrightarrow & \mathscr{C}/1 \\ C & \longmapsto & !_C : C \to 1 \\ c : C \to C' & \longmapsto & c \end{cases}$$

$$U : \begin{cases} \mathscr{C}/1 & \longrightarrow & \mathscr{C} \\ !_C : C \to 1 & \longmapsto & C \\ c : !_C \to !'_C & \longmapsto & c \end{cases}$$

Then U clearly is the inverse of F; thus F defines an isomorphism between $\mathscr{C}$ and $\mathscr{C}/1$. □

Another obvious fact is that any slice category has a terminal object:

Proposition 11.16. *In the slice category $\mathscr{C}/C$,* id_C *is the terminal object.*

Proposition 11.17. *Let $\mathscr{C}$ be a category and C be an object in $\mathscr{C}$.*
Consider the following diagram:

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "q"] & X_2 \arrow[d, "c_2"] \\ X_1 \arrow[r, "c_1"'] & C \end{tikzcd}$$

The previous diagram is a pullback in $\mathscr{C}$ $\Leftrightarrow$ $q = c_1 \circ p_1 = c_2 \circ p_2$ is a product in $\mathscr{C}/C$.

Proof. Just compare the universal properties of P (as a pullback) and q (as a product). □

Let us spend some time studying examples of slice categories, as they will turn crucial in our understanding of some properties of topoi.

Example 11.18 (Slice category in a preorder). Consider the preorder $(P, \leqslant)$. Let $p \in P$; what does slice category P/p look like? It is the following category:

Objects: The objects are the arrows $\mathrm{q} : q \to p$; that is, the objects are the pairs $\mathrm{q} = (q, p)$ such that $q \leqslant p$.

Morphisms: The arrows $\mathrm{x} : \mathrm{q} \to \mathrm{q}'$ are such that the following diagram commutes:

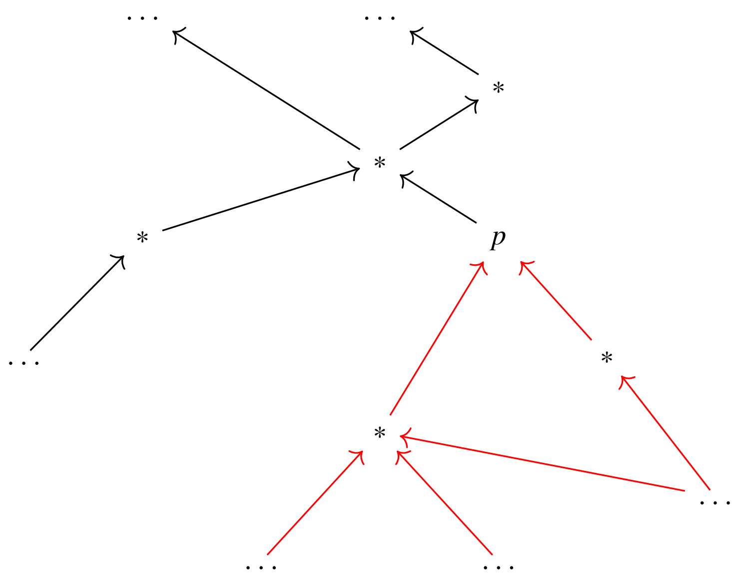
$$\begin{tikzcd} q \arrow[dr, "\mathrm{q}"] \arrow[dd, "\mathrm{x}"'] & \\ & p \\ q' \arrow[ur, "\mathrm{q}'"'] & \end{tikzcd}$$

That is, arrows $\mathrm{x} : \mathrm{q} \to \mathrm{q}'$ are simply pairs (q, q') such that $q \leqslant q' \leqslant p$.

Identities: The identity morphism of q : $q \to p$ is simply the reflexivity rule of a partial order.

Composition: The composition law for morphisms is the transitivity of the preorder.

In summary, the slice category of a preorder P/p is a truncated version of the preorder P, made with only the objects that are below p (which have arrows that target p) (in red in the following diagram).



Example 11.19 (Slice category in **Sets**). In **Sets**, a slice category is less obvious. Let A be a set.

If $A = 1$ then by Proposition 11.15, we have $\mathbf{Sets}/1 \cong \mathbf{Sets}$. If $A = 0 = \emptyset$ then $\mathbf{Sets}/0 \cong \mathscr{C}_1$, where $\mathscr{C}_1$ is the category with only one object and one identity morphism.

Now suppose A is not a trivial set: $A \neq 0$ and $A \neq 1$.

Let us start from the easy questions. What is the size of $\mathbf{Sets}/A$?

The objects of $\mathbf{Sets}/A$ are all the functions $a : X \to A$. As there are always $\mathrm{card}\,(A)^{\mathrm{card}(X)} > \mathrm{card}\,(X)$ functions between X and A, for each set X, there are $\mathrm{card}\,(A)^{\mathrm{card}(X)}$ arrows $X \to A$. Thus, $\mathbf{Sets}/A$ is a large category.

As for the morphisms; an arrow $x : a \to a'$ between $a : X \to A$ and $a' : X' \to A$ is an arrow $x : X \to X'$ with some additional properties (the commutative triangle). We deduce that $\mathrm{Hom}\,(a, a') \subset \mathrm{Hom}\,(X, X')$. Therefore, $\mathbf{Sets}/A$ is locally small, because **Sets** is.

Another question one may ask is: can we have two arrows $a : X \to A$ and $a' : X \to A$ and $a \neq a'$, that is, two different arrows from the same source? Consider the following functions:

$$\mathrm{mod}_2 : \left\{ \begin{array}{ccc} \mathbb{N} & \longrightarrow & 2 \\ n & \longmapsto & n \mod 2 \end{array} \right.$$

$$\mathrm{dom}_2 : \left\{ \begin{array}{ccc} \mathbb{N} & \longrightarrow & 2 \\ n & \longmapsto & (n+1) \mod 2 \end{array} \right.$$

where $2 = \{0, 1\}$ (the set-theoretic natural number 2). They are different functions and thus, different objects in $\mathbf{Sets}/2$.

There is also an arrow between them. The successor function:

$$\mathrm{succ} : \left\{ \begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{N} \\ n & \longmapsto & n+1 \end{array} \right.$$

is an arrow $\mathrm{succ} : \mathrm{mod}_2 \to \mathrm{dom}_2$ in $\mathbf{Sets}/2$.

Intuitively, the slice category $\mathbf{Sets}/A$ is a kind of "zoom" on how A "sees" **Sets**. This zoom somehow individualises the arrows that target A; in fact, mod_2 and dom_2 are indiscernible in **Sets** (they are arrows from the same source $\mathbb{N}$) while in $\mathbf{Sets}/2$, they are different objects.

