Lecture notes on category theory

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Categories and functors

1.1 Definition and first examples

Definition 1.1.1 *A category $\mathcal{C}$ consists of the following data:*

(1) A class Ob $\mathcal{C}$ whose elements A, B, C,... $\in \mathrm{Ob}\,\mathcal{C}$ are called objects;

(2) For every pair of objects A, B, a (possibly empty) set $\mathrm{Hom}_{\mathcal{C}}(A, B)$, whose elements are called morphisms from A to B;

(3) For every triple of objects A, B, C, a composition law

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

$$(f,\, g) \to g \circ f$$

(4) For every object A, a morphism $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$, called the identity on A

such that the following axioms hold:

(1) Associative law: given morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $h \in \mathrm{Hom}_{\mathcal{C}}(C, D$ the following equality holds:

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(2) Identity law: given a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ the following identities hold:

$$1_B \circ f = f = f \circ 1_A.$$

Remark 1.1.2 *Probably the most unfamiliar thing in Definition 1.1.1 is that the objects form a class, not a set. The reason for this choice is that we sometimes want to speak about the category of sets, having as objects all possible sets, and we run into a well-known* *set-theory problem namely that considering "the set of all sets" leads to a contradiction (Russell's paradox). One way to get around this issue is by using the Godel-Bernays set-theory which introduces the notion of a class. The connection between sets and classes is given by the following axiom: a class is a set if and only if it belongs to some (other) class. Informally, a class can be though of as a "big set".*

Definition 1.1.3 *A category $\mathcal{C}$ is called a small category if its class of objects Ob $\mathcal{C}$ is a set.*

Examples 1.1.4 *1) Any set X can be made into a category, called the discrete category on X and denoted by $\mathcal{C}_X$, as follows:*

$$\mathrm{Ob}\,\mathcal{C}_X := X$$
$$\mathrm{Hom}_{\mathcal{C}_X}(x,\, y) = \begin{cases} \emptyset & \textit{if} \;\; x \neq y \\ \{1_x\} & \textit{if} \;\; x = y \end{cases}, \text{ for every } x,\, y \in X.$$

The composition law comes down to $1_x \circ 1_x = 1_x$ for all $x \in X$.

2) More generally, any pre-ordered set[1] *(poset for short) $(X,\, \leqslant)$ defines a category PO$(X,\, \leqslant)$ as follows:*

$$\mathrm{Ob}\,\mathrm{PO}(X,\, \leqslant) := X$$
$$\mathrm{Hom}_{\mathrm{PO}(X,\, \leqslant)}(x,\, y) = \begin{cases} \emptyset & \textit{if} \;\; x \nleqslant y \\ \{u_{x,\,y}\} & \textit{if} \;\; x \leqslant y \end{cases}, \text{ for every } x,\, y \in X.$$

The composition of morphisms is given as follows $u_{y,\,z} \circ u_{x,\,y} = u_{x,\,z}$ while the identity on x is $u_{x,\,x}$.

3) A monoid $(M,\, \cdot)$ can be seen as a category $\mathcal{M}$ with a single object denoted by $*$ and the set of morphisms $\mathrm{Hom}_{\mathcal{M}}(*,\, *) = M$. The composition of morphisms in $\mathcal{M}$ is given by the multiplication of M and the identity on $*$ is just the unit 1_M.

*4) The category **Set** of sets has the class of all sets as objects while $\mathrm{Hom}_{\textbf{\textit{Set}}}(A,\, B)$ is the set of all functions from A to B. Composition is given by the usual composition of functions and the identity on any set A is the identity map 1_A.*

*5) **FinSet** is the category whose objects are finite sets, and where $\mathrm{Hom}_{\textbf{\textit{FinSet}}}(A,\, B)$ is just the set of all functions between the two finite sets A and B.*

*6) Consider **RelSet** to be the category defined as follows:*

$$\mathrm{Ob}\,\textbf{\textit{RelSet}} := \mathrm{Ob}\,\textbf{\textit{Set}}$$
$$\mathrm{Hom}_{\textbf{\textit{RelSet}}}(A,\, B) = \mathcal{P}(A \times B) = \{f \mid f \subseteq A \times B\}, \text{ for every } A,\, B \in \mathrm{Ob}\,\textbf{\textit{Set}}.$$

*The composition of morphisms in **RelSet** is defined as follows: given $f \subseteq A \times B$ and $g \subseteq B \times C$ we consider $g \circ f = \{(a,\, c) \in A \times C \mid \exists b \in B \,\text{such that}\, (a,\, b) \in f \,\text{and}\, (b,\, c) \in g\}$. Finally, the identity is defined as $1_A = \{(a,\, a) \mid a \in A\}$.*

[1] A set X is called *pre-ordered* if it is endowed with a binary relation $\leqslant$ which is reflexive and transitive.

7) ***Grp*** *is the category of groups, where Ob* ***Grp*** *is the class of all groups while* $\mathrm{Hom}_{\boldsymbol{Grp}}(A, B)$ *is the set of all group homomorphisms from* A *to* B*. Similarly,* ***Mon*** *denotes the category of monoids with monoid homomorphisms between them.*

8) ***SiGrp*** *denotes the category of simple*[2] *groups with group homomorphisms between them.*

9) ***Ab*** *is the category of abelian groups with group homomorphisms between them.*

10) ***Div*** *is the category of divisible*[3] *groups with group homomorphisms between them.*

11) ***Rng*** *is the category of rings with ring homomorphisms between them.*

12) ***Ring*** *(resp.* $\boldsymbol{Ring}^c$*) is the category of (commutative) unitary rings with unit preserving ring homomorphisms between them.*

13) ***Field*** *is the category of fields with field homomorphisms between them.*

14) *For a ring* R*, we denote by* ${}_R\mathcal{M}$ *the category of left* R*-modules with objects all left* R*-modules and morphisms between two* R*-modules given by module homomorphisms. If* R *is non-commutative one can also define analogously the category of right* R*-modules* $\mathcal{M}_R$*. In particular, if* K *is a field,* ${}_K\mathcal{M}$ *(resp.* ${}_K\mathcal{M}^{fd}$*) denotes the category of vector spaces (resp. finite dimensional vector spaces) over* K*.*

15) *For a commutative ring* R*, we denote by* $\boldsymbol{Alg}_R$ *the category of* R*-algebras together with algebra homomorphisms between them.*

16) ***Top*** *is the category of topological spaces where Ob* ***Top*** *is the class of all topological spaces while* $\mathrm{Hom}_{\boldsymbol{Top}}(A, B)$ *is the set of continuous functions between* A *and* B*.* $\boldsymbol{Top}_*$ *stands for the category of pointed topological spaces, that is the objects are pairs* (A, a_0) *where* A *is a topological space and* $a_0 \in A$ *while the morphisms between two such pairs* (A, a_0) *and* (B, b_0) *are just continuous functions* $f : A \to B$ *such that* $f(a_0) = b_0$*.*

17) ***Haus*** *(respectively* ***KHaus****) is the category of Hausdorff (compact) topological spaces where Ob* ***Haus*** *(Ob* ***KHaus****) is the class of all hausdorff (compact) topological spaces while* $\mathrm{Hom}_{\boldsymbol{Haus}}(A, B)$ *(*$\mathrm{Hom}_{\boldsymbol{KHaus}}(A, B)$*) is the set of continuous functions between* A *and* B*.*

18) *For a field* K*, we denote by* $\boldsymbol{Mat}_K$ *the category whose objects class is the set of positive integers* $\mathbb{N}$*. The morphisms in* $\boldsymbol{Mat}_K$ *between two objects* $m, n \in \mathbb{N}$ *are all* $n \times m$ *matrices with entries in* K *and the composition of morphisms is given by matrix multiplication:*

$$\mathrm{Hom}_{\boldsymbol{Mat}_K}(m, n) \times \mathrm{Hom}_{\boldsymbol{Mat}_K}(n, p) \to \mathrm{Hom}_{\boldsymbol{Mat}_K}(m, p)$$
$$(A, B) \mapsto BA$$

[2] A group is called *simple* if its only normal subgroups are the trivial group and the group itself.

[3] An abelian group $(G, +)$ is called *divisible* if for every positive integer n and every $g \in G$, there exists $h \in G$ such that $nh = g$.

The identity morphism on any $n \in \mathbb{N}$ is given by the $n \times n$ identity matrix, where the 0×0 identity matrix is by definition the zero matrix. Furthermore, by convention, if either integer m and n is zero, we have a unique $n \times m$ matrix called a null matrix.

Remark 1.1.5 *Notice that although we sometimes work with categories whose objects are sets, the morphisms need not be functions. This situation is best illustrated in Example 1.1.4, 6).*

Definition 1.1.6 *Let $\mathcal{C}$, $\mathcal{C}'$ be two categories. We shall say that $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ if the following conditions are satisfied:*

(i) $Ob\ \mathcal{C}' \subseteq Ob\ \mathcal{C}$;

(ii) $\mathrm{Hom}_{\mathcal{C}'}(A, B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, B)$ *for every* $A,\ B \in \mathrm{Ob}\,\mathcal{C}'$;

(iii) The composition of morphisms in $\mathcal{C}'$ is induced by the composition of morphisms in $\mathcal{C}$;

(iv) The identity morphisms in $\mathcal{C}'$ are identity morphisms in $\mathcal{C}$.

Moreover, $\mathcal{C}'$ is said to be full if for every pair $(A,\ B)$ of objects of $\mathcal{C}'$ we have:

$$\mathrm{Hom}_{\mathcal{C}'}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$$

Examples 1.1.7 *1) The category* ***FinSet*** *is a full subcategory of* ***Set****;*

2) The category ***Ab*** *is a full subcategory of* ***Grp****;*

3) The category ***Haus*** *is a full subcategory of* ***Top****;*

4) ***Ring*** *is a subcategory of* ***Rng*** *but not a full subcategory;*

5) ***Set*** *is a subcategory of* ***RelSet*** *but not a full subcategory.*

1.2 Special objects and morphisms in a category

Definition 1.2.1 *Let $\mathcal{C}$ be a category and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.*

1) f is called a monomorphism if for any $g_1,\ g_2 \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ such that $f \circ g_1 = f \circ g_2$ we have $g_1 = g_2$;

2) f is called an epimorphism if for any $h_1,\ h_2 \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ such that $h_1 \circ f = h_2 \circ f$ we have $h_1 = h_2$;

3) f is called an isomorphism if there exists $f' \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $f \circ f' = 1_B$ and $f' \circ f = 1_A$. In this case we say that A and B are isomorphic objects and we denote this by $A \cong B$.

Remark 1.2.2 *The notions of monomorphism and epimorphism are generalizations to arbitrary categories of the familiar injective and surjective functions from **Set**. However, although in **Set** monomorphisms (resp. epimorphisms) coincide with injective (resp. surjective) functions, this is no longer true in an arbitrary category whose objects and morphisms are sets and functions respectively (see Example 1.2.3, 2), 3)). Furthermore, a morphism that is both a monomorphism and an epimorphism need not be an isomorphism (see, for instance, Example 1.2.3, 4)).*

Examples 1.2.3 *1) In the categories **Set** of sets, **Grp** of groups, **Ab** of abelian groups, ${}_R\mathcal{M}$ of left R-modules, **Top** of topological spaces, monomorphisms (epimorphisms) coincide with the injective (surjective) morphisms. We will only prove that monomorphisms in **Set** coincide with injective maps. Indeed, suppose $f : A \to B$ is an injective map and g, $h : C \to A$ such that $f \circ h = f \circ g$. Then, we have $f\big(h(c)\big) = f\big(g(c)\big)$ for any $c \in C$ and since f is injective we get $h(c) = g(c)$ for any $c \in C$, i.e. $g = h$ as desired. Assume now that $f : A \to B$ is a monomorphism and let a, $a' \in A$ such that $f(a) = f(a')$. We denote by $i_a : \{*\} \to A$, respectively $i_{a'} : \{*\} \to A$ the maps given by $i_a(*) = a$, $i_{a'}(*) = a'$. Then we also have $f \circ i_a = f \circ i_{a'}$ and since f is a monomorphism we obtain $i_a = i_{a'}$. Therefore $a = a'$ and f is indeed injective.*

*2) In the category **Div** of divisible groups, the quotient map $q : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is obviously not injective but it is a monomorphism. Indeed, let G be another divisible group and f, $g : G \to \mathbb{Q}$ two morphisms of groups such that $q \circ f = q \circ g$. By denoting $h = f - g$ we obtain $q \circ h = 0$. Now for any $x \in G$ we have $q(h(x)) = 0$ and thus $h(x) \in \mathbb{Z}$. Suppose there exists some $x_0 \in G$ such that $h(x_0) \neq 0$. We can assume without loss of generality that $h(x_0) \in \mathbb{N}^*$. Since we work with divisible groups, we can find some $y_0 \in G$ such that $x_0 = 2h(x_0)y_0$. By applying h to the above equality yields:*

$$h(x_0) = 2h(x_0)h(y_0)$$

*which is an obvious contradiction since $h(x) \in \mathbb{Z}$ for all $x \in G$. Hence we get $h = 0$ which implies $f = g$ and we proved that q is indeed a monomorphism in **Div**.*

*3) In the category **Ring**^{c} of unitary commutative rings, the inclusion $i : \mathbb{Z} \to \mathbb{Q}$ is obviously not surjective but it is an epimorphism. Indeed, let R be another ring together with two ring morphisms f, $g : \mathbb{Q} \to R$ such that $f \circ i = g \circ i$. Consider now $z \in \mathbb{Z}^*$; then we have $1 = f(1) = f(z)f(1/z)$ and therefore $f(1/z) = 1/f(z)$. Similarly we can prove that $g(1/z) = 1/g(z)$ and since f and g coincide on $\mathbb{Z}$ we get $f(1/z) = g(1/z)$. Now for any $z' \in \mathbb{Z}$ we have:*

$$f(z'/z) = f(z')f(1/z) = g(z')g(1/z) = g(z'/z)$$

*Therefore $f = g$ which implies that i is an epimorphism in **Ring**^{c}.*

4) *It can be easily seen that the inclusion $i : \mathbb{Z} \to \mathbb{Q}$ is a monomorphism in the category* ***Ring***^{c} *of unitary commutative rings and also an epimorphism by Example 1.2.3, 3). Therefore, it provides an example of a morphism which is both a monomorphism and an epimorphism but not an isomorphism.*

5) *Let (T, τ) be a topological space such that τ is different from the discrete topology. Consider now the set T endowed with the discrete topology $\mathcal{P}(T)$. Then the identity $\mathrm{Id}_T : (T, \mathcal{P}(T)) \to (T, \tau)$ is obviously bijective and a continuous map between the two topological spaces. Therefore, Id_T is a morphism in* ***Top*** *but not an isomorphism although it is a bijective map.*

6) *In the category $PO(X, \leqslant)$ associated to a partially ordered set[4] $(X, \leqslant)$, any isomorphism is an identity morphism. Indeed suppose $f : x \to y$ is an isomorphism; this implies that $x \leqslant y$. If $g : y \to x$ is the inverse of f then we also have $y \leqslant x$. Due to the antisymmetry of $\leqslant$ we obtain $x = y$. Therefore $f : x \to x$ must be the identity on x.*

Definition 1.2.4 *Let $\mathcal{C}$ be a category and $A \in \mathrm{Ob}\,\mathcal{C}$.*

1) *We say that A is an initial object if the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ has exactly one element for each $B \in \mathrm{Ob}\,\mathcal{C}$;*

2) *We say that A is a final object if the set $\mathrm{Hom}_{\mathcal{C}}(C, A)$ has exactly one element for each $C \in \mathrm{Ob}\,\mathcal{C}$;*

3) *If $A \in \mathrm{Ob}\,\mathcal{C}$ is both an initial and a final object we say that A is a zero-object.*

Proposition 1.2.5 *If A and B are initial (final) objects in a category $\mathcal{C}$ then A is isomorphic to B.*

Proof: Since A is initial there exists a unique morphism $f : A \to B$ and a unique morphism from A to A which must be the identity 1_A. The same applies for B: there exists a unique morphism $g : B \to A$ and a unique morphism from B to B, namely the identity 1_B. Now remark that $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ and thus $g \circ f = 1_A$. Similarly we get $f \circ g = 1_B$ and we proved that A and B are isomorphic. The statement about final objects can be proved in a similar manner. □

Examples 1.2.6 1) *In the category* ***Set*** *of sets the initial object is the empty set while the final objects are the singletons, i.e. the one-element sets $\{x\}$. Thus* ***Set*** *has infinitely many final objects and they are all isomorphic;*

2) *The category* ***Set*** *of sets has no zero-objects. In the categories* ***Grp*** *of groups,* ***Ab*** *of abelian groups, ${}_R\mathcal{M}$ of modules, $\{0\}$ is a zero-object;*

[4] A set X is called *partially ordered* if it is endowed with a binary relation $\leqslant$ which is reflexive, antisymmetric and transitive.

3) *The category **Field** has neither an initial nor a final object since there are no morphisms between fields of different characteristics;*

4) *Let $(X, \leqslant)$ be a poset and $PO(X, \leqslant)$ the associated category (see Example 1.1.4, 2)). Then $PO(X, \leqslant)$ has an initial object if and only if $(X, \leqslant)$ has a least element (i.e. some element $0 \in X$ such that $0 \leq x$ for any $x \in X$). Similarly, $PO(X, \leqslant)$ has a final object if and only if $(X, \leqslant)$ has a greatest element (i.e. some element $1 \in X$ such that $x \leq 1$ for any $x \in X$).*

1.3 Some constructions of categories

In this section we provide several methods of constructing new categories.

Definition 1.3.1 *Given $\mathcal{C}$ a category, the dual (opposite) category of $\mathcal{C}$, denoted by $\mathcal{C}^{op}$, is obtained as follows:*

i) $\mathrm{Ob}\,\mathcal{C}^{op} = \mathrm{Ob}\,\mathcal{C}$;

ii) $\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$ *(i.e. the morphisms of $\mathcal{C}^{op}$ are those of $\mathcal{C}$ written in the reverse direction; in order to avoid confusion we write $f^{op} : A \to B$ for the morphism of $\mathcal{C}^{op}$ corresponding to the morphism $f : B \to A$ of $\mathcal{C}$);*

iii) The composition map

$$\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) \times \mathrm{Hom}_{\mathcal{C}^{op}}(B, C) \to \mathrm{Hom}_{\mathcal{C}^{op}}(A, C)$$

is defined as follows:

$$g^{op} \circ f^{op} = (f \circ g)^{op}, \text{ for all } f^{op} \in \mathrm{Hom}_{\mathcal{C}^{op}}(A, B),\; g^{op} \in \mathrm{Hom}_{\mathcal{C}^{op}}(B, C);$$

iv) The identities are the same as in $\mathcal{C}$.

