Elementary Logic

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github

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Information Update

Five Logicians Walk into a Bar

- **Waiter:** Do you all want beer?
- **1:** I don't know.
- **2:** I don't know.
- **3:** I don't know.
- **4:** I don't know.
- **5:** No.

The information content of a formula A is the set $\mathrm{Mod}(A)$ of its models. An update with new information B reduces the current set of models $\mathrm{Mod}(A)$ to the overlap of $\mathrm{Mod}(A)$ and $\mathrm{Mod}(B)$.

Unfaithful Husband Puzzle

Problem (Unfaithful Husband Puzzle)

1. *Every man in a village of* 100 *married couples has cheated on his wife.*
2. *Every wife in the village knows about the fidelity of every man in the village except for her own husband.*
3. *One day, the queen visits and announces that at least one husband has been unfaithful, and that any wife who discovers his husband's infidelity must kill him that very day.*
4. *What happens?*

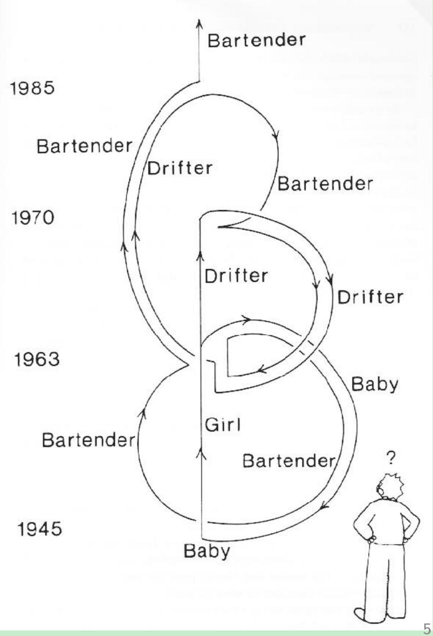
After a date, one says to the other: "Would you like to come up to my apartment to see my etchings?"

Test

Guess what $2/3$ of the average of your guesses will be, where the numbers are restricted to the real numbers between 0 and 100.

Predestination — All You Zombies — Heinlein

1945 一女婴被弃孤儿院。1963 长大后的女孩与一男子邂逅、怀孕。男子失踪。女孩产下一女婴后发现自己是双性人。女婴被偷。伤心的她变性成他。开始酗酒。1970 一酒保把他招募进时光穿梭联盟。为报复负心男，酒保带他飞回 1963。他邂逅一女孩并使其怀孕。酒保乘时光机前行 9 个多月偷走女婴，并将其送至 1945 的孤儿院，然后回 1963 把他带到 1985 的联盟基地。他受命飞回 1970，化装酒保去招募一个酒鬼。



Problem

周迅的前男友窦鹏是窦唯的堂弟；窦唯是王菲的前老公；周迅的前男友宋宁是高原的表弟；高原是窦唯的前任老婆；周迅的前男友李亚鹏是王菲的现任老公；周迅的前男友朴树的音乐制作人是张亚东；张亚东是王菲的前老公窦唯的妹妹窦颖的前老公，也是王菲的音乐制作人；张亚东是李亚鹏前女友瞿颖的现男友。

下列说法不正确的是:

1. 王菲周迅是情敌关系
2. 瞿颖王菲是情敌关系
3. 窦颖周迅是情敌关系
4. 瞿颖周迅是情敌关系

Gateway to Heaven

Problem (天堂之路)

- 你面前有左右两人守卫左右两门。
- 一人只说真话，一人只说假话。
- 一门通天堂，一门通地狱。
- 你只能向其中一人提一个“是/否”的问题。
- 怎么问出去天堂的路?

Hardest Logic Puzzle Ever

Problem (Hardest Logic Puzzle Ever)

- *Three gods, A, B, and C are called in some order, T, F, and R.*
- *T always speaks truly, F always speaks falsely (if he is certain he can; but if he is unable to lie with certainty, he responds like R), but whether R speaks truly or falsely (or whether R speaks at all) is completely random.*
- *Your task is to determine the identities of A, B, and C by asking 2 (3) yes/no questions; each question must be put to exactly one god.*
- *The gods understand English, but will answer in their own language, in which the words for 'yes' and 'no' are 'da' and 'ja' in some order. You don't know which word means which.*

HLPE — Solution

Solution (assume T and F can't predict R's answer)

1. *Directed to A:*
   Would you answer 'ja' to the question of whether you would answer with a word that means 'yes' in your language to the question of whether you and B would give the same answer to the question whether '$1+1=2$'? Q
2. *Directed to A or B we now know not to be R:*
   $Q[C/B]$

Solution (assume T and F can predict R's answer)

1. *Directed to A:*
   Would you answer 'ja' to the question of whether either:
   - *B isn't R and you are F, or*
   - *B is R and you would answer 'da' to Q?* Q
2. *Directed to A or B we now know not to be R:*
   $Q[C/B, Q'/Q]$ Q'

Contents

Outline

- Critical Thinking ✓
- History
- Term Logic
- Propositional Logic ✓
- Predicate Logic ✓
- Modal Logic
- Set Theory

Readings

1. P. J. Hurley: A Concise Introduction to Logic. — B
2. H. de Swart: Pilosophical and Mathematical Logic. — P
3. P. Smith: An Introduction to Formal Logic. — P
4. *P. Smith: Teach Yourself Logic.* — P
5. J. van Benthem: Logic in Action. — P
6. Open Logic Project. — P
7. H. Enderton: A Mathematical Introduction to Logic. — **L**
8. H. Ebbinghaus, J. Flum, W. Thomas: Mathematical Logic. — L
9. A. Nerode, R. A. Shore: Logic for Applications. — C
10. Yuri Manin: A Course in Mathematical Logic for Mathematicians.— M

Readings, Movies and More

Hofstadter's Law

It always takes longer than you expect, even when you take into account Hofstadter's Law.

- D. Hofstadter: Gödel, Escher, Bach
- Dangerous Knowledge
- The Imitation Game
- Philosophical Logic
- Philosophy of Logic
- Philosophy of Mathematics

- libgen
- sci-hub
- XX-Net
- ghelper
- Google Cloud
- JJQQKK

Advanced Readings

- Modal Logic
  - J. van Benthem: Modal Logic for Open Minds
  - P. Blackburn, M. de Rijke, Y. Venema: Modal Logic
- Set Theory
  - T. Jech: Set Theory
  - K. Kunen: Set Theory
- Recursion Theory
  - R. I. Soare: Turing Computability
  - A. Nies: Computability and Randomness
  - M. Li, P. Vitányi: An Introduction to Kolmogorov Complexity and Its Applications
- Model Theory
  - D. Marker: Model Theory
  - C. C. Chang, H. J. Keisler: Model theory
- Proof Theory
  - G. Takeuti: Proof Theory

Exams and Credits

- Question
- Discussion
- Exercises/Homework ✓
- Examination ✓
- Presentation
- Paper
- Techniques e.g. LaTeX / Coq ...
- ...

Homework

Google/Wikipedia/Stanford Encyclopedia/Internet Encyclopedia

- Leibniz, Cantor, Frege, Russell, Hilbert, Gödel, Tarski, Turing.
- finite, infinite, syntax, semantics, formal system, deduction, logical consequence, consistency, satisfiability, validity, soundness, completeness, compactness, decidability
- Philosophy of Logic, Philosophical Logic
- Logicism, Formalism, Intuitionism
- Hilbert's program
- Church-Turing thesis

Aim

- Critical thinking ✓
- Formalization of an argument ✓
- Demonstration of the validity of an argument ✓
- Object & Meta-language / Syntax & Semantics / Finite & Infinite / Countable & Uncountable / Induction & Recursion / Truth & Proof / Axiomatization / Theory / Soundness / Completeness / Compactness / Elementary Equivalent & Isomorphism / Representability / Definability / Categoricity / Decidability / Complexity / Expressiveness / Succinctness / Interpretability . . . $\surd$
- Formal Philosophy
- Understanding of the nature of mathematics
- Application in Math / CS / AI / Linguistics / Cognition / Physics / Information Theory / Game Theory / Social Science . . .
- Mathematical Logic

Contents

Logic vs other Disciplines

- Logic vs (Analytic) Philosophy.
  sense & reference / extension & intension / use & mention / truth & provability / mutual vs distributed vs common knowledge / knowledge update / belief revision / preference change / information flow / action & strategy / multi-agent interaction / counterfactual / causation / possible world / cross-world identity / essentialism / induction / ontological commitment / concept analysis / laws of thought / strength & limitation / paradoxes ...
  Peirce, Frege, Russell, Wittgenstein, Ramsey, Carnap, Quine, Putnam, Kripke, Chomsky, Gödel, Tarski, Turing ...
- Logic vs Mathematics.
  Logicism / Formalism / Intuitionism / Constructivism / Finitism / Structuralism / Homotopy Type Theory
- Logic vs Computer Science.

$$\frac{\text{Logic}}{\text{Computer Science}} \approx \frac{\text{Calculus}}{\text{Physics}}$$

Logic vs other Disciplines

- Logic vs Linguistics.
  Syntax, Semantics and Pragmatics of Natural Language
  Parsing as deduction (Lambek calculus)
- Logic vs Economics and Social Sciences.
  Epistemic Game Theory
  Social Choice Theory
  Decision Theory
- . . .

Logic vs CS

- Computer Architecture.
  Logic gates and digital circuit design ≈ Propositional Logic
- Programming Languages.
  Semantics of programming languages via methods of logic
  LISP ≈ λ-calculus
  Prolog ≈ First Order Logic + Recursion
  Typing ≈ Type Theory
- Theory of Computation and Computational Complexity.
  Models of computation (Turing machines, finite automata)
  Logic provides *complete problems* for complexity classes.
  Logical characterizations of complexity classes
  Descriptive Complexity
- General Problem Solver (SAT solvers).
- Automated Theorem Proving.

Logic vs CS

- Knowledge representation via logic rules.
- Common sense reasoning via Non-monotonic Logic.
- Fuzzy Control vs Fuzzy Logic and Multi-valued Logic.
- Relational Databases.
  SQL $\approx$ First Order Logic + Syntactic Sugar
- Software Engineering (Formal Specification and Verification).
  Extensive use of formal methods based on logic
  Temporal Logic, Dynamic Logic and Automata, Hoare Logic, Model Checking
- Multi-agent Systems.
  Epistemic Logic
- Semantic Web.
  Web Ontology Language (OWL) $\approx$ Description Logic

Branches of Logic

Mathematical Logic

- **First Order Logic**
- Set Theory
- Model Theory
- Proof Theory
- Recursion Theory

Computational Logic

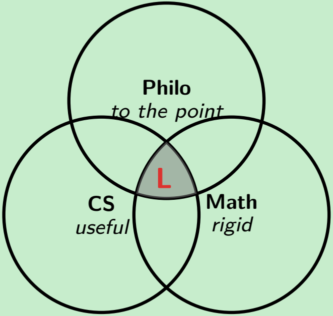
- Automata Theory
- Computational Complexity
- Finite Model Theory
- Model Checking
- Lambda Calculus
- Categorical Logic
- (Homotopy) Type Theory
- Theorem Proving
- Description Logic
- Dynamic Logic
- Temporal Logic
- Hoare Logic
- Inductive Logic
- Fuzzy Logic
- Non-monotonic Logic
- Computability Logic
- Default Logic
- Situation Calculus

Philosophical Logic

- Intuitionistic Logic
- Algebraic Logic
- Quantum Logic
- Modal Logic
- Epistemic Logic
- Doxastic Logic
- Preference Logic
- Provability Logic
- Hybrid Logic
- Free Logic
- Conditional Logic
- Relevance Logic
- Linear Logic
- Paraconsistent Logic
- Intensional Logic
- Partial Logic
- Diagrammatic Logic
- Deontic Logic

$\nabla(\odot\cdot\odot) = \odot\nabla\odot + \odot\nabla\odot$

- Logic is
  1. mainly philosophy by subject matter
  2. mainly mathematics by methodology
  3. mainly computer science by applications
- Logicians always want to be
  1. Philosophers of philosophers
  2. Mathematicians of mathematicians
  3. Computer scientists of computer scientists
- However, they often end up being
  1. Mathematicians to philosophers
  2. Computer scientists to mathematicians
  3. Philosophers to computer scientists



> *Between theology and science there is a no man's land, exposed to attack from both sides; this no man's land is philosophy.*
>
> — *Russell*

> *Philosophy is a 'catalyst' or 'spice' which makes the interdisciplinary mixture work. 'Philosophy-internal' issues seem like intellectual black holes: they absorb a lot of clever energy, but nothing ever seems to come out.*
>
> — *van Benthem*

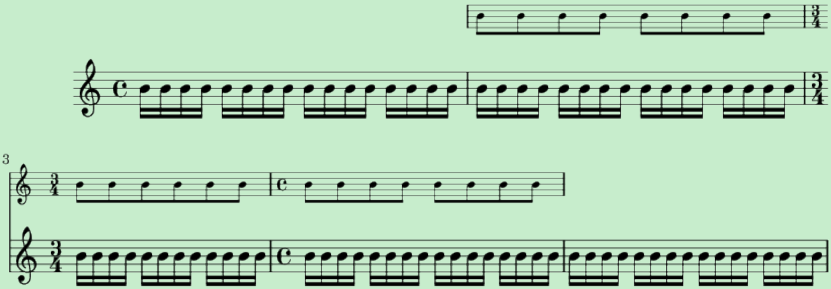
- Philosophy is a game with objectives and no rules.
- Logic is a game with rules and no objectives.

Logic is like love; a simple idea, but it can get complicated.

- 这 TM 也用证?
- 这 TM 也能证?

> *If Church says it's obvious, then everybody has seen it half an hour ago. If Weyl says it's obvious, von Neumann might be able to prove it. If Lefschetz says it's obvious, it's false.*
>
> — *Rosser*

The Music of Reason

How to *express* your thoughts precisely and succinctly?

The glory of the human spirit!

What are the extent and limits of reason?

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Aristotle(384-322 BC) — Term Logic

- Three Modes of Persuasion in Rhetoric: Ethos, Pathos, and Logos.
- Term Logic.
- Aristotle believed that any logical argument can, in principle, be broken down into a series of applications of a small number of syllogisms.
- Four Causes: material/formal/efficient/final

Sophistic vs Valid Argument

1. Nothing exists;
2. Even if something exists, nothing can be known about it;
3. Even if something can be known about it, knowledge about it can't be communicated to others;
4. Even if it can be communicated, it can't be understood.

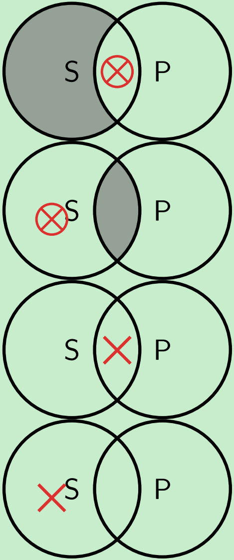
All men are mortal
Socrates is a man
Socrates is mortal

A: All S are P.

E: No S are P.

I: Some S are P.

O: Some S are not P.



Syllogism

$$\frac{\begin{array}{c}M\text{—}P\\ S\text{—}M\end{array}}{S\text{—}P} \qquad \frac{\begin{array}{c}P\text{—}M\\ S\text{—}M\end{array}}{S\text{—}P} \qquad \frac{\begin{array}{c}M\text{—}P\\ M\text{—}S\end{array}}{S\text{—}P} \qquad \frac{\begin{array}{c}P\text{—}M\\ M\text{—}S\end{array}}{S\text{—}P}$$

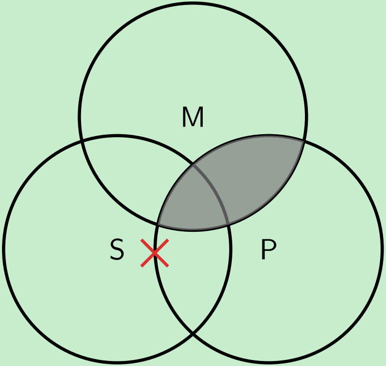
- Major term: the predicate of the conclusion.
- Minor term: the subject of the conclusion.
- Middle term: the third term
- Major premise: The premise that contains the major term
- Minor premise: The premise that contains the minor term

- 4 figure, $4^3 \times 4 = 256$ forms.
- 15 Boolean valid.
- 24 Aristotelean valid. (Existential Import)
- How to determine the valid syllogisms?
  1. Venn Diagrams
  2. Rules
  3. Boolean Algebra
  4. Axiomatization

Venn Diagram — Boolean Standpoint

1. label the circles of a three-circle Venn diagram with the syllogism's three terms.
2. diagram the two premises, and diagram the universal premise first if there is one universal and one particular.
3. in diagramming a particular proposition, put an × on a line if the premises do not determine on which side of the line it should go.
4. inspect the diagram to see if it supports the conclusion.

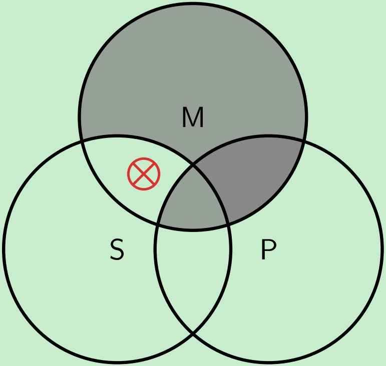
$$\begin{array}{c}\text{No } P \text{ are } M\\ \text{Some } S \text{ are not } M\\ \hline \text{Some } S \text{ are } P\end{array}$$



Venn Diagram — Aristotelean Standpoint

1. if a syllogism having universal premises and a particular conclusion is not valid from the Boolean standpoint, look to see if there is a Venn circle that is completely shaded except for one area. If there is, enter a ⊗ in that area.
2. if the syllogistic form is conditionally valid, determine if the ⊗ represents something that exists.

$$\begin{array}{c}\text{No } M \text{ are } P\\ \text{All } M \text{ are } S\\ \hline \text{Some } S \text{ are not } P\end{array}$$



Syllogistic Rules

	P undistributed	P distributed
S distributed	**A**: All S are P.	**E**: No S are P.
S undistributed	**I**: Some S are P.	**O**: Some S are not P.

1. the middle term must be distributed at least once.
2. any term that is distributed in the conclusion must be distributed in the premises.
3. the number of negative premises must be equal to the number of negative conclusions.
4. a particular conclusion requires a particular premise. (Existential Fallacy)

- Aristotle $1 – 3$
- Boole $1 – 4$

Example and Criticism

$$\begin{array}{c}\text{All men are intelligent}\\ \text{Women are not men}\\ \hline \text{Women are not intelligent}\end{array}$$

$$\begin{array}{c}\text{John does not read books}\\ \text{Students who like to learn read books}\\ \hline \text{John does not like to learn}\end{array}$$

$$\begin{array}{c}\text{Nothing is better than money}\\ \text{Philosophy is better than nothing}\\ \hline \text{Philosophy is better than money}\end{array}$$

$$\begin{array}{c}\text{Only man is rational}\\ \text{No woman is a man}\\ \hline \text{No woman is rational}\end{array}$$

$$\begin{array}{c}\text{No professors are ignorant}\\ \text{All ignorant people are vain}\\ \hline \text{No professors are vain}\end{array}$$

$$\begin{array}{c}\text{Everyone loves my baby}\\ \text{My baby loves only me}\\ \hline \text{I am my baby}\end{array}$$

Deduction/Induction/Abduction/Examplification

$$\begin{array}{c} M \longrightarrow P \\ S \longrightarrow M \\ \hline S \longrightarrow P \end{array}$$

$$\begin{array}{c} M \longrightarrow P \\ M \longrightarrow S \\ \hline S \longrightarrow P \end{array}$$

$$\begin{array}{c} H \longrightarrow E \\ \phantom{H \longrightarrow{}} E \\ \hline H \end{array}$$

$$\begin{array}{c} P \longrightarrow M \\ S \longrightarrow M \\ \hline S \longrightarrow P \end{array}$$

$$\begin{array}{c} H \longrightarrow E \\ \top \longrightarrow E \\ \hline \top \longrightarrow H \end{array}$$

$$\begin{array}{c} P \longrightarrow M \\ M \longrightarrow S \\ \hline S \longrightarrow P \end{array}$$

Abduction

1. 观察到恒星光谱红移。
2. 如果恒星在退行，那么恒星光谱红移就可以解释。
3. 如果整个宇宙在膨胀，那么恒星在离我们而去。
4. 如果宇宙起源于大爆炸，那么宇宙就会膨胀。
5. 因此，宇宙起源于大爆炸。

Leibniz 1646-1716

Don't argue. Calculate!

- Principle of Contradiction: Nothing can be and not be, but everything either is or is not.
- Principle of Sufficient Reason: Nothing is without a reason.
- Principle of Perfection: The real world is the best of all possible worlds.

In the beginning was the Logic.

As God calculates, so the world is made.

Leibniz

- The last “universal genius”, developed Calculus, refined binary number system, invented mechanical calculator that could perform addition, subtraction, multiplication and division.
- Leibniz was claimed (by Russell, Euler, Gödel, Weiner, Mandelbrot, Robinson, Chaitin) to be a precursor of *mathematical logic, topology, game theory, cybernetic theory, fractal geometry, non-standard analysis, algorithmic information theory and digital philosophy*.
- Wolfram: “Leibniz had the idea of encoding logical properties using numbers. He thought about associating every possible attribute of a thing with a prime number, then characterizing the thing by the product of the primes for its attributes — and then representing logical inference by arithmetic operations.”

Leibniz's Dream — Deduction

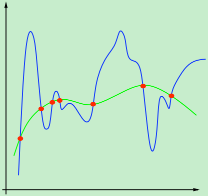
1. Characteristica Universalis & Calculus Ratiocinator.
   - i the coordination of knowledge in an encyclopedia — collect all present knowledge so we could sift through it for what is fundamental. With the set of ideas that it generated, we could formulate the *characteristica universalis*. (which form the alphabet of human thought).
   - ii ***characteristica universalis*** — a universal ideal language whose rules of composition directly expresses the structure of the world.

$$\text{sign} \rightleftarrows \text{idea}$$

$$\text{encyclopedia} \Rightarrow \text{fundamental principles} \Rightarrow \text{primitive notions}$$

   - iii ***calculus ratiocinator*** — the arrangement of all true propositions in an axiomatic system.
   - iv decision procedure. — an algorithm which, when applied to any formula of the *characteristica universalis*, would determine whether or not that formula were true. — a procedure for the rapid enlargement of knowledge. replace reasoning by computation. the art of invention. free mind from intuition.
   - v a proof that the *calculus ratiocinator* is consistent.

Leibniz's Dream — Induction



2. Compute all descriptions of possible worlds that can be expressed with the primitive notions. And the possible worlds will all have some propensity to exist.
3. Compute the probabilities of disputed hypotheses relative to the available data. As we learn more our probability assignments will asymptotically tend to a maximum for the real world, i.e., the possibility with the highest actual propensity.

Characteristica Universalis vs Calculus Ratiocinator

1. Characteristica Universalis — a universal language of human thought whose symbolic structure would reflect the structure of the world.
2. Calculus Ratiocinator — a method of symbolic calculation which would mirror the processes of human reasoning.