Now consider the functions:

$$\mathrm{mod}_3 : \begin{cases} \mathbb{N} & \longrightarrow & 3 \\ n & \longmapsto & n \mod 3 \end{cases}$$

$$\mathrm{mod}_2 : \begin{cases} \mathbb{N} & \longrightarrow & 3 \\ n & \longmapsto & n \mod 2 \end{cases}$$

where mod_2 is the obvious extension of $\mathrm{mod}_2 : \mathbb{N} \to 2$ to $\mathbb{N} \to 3$ (we could have used different symbols but it was only making the notations inconvenient).

Is there an arrow $f : \mathrm{mod}_2 \to \mathrm{mod}_3$? That is, a function such that $n \mod 2 = f(n) \mod 3$. The function $\mathrm{mod}_2 : \mathbb{N} \to \mathbb{N}$ (extension of mod_2) does the job, but there are an infinity of functions that would do the job as well, for example:

$$\begin{cases} \mathbb{N} & \longrightarrow & \mathbb{N} \\ n & \longmapsto & \begin{cases} 3k & \text{if } n = 2k \\ 3k+1 & \text{if } n = 2k+1 \end{cases} \end{cases}$$

What about an arrow $f : \mathrm{mod}_3 \to \mathrm{mod}_2$? That is, a function such that $n \mod 3 = f(n) \mod 2$. Such a function does not exist, because there is no m such that $m \mod 2 = 2 \mod 3 = 2$ (note that we are not considering $\mathbb{Z}/(2\mathbb{Z})$, so the modulo operation only applies once). We conclude that there is no arrow $\mathrm{mod}_3 \to \mathrm{mod}_2$.

What condition makes it possible to have an arrow between two functions $a : X \to A$ and $a' : X' \to A$?

The previous example becomes obvious once we see that $\mathrm{mod}_2(\mathbb{N}) \subsetneq \mathrm{mod}_3(\mathbb{N})$. Consequently, there cannot be arrows $\mathrm{mod}_3 \to \mathrm{mod}_2$. This seems to be the condition we are looking for. In fact, we can show that, in $\mathbf{Sets}/A$, there is an arrow $x : a \to a'$ between $a : X \to A$ and $a' : X' \to A$ if and only if $a(X) \subset a'(X')$. As a corollary, there are arrows $x : a \to a'$ and $x' : a' \to a$ if and only if $a(X) = a'(X')$.

In other words, the slice category induces a preorder $\lesssim$ on the functions that target A: for $a : X \to A$ and $a' : X' \to A$, we have $a \lesssim a' \Leftrightarrow a(X) \subset a'(X')$. This preorder gives the general structure of a slice category $\mathbf{Sets}/A$. We will see in another example (Example 11.32) another interpretation of a slice category in **Sets**.

Remark 11.20 (Slice of slice). Let $\mathscr{C}$ be a category and $f : C \to D$ be an arrow in $\mathscr{C}$.

Consider the category $\mathscr{C}/D$. The arrow $f : C \to D$ is an object in $\mathscr{C}/D$. We can keep "slicing" the category. Let's have a closer look at $(\mathscr{C}/D)/f$.

An object in $(\mathscr{C}/D)/f$ is an arrow $x : d \to f \in \mathrm{Mor}_{\mathscr{C}/D}$, that is, such that the following triangle commutes:

$$\begin{tikzcd} X \arrow[dr, "d"] \arrow[dd, "x"'] & \\ & D \\ C \arrow[ur, "f"'] & \end{tikzcd}$$

For $x : d \to f$ and $x' : d' \to f$, an arrow $u : x \to x'$ in $(\mathscr{C}/D)/f$ is an arrow $u : d \to d'$ such that the following diagram commutes:

$$\begin{tikzcd} d \arrow[dd, "u"'] \arrow[dr, "x"] & \\ & f \\ d' \arrow[ur, "x'"'] & \end{tikzcd}$$

But the arrows x and x' also make the following diagrams commute:

$$\begin{tikzcd} X \arrow[dd, "x"'] \arrow[dr, "d"] & \\ & D \\ C \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[dd, "x'"'] \arrow[dr, "d'"] & \\ & D \\ C \arrow[ur, "f"'] & \end{tikzcd}$$

Finally, $u : d \to d'$ is an arrow $u : X \to X'$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[rr, "u"] \arrow[dl, "d"'] \arrow[dr, "x"] & & X' \arrow[dl, "x'"'] \arrow[dr, "d'"] & \\ D & & C \arrow[ll, "f"] \arrow[rr, "f"'] & & D \end{tikzcd}$$

In summary, an object in $(\mathscr{C}/D)\,/f$ is an arrow $x : X \to C$, that is, an object in $\mathscr{C}/C$, and an arrow in $(\mathscr{C}/D)\,/f$ is in fact an arrow in $\mathscr{C}/C$.

Conversely, an arrow $x : X \to C$ (object in $\mathscr{C}/C$) easily converts to an arrow $x : x \circ f \to f$ (object in $(\mathscr{C}/D)\,/f$). Besides, an arrow $u : x \to x'$ in $\mathscr{C}/C$ gives the following diagram:

$$\begin{tikzcd} X \arrow[dd, "u"'] \arrow[dr, "x"] \arrow[drr, bend left, "f \circ x"] & & \\ & C \arrow[r, "f"] & D \\ X' \arrow[ur, "x'"] \arrow[urr, bend right, "f \circ x'"'] & & \end{tikzcd}$$

Thus, an arrow $u : x \to x'$ easily becomes an arrow $u : f \circ x \to f \circ x'$.

These two observations define two functors that are clearly inverses of each other. Consequently, we have proven the following proposition.

Proposition 11.21 (Slice of a slice is a slice)**.** *Let $\mathscr{C}$ be a category, and let $f : C \to D$ be an arrow in $\mathscr{C}$. Then, $(\mathscr{C}/D)\,/f \cong \mathscr{C}/C$.*

Proof. See Remark 11.20. □

Definition 11.22 (Composition functor / dependent sum)**.** Let $\mathscr{C}$ be a category with all pullbacks, and let $f : C \to D$ be an arrow in $\mathscr{C}$.

The *composition functor of f*, or *dependent sum relative to f*, written Σ_f, is the following functor:

$$\Sigma_f : \left\{ \begin{array}{ccc} \mathscr{C}/C & \longrightarrow & \mathscr{C}/D \\ c & \longmapsto & f \circ c \\ x : c \to c' & \longmapsto & x : f \circ c \to f \circ c' \end{array} \right.$$

Remark 11.23. Note that, in the definition of the composition functor, for $x : c \to c'$, we have the following diagram:

$$\begin{tikzcd}
X \arrow[d, "x"'] \arrow[dr, "c"] & \\
 & C \\
X' \arrow[ur, "c'"'] &
\end{tikzcd}$$

which gives, by composition by f:

$$\begin{tikzcd}
X \arrow[d, "x"'] \arrow[dr, "c"] & & \\
 & C \arrow[r, "f"] & D \\
X' \arrow[ur, "c'"'] & &
\end{tikzcd}$$

Consequently, an arrow $x \in \mathscr{C}/C$ is also an arrow $x \in \mathscr{C}/D$.