Example 1.3.2 *Let $PO(X, \leqslant)$ be the category associated to the poset $(X, \leqslant)$. Then $\mathrm{PO}(X, \leqslant)^{op} = \mathrm{PO}(X, \geqslant)$, where $\geqslant$ is the pre-order on X defined as follows: $x \geqslant y$ if and only if $y \leqslant x$.*

Theorem 1.3.3 (Duality principle) *Suppose that a statement expressing the existence of some objects or some morphisms or the equality of some composites is valid in every category. Then the dual statement (obtained by reversing the direction of every arrow and replacing every composite $f \circ g$ by the composite $g \circ f$) is also valid in every category.*

Proof: If S denotes the given statement and S^{op} denotes the dual statement then proving the statement S^{op} in a category $\mathcal{C}$ is equivalent to proving the statement S in the category $\mathcal{C}^{op}$ which is assumed to be valid. $\square$

Remarks 1.3.4 *It is straightforward to see that:*

1) $f^{op} : A \to B$ *is a monomorphism (resp. an epimorphism) in* $\mathcal{C}^{op}$ *if and only if* $f : B \to A$ *is an epimorphism (resp. a monomorphism) in* $\mathcal{C}$*;*

2) C *is an initial object (resp. a final object) in* $\mathcal{C}^{op}$ *if and only if* C *is a final object (resp. an initial object) in* $\mathcal{C}$*.*

Definition 1.3.5 *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories. We define the product category* $\mathcal{C} \times \mathcal{D}$ *in the following manner:*

i) $\mathrm{Ob}\big(\mathcal{C} \times \mathcal{D}\big) = \mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{D}$, *i.e. the objects of* $\mathcal{C} \times \mathcal{D}$ *are pairs of the form* $(C,\, D)$ *with* $C \in \mathrm{Ob}\,\mathcal{C}$ *and* $D \in \mathrm{Ob}\,\mathcal{D}$*;*

ii) $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}\big((C, D),\, (C',\, D')\big) = \mathrm{Hom}_{\mathcal{C}}(C, C') \times \mathrm{Hom}_{\mathcal{D}}(D, D')$*;*

iii) The composition map is defined as follows:

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}\big((C, D),\, (C', D')\big) \times \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}\big((C', D'),\, (C'', D'')\big) \to \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}\big((C, D),\, (C'', D'')\big)$$

$$(f',\, g') \circ (f,\, g) = (f' \circ f,\, g' \circ g)$$

for all $(f,\, g) \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}\big((C, D),\, (C', D')\big)$, $(f',\, g') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}\big((C', D'),\, (C'', D'')\big)$*;*

iv) $1_{(C,\, D)} = (1_C,\, 1_D)$*.*

Another way of constructing categories involves directed graphs. We start by reviewing these first.

Definition 1.3.6 *A graph consists of a class* $\mathcal{V}$ *whose elements are called vertices and for each pair* $(A,\, B) \in \mathcal{V} \times \mathcal{V}$ *a set* $\mathcal{E}(A,\, B)$ *whose elements are called edges. A graph is called small if* $\mathcal{V}$ *is a set.*
A path in a graph is a non-empty finite sequence $(A_1,\, f_1,\, A_2,\, \cdots,\, A_n)$ *alternating vertices and edges such that the first and the last terms are vertices and each edge* $f_i \in \mathcal{E}(A_i,\, A_{i+1})$*.*

Notice that every category is in particular a graph; this can be easily seen by leaving aside the composition law of the given category and by forgetting which morphisms are identities. Conversely, we will be able to construct a category out of a graph as follows:

Definition 1.3.7 *Let* $G = \big(\mathcal{V},\, \mathcal{E}\big)$ *be a small graph*[5]*. The free category on the graph* G*, denoted by* $\mathcal{G}$*, is constructed by considering:*

i) $\mathrm{Ob}\,\mathcal{G} = \mathcal{V}$ *as class of objects;*

[5] The smallness assumption on the graph G is needed is order for the paths between any two vertices to form a set.

ii) $\mathrm{Hom}_{\mathcal{G}}(A,\, B) = \mathcal{P}(A,\, B)$ *the set of paths between the vertices* A *and* B*, for any* A*,* $B \in \mathrm{Ob}\,\mathcal{G}$*;*

iii) *The composition is given by concatenation:*

$$(A_n,\, f_n,\, \cdots,\, A_m) \circ (A_1,\, f_1,\, \cdots,\, A_n) = (A_1,\, f_1,\, \cdots, A_n,\, f_n,\, \cdots,\, A_m);$$

iv) *The identity maps are given by the trivial paths.*

Example 1.3.8 *Let* G *be the graph depicted as follows:*

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3$$

Then the free category $\mathcal{G}$ *on the graph* G *is given as follows:*

$$\begin{aligned}
&\mathrm{Ob}\,\mathcal{G} = \{v_1,\, v_2, v_3\}\\
&\mathrm{Hom}_{\mathcal{G}}(v_1,\, v_1) = \{1_{v_1}\},\ \mathrm{Hom}_{\mathcal{G}}(v_1,\, v_2) = \{(v_1\, e_1\, v_2)\},\ \mathrm{Hom}_{\mathcal{G}}(v_1,\, v_3) = \{(v_1\, e_1\, v_2\, e_2\, v_3)\}\\
&\mathrm{Hom}_{\mathcal{G}}(v_2,\, v_1) = \emptyset,\ \mathrm{Hom}_{\mathcal{G}}(v_2,\, v_2) = \{1_{v_2}\},\ \mathrm{Hom}_{\mathcal{G}}(v_2,\, v_3) = \{(v_2\, e_2\, v_3)\}\\
&\mathrm{Hom}_{\mathcal{G}}(v_3,\, v_1) = \emptyset,\ \mathrm{Hom}_{\mathcal{G}}(v_3,\, v_2) = \emptyset,\ \mathrm{Hom}_{\mathcal{G}}(v_3,\, v_3) = \{1_{v_3}\}.
\end{aligned}$$

Definition 1.3.9 *Let* $\mathcal{C}$ *be a category. A diagram in* $\mathcal{C}$ *is a graph whose vertices and edges are objects and respectively morphisms of* $\mathcal{C}$*. A diagram in* $\mathcal{C}$ *will be called commutative if for each pair of vertices, every two paths between them are equal as morphisms.*

Example 1.3.10 *1) The following diagram is commutative if and only if* $g \circ f = h$*:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

2) The following diagram is commutative if and only if $g \circ f = h \circ k$*:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] & B \arrow[d, "g"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

3) The following diagram is commutative if and only if $h \circ g \circ f = k$*:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] & B \arrow[d, "g"] \\ C & D \arrow[l, "h"] \end{tikzcd}$$

The last construction we introduce is that of a quotient category; it involves an equivalence relation on the class of all morphisms of a given category.

Definition 1.3.11 *Let $\mathcal{C}$ be a category. An equivalence relation $\sim$ on the class of all morphisms* $\bigcup\limits_{A,\, B\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(A,\, B)$ *of $\mathcal{C}$ is called a congruence if the following conditions are fulfilled:*

i) If $f \in \mathrm{Hom}_{\mathcal{C}}(A,\, B)$ and $f \sim f'$ then $f' \in \mathrm{Hom}_{\mathcal{C}}(A,\, B)$;

ii) If $f \sim f'$, $g \sim g'$ and the composition $g \circ f$ exists then $g \circ f \sim g' \circ f'$.

Proposition 1.3.12 *Let $\sim$ be a congruence on a category $\mathcal{C}$ and denote by $\overline{f}$ the equivalence class of a morphism f of $\mathcal{C}$. Then $\mathcal{C}'$ defined below is a category called a quotient category of $\mathcal{C}$:*

i) $\mathrm{Ob}\,\mathcal{C}' = \mathrm{Ob}\,\mathcal{C}$;

ii) $\mathrm{Hom}_{\mathcal{C}'}(A,\, B) = \{\,\overline{f}\,|\, f \in \mathrm{Hom}_{\mathcal{C}}(A,\, B)\}$ *for any* $A,\, B \in \mathrm{Ob}\,\mathcal{C}'$;

iii) The composition map $\mathrm{Hom}_{\mathcal{C}'}(A,B) \times \mathrm{Hom}_{\mathcal{C}'}(B,C) \to \mathrm{Hom}_{\mathcal{C}'}(A,C)$ is defined as follows:

$$\overline{g} \circ \overline{f} = \overline{g \circ f}, \text{ for all } \overline{f} \in \mathrm{Hom}_{\mathcal{C}'}(A,B),\ \overline{g} \in \mathrm{Hom}_{\mathcal{C}'}(B,C);$$

iv) The identity on $A \in \mathrm{Ob}\,\mathcal{C}'$ is $\overline{1_A}$.

Proof: It can be easily seen from Definition 1.3.11, i), that $\sim$ induces a partition on each set $\mathrm{Hom}_{\mathcal{C}}(A,\, B)$ and therefore $\mathrm{Hom}_{\mathcal{C}'}(A,\, B)$ is also a set. Furthermore, Definition 1.3.11, ii), shows that the composition law in $\mathcal{C}$' is well-defined. □

1.4 Functors

Definition 1.4.1 *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A covariant functor (respectively contravariant functor) $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:*

(1) A mapping $A \mapsto F(A) : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$;

(2) For each pair of objects $A,\, B \in \mathrm{Ob}\,\mathcal{C}$, a mapping $f \mapsto F(f) : \mathrm{Hom}_{\mathcal{C}}(A,\, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A),\, F(B))$ *(respectively* $f \mapsto F(f) : \mathrm{Hom}_{\mathcal{C}}(A,\, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B),\, F(A))$*)*

subject to the following conditions:

(1) For every $A \in \mathrm{Ob}\,\mathcal{C}$ we have $F(1_A) = 1_{F(A)}$;

(2) For every $f \in \mathrm{Hom}_{\mathcal{C}}(A,\, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B,\, C)$ we have:

$$F(g \circ f) = F(g) \circ F(f) \text{ (respectively } F(g \circ f) = F(f) \circ F(g))$$

A functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ defined on the product of two categories is called a bifunctor (functor of two variables).

Throughout, the term *functor* will denote a *covariant functor*. Any reference to contravariant functors will be explicitly stated. It can be easily seen that $F : \mathcal{C} \to \mathcal{D}$ is a contravariant functor if and only if $F : \mathcal{C}^{op} \to \mathcal{D}$ (or $F : \mathcal{C} \to \mathcal{D}^{op}$) is a covariant functor.

Remark 1.4.2 *Note that the image*[6] *of a functor need not be a category. Indeed, consider the following two categories $\mathcal{C}$ and $\mathcal{D}$:*

$$\begin{tikzcd}
\mathcal{C}: & C_1 \arrow[loop above, "1_{C_1}"] \arrow[r, "f"] & C_2 \arrow[loop above, "1_{C_2}"] & C_3 \arrow[loop above, "1_{C_3}"] \arrow[r, "g"] & C_4 \arrow[loop above, "1_{C_4}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D}: & D_1 \arrow[loop above, "1_{D_1}"] \arrow[r, "h"] \arrow[rr, bend right, "k\circ h"'] & D_2 \arrow[loop above, "1_{D_2}"] \arrow[r, "k"] & D_3 \arrow[loop above, "1_{D_3}"]
\end{tikzcd}$$

The image of the functor $F\colon \mathcal{C} \to \mathcal{D}$ defined below is not a category:

$$F(C_1) = D_1,\ F(C_2) = F(C_3) = D_2,\ F(C_4) = D_3,$$
$$F(f) = h,\ F(g) = k, F(1_{C_1}) = 1_{D_1},\ F(1_{C_2}) = F(1_{C_3}) = 1_{D_2},\ F(1_{C_4}) = 1_{D_3}.$$

Indeed, note that the morphisms h and k are contained in the image of F while their composition $k \circ h$ is not.

Examples 1.4.3 *1) If $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ we can define the inclusion functor $i : \mathcal{C}' \to \mathcal{C}$ which sends every object as well as every morphism to itself. If $\mathcal{C}' = \mathcal{C}$ then i comes down to the identity functor $1_{\mathcal{C}}$ on $\mathcal{C}$.*

2) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $D_0 \in \operatorname{Ob}\mathcal{D}$ a fixed object. The constant functor at D_0 assigns to every object $C \in \operatorname{Ob}\mathcal{C}$ the object D_0 and to every morphism f in $\mathcal{C}$ the identity morphism 1_{D_0}.

3) If I is a small discrete category, then a functor $F : I \to \mathcal{C}$ is essentially nothing but a family of objects $(C_i)_{i\in I}$ indexed by I.

4) Having a category $\mathcal{C}$ and $C \in \operatorname{Ob}\mathcal{C}$ a fixed object allows us to define two functors, one of them being covariant and the other one contravariant. Indeed, define $\mathit{Hom}_{\mathcal{C}}(C, -)$, $\mathit{Hom}_{\mathcal{C}}(-, C) : \mathcal{C} \to \textbf{\textit{Set}}$ as follows:

i) $\mathit{Hom}_{\mathcal{C}}(C, A) = \operatorname{Hom}_{\mathcal{C}}(C, A) \in \operatorname{Ob}\textbf{\textit{Set}}$ for all $A \in \operatorname{Ob}\mathcal{C}$;
If $f \in \operatorname{Hom}_{\mathcal{C}}(A, B)$ then $\mathit{Hom}_{\mathcal{C}}(C, f) : \operatorname{Hom}_{\mathcal{C}}(C, A) \to \operatorname{Hom}_{\mathcal{C}}(C, B)$
is defined by $\mathit{Hom}_{\mathcal{C}}(C, f)(g) = f \circ g$, for all $g \in \operatorname{Hom}_{\mathcal{C}}(C, A)$.

[6]The *image of a functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of the class $\{F(C) \mid C \in \operatorname{Ob}\mathcal{C}\}$ together with all sets $\{F(f) \mid f \in \operatorname{Hom}_{\mathcal{C}}(A, B)\}$ for any $A, B \in \operatorname{Ob}\mathcal{C}$.

ii) $\mathcal{H}om_{\mathcal{C}}(A, C) = \mathrm{Hom}_{\mathcal{C}}(A, C) \in \mathrm{Ob}\,\boldsymbol{Set}$ for all $A \in \mathrm{Ob}\,\mathcal{C}$;
If $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ then $\mathcal{H}om_{\mathcal{C}}(f, C) : \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ is defined by $\mathcal{H}om_{\mathcal{C}}(f, C)(g) = g \circ f$, for all $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$.

5) *For any category $\mathcal{C}$ we can define the bifunctor $\mathcal{H}om_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \boldsymbol{Set}$ as follows:*

$\mathcal{H}om_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B) \in \mathrm{Ob}\,\boldsymbol{Set}$ for all $(A, B) \in \mathrm{Ob}\big(\mathcal{C}^{op} \times \mathcal{C}\big)$;
If $(f^{op}, g) \in \mathrm{Hom}_{\mathcal{C}^{op}\times\mathcal{C}}\big((A, B), (C, D)\big) = \mathrm{Hom}_{\mathcal{C}^{op}}(A, C) \times \mathrm{Hom}_{\mathcal{C}}(B, D)$ then $\mathcal{H}om_{\mathcal{C}}(f^{op}, g) : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(C, D)$ is defined by $\mathcal{H}om_{\mathcal{C}}(f^{op}, g)(h) = g \circ h \circ f$, for all $h \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

6) *For any set X we denote by $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$ the power set of X. We can define two power set functors $P : \boldsymbol{Set} \to \boldsymbol{Set}$ and respectively $P^c : \boldsymbol{Set} \to \boldsymbol{Set}$, the first one being covariant and the second one contravariant, as follows:*

i) $P : \boldsymbol{Set} \to \boldsymbol{Set}$, $P(A) = \mathcal{P}(A) \in \mathrm{Ob}\,\boldsymbol{Set}$, for all $A \in \mathrm{Ob}\,\boldsymbol{Set}$;
If $f \in \mathrm{Hom}_{\boldsymbol{Set}}(A, B)$ then $P(f) : \mathcal{P}(A) \to \mathcal{P}(B)$ is defined by $P(f)(U) = f(U)$, for all $U \subseteq A$.

ii) $P^c : \boldsymbol{Set} \to \boldsymbol{Set}$, $P^c(A) = \mathcal{P}(A) \in \mathrm{Ob}\,\boldsymbol{Set}$, for all $A \in \mathrm{Ob}\,\boldsymbol{Set}$;
If $f \in \mathrm{Hom}_{\boldsymbol{Set}}(A, B)$ then $P^c(f) : \mathcal{P}(B) \to \mathcal{P}(A)$ is defined by $P^c(f)(V) = f^{-1}(V)$, for all $V \subseteq B$.

7) *A special class of functors are the so-called forgetful functors. These are functors which "forget" (some of) the structure on objects of the domain category. For instance the categories in Example 1.1.4, 7)-13) allow for a forgetful functor to the category **Set** of sets which sends the objects of that category to the underlying set, and the homomorphisms to the underlying mapping between the underlying sets. Similarly, we have many other examples of forgetful functors which forget only some of the structure in objects of the domain category:*

$\boldsymbol{Rng} \to \boldsymbol{Ab}$, forgets about the product
$\boldsymbol{Ab} \to \boldsymbol{Grp}$, forgets about the commutativity
$\boldsymbol{Ring} \to \boldsymbol{Rng}$, forgets about the unit
$\boldsymbol{Top}_* \to \boldsymbol{Top}$, forgets about the point
$\boldsymbol{Haus} \to \boldsymbol{Top}$, forgets about the Hausdorff property

8) *Let $\mathcal{U} : \boldsymbol{Ring} \to \boldsymbol{Grp}$ be the functor defined as follows:*

$\mathcal{U}(R) =$ the group of invertible elements of R,
$\mathcal{U}(f) : \mathcal{U}(R) \to \mathcal{U}(S)$, $\mathcal{U}(f) = f_{|\mathcal{U}(R)}$, for any $f \in \mathrm{Hom}_{\boldsymbol{Ring}}(R, S)$.

9) *Let G, H be two groups and $\mathcal{G}$, respectively $\mathcal{H}$ the corresponding associated categories (see Example 1.1.4, 3)). Then, defining a functor $\mathcal{G} \to \mathcal{H}$ is the same as providing a group homomorphism $G \to H$.*

10) *Consider the field K as an object in ${}_K\mathcal{M}$ and denote the corresponding contravariant Hom functor $\mathit{Hom}_{{}_K\mathcal{M}}(-, K)$ by $(-)^* : {}_K\mathcal{M} \to$ **Set**. That is, we denote $V^* = \mathrm{Hom}_{{}_K\mathcal{M}}(V, K)$ and $\mathit{Hom}_{{}_K\mathcal{M}}(u, K) = u^*$. Now notice that given a vector space V the set V^* of linear maps from V to K is a vector space as follows: for any f, $g \in V^*$ and a, $b \in K$ the linear map $af + bg$ is defined by $(af + bg)(v) = af(v) + bg(v)$. Furthermore, given a linear map $u \colon V \to W$ it can be easily seen that $u^* \colon W^* \to V^*$ defined by $u^*(w) = w \circ u$ is also a liner map. Therefore, the functor $(-)^*$ maps into the category ${}_K\mathcal{M}$ and is called the dual space (contravariant) functor.*

11) *π_1 : $\boldsymbol{Top}_*$ $\to$ **Grp** denotes the functor which assigns to each pointed topological space (A, a_0) its fundamental group.*

Proposition 1.4.4 *Small categories and functors between them constitute a category which we will denote by **Cat**.*

Proof: Given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ we obtain a new functor $G \circ F : \mathcal{C} \to \mathcal{E}$ by pointwise composition. The composition law is obviously associative and the identity functor on a category is an identity for this composition.