Characteristica Universalis	**Calculus Ratiocinator**
Language as Medium	Language as Calculus
Semantics is ineffable	Semantics is possible
Interpretation can't be varied	Interpretation can be varied
Model theory impossible	Model theory possible
Only one world can be talked about	Possible worlds are possible
Only one domain of quantifiers	Domains of quantifiers can be different
Ontology is the central problem	Ontology conventional
Logical truths are about this world	Logical truth as truth in all possible worlds

Characteristica Universalis vs Calculus Ratiocinator

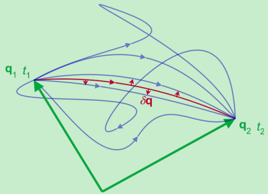
- For the *characteristica universalis* tradition, there is only one kind of human thinking logic must reflect. The meanings of the expressions of the language can't be defined. Its semantics can't be defined in that language itself without circularity, for this semantics is assumed in all its uses, and it can't be defined in a metalanguage, because there is no such language beyond our actual working language. A kind of one-world assumption is implicit in the idea of language as the universal medium.
- The *calculus ratiocinator* tradition applies logic "locally" leaving it up to the user to determine the universe of discourse in every concrete application, while the *characteristica universalis* tradition tends to apply logic to the fixed metaphysical universe that is supposed to include *all* that there is.

Leibniz's Metaphysics and Quantum Mechanics

Monadology	**Path Integral**
Amount of existence	Square of probability amplitude
Measure of necessity of individual possibility	Probability
Collision or competition of possibilities	Interference or summation of probability amplitudes
Coexisting or compatible essences	Superposition of coherent paths
Maximal degree of existence	Observed path

$$P = |\langle q_2, t_2 | q_1, t_1 \rangle|^2 \quad \langle q_2, t_2 | q_1, t_1 \rangle = \int_{q_1}^{q_2} \varphi[q] \mathcal{D}q$$

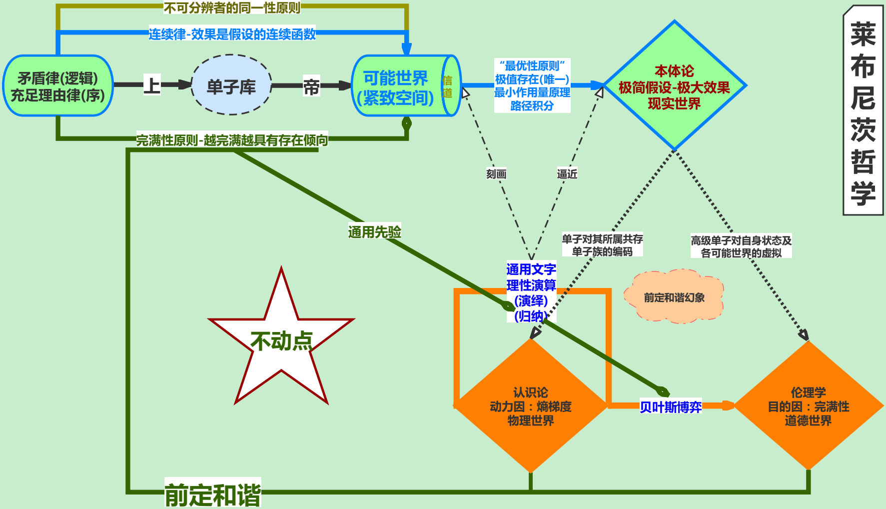
$$\varphi[q] \propto \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S[q]} \quad S[q] = \int_{t_1}^{t_2} L[q(t), \dot{q}(t)] \mathrm{d}t \quad \delta S = 0$$



- Probability of the actual path = maximum
- Action of the actual path = minimum

the absolute square of the sum of probability amplitudes over all possible paths

Leibniz's Program



不可分辨者的同一性原则

连续律-效果是假设的连续函数

矛盾律(逻辑)
充足理由律(序)

上

单子库

帝

可能世界
(紧致空间)

信
道

“最优性原则”
极值存在(唯一)
最小作用量原理
路径积分

本体论
极简假设-极大效果
现实世界

完满性原则-越完满越具有存在倾向

刻画

逼近

通用先验

单子对其所属共存
单子族的编码

高级单子对自身状态及
各可能世界的虚拟

通用文字
理性演算
(演绎)
(归纳)

前定和谐幻象

不动点

认识论
动力因：熵梯度
物理世界

贝叶斯博弈

伦理学
目的因：完满性
道德世界

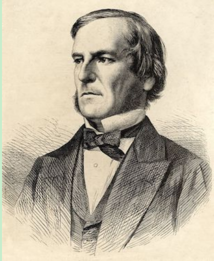
前定和谐

莱布尼茨哲学

Contents

Boole 1815-1864

- *The Laws of Thought.*
- Logic as Algebra.
- Propositional Logic.
- Algebra's strength emanates from the fact that the symbols that represent quantities and operations obey a small number of rules.



Cantor 1845-1918

- Mathematics $\rightsquigarrow$ Set Theory.
- Diagonalization.
- There are many different levels of infinity.
- Cantor set.
- Continuum Hypothesis (CH).
  How many points on the line?

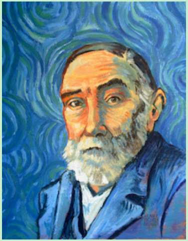
Frege 1848-1925 (+Peirce)

- *Begriffsschrift, a formal language of pure thought modelled upon that of arithmetic.*
- Predicate Logic. (Relation & Quantification)
  (Every boy loves some girl.)

$$\frac{\text{subject}}{\text{predicate}} \approx \frac{\text{argument}}{\text{function}}$$

- Philosophy of Language.
  The evening star is the morning star. (venus)

$\boxed{\text{Logicism}}$ Mathematics $\rightsquigarrow$ Logic.[1]



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[1] Frege: The Foundations of Arithmetic.

Russell 1872-1970

- Russell Paradox.
  (3^{ed} crisis of the Foundations of Mathematics)
- Theory of Descriptions.
  (The present King of France is not bald.)
- Type Theory.
- *Principia Mathematica.*

No barber shaves exactly those who do not shave themselves.[2]

---

[2] Russell: On denoting.

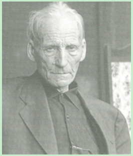
Intuitionism

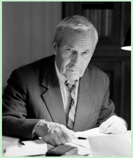
- Impredicativism. (*Poincaré*, Russell)
  Vicious circle principle: No entity can be defined only in terms of a totality to which this entity belongs.
- Intuitionism Logic $\rightsquigarrow$ Mathematics $\rightsquigarrow$ Mental construction.
  (Kronecker, *Brouwer*, Heyting, *Kolmogorov*, Weyl)
  - Potential infinity vs actual infinity.
  - To be is to be constructed by intuition.
  - Law of excluded middle.×
  - Non-constructive proof.×

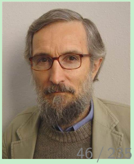
  (There exist two irrational numbers x and y s.t. x^y is rational.)

  $$\sqrt{2} \quad \log_2 9$$

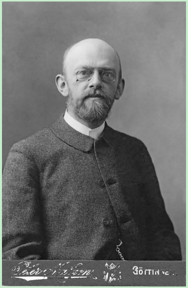
  "God created the integers, all the rest is the work of man."
- Constructive Mathematics. (Bishop, *Martin-Löf*)







Hilbert 1862-1943



- **Formal** Axiomatization of Geometry.
  The consistency of geometry relative to arithmetic.
  (Klein: Non-Euclidean relative to Euclidean)
  (natural/integer/rational/real/complex)
- Hilbert's $23/24$ problems. $(1^{\text{st}}, 2^{\text{nd}}, 10^{\text{th}}, 24^{\text{th}})$
- Meta-mathematics — Proof Theory.
- Formalism Mathematics $\rightsquigarrow$ Symbolic Game.

- Axioms are the implicit definitions of the concepts.
- One must be able to say 'table, chair, beer-mug' each time in place of 'point, line, plane'.
- Mathematics is a game played according to certain rules with meaningless marks on paper.
- We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is *no ignorabimus*.
- We must know; We will know.

Hilbert's Program

When we consider the axiomatization of logic more closely we soon recognize that the question of the consistency of the integers and of sets is not one that stands alone, but that it belongs to a vast domain of difficult epistemological questions which have a specifically mathematical tint: e.g., the problem of the solvability in principle of every mathematical question, the problem of the subsequent checkability of the results of a mathematical investigation, the question of a criterion of simplicity for mathematical proofs, the question of the relationship between content and formalism in mathematics and logic, and finally the problem of the decidability of a mathematical question in a finite number of operations.

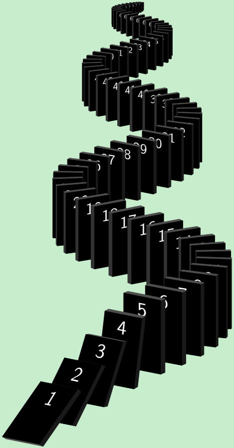
— Hilbert

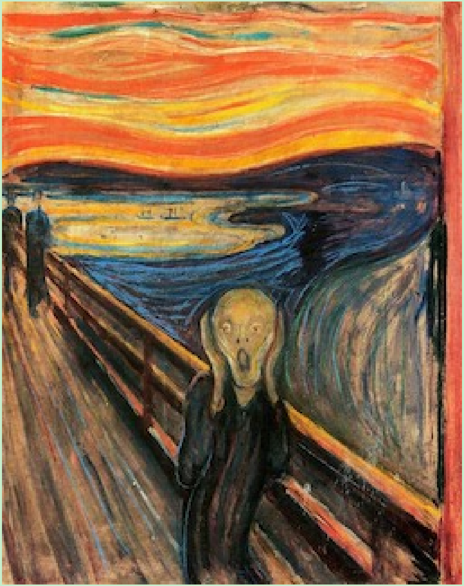
All of this that's happening now with the computer taking over the world, the digitalization of our society, of information in human society, is the result of a philosophical question that was raised by Hilbert at the beginning of the century.

— Chaitin

Contents

Leibniz & Hilbert — Dream Shattered…





Gödel 1906-1978

“I am unprovable.”[3]

- Completeness.
  I think (consistently), therefore I am.
  (Consistency implies existence.)
- Incompleteness.
  1. provable $<$ true
  2. un-self-aware
- Consistency of AC and CH.

---

[3] Gödel: On formally undecidable propositions of Principia Mathematica and related systems.

Tarski 1901-1983

“snow is white” is true iff snow is white.

“I am false.”[4]

Model Theory

Undefinability of Truth

Arithetical truth can’t be defined in arithmetic.

The theory of real closed fields / elementary geometry is complete and decidable.

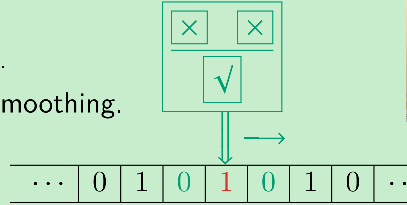
Banach-Tarski Paradox

---

[4] Tarski: On the Concept of Truth in Formalized Languages.
Tarski: The Semantic Conception of Truth and the Foundations of Semantics.

Turing 1912-1954

- Universal Turing Machine.
- Church-Turing Thesis.
- Halting Problem.
- Undecidability.
- Oracle Machine.
- Computable Absolutely Normal Number.
- Turing Test.
- Morphogenesis.
- Good-Turing Smoothing.
- Enigma.



What is "effective procedure"?[5] — Recursion Theory

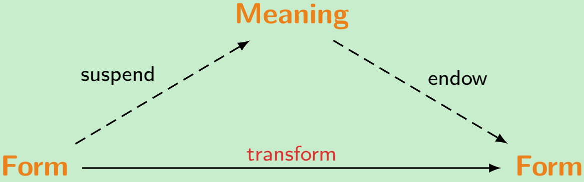
---

[5] Turing: On computable numbers, with an application to the Entscheidungsproblem.

Contents

Logic → Truth

> *Truth points the way for logic, just as beauty does for aesthetics, and goodness for ethics.*
>
> — *Frege*



Meaning

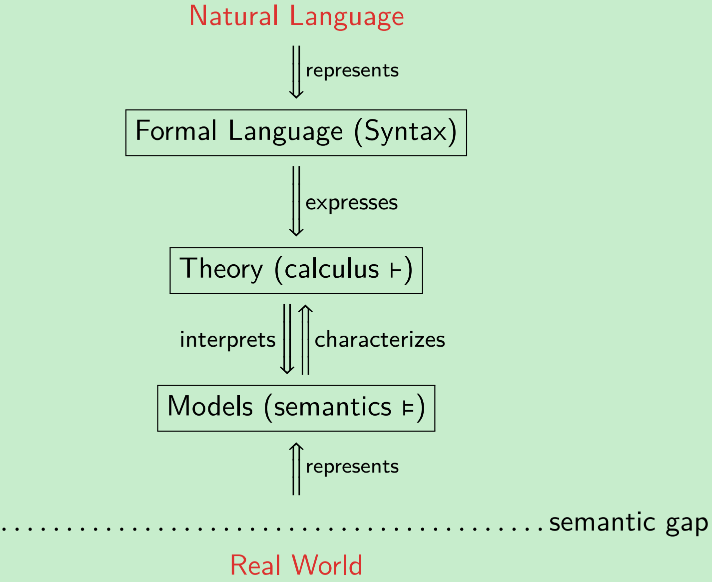
suspend

endow

Form

transform

Form



Natural Language

represents

Formal Language (Syntax)

expresses

Theory (calculus $\vdash$)

interprets

characterizes

Models (semantics $\models$)

represents

. . . . semantic gap

Real World

Contents

Propositional Logic

- Language.
  Building blocks of propositional logic language.
- Syntax.
  Propositional symbols and propositional formulae.
- Semantics.
  Assign “meaning” to propositional formulae by first assigning “meaning” to propositional symbols.
- Calculus.
  Axioms and inference rules.

Syntax

Language

$$\mathscr{L}^0 := \{\neg, \wedge, \vee, \to, \leftrightarrow, (,)\} \cup \mathcal{P}$$

where $\mathcal{P} := \{p_1, \ldots, p_n, (\ldots)\}$.

Well-Formed Formula wff

$$A ::= p \mid (\neg A) \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid (A \leftrightarrow A)$$

- $\bot := (A \wedge (\neg A))$
- $\top := (\neg \bot)$

Example

- Lily is (not) beautiful.
- If wishes are horses, then beggars will ride.
- Lily is beautiful and/or/iff 2 is not a prime number.

Well-Formed Formula

Definition (Formula-Building Operator)

$$
\begin{aligned}
\mathcal{E}_{\neg}(A) &:= (\neg A)\\
\mathcal{E}_{\wedge}(A,B) &:= (A \wedge B)\\
\mathcal{E}_{\vee}(A,B) &:= (A \vee B)\\
\mathcal{E}_{\rightarrow}(A,B) &:= (A \rightarrow B)\\
\mathcal{E}_{\leftrightarrow}(A,B) &:= (A \leftrightarrow B)
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\mathcal{E}_{\neg}(A) &:= \neg A\\
\mathcal{E}_{\wedge}(A,B) &:= \wedge AB\\
\mathcal{E}_{\vee}(A,B) &:= \vee AB\\
\mathcal{E}_{\rightarrow}(A,B) &:= \rightarrow AB\\
\mathcal{E}_{\leftrightarrow}(A,B) &:= \leftrightarrow AB
\end{aligned}
}
$$

Well-Formed Formula

Definition (Construction Sequence)

A construction sequence $(C_1, \ldots, C_n)$ is a finite sequence of expressions s.t. for each $i \leq n$ we have at least one of

$$C_i = p_i \quad \text{for some } i$$

$$C_i = (\neg C_j) \quad \text{for some } j$$

$$C_i = (C_j \star C_k) \quad \text{for some } j < i, k < i, \text{ where } \star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}.$$

Definition (Well-Formed Formula)

A formula A is a well-formed formula (wff) iff there is some construction sequence $(C_1, \ldots, C_n)$ and $C_n = A$.

$$\begin{aligned} \mathrm{wff}_0 &:= \{p_1, p_2, \ldots\} \\ \mathrm{wff}_{n+1} &:= \mathrm{wff}_n \cup \left\{(\neg A) : A \in \mathrm{wff}_n\right\} \cup \left\{(A \rightarrow B) : A, B \in \mathrm{wff}_n\right\} \\ \mathrm{wff}_* &:= \bigcup_{n \in \mathbb{N}} \mathrm{wff}_n \end{aligned}$$

Generation — Bottom Up vs Top Down

Problem

Given a class $\mathcal{F}$ of functions over U, how to generate a certain subset of U by starting with some initial elements $B \subset U$?

Bottom Up

$$\begin{aligned} C_0 &:= B \\ C_{n+1} &:= C_n \cup \bigcup_{f \in \mathcal{F}} \{f(\mathbf{x}) : \mathbf{x} \in C_n\} \qquad \deg(\mathbf{x}) := \mu n\,[\mathbf{x} \in C_n] \\ C_* &:= \bigcup_{n \in \mathbb{N}} C_n \end{aligned}$$

Top Down

- A set S is closed under a function f if for all $\mathbf{x}$: $\mathbf{x} \in S \to f(\mathbf{x}) \in S$.
- A set S is inductive if $B \subset S$ and for all $f \in \mathcal{F}$: S is closed under f.
- $C^* := \bigcap \{S : S \text{ is inductive}\}$

Bottom Up vs Top Down

How many bottles of beer can you buy with $\$10$?

- $\$2$ can buy 1 bottle of beer.
- 4 bottle caps can be exchanged for 1 bottle of beer.
- 2 empty bottles can be exchanged for 1 bottle of beer.

Generation — Bottom Up vs Top Down

Example

Let $B := \{0\}, \mathcal{F} := \{S, P\}, S(x) := x + 1, P(x) := x - 1$

$$C_* = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

There is more than one way of obtaining a member of C_*, e.g. $1 = S(0) = S(P(S(0)))$.

Theorem (Bottom up and Top down)

$$C_* = C^*$$

Proof.

$(C^* \subset C_*)$: to show C_* is inductive.
$(C_* \subset C^*)$: consider $x \in C_*$ and a construction sequence $(x_1, \dots, x_n)$ for x. First $x_1 \in B \subset C^*$. If for all $j < i$ we have $x_j \in C^*$, then $x_i \in C^*$. By induction, $x_1, \dots, x_n \in C^*$.

Induction Principle for wff

Theorem (Induction Principle)

Let P be a property of formulae, satisfying

- *every atomic formula has property P, and*
- *property P is closed under all the formula-building operations,*

then every formula has property P.

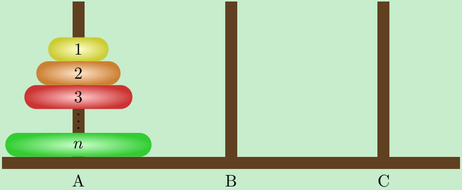
Proof.

$$\mathrm{wff}_* = \mathrm{wff}^* \subset P$$

$$P(0) \land \forall k \in \mathbb{N}(P(k) \to P(k+1)) \to \forall n \in \mathbb{N} P(n)$$

$$P(k) := P(\mathrm{wff}_k)$$

Induction vs Recursion



$P(n)$:= "n rings needs $2^n - 1$ moves."

1. If ever you leave milk one day, be sure and leave it the next day as well.
2. Leave milk today.

Leave milk today and read this note again tomorrow.

Subformula

Definition (Subformula)

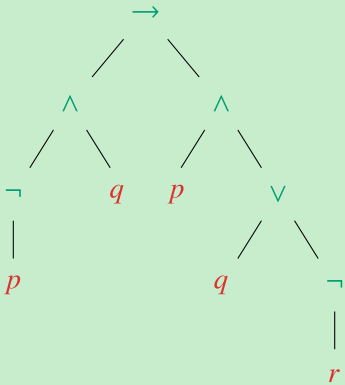
The set $\mathrm{Sub}(A)$ of subformulae of a wff A is the smallest set Γ that satisfies

1. $A \in \Gamma$
2. $\neg B \in \Gamma \implies B \in \Gamma$
3. $B \to C \in \Gamma \implies B, C \in \Gamma$

$$\mathrm{Sub}(A) \coloneqq \begin{cases} A & \text{if } A = p \\ \{A\} \cup \mathrm{Sub}(B) & \text{if } A = \neg B \\ \{A\} \cup \mathrm{Sub}(B) \cup \mathrm{Sub}(C) & \text{if } A = B \to C \end{cases}$$

Unique Readability, Unique Tree

$$((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r)))$$



subformula vs subtree

Balanced-Parentheses

Corollary (Balanced-Parentheses)

In any wff, the number of left parentheses is equal to the number of right parentheses.

Proof.

Let S be the set of “balanced” wffs.
Base step: the propositional symbols have zero parentheses.
Inductive step: obvious.

Left-Weighted-Parentheses

Lemma

Any proper initial segment of a wff contains an excess of left parentheses. Thus no proper initial segment of a wff can itself be a wff.

Proof.

Consider $A = (C \wedge D)$. The proper initial segments of $(C \wedge D)$ are the following:

1. $($
2. $(C_0$ [inductive hypothesis]
3. $(C$ [balanced-parentheses]
4. $(C\wedge$ [balanced-parentheses]
5. $(C \wedge D_0$ [inductive hypothesis]
6. $(C \wedge D$ [balanced-parentheses]

Unique Readability

Theorem (Unique Readability Theorem)

The five formula-building operations, when restricted to the set of wffs,

1. *have ranges that are disjoint from each other and from the set of proposition symbols, and*
2. *are injective.*

Proof.

To show $\mathcal{E}_\wedge$ is injective.

$$
\begin{array}{cc}
(A \wedge B) = (C \wedge D) & \\
\Downarrow & \\
A \wedge B) = C \wedge D) & \\
\Downarrow & \\
A = C & [Lemma]
\end{array}
$$

then it follows $B = D$.
Similarly, we can prove

$$(A \wedge B) \neq (C \rightarrow D)$$

Omitting Parentheses

1. The outermost parentheses need not be explicitly mentioned.
2. We order the boolean connectives according to decreasing binding strength: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.
3. Where one connective symbol is used repeatedly, grouping is to the right.

$$1 + 2 * 3$$

Contents

Assignment

- A truth assignment for $\mathscr{L}^0$ is a function

$$v : \mathcal{P} \longrightarrow \{0,1\}$$

- Such a truth assignment can be uniquely extended to $\overline{v} : \mathrm{wff} \longrightarrow \{0,1\}$ satisfying the following condition:

1. $\overline{v}(p) = v(p)$ for $p \in \mathcal{P}$
2. $\overline{v}(\neg A) = 1 - \overline{v}(A)$
3. $\overline{v}(A \wedge B) = \min\{\overline{v}(A), \overline{v}(B)\}$
4. $\overline{v}(A \vee B) = \max\{\overline{v}(A), \overline{v}(B)\}$
5. $\overline{v}(A \longrightarrow B) = 1 - \overline{v}(A) + \overline{v}(A) \cdot \overline{v}(B)$
6. $\overline{v}(A \leftrightarrow B) = \overline{v}(A) \cdot \overline{v}(B) + (1 - \overline{v}(A)) \cdot (1 - \overline{v}(B))$

Freeness vs Unique Readability

Definition

The set C is freely generated from B by a class of functions $\mathcal{F}$ iff in addition to the requirements for being generated, the following conditions hold:

1. for every $f \in \mathcal{F}$: $f\restriction_C$ is injective.
2. the range of $f\restriction_C$ for all $f \in \mathcal{F}$, and the set B are pairwise disjoint.

Recursion Theorem

Theorem (Recursion Theorem)

Assume that C is freely generated from B by $\mathcal{F}$, and for every $f \in \mathcal{F}$ we have $F_f : V^n \to V$, where $n = \mathrm{arity}(f)$. Then for every function $h : B \to V$, there exists a unique function $\overline{h} : C \to V$ s.t.

1. $\overline{h}\restriction_B = h$
2. *for all $f \in \mathcal{F}$ and all $x_1, \ldots, x_n \in C$:*

$$\overline{h}(f(x_1, \ldots, x_n)) = F_f\left(\overline{h}(x_1), \ldots, \overline{h}(x_n)\right)$$

$$\begin{tikzcd} B \arrow[r] \arrow[dr, "h"'] & C \arrow[d, dashed, "\overline{h}"] \\ & V \end{tikzcd}$$

- h tells you how to color the initial elements in B;
- F_f tells you how to convert the color of $\mathbf{x}$ into the color of $f(\mathbf{x})$.