Definition 11.24 (Pullback functor)**.** Let $\mathscr{C}$ be a category with all pullbacks, and let $f : C \to D$ be an arrow in $\mathscr{C}$.

The *pullback functor* f^* is the following functor:

$$f^* : \begin{cases} \mathscr{C}/D & \longrightarrow & \mathscr{C}/C \\ d & \longmapsto & f^*(d) \\ x : d \to d' & \longmapsto & f^*(x) \end{cases}$$

where, for $d : X \to D$, $f^*(d)$ is such that the following square is a pullback:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "f^*(d)"'] \arrow[dr, phantom, "\checkmark"] & X \arrow[d, "d"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}$$

and for an arrow $x : d \to d'$, $f^*(x)$ is the unique arrow $P \to P'$ between pullbacks such that the following diagram commutes:

$$\begin{tikzcd}
P \arrow[rr] \arrow[dr, "f^*(x)"] \arrow[ddr, "f^*(d)"'] & & X \arrow[dr, "x"] \arrow[ddr, dotted, "d'"] & \\
 & P' \arrow[rr] \arrow[d, "f^*(d')"] & & X' \arrow[d, "d"] \\
 & C \arrow[rr, "f"'] & & D
\end{tikzcd}$$

Proposition 11.25. *Let $\mathscr{C}$ be a category with pullbacks. Let $f : A \to B$ be an arrow in $\mathscr{C}$.*

Then, $\Sigma_f \dashv f^*$; that is, the composition functor is left adjoint to the pullback functor.

Proof. Let $c : X \to C$ and $d : Y \to D$. We have to check that there is a natural isomorphism:

$$\mathrm{Hom}_{\mathscr{C}/D}\left(\Sigma_f(c), d\right) \cong \mathrm{Hom}_{\mathscr{C}/C}\left(c, f^*(d)\right)$$

Let $u \in \mathrm{Hom}_{\mathscr{C}}\left(\Sigma_f(c), d\right)$; the following square commutes:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "c"'] \arrow[dr, phantom, "\checkmark"] & Y \arrow[d, "d"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}$$

By definition of $f^*(d)$, the following diagram is a pullback; as a consequence, there is a unique $v : X \to P$ such that the diagram commutes:

$$\begin{tikzcd}
X \arrow[dr, "v"] \arrow[drr, "u"] \arrow[ddr, "c"'] & & \\
& P \arrow[r, "p"'] \arrow[d, "f^*(d)"] \arrow[dr, phantom, "\checkmark"] & Y \arrow[d, "d"] \\
& C \arrow[r, "f"'] & D
\end{tikzcd}$$

So, for all $u \in \mathrm{Hom}_{\mathscr{C}/D}\left(\Sigma_f(c), d\right)$, there is a unique $v \in \mathrm{Hom}_{\mathscr{C}/C}(c, f^*(d))$ such that the above diagram commutes; in other words, the mapping $\varphi_{c,d} : u \mapsto v$ is a bijection[4].

We now have to check the naturality in c and d.

Let $x : c' \to c$ be an arrow in $\mathscr{C}/C$; thus x makes the following diagram commute:

$$\begin{tikzcd}
X' \arrow[rr, "x"] \arrow[dr, "c'"'] & & X \arrow[dl, "c"] \\
& C &
\end{tikzcd}$$

We have to check whether the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}/D}\left(\Sigma_f(c), d\right) \arrow[r, "\varphi_{c,d}"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}/D}\left(\Sigma_f(x), d\right)"'] \arrow[dr, phantom, "\checkmark"] & \mathrm{Hom}_{\mathscr{C}/C}(c, f^*(d)) \arrow[d, "\mathrm{Hom}_{\mathscr{C}/C}(x, f^*(d))"] \\
\mathrm{Hom}_{\mathscr{C}/D}\left(\Sigma_f(c'), d\right) \arrow[r, "\varphi_{c',d}"'] & \mathrm{Hom}_{\mathscr{C}/C}(c', f^*(d))
\end{tikzcd}$$

Let $u \in \mathrm{Hom}_{\mathscr{C}/D}\left(\Sigma_f(c), d\right)$; u makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[rr, "u"] \arrow[dr, "\Sigma_f(c)"'] & & Y \arrow[dl, "d"] \\
& D &
\end{tikzcd}$$

So:

[4] In fact the formula establishes the reverse bijection, but this bijection will do.

$$\mathrm{Hom}_{\mathscr{C}/C}\left(x, f^*(d)\right) \circ \varphi_{c,d}(u) = v \circ x$$

where v is the unique arrow $X \to P$ that makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[dr, "v"] \arrow[drr, "u"] \arrow[ddr, "c"'] & & \\
& P \arrow[r, "p"] \arrow[d, "f^*(d)"] \arrow[dr, phantom, "\checkmark"] & Y \arrow[d, "d"] \\
& C \arrow[r, "f"'] & D
\end{tikzcd} \tag{30}$$

Then:

$$\begin{aligned}
\varphi_{c',d} \circ \mathrm{Hom}_{\mathscr{C}/C}\left(\Sigma_f(x), d\right)(u) &= \varphi_{c',d}(u \circ x) \\
&= v'
\end{aligned}$$

where v' is the unique arrow $X' \to P$ such that $p \circ v' = u \circ x$ and $f^*(d) \circ v' = c'$, as in the following diagram:

$$\begin{tikzcd}
X' \arrow[dr, dashed, "x"] \arrow[ddrr, bend left, "v'"] \arrow[ddrrr, bend left, "u \circ x"] \arrow[dddrr, bend right, "c'"'] & & & \\
& X \arrow[dr, dashed, "v"'] \arrow[drr, dashed, "u"] \arrow[ddr, dashed, "c"'] & & \\
& & P \arrow[r, "p"'] \arrow[d, "f^*(d)"] \arrow[dr, phantom, "\checkmark"] & Y \arrow[d, "d"] \\
& & C \arrow[r, "f"'] & D
\end{tikzcd} \tag{31}$$

By chasing diagram 31 and using the equations given by diagram 30, we see that:

$$\begin{aligned}
f \circ c' &= f \circ c \circ x = d \circ u \circ x \\
f^*(d) \circ v' &= c' = f^*(d) \circ v \circ x \\
p \circ v' &= u \circ x = p \circ v \circ x
\end{aligned}$$

By unicity of v', we must have $v' = v \circ x$, hence the equality:

$$\begin{gathered}
v' = v \circ x \\
\varphi_{c',d} \circ \mathrm{Hom}_{\mathscr{C}/C}\left(\Sigma_f(x), d\right)(u) = \mathrm{Hom}_{\mathscr{C}/C}\left(x, f^*(d)\right) \circ \varphi_{c,d}(u)
\end{gathered}$$

Which gives the naturality in c. The naturality in d is very similar and is left to the reader. $\square$

The adjunction $\Sigma_f \dashv f^*$ may sometimes be completed with a third functor Π_f, called the dependent product functor, that is right adjoint to the pullback functor f^*. However, this does not occur often; the existence of this right adjoint depends on some property of the category $\mathscr{C}$.