Finally, if $\mathcal{C}$ and $\mathcal{D}$ are small categories, i.e. $\mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Ob}\,\mathcal{D}$ are sets, then $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ is also a set. □

Comma categories

Definition 1.4.5 *Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be two functors. The comma-category $(F \downarrow G)$ is defined as follows:*

1) *The objects are triples (A, f, B) consisting of two objects $A \in \mathrm{Ob}\,\mathcal{A}$, $B \in \mathrm{Ob}\,\mathcal{B}$ and a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(F(A), G(B))$;*

2) *A morphism in $(F \downarrow G)$ from (A, f, B) to (A', f', B') is a pair (a, b), where $a \in \mathrm{Hom}_{\mathcal{A}}(A, A')$, $b \in \mathrm{Hom}_{\mathcal{B}}(B, B')$ such that the following diagram is commutative:*

$$\begin{tikzcd} F(A) \arrow[r, "F(a)"] \arrow[d, "f"'] & F(A') \arrow[d, "f'"] \\ G(B) \arrow[r, "G(b)"'] & G(B') \end{tikzcd} \qquad \text{i.e. } G(b) \circ f = f' \circ F(a).$$

3) *The composition law in $(F \downarrow G)$ is that induced by the composition laws of $\mathcal{A}$ and respectively $\mathcal{B}$, i.e.:*

$$(a, b) \circ (a', b') = (a \circ a', b \circ b')$$

4) The identities are $1_{(A,\,f,\,B)} = (1_A,\,1_B)$.

Examples 1.4.6 *1) Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be discrete categories with only one object and take* F *and* G *to be objects* C *and* C' *of* $\mathcal{C}$*. Then* $(F \downarrow G) = (C \downarrow C')$ *is the category with objects all morphisms* $f : C \to C'$ *in* $\mathcal{C}$ *and whose morphisms are just the identity maps on each object. In other words,* $(C \downarrow C')$ *is the discrete category on the set* $\mathrm{Hom}_{\mathcal{C}}(C,\,C')$*;*

2) Let $\mathcal{A}$ *be the discrete category with only one object and take* F *to be the object* X *of* $\mathcal{C}$*. For any functor* $G : \mathcal{B} \to \mathcal{C}$ *the comma-category* $(F \downarrow G) = (X \downarrow G)$ *is defined as follows:*

i) The objects of $(X \downarrow G)$ *are pairs* $(f,\,Y)$ *where* $Y \in \mathrm{Ob}\,\mathcal{B}$*,* $f \in \mathrm{Hom}_{\mathcal{C}}(X,\,G(Y))$*;*

ii) A morphism between two objects $(f,\,Y)$ *and* $(f',\,Y')$ *is a morphism* $h \in \mathrm{Hom}_{\mathcal{B}}(Y,\,Y')$ *such that the following diagram is commutative:*

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ G(Y) \arrow[rr, "G(h)"'] & & G(Y') \end{tikzcd}$$

i.e. $G(h) \circ f = f'$.

iii) The composition of morphisms in $(X \downarrow G)$ *is given by the composition in* $\mathcal{B}$ *of the base morphisms* h *and the identities are* $1_{(f,\,Y)} = 1_Y$ *for any* $Y \in \mathrm{Ob}\,\mathcal{B}$*.*

Similarly, we can define the comma category $(F \downarrow X)$ *by considering* $\mathcal{B}$ *to be the discrete category with only one object and letting* G *be the object* X*. It can be easily seen that* $(F \downarrow X) = (X \downarrow F)^{op}$*.*

3) Moreover, if we consider $\mathcal{B} = \mathcal{C}$ *and* $G = 1_{\mathcal{C}}$ *the identity functor on* $\mathcal{C}$ *in the previous example then in this case the comma-category* $(F \downarrow G) = (A \downarrow \mathcal{C})$ *is called the category of objects under* A*. Similarly, we can define* $(\mathcal{C} \downarrow A)$ *the comma category of objects over* A *as* $(\mathcal{C} \downarrow A) = (A \downarrow \mathcal{C})^{op}$*.*

Definition 1.4.7 *A functor* $F : \mathcal{C} \to \mathcal{D}$ *is called an isomorphism of categories if there exists another functor* $G : \mathcal{D} \to \mathcal{C}$ *such that* $F \circ G = 1_{\mathcal{D}}$ *and* $G \circ F = 1_{\mathcal{C}}$*.* G *is called the inverse of* F*.*

Remark 1.4.8 1) *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor and for each* $A,\,B \in \mathrm{Ob}\,\mathcal{C}$ *consider the following induced mapping:*

$$\mathcal{F}_{A,\,B} : \mathrm{Hom}_{\mathcal{C}}(A,\,B) \to \mathrm{Hom}_{\mathcal{D}}\big(F(A),\,F(B)\big), \ \ f \mapsto F(f).$$

If F *is an isomorphism of categories with inverse* $G : \mathcal{D} \to \mathcal{C}$ *then each* $\mathcal{F}_{A,\,B}$ *is a bijective map with inverse given by* $\mathcal{G}_{F(A),\,F(B)}$*, where:*

$$\mathcal{G}_{F(A),\,F(B)} : \mathrm{Hom}_{\mathcal{D}}\big(F(A),\,F(B)\big) \to \mathrm{Hom}_{\mathcal{C}}(A,\,B), \ \ \mathcal{G}_{F(A),\,F(B)}(g) = G(g)$$

for all $g \in \mathrm{Hom}_{\mathcal{D}}\big(F(A),\, F(B)\big)$*.*

2) *In particular, an isomorphism of categories takes initial (resp. final) objects to initial (resp. final) objects. Indeed, it follows from the above discussion that there is a bijection between the sets* $\mathrm{Hom}_{\mathcal{C}}(A,\, G(D))$ *and respectively* $\mathrm{Hom}_{\mathcal{D}}(F(A),\, D)$*; hence, if* A *is an initial object in* $\mathcal{C}$ *then* $F(A)$ *is an initial object in* $\mathcal{D}$*.*

Examples 1.4.9 *1) The forgetful functor* $F : {}_{\mathbb{Z}}\mathcal{M} \to \boldsymbol{Ab}$ *is an isomorphism of categories. Indeed, the inverse of* F *is the functor* $G : \boldsymbol{Ab} \to {}_{\mathbb{Z}}\mathcal{M}$ *defined by* $G(M) = M$*,* $G(u) = u$*, where* $M \in \boldsymbol{Ab}$ *has a left* $\mathbb{Z}$*-module structure as follows:*

$$t \cdot m = \begin{cases} \underbrace{m + m + \cdots + m}_{\text{t times}} & \textit{if} \quad t > 0 \\ 0_M & \textit{if} \quad t = 0 \\ \underbrace{-m - m - \cdots - m}_{-\text{t times}} & \textit{if} \quad t < 0 \end{cases}, \text{ for every } t \in \mathbb{Z},\ m \in M.$$

2) Let R *be a ring and denote by* R^{op} *the opposite ring*[7]*. Then we have an isomorphism of categories* $F : {}_R\mathcal{M} \to \mathcal{M}_{R^{op}}$ *given by:*

$$\begin{aligned} &F(M) = M \in \mathcal{M}_{R^{op}} \ \textit{via} \ m * r = rm, \text{ for any } m \in M,\, r \in R \\ &F(u) = u \end{aligned}$$

with the inverse constructed in the same manner.

3) $\boldsymbol{Set} \not\cong \boldsymbol{Set}^{op}$*. Indeed, recall that* $\boldsymbol{Set}$ *has one initial object, namely the empty set, and infinitely many final objects, the singletons. Therefore, in* $\boldsymbol{Set}^{op}$ *we have infinitely many initial objects and one final object. The conclusion now follows since a potential isomorphism between the two categories should take initial (final) objects to initial (final) objects.*

The following notions, although much weaker than isomorphism, will also be useful.

Definition 1.4.10 *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor and for each* $A,\, B \in \mathrm{Ob}\,\mathcal{C}$ *consider the following induced mapping:*

$$\mathcal{F}_{A,\,B} : \mathrm{Hom}_{\mathcal{C}}(A,\, B) \to \mathrm{Hom}_{\mathcal{D}}\big(F(A),\, F(B)\big), \ \ f \mapsto F(f) \tag{1.1}$$

1) The functor F *is called faithful if the aforementioned mappings are injective for all* $A,\, B \in \mathrm{Ob}\,\mathcal{C}$*;*

2) The functor F *is called full if the aforementioned mappings are surjective for all* $A,\, B \in \mathrm{Ob}\,\mathcal{C}$*;*

[7] In R^{op} we have $(R^{op}, +) = (R, +)$ and the multiplication is given by $r \cdot_{op} r' = r'r$, for all r, $r' \in (R^{op}, +)$.

3) The functor F is called essentially surjective (or dense) if each object $D \in \mathcal{D}$ is isomorphic to an object of the form $F(C)$ for some $C \in \mathrm{Ob}\,\mathcal{C}$.

Examples 1.4.11 *1) The inclusion functor is automatically faithful. If the subcategory is full then the inclusion functor is also full;*

2) The inclusion functor $F : \boldsymbol{Ab} \to \boldsymbol{Grp}$ is faithful and full;

3) The forgetful functor $U : \boldsymbol{Grp} \to \boldsymbol{Set}$ is faithful but not full;

4) Let G_1 and G_2 be two groups and $\mathcal{G}_1$, respectively $\mathcal{G}_2$, the corresponding categories (see Example 1.1.4, 3)). Any morphism of groups $f : G_1 \to G_2$ which is surjective but not an isomorphism gives rise to a functor from $\mathcal{G}_1$ to $\mathcal{G}_2$ which is full but not faithful.

Definition 1.4.12 *A category $\mathcal{C}$ is said to be concrete if there exists a faithful functor $F : \mathcal{C} \to \boldsymbol{Set}$.*

Examples 1.4.13 *The categories **FinSet**, **Grp**, **Ab**, **Rng**, **Ring**, **Top**, ${}_R\mathcal{M}$ are all concrete categories due to the existence of forgetful functors from any of the above categories to **Set** which are obviously faithful.*

In fact, we have a lot more examples of concrete categories as it can be seen from the next result.

Theorem 1.4.14 *Any small category is concrete.*

Proof: For any small category $\mathcal{C}$ we construct a faithful functor $F : \mathcal{C} \to \mathbf{Set}$ as follows. Given $C \in \mathrm{Ob}\,\mathcal{C}$ and $u \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ we define:

$$F(C) = \{(Y, \alpha) \mid Y \in \mathrm{Ob}\,\mathcal{C},\ \alpha \in \mathrm{Hom}_{\mathcal{C}}(Y, C)\} \in \mathrm{Ob}\,\mathbf{Set};$$
$$F(u) : F(C) \to F(C'),\ \ F(u)(Y, \alpha) = (Y, u \circ \alpha)$$

It is straightforward to see that F defined above is a functor; we only point out that $F(C)$ is a set due to the fact that $\mathcal{C}$ is a small category. We will prove that it is faithful. Indeed, if $u_1, u_2 \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ such that $F(u_1) = F(u_2)$ we obtain $(Y, u_1 \circ \alpha) = (Y, u_2 \circ \alpha)$ for any $(Y, \alpha) \in F(C)$. Now for $(C, 1_C) \in F(C)$ we get $u_1 = u_2$ as desired. □

Remark 1.4.15 *Not every category admits a faithful functor to **Set**. However, there are no elementary examples of this fact. The interested reader may find such an example in [4] where it is shown that the homotopy category of pointed spaces is not concrete.*

Definition 1.4.16 *1) A functor F preserves a property P of morphisms if whenever f has the property P so does $F(f)$;*

2) A functor G reflects a property P of morphisms if whenever $G(f)$ has the property P so does f.

Proposition 1.4.17 *1) Any functor preserves isomorphisms;*

2) Any full and faithful functor reflects isomorphisms;

3) Any full and faithful functor reflects initial and final objects;

4) Any faithful functor reflects monomorphisms and epimorphisms.

Proof: The first assertion is straightforward and we leave it to the reader.

2) Let $F : \mathcal{C} \to \mathcal{D}$ be a full and faithful functor and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that $F(f)$ is an isomorphism in $\mathcal{D}$. Then, there exists $h \in \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$ such that:

$$F(f) \circ h = 1_{F(B)} \text{ and } h \circ F(f) = 1_{F(A)}.$$

Since F is full we can find $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $F(g) = h$. Therefore, the above identities came down to:

$$1_{F(B)} = F(f) \circ F(g) = F(f \circ g) \text{ and } 1_{F(A)} = F(g) \circ F(f) = F(g \circ f).$$

Now F is faithful and $F(1_A) = 1_{F(A)}$, respectively $F(1_B) = 1_{F(B)}$ yield $f \circ g = 1_B$ and $g \circ f = 1_A$ as desired.

3) Let $F : \mathcal{C} \to \mathcal{D}$ be a full and faithful functor and $I, C \in \mathrm{Ob}\,\mathcal{C}$. We have the following bijection of sets:

$$\mathrm{Hom}_{\mathcal{C}}(I, C) \cong \mathrm{Hom}_{\mathcal{D}}(F(I), F(C))$$

If $F(I)$ is an initial object in $\mathcal{D}$ then $1 = |\mathrm{Hom}_{\mathcal{D}}(F(I), F(C))|$ for any $C \in \mathrm{Ob}\,\mathcal{C}$ and the above isomorphism gives $1 = |\mathrm{Hom}_{\mathcal{C}}(I, C)|$, i.e. I is an initial object in $\mathcal{C}$. The last statement follows in a similar manner.

4) Let $F : \mathcal{C} \to \mathcal{D}$ be a faithful functor and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that $F(f)$ is a monomorphism in $\mathcal{D}$. Consider now $C \in \mathrm{Ob}\,\mathcal{C}$ and $g, h : C \to A$ such that $f \circ g = f \circ h$. By applying F to this equality gives $F(f) \circ F(g) = F(f) \circ F(h)$. Since $F(f)$ is a monomorphism this implies $F(g) = F(h)$ and by the faithfulness of F we get $g = h$ as desired. A similar argument proves that F reflects epimorphisms as well. $\square$

Corollary 1.4.18 *Let $\mathcal{C}$ be a concrete category, $F : \mathcal{C} \to$ **Set** the corresponding faithful functor and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.*

1) If $F(f)$ is injective then f is a monomorphism in $\mathcal{C}$;

2) If $F(f)$ is surjective then f is an epimorphism in $\mathcal{C}$.

Proof: 1) We already proved that monomorphisms in **Set** coincide with injective maps (see Example 1.2.3). Since $F(f)$ is a monomorphism in **Set** and F is faithful it follows by Proposition 1.4.17, 4) that f is also a monomorphism. The second statement follows by duality. $\square$

Remark 1.4.19 *Each of the following categories* $\boldsymbol{Grp}$, $\boldsymbol{Ab}$, $\boldsymbol{Rng}$, $\boldsymbol{Ring}$, $\boldsymbol{Ring}^c$, $\boldsymbol{Div}$, ${}_R\mathcal{M}$, $\boldsymbol{Top}$ *allow for a forgetful functor into* $\boldsymbol{Set}$. *Hence, we can conclude that in the above mentioned categories all injective maps are monomorphisms and respectively all surjective maps are epimorphisms. However, the converse is not necessarily true as it can be seen from Example 1.2.3, 2) and 3) respectively.*

1.5 Natural transformations. Representable functors

Definition 1.5.1 *Let* $F, G : \mathcal{C} \to \mathcal{D}$ *be two functors. A natural transformation* $\alpha : F \to G$ *consists of a family of morphisms* $\big(\alpha_C : F(C) \to G(C)\big)_{C \in \operatorname{Ob}\mathcal{C}}$ *in* $\mathcal{D}$ *such that for every morphism* $f \in \operatorname{Hom}_{\mathcal{C}}(C, C')$ *the following diagram is commutative:*

$$\begin{tikzcd} F(C) \arrow[r, "\alpha_C"] \arrow[d, "F(f)"'] & G(C) \arrow[d, "G(f)"] \\ F(C') \arrow[r, "\alpha_{C'}"'] & G(C') \end{tikzcd} \qquad \text{i.e. } \alpha_{C'} \circ F(f) = G(f) \circ \alpha_C. \tag{1.2}$$

Assuming it exists, we denote by $\operatorname{Nat}(F, G)$ *the class of all natural transformations between* F *and* G.

If all components α_C *are isomorphisms then* α *is called a natural isomorphism. In this case we say that the functors* F *and* G *are naturally isomorphic and we use the notation* $F \cong G$.

Examples 1.5.2 *1) For any functor* $F : \mathcal{C} \to \mathcal{D}$ *we have a natural transformation* $1_F : F \to F$ *called the identity natural transformation defined by* $1_F = \big(1_{F(C)}\big)_{C \in \operatorname{Ob}\mathcal{C}}$.

2) If $F, G : \mathcal{C} \to \mathcal{D}$ *are functors and* $\alpha : F \to G$ *is a natural isomorphism then* $\alpha^{-1} : G \to F$ *defined by:*

$$\alpha_C^{-1} = \big(\alpha_C\big)^{-1} \text{ for all } C \in \operatorname{Ob}\mathcal{C}$$

is obviously also a natural isomorphism.

3) If $F, G, H : \mathcal{C} \to \mathcal{D}$ *are functors and* $\alpha : F \to G$, $\beta : G \to H$ *are natural transformations then we can define a new natural transformation* $\beta \circ \alpha : F \to H$ *by the formula:*

$$(\beta \circ \alpha)_C : F(C) \to H(C), \ (\beta \circ \alpha)_C = \beta_C \circ \alpha_C \text{ for all } C \in \operatorname{Ob}\mathcal{C}.$$

4) If $F, G : \mathcal{C} \to \mathcal{D}$, $H : \mathcal{D} \to \mathcal{E}$ *are functors and* $\alpha : F \to G$ *is a natural transformation then* $H\alpha : HF \to HG$ *defined by:*

$$(H\alpha)_C = H(\alpha_C) \text{ for all } C \in \operatorname{Ob}\mathcal{C}$$

is also a natural transformation. Similarly, if $K : \mathcal{B} \to \mathcal{C}$ is a functor then $\alpha_K : FK \to GK$ defined by:

$$(\alpha_K)_B = \alpha_{K(B)} \text{ for all } B \in \operatorname{Ob}\mathcal{B}$$

is also a natural transformation. Furthermore, if α is a natural isomorphism then both $H\alpha$ and α_K are natural isomorphisms.