Danger! F_f is saying “green” but F_g is saying “red” for the same point.

Truth Table & Truth/Boolean Function

p	$\neg p$
0	1
1	0

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Example

- If $0 = 1$, then Russell is God.
- Snow is white iff $1 + 1 = 2$.

Material Implication vs Cognition

Which cards must be turned over to test the idea that if a card shows an even number on one face, then its opposite face is red?

No drinking under 18!

Tautology

If lily is beautiful, then the fact that 2 is a prime number implies lily is beautiful.

p	q	$q \rightarrow p$	$p \rightarrow q \rightarrow p$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

2^n truth assignments for a set of n propositional symbols.

- $\nu \vDash A$ if $\overline{\nu}(A) = 1$.
- Logical Consequence. $\Gamma \vDash A$ if for any truth assignment ν s.t. (for all $B \in \Gamma : \nu \vDash B$) $\implies \nu \vDash A$.
- Tautology. $\vDash A$ if $\emptyset \vDash A$.

$$\models (p \to q \to r) \to (p \to q) \to p \to r$$

p	q	r	$q \to r$	$p \to q \to r$	$p \to q$	$p \to r$	$(p \to q) \to p \to r$	$(p \to q \to r) \to (p \to q) \to p \to r$
0	0	0						1
0	0	1						1
0	1	0						1
0	1	1						1
1	0	0						1
1	0	1						1
1	1	0						1
1	1	1						1

Truth Table — Simplification for Tautology

(	p	$\rightarrow$	q	$\rightarrow$	r	)	$\rightarrow$	(	p	$\rightarrow$	q	)	$\rightarrow$	p	$\rightarrow$	r
							$\underline{\mathbf{0}}$									
		1					**0**						$\underline{\mathbf{0}}$			
		1					**0**			1			**0**		$\underline{\mathbf{0}}$	
	1	1			0		**0**		1	$\underline{\mathbf{1}}$			**0**	1	**0**	0
	1	$\underline{\mathbf{1}}$	1		0		**0**		1	**1**	1		**0**	1	**0**	0
	1	**1**	1	$\underline{\mathbf{1}}$	0		**0**		1	**1**	1		**0**	1	**0**	0
				×												

Exercises — Translation

1. The answer is 3 or 6.
2. I am not good at logic.
3. If you can't say it clearly, you don't understand it yourself.
4. You understand something only if you can formalize it.
5. I will go out unless it rains.
6. You can pay by credit card or cheque.
7. Neither Sarah nor Peter was to blame for the mistake.
8. I want to buy either a new desktop computer or a laptop, but I have neither the cash nor the credit I need.
9. If I get in the lift then it breaks, and/or if you get in then the lift breaks. (?) (Natural language is ambiguous!)
10. If we both get in the lift, then the lift breaks.
11. $p \vee q \to r \mathrel{\text{⫤}\!\!\models} (p \to r) \wedge (q \to r)$
12. $p \wedge q \to r \mathrel{\text{⫤}\!\!\models} (p \to r) \vee (q \to r)$

Example

◎ô◎

1. The programmer's wife tells him: "Run to the store and pick up a loaf of bread. If they have eggs, get a dozen."
2. The programmer comes home with 12 loaves of bread.
3. "Why did you buy 12 loaves of bread!?", his wife screamed.
4. "Because they had eggs!"

- wife.

$$q \wedge (p \to r)$$

- programmer.

$$(\neg p \to q) \wedge (p \to s)$$

Exercises — Validity

1. $p \vee q \mathrel{=\!\!\!|\!\!\models} \neg p \to q \mathrel{=\!\!\!|\!\!\models} (p \to q) \to q$
2. $p \wedge q \mathrel{=\!\!\!|\!\!\models} \neg(p \to \neg q)$
3. $p \leftrightarrow q \mathrel{=\!\!\!|\!\!\models} (p \to q) \wedge (q \to p)$
4. $p \wedge q \mathrel{=\!\!\!|\!\!\models} \neg(\neg p \vee \neg q)$
5. $p \to q \to r \mathrel{=\!\!\!|\!\!\models} (p \wedge q) \to r$
6. $p \to q \mathrel{=\!\!\!|\!\!\models} \neg q \to \neg p$
7. $p \wedge (q \vee r) \mathrel{=\!\!\!|\!\!\models} (p \wedge q) \vee (p \wedge r)$
8. $p \vee (q \wedge r) \mathrel{=\!\!\!|\!\!\models} (p \vee q) \wedge (p \vee r)$
9. $\neg(p \vee q) \mathrel{=\!\!\!|\!\!\models} \neg p \wedge \neg q$
10. $\neg(p \wedge q) \mathrel{=\!\!\!|\!\!\models} \neg p \vee \neg q$
11. $p \mathrel{=\!\!\!|\!\!\models} p \vee (p \wedge q)$
12. $p \mathrel{=\!\!\!|\!\!\models} p \wedge (p \vee q)$

1. $\neg\neg p \to p$
2. $p \to \neg\neg p$
3. $p \vee \neg p$
4. $\neg(p \wedge \neg p)$
5. $p \wedge \neg p \to q$
6. $(p \to q) \wedge (\neg p \to q) \to q$
7. $(p \to q) \wedge (p \to \neg q) \to \neg p$
8. $(\neg p \to q) \wedge (\neg p \to \neg q) \to p$
9. $((p \to q) \to p) \to p$

1. $\Gamma, A \models B \iff \Gamma \models A \to B$
2. $A \mathrel{=\!\!\!|\!\!\models} B \iff \models A \leftrightarrow B$
3. $A \vee B, \neg A \vee C \models B \vee C$

$$\begin{array}{r} p \to q \\ q \\ \hline p \end{array} \qquad\qquad \begin{array}{l} p \to q \\ \neg p \\ \hline \neg q \end{array} \qquad\qquad \begin{array}{c} p \lor q \\ p \\ \hline \neg q \end{array}$$

$$\begin{array}{l} \text{I think, therefore I am} \\ \text{I do not think} \\ \hline \text{Therefore I am not} \end{array}$$

$$\begin{array}{c} \text{Mickey is murdered by Tom or Jerry} \\ \text{Tom is the killer} \\ \hline \text{Jerry is innocent} \end{array}$$

> *By all means marry; if you get a good wife, you'll be happy. If you get a bad one, you'll become a philosopher.*
>
> — *Socrates*

Example

明·浮白斋主人《雅谑》

叶衡罢相归，一日病，问诸客曰:“我且死，但未知死后佳否?”一士曰:“甚佳”。叶惊问曰:“何以知之?”答曰:**“使死而不佳，死者皆逃回矣。一死不返，以是知其佳也。”**

好货不贱，贱货不好。

痞子蔡《第一次的亲密接触》

1. 如果把整个太平洋的水倒出，也浇不灭我对你爱情的火焰。整个太平洋的水倒得出吗？不行。所以，我不爱你。
2. 如果把整个浴缸的水倒出，也浇不灭我对你爱情的火焰。整个浴缸的水倒得出吗？可以。所以，是的，我爱你。

Example

- 如果你工作，就能挣钱；如果你赋闲在家，就能悠然自在。你要么工作要么赋闲，总之，你能挣钱或者能悠然自在。
- 如果你工作，就不能悠然自在；如果你赋闲在家，就不能挣钱。你要么工作要么赋闲，总之，你不能悠然自在或者不能挣钱。

$$p \to r, q \to s \vDash p \lor q \to r \lor s$$

$$p \to \neg s, q \to \neg r \vDash p \lor q \to \neg s \lor \neg r$$

- 老婆婆有俩儿子，老大卖阳伞，老二卖雨伞，晴天雨伞不好卖，雨天阳伞不好卖……
- 被困失火的高楼，走楼梯会被烧死，跳窗会摔死……

Example

诉讼悖论

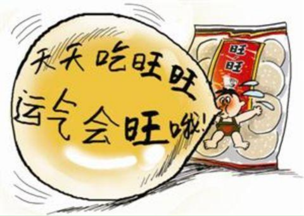
- 曾有师生签订合同：上学期间不收费，学生毕业打赢第一场官司后交学费。
- 可学生毕业后并未从事律师职业，于是老师威胁起诉学生。
- 老师说：如果我赢了，根据法庭判决，你必须交学费；如果你赢了，根据合同，你也必须交学费。要么我赢要么你赢，你都必须交学费。
- 学生说：如果我赢了，根据法庭判决，我不用交学费；如果你赢了，根据合同，我不用交学费。要么我赢要么你赢，我都不用交学费。

$$w \to p, \neg w \to p, w \vee \neg w \vDash p$$

$$w \wedge j \to p, \neg w \wedge c \to p, w \vee \neg w \overset{?}{\vDash} p$$

$$\neg w \wedge j \to \neg p, w \wedge c \to \neg p, w \vee \neg w \overset{?}{\vDash} \neg p$$

$$w \wedge j \to p, \neg w \wedge c \to p, (w \wedge j) \vee (\neg w \wedge c) \vDash p$$



The Crocodile Dilemma

The Crocodile Dilemma

I will return your child iff you can correctly predict what I will do next.

$$x = ? \implies \models (x \leftrightarrow r) \rightarrow r$$

r	$(\neg r \leftrightarrow r) \rightarrow r$
0	1
1	1

$$((r \vee \neg r) \leftrightarrow r) \rightarrow r$$

怎么得大奖?

Problem (怎么得大奖？)

- 说真话得一个大奖或一个小奖。
- 说假话不得奖。

- b: 我会得大奖。
- s: 我会得小奖。

怎么得大奖?

Problem (怎么得大奖?)

- 说真话得一个大奖或一个小奖。
- 说假话不得奖。

- b: 我会得大奖。
- s: 我会得小奖。

$$x = ? \implies \models (x \leftrightarrow b \vee s) \to b$$

b	s	$(\neg b \wedge \neg s \leftrightarrow b \vee s) \to b$	$(\neg s \leftrightarrow b \vee s) \to b$	$((s \to b) \leftrightarrow b \vee s) \to b$
0	0	1	1	1
0	1	1	1	1
1	0	1	1	1
1	1	1	1	1

Gateway to Heaven

Problem (天堂之路)

- 你面前有左右两人守卫左右两门。
- 一人只说真话，一人只说假话。
- 一门通天堂，一门通地狱。
- 你只能向其中一人提一个“是/否”的问题。
- 怎么问出去天堂的路?

$$x = ? \implies \vDash (p \to (x \leftrightarrow q)) \land (\neg p \to (x \leftrightarrow \neg q))$$

- p: 你说真话。
- q: 左门通天堂。

p	q	$(p \land q) \lor (\neg p \land \neg q)$	report	A
0	0	1	0	1
0	1	0	1	1
1	0	0	0	1
1	1	1	1	1

Proof by Contradiction

$$(\neg p \to q) \wedge (\neg p \to \neg q) \to p$$

Theorem (Euclid's Theorem)

There are infinitely many prime numbers.

Proof.

Assume to the contrary that $\mathbb{P} := \{p_1, p_2, \ldots, p_k\}$ is finite. Let $N := \prod_{i=1}^{k} p_i$.
$\exists p \in \mathbb{P} : p \mid (N+1) \ \& \ p \mid N \implies p \mid 1.$

Semantic Equivalence

- Semantic equivalence is an equivalence relation between formulae.
- Semantic equivalence is compatible with operators.

$$A \mathrel{\exists\!\models} A' \implies \neg A \mathrel{\exists\!\models} \neg A'$$

$$\left.\begin{array}{l} A \mathrel{\exists\!\models} A' \\ B \mathrel{\exists\!\models} B' \end{array}\right\} \implies A \star B \mathrel{\exists\!\models} A' \star B' \quad \text{where} \quad \star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$$

- Equivalence relation + Compatible with Operators = Congruence relation

Substitution

$$\begin{aligned}
p[C_1/p_1,\ldots,C_n/p_n] &\coloneqq \begin{cases} C_i & \text{if } p = p_i \text{ for some } 1 \le i \le n \\ p & \text{otherwise} \end{cases} \\
(\neg A)[C_1/p_1,\ldots,C_n/p_n] &\coloneqq \neg A[C_1/p_1,\ldots,C_n/p_n] \\
(A \to B)[C_1/p_1,\ldots,C_n/p_n] &\coloneqq A[C_1/p_1,\ldots,C_n/p_n] \to B[C_1/p_1,\ldots,C_n/p_n]
\end{aligned}$$

Theorem

Consider a wff A and a sequence $C_1,\ldots,C_n$ of wffs.

1. *Let ν be a truth assignment for the set of all propositional symbols. Define μ to be the truth assignment for which $\mu(p_i) = \overline{\nu}(C_i)$. Then $\overline{\mu}(A) = \overline{\nu}\left(A[C_1/p_1,\ldots,C_n/p_n]\right)$.*
2. $\vDash A \implies \vDash A[C_1/p_1,\ldots,C_n/p_n]$

Example

$$\vDash p \vee \neg p \implies \vDash (p \wedge \neg p) \vee \neg(p \wedge \neg p)$$

Duality

Theorem

Let A be a wff whose only connectives are $\neg, \wedge, \vee$. Let A^* be the result of interchanging $\wedge$ and $\vee$ and replacing each propositional symbol by its negation. Then $\neg A \mathrel{\text{⊨⊨}} A^*$.

Proof.

Prove by induction.

- $A = p_i$
- $A = \neg B$
- $A = B \wedge C$
- $A = B \vee C$

Contents

Connectives

Would we gain anything by adding more connectives to the language?

Exclusive Disjunction

$$
\begin{aligned}
\nu(p \oplus q) = \nu(p) + \nu(q) \quad \mod 2 \\
\Downarrow \\
p \oplus q &\mathrel{\text{⫤⊨}} (\neg p \wedge q) \vee (p \wedge \neg q) \\
&\mathrel{\text{⫤⊨}} (p \vee q) \wedge (\neg p \vee \neg q) \\
&\mathrel{\text{⫤⊨}} \neg(p \leftrightarrow q)
\end{aligned}
$$

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

$$\frac{p \oplus q \quad p}{\neg q} \qquad \frac{p \oplus q \quad \neg p}{q} \qquad \frac{p}{p \oplus q}\,?$$

Example

Example

Let $\#$ be a three-place proposition connective.
The interpretation of $\#$ is given by

$$\nu(\#(p,q,r)) = \left\lfloor \frac{\nu(p)+\nu(q)+\nu(r)}{2} \right\rfloor$$

then

$$\#(p,q,r) \mathrel{\text{⟚}} (p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$$

p	q	r	$\#(p,q,r)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Truth Table & Truth/Boolean Function

A truth assignment for $\mathscr{L}^0$ is a function $v : \mathcal{P} \to \{0, 1\}$.

p	$\neg p$
0	1
1	0

p	q	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$	$\cdots$
0	0	0	0	1	1	$\cdots$
0	1	0	1	1	0	$\cdots$
1	0	0	1	0	0	$\cdots$
1	1	1	1	1	1	$\cdots$

A n-place truth/Boolean function is a function $F : \{0, 1\}^n \to \{0, 1\}$.

x	$F_\neg(x)$
0	1
1	0

x	y	$F_\wedge(x, y)$	$F_\vee(x, y)$	$F_\to(x, y)$	$F_\leftrightarrow(x, y)$	$\cdots$
0	0	0	0	1	1	$\cdots$
0	1	0	1	1	0	$\cdots$
1	0	0	1	0	0	$\cdots$
1	1	1	1	1	1	$\cdots$

There are 2^{2^n} distinct truth functions with n places.

Truth Table & Truth/Boolean Function

$$v : \mathcal{P} \to \{0,1\}$$

$$F : \{0,1\}^n \to \{0,1\}$$

$v(p_1), \ldots, v(p_n)$		$x_1, \ldots, x_n$	$F_A(x_1, \ldots, x_n)$		$\overline{v}(A)$
$v_1(p_1), \ldots, v_1(p_n)$	$=$	$0, \ldots, 0$	$F_A(0, \ldots, 0)$	$=$	$\overline{v}_1(A)$
$\vdots$		$\vdots$	$\vdots$		$\vdots$
$v_{2^n}(p_1), \ldots, v_{2^n}(p_n)$	$=$	$1, \ldots, 1$	$F_A(1, \ldots, 1)$	$=$	$\overline{v}_{2^n}(A)$

Definition

Suppose A is a wff whose propositional symbols are $p_1, \ldots, p_n$.
A truth function $F : \{0,1\}^n \to \{0,1\}$ represented by A is

$$F_A(v(p_1), \ldots, v(p_n)) = \overline{v}(A)$$

$$A \mathrel{\Dashv\vDash} B \iff F_A = F_B$$

Theorem (Post1921)

Every truth function $F : \{0,1\}^n \to \{0,1\}$ *can be represented by some wff whose only connectives are* $\neg, \wedge, \vee$*.*

Proof.

$$p_i^{x_i} \coloneqq \begin{cases} p_i & \text{if } x_i = 1 \\ \neg p_i & \text{otherwise} \end{cases}$$

Case1: $F(\mathbf{x}) = 0$ for all $\mathbf{x} \in \{0,1\}^n$.
Let $A \coloneqq p \wedge \neg p$.
Case2:

$$A \coloneqq \bigvee_{\mathbf{x}:F(\mathbf{x})=1} \bigwedge_{i=1}^{n} p_i^{x_i}$$

Case1: $F(\mathbf{x}) = 1$ for all $\mathbf{x} \in \{0,1\}^n$.
Let $B \coloneqq p \vee \neg p$.
Case2:

$$B \coloneqq \bigwedge_{\mathbf{x}:F(\mathbf{x})=0} \bigvee_{i=1}^{n} p_i^{1-x_i}$$

Normal Form

Corollary

Every wff which is not a contradiction is logically equivalent to a formula of disjunctive normal form (DNF):

$$\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \pm p_{ij}$$

Corollary

Every wff which is not a tautology is logically equivalent to a formula of conjunctive normal form (CNF):

$$\bigwedge_{i=1}^{m} \bigvee_{j=1}^{n} \pm p_{ij}$$

Proof.

$$\neg A \;\text{⫤⊨}\; \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \pm p_{ij} \implies A \;\text{⫤⊨}\; \neg\left(\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \pm p_{ij}\right) \;\text{⫤⊨}\; \bigwedge_{i=1}^{m} \bigvee_{j=1}^{n} \mp p_{ij}$$

CNF Transformation

subformula	replaced by
$A \leftrightarrow B$	$(\neg A \vee B) \wedge (\neg B \vee A)$
$A \rightarrow B$	$\neg A \vee B$
$\neg(A \wedge B)$	$\neg A \vee \neg B$
$\neg(A \vee B)$	$\neg A \wedge \neg B$
$\neg\neg A$	A
$(A_1 \wedge \cdots \wedge A_n) \vee B$	$(A_1 \vee B) \wedge \cdots \wedge (A_n \vee B)$

Adequate Sets of Connectives

Definition

A set of connectives is adequate if every truth function can be represented by a wff containing only connectives from that set.

- $\{\neg, \wedge, \vee\}$
- $\{\neg, \wedge\}; \{\neg, \vee\}; \{\neg, \rightarrow\}; \{\bot, \rightarrow\}$
- $\{\uparrow\}; \{\downarrow\}$
- $\{\wedge, \vee, \rightarrow, \leftrightarrow\}; \{\neg, \leftrightarrow\}$ not adequate.

p	$\bot$
0	0
1	0

p	q	$p \uparrow q$	$p \downarrow q$
0	0	1	1
0	1	1	0
1	0	1	0
1	1	0	0

$$\begin{aligned}
\bot &\coloneqq p \wedge \neg p \\
p \uparrow q &\coloneqq \neg(p \wedge q) \\
p \downarrow q &\coloneqq \neg(p \vee q) \\
\neg p &\coloneqq p \uparrow p \\
p \wedge q &\coloneqq (p \uparrow q) \uparrow (p \uparrow q) \\
p \vee q &\coloneqq (p \uparrow p) \uparrow (q \uparrow q)
\end{aligned}$$

3-valued Logics

p	$\neg p$
0	1
u	u
1	0

$\wedge$	0	u	1
0	0	u	0
u	u	u	u
1	0	u	1

$\vee$	0	u	1
0	0	u	1
u	u	u	u
1	1	u	1

$\rightarrow$	0	u	1
0	1	u	1
u	u	u	u
1	0	u	1

$\leftrightarrow$	0	u	1
0	1	u	0
u	u	u	u
1	0	u	1

Table: Bochvar: u as “meaningless”

p	$\neg p$
0	1
u	u
1	0

$\wedge$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

$\vee$	0	u	1
0	0	u	1
u	u	u	1
1	1	1	1

$\rightarrow$	0	u	1
0	1	1	1
u	u	u	1
1	0	u	1

$\leftrightarrow$	0	u	1
0	1	u	0
u	u	u	u
1	0	u	1

Table: Kleene: u as “undefined”

p	$\neg p$
0	1
u	u
1	0

$\wedge$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

$\vee$	0	u	1
0	0	u	1
u	u	u	1
1	1	1	1

$\rightarrow$	0	u	1
0	1	1	1
u	u	1	1
1	0	u	1

$\leftrightarrow$	0	u	1
0	1	u	0
u	u	1	u
1	0	u	1

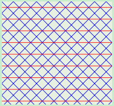
Table: Lukasiewicz: u as “possible”

Contents

Why Study Formal System?

Why truth tables are not sufficient?

- Exponential size
  - How many times would you have to fold a piece of paper($0.1mm$) onto itself to reach the Moon?
  - Common Ancestors of All Humans
    (1) Someone alive $1000BC$ is an ancestor of everyone alive today;
    (2) Everyone alive $2000BC$ is either an ancestor of nobody alive today or of everyone alive today;
    (3) Most of the people you are descended from are no more genetically related to you than strangers are.
    (4) Even if everyone alive today had exactly the same set of ancestors from $2000BC$, the distribution of one's ancestors from that population could be very different.
- Inapplicability beyond Boolean connectives.



Formal System = Axiom + Inference Rule

Axiom Schema

1. $A \rightarrow B \rightarrow A$
2. $(A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$
3. $(\neg A \rightarrow \neg B) \rightarrow (\neg A \rightarrow B) \rightarrow A$

Inference Rule

$$\frac{A \quad A \rightarrow B}{B}\ [\mathrm{MP}]$$

Deduction / Proof

This sentence can never be proved.

What is “proof”?

Definition (Deduction)

A deduction from Γ is a sequence of wff $(C_1, \ldots, C_n)$ s.t. for $k \leq n$, either

1. C_k is an axiom, or
2. $C_k \in \Gamma$, or
3. for some $i < k$ and $j < k$, $C_i = C_j \to C_k$.

- $\Gamma \vdash A$ if A is the last member of some deduction from Γ.
- $\vdash A := \emptyset \vdash A$

A mathematician’s house is on fire. His wife puts it out with a bucket of water. Then there is a gas leak. The mathematician lights it on fire.

Example

Theorem

$$\vdash p \to p$$

Proof.

1. $p \to (p \to p) \to p$ A1
2. $(p \to (p \to p) \to p) \to (p \to p \to p) \to p \to p$ A2
3. $(p \to p \to p) \to p \to p$ 1,2 MP
4. $p \to p \to p$ A1
5. $p \to p$ 3,4 MP

Example

Theorem

$$\vdash (\neg p \to p) \to p$$

Proof.

1. $(\neg p \to \neg p) \to (\neg p \to p) \to p$ A3
2. $\neg p \to \neg p$
3. $(\neg p \to p) \to p$ 1,2 MP

Example

Theorem

$$p \to q, q \to r \vdash p \to r$$

Proof.