Definition 11.26 (Locally Cartesian closed)**.** A category $\mathscr{C}$ is called *locally Cartesian closed* whenever, for all object $C \in \mathscr{C}$, the slice category $\mathscr{C}/C$ is Cartesian closed.

From the definition and Proposition 11.15, it is easy to see that:

Proposition 11.27. *If $\mathscr{C}$ is locally Cartesian closed and has a terminal object, then $\mathscr{C}$ is Cartesian closed.*

Another result that is easy to see is the following:

Proposition 11.28. *Let $\mathscr{C}$ be a category.*
If $\mathscr{C}$ is locally Cartesian closed, then so is every slice of $\mathscr{C}$.

Proof. For all $C \in \mathscr{C}$, $\mathscr{C}/C$ is Cartesian closed. Then keep slicing the category by $c : X \to C$; we have $(\mathscr{C}/C)/c \cong (\mathscr{C}/X)$ (by Proposition 11.21), and $\mathscr{C}/X$ is Cartesian closed, hence the result. □

The property that matters to us now is the following:

Proposition 11.29. *Let $\mathscr{C}$ be a category with all pullbacks.*
Then, $\mathscr{C}$ is locally Cartesian closed $\Leftrightarrow$ for all arrow f, the pullback functor f^* has a right adjoint Π_f.

Proof. **[Proof of $\Leftarrow$]** Let $f : C \to D$ be an arrow, and let Π_f be its right adjoint.

We have to find the terminal object, the products and the exponentials in $\mathscr{C}/D$. By Proposition 11.16, we know that the terminal object in $\mathscr{C}/D$ is id_D. By Proposition 11.17, as $\mathscr{C}$ has all pullbacks, we know that $\mathscr{C}/D$ has all products.

Consider the following pullback:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f^*(d)"'] \arrow[dr, phantom, "\checkmark"] & X \arrow[d, "d"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

By Proposition 11.25, the composition functor Σ_f is left adjoint to the pullback functor f^*. By Proposition 11.17, the pullback of f and d in $\mathscr{C}$ corresponds to a product in $\mathscr{C}/D$. Then:

$$f \times d = f \circ f^*(d) = \Sigma_f\ (f^*(d))$$

We deduce the following equivalence of hom-sets:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}/D}\,(f \times d, u) &= \mathrm{Hom}_{\mathscr{C}/D}\,\left(\Sigma_f\ (f^*(d))\,, u\right) \\ &\cong \mathrm{Hom}_{\mathscr{C}/C}\,(f^*(d), f^*(u)) \\ &\cong \mathrm{Hom}_{\mathscr{C}/D}\,\left(d, \Pi_f\ (f^*(u))\right) \end{aligned}$$

Then, in $\mathscr{C}/D$, by Proposition 6.23 (exponential is right adjoint to product), the exponential u^f can only be $u^f = \Pi_f\ (f^*(u))$. As such an exponential always exists (because f^* always does, and Π_f does by assumption), $\mathscr{C}/D$ is Cartesian closed.

Thus, $\mathscr{C}$ is locally Cartesian closed.

[Proof of $\Rightarrow$] Assume that $\mathscr{C}$ is locally Cartesian closed. Then each slice category is Cartesian closed, so each slice category $\mathscr{C}/D$ of $\mathscr{C}$ has products. By Proposition 11.21, a slice of a slice is a slice. By the same reasonning, each slice category $(\mathscr{C}/D)\,/f$ of $\mathscr{C}/D$ has products.

By Proposition 11.17, each product in $(\mathscr{C}/D)\,/f$ is a pullback in $\mathscr{C}/D$, so $\mathscr{C}/D$ has all pullbacks. By Proposition 11.16, each slice category $\mathscr{C}/D$ has a terminal object id_D.

Each slice category $\mathscr{C}/D$ has pullbacks and a terminal object, so by Proposition 7.34, it has all finite limits. By Proposition 8.27, as every slice category has finite limits and exponentials, we deduce that the pullback functor f^* (in $\mathscr{C}$, that is the "product" functor in $\mathscr{C}/D$) has a right adjoint. □

Corollary 11.30. *If $\mathscr{C}$ has all pullbacks and is locally Cartesian closed, then each slice category $\mathscr{C}/C$ has finite limits.*

Definition 11.31 (Dependent product)**.** Let $\mathscr{C}$ be a category with all pullbacks and locally Cartesian closed. Let $f : C \to D$ be an arrow in $\mathscr{C}$.

The *dependent product* Π_f is the right adjoint to the pullback functor f^*.

Before studying more properties, maybe we should take a break and look at how this functor behaves.

Example 11.32 (Slice category in **Sets**, again). Consider **Sets** and some function $f : C \to D$.

Let $c : X \to C$ be an object in **Sets**$/C$. It is a function, indeed, but the point of view that makes more sense in this context is the following. For all $y \in C$, we can define the set $c^{-1}(y) = \{x \in X \mid c(x) = y\}$. In this case, the function $c : X \to C$ becomes an C-indexed set $\left(c^{-1}(y)\right)_{y\in C}$ where $X = \sum_{y\in C} c^{-1}(y)$ (coproduct). Then, a morphism $h : c \to c'$ in **Sets**$/C$ is a function between C-indexed sets, such that $h = \sum_{y\in C} h_y : c^{-1}(y) \to c'^{-1}(y)$.

Note that this point of view explains why there cannot be any arrow from $c : X \to C$ to $c' : X' \to C$ whenever $c'(X') \subsetneq c(X)$. In fact, take $y \in c(X)\backslash c'(X')$; we have $c'^{-1}(y) = \emptyset$, so the function $h_y : c^{-1}(y) \to c'^{-1}(y)$ is a function to the empty set, what does not exist unless $c^{-1}(y) = \emptyset$ too.