5) *Let K be a field, $V \in {}_K\mathcal{M}$ and V^* its dual vector space (see Example 1.4.3, 10)). Define a functor $F\colon {}_K\mathcal{M} \to {}_K\mathcal{M}$ by $F(V) = V^{**}$, $F(u) = u^{**}$ for all vector spaces V and linear maps u. We can define a natural transformation $\eta\colon 1_{{}_K\mathcal{M}} \to F$ as follows: for a vector space V, let $\eta_V\colon V \to V^{**}$ be the linear map given by $\eta_V(v)(f) = f(v)$ for all $v \in V$ and $f \in V^*$. Indeed, we will show that for any $u \in \operatorname{Hom}_{{}_K\mathcal{M}}(V, W)$ the following diagram is commutative:*

$$\begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[d, "u"'] & V^{**} \arrow[d, "u^{**}"] \\ W \arrow[r, "\eta_W"'] & W^{**} \end{tikzcd}$$

To this end, for any $v \in V$ and $f \in W^*$ we have:

$$\big[u^{**} \circ \eta_V(v)\big](f) = \big[\eta_V(v) \circ u^*\big](f) = \eta_V(v)\big(u^*(f)\big) = u^*(f)(v) = \big(f \circ u\big)(v)$$
$$= f\big(u(v)\big) = \eta_W\big(u(v)\big)(f) = \big[(\eta_W \circ u)(v)\big](f).$$

6) *Let G_1 and G_2 be two groups, $\mathcal{G}_1$, respectively $\mathcal{G}_2$ the corresponding categories (see Example 1.1.4, 3)) and F, $H : \mathcal{G}_1 \to \mathcal{G}_2$ two functors. We denote by f_F, respectively f_H the morphisms of groups from G_1 to G_2 corresponding to the two functors F and H (see Example 1.4.3, 8)). Denote $\operatorname{Ob}\mathcal{G}_1 = \{*\}$ and $\operatorname{Ob}\mathcal{G}_2 = \{\star\}$. Then, since $F(*) = \star$ and $H(*) = \star$, a natural transformation $\varphi : F \to H$ is completely determined by a morphism $\varphi_* : \star \to \star$ in $\mathcal{G}_2$ (i.e. an element of the group G_2) which makes the following diagram commute for all morphisms $t : * \to *$ in $\mathcal{G}_1$ (i.e. an element of the group G_1):*

$$\begin{tikzcd} F(*) = \star \arrow[r, "\varphi_*"] \arrow[d, "F(t)"'] & H(*) = \star \arrow[d, "H(t)"] \\ F(*) = \star \arrow[r, "\varphi_*"'] & H(*) = \star \end{tikzcd}$$

i.e. $H(t) \circ \varphi_* = \varphi_* \circ F(t)$. Having in mind that the composition of morphisms in $\mathcal{G}_2$ is given by the multiplication in the group G_2, we can conclude that the natural transformations from F to H are in bijection with elements $g \in G_2$ such that for any $g' \in G_1$ we have $f_H(g')g = gf_F(g')$. Hence, we have an isomorphism of sets $\operatorname{Nat}(F, H) \cong \{g \in G_2 \mid f_H(g')g = gf_F(g'),\ \text{for all } g' \in G_1\}$.

Proposition 1.5.3 *Let I and $\mathcal{C}$ be two categories. If I is a small category then the functors from I to $\mathcal{C}$ and the natural transformations between them form a category, called functor category, which we will denote by* $\mathrm{Fun}(\mathrm{I}, \mathcal{C})$. *If $\mathcal{C}$ is also small then* $\mathrm{Fun}(\mathrm{I}, \mathcal{C})$ *is small.*

Proof: The composition of natural transformations is defined as in Example 1.5.2, 3). This composition law is obviously associative and the identity at each functor F is just the identity natural transformation (see Example 1.5.2, 1)).

Finally, remark that for any two functors $F,\ G : I \to \mathcal{C}$, a natural transformation $\eta : F \to G$ is determined by a map

$$\eta : \mathrm{Ob}\, I \to \bigcup_{i \in \mathrm{Ob}\, I} \mathrm{Hom}_{\mathcal{C}}\big(F(i),\, G(i)\big).$$

Now since $\mathrm{Ob}\, I$ and $\mathrm{Hom}_{\mathcal{C}}\big(F(i),\, G(i)\big)$ are both sets it follows that the class of all natural transformations from F to G is actually a set. Hence $\mathrm{Fun}(\mathrm{I}, \mathcal{C})$ is indeed a category. $\square$

Remark 1.5.4 *Let I be a small category. Then, for any category $\mathcal{C}$ we can define a functor $\Delta : \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$, called* the diagonal functor, *which assigns to each $C \in \mathrm{Ob}\, \mathcal{C}$ the constant functor $\Delta(C) : I \to \mathcal{C}$ at C (the functor that sends each object of the category I to C and and each morphism of I to the identity 1_C, see Example 1.4.3, 2)) and to each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(C,\, D)$ the natural transformation $\eta : \Delta(C) \to \Delta(D)$ given by $\eta_i = f$ for all $i \in \mathrm{Ob}\, I$.*

Definition 1.5.5 *Let $\mathcal{C}$ be a category. We say that a functor $F : \mathcal{C} \to$ **Set** is representable if there exist $C \in \mathrm{Ob}\, \mathcal{C}$ and a natural isomorphism $F \cong \mathit{Hom}_{\mathcal{C}}(C, -)$. A contravariant functor $G : \mathcal{C} \to$ **Set** is representable if there exists $C \in \mathrm{Ob}\, \mathcal{C}$ such that $G \cong \mathit{Hom}_{\mathcal{C}}(-, C)$. In this case, C is called the representing object.*

Remark 1.5.6 *More precisely, Definition 1.5.5 reads as follows: there exist an object $C \in \mathrm{Ob}\, \mathcal{C}$ and a family of isomorphisms $\Big(\alpha_A : \mathrm{Hom}_{\mathcal{C}}(C,\, A) \to F(A)\Big)_{A \in \mathrm{Ob}\, \mathcal{C}}$ in **Set** (set bijections) such that for any $f \in \mathrm{Hom}_{\mathcal{C}}(X,\, Y)$ the following diagram is commutative:*

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C,\, X) \arrow[r, "\alpha_X"] \arrow[d, "\mathit{Hom}_{\mathcal{C}}(C,\, f)"'] & F(X) \arrow[d, "F(f)"] \\ \mathrm{Hom}_{\mathcal{C}}(C,\, Y) \arrow[r, "\alpha_Y"'] & F(Y) \end{tikzcd}$$

i.e. $F(f) \circ \alpha_X = \alpha_Y \circ \mathit{Hom}_{\mathcal{C}}(C,\, f)$.

Examples 1.5.7 *1) The forgetful functor U : **Grp** $\to$ **Set** is representable and the representing object is $(\mathbb{Z}, +)$. Indeed, for any $X \in \mathrm{Ob}\,$**Grp** and any $g \in \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z},\, X)$ we define $\alpha_X : \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z},\, X) \to U(X) = X$ by $\alpha_X(g) = g(1)$.*

The above diagram is now obviously commutative for any $f \in \mathrm{Hom}_{\boldsymbol{Grp}}(X, Y)$:

$$\begin{tikzcd} \mathrm{Hom}_{\boldsymbol{Grp}}(\mathbb{Z}, X) \arrow[r, "\alpha_X"] \arrow[d, "\mathit{Hom}_{\boldsymbol{Grp}}(\mathbb{Z}, f)"'] & U(X) = X \arrow[d, "U(f)=f"] \\ \mathrm{Hom}_{\boldsymbol{Grp}}(\mathbb{Z}, Y) \arrow[r, "\alpha_Y"'] & U(Y) = Y \end{tikzcd}$$

Indeed, for any $g \in \mathrm{Hom}_{\boldsymbol{Grp}}(\mathbb{Z}, X)$ we have $\alpha_Y \circ \mathit{Hom}_{\boldsymbol{Grp}}(\mathbb{Z}, f)(g) = \alpha_Y\big(f \circ g\big) = (f \circ g)(1)$ and $f \circ \alpha_X(g) = f(g(1))$.

2) *The forgetful functor $U : \boldsymbol{Top} \to \boldsymbol{Set}$ is representable and the representing object is any singleton topological space $\{x_0\}$ (with the discrete topology). To this end, for any $X \in \mathrm{Ob}\, \boldsymbol{Top}$ and any $h \in \mathrm{Hom}_{\boldsymbol{Top}}(\{x_0\}, X)$ we define $\alpha_X : \mathrm{Hom}_{\boldsymbol{Top}}(\{x_0\}, X) \to U(X) = X$ by $\alpha_X(h) = h(x_0) \in X$. It can be easily seen that the above diagram is commutative for any $f \in \mathrm{Hom}_{\boldsymbol{Top}}(X, Y)$:*

$$\begin{tikzcd} \mathrm{Hom}_{\boldsymbol{Top}}(\{x_0\}, X) \arrow[r, "\alpha_X"] \arrow[d, "\mathit{Hom}_{\boldsymbol{Top}}(\{x_0\}, f)"'] & U(X) = X \arrow[d, "U(f)=f"] \\ \mathrm{Hom}_{\boldsymbol{Top}}(\{x_0\}, Y) \arrow[r, "\alpha_Y"'] & U(Y) = Y \end{tikzcd}$$

Indeed, for any $h \in \mathrm{Hom}_{\boldsymbol{Top}}(\{x_0\}, X)$ we have $f \circ \alpha_X(h) = f(h(x_0))$ and $\alpha_Y \circ \mathit{Hom}_{\boldsymbol{Top}}(\{x_0\}, f)(h) = \alpha_Y\big(f \circ h\big) = (f \circ h)(x_0)$.

3) *The constant functor $F : \mathcal{C} \to \boldsymbol{Set}$ which sends every object of $\mathcal{C}$ to the singleton $\{x_0\}$ and every morphism in $\mathcal{C}$ to the identity map on $\{x_0\}$ is representable if and only if $\mathcal{C}$ has an initial object. Moreover, in this case the representing object is the initial object. Indeed, suppose the functor is represented by $I \in \mathrm{Ob}\,\mathcal{C}$. Then, for any $C \in \mathrm{Ob}\,\mathcal{C}$ we have an isomorphism in $\boldsymbol{Set}$ denoted by $\alpha_C : \mathrm{Hom}_{\mathcal{C}}(I, C) \to \{x_0\}$. This implies that $\mathrm{Hom}_{\mathcal{C}}(I, C)$ has exactly one element for any $C \in \mathrm{Ob}\,\mathcal{C}$, i.e. I is initial in $\mathcal{C}$.*

4) *The power-set functor $\mathcal{P} : \boldsymbol{Set} \to \boldsymbol{Set}$ is not representable. Indeed, assume that $\mathcal{P}$ is representable. Let A be the representing object and $\tau : \mathrm{Hom}_{\boldsymbol{Set}}(A, -) \to \mathcal{P}$ the corresponding natural isomorphism. Consider $\{*\}$ to be an arbitrary singleton. Since τ is a natural isomorphism we obtain a bijective map $\tau : \mathrm{Hom}_{\boldsymbol{Set}}(A, \{*\}) \to \mathcal{P}(\{*\})$. This leads to a contradiction since $|\mathrm{Hom}_{\boldsymbol{Set}}(A, \{*\})| = 1$ and $|\mathcal{P}(\{*\})| = 2$. Therefore, $\mathcal{P}$ is not representable.*

The following result provides an important criterion for deciding whether a functor is representable or not.

Proposition 1.5.8 *Let $F : \mathcal{C} \to \boldsymbol{Set}$ be a functor. Then F is representable if and only if there exists a pair (A, a) with $A \in \mathrm{Ob}\,\mathcal{C}$ and $a \in F(A)$ satisfying the following property: for any other pair (B, b) with $B \in \mathrm{Ob}\,\mathcal{C}$ and $b \in F(B)$ there exists a unique $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that $F(f)(a) = b$. In this case (A, a) is called a representing pair.*

Proof: Suppose first that F is representable, i.e. there exists a natural isomorphism $\varphi : \mathtt{Hom}_{\mathcal{C}}(A, -) \to F$ for some $A \in \operatorname{Ob}\mathcal{C}$. In particular we have a bijection of sets $\varphi_A : \operatorname{Hom}_{\mathcal{C}}(A, A) \to F(A)$ and we denote by $a = \varphi_A(1_A) \in F(A)$. We will show that (A, a) is a representing pair. Indeed, consider $B \in \operatorname{Ob}\mathcal{C}$ and $b \in F(B)$. As before, we have a bijective map $\varphi_B : \operatorname{Hom}_{\mathcal{C}}(A, B) \to F(B)$ so there exists a unique $f \in \operatorname{Hom}_{\mathcal{C}}(A, B)$ such that $\varphi_B(f) = b$. We are left to prove that $F(f)(a) = b$. Since φ is a natural transformation, the following diagram is commutative:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}}(A, A) \arrow[r, "\varphi_A"] \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(A, f)"'] & F(A) \arrow[d, "F(f)"] \\
\operatorname{Hom}_{\mathcal{C}}(A, B) \arrow[r, "\varphi_B"'] & F(B)
\end{tikzcd}$$

i.e. $\varphi_B \circ \mathtt{Hom}_{\mathcal{C}}(A, f) = F(f) \circ \varphi_A$. This yields $F(f) = \varphi_B \circ \mathtt{Hom}_{\mathcal{C}}(A, f) \circ \varphi_A^{-1}$ and we obtain:

$$\begin{aligned}
F(f)(a) &= \varphi_B \circ \mathtt{Hom}_{\mathcal{C}}(A, f) \circ \varphi_A^{-1}(a) \\
&= \varphi_B \circ \mathtt{Hom}_{\mathcal{C}}(A, f)(1_A) \\
&= \varphi_B\big(f \circ 1_A\big) \\
&= \varphi_B(f) = b
\end{aligned}$$

Assume now that (A, a) is a representing pair. Let $\psi : \mathtt{Hom}_{\mathcal{C}}(A, -) \to F$ be the natural isomorphism defined as follows for any $B \in \operatorname{Ob}\mathcal{C}$:

$$\psi_B : \operatorname{Hom}_{\mathcal{C}}(A, B) \to F(B), \ \ \psi_B(f) = F(f)(a), \ \ f \in \operatorname{Hom}_{\mathcal{C}}(A, B).$$

The property assumed to be satisfied by (A, a) implies that each such map ψ_B is bijective. The proof will be finished once we show that ψ is a natural transformation, i.e. for any $g \in \operatorname{Hom}_{\mathcal{C}}(B, C)$ the following diagram is commutative:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}}(A, B) \arrow[r, "\psi_B"] \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(A, g)"'] & F(B) \arrow[d, "F(g)"] \\
\operatorname{Hom}_{\mathcal{C}}(A, C) \arrow[r, "\psi_C"'] & F(C)
\end{tikzcd}$$

Indeed, for any $h \in \operatorname{Hom}_{\mathcal{C}}(A, B)$ we have:

$$F(g)\big(\psi_B(h)\big) = F(g)\big(F(h)(a)\big) = F(g \circ h)(a) = \psi_C(g \circ h) = \psi_C \circ \mathtt{Hom}_{\mathcal{C}}(A, g)(h)$$

as desired. This finishes the proof. $\square$

Example 1.5.9 *Let A be a group and $H \trianglelefteq A$ a normal subgroup. Consider the functor $F : \boldsymbol{Grp} \to \boldsymbol{Set}$ defined as follows:*

$$\begin{aligned}
&F(G) = \{f \in \operatorname{Hom}_{\boldsymbol{Grp}}(A, G) \mid H \subseteq \ker f\} \\
&F(u)(g) = u \circ g
\end{aligned}$$

for any $G \in \mathrm{Ob}\,\boldsymbol{Grp}$ and any $u \in \mathrm{Hom}_{\boldsymbol{Grp}}(G, G')$, $g \in F(G)$. Then F is representable and $\big(A/H, \pi : A \to A/H\big)$ is the representing pair, where π is the canonical projection. Indeed, consider another pair $(G, f : A \to G)$ with $G \in \mathrm{Ob}\,\boldsymbol{Grp}$ and $f \in F(G)$.

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "f"'] & A/H \arrow[dl, "\overline{f}"] \\ G & \end{tikzcd}$$

Since $H \subseteq \ker f$, the universal property of the quotient group A/H yields a unique $\overline{f} \in \mathrm{Hom}_{\boldsymbol{Grp}}(A/H, G)$ such that the above diagram is commutative, i.e. $\overline{f} \circ \pi = f$. The last equality is equivalent to $F(\overline{f})(\pi) = f$ and the desired conclusion now follows from Proposition 1.5.8.

1.6 Yoneda's lemma

Having defined natural transformations, the next obvious question to ask is whether there is a general strategy for computing them. Yoneda's lemma provides a useful result for the case where one of the functors is a Hom functor.

Theorem 1.6.1 (Yoneda's lemma) *Let $F : \mathcal{C} \to \boldsymbol{Set}$ be a functor and $C \in \mathrm{Ob}\,\mathcal{C}$. Then we have a bijection between the natural transformations from $\mathtt{Hom}_{\mathcal{C}}(C, -)$ to F and the elements of the set $F(C)$ given by:*

$$\pi : \mathrm{Nat}\big(\mathtt{Hom}_{\mathcal{C}}(C, -),\, F\big) \to F(C),\;\; \pi(\varphi) = \varphi_C(1_C) \in F(C)$$

for any natural transformation $\varphi : \mathtt{Hom}_{\mathcal{C}}(C, -) \to F$.

Proof: Consider $\tau : F(C) \to \mathrm{Nat}\big(\mathtt{Hom}_{\mathcal{C}}(C, -),\, F\big)$ defined for every $x \in F(C)$ by:

$$\tau(x) = h^x \text{ where } \big(h^x\big)_D(f) = F(f)(x) \tag{1.3}$$

for every $D \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(C, D)$. First we have to check that h^x is a natural transformation. This comes down to proving that given a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ the following diagram is commutative:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C, A) \arrow[r, "(h^x)_A"] \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(C, f)"'] & F(A) \arrow[d, "F(f)"] \\ \mathrm{Hom}_{\mathcal{C}}(C, B) \arrow[r, "(h^x)_B"'] & F(B) \end{tikzcd}$$

Indeed, for any $g \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ we have:

$$\begin{aligned} (h^x)_B\big(\mathtt{Hom}_{\mathcal{C}}(C, f)\big)(g) &= (h^x)_B(f \circ g) \\ &= F(f \circ g)(x) = F(f) \circ F(g)(x) \\ &= F(f)(h^x)_A(g) \end{aligned}$$

Thus h^x is a natural transformation. The proof will be finished once we show that π and τ are inverses to each other. To start with, consider $x \in F(C)$. Then we have:

$$\begin{aligned}(\pi \circ \tau)(x) &= \pi\big(\tau(x)\big) = \pi\big(h^x\big) \\ &= \big(h^x\big)_C(1_C) = F(1_C)(x) \\ &= 1_{F(C)}(x) = x\end{aligned}$$

Thus $\pi \circ \tau = 1_{F(C)}$. Consider now a natural transformation $\varphi : \mathtt{Hom}_{\mathcal{C}}(C, -) \to F$. We want to prove that $(\tau \circ \pi)(\varphi) = \varphi$. Indeed, as φ is a natural transformation, for every $D \in \mathrm{Ob}\,\mathcal{C}$ and every $f \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ the following diagram is commutative:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C, C) \arrow[r, "\varphi_C"] \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(C, f)"'] & F(C) \arrow[d, "F(f)"] \\ \mathrm{Hom}_{\mathcal{C}}(C, D) \arrow[r, "\varphi_D"'] & F(D) \end{tikzcd}$$

In particular, by evaluating the above diagram at $1_C \in \mathrm{Hom}_{\mathcal{C}}(C, C)$ we obtain:

$$F(f)\big(\varphi_C(1_C)\big) = \varphi_D(f) \tag{1.4}$$

Hence for every $D \in \mathrm{Ob}\,\mathcal{C}$ and every $f \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ we have:

$$\begin{aligned}\tau\big(\pi(\varphi)\big)_D(f) &= \tau\big(\varphi_C(1_C)\big)_D(f) \\ &= F(f)\big(\varphi_C(1_C)\big) \overset{(1.4)}{=} \varphi_D(f)\end{aligned}$$

which implies $\tau \circ \pi(\varphi) = \varphi$ and the proof is now complete. $\square$

Proposition 1.6.2 *Let $\mathcal{C}$ be a category and C, $D \in \mathrm{Ob}\,\mathcal{C}$. Then C and D are isomorphic if and only if the functors $\mathtt{Hom}_{\mathcal{C}}(C, -)$ and $\mathtt{Hom}_{\mathcal{C}}(D, -)$ are naturally isomorphic.*

Proof: Suppose first that C and D are isomorphic objects in $\mathcal{C}$ and consider $\phi \in \mathrm{Hom}_{\mathcal{C}}(D, C)$ to be an isomorphism. We apply the Yoneda lemma for $F = \mathtt{Hom}_{\mathcal{C}}(D, -)$ and we obtain a bijection of sets $\tau : \mathrm{Hom}_{\mathcal{C}}(D, C) \to \mathrm{Nat}\big(\mathtt{Hom}_{\mathcal{C}}(C, -), \mathtt{Hom}_{\mathcal{C}}(D, -)\big)$ as defined in (1.3). In particular, we have a natural transformation $\tau(\phi) = h^\phi : \mathtt{Hom}_{\mathcal{C}}(C, -) \to \mathtt{Hom}_{\mathcal{C}}(D, -)$ given by $\big(h^\phi\big)_X(f) = \mathrm{Hom}_{\mathcal{C}}(D, f)(\phi) = f \circ \phi$ for all $X \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(C, X)$. We will prove that $\big(h^\phi\big)_X$ is a set bijection for every $X \in \mathrm{Ob}\,\mathcal{C}$. To this end, we define $\big(\mu^\phi\big)_X$: $\mathrm{Hom}_{\mathcal{C}}(D, X) \to \mathrm{Hom}_{\mathcal{C}}(C, X)$ by $\big(\mu^\phi\big)_X(g) = g \circ \phi^{-1}$ for any $g \in \mathrm{Hom}_{\mathcal{C}}(D, X)$. We will see that $\big(\mu^\phi\big)_X$ is the inverse of $\big(h^\phi\big)_X$. Indeed, for any $g \in \mathrm{Hom}_{\mathcal{C}}(D, X)$ and any $f \in \mathrm{Hom}_{\mathcal{C}}(C, X)$ we have:

$$\begin{aligned}\big(h^\phi\big)_X \circ \big(\mu^\phi\big)_X(g) = \big(h^\phi\big)_X\big(g \circ \phi^{-1}\big) = g \circ \phi^{-1} \circ \phi = g \\ \big(\mu^\phi\big)_X \circ \big(h^\phi\big)_X(f) = \big(\mu^\phi\big)_X\big(f \circ \phi\big) = f \circ \phi \circ \phi^{-1} = f\end{aligned}$$

Hence, $\tau(\phi) : \mathtt{Hom}_{\mathcal{C}}(C, -) \to \mathtt{Hom}_{\mathcal{C}}(D, -)$ is a natural isomorphism, as desired.