1. $(q \to r) \to (p \to q \to r)$ A1
2. $q \to r$ Premise
3. $p \to q \to r$ 1,2 MP
4. $(p \to q \to r) \to (p \to q) \to p \to r$ A2
5. $(p \to q) \to p \to r$ 4,3 MP
6. $p \to q$ Premise
7. $p \to r$ 5,6 MP

Example — Curry's Paradox $^{\circledcirc}\hat{\text{o}}^{\circledcirc}$

If this sentence is true, then God exists.

$$p \leftrightarrow (p \to q) \vdash q$$

Proof.

1. $p \leftrightarrow (p \to q)$
2. $p \to p \to q$
3. $(p \to p) \to p \to q$
4. $p \to q$
5. p
6. q

1. 甲：如果我没说错，那么上帝存在。
2. 乙：**如果你没说错，**那么上帝存在。
3. 甲：你承认我没说错了？
4. 乙：当然。
5. 甲：可见我没说错。你已经承认：**如果我没说错，那么上帝存在。**所以，上帝存在。

$\boxed{\text{This sentence is false, and God does not exist.}}$

Curry's Paradox — How to Flirt with a Beauty $^{\heartsuit}\circledcirc^{\heartsuit}$

Smullyan Flirts with a Beauty $^{\heartsuit}\circledcirc^{\heartsuit}$

1. "I am to make a statement. If it is true, would you give me your autograph?"
2. "I don't see why not."
3. "If it is false, do not give me your autograph."
4. "Alright."
5. Then Smullyan said such a sentence that she have to give him a kiss.

$$x = ? \implies \vDash (a \leftrightarrow x) \to k$$

Hi 美女，问你个问题呗

如果我问你“你能做我女朋友吗”，那么你的答案能否和**这个问题本身**的答案一样?

Deduction Theorem

Theorem (Deduction Theorem)

$$\Gamma, A \vdash B \implies \Gamma \vdash A \to B$$

Proof.

Prove by induction on the length of the deduction sequence $(C_1, \ldots, C_n)$ of B from $\Gamma \cup \{A\}$.

Base step $n = 1$:

case1. B is an axiom. (use Axiom1.)

case2. $B \in \Gamma$.

case3. $B = A$.

Inductive step $n > 1$:

case1. B is either an axiom, or $B \in \Gamma$, or $B = A$.

case2. $C_i = C_j \to B$

$\Gamma, A \vdash C_j \implies \Gamma \vdash A \to C_j$

$\Gamma, A \vdash C_j \to B \implies \Gamma \vdash A \to C_j \to B$

$\Gamma \vdash A \to B$

Equivalent Replacement

Theorem

Suppose $B \in \mathrm{Sub}(A)$, and A^* arises from the wff A by replacing one or more occurrences of B in A by C. Then

$$B \leftrightarrow C \vdash A \leftrightarrow A^*$$

Proof.

Prove by induction on the number of connective of A.

Example

◎ô◎

1. A logician's wife is having a baby.
2. The doctor immediately hands the newborn to the dad.
3. His wife asks impatiently: "So, is it a boy or a girl"?
4. The logician replies: "yes".

- wife.

$$p?$$

- logician.

$$\left.\begin{array}{r} p \vee q \\ q \leftrightarrow \neg p \end{array}\right\} \Longrightarrow p \vee \neg p \qquad \checkmark$$

Formal System — Variant

Axiom Schema

1. $A \to B \to A$
2. $(A \to B \to C) \to (A \to B) \to A \to C$
3. $A \wedge B \to A$
4. $A \wedge B \to B$
5. $(A \to B) \to (A \to C) \to A \to B \wedge C$
6. $A \to A \vee B$
7. $B \to A \vee B$
8. $(A \to C) \to (B \to C) \to A \vee B \to C$
9. $(A \to \neg B) \to (A \to B) \to \neg A$
10. $\neg A \to A \to B$
11. $\neg\neg A \to A$

Reference Rule

$$\frac{A \quad A \to B}{B}\ [\mathrm{MP}]$$

$$p' := \neg\neg p$$

$$(A \star B)' := \neg\neg(A' \star B')$$

where $\star \in \{\wedge, \vee, \to\}$

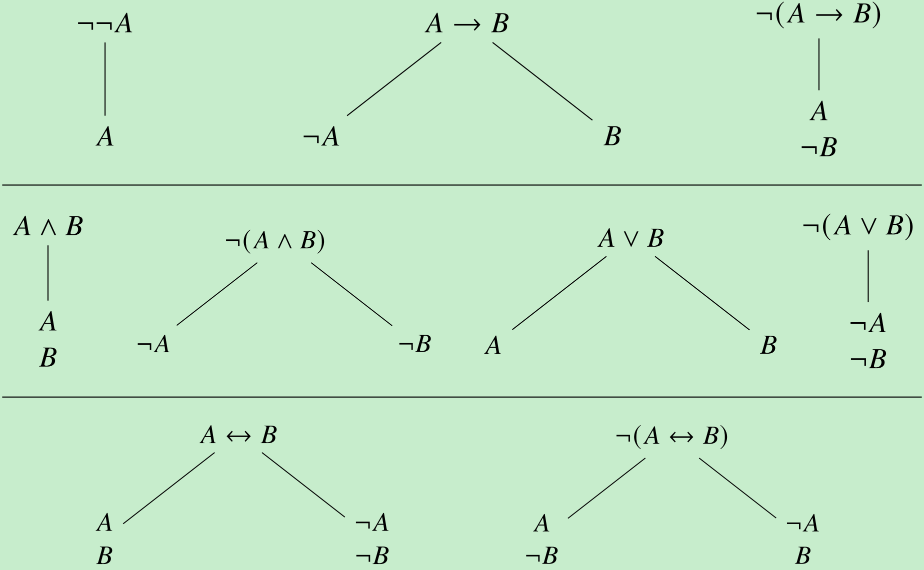
$$\Gamma' := \{A' : A \in \Gamma\}$$

$$\Gamma \vdash_{\mathrm{C}} A \iff \Gamma' \vdash_{\mathrm{I}} A'$$

1–8+MP=**P**ositive Calculus **P**+9=**M**inimal Calculus

M+10=**I**ntuitionistic Calculus **I**+11=**C**lassical Calculus

Tree Method for Propositional Logic



$\neg\neg A$

A

$A \rightarrow B$

$\neg A$ B

$\neg(A \rightarrow B)$

A
$\neg B$

$A \wedge B$

A
B

$\neg(A \wedge B)$

$\neg A$ $\neg B$

$A \vee B$

A B

$\neg(A \vee B)$

$\neg A$
$\neg B$

$A \leftrightarrow B$

A
B

$\neg A$
$\neg B$

$\neg(A \leftrightarrow B)$

A
$\neg B$

$\neg A$
B

✓

Instructions for Tree Construction

- A *literal* is an atomic formula or its negation.
- When a non-literal wff has been fully unpacked, check it with ✓

1. Start with premises and the negation of the conclusion.
2. Inspect each open path for an occurrance of a wff and its negation. If these occur, close the path with ×.
3. If there is no unchecked non-literal wff on any open path, then stop!
4. Otherwise, unpack any unchecked non-literal wff on any open path.
5. Goto ②.

- *Closed branch.* A branch is closed if it contains a wff and its negation.
- *Closed tree.* A tree is closed if all its branches are closed.
- *Open branch.* A branch is open if it is not closed and no rule can be applied.
- *Open tree.* A tree is open if it has at least one open branch.

Tactics

- Try to apply "non-branching" rules first, in order to reduce the number of branches.
- Try to close off branches as quickly as possible.

Definition (Deduction)

$A_1, \ldots, A_n \vdash B$ iff there exists a *closed tree* from $\{A_1, \ldots, A_n, \neg B\}$.

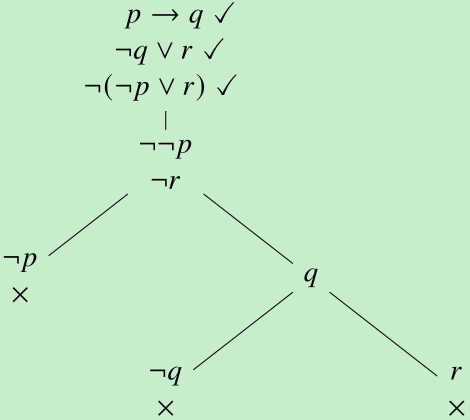
Theorem (Soundness & Completeness Theorem)

$$A_1, \ldots, A_n \vdash B \iff A_1, \ldots, A_n \vDash B$$

Remark: If an inference with propositional formulae is not valid, then its tree will have at least one open branch. The tree method can generate every counterexample of an invalid inference in propositional logic.

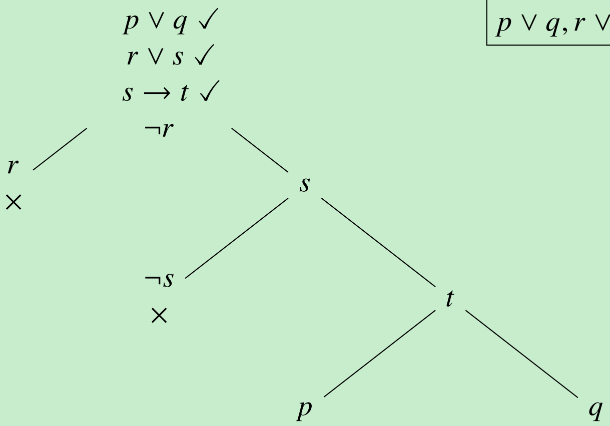
Examples — Tree Method

$\boxed{p \rightarrow q, \neg q \vee r \vdash \neg p \vee r}$



An open branch corresponds to a valuation

$$\boxed{p \vee q, r \vee s, s \to t \overset{?}{\models} r}$$



$p \vee q$ ✓

$r \vee s$ ✓

$s \to t$ ✓

$\neg r$

r

×

s

$\neg s$

×

t

p

q

$v(r) = 0, \quad v(s) = 1, \quad v(t) = 1 \quad v(p) = 1 \quad v(q) = 1$ or 0

$v(r) = 0, \quad v(s) = 1, \quad v(t) = 1 \quad v(q) = 1 \quad v(p) = 1$ or 0

$$v \models p \vee q, \quad v \models r \vee s, \quad v \models s \to t, \quad v \not\models r$$

Don't just read it; fight it!
Ask your own questions,
look for your own examples,
discover your own proofs.
Is the hypothesis necessary?
Is the converse true?
What happens in the classical special case?
What about the degenerate cases?
Where does the proof use the hypothesis?

Exercises — Tree Method

1. $p \to (\neg q \to q) \vdash p \to q$
2. $(p \to r) \land (q \to r) \vdash p \lor q \to r$
3. $(p \to q) \land (r \to s) \vdash \neg q \land r \to \neg q \land s$
4. $\Big(\big((p \to q) \to (\neg r \to \neg s)\big) \to r\Big) \to t \vdash (t \to p) \to s \to p$
5. $(p \to q) \lor (q \to r)$
6. $(p \to q) \to (\neg p \to q) \to q$
7. $((p \to q) \to p) \to p$
8. $(p \to q) \land (r \to s) \to p \lor r \to q \lor s$
9. $(p \to q) \land r \to \neg(p \land r) \lor (q \land r)$
10. $(p \leftrightarrow (p \to q)) \to q$
11. $\neg(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow q)$

Exercises — Tree Method

Decide whether the following inferences are valid or not. If not, provide a counterexample.

1. $(p \vee q) \wedge r \overset{?}{\vDash} p \vee (q \wedge r)$
2. $p \vee (q \wedge r) \overset{?}{\vDash} (p \vee q) \wedge r$
3. $p \leftrightarrow (q \rightarrow r) \overset{?}{\vDash} (p \leftrightarrow q) \rightarrow r$
4. $(p \leftrightarrow q) \rightarrow r \overset{?}{\vDash} p \leftrightarrow (q \rightarrow r)$
5. $\neg(p \rightarrow q \wedge r), r \rightarrow p \wedge q \overset{?}{\vDash} \neg r$
6. $p \rightarrow (q \wedge r), \neg(p \vee q \rightarrow r) \overset{?}{\vDash} p$
7. $p \rightarrow q, r \rightarrow s, p \vee r, \neg(q \wedge s) \overset{?}{\vDash} (q \rightarrow p) \wedge (s \rightarrow r)$
8. If God does not exist, then it's not the case that *if I pray, my prayers will be answered*; and I don't pray; so God exists.

Contents

Independence

Definition (Independence)

An axiom A in Γ is independent if $\Gamma \setminus \{A\} \nvdash A$.

Find some property that makes the axiom false and the propositions deduced from the other axioms true.

- $\nVdash A$
- for all B, $\Gamma \setminus \{A\} \vdash B \implies \Vdash B$

Theorem

Axiom3 *is independent of Axiom*1 *and Axiom*2*.*

p	$\neg p$
0	0
1	0

$\to$	0	1
0	1	1
1	0	1

Let $\nu(p) = 0$ and $\nu(q) = 1$, then $\nVdash (\neg p \to \neg q) \to (\neg p \to q) \to p$.

Independence

Axiom1 and Axiom2 axiomatizes the conditional ($\rightarrow$) fragment of intuitionistic propositional logic. To axiomatize the conditional fragment of classical logic, we also need *Peirce's law*: $((p \rightarrow q) \rightarrow p) \rightarrow p$.

Theorem

Peirce's law is independent of Axiom1 *and Axiom*2*.*

$\rightarrow$	1	2	3
1	1	2	3
2	1	1	3
3	1	1	1

Here we interpret 1 as "true", 3 as "false", and 2 as "maybe".
Let $\nu(p) = 2$ and $\nu(q) = 3$, then $\nu(((p \rightarrow q) \rightarrow p) \rightarrow p) = 2$.

Model & Semantic Consequence

- $\mathrm{Mod}(A) \coloneqq \{v : v \vDash A\}$
- $\mathrm{Mod}(\Gamma) \coloneqq \bigcap_{A \in \Gamma} \mathrm{Mod}(A)$
- $\mathrm{Th}(v) \coloneqq \{A : v \vDash A\}$
- $\mathrm{Th}(\mathcal{K}) \coloneqq \bigcap_{v \in \mathcal{K}} \mathrm{Th}(v)$
- $\mathrm{Cn}(\Gamma) \coloneqq \{A : \Gamma \vDash A\}$

- $\Gamma \subset \Gamma' \implies \mathrm{Mod}(\Gamma') \subset \mathrm{Mod}(\Gamma)$
- $\mathcal{K} \subset \mathcal{K}' \implies \mathrm{Th}(\mathcal{K}') \subset \mathrm{Th}(\mathcal{K})$
- $\Gamma \subset \mathrm{Th}(\mathrm{Mod}(\Gamma))$
- $\mathcal{K} \subset \mathrm{Mod}(\mathrm{Th}(\mathcal{K}))$
- $\mathrm{Mod}(\Gamma) = \mathrm{Mod}(\mathrm{Th}(\mathrm{Mod}(\Gamma)))$
- $\mathrm{Th}(\mathcal{K}) = \mathrm{Th}(\mathrm{Mod}(\mathrm{Th}(\mathcal{K})))$
- $\mathrm{Cn}(\Gamma) = \mathrm{Th}(\mathrm{Mod}(\Gamma))$
- $\Gamma \subset \Gamma' \implies \mathrm{Cn}(\Gamma) \subset \mathrm{Cn}(\Gamma')$
- $\mathrm{Cn}(\mathrm{Cn}(\Gamma)) = \mathrm{Cn}(\Gamma)$

Consistency & Satisfiability

- Γ is consistent if $\Gamma \nvdash \bot$.
- Γ is Post-consistent if there is some wff $A : \Gamma \nvdash A$.

$\boxed{\Gamma \text{ is consistent iff it is Post-consistent.}}$

- Γ is maximal if for every wff A, either $A \in \Gamma$ or $\neg A \in \Gamma$.
- Γ is maximal consistent if it is both consistent and maximal.
- Γ is satisfiable if $\mathrm{Mod}(\Gamma) \neq \emptyset$.
- Γ is finitely satisfiable if every finite subset of Γ is satisfiable.

- If Γ is consistent and $\Gamma \vdash A$, then $\Gamma \cup \{A\}$ is consistent.
- $\Gamma \cup \{\neg A\}$ is inconsistent iff $\Gamma \vdash A$.
- If Γ is maximal consistent, then $A \notin \Gamma \implies \Gamma \cup \{A\}$ is inconsistent.

Soundness Theorem

Theorem (Soundness Theorem)

$$\Gamma \vdash A \implies \Gamma \vDash A$$

Proof.

Prove by induction on the length of the deduction sequence.

Case1: A is an axiom. (truth table)

Case2: $A \in \Gamma$

Case3:

$$\left.\begin{array}{l}\Gamma \vDash C_j \\ \Gamma \vDash C_j \to A\end{array}\right\} \implies \Gamma \vDash A$$

Corollary

Any satisfiable set of wffs is consistent.

Compactness Theorem

Theorem (Compactness Theorem)

A set of wffs is satisfiable iff it is finitely satisfiable.

如果语言可以说无穷析取，则没有紧致性。$\left\{\bigvee_{i=1}^{\infty} p_i, \neg p_1, \neg p_2, \ldots\right\}$

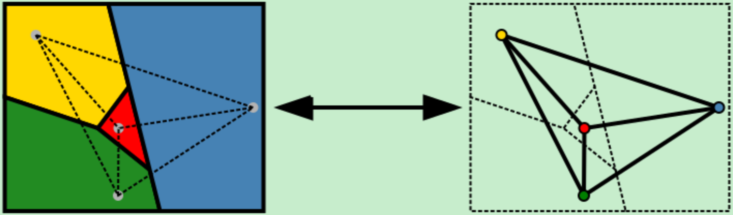
Corollary

If $\Gamma \vDash A$*, then there is a finite* $\Gamma_0 \subset \Gamma$ *s.t.* $\Gamma_0 \vDash A$*.*

Proof.

$$\begin{aligned} \Gamma_0 \nvDash A \text{ for any } \Gamma_0 \subset \Gamma &\implies \Gamma_0 \cup \{\neg A\} \text{ is satisfiable for any } \Gamma_0 \subset \Gamma \\ &\implies \Gamma \cup \{\neg A\} \text{ is satisfiable} \\ &\implies \Gamma \nvDash A \end{aligned}$$

Applications of Compactness



An infinite graph (V, E) is n-colorable iff every finite subgraph of (V, E) is n-colorable.

Proof.

Take $\left\{p_v^i : v \in V, 1 \leq i \leq n\right\}$ as the set of atoms.

$\Gamma \coloneqq \left\{p_v^1 \vee \cdots \vee p_v^n : v \in V\right\} \cup \left\{\neg\left(p_v^i \wedge p_v^j\right) : v \in V, 1 \leq i < j \leq n\right\} \cup$
$\left\{\neg\left(p_v^i \wedge p_w^i\right) : (v, w) \in E, 1 \leq i \leq n\right\}$

Proof of Compactness Theorem

Proof.

part1. Extend the finitely satisfiable set Γ to a maximal finitely satisfiable set Δ.

Let $\langle A_i : i \in \mathbb{N} \rangle$ be a fixed enumeration of the wffs.

$$\begin{aligned} \Delta_0 &:= \Gamma \\ \Delta_{n+1} &:= \begin{cases} \Delta_n \cup \{A_n\} & \text{if } \Delta_n \cup \{A_n\} \text{ is finitely satisfiable} \\ \Delta_n \cup \{\neg A_n\} & \text{otherwise} \end{cases} \\ \Delta &:= \bigcup_{n \in \mathbb{N}} \Delta_n \end{aligned}$$

part2. Define a truth assignment that satisfies Γ.

$$\nu(p) := \begin{cases} 1 & \text{if } p \in \Delta \\ 0 & \text{otherwise} \end{cases} \implies \big(\nu \vDash A \iff A \in \Delta\big)$$

“Compactness Theorem”

Theorem (“Compactness Theorem”)

Γ *is consistent iff every finite subset of* Γ *is consistent.*

Proof.

Suppose $\Gamma \vdash A$ and $\Gamma \vdash \neg A$.
Then there is a deduction sequence $(C_1, \ldots, C_n)$ of A from Γ, and a deduction sequence $(D_1, \ldots, D_m)$ of $\neg A$ from Γ.
Let $\Sigma_0 \coloneqq \{C_i \in \Gamma : 1 \leq i \leq n\}$ and $\Sigma_1 \coloneqq \{D_i \in \Gamma : 1 \leq i \leq m\}$.
The finite set $\Sigma \coloneqq \Sigma_0 \cup \Sigma_1$ is inconsistent.

Weak Completeness Theorem

Lemma

Let A *be a wff whose only propositional symbols are* $p_1, \ldots, p_n$. *Let*

$$p_i^\nu := \begin{cases} p_i & \text{if } \nu \vDash p_i \\ \neg p_i & \text{otherwise} \end{cases} \qquad A^\nu := \begin{cases} A & \text{if } \nu \vDash A \\ \neg A & \text{otherwise} \end{cases}$$

then $p_1^\nu, \ldots, p_n^\nu \vdash A^\nu$.

Weak Completeness Theorem $\vDash A \implies \vdash A$

$$\mu(p) := \begin{cases} 1 - \nu(p) & \text{if } p = p_n \\ \nu(p) & \text{otherwise} \end{cases}$$

$$\left.\begin{array}{r} p_1^\nu, \ldots, p_{n-1}^\nu, p_n^\nu \vdash A \\ p_1^\mu, \ldots, p_{n-1}^\mu, p_n^\mu \vdash A \end{array}\right\} \implies p_1^\nu, \ldots, p_{n-1}^\nu \vdash A$$

Completeness Theorem

$$\models A \iff \vdash A$$

$$+$$

Compactness

$$\Downarrow$$

$$\Gamma \models A \iff \Gamma \vdash A$$

Completeness Theorem — Post1921

Theorem (Completeness Theorem)

$$\Gamma \vDash A \implies \Gamma \vdash A$$

Corollary

Any consistent set of wffs is satisfiable.

$$\begin{tikzcd}
\Gamma \vDash A \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & \Gamma \vdash A \arrow[d, Leftrightarrow] \\
\begin{array}{c}\Gamma \cup \{\neg A\} \\ \text{unsatisfiable}\end{array} \arrow[r, Leftrightarrow] & \begin{array}{c}\Gamma \cup \{\neg A\} \\ \text{inconsistent}\end{array}
\end{tikzcd}$$

Corollary (Compactness Theorem)

A set of wffs is satisfiable iff it is finitely satisfiable.

Proof of Completeness Theorem

Proof.

step1. Extend the consistent set Γ to a maximal consistent set Δ.
Let $\langle A_i : i \in \mathbb{N} \rangle$ be a fixed enumeration of the wffs.

$$\begin{aligned}
\Delta_0 &:= \Gamma \\
\Delta_{n+1} &:= \begin{cases} \Delta_n \cup \{A_n\} & \text{if } \Delta_n \cup \{A_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg A_n\} & \text{otherwise} \end{cases} \\
\Delta &:= \bigcup_{n \in \mathbb{N}} \Delta_n
\end{aligned}$$

step2. Define a truth assignment that satisfies Γ.

$$\nu(p) := \begin{cases} 1 & \text{if } p \in \Delta \\ 0 & \text{otherwise} \end{cases} \implies \big(\nu \vDash A \iff A \in \Delta\big)$$

Decidability — Post1921

Theorem

There is an effective procedure that, given any expression, will decide whether or not it is a wff.

Theorem

There is an effective procedure that, given a finite set $\Gamma \cup \{A\}$ *of wffs, will decide whether or not* $\Gamma \vDash A$*.*

Theorem

If Γ *is a decidable set of wffs, then the set of logical consequences of* Γ *is recursively enumerable.*

Post 1897-1954

- Truth table
- Completeness of propositional logic
- Post machine
- Post canonical system
- Post correspondence problem
- Post problem

Theory & Axiomatization

What is “theory”?