What about Σ_f? If $c : X \to C \in \mathscr{C}/C$, then $\Sigma_f(c) = f \circ c : X \to D$; that is, $\Sigma_f(c)$ can be seen as the D-indexed set $\left((f \circ c)^{-1}(y)\right)_{y\in D}$. The function between C-indexed sets $h : c \to c' \simeq h : \sum_{y\in C} h_y : c^{-1}(y) \to c'^{-1}(y)$ (morphism in $\mathscr{C}/C$) is sent to $\Sigma_f(h) : f \circ c \to f \circ c' \simeq \Sigma_f(h) : \sum_{y\in D} (f \circ h)_y : (f \circ c)^{-1}(y) \to (f \circ c')^{-1}(y)$. Beware of the notation: $\sum_{y\in D}$ is a coproduct symbol.

As for Π_f. Let $y \in D$ and $c : X \to C$ be an object in $\mathscr{C}/C$. We call *partial section of f for y along c* any function $s : f^{-1}(y) \to X$ such that the following diagram commutes:

$$
\begin{tikzcd}
f^{-1}(y) \arrow[r, "s"] \arrow[dr, "\subseteq"'] & X \arrow[d, "c"] \\
& C
\end{tikzcd}
$$

that is, for all $x \in f^{-1}(y)$, $c \circ s(x) = x$. Note that s is essentially a right inverse of c on the reverse image of $y \in D$ by f, and such an s may not be unique. Also, a partial section along $c : X \to C$ requires that $f^{-1}(y) \subset c(X)$; so f may not have partial sections for all y along c, for example if $f^{-1}(y)\backslash c(X) \neq \emptyset$.

Awodey [1, p233, below Proposition 9.18] states that $\Pi_f(c) : S \to D$ where S is:

$$
\begin{aligned}
&S = \left\{s : f^{-1}(y) \to X \mid y \in D \text{ and } s \text{ is a partial section of } f \text{ for } y \text{ along } c\right\}\\
&S \subset \bigcup_{y\in D} \mathrm{Hom}\left(f^{-1}(y), X\right)
\end{aligned}
$$

and for all $s : f^{-1}(y) \to X \in S$, $\Pi_f(c)(s) = y$ (that is, $\Pi_f(c)$ "projects" a partial section $s : f^{-1}(y) \to X$ to the base of the inverse image on which the section takes place).

Finally, the pullback functor is easier to see. Remember that, in **Sets**, the pullback between $f : C \longrightarrow D$ and $d : X \longrightarrow D$ is the set $C \times_D X = \{(c,x) \in C \times X \mid f(c) = d(x)\}$ (cf. Example 6.51). The pullback functor f^* sends $d : X \longrightarrow D$ to the projection of $C \times_D X$ to C:

$$f^*(d) : \begin{cases} C \times_D X & \longrightarrow & C \\ (c,x) & \longmapsto & c \end{cases}$$

One may wonder where d appears in the construction of $f^*(d)$; just remember the above definition of $C \times_D X$ in **Sets**.

Proposition 11.33. *Let $\mathscr{E}$ be a topos. Then every slice of $\mathscr{E}$ is a topos as well.*

Proof. Let $E \in \mathrm{Ob}_{\mathscr{E}}$. By Corollary 11.30, the slice category $\mathscr{E}/E$ has finite limits. We now have to show that every object has a power object.

So, the proof can be found in [10, Chapter IV, Section 7, Theorem 1, p190] and in [9, Chapter 5, Section 2, Theorem 2.1, p149], but I don't want to spend too much time on it. The proofs are very long. We will just admit this proposition. □

Corollary 11.34. *A topos is locally Cartesian closed.*

Proof. In fact, as every slice of a topos is a topos, then in particular, each slice of a topos is Cartesian closed. □

The fact that a topos is locally Cartesian closed is crucial in theoretical computer science and logic, because it means that any topos has an internal type theory.

Let us sum up the properties of a topos that we have seen:

Theorem 11.35 (Properties of a topos)**.** *Let $\mathscr{E}$ be a topos. Then $\mathscr{E}$ has all the following properties:*

- *It has all finite limits*
- *It has all finite colimits*
- *It has all exponentials*
- *Every object has a power object*
- *It has a subobject classifier*
- *It is Cartesian closed*
- *It is locally Cartesian closed*
- *Its slices are Cartesian closed*
- *Its slices are locally Cartesian closed*
- *Its slices are topoi*
- *Its isomorphisms are exactly the monic/epic*
- *It has all epi-mono factorisations*

12. Presheaves, sheaves, sheaf topoi

Incomplete section. We introduce the category of presheaves and prove it's a topos, but the Crash Course stops here for now (2019-01-18).

For now we have only see one example of topos. Let us introduce another example.

Definition 12.1 (Presheaf)**.** Let $\mathscr{C}$ be a small category. A *presheaf on* $\mathscr{C}$ is a functor $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$.

The *presheaf category*, written $\left[\mathscr{C}^{\mathrm{op}}, \mathbf{Sets}\right]$, $\mathbf{Sets}^{\mathscr{C}^{\mathrm{op}}}$ or simply $\mathbf{PSh}\left(\mathscr{C}\right)$, is the functor category $\mathbf{Func}\left(\mathscr{C}^{\mathrm{op}}, \mathbf{Sets}\right)$.

The following propositions will aim at proving that the presheaf category $\mathbf{PSh}\left(\mathscr{C}\right)$ is a topos.

Lemma 12.2. *Let $\mathscr{C}$ be a small category. Then for all diagram $D : \mathscr{I} \longrightarrow \mathbf{PSh}\left(\mathscr{C}\right)$, we have* $\mathrm{Lim}\left(D\right)\left(C\right) \cong \mathrm{Lim}\left(D(-)(C)\right)$*; in other words, limits in a presheaf category are computed objectwise.*

Proof. Let $D : \mathscr{I} \longrightarrow \mathbf{PSh}\left(\mathscr{C}\right)$ be a diagram in $\mathbf{PSh}\left(\mathscr{C}\right)$. The proof lies on the (admitted) fact that **Sets** has all small limits (admitted in Example 8.12).

Note that there is a canonical equivalence of categories: $\mathbf{Func}\left(\mathscr{I}, \mathbf{Func}\left(\mathscr{C}^{\mathrm{op}}, \mathbf{Sets}\right)\right) \cong \mathbf{Func}\left(\mathscr{I} \times \mathscr{C}^{\mathrm{op}}, \mathbf{Sets}\right)$. Call $\widehat{D}$ the equivalent diagram $\widehat{D} : \mathscr{I} \times \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$.