Conversely, let $\alpha : \mathtt{Hom}_{\mathcal{C}}(C, -) \to \mathtt{Hom}_{\mathcal{C}}(D, -)$ be a natural isomorphism. Denote by $u := \alpha_C(1_C) \in \mathrm{Hom}_{\mathcal{C}}(D, C)$; we will prove that u is an isomorphism. We start by pointing out that $\alpha_D : \mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathrm{Hom}_{\mathcal{C}}(D, D)$ is a bijection and since $1_D \in \mathrm{Hom}_{\mathcal{C}}(D, D)$ we can find $v \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ such that $\alpha_D(v) = 1_D$.

Now as α is a natural transformation, the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(C, D) \arrow[r, "\alpha_D"] \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(C{,}u)"'] & \mathrm{Hom}_{\mathcal{C}}(D, D) \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(D{,}u)"] \\
\mathrm{Hom}_{\mathcal{C}}(C, C) \arrow[r, "\alpha_C"'] & \mathrm{Hom}_{\mathcal{C}}(D, C)
\end{tikzcd}$$

i.e. $\mathtt{Hom}_{\mathcal{C}}(D, u) \circ \alpha_D = \alpha_C \circ \mathtt{Hom}_{\mathcal{C}}(C, u)$. By evaluating this diagram at $v \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ and using $\alpha_D(v) = 1_D$ we obtain:

$$\begin{aligned}
& \mathtt{Hom}_{\mathcal{C}}(D, u) \circ \alpha_D \circ v = \alpha_C \circ \mathtt{Hom}_{\mathcal{C}}(C, u) \circ v \\
\Leftrightarrow \quad & u \circ \alpha_D(v) = \alpha_C(u \circ v) \\
\Leftrightarrow \quad & u = \alpha_C(u \circ v)
\end{aligned}$$

Since we also have $u = \alpha_C(1_C)$ and α_C is a set bijection we get $u \circ v = 1_C$.

Again because α is a natural transformation we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(C, C) \arrow[r, "\alpha_C"] \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(C{,}v)"'] & \mathrm{Hom}_{\mathcal{C}}(D, C) \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(D{,}v)"] \\
\mathrm{Hom}_{\mathcal{C}}(C, D) \arrow[r, "\alpha_D"'] & \mathrm{Hom}_{\mathcal{C}}(D, D)
\end{tikzcd}$$

Evaluating the above diagram at $1_C \in \mathrm{Hom}_{\mathcal{C}}(C, C)$ and using $\alpha_D(v) = 1_D$ we obtain:

$$\begin{aligned}
& \mathtt{Hom}_{\mathcal{C}}(D, v) \circ \alpha_C \circ 1_C = \alpha_D \circ \mathtt{Hom}_{\mathcal{C}}(C, v) \circ 1_C \\
& \Leftrightarrow v \circ \alpha_C(1_C) = \alpha_D(v \circ 1_C) \\
& \Leftrightarrow v \circ u = 1_D
\end{aligned}$$

therefore v is the inverse of u and the proof is now finished. $\square$

In light of the above result we obtain:

Corollary 1.6.3 *Representing objects are unique up to isomorphism.*

1.7 Exercises

1. (a) Is **Grp** a subcategory of **Set**?
   (b) Is **Ring** a subcategory of **Ab**?

2. Let $I \in \operatorname{Ob}\mathcal{C}$ be an initial object. Prove that any monomorphism $m : C \to I$ is an isomorphism.

3. Let $\mathcal{C}$ be a category and $f \in \operatorname{Hom}_{\mathcal{C}}(A, B)$, $g \in \operatorname{Hom}_{\mathcal{C}}(B, C)$. Prove that:

   (a) If both f and g are monomorphisms (epimorphisms) then $g \circ f$ is a monomorphism (epimorphism);
   (b) If the composition $g \circ f$ is a monomorphism then f is a monomorphism;
   (c) If the composition $g \circ f$ is an epimorphism then g is an epimorphism.

4. If $A \in \operatorname{Ob}\mathcal{C}$ is an initial object and $B \in \operatorname{Ob}\mathcal{C}$ such that A and B are isomorphic then B is also an initial object in $\mathcal{C}$.

5. Prove that any group regarded as a one-object category is isomorphic to its opposite. Is the assertion still true for monoids?

6. a) Give examples to show that functors do not necessarily preserve or reflect monomorphisms and epimorphisms;
   b) Give an example of a functor which reflects isomorphisms without being full and faithful.

7. Let G be a group and consider $\mathcal{G}$ to be the corresponding category. For a normal subgroup H of G we define the following relation on the morphisms of $\mathcal{G}$:

$$x \equiv y \text{ if and only if } xy^{-1} \in H.$$

   Prove that $\equiv$ is a congruence relation and describe the corresponding quotient category.

8. Let $F : \mathcal{C} \to \mathcal{D}$ be a full and faithful functor. If $C, D \in \operatorname{Ob}\mathcal{C}$ such that $F(C) \cong F(D)$ then $C \cong D$.

9. a) Construct a faithful functor which is not injective on objects/morphisms;
   b) Construct a full functor which is not surjective on objects/morphisms.

10. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors such that GF is naturally isomorphic to $1_{\mathcal{C}}$. Prove that F is faithful.

11. Show that if two functors $F, G : \mathcal{C} \to \mathcal{D}$ are naturally isomorphic then F is full (resp. faithful) if and only if G is full (resp. faithful.)

12. Let $U : \mathbf{Mon} \to \mathbf{Set}$ be the forgetful functor and define $U \times U : \mathbf{Mon} \to \mathbf{Set}$ as follows:

$(U \times U)(M) = U(M) \times U(M)$ for all $M \in \operatorname{Ob} \mathbf{Mon}$;
$(U \times U)(f)(m, m') = (f(m), f(m'))$ for all $f \in \operatorname{Hom}_{\mathbf{Mon}}(M, N)$, $m, m' \in U(M)$.

Prove that $\gamma : U \times U \to U$, defined by $\gamma_M = m_M$ for all $M \in \operatorname{Ob} \mathbf{Mon}$ is a natural transformation, where m_M denotes the multiplication of the monoid M.

13. Let $F, G : \mathcal{C} \to \mathcal{D}$, $H, K : \mathcal{D} \to \mathcal{E}$ be functors and $\alpha : F \to G$, $\beta : H \to K$ natural transformations.

   (a) Prove that $\beta \star \alpha : HF \to KG$ defined by $(\beta \star \alpha)_C = \beta_{G(C)} \circ H\big(\alpha_C\big)$ for all $C \in \operatorname{Ob} \mathcal{C}$ is a natural transformation;

   (b) Show that for all $C \in \operatorname{Ob} \mathcal{C}$ we have $K\big(\alpha_C\big) \circ \beta_{F(C)} = \beta_{G(C)} \circ H\big(\alpha_C\big)$.

14. Let G be a group and consider $\mathcal{G}$ to be the corresponding category. Describe $\operatorname{Nat}\big(1_{\mathcal{G}}, 1_{\mathcal{G}}\big)$.

15. Let $(X, \leqslant)$ and $(Y, \ll)$ two posets and $\operatorname{PO}(X, \leqslant)$, respectively $\operatorname{PO}(Y, \ll)$ the associated categories. Describe the functors between these two categories. If F, G:$\operatorname{PO}(X, \leqslant) \to \operatorname{PO}(Y, \ll)$ are two such functors, describe $\operatorname{Nat}\big(F, G\big)$.

16. Let $\mathcal{C}$ be a category. Construct a functor $F : \mathcal{C} \to \mathbf{Set}$ which is not representable.

17. Prove that the contravariant power-set functor $P^c : \mathbf{Set} \to \mathbf{Set}$ is representable.

18. Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathbf{Set}$ a functor. Prove that:

   (a) If F is representable then it preserves monomorphisms;

   (b) If F is contravariant representable then it maps epimorphisms to monomorphisms.

19. Let $G \in \operatorname{Ob} \mathbf{Grp}$. Prove that the functor $F_G : \mathbf{Grp} \to \mathbf{Set}$ defined as follows for any $H \in \operatorname{Ob} \mathbf{Grp}$, $f \in \operatorname{Hom}_{\mathbf{Grp}}(G, H)$:

$$F_G(H) = \{f \in \operatorname{Hom}_{\mathbf{Set}}(G, H) \mid f \text{ is an antimorphism of groups}\};$$
$$F(f)(g) = f \circ g, \; g \in F_G(H)$$

   is representable.

20. A family $(G_i)_{i \in I}$ of objects in a category $\mathcal{C}$, where I is a set, is called a *set of generators* if for each pair of distinct morphisms $f, g \in \operatorname{Hom}_{\mathcal{C}}(A, B)$ there exists some $i_0 \in I$ and a morphism $h \in \operatorname{Hom}_{\mathcal{C}}(G_{i_0}, A)$ such that $f \circ h \neq g \circ h$. If the set of generators contains only one object G, then G will be called *generator*.

   Show that for any small category $\mathcal{C}$ the category $\operatorname{Fun}(\mathcal{C}, \mathbf{Set})$ has a set of generators.

2

Limits and colimits

Limits and colimits are fundamental notions in category theory. Many well-known constructions from different fields of mathematics such as the free product of groups, the tensor product of (co)algebras or the direct sum of modules are in fact special instances of this very general concepts. In fact, we will see that the initial/final objects introduced in the previous chapter are also special cases of colimits, respectively limits.

2.1 (Co)products, (co)equalizers, pullbacks and pushouts

In order to get a better understanding of (co)limits we will introduce them gradually starting with some generic special cases, namely (co)products, (co)equalizers, pullbacks and pushouts.

Definition 2.1.1 *Let I be a set and $\big(P_i\big)_{i\in I}$ a family of objects in a category $\mathcal{C}$. A pair $\big(\prod_{i\in I} P_i, (p_i)_{i\in I}\big)$ where $\prod_{i\in I} P_i \in \operatorname{Ob}\mathcal{C}$ and $p_j : \prod_{i\in I} P_i \to P_j$ are morphisms in $\mathcal{C}$ for all $j \in I$ is called the product of the family $(P_i)_{i\in I}$ if for any other pair $\big(P, (f_i)_{i\in I}\big)$ where $P \in \operatorname{Ob}\mathcal{C}$ and $f_j : P \to P_j$ are morphisms in $\mathcal{C}$ for all $j \in I$ there exists a unique morphism $f : P \to \prod_{i\in I} P_i$ in $\mathcal{C}$ such that the following diagram commutes for all $j \in I$:*

$$\begin{tikzcd} P \arrow[d, "f"'] \arrow[dr, "f_j"] & \\ \prod_{i\in I} P_i \arrow[r, "p_j"'] & P_j \end{tikzcd}$$

i.e. $p_j \circ f = f_j$.

We say that $\mathcal{C}$ is a category with (finite) products or that $\mathcal{C}$ has (finite) products if there is a product in $\mathcal{C}$ for any (finite) family $\big(P_i\big)_{i\in I}$ of objects.

Coproducts are the dual notion of products, i.e. the coproduct of the family $\big(X_i\big)_{i\in I}$ of objects of a category $\mathcal{C}$ is defined to be the product of the same family of objects in the dual category $\mathcal{C}^{op}$. This comes down to the following:

Definition 2.1.2 *Let I be a set and $\left(Q_i\right)_{i\in I}$ a family of objects in a category $\mathcal{C}$. A pair $\left(\coprod_{i\in I} Q_i, (q_i)_{i\in I}\right)$ where $\coprod_{i\in I} Q_i \in \operatorname{Ob}\mathcal{C}$ and $q_j : Q_j \to \coprod_{i\in I} Q_i$ are morphisms in $\mathcal{C}$ for all $j \in I$ is called the coproduct of the family $(Q_i)_{i\in I}$ if for any other pair $\left(Q, (f_i)_{i\in I}\right)$ where $Q \in \operatorname{Ob}\mathcal{C}$ and $f_j : Q_j \to Q$ are morphisms in $\mathcal{C}$ for all $j \in I$ there exists a unique morphism $f : \coprod_{i\in I} Q_i \to Q$ in $\mathcal{C}$ such that the following diagram commutes for all $j \in I$:*

$$\begin{tikzcd} Q_j \arrow[r, "q_j"] \arrow[dr, "f_j"'] & \coprod_{i\in I} Q_i \arrow[d, "f"] \\ & Q \end{tikzcd} \qquad \text{i.e. } f \circ q_j = f_j.$$

We say that $\mathcal{C}$ is a category with (finite) coproducts or that $\mathcal{C}$ has (finite) coproducts if there is a coproduct in $\mathcal{C}$ for any (finite) family $\left(P_i\right)_{i\in I}$ of objects.

Proposition 2.1.3 *When it exists, the (co)product of a family of objects is unique up to isomorphism.*

Proof: We will only prove the uniqueness up to isomorphism of the products. Let $\mathcal{C}$ be a category and consider $\left(P, (p_i)_{i\in I}\right)$, respectively $\left(\overline{P}, (\overline{p}_i)_{i\in I}\right)$, two products of the same family of objects $\left(P_i\right)_{i\in I}$. Since $\left(\overline{P}, (\overline{p}_i)_{i\in I}\right)$ is a product there exists a unique $f \in \operatorname{Hom}_{\mathcal{C}}(P, \overline{P})$ such that for any $j \in I$ we have:

$$\overline{p}_j \circ f = p_j \tag{2.1}$$

Similarly, as $\left(P, (p_i)_{i\in I}\right)$ is also a product we obtain a unique $g \in \operatorname{Hom}_{\mathcal{C}}(\overline{P}, P)$ such that for any $j \in I$ we have:

$$p_j \circ g = \overline{p}_j \tag{2.2}$$

$$\begin{tikzcd} P \arrow[d, "f"] \arrow[dr, "p_j"] \arrow[dd, bend right=60, "1_P"'] & \\ \overline{P} \arrow[r, "\overline{p}_j"] \arrow[d, "g"] & P_j \\ P \arrow[ur, "p_j"'] & \end{tikzcd}$$

By composing the equality in (2.2) with f on the right and using (2.1) we obtain:

$$p_j \circ (g \circ f) = \overline{p}_j \circ f \overset{(2.1)}{=} p_j \tag{2.3}$$

for any $j \in I$. Applying Definition 2.1.1 to the pair $\left(P, (p_i)_{i\in I}\right)$ seen both as a product and as the other pair, yields a unique $h \in \operatorname{Hom}_{\mathcal{C}}(P, P)$ such that $p_j \circ h = p_j$ for any $j \in I$. By the uniqueness of h we must have $h = 1_P$. Moreover, since by (2.3) the map $g \circ f$ also fulfills the above identity we obtain $g \circ f = 1_P$. In the same manner we obtain $f \circ g = 1_{\overline{P}}$ and therefore P and $\overline{P}$ are isomorphic. $\square$

Examples 2.1.4 *1) The product of any family of objects in* ***Set****,* ***Grp****,* ***Ab****, ${}_R\mathcal{M}$,* ***Ring*** *is given by the cartesian product of the underlaying sets endowed with componentwise operations together with the canonical projections. For instance, in the category* ***Grp*** *for a family $\left(G_i\right)_{i\in I}$ of groups the product $\left(\prod_{i\in I} G_i, (\pi_i)_{i\in I}\right)$ is given as follows:*

$$\prod_{i\in I} G_i = \{(g_i)_{i\in I} \mid g_i \in G_i\},\ \left((g_i)_{i\in I}\right)\cdot\left((h_i)_{i\in I}\right)= \left((g_i \cdot_i h_i)_{i\in I}\right)$$

$$\pi_{i_0} : \prod_{i\in I} G_i \to G_{i_0},\ \pi_{i_0}\left((g_i)_{i\in I}\right)= g_{i_0} \text{ for all } i_0 \in I,$$

where $\cdot_i$ denotes the group multiplication in G_i.