- A set Γ of sentences is a theory if $\Gamma = \mathrm{Cn}(\Gamma)$.
- A theory Γ is complete if for every sentence A, either $A \in \Gamma$ or $\neg A \in \Gamma$.
- A theory Γ is axiomatizable if there is a decidable set Σ of sentences s.t. $\Gamma = \mathrm{Cn}(\Sigma)$.
- A theory Γ is finitely axiomatizable if $\Gamma = \mathrm{Cn}(\Sigma)$ for some finite set Σ of sentences.

Model Checking & Satisfiability Checking & Validity Checking[6]

- Given a model ν and a formula A. Is $\nu \vDash A$? —P
- Given a formula A. Is there a model ν s.t. $\nu \vDash A$? —NP
- Given a sentence A. Is $\vDash A$?

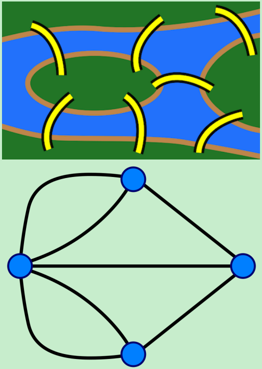


Figure: *Eulerian Circle(P)*

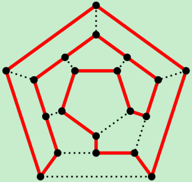


Figure: *Hamiltonian Circle(NPC)*

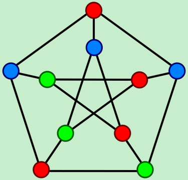
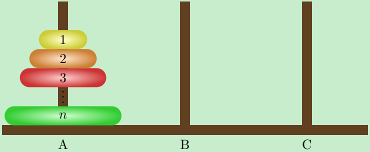


Figure: *Graph Coloring(NPC)*



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[6] Aaronson: Why philosophers should care about computational complexity.

Contents

Party and Friends

Problem

- *We want to throw a party for Tweety, Gentoo and Tux.*
- *But they have different circles of friends and dislike some.*
- *Tweety tells you that he would like to see either his friend Kimmy or not to meet Gentoo's Alice, but not both.*
- *But Gentoo proposes to invite Alice or Harry or both.*
- *Tux, however, does not like Harry and Kimmy too much, so he suggests to exclude at least one of them.*

Party and Friends

Problem

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- *Tux, however, does not like Harry and Kimmy too much, so he suggests to exclude at least one of them.*

Solution

$$(K \vee \neg A) \wedge \neg(K \wedge \neg A) \wedge (A \vee H) \wedge (\neg H \vee \neg K)$$

Sudoku

	8	**6**				**2**	**9**	
4			**1**		**5**			**8**
7				**9**				**4**
1								**9**
	5						**1**	
		8				**3**		
			5		**9**			
				2				

$p(i, j, n)$:= the cell in row i and column j contains the number n

- Every row/column contains every number.

$$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, n) \qquad \bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i, j, n)$$

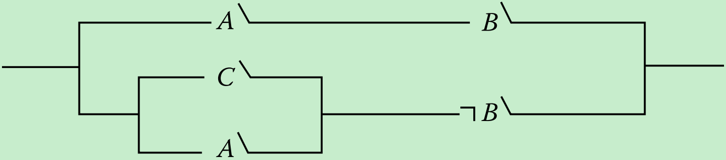
- Every 3×3 block contains every number.

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r + i, 3s + j, n)$$

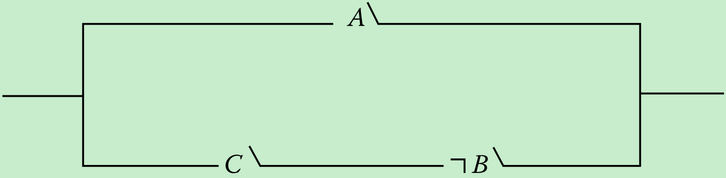
- No cell contains more than one number.
  for all $1 \leq i, j, n, n' \leq 9$ and $n \neq n'$:

$$p(i, j, n) \rightarrow \neg p(i, j, n')$$

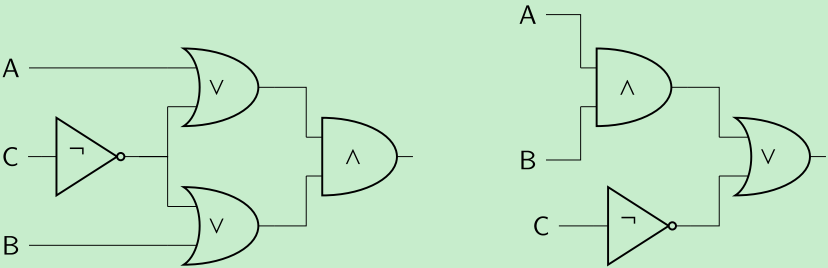
Shannon — Digital Circuit Design



$$(A \wedge B) \vee ((C \vee A) \wedge \neg B) \dashv\vdash A \vee (C \wedge \neg B)$$

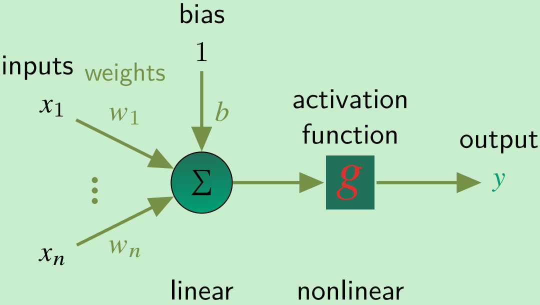


Shannon — Digital Circuit Design

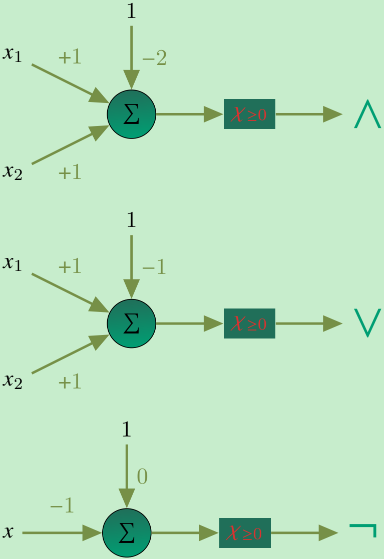


$$(A \vee \neg C) \wedge (B \vee \neg C) \boxminus (A \wedge B) \vee \neg C$$

McCulloch-Pitts Artificial Neural Network



$$y = g\left(\sum_{i=1}^{n} w_i x_i + b\right)$$



《三体》

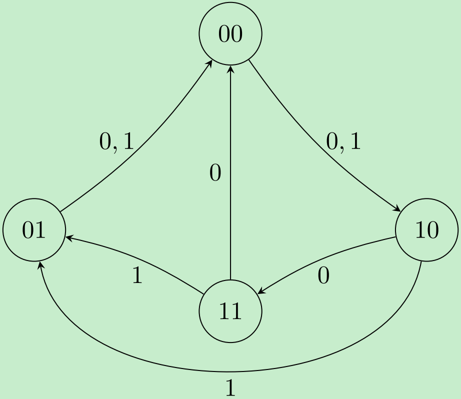
- 秦始皇：朕当然需要预测太阳的运行，但你们让我集结三千万大军，至少要首先向朕演示一下这种计算如何进行吧。
- 冯诺依曼：陛下，请给我三个士兵，我将为您演示。……
- 秦始皇：他们不需要学更多的东西了吗?
- 冯诺依曼：不需要，我们组建一千万个这样的门部件，再将这些部件组合成一个系统，这个系统就能进行我们所需要的运算，解出那些预测太阳运行的微分方程。

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

$$\begin{aligned} w_1 \cdot 0 + w_2 \cdot 0 + b &< 0 & b &< 0 \\ w_1 \cdot 0 + w_2 \cdot 1 + b &\geq 0 & w_2 + b &\geq 0 \\ w_1 \cdot 1 + w_2 \cdot 0 + b &\geq 0 & w_1 + b &\geq 0 \\ w_1 \cdot 1 + w_2 \cdot 1 + b &< 0 & w_1 + w_2 + b &< 0 \end{aligned}$$

A simple single-layer perception can't solve nonlinearly separable problems.

Finite State Automaton

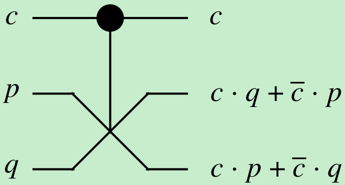


y_1	y_2	x	y_1^+	y_2^+
0	0	0	1	0
0	0	1	1	0
0	1	0	0	0
0	1	1	0	0
1	0	0	1	1
1	0	1	0	1
1	1	0	0	0
1	1	1	0	1

$$y_1^+ = \bar{y}_1\bar{y}_2 + \bar{x}\bar{y}_2$$
$$y_2^+ = y_1\bar{y}_2 + xy_1$$

Reversible Computing — Fredkin Gate: CSWAP

c	p	q	x	y	z
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	1	0
1	1	0	1	0	1
1	1	1	1	1	1



transmit the first bit unchanged and swap the last two bits iff the first bit is 1.

$$f : (c, p, q) \mapsto (c, c \cdot q + \overline{c} \cdot p, c \cdot p + \overline{c} \cdot q)$$

$$\neg \quad p = 0 \;\; \& \;\; q = 1 \implies z = \overline{c}$$

$$\wedge \quad q = 0 \implies z = c \cdot p$$

Exercise

宝藏在哪里?

你面前有三扇门，只有一扇门后是宝藏。门上各有一句话，只有一扇门上的是真话。

1. 宝藏不在这儿。
2. 宝藏不在这儿。
3. 宝藏在②号门。

- ① $\neg t_1$; ② $\neg t_2$; ③ t_2.
- 只有一扇门上的是真话。
  $(\neg t_1 \wedge \neg\neg t_2 \wedge \neg t_2) \vee (\neg\neg t_1 \wedge \neg t_2 \wedge \neg t_2) \vee (\neg\neg t_1 \wedge \neg\neg t_2 \wedge t_2)$
- 只有一扇门后是宝藏。
  $(t_1 \wedge \neg t_2 \wedge \neg t_3) \vee (\neg t_1 \wedge t_2 \wedge \neg t_3) \vee (\neg t_1 \wedge \neg t_2 \wedge t_3)$

Exercise

谁是凶手?

一起凶杀案有三个嫌疑人：小白、大黄和老王。

1. 至少有一人是凶手，但不可能三人同时犯罪。
2. 如果小白是凶手，那么老王是同犯。
3. 如果大黄不是凶手，那么老王也不是。

谁是窃贼?

1. 钱要么是甲偷的要么是乙偷的。
2. 如果是甲偷的，则偷窃时间不会在午夜前。
3. 如果乙的证词正确，则午夜时灯光未灭。
4. 如果乙的证词不正确，则偷窃发生在午夜前。
5. 午夜时没有灯光。

Exercise

哪个部落的?

一个岛上有 T、F 两个部落，T 部落的居民只说真话，F 部落的居民只说谎。你在岛上遇到了小白、大黄、老王三个土著。

1. 小白:“如果老王说谎，我或大黄说的就是真话”。
2. 大黄:“只要小白或老王说真话，那么，我们三人中有且只有一人说真话是不可能的”。
3. 老王:“小白或大黄说谎当且仅当小白或我说真话”。

我在做什么?

1. 如果我不在打网球，那就在看网球。
2. 如果我不在看网球，那就在读网球杂志。
3. 但我不能同时做两件以上的事。

Summary

- Syntax
- Semantics
- Formal System
- Expressiveness / Succinctness
- Satisfiability / Validity
- Soundness / Completeness / Compactness
- Decidability / Computational Complexity
- $\vdots$

Contents

Why Study Predicate Logic?

- Propositional logic assumes the world contains facts.
- Predicate logic assumes the world contains
  - Objects: people, houses, numbers, colors, baseball games, wars, . . .
  - Relations: red, round, prime, brother of, bigger than, part of, between, fall in love with, . . .
  - Functions: father of, best friend, one more than, plus, . . .
- Expressive power.

Example

◎ô◎

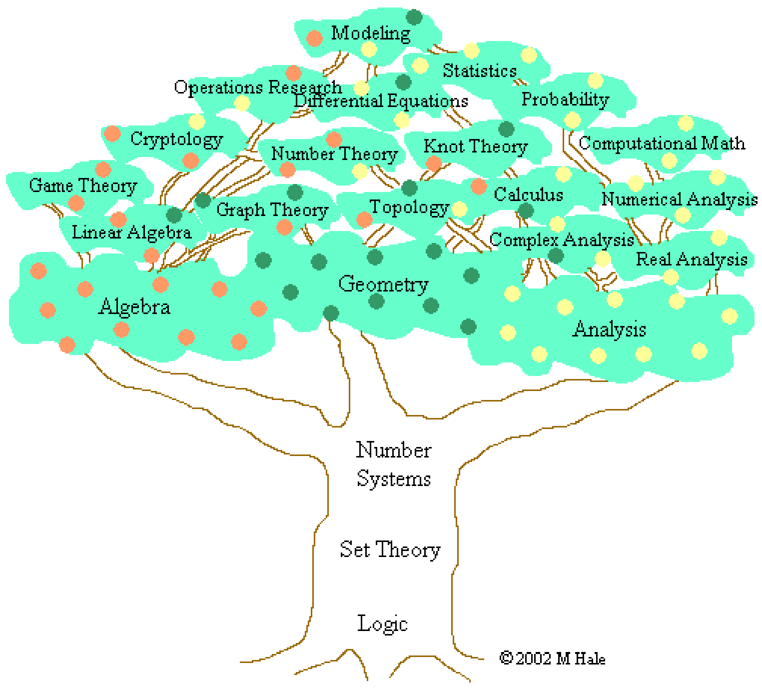
What will a logician choose: an egg or eternal bliss in the afterlife? An egg! Because nothing is better than eternal bliss in the afterlife, and an egg is better than nothing.

$$b < 0 < e \implies b < e$$

$$\neg\exists x(x > b) \implies 0 \not> b$$

◎ô◎

No cat has eight tails. A cat has one tail more than no cat. Therefore, a cat has nine tails.


Modeling
Statistics
Operations Research
Differential Equations
Probability
Cryptology
Number Theory
Knot Theory
Computational Math
Game Theory
Graph Theory
Topology
Calculus
Numerical Analysis
Linear Algebra
Complex Analysis
Real Analysis
Geometry
Algebra
Analysis
Number Systems
Set Theory
Logic
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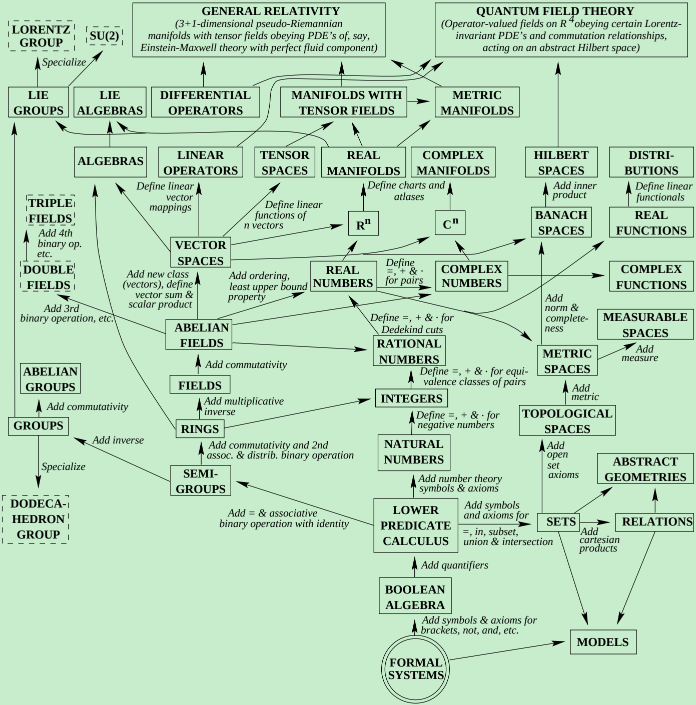


GENERAL RELATIVITY
(3+1-dimensional pseudo-Riemannian manifolds with tensor fields obeying PDE's of, say, Einstein-Maxwell theory with perfect fluid component)

QUANTUM FIELD THEORY
(Operator-valued fields on R^4 obeying certain Lorentz-invariant PDE's and commutation relationships, acting on an abstract Hilbert space)

LORENTZ GROUP

SU(2)

Specialize

LIE GROUPS

LIE ALGEBRAS

DIFFERENTIAL OPERATORS

MANIFOLDS WITH TENSOR FIELDS

METRIC MANIFOLDS

ALGEBRAS

LINEAR OPERATORS

TENSOR SPACES

REAL MANIFOLDS

COMPLEX MANIFOLDS

HILBERT SPACES

DISTRI-BUTIONS

Define linear vector mappings

Define linear functions of n vectors

Define charts and atlases

Add inner product

Define linear functionals

TRIPLE FIELDS

Add 4th binary op. etc.

DOUBLE FIELDS

$\mathbf{R^n}$

$\mathbf{C^n}$

BANACH SPACES

REAL FUNCTIONS

VECTOR SPACES

REAL NUMBERS

Define =, + & · for pairs

COMPLEX NUMBERS

COMPLEX FUNCTIONS

Add new class (vectors), define vector sum & scalar product

Add ordering, least upper bound property

Add 3rd binary operation, etc.

ABELIAN FIELDS

Define =, + & · for Dedekind cuts

Add norm & complete-ness

MEASURABLE SPACES

RATIONAL NUMBERS

METRIC SPACES

Add measure

Add commutativity

Define =, + & · for equi-valence classes of pairs

ABELIAN GROUPS

FIELDS

Add metric

Add commutativity

Add multiplicative inverse

INTEGERS

TOPOLOGICAL SPACES

GROUPS

RINGS

Define =, + & · for negative numbers

Add inverse

Add commutativity and 2nd assoc. & distrib. binary operation

NATURAL NUMBERS

Add open set axioms

ABSTRACT GEOMETRIES

Specialize

SEMI-GROUPS

Add number theory symbols & axioms

DODECA-HEDRON GROUP

Add = & associative binary operation with identity

LOWER PREDICATE CALCULUS

Add symbols and axioms for =, in, subset, union & intersection

SETS

Add cartesian products

RELATIONS

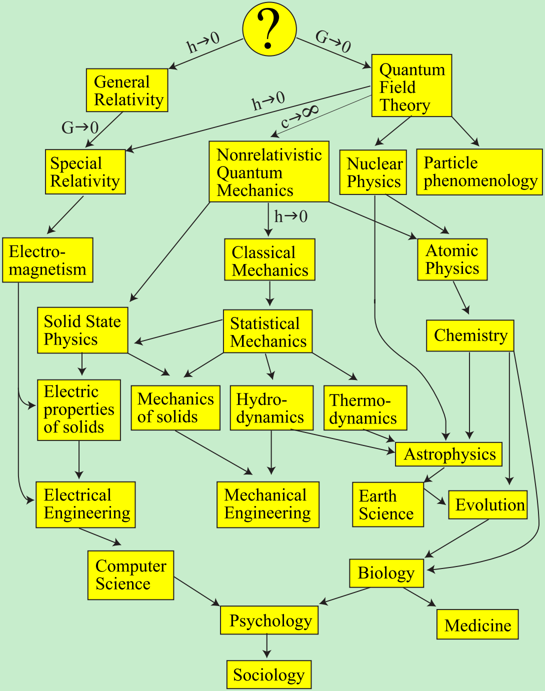
Add quantifiers

BOOLEAN ALGEBRA

Add symbols & axioms for brackets, not, and, etc.

MODELS

FORMAL SYSTEMS



?

h→0

G→0

General Relativity

Quantum Field Theory

h→0

c→∞

G→0

Special Relativity

Nonrelativistic Quantum Mechanics

Nuclear Physics

Particle phenomenology

h→0

Electro-magnetism

Classical Mechanics

Atomic Physics

Solid State Physics

Statistical Mechanics

Chemistry

Electric properties of solids

Mechanics of solids

Hydro-dynamics

Thermo-dynamics

Astrophysics

Electrical Engineering

Mechanical Engineering

Earth Science

Evolution

Computer Science

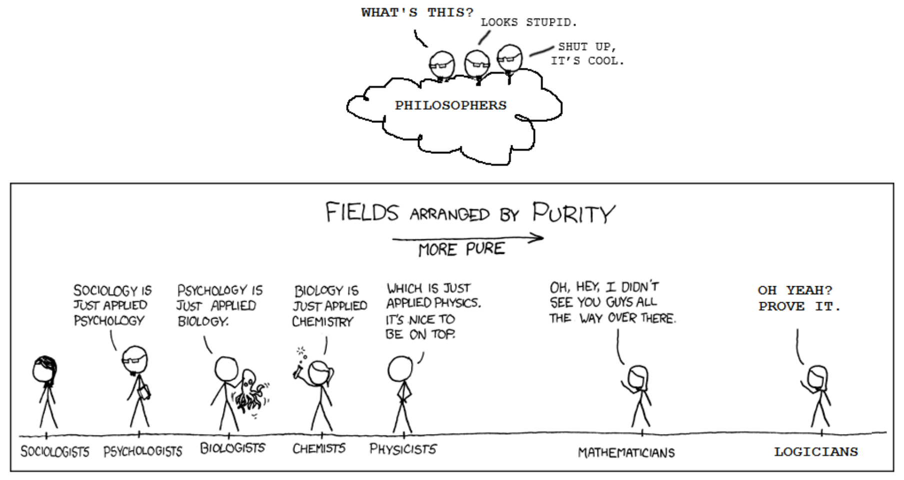
Biology

Psychology

Medicine

Sociology

Reductionism ≠ Emergence



Contents

Syntax

Language

$$\mathscr{L}^1 := \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, =, (,)\} \cup \mathcal{V} \cup \overbrace{\mathcal{F} \cup \mathcal{Q}}^{\text{signature}}$$

where

$$\mathcal{V} := \{x_i : i \in \mathbb{N}\}$$

$$\mathcal{F} := \bigcup_{k \in \mathbb{N}} \mathcal{F}^k \quad \mathcal{F}^k := \left\{f_1^k, \ldots, f_n^k, (\ldots)\right\}$$

$$\mathcal{Q} := \bigcup_{k \in \mathbb{N}} \mathcal{Q}^k \quad \mathcal{Q}^k := \left\{P_1^k, \ldots, P_n^k, (\ldots)\right\}$$

f^k is a k-place function symbol.
P^k is a k-place predicate symbol.
A 0-place function symbol f^0 is called constant.
A 0-place predicate symbol P^0 is called (atomic) proposition.

Term & Formula

Term $\mathcal{T}$

$$t ::= x \mid c \mid f(t, \ldots, t)$$

where $x \in \mathcal{V}$ and $f \in \mathcal{F}$.

- $\mathcal{T}$ is freely generated from $\mathcal{V}$ by $\mathcal{F}$.

Well-Formed Formula wff

$$A ::= \overbrace{t = t \mid P(t, \ldots, t)}^{\text{atomic formula}} \mid \neg A \mid A \land A \mid A \lor A \mid A \to A \mid A \leftrightarrow A \mid \forall x A \mid \exists x A$$

where $t \in \mathcal{T}$ and $P \in \mathcal{Q}$.

- wff is freely generated from atomic formulae by connective and quantifier operators.