For a given $C \in \mathscr{C}^{\mathrm{op}}$, $\widehat{D}(-, C)$ is a diagram from $\mathscr{I} \longrightarrow \mathbf{Sets}$, and as **Sets** has all small limits, it has a limit $\left(\mathrm{Lim}\left(\widehat{D}(-, C)\right), \varepsilon_{D,C}\right)$. Then, if $c : C \longrightarrow C' \in \mathscr{C}$, then, as $\widehat{D}$ is a (contravariant) functor, there is an arrow: $\widehat{D}(-, c) : \widehat{D}(-, C') \longrightarrow \widehat{D}(-, C)$ between cones to D and by property of limits, there is a unique arrow $\mathrm{Lim}\left(\widehat{D}(-, c)\right) : \mathrm{Lim}\left(\widehat{D}(-, C')\right) \longrightarrow \mathrm{Lim}\left(\widehat{D}(-, C)\right)$, such that the following diagram commutes:

$$
\begin{tikzcd}
C \arrow[d, "c"] & \rightsquigarrow & C' \arrow[r, "\varepsilon_{D,C'}"] \arrow[d, "c"'] \arrow[dr, phantom, "\checkmark"] & \mathrm{Lim}\left(\widehat{D}(-, C')\right) \arrow[d, "\mathrm{Lim}\left(\widehat{D}(-,c)\right)"] \\
C' & & C \arrow[r, "\varepsilon_{D,C}"'] & \mathrm{Lim}\left(\widehat{D}(-, C)\right)
\end{tikzcd}
$$

Note that we are considering the right-hand square diagram in $\mathscr{C}^{\mathrm{op}}$; we have the naturality of $\varepsilon_{D,C}$ in C.

There remains to show that $\mathrm{Lim}\left(\widehat{D}(-, C)\right)$ is indeed a limit. Let $\alpha : \Delta(P) \longrightarrow D$ be a cone to D. Here, the diagonal functor is: $\Delta : \mathbf{PSh}\left(\mathscr{C}\right) \longrightarrow \mathbf{PSh}\left(\mathscr{C}\right)^{\mathscr{I}}$. We are looking for a unique $\gamma : P \longrightarrow \mathrm{Lim}\left(\widehat{D}(-, -)\right)$ such that, for all objects $i \in \mathscr{I}$ and $C \in \mathscr{C}$, the following diagram commutes: $\alpha_i(C) = \varepsilon_{D,C,i} \circ \gamma(C)$.

$$
\begin{tikzcd}
 & D(i)(C) \arrow[d, "\gamma(C)"] \\
\Delta(P)(i)(C) \arrow[ur, "\alpha_i(C)"] \arrow[r, "\varepsilon_{D,C,i}"'] & \mathrm{Lim}\left(\widehat{D}(-, C)\right)
\end{tikzcd}
$$

Such a (unique) $\gamma(C)$ always exists due to the universal property of limits in **Sets**. We only have to check that this $\gamma = (\gamma(C))_{C \in \mathscr{C}}$ is natural in C. It is due to the naturality of the other natural transformations α and $\varepsilon_{D,-}$. Finally, γ is unique due to the uniqueness of each $\gamma(C)$.

Thus, for every diagram $D : \mathscr{I} \longrightarrow \mathbf{PSh}\left(\mathscr{C}\right)$, there is a limit, and for all $C \in \mathscr{C}$, $\mathrm{Lim}\left(D\right)\left(C\right) \cong \mathrm{Lim}\left(\widehat{D}(-, C)\right) \cong \mathrm{Lim}\left(D(-)(C)\right)$. $\square$

In fact, we proved something stronger:

Corollary 12.3. **PSh** $(\mathscr{C})$ *has all small limits.*

Corollary 12.4. *For all* $P, Q \in$ **PSh** $(\mathscr{C})$*,* $P \times Q$ *is the functor:*

$$P \times Q : \begin{cases} \mathscr{C}^{op} & \longrightarrow & \mathbf{Sets} \\ C & \longmapsto & P(C) \times Q(C) \\ c & \longmapsto & P(c) \times Q(c) \end{cases}$$

Corollary 12.5. *The terminal object in* **PSh** $(\mathscr{C})$ *is the constant presheaf* $\Delta(1)$ *where* Δ *is the diagonal functor* $\Delta : \mathbf{Sets} \to \mathbf{PSh}(\mathscr{C})$.

Proposition 12.6. *Let* $\alpha : P \to Q$ *be a morphism between two presheaves in* **PSh** $(\mathscr{C})$.
Then, α *is monic* $\Leftrightarrow$ *for all* $X \in \mathscr{X}$*,* $\alpha_X : F(X) \to G(X)$ *is monic.*

Proof. The proof of $\Leftarrow$ has already been given in Proposition 1.38.

Conversely, suppose that α is monic. The characterisation of monics by pullbacks (Proposition 6.54 states that the pullback of α with itself is $(P, \mathrm{id}_P, \mathrm{id}_P)$. As limits, and thus pullbacks, are computed objectwise, we deduce that the pullback of each C-component $\alpha_C : P(C) \to Q(C)$ with itself is also a triple $\left(P(C), \mathrm{id}_{P(C)}, \mathrm{id}_{P(C)}\right)$, making each component monic. $\square$

Remark 12.7. We know that a presheaf category has all small limits. In particular, it has all binary products, so maybe it has exponentials.

Let $P, Q \in$ **PSh** $(\mathscr{C})$ be presheaves. Suppose their exponential Q^P exists; let us study it.

By adjunction product/exponential (Proposition 6.23), we know that $\mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(X \times P, Q) \cong \mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}\left(X, Q^P\right)$. By Yoneda lemma, we have:

$$\mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}\left(\mathrm{Hom}_{\mathscr{C}}(-, C), Q^P\right) \cong Q^P(C)$$

which defines the functor Q^P as:

$$Q^P(C) \cong \mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}\left(\mathrm{Hom}_{\mathscr{C}}(-, C) \times P, Q\right)$$

We have to check that this actually defines an exponential; that is, for every $f : X \times P \to Q$, there is a unique $\hat{f} : X \to Q^P$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "\hat{f}"] & P \times X \arrow[d, "\mathrm{id}_B \times \hat{f}"'] \arrow[dr, "f"] & \\ Q^P & P \times Q^P \arrow[r, "\mathrm{eval}"'] & Q \end{tikzcd}$$

Note that $f : X \times P \to Q$ is a natural transformation, and as limits (hence, producs) are computed objectwise, the previous diagram becomes:

$$\begin{tikzcd} X(C) \arrow[d, "\hat{f}_C"] & P(C) \times X(C) \arrow[d, "\mathrm{id}_{B,C} \times \hat{f}_C"'] \arrow[dr, "f_C"] & \\ Q^P(C) & P(C) \times Q^P(C) \arrow[r, "\mathrm{eval}_C"'] & Q(C) \end{tikzcd}$$

We define the natural transformation eval_C as:

$$\mathrm{eval}_C : \begin{cases} P(C) \times \mathrm{Nat}\left(\mathrm{Hom}_{\mathscr{C}}(-, C) \times P, Q\right) & \longrightarrow & Q(C) \\ (p, \alpha) & \longmapsto & \alpha_C\left(\mathrm{id}_C, p\right) \end{cases}$$

Note that eval is the counit of the adjunction $\mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(X \times P, Q) \cong \mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(X, Q^P)$.