2) *The product of any family of objects in* ***Top*** *is given by the cartesian product of the underlaying sets endowed with the product topology together with the canonical projections;*

3) *Let $PO(X, \leqslant)$ be the category corresponding to the pre-ordered set $(X, \leqslant)$ (as defined in Example 1.1.4, 2)) and $(x_i)_{i\in I}$ a family of objects in $PO(X, \leqslant)$ indexed by a set I, i.e. $x_i \in X$ for any $i \in I$. If it exists, the product of this family is a pair $\left(p, (\pi_i)_{i\in I}\right)$ where $p \in X$ and $\pi_i : p \to x_i$ are morphisms in $PO(X, \leqslant)$. This comes down to $p \leqslant x_i$ for any $i \in I$. Moreover, for any other pair $\left(q, (u_i)_{i\in I}\right)$, where $q \in X$ and $u_i : q \to x_i$ are morphisms in $PO(X, \leqslant)$, there exists a morphism $f : q \to p$. In other words, for any $q \in X$ such that $q \leqslant x_i$ for any $i \in I$ we also have $q \leqslant p$. Therefore, p is precisely the infimum (if it exists) of the family $(x_i)_{i\in I}$;*

4) *The category* ***SiGrp*** *of simple groups does not admit products. Indeed, suppose this category admits products and consider H, K to be simple groups such that H, $K \neq \{1\}$. Let $\left(X, (p, q)\right)$ be the product in* ***SiGrp*** *of H and K. In particular, X is a simple group and $p : X \to H$, $q : X \to K$ are group homomorphisms. Consider now the pair $\left(H, (\mathrm{Id}_H, 0_K)\right)$ where Id_H is the identity homomorphism on H while $0_K : H \to K$ denotes the group homomorphism defined by $0_K(h) = 1_K$ for all $h \in H$. By Definition 2.1.1 there exists a unique homomorphism of groups $f : H \to X$ such that $p \circ f = \mathrm{Id}_H$ and $q \circ f = 0_K$. From $p \circ f = \mathrm{Id}_H$ it follows that f is injective and $q \circ f = 0_K$ implies that $\mathrm{Im}(f) \subseteq \ker(q) \subseteq X$. Putting all together we have $\{1\} \neq H \cong \mathrm{Im}(f) \subseteq \ker(q) \trianglelefteq X$ and since X is a simple group we must have $\ker(q) = X$, i.e. $q = 0_K$. Next, we consider the pair $\left(K, (0_H, \mathrm{Id}_K)\right)$ where Id_K is the identity homomorphism on K while $0_H : K \to H$ denotes the group homomorphism defined by $0_H(k) = 1_H$ for all $k \in K$. Using again Definition 2.1.1 yields a unique homomorphism of groups $g : K \to X$ such that $p \circ g = 0_H$ and $q \circ g = \mathrm{Id}_K$. Since $q = 0_K$ the last equality gives $K = \{1\}$ which is a contradiction;*

5) *In* ***Set****, the coproduct of a family $(X_i)_{i\in I}$ is just its disjoint union, i.e. the union of the sets $X_i' = X_i \times \{i\}$. Thus, the coproduct of the family $(X_i)_{i\in I}$ is the pair $\left(\coprod_{i\in I} X_i, (q_i)_{i\in I}\right)$ where $\coprod_{i\in I} X_i = \{(x, i) \mid i \in I, x \in X_i\}$ and $q_j : X_j \to \coprod_{i\in I} X_i$, $q_j(x) = (x, j)$ for all $j \in I$. Indeed, given a set Q together with a collection of*

maps $f_j : X_j \to Q$, define $f : \coprod_{i\in I} X_i \to Q$ by considering $f\big((x,j)\big) = f_j(x)$ for any $(x,j) \in X'_j \subset \coprod_{i\in I} X_i$. Since each (x,j) lies inside a unique copy of X'_j, the above map is well-defined.

$$\begin{tikzcd} X_j \arrow[r, "q_j"] \arrow[dr, "f_j"'] & \coprod_{i\in I} X_i \arrow[d, "f"] \\ & Q \end{tikzcd}$$

6) *The coproduct of any family $(X_i)_{i\in I}$ of objects in **Top** is given by the disjoin union of the underlaying sets $\big(\coprod_{i\in I} X_i, (q_i)_{i\in I}\big)$ constructed in the previous example endowed with the finest topology for which all maps q_i are continuous;*

7) *For other categories such as **Grp** the coproducts are more complicated then the products and the construction does not rely on the one performed in **Set**. This is basically because unions do not usually preserve operations (for instance, the union of an arbitrary family of groups is not necessarily a group). In group theory, the construction which gives the coproducts is called free product of groups but we will not discuss it here. Instead, we will indicate the coproducts in **Ab**. In this case for any family $(A_i)_{i\in I}$ of abelian groups the coproduct $\big(\coprod_{i\in I} A_i, (q_i)_{i\in I}\big)$ is given by their direct sum with componentwise multiplication law. More precisely, we have:*

$$\coprod_{i\in I} A_i = \{(a_i)_{i\in I} \mid a_i \in A_i,\ \{i \mid a_i \neq 0\} \,\text{is finite}\}$$

$$q_{i_0} : A_{i_0} \to \coprod_{i\in I} A_i,\ q_{i_0}(a) = (a_i)_{i\in I}$$

where $a_{i_0} = a$ and $a_j = 0$ for all $j \neq i_0$. Indeed, consider the pair $\big(H, (f_i)_{i\in I}\big)$ where H is an abelian group and $f_j : A_j \to H$ are group homomorphisms for all $j \in I$. Then, the unique homomorphism of groups $f : \coprod_{i\in I} A_i \to H$ which makes the following diagram commutative for all $j \in I$:

$$\begin{tikzcd} A_j \arrow[r, "q_j"] \arrow[dr, "f_j"'] & \coprod_{i\in I} A_i \arrow[d, "f"] \\ & H \end{tikzcd}$$

is given by $f\big((a_i)_{i\in I}\big) = \sum_{i\in I} f_i(a_i)$. Note that the sum in the right-hand side makes sense without concern for convergence since it contains only finitely many non-zero terms. Suppose now that $g : \coprod_{i\in I} A_i \to H$ is another group homomorphism such that $g \circ q_j = f_j$ for all $j \in I$. Then $(f-g)\circ q_j$ is the zero map from A_j to H and thus the image of q_j is contained in $\ker(f-g)$ for all $j \in I$. Now remark that any element in $\coprod_{i\in I} A_i$ is a sum of finitely many elements of the form $q_j(a_j)$. Therefore, since $\ker(f-g)$ is closed under finite sums as a consequence of being a

subgroup in $\coprod_{i\in I} A_i$, we obtain that $\ker(f-g) = \coprod_{i\in I} A_i$ *i.e.* $f = g$ *and the proof is now complete.*
An analogous description of coproducts holds for the category ${}_R\mathcal{M}$ of modules over a ring R.

8) *The category* ***Field*** *does not have products nor coproducts. Indeed, consider the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$ of integers modulo 2, respectively 3 and suppose $(K, (i, j))$ to be their product in* ***Field****. In particular, since $i : K \to \mathbb{Z}_2$ and $j : K \to \mathbb{Z}_3$ are morphisms of fields they are injective. Thus, in K we have both $1 + 1 = 0$ and $1+1+1 = 0$ which yields $1 = 0$, an obvious contradiction. Similarly one can prove that* ***Field*** *does not have coproducts either.*

9) *Let $PO(X, \leqslant)$ be the category corresponding to the pre-ordered set $(X, \leqslant)$. Then, the coproduct of a family $(x_i)_{i\in I}$ is just its supremum (if it exists).*

Another important example of a limit is an equalizer.

Definition 2.1.5 *An equalizer of the morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a pair (E, p) where $E \in \mathrm{Ob}\,\mathcal{C}$ and $p \in \mathrm{Hom}_{\mathcal{C}}(E, X)$ such that $f \circ p = g \circ p$ and for any other pair (E', p'), where $E' \in \mathrm{Ob}\,\mathcal{C}$ and $p' \in \mathrm{Hom}_{\mathcal{C}}(E', X)$ such that $f \circ p' = g \circ p'$ there exists a unique $u \in \mathrm{Hom}_{\mathcal{C}}(E', E)$ which makes the following diagram commute:*

$$\begin{tikzcd} E \arrow[r, "p"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ E' \arrow[u, "u"] \arrow[ur, "p'"'] & & \end{tikzcd} \qquad \text{i.e. } p \circ u = p'.$$

Examples 2.1.6 *In* ***Set***, ***Grp***, ***Ab***, ***Top***, *${}_R\mathcal{M}$ the equalizer of two morphisms $f, g : X \to Y$ is given by the pair (E, i), where $E = \{x \in X \mid f(x) = g(x)\}$ (endowed with the structure induced by that of X) and $i : E \to X$ is the canonical inclusion. Indeed, consider for instance $f, g \in \mathrm{Hom}_{\textbf{\textit{Set}}}(X, Y)$ and suppose $j : E' \to X$ is a morphism in* ***Set*** *such that $f \circ j = g \circ j$.*

$$\begin{tikzcd} E \arrow[r, "i"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ E' \arrow[u, "u"] \arrow[ur, "j"'] & & \end{tikzcd}$$

Then $\mathrm{Im}(j) \subseteq E$ and the unique map $u : E' \to E$ such that $i \circ u = j$ is given by $u(e) = j(e)$ for all $e \in E'$.

Proposition 2.1.7 *If (E, p) is the equalizer of the pair of morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ in a category $\mathcal{C}$, then p is a monomorphism.*

Proof: Consider $h_1,\, h_2 : E' \to E$ such that $p \circ h_1 = p \circ h_2 := h$. First notice that $f \circ h = f \circ (p \circ h_1) = (f \circ p) \circ h_1 = (g \circ p) \circ h_1 = g \circ (p \circ h_1) = g \circ h$. Since p is the equalizer of f and g, there exists a unique morphism $u : E' \to E$ such that $p \circ u = h$.

$$\begin{tikzcd}
E \arrow[r, "p"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
E' \arrow[u, "u"] \arrow[ur, "h"'] & &
\end{tikzcd}$$

Now notice that both maps $h_1,\, h_2 : E' \to E$ fulfill the above equality. Due to the uniqueness of u we obtain $u = h_1 = h_2$ as desired. $\square$

The dual notion of an equalizer is the notion of a coequalizer.

Definition 2.1.8 *A coequalizer of the morphisms $f,\, g \in \mathrm{Hom}_{\mathcal{C}}(X,\, Y)$ is a pair (Q, q) where $Q \in \mathrm{Ob}\,\mathcal{C}$ and $q \in \mathrm{Hom}_{\mathcal{C}}(Y,\, Q)$ such that $q \circ f = q \circ g$ and for any other pair $(Q',\, q')$ where $Q' \in \mathrm{Ob}\,\mathcal{C}$ and $q' \in \mathrm{Hom}_{\mathcal{C}}(Y,\, Q')$ such that $q' \circ f = q' \circ g$ there exists a unique $v \in \mathrm{Hom}_{\mathcal{C}}(Q,\, Q')$ which makes the following diagram commute:*

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "q'"'] & Q \arrow[d, "v"] \\
& & Q'
\end{tikzcd} \qquad \text{i.e. } v \circ q = q'.$$

As the next result is just the dual of Proposition 2.1.7 we leave the proof to the reader:

Proposition 2.1.9 *If $(Q,\, q)$ is the coequalizer of the pair of morphisms $f,\, g : X \to Y$ in a category $\mathcal{C}$ then q is an epimorphism.*

Example 2.1.10 *Let $f,\, g \in \mathrm{Hom}_{\boldsymbol{Grp}}(X,\, Y)$ and consider $Z = \{f(x)g(x)^{-1} \mid x \in X\} \subseteq Y$. If we denote by $H = \cap_{Z \subseteq N \trianglelefteq Y} N$ the normal subgroup generated by Z then $(Y/H, \pi)$ is the coequalizer of the pair of morphisms $(f,\, g)$ in the category **Grp** of groups, where $\pi : Y \to Y/H$ is the canonical projection. Indeed, since $f(x)g(x)^{-1} \in Z \subseteq H$ for all $x \in X$, we have $\widehat{f(x)} = \widehat{g(x)}$ in Y/H which comes down to $\pi \circ f = \pi \circ g$. Consider now $q' \in \mathrm{Hom}_{\boldsymbol{Grp}}(Y,\, Q')$ such that $q' \circ f = q' \circ g$. This yields $q'\big(f(x)g(x)^{-1}\big) = 1$ for all $x \in X$. Therefore, we have $Z \subseteq \ker(q') \trianglelefteq Y$ and thus $H \subseteq \ker(q')$. Now from the universal property of the quotient group Y/H we obtain an unique morphism $v \in \mathrm{Hom}_{\boldsymbol{Grp}}\big(Y/H,\, Q'\big)$ such that the following diagram is commutative:*

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\pi"] \arrow[dr, "q'"'] & Y/H \arrow[d, "v"] \\
& & Q'
\end{tikzcd}$$

*i.e. $(Y/H, \pi)$ is the coequalizer of the pair of morphisms $(f,\, g)$ in **Grp**.*

Proposition 2.1.11 *When it exists, the (co)equalizer of two morphisms is unique up to isomorphism.*

The last examples we provide before introducing (co)limits are pullbacks and respectively pushouts.

Definition 2.1.12 *Let $\mathcal{C}$ be a category and $f \in \mathrm{Hom}_{\mathcal{C}}(B,\, A)$, $g \in \mathrm{Hom}_{\mathcal{C}}(C,\, A)$. A pullback of $(f,\, g)$ is a triple $(P,\, f',\, g')$ where:*

1) $P \in \mathrm{Ob}\,\mathcal{C}$;

2) $f' \in \mathrm{Hom}_{\mathcal{C}}(P,\, C)$, $g' \in \mathrm{Hom}_{\mathcal{C}}(P,\, B)$ such that $f \circ g' = g \circ f'$,

and for any other triple $(P',\, f'',\, g'')$ where

1) $P' \in \mathrm{Ob}\,\mathcal{C}$;

2) $f'' \in \mathrm{Hom}_{\mathcal{C}}(P',\, C)$, $g'' \in \mathrm{Hom}_{\mathcal{C}}(P',\, B)$ such that $f \circ g'' = g \circ f''$,

there exists a unique $\Theta \in \mathrm{Hom}_{\mathcal{C}}(P',\, P)$ such that $f'' = f' \circ \Theta$ and $g'' = g' \circ \Theta$. The complete picture is captured by the diagram below:

$$\begin{tikzcd}
P' \arrow[dr, "\Theta"] \arrow[drr, bend left, "f''"] \arrow[ddr, bend right, "g''"'] & & \\
& P \arrow[r, "f'"'] \arrow[d, "g'"] & C \arrow[d, "g"'] \\
& B \arrow[r, "f"] & A
\end{tikzcd}$$

Example 2.1.13 *In **Set**, **Grp**, ${}_R\mathcal{M}$ the pullback of two morphisms $f : B \to A$, $g : C \to A$ is given by the triple $(B \times_A C,\, f',\, g')$, where $B \times_A C = \{(b,\, c) \in B \times C \mid f(b) = g(c)\}$ (endowed with the induced structure) and $f' : B \times_A C \to C$, $g' : B \times_A C \to B$ are given by:*

$$f'(b,\, c) = c, \qquad g'(b,\, c) = b$$

*We will only treat here the case of the category **Set**. First notice that $f \circ g' = g \circ f'$. Consider now $P' \in \mathrm{Ob}\,\boldsymbol{Set}$ and $f'' \in \mathrm{Hom}_{\boldsymbol{Set}}(P',\, C)$, $g'' \in \mathrm{Hom}_{\boldsymbol{Set}}(P',\, B)$ such that $f \circ g'' = g \circ f''$. Then $(g''(x),\, f''(x)) \in B \times_A C$ for all $x \in P'$ and we can define $\Theta : P' \to B \times_A C$ by $\Theta(x) = (g''(x),\, f''(x))$. Moreover, we have:*

$$f' \circ \Theta(x) = f'\big(g''(x),\, f''(x)\big) = f''(x)$$
$$g' \circ \Theta(x) = g'\big(g''(x),\, f''(x)\big) = g''(x)$$

Obviously, Θ is the unique morphism which renders the diagram below commutative:

$$\begin{tikzcd}
P' \arrow[dr, "\Theta"] \arrow[drr, bend left, "f''"] \arrow[ddr, bend right, "g''"'] & & \\
& B \times_A C \arrow[r, "f'"'] \arrow[d, "g'"] & C \arrow[d, "g"'] \\
& B \arrow[r, "f"] & A
\end{tikzcd}$$

Definition 2.1.14 *Let $\mathcal{C}$ be a category and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(A, C)$. A pushout of (f, g) is a triple (P, f', g') where:*

1) $P \in \mathrm{Ob}\, \mathcal{C}$;

2) $f' \in \mathrm{Hom}_{\mathcal{C}}(C, P)$, $g' \in \mathrm{Hom}_{\mathcal{C}}(B, P)$ such that $g' \circ f = f' \circ g$,

and for any other triple (P', f'', g'') where

1) $P' \in \mathrm{Ob}\, \mathcal{C}$;

2) $f'' \in \mathrm{Hom}_{\mathcal{C}}(C, P')$, $g'' \in \mathrm{Hom}_{\mathcal{C}}(B, P')$ such that $g'' \circ f = f'' \circ g$,

there exists a unique $\Theta \in \mathrm{Hom}_{\mathcal{C}}(P, P')$ such that $f'' = \Theta \circ f'$ and $g'' = \Theta \circ g'$.
The complete picture is captured by the diagram below:

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[d, "g"] & B \arrow[d, "g'"'] \arrow[ddr, bend left, "g''"] & \\
C \arrow[r, "f'"] \arrow[drr, bend right, "f''"'] & P \arrow[dr, "\Theta"] & \\
& & P'
\end{tikzcd}$$

Examples 2.1.15 1) *Given $f \in \mathrm{Hom}_{_R\mathcal{M}}(A, B)$, $g \in \mathrm{Hom}_{_R\mathcal{M}}(A, C)$, the triple $(B \times C/S, f', g')$ is the pushout in ${}_R\mathcal{M}$ of the morphisms above, where $S = \{(f(a), -g(a)) \mid a \in A\}$, $B \times C/S$ denotes the corresponding quotient module and $f' : C \to B \times C/S$, $g' : B \to B \times C/S$ are given for any $b \in B$ and $c \in C$ as follows:*

$$f'(c) = \overline{(0, c)}, \qquad g'(b) = \overline{(b, 0)}$$

Indeed, since $\big(f(a), 0\big) - \big(0, g(a)\big) = \big(f(a), -g(a)\big) \in S$ we get $\overline{\big(f(a), 0\big)} = \overline{\big(0, g(a)\big)}$ and thus $g' \circ f = f' \circ g$. Consider now $P' \in \mathrm{Ob}\, {}_R\mathcal{M}$ and $f'' \in \mathrm{Hom}_{_R\mathcal{M}}(C, P')$, $g'' \in \mathrm{Hom}_{_R\mathcal{M}}(B, P')$ such that $g'' \circ f = f'' \circ g$. The map defined for all $(b, c) \in B \times C$ as follows:

$$\chi : B \times C \to P', \qquad \chi(b, c) = g''(b) + f''(c)$$

is a morphism in ${}_R\mathcal{M}$ and moreover, $\chi(S) = 0$ since we have:

$$\chi\big(f(a),\,-g(a)\big) = g''(f(a)) - f''(g(a)) = 0$$

for all $a \in A$. Now the universal property of the quotient module yields a unique morphism $\Theta \in \mathrm{Hom}_{{}_R\mathcal{M}}(B \times C/S,\, P')$ such that $\Theta\big(\overline{(b,\, c)}\big) = g''(b) + f''(c)$ for all $(b,\, c) \in B \times C$. Moreover, we have:

$$\begin{aligned}(\Theta \circ g')(b) &= \Theta\big(\overline{(b,\, 0)}\big) = g''(b)\\ (\Theta \circ f')(c) &= \Theta\big(\overline{(0,\, c)}\big) = f''(c)\end{aligned}$$

i.e. the following diagram is commutative:

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[d, "g"] & B \arrow[d, "g'"'] \arrow[ddr, bend left, "g''"] & \\
C \arrow[r, "f'"] \arrow[drr, bend right, "f''"'] & B \times C/S \arrow[dr, "\Theta"] & \\
 & & P'
\end{tikzcd}$$

We are left to prove the uniqueness of Θ. Let $\Upsilon : B \times C/S \to P'$ such that $\Upsilon \circ g' = g''$ and $\Upsilon \circ f' = f''$. Then, for all $(b,\, c) \in B \times C$ we have:

$$\Upsilon\big(\overline{(b,\, c)}\big) = \Upsilon\big(\overline{(b,\, 0)}\big) + \Upsilon\big(\overline{(0,\, c)}\big) = \Upsilon \circ g'(b) + \Upsilon \circ f'(c) = g''(b) + f''(c) = \Theta\big(\overline{(b,\, c)}\big)$$

2) *Let $PO(X,\, \leqslant)$ be the category corresponding to the pre-ordered set $(X,\, \leqslant)$ and $a,\, b,\, c \in X$ such that $a \leq b$ and $a \leq c$. If it exists, the pushout of the above maps is some element $p \in X$ satisfying:*

- *$b \leq p$ and $c \leq p$;*
- *for any $x \in X$ such that $b \leq x$ and $c \leq x$ we have $p \leq x$.*

Therefore, if it exists, the pushout of the maps above is given by $p = \sup\{b,\, c\}$. Similarly, it can be easily seen that the pullback of two maps $b \leq a$ and $c \leq a$, if it exists, is given by $\inf\{b,\, c\}$.