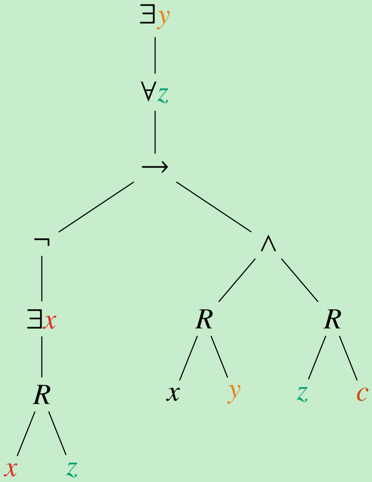
Syntax

- $A \wedge B := \neg(A \rightarrow \neg B)$
- $A \vee B := \neg A \rightarrow B$
- $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$
- $\exists x A := \neg \forall x \neg A$
- $\bot := A \wedge \neg A$
- $\top := \neg \bot$

- Bottom up and Top down definitions of terms, subterms, wffs and subformulae.
- Induction Principle for terms and wffs.
- Unique readability theorem for terms and wffs.
- Omitting Parenthesis.
  1). outermost parentheses. 2). $\neg, \forall, \exists, \wedge, \vee, \rightarrow, \leftrightarrow$ 3). group to the right.

Freedom & Bondage

$$\exists y \forall z (\neg \exists x R x z \to R x y \wedge R z c)$$



$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p^s}}$$

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

$$\left(\sum_{x \in \mathcal{X}} |P(x) - Q(x)| \right)^2 \leq 2 \sum_{x \in \mathcal{X}} P(x) \ln \frac{P(x)}{Q(x)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\mathrm{d}t = f(x)$$

$$\int_0^t \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x$$

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-2\pi \mathrm{i} x \xi} \mathrm{d}x$$

Freedom & Bondage

Definition (Free Variable of a Term)

$$\mathrm{Fv}(t) \coloneqq \begin{cases} x & \text{if } t = x \\ \emptyset & \text{if } t = c \\ \mathrm{Fv}(t_1) \cup \cdots \cup \mathrm{Fv}(t_n) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Definition (Free Variable of a wff)

$$\mathrm{Fv}(A) \coloneqq \begin{cases} \mathrm{Fv}(t_1) \cup \mathrm{Fv}(t_2) & \text{if } A = t_1 = t_2 \\ \mathrm{Fv}(t_1) \cup \cdots \cup \mathrm{Fv}(t_n) & \text{if } A = P(t_1, \ldots, t_n) \\ \mathrm{Fv}(B) & \text{if } A = \neg B \\ \mathrm{Fv}(B) \cup \mathrm{Fv}(C) & \text{if } A = B \to C \\ \mathrm{Fv}(B) \setminus \{x\} & \text{if } A = \forall x B \end{cases}$$

Freedom & Bondage

Definition (Bound Variable)

$$\mathrm{Bv}(A) \coloneqq \begin{cases} \emptyset & \text{if } A = t_1 = t_2 \\ \emptyset & \text{if } A = P(t_1, \ldots, t_n) \\ \mathrm{Bv}(B) & \text{if } A = \neg B \\ \mathrm{Bv}(B) \cup \mathrm{Bv}(C) & \text{if } A = B \rightarrow C \\ \mathrm{Bv}(B) \cup \{x\} & \text{if } A = \forall x B \end{cases}$$

- t is a ground (closed) term if $\mathrm{Fv}(t) = \emptyset$.
- A is a sentence (closed formula) if $\mathrm{Fv}(A) = \emptyset$.
- A is an open formula if $\mathrm{Bv}(A) = \emptyset$.

Example: $c = d$ is clopen.

Translation

How to ‘speak’ the language of first order logic?

1. 敌人的敌人是朋友。

$$\forall xyz(Exy \wedge Eyz \to Fxz)$$

2. 朋友之间要么都吸烟要么都不吸烟。

$$\forall xy\big(Fxy \to (Sx \leftrightarrow Sy)\big)$$

3. 既没有朋友又没有敌人是寂寞的。

$$\forall x(\neg\exists yFxy \wedge \neg\exists zExz \to Lx)$$

4. 最可怕的敌人是最亲密的朋友。

$$\forall xy\big(Exy \wedge \forall z(Exz \to Tyz) \to Fxy \wedge \forall z(Fxz \to Cyz)\big)$$

5. 如果大鱼比小鱼游得快，那么，有最大的鱼就有游得最快的鱼。
$\forall xy(Fx \wedge Fy \wedge Bxy \to Sxy) \to \exists x(Fx \wedge \forall y(Fy \to Bxy)) \to$
$\exists x(Fx \wedge \forall y(Fy \to Sxy))$

Translation

1. **A**: $\forall x(Sx \to Px)$
2. **E**: $\forall x(Sx \to \neg Px)$
3. **I**: $\exists x(Sx \land Px)$
4. **O**: $\exists x(Sx \land \neg Px)$
5. Every boy loves some girl.
$$\forall x\big(Bx \to \exists y(Gy \land Lxy)\big)$$
6. Whoever has a father has a mother.
$$\forall x(\exists y Fyx \to \exists y Myx)$$
7. Grandmother is mother's mother.
$$\forall xy\big(Gxy \leftrightarrow \exists z(Mxz \land Mzy)\big)$$
$$\forall xy\big(x = Gy \leftrightarrow \exists z(x = Mz \land z = My)\big)$$
8. There are n elements.
$$\exists x_1 \ldots x_n \left(\bigwedge_{1 \le i < j \le n} x_i \neq x_j \land \forall x \left(\bigvee_{i=1}^{n} x = x_i \right)\right)$$

Translation

1. $\text{Cogito}(i) \to \exists x(x = i)$ *Descartes*
2. $\exists x(x = i) \lor \neg\exists x(x = i)$ *Shakespeare*
3. $\forall x\big(\text{Month}(x) \to \text{Crueler}(\text{april}, x)\big)$ *Eliot*
4. $\forall x(\neg\,\text{Weep}(x) \to \neg\,\text{See}(x))$ *Hugo*
5. $\forall x\big(\text{Time}(x) \to \text{Better}(t, x)\big) \land \forall x\big(\text{Time}(x) \to \text{Better}(x, t)\big)$ *Dickens*
6. $\exists p\Big(\text{Child}(p) \land \neg\,\text{Grow}(p) \land \forall x\big(\text{Child}(x) \land x \neq p \to \text{Grow}(x)\big)\Big)$ *Barrie*
7. $\forall xy\big(Fx \land Fy \to (Hx \land Hy \to Axy) \land (\neg Hx \land \neg Hy \to \neg Axy)\big)$ *Tolstoi*
8. $\exists t\forall x\,\text{Fool}(x, t) \land \exists x\forall t\,\text{Fool}(x, t) \land \neg\forall x\forall t\,\text{Fool}(x, t)$ *Lincoln*
9. $\forall x\big(\text{Problem}(x) \land \text{Philo}(x) \land \text{Serious}(x) \leftrightarrow x = \text{suicide}\big)$ *Camus*
10. $\forall x\big(\text{Feather}(x) \land \text{Perch}(x, \text{soul}) \leftrightarrow x = \text{hope}\big)$ *Dickinson*
11. $\forall x\Big(\text{Enter}(x) \to \forall y\big(\text{Hope}(y) \to \text{Abandon}(x, y)\big)\Big)$ *Dante*
12. $\exists x\forall y\big(\text{For}(y, x) \land \text{For}(x, y)\big)?$ *Dumas*
13. $\exists x\big(\text{Fear}(\text{we}, x) \leftrightarrow x = \text{Fear}\big)?$ *Roosevelt*
14. $\forall xy(Ax \land Ay \to Exy) \land \exists xy\big(Ax \land Ay \land [\![Exx]\!] > [\![Eyy]\!]\big)?$ *Orwell*

1. Cogito, ergo sum. (I think, therefore I am.) *Descartes*
2. To be or not to be. *Shakespeare*
3. April is the cruellest month. *Eliot*
4. Those who do not weep, do not see. *Hugo*
5. It was the best of times, it was the worst of times. *Dickens*
6. All Children, except one, grow up. *Barrie*
7. All happy families are alike; each unhappy family is unhappy in it's own way. *Tolstoi*
8. You can fool all the people some of the time, and some of the people all the time, but you can't fool all the people all the time. *Lincoln*
9. There is but one truly serious philosophical problem and that is suicide. *Camus*
10. Hope is the thing with feathers that perches in the soul. *Dickinson*
11. All hope abandon, all you who enter here. *Dante*
12. One for all and all for one. *Dumas*
13. The only thing we have to fear is fear itself. *Roosevelt*
14. All animals are equal, but some animals are more equal than others. *Orwell*

Exercises — Translation

1. If you can't solve a problem, then there is an easier problem that you can't solve.
2. Men *and* women are welcome to apply.
3. *None but* ripe bananas are edible.
4. *Only* Socrates and Plato are human.
5. *All but* Socrates and Plato are human.
6. Every boy loves *at least* two girls.
7. Adams can't do *every* job right.
8. Adams can't do *any* job right.
9. *Not all* that glitters are gold.
10. Every farmer who owns a donkey is happy.
11. Every farmer who owns a donkey beats it.
12. All even numbers are divisible by 2, but *only some* are divisible by 4.

Exercises — Translation

1. Everyone alive $2000BC$ is either an ancestor of nobody alive today or of everyone alive today.
2. John hates all people who do not hate themselves.
3. No barber shaves exactly those who do not shave themselves.
4. Andy and Bob have the same maternal grandmother. $\text{mother}(x, y)$
5. Anyone who loves *two* different girls is Tony.
6. There is *exactly* one sun.
7. Socrates' wife *has* a face that *only* her mother could love.
8. If dogs are animals, every head of a dog is the head of an animal.
9. Someone *other than the girl* who loves Bob is stupid.
10. Morris only loves *the girl* who loves him.
11. *The one* who loves Alice is *the one* she loves.
12. *The shortest* English speaker loves *the tallest* English speaker.

Translation

$$\lim_{n\to\infty} a_n = a \iff \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \big(|a_n - a| < \varepsilon\big)$$

$$\lim_{x\to c} f(x) \uparrow \iff \forall y \in \mathbb{R} \exists \varepsilon > 0 \forall \delta > 0 \exists x \in \mathbb{R} \big(0 < |x-c| < \delta \wedge |f(x)-y| \geq \varepsilon\big)$$

continuity vs uniform continuity

$$\forall x \in \mathbb{R} \forall \varepsilon > 0 \exists \delta > 0 \forall y \in \mathbb{R} \big(|x - y| < \delta \to |f(x) - f(y)| < \varepsilon\big)$$

$$\forall \varepsilon > 0 \exists \delta > 0 \forall xy \in \mathbb{R} \big(|x - y| < \delta \to |f(x) - f(y)| < \varepsilon\big)$$

Translation

1. $\exists x\Big(Gx \land \forall y\big(By \land \forall z(Gz \land z \neq x \to \neg Lzy) \to Lxy\big)\Big) \to \forall x\Big(Bx \to \exists y\big(Gy \land Lyx\big)\Big)$
2. $\forall xy\Big(\big(Gx \land \forall y(By \to \neg Lxy)\big) \land \big(Gy \land \exists x(Bx \land Lyx)\big) \to \neg Lxy\Big)$

Translation

1. $\exists x\Big(Gx \land \forall y\big(By \land \forall z(Gz \land z \neq x \to \neg Lzy) \to Lxy\big)\Big) \to \forall x\Big(Bx \to \exists y\big(Gy \land Lyx\big)\Big)$
2. $\forall xy\Big(\big(Gx \land \forall y(By \to \neg Lxy)\big) \land \big(Gy \land \exists x(Bx \land Lyx)\big) \to \neg Lxy\Big)$

安得圣母爱渣男，大庇天下雄性有红颜！

相信我，我肯定能找到一种你不屑于理解的语言来试图跟你对（zhuang）话（B）的。

No girl who does not love a boy loves a girl who loves a boy.

没有一个不爱男孩的女孩会爱上一个爱着某个男孩的女孩。

女同不爱女异。

$\forall x\forall y(((Gx \land \forall v(Bv \to \neg Lxv)) \land (Gy \land \exists z(Bz \land Lyz))) \to \neg Lxy).$

Exercises — Translation

1. Only the bishop gave the monkey the banana.
2. The only bishop gave the monkey the banana.
3. The bishop only gave the monkey the banana.
4. The bishop gave only the monkey the banana.
5. The bishop gave the only monkey the banana.
6. The bishop gave the monkey only the banana.
7. The bishop gave the monkey the only banana.
8. The bishop gave the monkey the banana only.

Substitution and Substitutable

Definition (Substitution in a term/formula)

$$=, P, \neg, \to \cdots$$

$$(\forall y B)[t/x] \coloneqq \begin{cases} \forall y B[t/x] & \text{if } y \neq x \\ \forall y B & \text{if } y = x \end{cases}$$

Definition (Substitutable)

t is substitutable for x in A:

$$=, P, \neg, \to \cdots$$

$A = \forall y B$ iff either

1. $x \notin \mathrm{Fv}(A)$ or
2. $y \notin \mathrm{Fv}(t)$ and t is substitutable for x in B.

Prevent the variables in t from being captured by a quantifier in A.

$$A = \exists y(x \neq y) \quad t = y \quad A[t/x]?$$

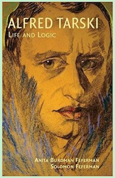
Contents

Philosophy

- No entity without identity. — Quine's standards of ontological admissibility
- To be is to be the value of a bound variable. — Quine's criterion of ontological commitments
- To be is to be constructed by intuition. — Brouwer
- To be true is to be provable. — Kolmogorov
- "p" is true iff p. — Tarski's "T-schema"

What is "truth" — Are all truths knowable?

1. *formally correct* $\forall x\big(T(x) \leftrightarrow A(x)\big)$
2. *materially adequate* $A(s) \leftrightarrow p$
   where 's' is the name of a sentence of $\mathscr{L}$, and 'p' is the translation of this sentence in $\mathscr{L}'$.



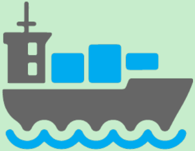
Structure

A structure over the signature is a pair $\mathcal{M} := (M, I)$, where M is a non-empty set, and I is a mapping which

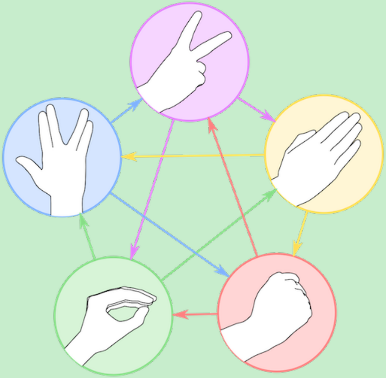
- assigns to each constant symbol c an element $I(c) \in M$,
- assigns to each function symbol f^k a k-ary function $I(f^k) : M^k \longrightarrow M$,
- assigns to each predicate symbol P^k a k-ary relation $I(P^k) \subset M^k$.

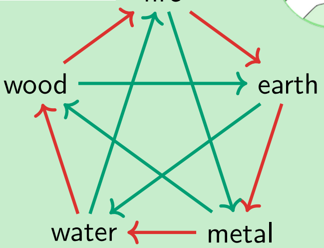
We write $\mathcal{M} = \left(M, c^{\mathcal{M}}, f^{\mathcal{M}}, P^{\mathcal{M}}\right)$ for convenience.

The 'elements' of the structure have no properties other than those relating them to other 'elements' of the same structure.



Structure





fire

wood

earth

water

metal

Interpretation

An interpretation $(\mathcal{M}, \nu)$ is a structure $\mathcal{M}$ with a variable assignment $\nu : \mathcal{V} \to M$.
We extend ν to $\overline{\nu} : \mathcal{T} \to M$ by recursion as follows:

- $\overline{\nu}(x) := \nu(x)$
- $\overline{\nu}(c) := c^{\mathcal{M}}$
- $\overline{\nu}(f(t_1, \ldots, f_n)) := f^{\mathcal{M}}(\overline{\nu}(t_1), \ldots, \overline{\nu}(t_n))$

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\overline{\nu}"] \arrow[d, "\mathcal{E}_f"'] & M \arrow[d, "f^{\mathcal{M}}"] \\
\mathcal{T} \arrow[r, "\overline{\nu}"'] & M
\end{tikzcd}$$

Tarski's Definition of Truth

Definition ($\mathcal{M}, \nu \models A$)

- $\mathcal{M}, \nu \models t_1 = t_2$ if $\overline{\nu}(t_1) = \overline{\nu}(t_2)$
- $\mathcal{M}, \nu \models P(t_1, \ldots, t_n)$ if $(\overline{\nu}(t_1), \ldots, \overline{\nu}(t_n)) \in P^{\mathcal{M}}$
- $\mathcal{M}, \nu \models \neg A$ if $\mathcal{M}, \nu \not\models A$
- $\mathcal{M}, \nu \models A \to B$ if $\mathcal{M}, \nu \not\models A$ or $\mathcal{M}, \nu \models B$
- $\mathcal{M}, \nu \models \forall x A$ if for every $a \in M : \mathcal{M}, \nu(a/x) \models A$ where

$$\nu(a/x)(y) := \begin{cases} \nu(y) & \text{if } y \neq x \\ a & \text{otherwise} \end{cases}$$

or, $\mathcal{M}, \nu \models \forall x A$ if for all $\nu' \sim_x \nu : \mathcal{M}, \nu' \models A$.
where $\nu' \sim_x \nu$ if for all $y \neq x : \nu'(y) = \nu(y)$.

> *To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, or of what is not that it is not, is true.*
>
> — *Aristotle*

Tarski's Definition of Truth

Let h map atomic formulae to variable assignments $\mathrm{P}\left(M^{\mathcal{V}}\right)$.

- $h(t_1 = t_2) = \{\nu : \overline{\nu}(t_1) = \overline{\nu}(t_2)\}$
- $h(P(t_1, \ldots, t_k)) = \left\{\nu : (\overline{\nu}(t_1), \ldots, \overline{\nu}(t_n)) \in P^{\mathcal{M}}\right\}$

We extend h to $\overline{h} : \mathrm{wff} \to \mathrm{P}\left(M^{\mathcal{V}}\right)$ by recursion as follows:

1. $\overline{h}(A) = h(A)$ for atomic A
2. $\overline{h}(\neg A) = M^{\mathcal{V}} \setminus \overline{h}(A)$
3. $\overline{h}(A \to B) = \left(M^{\mathcal{V}} \setminus \overline{h}(A)\right) \cup \overline{h}(B)$
4. $\overline{h}(\forall x A) = \bigcap_{a \in M} \left\{\nu : \nu(a/x) \in \overline{h}(A)\right\}$

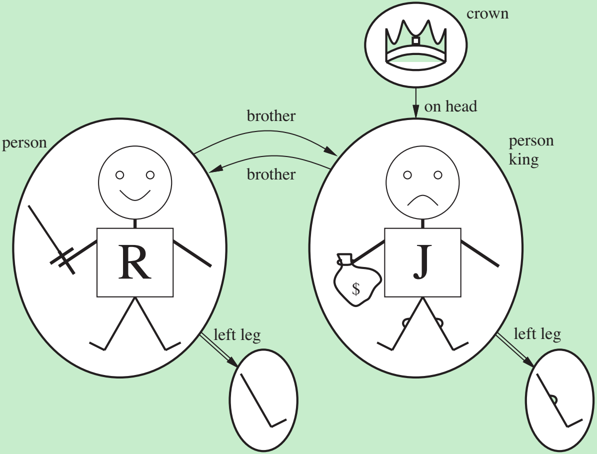
$$\mathcal{M}, \nu \vDash A \;\; := \;\; \nu \in \overline{h}(A)$$

Tarski's Definition of Truth

- $\mathcal{M} \vDash A$ if for all $v : \mathcal{M}, v \vDash A$. (True)
- $\mathcal{M}, v \vDash \Gamma$ if for all $A \in \Gamma : \mathcal{M}, v \vDash A$.
- $\mathcal{M} \vDash \Gamma$ if for all $A \in \Gamma : \mathcal{M} \vDash A$.
- $\Gamma \vDash A$ if for all $\mathcal{M}, v : \mathcal{M}, v \vDash \Gamma \implies \mathcal{M}, v \vDash A$.
- $\Gamma \vDash^* A$ if for all $\mathcal{M} : \mathcal{M} \vDash \Gamma \implies \mathcal{M} \vDash A$.
- $\vDash A$ if $\emptyset \vDash A$. (Valid)
- A is satisfiable if there exists $\mathcal{M}, v$ s.t. $\mathcal{M}, v \vDash A$.

$$Px \vDash \forall x Px \quad ?$$

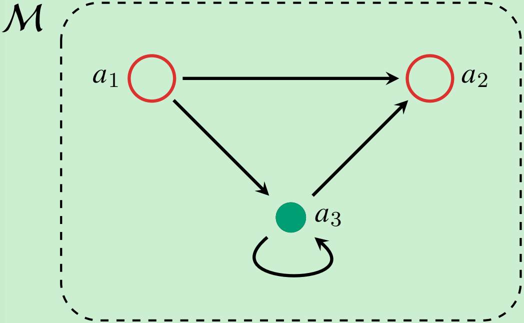
$$Px \vDash^* \forall x Px \quad ?$$



- $\text{brother}(\mathsf{Richard}, \mathsf{John})$
- $\neg\,\text{king}(\mathsf{Richard}) \rightarrow \text{king}(\mathsf{John})$
- $\text{king}(\mathsf{John}) \wedge \exists x\big(\,\text{crown}(x) \wedge \text{on-head}(x, \mathsf{John})\big)$
- $\neg\,\text{brother}\,\big(\,\text{left-leg}(\mathsf{Richard}), \mathsf{John}\big)$
- $\forall x\big(\,\text{king}(x) \rightarrow \text{person}(x)\big)$
- $\text{length}\,\big(\,\text{left-leg}\,\mathsf{Richard})\big) > \text{length}\,\big(\,\text{left-leg}(\mathsf{John})\big)$

Example

Example



- $M = \{a_1, a_2, a_3\}$
- $c^{\mathcal{M}} = a_3$
- $P^{\mathcal{M}} = \{a_1, a_2\}$
- $R^{\mathcal{M}} = \{(a_1, a_2), (a_1, a_3), (a_3, a_2), (a_3, a_3)\}$

- $c^{\mathcal{M}}$: green point
- $P^{\mathcal{M}}$: red circles
- $R^{\mathcal{M}}$: arrows

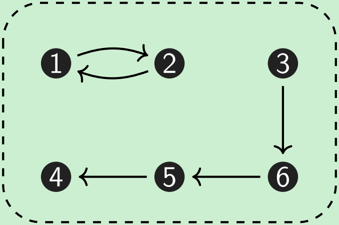
- $\mathcal{M} \overset{?}{\vDash} Pc$
- $\mathcal{M} \overset{?}{\vDash} Pc \vee Rcc$
- $\mathcal{M} \overset{?}{\vDash} \forall x(Px \vee Rxx)$
- $\mathcal{M} \overset{?}{\vDash} \exists x \forall y(y = x \vee Rxy)$
- $\mathcal{M}, \nu \overset{?}{\vDash} Rxy \longrightarrow Rcy$
  where $\nu(x) = a_1, \nu(y) = a_3$.

Example

Example

$$\forall xyz(Rxy \land Ryz \to Rxz)$$

What arrows are missing to make the following a model?



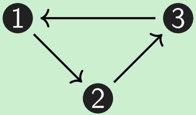
(Add only those arrows that are really needed.)

Counter Model

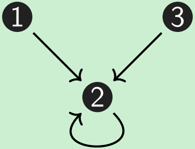
Counter Model

$\forall x \exists y Rxy \nvDash \exists x \forall y Rxy$

$\forall x \exists y Rxy \nvDash \exists y \forall x Rxy$



$\exists y \forall x Rxy \nvDash \forall y \exists x Rxy$



Is there a finite counter model?