Then, note that eval_C is defined such that $\mathrm{eval}_C \circ \left(\mathrm{id}_P \times \hat{f}\right)_C (x, p) = \mathrm{eval}_C\left(p, \hat{f}_C(x)\right) = \hat{f}_C(x)(\mathrm{id}_C, p) = f_C(x, p)$ by commutativity of the previous diagrams.

We now have to find the expression of $\hat{f}$. For now we focus on $\hat{f}_C(x)$. We already know that $\hat{f}_C(x)(\mathrm{id}_C, p) = f_C(x, p)$. For $c : C' \to C$, the following diagram commutes:

$$\begin{tikzcd}
C' \arrow[d, "c"] & \mathrm{Hom}_{\mathscr{C}}(C, C) \times P(C) \arrow[r, "\hat{f}_C(x)_C"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(c,C)\times P(c)"'] \arrow[dr, phantom, "\checkmark"] & Q(C) \arrow[d, "Q(c)"] \\
C \arrow[ur, phantom, "\rightsquigarrow"] & \mathrm{Hom}_{\mathscr{C}}(C', C) \times P(C') \arrow[r, "\hat{f}_C(x)_{C'}"'] & Q(C')
\end{tikzcd}$$

that is:

$$Q(c) \circ \hat{f}_C(x)_C(u, p) = \hat{f}_C(x)_{C'} \circ (\mathrm{Hom}_{\mathscr{C}}(c,C) \times P(c))(u, p)$$

and in particular, when $u = \mathrm{id}_C$:

$$\begin{aligned}
Q(c) \circ \hat{f}_C(x)_C(\mathrm{id}_C, p) &= \hat{f}_C(x)_{C'} \circ (\mathrm{Hom}_{\mathscr{C}}(c,C) \times P(c))(\mathrm{id}_C, p) \\
Q(c)(f_C(c, p)) &= \hat{f}_C(x)_{C'}(c, P(c)(p))
\end{aligned}$$

which defines $\hat{f}_C(x)_{C'}(c, p')$ on pairs (c, p') such that $p' = P(c)(p)$ (which is enough for our purposes).
The naturality of $\hat{f}_C(x)$ in C is immediate.
Finally, we have defined the exponential in $\mathbf{PSh}(\mathscr{C})$.

Definition 12.8 (Exponential in a presheaf category)**.** Let P, Q be two presheaves in $\mathbf{PSh}(\mathscr{C})$. Their exponential Q^P is defined as:

$$Q^P : \begin{cases} \mathscr{C}^{\mathrm{op}} & \longrightarrow & \mathbf{Sets} \\ C & \longmapsto & \mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(\mathrm{Hom}_{\mathscr{C}}(-, C) \times P, Q) \\ c & \longmapsto & \mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(\mathrm{Hom}_{\mathscr{C}}(-, c) \times P, Q) \end{cases}$$

Lemma 12.9. *A presheaf category* $\mathbf{PSh}(\mathscr{C})$ *has all exponentials.*

Proof. The construction in Remark 12.7 holds for any preasheaves P and Q. □

Corollary 12.10. *A presheaf category is Cartesian closed.*

We could have defined the exponential in a presheaf category, and then prove that the presheaf it defines actually is an exponential, but we prefered showing how the definition naturally made sense.

So, we have all small (thus finite) limits, and all exponentials. The only thing missing is the subobject classifier. To this extend, we define:

Definition 12.11 (Sieve [1])**.** Let $\mathscr{C}$ be a small category, and let C be an object in $\mathscr{C}$.
A *sieve on* C is a set $S \in \mathbf{Sets}$ such that:

$$\begin{aligned}
S = &\{f : X \to C \mid \text{for some arrows } f : X \to C\} \\
&\cup \{f \circ g : Y \to C \mid Y \in \mathscr{C}, g : Y \to X \text{ and } f \in S\}
\end{aligned}$$

In other words, S is a set of (some) arrows of $\mathscr{C}$ with codomain C (left-hand part of the union), stable by precomposition (right-hand part of the union), that is, for all $g : Y \to X$ and $f \in S$, we have $f \circ g \in S$. Note that S doesn't necessarily contain all arrows $X \to C$.

Definition 12.12 (Sieve presheaf)**.** Let $\mathscr{C}$ be a small category.

For $C \in \mathscr{C}$, we define $\mathrm{Sieve}(C) = \{S \in \mathbf{Sets} \mid S \text{ is a sieve on } C\}$.

For $c : C \to C' \in \mathscr{C}$, we define:

$$\mathrm{Sieve}(c) : \begin{cases} \mathrm{Sieve}(C') & \longrightarrow & \mathrm{Sieve}(C) \\ S & \longmapsto & \{g : X \to C \mid c \circ g \in S\} \end{cases}$$

The *sieve presheaf*[5], written Sieve, is the following contravariant functor:

$$\mathrm{Sieve} : \begin{cases} \mathscr{C} & \longrightarrow & \mathbf{Sets} \\ C & \longmapsto & \mathrm{Sieve}(C) \\ c : C \to C' & \longmapsto & \mathrm{Sieve}(c) \end{cases}$$

Lemma 12.13. *Let $\mathscr{C}$ be a small category and let* $\mathrm{Sieve} : \mathscr{C}^{op} \to \mathbf{Sets}$ *be its sieve presheaf. And let $\Delta(1)$ be the terminal object in* $\mathbf{PSh}\,(\mathscr{C})$.

Then there exists $t : \Delta(1) \to \mathrm{Sieve}$ such that (Sieve, t) is a subobject classifier of $\mathbf{PSh}\,(\mathscr{C})$.

Proof. The morphism (natural transformation) t is:

$$t = \left(t_C : \begin{cases} 1 & \longrightarrow & \mathrm{Sieve}(C) \\ x & \longmapsto & \{f : X \to C \mid X \in \mathscr{C}, f : X \to C\} \end{cases} \right)_{C \in \mathscr{C}}$$

That is, t_C is the function that selects the (unique) sieve that contains all arrows whose codomain is C (remember that sieves need not contain all arrows). The naturality of t is quite obvious once we remember that $\Delta(1)(C) = 1$.

We now have to check that (Sieve, t) is a subobject classifier. Awodey describes the classifying arrow of $\pi : Q \to P$ as α such that:

$$\alpha_C : \begin{cases} P(C) & \longrightarrow & \mathrm{Sieve}(C) \\ x & \longmapsto & \{f : X \to C \mid P(f)(x) \in Q(X)\} \end{cases}$$

but I am having trouble finding out that this defines a pullback. I leave the proof for now. □

Proposition 12.14. *Let $\mathscr{C}$ be a small category. Then* $\mathbf{PSh}\,(\mathscr{C})$ *is a topos.*

Proof. By Corollary 12.3, a presheaf category has all small limits; in particular, it has all finite limits. By Lemma 12.9, a presheaf category has all exponentials. Finally, by Lemma 12.13, it has a subobject classifier. Consequently, $\mathbf{PSh}\,(\mathscr{C})$ deserves its title of topos. □

[5]Note that this name is not standard.