Proposition 2.1.16 *When it exists, the pullback (resp. pushout) of two morphisms is unique up to isomorphism.*

2.2 (Co)limit of a functor. (Co)complete categories

Following the general pattern induced by the previous constructions (i.e. products, equalizers and pullbacks) we can now introduce the concept which unifies all the above, namely that of a limit.

Definition 2.2.1 *Let $F : \mathcal{D} \to \mathcal{C}$ be a functor*[1]*. A cone on F consists of the following:*

1. *$C \in \mathrm{Ob}\,\mathcal{C}$,*
2. *for every $D \in \mathrm{Ob}\,\mathcal{D}$, a morphism $s_D \in \mathrm{Hom}_{\mathcal{C}}(C,\, F(D))$,*

such that for any morphism $d \in \mathrm{Hom}_{\mathcal{D}}(D,\, D')$ the following diagram is commutative:

$$\begin{tikzcd} & C \arrow[dl, "s_D"'] \arrow[dr, "s_{D'}"] & \\ F(D) \arrow[rr, "F(d)"'] & & F(D') \end{tikzcd} \qquad \text{i.e. } F(d) \circ s_D = s_{D'}.$$

Definition 2.2.2 *A morphism between two cones $\big(C,\, (s_D)_{D \in \mathrm{Ob}\,\mathcal{D}}\big)$ and $\big(\overline{C},\, (r_D)_{D \in \mathrm{Ob}\,\mathcal{D}}\big)$ on a functor $F : \mathcal{D} \to \mathcal{C}$ is a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(C,\, \overline{C})$ such that the following diagram is commutative for any $D \in \mathrm{Ob}\,\mathcal{D}$:*

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "s_D"'] & & \overline{C} \arrow[dl, "r_D"] \\ & F(D) & \end{tikzcd} \qquad \text{i.e. } r_D \circ f = s_D.$$

Remark 2.2.3 *The cones on a given functor together with morphisms between them as defined above form a category.*

Definition 2.2.4 *A limit for a given functor $F : \mathcal{D} \to \mathcal{C}$ is a final object in the category of cones on F, i.e. a cone $\big(\lim F, (p_D)_{D \in \mathrm{Ob}\,\mathcal{D}}\big)$ on F such that for any other cone $\big(C,\, (s_D)_{D \in \mathrm{Ob}\,\mathcal{D}}\big)$ on F there exists a unique morphism $f \in \mathrm{Hom}_{\mathcal{C}}(C,\, \lim F)$ such that the following diagram is commutative for any $D \in \mathrm{Ob}\,\mathcal{D}$:*

$$\begin{tikzcd} \lim F \arrow[r, "p_D"] & F(D) \\ C \arrow[u, "f"] \arrow[ur, "s_D"'] & \end{tikzcd} \qquad \text{i.e. } p_D \circ f = s_D.$$

A category $\mathcal{C}$ has (small) limits if any functor $F : \mathcal{D} \to \mathcal{C}$ has a limit for any (small) category $\mathcal{D}$. We say that a category $\mathcal{C}$ is complete if it has small limits.

The following result is just an easy consequence of the above definition:

Proposition 2.2.5 *Let $\big(\lim F, (p_D)_{D \in \mathrm{Ob}\,\mathcal{D}}\big)$ be the limit of the functor $F : \mathcal{D} \to \mathcal{C}$ and $C \in \mathrm{Ob}\,\mathcal{C}$. If $f,\, g \in \mathrm{Hom}_{\mathcal{C}}(C,\, \lim F)$ such that $p_D \circ f = p_D \circ g$ for all $D \in \mathrm{Ob}\,\mathcal{D}$ then $f = g$.*

[1] The category $\mathcal{D}$ will almost always be considered small.

Proof: To start with, remark that $\big(C, (p_D \circ f)_{D \in \mathcal{D}}\big)$ is a cone on the functor F, i.e. $F(d) \circ p_D \circ f = p_{D'} \circ f$ for any $d \in \mathrm{Hom}_{\mathcal{D}}(D, D')$. Indeed, since $\big(\lim F, (p_D)_{D \in \mathrm{Ob}\,\mathcal{D}}\big)$ is the limit of the functor F (and in particular a cone on F) the following diagram is commutative:

$$\begin{tikzcd} & \lim F \arrow[dl, "p_D"'] \arrow[dr, "p_{D'}"] & \\ F(D) \arrow[rr, "F(d)"'] & & F(D') \end{tikzcd}$$

i.e. $F(d) \circ p_D = p_{D'}$ and the conclusion follows by simply composing the last equality to the right by f. Now since $\big(\lim F, (p_D)_{D \in \mathrm{Ob}\,\mathcal{D}}\big)$ is the limit of the functor F there exists a unique morphism $h \in \mathrm{Hom}_{\mathcal{C}}(C, \lim F)$ which makes the following diagram commutative:

$$\begin{tikzcd} \lim F \arrow[r, "p_D"] & F(D) \\ C \arrow[u, "h"] \arrow[ur, "p_D \circ f"'] & \end{tikzcd} \qquad \text{i.e. } p_D \circ h = p_D \circ f.$$

As both morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(C, \lim F)$ render the above diagram commutative, the desired conclusion follows. □

Although colimits are just duals of limits, due to the importance of these constructions in category theory we will write down explicitly the corresponding definitions.

Definition 2.2.6 *Let $F : \mathcal{D} \to \mathcal{C}$ be a functor. A cocone on F consists of the following:*

1. *$C \in \mathrm{Ob}\,\mathcal{C}$,*
2. *for every $D \in \mathrm{Ob}\,\mathcal{D}$, a morphism $t_D \in \mathrm{Hom}_{\mathcal{C}}(F(D), C)$,*

such that for any morphism $d \in \mathrm{Hom}_{\mathcal{D}}(D, D')$ the following diagram is commutative:

$$\begin{tikzcd} & C & \\ F(D) \arrow[ur, "t_D"] \arrow[rr, "F(d)"'] & & F(D') \arrow[ul, "t_{D'}"'] \end{tikzcd} \qquad \text{i.e. } t_{D'} \circ F(d) = t_D.$$

Definition 2.2.7 *A morphism between two cocones $\big(C, (t_D)_{D \in \mathrm{Ob}\,\mathcal{D}}\big)$ and $\big(\overline{C}, (u_D)_{D \in \mathrm{Ob}\,\mathcal{D}}\big)$ on a functor $F : \mathcal{D} \to \mathcal{C}$ is a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(C, \overline{C})$ such that the following diagram is commutative for any $D \in \mathrm{Ob}\,\mathcal{D}$:*

$$\begin{tikzcd} C \arrow[rr, "f"] & & \overline{C} \\ & F(D) \arrow[ul, "t_D"] \arrow[ur, "u_D"'] & \end{tikzcd} \qquad \text{i.e. } f \circ t_D = u_D.$$

Remark 2.2.8 *The cocones on a given functor together with morphisms between them as defined above form a category.*

Definition 2.2.9 *A colimit for a given functor $F : \mathcal{D} \to \mathcal{C}$ is an initial object in the category of cocones on F, i.e. a cocone $\big(\operatorname{colim} F, (q_D)_{D \in \operatorname{Ob} \mathcal{D}}\big)$ on F such that for every other cocone $\big(C, (t_D)_{D \in \operatorname{Ob} \mathcal{D}}\big)$ on F there exists a unique morphism $f \in \operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} F, C)$ such that the following diagram is commutative for any $D \in \operatorname{Ob} \mathcal{D}$:*

$$\begin{tikzcd} F(D) \arrow[r, "q_D"] \arrow[dr, "t_D"'] & \operatorname{colim} F \arrow[d, "f"] \\ & C \end{tikzcd} \qquad \text{i.e. } f \circ q_D = t_D.$$

A category $\mathcal{C}$ has (small) colimits if any functor $F : \mathcal{D} \to \mathcal{C}$ has a colimit for any (small) category $\mathcal{D}$. We say that a category $\mathcal{C}$ is cocomplete if it has small colimits.

Remark 2.2.10 *In the sequel we will focus mainly on small (co)limits. Considering limits of functors $F : \mathcal{D} \to \mathcal{C}$ for arbitrary categories $\mathcal{D}$ leads to set-theoretical issues which exceed the purpose of these notes.*

Dual to the case of limits, we have the following:

Proposition 2.2.11 *Let $\big(\operatorname{colim} F, (q_D)_{D \in \operatorname{Ob} \mathcal{D}}\big)$ be the colimit of the functor $F : \mathcal{D} \to \mathcal{C}$ and $C \in \operatorname{Ob} \mathcal{C}$. If $f,\, g \in \operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} F,\, C)$ such that $f \circ q_D = g \circ q_D$ for all $D \in \operatorname{Ob} \mathcal{D}$ then $f = g$.*

The uniqueness up to isomorphism of initial and respectively final objects in a category (see Proposition 1.2.5) implies:

Proposition 2.2.12 *When it exists, the (co)limit of a functor is unique up to isomorphism.*

Remark 2.2.13 *Taking (co)limits yields a functor. Indeed, let I be a small category and $\mathcal{C}$ a category such that every functor $F : I \to \mathcal{C}$ has a limit. Then $\mathit{lim} : \operatorname{Fun}(I, \mathcal{C}) \to \mathcal{C}$ defined below is a functor:*

$$\mathit{lim}(F) = \lim F, \text{ for all functors } F : I \to \mathcal{C},$$
$$\mathit{lim}(\alpha) = \overline{\alpha}, \text{ for all natural transformations } \alpha : F \to G,\ F, G : I \to \mathcal{C} \text{ functors},$$

where $\big(\lim F, (p_i)_{i \in \operatorname{Ob} I}\big)$ and $\big(\lim G, (s_i)_{i \in \operatorname{Ob} I}\big)$ are the limits of F and G respectively and $\overline{\alpha} \in \operatorname{Hom}_{\mathcal{C}}(\lim F,\, \lim G)$ is the unique morphism which makes the following diagram commute for all $i \in \operatorname{Ob} I$:

$$\begin{tikzcd} \lim G \arrow[r, "s_i"] & G(i) \\ \lim F \arrow[u, "\overline{\alpha}"] \arrow[ur, "\alpha_i \circ p_i"'] & \end{tikzcd}$$

Examples 2.2.14 1) *Consider ϕ the empty functor from the empty category to $\mathcal{C}$. Then the limit of ϕ, if it exists, is just the final object in $\mathcal{C}$. Analogously, the colimit of ϕ, if it exists, is just the initial object in $\mathcal{C}$.*

2) *Take I to be a small discrete category. Then a functor $F : I \to \mathcal{C}$ is essentially nothing but a family of objects $(C_i)_{i\in I}$ in $\mathcal{C}$ indexed by the set I and a (co)limit of F, if it exists, is just a (co)product in $\mathcal{C}$ of the family $(C_i)_{i\in I}$.*

3) *Consider I to be a category with two objects A_1 and A_2 and four morphisms 1_{A_1}, 1_{A_2}, u, v, where u, $v \in \mathrm{Hom_I}(A_1, A_2)$ and define the functor $F : I \to \mathcal{C}$ as follows:*

$$F(A_1) = X, \quad F(A_2) = Y, \quad F(u) = f, \quad F(v) = g$$

The (co)limit of the functor F defined above, if it exists, is nothing but the (co)equalizer of the pair of morphisms f, $g : X \to Y$ in $\mathcal{C}$.

4) *Consider I to be a category with three objects A_1, A_2, A_3 and five morphisms 1_{A_1}, 1_{A_2}, 1_{A_3}, u, v, where $u \in \mathrm{Hom_I}(A_1, A_3)$, $v \in \mathrm{Hom_I}(A_2, A_3)$ and define the functor $F : I \to \mathcal{C}$ as follows:*

$$F(A_1) = X, \quad F(A_2) = Y, \quad F(A_3) = Z, \quad F(u) = f, \quad F(v) = g$$

The limit of the functor F defined above, if it exists, is nothing but the pullback of the pair of morphisms $f : X \to Z$, $g : Y \to Z$ in $\mathcal{C}$. In a similar manner pushouts can be obtained as a special case of colimits.

2.3 (Co)limits by (co)equalizers and (co)products

(Co)products and (co)equalizers are perhaps the most important among special cases of (co)limits. This is due to the fact that all (co)limits can be constructed out of (co)products and (co)equalizers. We start by presenting an example which hints at this construction.

Example 2.3.1 Set is a complete category. Indeed, let I be a small category and $F : I \to \mathbf{Set}$ a functor. Then $\big(\lim F,\, (p_k)_{k\in \mathrm{Ob} I}\big)$ is the limit of F, where:

$$\lim F = \{(x_k)_{k\in \mathrm{Ob} I} \in \prod_{k\in \mathrm{Ob} I} F(k) \mid F(f)(x_i) = x_j \text{ for all } f \in \mathrm{Hom}_I(i,\, j)\}$$
$$p_j : \lim F \to F(j),\ p_j((x_k)_{k\in \mathrm{Ob} I}) = x_j, \text{ for all } j \in \mathrm{Ob}\, I.$$

We start by proving that $\big(\lim F,\, (p_k)_{k\in \mathrm{Ob} I}\big)$ is a cone on F. Indeed, for any $f \in \mathrm{Hom}_I(i,\, j)$ we have $F(f)(x_i) = x_j$ for all $(x_k)_{k\in \mathrm{Ob}\, I} \in \lim F$ which can be written equivalently as $F(f)\big(p_i((x_k)_{k\in \mathrm{Ob} I})\big) = p_j\big((x_k)_{k\in \mathrm{Ob} I}\big)$. Thus we obtain $F(f) \circ p_i = p_j$, as desired.

Assume now that $\big(C,\, (s_k)_{k\in \mathrm{Ob}\, I}\big)$ is another cone on F, i.e. $C \in \mathrm{Ob}\,\mathbf{Set}$ and $s_k \in \mathrm{Hom}_{\mathbf{Set}}(C,\, F(k))$ such that $F(f) \circ s_i = s_j$ for all $f \in \mathrm{Hom}_I(i,\, j)$. Now recall that

$\Big(\prod_{k\in \mathrm{Ob}I} F(k),\, (p_k)_{k\in \mathrm{Ob}I}\Big)$ is the product in **Set** of the family $\big((F(k))_{k\in \mathrm{Ob}I}\big)$; hence, there exists a unique morphism $g : C \to \prod_{ik\in \mathrm{Ob}I} F(k)$ in **Set** such that the following diagram is commutative for all $j \in \mathrm{Ob}I$:

$$\begin{tikzcd} C \arrow[d, "g"'] \arrow[drr, "s_j"] & & \\ \prod_{k\in \mathrm{Ob}I} F(k) \arrow[rr, "p_j"'] & & F(j) \end{tikzcd} \qquad \text{i.e. } p_j \circ g = s_j.$$

We are left to prove that $\mathrm{Im}g \subseteq \lim F$. To this end, notice that $g(c) = \big(s_k(c)\big)_{k\in \mathrm{Ob}\, I}$ for all $c \in C$. Moreover, since $F(f) \circ s_i = s_j$ for any $f \in \mathrm{Hom}_I(i, j)$ we get $F(f)\big(s_i(c)\big) = s_j(c)$ for all $c \in C$. Thus $g(c) = \big(s_k(c)\big)_{k\in \mathrm{Ob}\, I} \in \lim F$ for all $c \in C$.