Exercise (Counter Model)

Give a counter model for

1. $\forall x \exists y Rxy \wedge \forall xyz(Rxy \wedge Ryz \rightarrow Rxz) \not\models \exists x Rxx$
2. $\forall x \exists y Rxy \wedge \forall xyz(Rxy \wedge Ryz \rightarrow Rxz) \not\models \exists xy(Rxy \wedge Ryx)$

$$\frac{\begin{array}{c}\text{Everybody loves somebody}\\ \text{Everybody loves all persons who are loved by his loved ones}\end{array}}{\text{There is at least a pair of persons who love each other}}$$

$$(\mathbb{Z}, <)$$

Mistakes to Avoid

$$\forall x(Bx \to Sx)$$

$$\exists x(Bx \land Sx)$$

- $\forall x(Bx \land Sx)$
  Everyone is a boy and everyone is smart.
- $\exists x(Bx \to Sx)$
  It is true if there is anyone who is not a boy.

Coincidence Lemma

Lemma (Coincidence Lemma)

Assume $v_1, v_2 : \mathcal{V} \to M$*, and for all* $x \in \mathrm{Fv}(A) : v_1(x) = v_2(x)$*. Then*

$$\mathcal{M}, v_1 \vDash A \iff \mathcal{M}, v_2 \vDash A$$

- If A is a sentence, then either $\mathcal{M} \vDash A$ or $\mathcal{M} \vDash \neg A$.
- $\mathcal{M} \vDash A \implies \mathcal{M} \vDash \forall x A$
- **Notation**: If $\mathrm{Fv}(A) \subset \{x_1, \ldots, x_n\}$, then we write $\mathcal{M} \vDash A[a_1, \ldots, a_n]$ to mean $\mathcal{M}, v \vDash A$ for some (equivalently any) assignment v s.t. $v(x_i) = a_i$ for $1 \leq i \leq n$.

Substitution Lemma

Lemma (Substitution Lemma)

- $\nu\left(s[t/x]\right) = \nu(\nu(t)/x)(s)$
- *If the term t is substitutable for the variable x in the wff A, then* $\mathcal{M}, \nu \vDash A[t/x] \iff \mathcal{M}, \nu(\nu(t)/x) \vDash A$

$$\begin{tikzcd}
s \arrow[rr, "{[t/x]}"] \arrow[dr, "\overline{\nu(\overline{\nu}(t)/x)}"'] & & s[t/x] \arrow[dl, "\overline{\nu}"] \\
& M &
\end{tikzcd}$$

$$\mathscr{L}_M := \mathscr{L} \cup C_M \quad \text{where} \quad C_M := \{c_a : a \in M\}$$

$$\mathcal{M}, \nu \vDash A[c_a/x] \iff \mathcal{M}, \nu(a/x) \vDash A$$

We abbreviate $\mathcal{M}, \nu \vDash A[c_a/x]$ by $\mathcal{M}, \nu \vDash A[a]$.

$$\mathcal{M}, \nu \vDash \forall x A \iff \text{for every } a \in M : \mathcal{M}, \nu \vDash A[a]$$

Equivalent Replacement

Lemma

Suppose $B \in \mathrm{Sub}(A)$, and A^* arises from A by replacing zero or more occurrences of B by C. Then

$$\vDash B \leftrightarrow C \implies \vDash A \leftrightarrow A^*$$

Alphabetic Variant

Definition (Alphabetic Variant)

If $y \notin \mathrm{Fv}(A)$, and y is substitutable for x in A, we say that $\forall y A[y/x]$ is an alphabetic variant of $\forall x A$.

Theorem

If $\forall y A[y/x]$ is an alphabetic variant of $\forall x A$, then

$$\vDash \forall x A \leftrightarrow \forall y A[y/x]$$

If $y \notin \mathrm{Fv}(A)$, then $A[y/x][x/y] = A$.

- **Convention**: When we write $A[t/x]$ we assume that t is substitutable for x in A. — *For any formula A and a finite number of variables $y_1, \ldots, y_n$ (occurring in t), we can always find a logically equivalent alphabetic variant A^* of A s.t. $y_1, \ldots, y_n$ do not occur bound in A^*.*

Equality and Equivalence

Lemma

Suppose $\mathrm{Fv}(t) \cup \mathrm{Fv}(s) \subset \{x_1, \ldots, x_n\}$*, and* A^* *arises from the wff* A *by replacing one occurrence of* t *in* A *by* s*. Then*

$$\vDash \forall x_1 \ldots x_n (t = s) \to (A \leftrightarrow A^*)$$

$$\mathcal{M} \vDash t = s \implies \mathcal{M} \vDash A \leftrightarrow A^*$$

Lemma

Suppose $\mathrm{Fv}(B) \cup \mathrm{Fv}(C) \subset \{x_1, \ldots, x_n\}$*, and* A^* *arises from the wff* A *by replacing one occurrence of* B *in* A *by* C*. Then*

$$\vDash \forall x_1 \ldots x_n (B \leftrightarrow C) \to (A \leftrightarrow A^*)$$

$$\mathcal{M} \vDash B \leftrightarrow C \implies \mathcal{M} \vDash A \leftrightarrow A^*$$

Remark

- $\vDash \forall x(Px \leftrightarrow Qx) \to (\forall x Px \leftrightarrow \forall x Qx)$
  $\nvDash (Px \leftrightarrow Qx) \to (\forall x Px \leftrightarrow \forall x Qx)$
- $\mathcal{M}, v \vDash t = s \;\not\Longrightarrow\; \mathcal{M}, v \vDash A \leftrightarrow A^*$
- $\mathcal{M}, v \vDash B \leftrightarrow C \;\not\Longrightarrow\; \mathcal{M}, v \vDash A \leftrightarrow A^*$

$$B = Px, \quad C = Py, \quad A = \forall x Px, \quad A^* = \forall x Py$$

Valid Formulas — Example

$$\forall x A \rightarrow A[t/x]$$

$$\neg \forall x A \leftrightarrow \exists x \neg A$$

$$\forall x (A \land B) \leftrightarrow \forall x A \land \forall x B$$

$$\exists x (A \lor B) \leftrightarrow \exists x A \lor \exists x B$$

$$\forall x (A \rightarrow B) \rightarrow \forall x A \rightarrow \forall x B$$

$$\forall x y A \leftrightarrow \forall y x A$$

$$\exists x \forall y A \rightarrow \forall y \exists x A$$

$$\forall x (A \leftrightarrow B) \rightarrow (\forall x A \leftrightarrow \forall x B)$$

$$(\forall x A \rightarrow \exists x B) \leftrightarrow \exists x (A \rightarrow B)$$

$$A[t/x] \rightarrow \exists x A$$

$$\neg \exists x A \leftrightarrow \forall x \neg A$$

$$\forall x A \lor \forall x B \rightarrow \forall x (A \lor B)$$

$$\exists x (A \land B) \rightarrow \exists x A \land \exists x B$$

$$\forall x (A \rightarrow B) \rightarrow \exists x A \rightarrow \exists x B$$

$$\exists x y A \leftrightarrow \exists y x A$$

Valid Formulas — Example

$x \notin \mathrm{Fv}(A)$:

$A \leftrightarrow \forall x A$	$A \leftrightarrow \exists x A$
$\forall x(A \vee B) \leftrightarrow A \vee \forall x B$	$\exists x(A \vee B) \leftrightarrow A \vee \exists x B$
$\forall x(A \wedge B) \leftrightarrow A \wedge \forall x B$	$\exists x(A \wedge B) \leftrightarrow A \wedge \exists x B$
$\forall x(A \longrightarrow B) \leftrightarrow (A \longrightarrow \forall x B)$	$\exists x(A \longrightarrow B) \leftrightarrow (A \longrightarrow \exists x B)$
$\forall x(B \longrightarrow A) \leftrightarrow (\exists x B \longrightarrow A)$	$\exists x(B \longrightarrow A) \leftrightarrow (\forall x B \longrightarrow A)$
	$\exists x(A \longrightarrow \forall x A)$

Example

$$\boxed{\forall x A \to A[t/x]}$$

$\mathcal{M}, \nu \vDash \forall x A \implies$ for all $a \in M : \mathcal{M}, \nu(a/x) \vDash A \implies \mathcal{M}, \nu(\nu(t)/x) \vDash A$
According to Substitution Lemma, $\mathcal{M}, \nu \vDash A[t/x]$.

$$\boxed{\forall x(B \to A) \to (\exists x B \to A) \quad \text{where } x \notin \mathrm{Fv}(A)}$$

Assume $\mathcal{M}, \nu \vDash \exists x B$ and $\mathcal{M}, \nu \nvDash A$. Then there exists $a \in M$ s.t. $\mathcal{M}, \nu(a/x) \vDash B$. According to Coincidence Lemma and $x \notin \mathrm{Fv}(A)$, we have $\mathcal{M}, \nu(a/x) \nvDash A$. Therefore $\mathcal{M}, \nu(a/x) \nvDash B \to A$. This contradicts $\mathcal{M}, \nu \vDash \forall x(B \to A)$.

$$\boxed{(\exists x B \to A) \to \forall x(B \to A) \quad \text{where } x \notin \mathrm{Fv}(A)}$$

$\mathcal{M}, \nu \vDash \exists x B \to A \implies \mathcal{M}, \nu \nvDash \exists x B$ or $\mathcal{M}, \nu \vDash A$.
If $\mathcal{M}, \nu \nvDash \exists x B$, then for all $a \in M$, $\mathcal{M}, \nu(a/x) \nvDash B$. It follows that $\mathcal{M}, \nu(a/x) \vDash B \to A$. Therefore $\mathcal{M}, \nu \vDash \forall x(B \to A)$.
If $\mathcal{M}, \nu \vDash A$, then according to Coincidence Lemma and $x \notin \mathrm{Fv}(A)$, for all $a \in M$, $\mathcal{M}, \nu(a/x) \vDash A$. It follows that $\mathcal{M}, \nu(a/x) \vDash B \to A$. Therefore $\mathcal{M}, \nu \vDash \forall x(B \to A)$.

$$(\forall x B \to A) \leftrightarrow \exists x (B \to A)$$

$$\operatorname{diam}(X) \coloneqq \sup \left\{ |x - y| : x, y \in X \right\}$$

$$\big(\forall x \in X\ |x| \leq 1\big) \to \operatorname{diam}(X) \leq 2$$

$$\updownarrow ?$$

$$\exists x \in X\ \big(|x| \leq 1 \to \operatorname{diam}(X) \leq 2\big)?$$

Valid Formulas — Example

$$
\begin{aligned}
& t = t \\
& t = s \rightarrow s = t \\
& t = s \rightarrow s = r \rightarrow t = r \\
& t_1 = s_1 \rightarrow \cdots \rightarrow t_n = s_n \rightarrow f(t_1, \ldots, t_n) = f(s_1, \ldots, t_n) \\
& t_1 = s_1 \rightarrow \cdots \rightarrow t_n = s_n \rightarrow \big(P(t_1, \ldots, t_n) \leftrightarrow P(s_1, \ldots, s_n)\big) \\
& t = s \rightarrow r[t/x] = r[s/x] \\
& t = s \rightarrow \big(A[t/x] \leftrightarrow A[s/x]\big)
\end{aligned}
$$

Valid Formulas — Example

$x \notin \mathrm{Fv}(t)$:

$$\exists x(x = t)$$

$$A[t/x] \leftrightarrow \exists x(x = t \land A)$$

$$A[t/x] \leftrightarrow \forall x(x = t \rightarrow A)$$

Example

$$\boxed{A[t/x] \to \forall x(x = t \to A) \quad \text{where } x \notin \mathrm{Fv}(t)}$$

$\mathcal{M}, \nu \vDash A[t/x] \implies \mathcal{M}, \nu(\nu(t)/x) \vDash A$
Assume $\nu(t) = b$. Then for all $a \in M$, ether $a = b$ or $a \neq b$.
If $a = b$, then $\mathcal{M}, \nu(a/x) \vDash A$.
If $a \neq b$, then $\nu(a/x)(x) = a \neq b = \nu(t) = \nu(a/x)(t)$. So $\mathcal{M}, \nu(a/x) \nvDash x = t$.
Therefore we have $\mathcal{M}, \nu(a/x) \vDash x = t \to A$ for all $a \in M$.

$$\boxed{\forall x(x = t \to A) \to A[t/x] \quad \text{where } x \notin \mathrm{Fv}(t)}$$

$\mathcal{M}, \nu \vDash \forall x(x = t \to A) \implies$ for all $a \in M : \mathcal{M}, \nu(a/x) \vDash x = t \to A$
Let $\nu(t) = b$. Then $\mathcal{M}, \nu(b/x) \vDash x = t \to A$.
$\nu(b/x)(x) = b = \nu(t) = \nu(b/x)(t) \implies \mathcal{M}, \nu(b/x) \vDash x = t$
Therefore $\mathcal{M}, \nu(b/x) \vDash A$. By Substitution Lemma, $\mathcal{M}, \nu \vDash A[t/x]$.

Application — Game

Theorem (Zermelo's Theorem)

Every finite game of perfect information with no tie is determined.

Proof.

First, color those end nodes black that are wins for player 1, and color the other end nodes white, being the wins for 2. Then

- if player 1 is to move, and at least one child is black, color it black; if all children are white, color it white.
- if player 2 is to move, and at least one child is white, color it white; if all children are black, color it black.

Proof.

$$\exists x_1 \forall y_1 \ldots \exists x_n \forall y_n A \vee \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \neg A$$

where A states that a final position is reached where player 1 wins.

Contents

Formal Systems

- Hilbert System
- Tree Method
- Natural Deduction
- Sequent Calculus
- Resolution
- ...

Hilbert System = Axiom + Inference Rule

Axiom Schema

1. $A \to B \to A$
2. $(A \to B \to C) \to (A \to B) \to A \to C$
3. $(\neg A \to \neg B) \to (\neg A \to B) \to A$
4. $\forall x(A \to B) \to \forall xA \to \forall xB$
5. $\forall xA \to A[t/x]$ where t is substitutable for x in A.
6. $A \to \forall xA$ where $x \notin \mathrm{Fv}(A)$.
7. $x = x$
8. $x = y \to A \to A'$ where A is atomic and A' is obtained from A by replacing x in zero or more places by y.
9. $\forall x_1 \ldots x_n A$ where $n \geq 0$ and A is any axiom of the preceding groups.

Inference Rule

$$\frac{A \quad A \to B}{B}\,[\mathrm{MP}]$$

Example

Theorem

$$A \vdash \exists x A$$

Proof.

1. $(\forall x \neg A \to \neg A) \to A \to \neg \forall x \neg A$ — Tautology
2. $\forall x \neg A \to \neg A$ — A5
3. $A \to \neg \forall x \neg A$ — 1,2 MP
4. A — Premise
5. $\neg \forall x \neg A$ — 3,4 MP
6. $\exists x\, A$ — Definition of $\exists$

Deduction Theorem

Theorem (Deduction Theorem1)

$$\Gamma, A \vdash B \implies \Gamma \vdash A \to B$$

Inference Rule

$$\frac{A}{\forall x A} \text{ [G]}$$

What if we remove Axiom9 and add the rule of generalization to Hilbert System?

Theorem (Deduction Theorem2)

If $\Gamma, A \vdash B$, where the rule of generalization is not applied to the free variables of A, then $\Gamma \vdash A \to B$.

Meta-properties

- $\models A[B_1/p_1, \ldots, B_n/p_n]$ where $A \in \mathscr{L}^0$, $B_1, \ldots, B_n \in \mathscr{L}^1$. tautology
- $\Gamma, A \vdash B \wedge \neg B \implies \Gamma \vdash \neg A$ reductio ad absurdum
- $\Gamma, \neg A \vdash B \;\&\; \Gamma, \neg A \vdash \neg B \implies \Gamma \vdash A$ proof by contradiction
- $\Gamma, A \vdash \neg B \iff \Gamma, B \vdash \neg A$ contraposition
- $t = s \vdash r[t/x] = r[s/x]$ substitution
- $t = s \vdash A[t/x] \leftrightarrow A[s/x]$ substitution
- $\vdash B \leftrightarrow C \implies \vdash A \leftrightarrow A^*$ where A^* arises from A by replacing one or more occurrences of B in A by C. equivalent replacement
- $\vdash \forall x A \iff \vdash \forall y A[y/x]$ alphabetic variant

Meta-properties

- $\Gamma \vdash A[t/x] \implies \Gamma \vdash \exists x A$ $\exists R$
- $\Gamma, A[t/x] \vdash B \implies \Gamma, \forall x A \vdash B$ $\forall L$
- $\Gamma, A \vdash B \;\; \& \;\; x \notin \mathrm{Fv}(\Gamma, B) \implies \Gamma, \exists x A \vdash B$ $\exists L$
- $\Gamma \vdash A \;\; \& \;\; x \notin \mathrm{Fv}(\Gamma) \implies \Gamma \vdash \forall x A$ $\forall R$
- $\Gamma, A[y/x] \vdash B \;\; \& \;\; y \notin \mathrm{Fv}(\Gamma, \exists x A, B) \implies \Gamma, \exists x A \vdash B$ $\exists L$
- $\Gamma \vdash A[y/x] \;\; \& \;\; y \notin \mathrm{Fv}(\Gamma, \forall x A) \implies \Gamma \vdash \forall x A$ $\forall R$
- $\Gamma, A[a/x] \vdash B \;\; \& \;\; a \notin \mathrm{Cst}(\Gamma, \exists x A, B) \implies \Gamma, \exists x A \vdash B$ $\exists L$
- $\Gamma \vdash A[a/x] \;\; \& \;\; a \notin \mathrm{Cst}(\Gamma, \forall x A) \implies \Gamma \vdash \forall x A$ $\forall R$
- $\Gamma \vdash A \;\; \& \;\; a \notin \mathrm{Cst}(\Gamma) \;\; \& \;\; x \notin \mathrm{Fv}(A) \implies \Gamma \vdash \forall x A[x/a]$

Alphabetic Variant

Theorem (Existence of Alphabetic Variants)

Let A be a formula, t a term, and x a variable. Then we can find a formula A^* which differs from A only in the choice of quantified variables s.t.

1. $A \dashv\vdash A^*$
2. *t is substitutable for x in A^*.*

Strategy

$\to$

- $\Gamma \vdash A \to B \;\Longleftarrow\; \Gamma, A \vdash B$

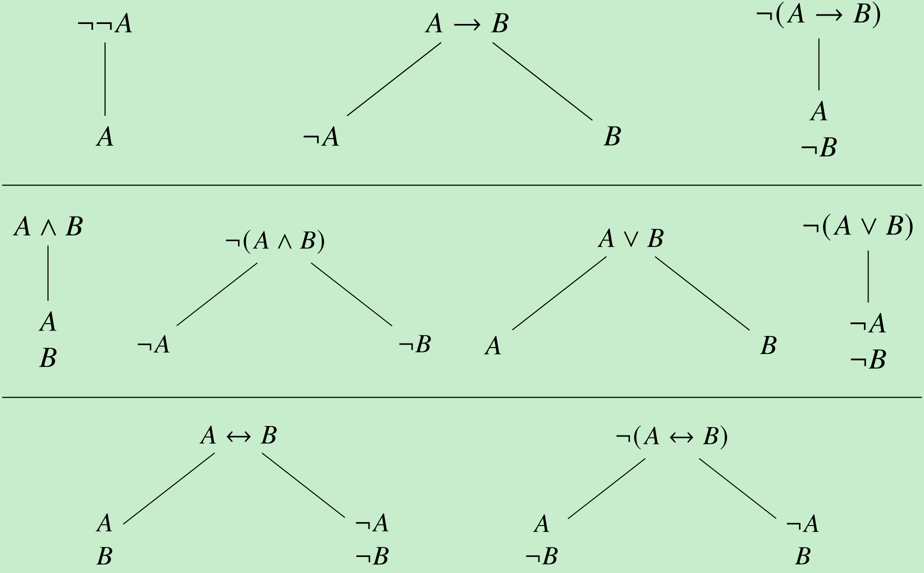
$\forall$

1. if $x \notin \mathrm{Fv}(\Gamma)$, $\Gamma \vdash \forall x A \;\Longleftarrow\; \Gamma \vdash A$
2. if $x \in \mathrm{Fv}(\Gamma)$,
   $\Gamma \vdash \forall x A \;\Longleftarrow\; \Gamma \vdash \forall y A[y/x] \;\Longleftarrow\; \Gamma \vdash A[y/x]$ for some new y.

$\neg$

1. $(\neg \to)$ $\quad \Gamma \vdash \neg(A \to B) \;\Longleftarrow\; \Gamma \vdash A \;\;\&\;\; \Gamma \vdash \neg B$
2. $(\neg\neg)$ $\quad \Gamma \vdash \neg\neg A \;\Longleftarrow\; \Gamma \vdash A$
3. $(\neg\forall)$ $\quad \Gamma \vdash \neg\forall x A \;\Longleftarrow\; \Gamma \vdash \neg A[t/x]$
   Unfortunately this is not always possible. Try contraposition, reductio ad absurdum or prove by contradiction. . .

Tree Method for Propositional Logic



$\neg\neg A$

A

$A \rightarrow B$

$\neg A$ B

$\neg(A \rightarrow B)$

A
$\neg B$

$A \wedge B$

A
B

$\neg(A \wedge B)$

$\neg A$ $\neg B$

$A \vee B$

A B

$\neg(A \vee B)$

$\neg A$
$\neg B$

$A \leftrightarrow B$

A
B

$\neg A$
$\neg B$

$\neg(A \leftrightarrow B)$

A
$\neg B$

$\neg A$
B

✓

Tree Method for Predicate Logic I

Ground Tree:

$\forall x A$

|

$A[t/x]$

where t is a ground term.

$\exists x A$ ✓

|

$A(a)$

where a is a new constant.

$\neg \forall x A$ ✓

|

$\exists x \neg A$

$\neg \exists x A$ ✓

|

$\forall x \neg A$

Tree Method for Predicate Logic II

Tree Method with Unification:

$$\forall x A \checkmark$$

$$A[x_i/x]$$

where x_i is a new variable.

$$\exists x A \checkmark$$

$$A[f(x_1, \ldots, x_m)/x]$$

where f is a new function and $\{x_1, \ldots, x_m\} = \mathrm{Fv}(\exists x A)$.

$$\neg\forall x A \checkmark$$

$$\exists x \neg A$$

$$\neg\exists x A \checkmark$$

$$\forall x \neg A$$

Tree Method with Unification

- when expanding a universally quantified formula, do not choose a specific term but a rigid variable as a placeholder.
- choose the term only when it is clear it allows closing a branch.

rigid variable=same value in the whole tree

- variables can assigned to closed terms, like $x_1 = a$.
- can also be assigned to unclosed terms, like $x_1 = f(x_2)$.
- make literals one the opposite of the other.
- using terms as unspecified as possible — Given literals A and $\neg B$ on the same branch, take the most general unifier of A and B.

Unifier

- A substitution σ is a *unifier* for a set Γ of formulae if for every $A, B \in \Gamma : A\sigma = B\sigma$.
- A unifier σ is a *most general unifier* for Γ if for each unifier θ there exists a substitution λ s.t. $\theta = \sigma\lambda$.

$$\sigma := \{t_1/x_1, \ldots, t_m/x_m\} \qquad \lambda := \{s_1/y_1, \ldots, s_n/y_n\}$$

$$\sigma\lambda = \left\{t_1\lambda/x_1, \ldots, t_m\lambda/x_m,\ s_1/y_1, \ldots, s_n/y_n\right\} \setminus \left\{s_i/y_i : y_i \in \{x_1, \ldots, x_m\}\right\}$$

- $(A\sigma)\lambda = A(\sigma\lambda)$ and $(t\sigma)\lambda = t(\sigma\lambda)$
- $(\sigma\lambda)\theta = \sigma(\lambda\theta)$

Tree Method for Predicate Logic

$$\begin{array}{c} A(x) \\ x = y \\ \vert \\ A(y) \end{array} \qquad\qquad \begin{array}{c} A(x) \\ y = x \\ \vert \\ A(y) \end{array}$$

where $A(y)$ arises from the wff $A(x)$ by replacing one or more occurrences of x by y.