13. David's riddles

Proposition 13.1. *Consider $\mathscr{C}$ and its slice $\mathscr{C}/C$.*

The coproduct of $a_1 : A_1 \to C$ and $a_2 : A_2 \to C$ (objects of the slice) is $a_1 + a_2 : A_1 + A_2 \to C$, where $a_1 + a_2$ is the unique arrow from the UMP of the coproduct.

Proof. The proof is mainly diagram chase.

Let $A_1 + A_2$ be the coproduct of A_1 and A_2, the following diagram commutes for a unique arrow u:

$$\begin{tikzcd} & C & \\ A_1 \arrow[ur, "a_1"] \arrow[r, "i_1"'] & A_1 + A_2 \arrow[u, "u"] & A_2 \arrow[ul, "a_2"'] \arrow[l, "i_2"] \end{tikzcd} \tag{32}$$

This ensures that the inclusion maps i_1 and i_2 actually are arrows in $\mathscr{C}/C$.

We denote this $u = a_1 + a_2$. We now have to check that this actually defines a coproduct. Let P as in the diagram:

$$\begin{tikzcd} & & A_1 + A_2 \arrow[dl, "v"'] \arrow[ddd, dashed, "a_1+a_2"'] & \\ A_1 \arrow[urr, "i_1"] \arrow[dr, "p_1"] \arrow[ddrr, bend right, "a_1"'] & & & A_2 \arrow[ul, "i_2"'] \arrow[dll, "p_2"] \arrow[ddl, "a_2"] \\ & P \arrow[dr, "p"] & & \\ & & C & \end{tikzcd}$$

The fact that $A_1 + A_2$ is a coproduct in $\mathscr{C}$ gives that unique arrow $v : A_1 + A_2 \to P$ such that $v \circ i_1 = p_1$ and $v \circ i_2 = p_2$. We than have to check that v actually is an arrow in $\mathscr{C}/C$, that is, that $p \circ v = a_1 + a_2$. In fact, $a_1 + a_2$ is the unique arrow $A_1 + A_2 \to C$ that makes Diagram 32 commute; $p \circ v$ also does, so the equality must hold. □

Proposition 13.2 (Coproduct of pullbacks is a pullback)**.** *Let $\mathscr{C}$ be locally Cartesian closed.*

Consider the following diagram:

$$\begin{tikzcd} A_1 \arrow[r, "b_1"] \arrow[d, "a_1"'] & B_1 \arrow[d, "d_1"] & & A_2 \arrow[r, "b_2"] \arrow[d, "a_2"'] & B_2 \arrow[d, "d_2"] & & A_1 + A_2 \arrow[r, "b_1+b_2"] \arrow[d, "a_1+a_2"'] & B_1 + B_2 \arrow[d, "d_1+d_2"] \\ C \arrow[r, "c"'] & D \arrow[r, phantom, "+"] & & C \arrow[r, "c"'] & D \arrow[r, phantom, "\Rightarrow"] & & C \arrow[r, "c"'] & D \end{tikzcd}$$

If the left-hand squares are pullbacks, then the right-hand square is a pullback.

Proof. The category $\mathscr{C}$ is locally Cartesian closed. Then the pullback functor $c^* : \mathscr{C}/D \to \mathscr{C}/C$ based on $c : C \to D$ has a right adjoint. Thus, it is a left adjoint. By Proposition 8.10, left adjoints preserve colimits, so in particular, coproducts. In consequence:

$$c^*(d_1 + d_2) = c^*(d_1) + c^*(d_2) = a_1 + a_2 \tag{33}$$

which gives the result. □

14. To do

Des sections en plus :

1. Monads and comonads (juste après les adjoints). C'est assez facile, il y a plein d'exemples.
2. Kan extensions
3. Finir le bestiaires sur les examples of adjunctions
4. Finir la section sur les sheaf topoi
5. Bestiaire de catégories ? (Lister des catégories importantes, est citer leurs propriétés: par exemple, Sets: a les limites, les colimites, est un topos, a un subobject classifier, y'a une épi-mono factorisation, donner des équivalences... Et donner une source pour chaque propriété)

Les exemples d'adjonctions:

1. L'adjonction $\exists \dashv ?? \dashv \forall$
2. I like this exercise
3. Infinite chain of adjoint functors : (un peu compliqué)
4. Une chaine rigolote d'adjoints : cf cahier, le 15/05/2019.
5. Encore plus d'exemples d'adjunctions
6. Beautiful examples of adjunctions
7. Adjunctions in fundamental theorems

Sur les topoi:

1. Natural numbers objects + consturction of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$... Cf David's draft on discrete time behaviour type (given while at MIT): the corresponding sheaves are constant, except for reals.
2. Présenter les topoi de sheaves, sur une topologie, sur une Heyting algebra.
3. Préciser topos $\supset$ presheaf topos $\supset$ sheaf topos = Gorthendieck topos (+ presheaf topos is the sheaf topos for the trivial topology)
4. Bon. J'adore les topoi, du coup peutêtre que je ne me rends pas compte si je veux en rajouter trop ou pas. Donc voilà. Une fois qu'on a un topos (de presheaves par exemple), on peut définir une logique. Puis on peut définir des modalities (local operators?) qui sont des arrows $\Omega \to \Omega$ qui vérifient quelques propriétés. Chaque modality définit un subtopos dont la logique vérifie cette modalité (cf. Seven sketches, chap 7) Cf aussi ce post

Autres:

1. ETCS ? Elementary Theory of the Category of Sets

Sur la forme :

1. Donner des noms aux théorèmes (dire en gros ce qu'ils disent)
2. Présentation d'une catégorie pour les enfants ?

3. Harmoniser les notations: est-ce que la catégorie par défaut est $\mathscr{C}$ ou $\mathscr{X}$? Est-ce que l'object par défaut est C ou X ? Est-ce que l'adjunction par défaut est U^* ou F ?

4. Comment lire un diagramme commutatif

5. Mettre des exemples partout!

   (a) A locally Cartesian closed category that is not Cartesian closed (LH, category of local homeomorphismes, doesn't have a terminal object), cf https://en.wikipedia.org/wiki/Cartesian_closed_category#Examples

   (b) Examples of topoi: FinSets, V_ω, V_α for α limit ordinal (small topos), cf: https://math.stackexchange.com/questions/116023/example-of-a-small-topos

   (c) The category of topological spaces lacks some properties: https://ncatlab.org/nlab/show/nice+category+of+spaces

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Symbols

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