Deduction & Tactics

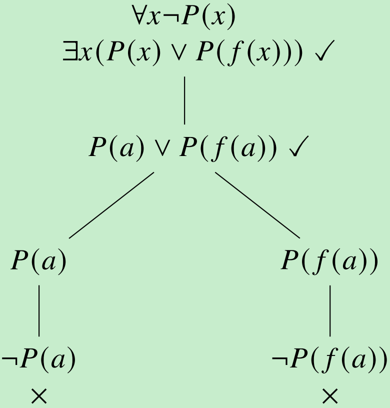
Definition (Deduction)

$A_1, \ldots, A_n \vdash B$ iff there exists a closed tree from $\{A_1, \ldots, A_n, \neg B\}$.

- Try to apply “non-branching” rules first, in order to reduce the number of branches.
- Try to close off branches as quickly as possible.
- Deal with negated quantifiers first.
- Instantiate existentials before universals.

Example — Ground Tree

$$\boxed{\left\{\forall x \neg P(x), \exists x (P(x) \lor P(f(x)))\right\} \text{ is unsatisfiable.}}$$



$\forall x \neg P(x)$

$\exists x (P(x) \lor P(f(x)))$ ✓

$P(a) \lor P(f(a))$ ✓

$P(a)$

$\neg P(a)$

$\times$

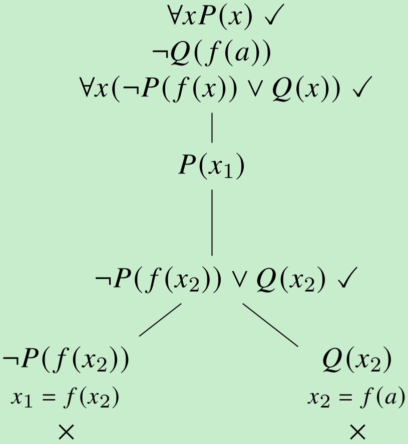
$P(f(a))$

$\neg P(f(a))$

$\times$

Example — Tree Method with Unification

$\boxed{\left\{\forall x P(x), \neg Q(f(a)), \forall x(\neg P(f(x)) \vee Q(x))\right\} \text{ is unsatisfiable.}}$



$\forall x P(x)$ ✓

$\neg Q(f(a))$

$\forall x(\neg P(f(x)) \vee Q(x))$ ✓

$P(x_1)$

$\neg P(f(x_2)) \vee Q(x_2)$ ✓

$\neg P(f(x_2))$

$x_1 = f(x_2)$

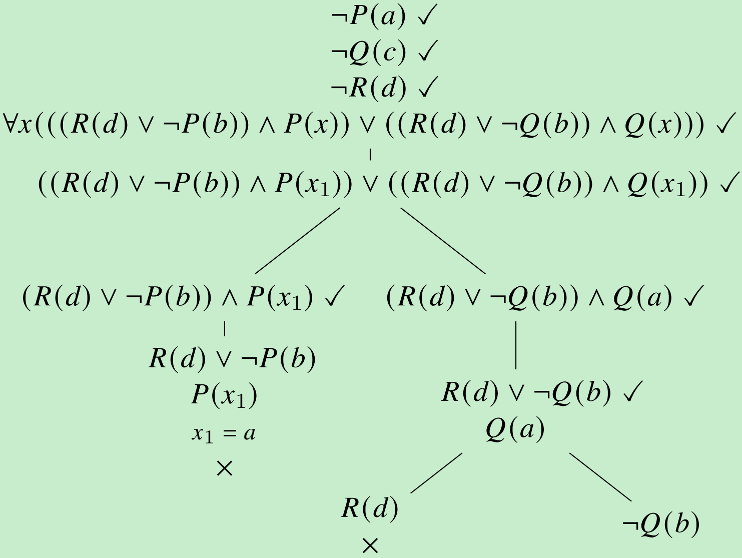
×

$Q(x_2)$

$x_2 = f(a)$

×

Unification — Greedy Unification (incomplete)



$\neg P(a)$ ✓

$\neg Q(c)$ ✓

$\neg R(d)$ ✓

$\forall x(((R(d) \lor \neg P(b)) \land P(x)) \lor ((R(d) \lor \neg Q(b)) \land Q(x)))$ ✓

$((R(d) \lor \neg P(b)) \land P(x_1)) \lor ((R(d) \lor \neg Q(b)) \land Q(x_1))$ ✓

$(R(d) \lor \neg P(b)) \land P(x_1)$ ✓

$R(d) \lor \neg P(b)$

$P(x_1)$

$x_1 = a$

×

$(R(d) \lor \neg Q(b)) \land Q(a)$ ✓

$R(d) \lor \neg Q(b)$ ✓

$Q(a)$

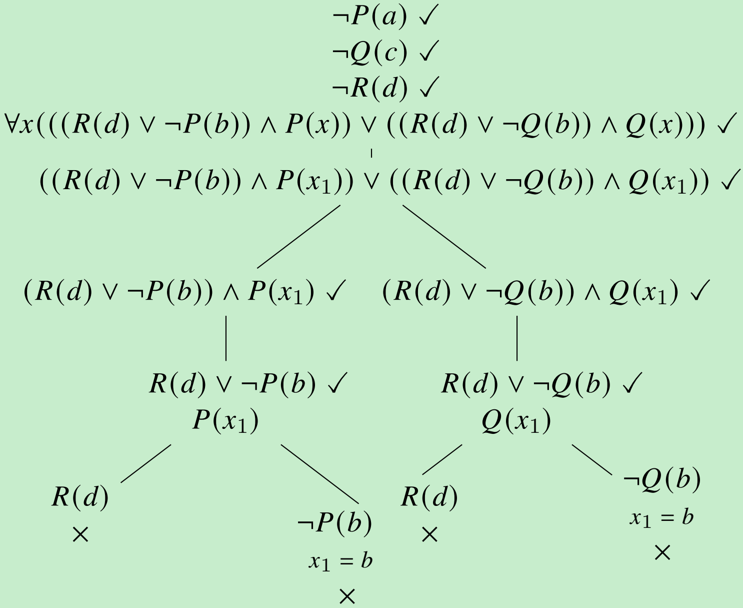
$R(d)$

×

$\neg Q(b)$

Applying unification as soon as a branch can be closed by lead to incompleteness.

Unification — Final Closure



$$\neg P(a) \checkmark$$

$$\neg Q(c) \checkmark$$

$$\neg R(d) \checkmark$$

$$\forall x(((R(d) \vee \neg P(b)) \wedge P(x)) \vee ((R(d) \vee \neg Q(b)) \wedge Q(x))) \checkmark$$

$$((R(d) \vee \neg P(b)) \wedge P(x_1)) \vee ((R(d) \vee \neg Q(b)) \wedge Q(x_1)) \checkmark$$

$$(R(d) \vee \neg P(b)) \wedge P(x_1) \checkmark \qquad (R(d) \vee \neg Q(b)) \wedge Q(x_1) \checkmark$$

$$R(d) \vee \neg P(b) \checkmark \qquad R(d) \vee \neg Q(b) \checkmark$$

$$P(x_1) \qquad Q(x_1)$$

$$R(d) \quad \neg P(b) \quad R(d) \quad \neg Q(b)$$

$$\times \quad x_1 = b \quad \times \quad x_1 = b$$

$$\times \qquad \times$$

Unification is applied only when it closes all open branches at the same time.

Example — Unification vs Ground

There is someone such that if he is drinking, then everyone is drinking.

$$\boxed{\vdash \exists x\big(A(x) \to \forall x A(x)\big)}$$

$$\neg\exists x(A(x) \to \forall x A(x))\ \checkmark$$

$$\forall x\neg(A(x) \to \forall x A(x))\ \checkmark$$

$$\neg(A(x_1) \to \forall x A(x))\ \checkmark$$

$$A(x_1)$$

$$\neg\forall x A(x)\ \checkmark$$

$$\neg A(a)$$

$$x_1 = a$$

$$\times$$

$$\forall x\neg(A(x) \to \forall x A(x))$$

$$\neg(A(a) \to \forall x A(x))\ \checkmark$$

$$A(a)$$

$$\neg\forall x A(x)\ \checkmark$$

$$\neg A(b)$$

$$\neg(A(b) \to \forall x A(x))\ \checkmark$$

$$A(b)$$

$$\neg\forall x A(x)$$

$$\times$$

Soundness & Completeness

Theorem (Soundness Theorem)

If the tree closes, the set is unsatisfiable.

Theorem (Completeness Theorem)

If a set is unsatisfiable, there exists a closed tree from it.

$$A_1, \ldots, A_n \vdash B \iff A_1, \ldots, A_n \vDash B$$

Remark: If an inference with predicate wff is not valid and its counterexample is an infinite model, the tree will not find it. The tree method can't generate every counterexample of an invalid inference in predicate logic.

Exercises — Tree Method

1. $\forall x(Px \to Qx) \to \exists x Px \to \exists x Qx$
2. $\exists x \forall y Rxy \to \forall y \exists x Rxy$
3. $\exists x(Px \land Qx) \to \exists x Px \land \exists x Qx$
4. $\forall x\big(A \lor B(x)\big) \to A \lor \forall x B(x)$ where $x \notin \mathrm{Fv}(A)$
5. $\exists x\Big(\big(Px \land \forall y(Py \to y = x)\big) \land Qx\Big) \dashv\vdash \exists x \forall y\Big(\big(Py \leftrightarrow y = x\big) \land Qx\Big)$
6. $\exists x\big(Px \land \forall y(Py \to y = x)\big) \land \exists x\big(Qx \land \forall y(Qy \to y = x)\big) \land \neg\exists x(Px \land Qx) \to \exists xy\big(x \neq y \land (Px \lor Qx) \land (Py \lor Qy) \land \forall z(Pz \lor Qz \to z = x \lor z = y)\big)$

$*\mathbf{54 \cdot 43}.\ \vdash\!:\ .\alpha, \beta \in 1.\ \supset\!:\ \alpha \cap \beta = \Lambda.\ \equiv .\alpha \cup \beta \in 2$

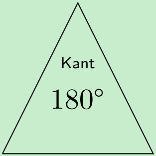
Dem.

$$\begin{aligned}
&\vdash .*54\cdot 26.\ \supset\vdash\!:\ .\alpha = \iota' x.\beta = \iota' y.\ \supset\!:\ \alpha \cup \beta \in 2.\ \equiv .x \neq y. \\
&[*51 \cdot 231] \qquad\qquad\qquad\qquad\qquad\qquad \equiv .t' x \cap \iota' y = \Lambda. \\
&[*13 \cdot 12] \qquad\qquad\qquad\qquad\qquad\qquad \equiv .\alpha \cap \beta = \Lambda \qquad (1) \\
&\vdash .(1).*11\cdot 11\cdot 35.\ \supset \\
&\qquad \vdash\!:\ .(\exists x, y).\alpha = \iota' x.\beta = \iota' y.\ \supset\!:\ \alpha \cup \beta \in 2.\ \equiv .\alpha \cap \beta = \Lambda \qquad (2) \\
&\vdash .(2).*11\cdot 54.*52\cdot 1.\ \supset\vdash .Prop
\end{aligned}$$

From this proposition it will follow, when arithmetical addition has been defined, that $\mathbf{1 + 1 = 2}$.

Philosophy of Math: is math synthetic a priori?

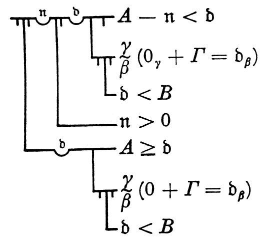
- Descartes: we can be certain about how things seem to us from the inside; but how to build up to the external world?
- Hume: we can't. (i) Knowledge of the external world requires knowledge of causation. (ii) Causal statements are synthetic, and so can be known only a posteriori. (iii) Causal statements can't be known a posteriori, because we don't perceive causation itself and can't noncircularly argue that the future will resemble the past.
- Kant: we can know facts about causation a priori, even though they are synthetic, because facts about causation are constituted partly by how the world is in itself, and partly by our minds' operation; and we can know a priori the rules by which our mind operates.



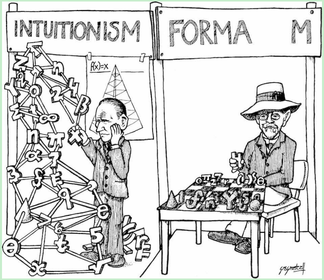
Kant: Mathematics is synthetic a priori.
Frege: Mathematics is analytic.

Frege

- Arithmetic laws are analytic judgements, and hence a priori. Arithmetic is a developed logic. The application of arithmetic to natural science is logical processing of observed facts; calculation is deduction.
- If the task of philosophy is to break the domination of words over the human mind by freeing thought from the mask of existing means of expression, then my ideography would become a useful instrument in the hands of philosophers.
- Every good mathematician is at least half a philosopher, and every good philosopher is at least half a mathematician.

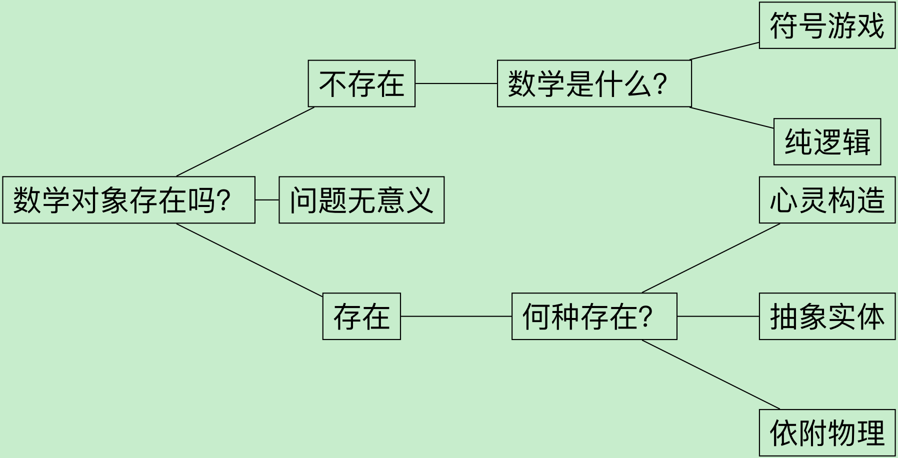


Philosophy of Math: Logicism/Intuitionism/Formalism



Logicism	Intuitionism	Formalism
$\frac{\text{Mathematics}}{\text{Logic}}$	$\frac{\frac{\text{Logic}}{\text{Mathematics}}}{\text{Mind}}$	$\frac{\text{Mathematics}}{\text{Game}}$
Realism	Conceptualism	Nominalism

数学哲学



数学对象存在吗?

不存在

数学是什么?

符号游戏

纯逻辑

问题无意义

存在

何种存在?

心灵构造

抽象实体

依附物理

Figure: 形式主义/逻辑主义/直觉主义/柏拉图主义/物理主义

Is there more than one mathematical universe?

Exercises — Tree Method

1. Nobody trusts *exactly* those who have no mutual trust with anybody.
2. If dogs are animals, every head of a dog is the head of an animal.
3. Every non-analytic, meaningful proposition is either verifiable or falsifiable. Philosophical propositions are neither analytic nor verifiable or falsifiable. Therefore, they are meaningless.
4. No girl loves any sexist pig. Caroline is a girl who loves whoever loves her. Henry loves Caroline. Thus Henry isn't a sexist pig.
5. *The* present king of France is bald. Bald men are sexy. Hence whoever is a present King of France is sexy.
6. *Only* Russell is a great philosopher. Wittgenstein is a great philosopher who smokes. So Russell smokes.
7. Everyone is afraid of Dracula. Dracula is afraid *only* of me. Therefore, I am Dracula.
8. Everyone loves a *lover*(*anyone who loves somebody*). Romeo loves Juliet. Therefore, I love you.
9. Everyone loves a *lover*(*anyone who loves somebody*); hence if someone is a lover, everyone loves everyone!

Exercises — Tree Method

1. I am a philosopher. A philosopher can *only* be appreciated by philosophers. No philosopher is without some eccentricity. I sing rock. Every eccentric rock singer is appreciated by some girl. Eccentrics are conceited. Therefore, some girl is conceited.
2. Any philosopher admires some logician. Some students admire *only* film stars. No film stars are logicians. Therefore not all students are philosophers.
3. If anyone speaks to anyone, then someone introduces them; no one introduces anyone to anyone unless he knows them both; everyone speaks to Frank; therefore everyone is introduced to Frank by someone who knows him.
4. Whoever stole the goods, knew the safe combination. Someone stole the goods, and *only* Jack knew the safe combination. Hence Jack stole the goods.
5. *No one but* Alice and Bette (*who are different people*) admires Carl. All and only those who admire Carl love him. Hence *exactly* two people love Carl.

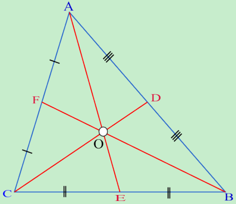
Application — Minesweeper

- There are exactly n mines in the game.
- If a cell contains the number 1, then there is exactly one mine in the adjacent cells.

  $$\forall x(\mathrm{contain}(x,1) \rightarrow$$
  $$\exists y(\mathrm{adj}(x,y) \wedge \mathrm{mine}(y) \wedge \forall z(\mathrm{adj}(x,z) \wedge \mathrm{mine}(z) \rightarrow z = y)))$$
- . . .

Russell's Theory of Descriptions

1. The substitution of identicals.
   "The morning star is the evening star."
2. The law of the excluded middle.
   "The present King of France is bald." or
   "The present King of France is not bald."
3. The problem of negative existentials.
   "The round square is round."



Russell's Theory of Descriptions

$$
\begin{aligned}
B(\iota_x A) &:= \exists! x A \land \exists x(A \land B) \\
&\dashv\vdash \exists x \forall y \Big(\big(A(y) \leftrightarrow y = x \big) \land B(x) \Big)
\end{aligned}
$$

The round square does not exist. $B(\iota_x A) \lor (\neg B)(\iota_x A)$?

$$\exists x \forall y \Big(\big(Ry \land Sy \leftrightarrow y = x \big) \land \neg Ex \Big) \quad (\neg B)(\iota_x A) \ ?$$

$$\neg \exists x \forall y \Big(\big(Ry \land Sy \leftrightarrow y = x \big) \land Ex \Big) \quad \neg B(\iota_x A) \ ?$$

$$Ex \overset{?}{:=} \exists P \big(Px \land \exists y \neg Py \big)$$

$$\iota_x A = \iota_x A \ ? \qquad \forall x B \longrightarrow B(\iota_x A) \ ?$$

$$B(\iota_x^y A) := \big(\exists! x A \longrightarrow \exists x (A \land B) \big) \land \big(\neg \exists! x A \longrightarrow B[y/x] \big)$$

$$\vdash \forall x B \longrightarrow B(\iota_x^y A)$$

Russell's Theory of Descriptions & Church's λ-Abstraction

$$\nu(\iota_x A) = \begin{cases} a & \text{if there is a unique } a \in M : \mathcal{M}, \nu(a/x) \vDash A \\ \uparrow & \text{otherwise} \end{cases}$$

$$\begin{cases} \mathcal{M}, \nu \vDash (\lambda x.A)t \iff \mathcal{M}, \nu \vDash A[t/x] & \text{if } \nu(t)\downarrow \\ \mathcal{M}, \nu \nvDash (\lambda x.A)t & \text{if } \nu(t)\uparrow \end{cases}$$

The present King of France is not bald.

$$(\lambda x.\neg Bx)\iota_x Kx$$

It's not the case that the present King of France is bald.

$$\neg(\lambda x.Bx)\iota_x Kx$$

Crossing the street without looking is dangerous.

$$\mathbf{D}(\lambda x(Cx \wedge \neg Lx))$$

- The logical form of a statement may differ from its grammatical form.
- (Contextuality Principle.) Never ask for the meaning of a phrase in isolation, but only in the context of some meaningful fragment of a text.
- The method of contextual definition, which the theory of descriptions exemplifies, was inspired by the nineteenth-century rigorization of analysis.

Berkeley: 2^{nd} crisis of the Foundations of Mathematics

For $f(x) = x^2$,

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{f(x+\mathrm{d}x) - f(x)}{\mathrm{d}x} = \frac{(x+\mathrm{d}x)^2 - x^2}{\mathrm{d}x} = \frac{2x\mathrm{d}x + (\mathrm{d}x)^2}{\mathrm{d}x} = 2x + \mathrm{d}x = 2x$$

$\dfrac{\mathrm{d}}{\mathrm{d}x}$ should be explained as a whole.

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Expressive Limitation of First Order Language

- Most boys are funny.
- Some critics admire only one another.

$$\exists X\Big(\exists x Xx \wedge \forall x(Xx \to Cx) \wedge \forall xy\big(Xx \wedge A(x,y) \to Xy \wedge x \neq y\big)\Big)$$

- There are some gunslingers each of whom has shot the right foot of at least one of the others.

$$\exists X\Big(\exists x Xx \wedge \forall x(Xx \to Gx) \wedge \forall x\big(Xx \to \exists y(Xy \wedge y \neq x \wedge Sxy)\big)\Big)$$

- Least Number Principle.

$$\forall X\Big(\exists x Xx \wedge \forall x(Xx \to Nx) \to \exists x\big(Xx \wedge \forall y(Xy \wedge y \neq x \to x < y)\big)\Big)$$

- A linear order $(P, <)$ is *complete* if every non-empty subset of P that is bounded above has a supremum in P.

$$\forall X\Bigg(\exists x Xx \wedge \exists y \forall x(Xx \to x \leq y) \to \exists y\Big(\forall x(Xx \to x \leq$$

$$y) \wedge \forall z\big(\forall x(Xx \to x \leq z) \to y \leq z\big)\Big)\Bigg)$$

Logicism & Logical Positivism

- Mathematics could be reduced to logic.
- Science could be reduced to logical compounds of statements about sense data.
- Only statements verifiable through observation or logical proof are meaningful.
- If all you have is a hammer, everything looks like a nail.
- The new logical resources provided by Frege and Russell had both tempted the positivists to conjecture more than they could prove and made it clear to them that proof of their conjecture was impossible.
- Few if any philosophical schools before the positivists had even stated their aims with sufficient clarity to make it possible to see that they were unachievable.

Elimination of Metaphysics?

一个陈述的意义在于它的证实方法。形而上学陈述不能被证实，毫无意义。那么留给哲学的还有什么呢？一种方法：逻辑分析法。逻辑分析的消极应用是清除无意义的词和陈述，积极应用是澄清有意义的概念和命题，为经验科学和数学奠基。形而上学家相信自己是在攸关真假的领域里前行，却未断言任何东西。他们只是试图表达一点儿人生态度。艺术是表达人生态度的恰当手段。抒情诗人并不企图在自己的诗里驳倒其他抒情诗人诗里的陈述，但形而上学家却用论证维护他的陈述。形而上学家是没有艺术才能的艺术家，有的是在理论环境里工作的爱好，却既不在科学领域里发挥这种爱好，又不能满足艺术表达的要求，倒是混淆了这两个方面，创造出一种对知识既无贡献、对人生态度的表达又不相宜的东西。[a]

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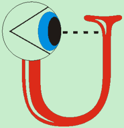
[a] 卡尔纳普：通过语言的逻辑分析清除形而上学

Mystics exult in mystery and want it to stay mysterious. Scientists exult in mystery for a different reason: it gives them something to do.

— *Dawkins*



Thanks