Remark 2.3.2 *Note that we have constructed the limit of a functor $F : I \to$ **Set**, for any small category I, as a subset of the product $\prod_{i\in \mathrm{Ob}I} F(i)$ satisfying some equations. In fact, it can be easily seen that for any $(x_i)_{i\in \mathrm{Ob}I} \in \prod_{i\in \mathrm{Ob}I} F(i)$ we have $(x_i)_{i\in \mathrm{Ob}I} \in L$ if and only if $F(f)(x_t) = F(g)(x_s)$ for all $f \in \mathrm{Hom}_I(t, u)$ and $g \in \mathrm{Hom}_I(s, u)$. This suggests that an equalizer is being taken in order to construct the limit. The next result shows that this procedure can be generalized to arbitrary categories allowing for the construction of limits out of products and equalizers.*

Theorem 2.3.3 *A category $\mathcal{C}$ is (co)complete if and only if it has (co)products and (co)equalizers.*

Proof: We will only prove the assertion regarding completeness. Obviously, if a category is complete then it has products and equalizers by Example 2.2.14. Conversely, assume the category $\mathcal{C}$ has products and equalizers and let $F : I \to \mathcal{C}$ be a functor, where I is a small category. For any morphism f in I we will denote by $s(f)$ the domain of f and by $t(f)$ the codomain of f, i.e. $f \in \mathrm{Hom}_I(s(f), t(f))$. We start by constructing the products in $\mathcal{C}$ for the families of objects $(F(i))_{i\in \mathrm{Ob}I}$ and respectively $\big(F(t(f))\big)_{f\in \mathrm{Hom}_I(s(f),\, t(f))}$:

$$\Big(\prod_i F(i),\, (u_i)_{i\in \mathrm{Ob}\, I}\Big), \qquad \Big(\prod_f F(t(f)),\, (v_{t(f)})_{f\in \mathrm{Hom}_I(s(f),\, t(f))}\Big)$$

Since $\big(\prod_f F(t(f)),\, (v_{t(f)})_{f\in \mathrm{Hom}_I(s(f),\, t(f))}\big)$ is the product in $\mathcal{C}$ of the family of objects $\big(F(t(f))\big)_{f\in \mathrm{Hom}_I(s(f),\, t(f))}$ there exist two unique morphisms in $\mathcal{C}$

$$\alpha,\, \beta : \prod_i F(i) \to \prod_f F(t(f))$$

such that for any $g \in \mathrm{Hom}_I(s(g),\, t(g))$ we have:

$$\begin{aligned} v_{t(g)} \circ \alpha &= u_{t(g)} && (2.4) \\ v_{t(g)} \circ \beta &= F(g) \circ u_{s(g)} && (2.5) \end{aligned}$$

Now consider $(L,\, l)$ to be the equalizer in $\mathcal{C}$ of the pair of morphisms $(\alpha,\, \beta)$.

$$\begin{tikzcd}
M \arrow[r, "q_j"] \arrow[dr, "q'"] \arrow[d, "q"'] & F(j) & F(t(g)) \\
L \arrow[r, "l"] & \prod_i F(i) \arrow[u, "u_j"'] \arrow[ur, "u_{t(g)}"] \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "u_{s(g)}"'] & \prod_f F(t(f)) \arrow[u, "v_{t(g)}"'] \arrow[d, "v_{t(g)}"] \\
& F(s(g)) \arrow[r, "F(g)"'] & F(t(g))
\end{tikzcd}$$

We will prove that $\big(L,\, (p_i := u_i \circ l)_{i \in \mathrm{Ob}\, I}\big)$ is the limit of the functor F. First we prove that $\big(L,\, (p_i)_{i \in \mathrm{Ob}\, I}\big)$ is a cone on F. Indeed, if $r \in \mathrm{Hom}_I(s(r),\, t(r))$ we have:

$$F(r) \circ p_{s(r)} = \underline{F(r) \circ u_{s(r)}} \circ l \overset{(2.5)}{=} v_{t(r)} \circ \beta \circ l = \underline{v_{t(r)} \circ \alpha} \circ l \overset{(2.4)}{=} u_{t(r)} \circ l = p_{t(r)}$$

Moreover, consider $\big(M,\, (q_i)_{i \in \mathrm{Ob}\, I}\big)$ another cone on F. Since $\Big(\prod_i F(i),\, (u_i)_{i \in \mathrm{Ob}\, I}\Big)$ is the product in $\mathcal{C}$ of the family of objects $(F(i))_{i \in \mathrm{Ob} I}$, there exists a unique morphism $q' : M \to \prod_i F(i)$ in $\mathcal{C}$ such that for any $j \in \mathrm{Ob}\, I$ we have:

$$\begin{tikzcd}
M \arrow[d, "q'"'] \arrow[dr, "q_j"] & \\
\prod\limits_{i \in \mathrm{Ob} I} F(i) \arrow[r, "u_j"'] & F(j)
\end{tikzcd} \qquad \text{i.e.} \quad u_j \circ q' = q_j \qquad (2.6)$$

Now for any $r \in \mathrm{Hom}_I(s(r),\, t(r))$ we have:

$$\underline{v_{t(r)} \circ \alpha} \circ q' \overset{(2.4)}{=} \underline{u_{t(r)} \circ q'} \overset{(2.6)}{=} q_{t(r)} = F(r) \circ \underline{q_{s(r)}} \overset{(2.6)}{=} \underline{F(r) \circ u_{s(r)}} \circ q' \overset{(2.5)}{=} v_{t(r)} \circ \beta \circ q'$$

where in the third equality we used the fact that $\big(M,\, (q_i)_{i \in \mathrm{Ob}\, I}\big)$ is a cone on F. Therefore we have $v_{t(r)} \circ \alpha \circ q' = v_{t(r)} \circ \beta \circ q'$ for any $r \in \mathrm{Hom}_I(s(r),\, t(r))$ and according to Proposition 2.2.5 we obtain $\alpha \circ q' = \beta \circ q'$. Since $(L,\, l)$ is the equalizer of the pair of morphisms $(\alpha,\, \beta)$ in $\mathcal{C}$ we obtain a unique morphism $q : M \to L$ such that:

$$l \circ q = q' \qquad (2.7)$$

It turns out that q is the unique morphism in $\mathcal{C}$ which makes the following diagram commute for all $j \in \mathrm{Ob}\, I$:

$$\begin{tikzcd}
M \arrow[dd, shift right, "\overline{q}"'] \arrow[dd, shift left, "q"] \arrow[drr, "q_j"] & & \\
& & F(j) \\
L \arrow[urr, "p_j"'] & &
\end{tikzcd}$$

Indeed, for any $j \in \operatorname{Ob} I$ we have: $\underline{p_j} \circ q = u_j \circ \underline{l \circ q} \overset{(2.7)}{=} u_j \circ q' \overset{(2.6)}{=} q_j$. Finally, we are left to prove the uniqueness of q. To this end, assume $\overline{q} \in \operatorname{Hom}_{\mathcal{C}}(M,\, L)$ is another morphism such that $p_j \circ \overline{q} = q_j$ for all $j \in \operatorname{Ob} I$. Hence, we obtain $u_j \circ l \circ \overline{q} = q_j$ for all $j \in \operatorname{Ob} I$ and since q' is the unique morphism in $\mathcal{C}$ which makes diagram (2.6) commute we obtain $l \circ \overline{q} = q'$. Now the uniqueness of the morphism in $\mathcal{C}$ for which (2.7) holds implies $\overline{q} = q$, as desired. $\square$

2.4 (Co)limit preserving functors

Definition 2.4.1 *A functor $F : \mathcal{C} \to \mathcal{D}$ preserves (small) limits/colimits when for every functor $G : I \to \mathcal{C}$, where I is a (small) category, if $\big(L,\, (p_i)_{i \in \mathrm{Ob} I}\big)$ is the limit/colimit of G then $\big(F(L),\, (F(p_i))_{i \in \mathrm{Ob} I}\big)$ is the limit/colimit of $F \circ G$.*

As a consequence of Theorem 2.3.3 we obtain the following:

Proposition 2.4.2 *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories such that $\mathcal{C}$ is (co)complete. A functor $F : \mathcal{C} \to \mathcal{D}$ preserves small (co)limits if and only if it preserves (co)products and (co)equalizers.*

The following straightforward lemma will be useful in the sequel:

Lemma 2.4.3 *Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon I \to \mathcal{C}$ be two functors. If $\big(X,\, (s_i)_{i \in \mathrm{Ob}\, I}\big)$ is a (co)cone on G, then $\big(F(X),\, (F(s_i))_{i \in \mathrm{Ob}\, I}\big)$ is a (co)cone on $F \circ G \colon I \to \mathcal{D}$.*

Proof: We will only prove the statement regarding cones. To this end consider $f \in \operatorname{Hom}_I(i,\, j)$. The proof will be finished once we show that the following diagram is commutative:

$$\begin{tikzcd}
& F(X) \arrow[dl, "F(s_i)"'] \arrow[dr, "F(s_j)"] & \\
F \circ G(i) \arrow[rr, "F\circ G(f)"'] & & F \circ G(j)
\end{tikzcd} \qquad \text{i.e. } F \circ G(f) \circ F(s_i) = F(s_j) \tag{2.8}$$

Indeed, as $\big(X,\,(s_i)_{i\in \mathrm{Ob}\,I}\big)$ is a cone on G, the following diagram is commutative:

$$\begin{tikzcd} & X \arrow[dl, "s_i"'] \arrow[dr, "s_j"] & \\ G(i) \arrow[rr, "G(f)"'] & & G(j) \end{tikzcd} \qquad \text{i.e. } G(f)\circ s_i = s_j \tag{2.9}$$

Now it is straightforward to see that (2.8) holds true just by applying F to the identity (2.9). $\square$

One of the main examples of functors which preserve limits are the hom functors.

Theorem 2.4.4 *Let $\mathcal{C}$ be a category and $C \in \mathrm{Ob}\,\mathcal{C}$. Then, the hom functor $\mathtt{Hom}_{\mathcal{C}}(C,\,-) : \mathcal{C} \to \boldsymbol{Set}$ preserves small limits.*

Proof: Consider a functor $G : I \to \mathcal{C}$, where I is a small category, and $\big(L,\,(p_i)_{i\in \mathrm{Ob}\,I}\big)$ its limit. The proof will be finished once we prove that $\big(\mathrm{Hom}_{\mathcal{C}}(C,\,L),\,(\mathtt{Hom}_{\mathcal{C}}(C,\,p_i))_{i\in \mathrm{Ob}\,I}\big)$ is the limit of the functor $\mathtt{Hom}_{\mathcal{C}}(C,\,G(-)) : I \to \mathbf{Set}$. To start with, by Lemma 2.4.3 we obtain that $\big(\mathrm{Hom}_{\mathcal{C}}(C,\,L),\,(\mathtt{Hom}_{\mathcal{C}}(C,\,p_i))_{i\in \mathrm{Ob}\,I}\big)$ is a cone on $\mathtt{Hom}_{\mathcal{C}}(C,\,G(-))$. Consider now $\big(M,\,(q_i)_{i\in \mathrm{Ob}\,I}\big)$ another cone on $\mathtt{Hom}_{\mathcal{C}}(C,\,G(-))$, where $M \in \mathrm{Ob}\,\mathbf{Set}$ and $q_i \in \mathrm{Hom}_{\mathbf{Set}}\big(M,\,\mathrm{Hom}_{\mathcal{C}}(C,\,G(i))\big)$ for all $i \in \mathrm{Ob}I$. Hence, for any $f \in \mathrm{Hom}_I(i,\,j)$ we have $\mathtt{Hom}_{\mathcal{C}}(C,\,G(f))(q_i) = q_j$ which is equivalent to $G(f)\circ q_i = q_j$. In particular, for all $m \in M$ we have $G(f)\circ q_i(m) = q_j(m)$. This implies that for each $m \in M$, $\big(C,\,(q_i(m))_{i\in \mathrm{Ob}\,I}\big)$ is a cone on G, where $q_i(m) \in \mathrm{Hom}_{\mathcal{C}}(C, G(i))$ for all $i \in \mathrm{Ob}\,I$. As $\big(L,\,(p_i)_{i\in \mathrm{Ob}\,I}\big)$ is the limit of G, it yields a unique morphism $q(m) \in \mathrm{Hom}_{\mathcal{C}}(C,\,L)$ such that the following diagram is commutative for all $i \in \mathrm{Ob}\,I$:

$$\begin{tikzcd} L \arrow[r, "p_i"] & G(i) \\ C \arrow[u, "q(m)"] \arrow[ur, "q_i(m)"'] & \end{tikzcd} \qquad \text{i.e. } p_i \circ q(m) = q_i(m)$$

Putting all together we have defined a function $q : M \to \mathrm{Hom}_{\mathcal{C}}(C,\,L)$ (i.e. a morphism in **Set**) satisfying $\mathtt{Hom}_{\mathcal{C}}(C,\,p_i)\circ q = q_i$ for any $i \in \mathrm{Ob}\,I$, i.e. the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C,\,L) \arrow[r, "\mathtt{Hom}_{\mathcal{C}}(C,p_i)"] & \mathrm{Hom}_{\mathcal{C}}(C,\,G(i)) \\ M \arrow[u, "q"] \arrow[ur, "q_i"'] & \end{tikzcd}$$

Finally, the uniqueness of q follows from that of the $q(m)$'s. $\square$

Regarding the contravariant hom functor, which reverses the direction of morphisms, we have the following:

Theorem 2.4.5 *Let $\mathcal{C}$ be a category and $C \in \mathrm{Ob}\,\mathcal{C}$. The contravariant hom functor $\mathit{Hom}_{\mathcal{C}}(-, C) : \mathcal{C} \to \textbf{\textit{Set}}$ maps existing small colimits to small limits.*

Definition 2.4.6 *A functor $F : \mathcal{C} \to \mathcal{D}$ reflects (small) limits/colimits when for every functor $G : I \to \mathcal{C}$, where I is a (small) category, and every cone/cocone $\big(L, (p_i)_{i\in \mathrm{Ob}\, I}\big)$ on G, if $\big(F(L), (F(p_i))_{i\in \mathrm{Ob}\, I}\big)$ is the limit/colimit of $F \circ G$, then $\big(L, (p_i)_{i\in \mathrm{Ob}\, I}\big)$ is the limit/colimit of G.*

Theorem 2.4.7 *A full and faithful functor $F : \mathcal{C} \to \mathcal{D}$ reflects small (co)limits.*

Proof: Let $G : I \to \mathcal{C}$ be a functor where I is a small category and consider $\big(L, (p_i)_{i\in \mathrm{Ob}\, I}\big)$ to be a cone on G such that $\big(F(L), (F(p_i))_{i\in \mathrm{Ob}\, I}\big)$ is the limit of $F \circ G$. We will prove that $\big(L, (p_i)_{i\in \mathrm{Ob}\, I}\big)$ is also a limit of G. Indeed, if $\big(M, (q_i)_{i\in \mathrm{Ob}\, I}\big)$ is another cone on G then using Lemma 2.4.3 we obtain that $\big(F(M), (F(q_i))_{i\in \mathrm{Ob}\, I}\big)$ is a cone on $F \circ G$. Therefore, we have a unique morphism $f \in \mathrm{Hom}_{\mathcal{D}}(F(M), F(L))$ such that the following diagram is commutative for all $i \in \mathrm{Ob}\, I$:

$$\begin{tikzcd} F(L) \arrow[r, "F(p_i)"] & (F \circ G)(i) \\ F(M) \arrow[u, "f"] \arrow[ur, "F(q_i)"'] & \end{tikzcd} \qquad \text{i.e. } F(p_i) \circ f = F(q_i) \tag{2.10}$$

Since F is full and faithful there exists a unique morphism $\overline{f} \in \mathrm{Hom}_{\mathcal{C}}(M, L)$ such that $F(\overline{f}) = f$. Then (2.10) comes down to $F(q_i) = F(p_i) \circ F(\overline{f})$ and since F is faithful we obtain $q_i = p_i \circ \overline{f}$ for all $i \in \mathrm{Ob}\, I$, i.e. the following diagram is commutative:

$$\begin{tikzcd} L \arrow[r, "p_i"] & G(i) \\ M \arrow[u, "\overline{f}"] \arrow[ur, "q_i"'] & \end{tikzcd}$$

We are left to prove that $\overline{f}$ is the unique morphism which makes the above diagram commutative. To this end, assume that $g \in \mathrm{Hom}_{\mathcal{C}}(M, L)$ is another morphism such that $q_i = p_i \circ g$ for all $i \in \mathrm{Ob}\, I$. This implies $F(p_i) \circ F(g) = F(q_i)$ for all $i \in \mathrm{Ob}\, I$ and since f is the unique morphism which makes diagram (2.10) commute we obtain $F(g) = f$. Now recall that we also have $F(\overline{f}) = f$ and since F is faithful we arrive at $g = \overline{f}$, as desired. Therefore $\big(L, (p_i)_{i\in \mathrm{Ob}\, I}\big)$ is a final object in the category of cones on G as desired. The statement concerning colimits can be proved in much the same fashion as above. $\square$

Proposition 2.4.8 *Let $F : \mathcal{C} \to \mathcal{D}$ be a (co)limit preserving functor. If $\mathcal{C}$ is (co)complete and F reflects isomorphisms then F also reflects small (co)limits.*

Proof: Consider a functor $G : I \to \mathcal{C}$, where I is a small category, and let $\big(M, (q_i)_{i\in \mathrm{Ob}\, I}\big)$ be a cone on G such that $\big(F(M), (F(q_i))_{i\in \mathrm{Ob}\, I}\big)$ is the limit of $F \circ G$. The proof will be finished once we show that $\big(M,\, (q_i)_{i \in \operatorname{Ob} I}\big)$ is the limit of G. According to the completeness assumption on $\mathcal{C}$ the functor G has a limit, say $\big(L,\, (p_i)_{i \in \operatorname{Ob} I}\big)$. Thus, there exists a unique morphism $f \in \operatorname{Hom}_{\mathcal{C}}(M,\, L)$ such that the following diagram is commutative for all $i \in \operatorname{Ob} I$:

$$\begin{tikzcd} L \arrow[r, "p_i"] & G(i) \\ M \arrow[u, "f"] \arrow[ur, "q_i"'] & \end{tikzcd} \qquad \text{i.e. } p_i \circ f = q_i \tag{2.11}$$

In particular, this implies that:

$$F(p_i) \circ F(f) = F(q_i), \text{ for all } i \in \operatorname{Ob} I.$$

Since F is a limit preserving functor then $\big(F(L),\, (F(p_i))_{i \in \operatorname{Ob} I}\big)$ is also a limit of $F \circ G$. Exactly as in the proof of Proposition 2.1.3, one can show that there exists a unique isomorphism $g \in \operatorname{Hom}_{\mathcal{D}}(F(M),\, F(L))$ such that the following diagram is commutative for all $i \in \operatorname{Ob} I$:

$$\begin{tikzcd} F(L) \arrow[r, "F(p_i)"] & F \circ G(i) \\ F(M) \arrow[u, "g"] \arrow[ur, "F(q_i)"'] & \end{tikzcd} \qquad \text{i.e. } F(p_i) \circ g = F(q_i)$$

Hence $F(f) = g$ is an isomorphism in $\mathcal{D}$. Our assumption implies that f is an isomorphism in $\mathcal{C}$ and thus $\big(M,\, (q_i)_{i \in \operatorname{Ob} I}\big)$ is also a limit of G as desired. $\square$

Examples 2.4.9 1) The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ preserves products and equalizers (see Example 2.1.4 2) and Example 2.1.6). Therefore U preserves limits.

2) The category $\mathbf{Ab}$ is complete and the forgetful functor $U : \mathbf{Ab} \to \mathbf{Grp}$ preserves limits and reflects isomorphisms. Thus, according to Proposition 2.4.8, U also reflects limits.

2.5 Exercises

1. Let $\mathcal{C}$ be a category and $X \in \operatorname{Ob}\mathcal{C}$. Prove that:

   - If $\mathcal{C}$ has binary products then the category $(X \downarrow \mathcal{C})$ of objects under X has binary products;
   - If $\mathcal{C}$ has binary coproducts then the category $(\mathcal{C} \downarrow X)$ of objects over X has binary coproducts.

2. Let $\mathcal{C}$ be a category, $f,\, g \in \operatorname{Hom}_{\mathcal{C}}(A,\, B)$ and $(E,\, e)$ the equalizer of the pair $(f,\, g)$. Then the following are equivalent:

   a) $f = g$;

   b) e is an epimorphism;

   c) e is an isomorphism;

   d) 1_A is the equalizer of $(f,\, g)$.

3. Consider the following diagram:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend left, "u"] \arrow[r, "h"] & C \arrow[l, bend left, "v"] \end{tikzcd}$$

   where $h \circ f = h \circ g$, $h \circ v = 1_C$, $g \circ u = 1_B$, $f \circ u = v \circ h$. Prove that $(C,\, h)$ is the coequalizer of the pair of morphisms $(f,\, g)$.

4. Let $\mathcal{C}$ be a category and $f \in \operatorname{Hom}_{\mathcal{C}}(A,\, B)$. Prove that:

   - f is a monomorphism if and only if $(B,\, 1_B,\, 1_B)$ is the pullback of the pair $(f,\, f)$;
   - f is an epimorphism if and only if $(B,\, 1_B,\, 1_B)$ is the pushout of the pair $(f,\, f)$.

5. Let $\mathcal{C}$ be a complete category and $f,\, g \in \operatorname{Hom}_{\mathcal{C}}(X,\, Y)$. Show that if $(E,\, p)$ is the equalizer of the pair $(f,\, g)$ then $(E,\, p,\, p)$ is the pullback of the pair $(\overline{f},\, \overline{g})$, where $\overline{f},\, \overline{g} \in \operatorname{Hom}_{\mathcal{C}}(X,\, X \times Y)$, are given by $\overline{f}(x) = (x,\, f(x))$ and $\overline{g}(x) = (x,\, g(x))$ for all $x \in X$.

6. Let $\mathcal{C}$ be a category, I a set, $(A_i)_{i\in I}$ a family of objects in $\mathcal{C}$ and the functor $F = \prod_{i\in I} \mathtt{Hom}_{\mathcal{C}}(A_i,\, -) : \mathcal{C} \to \mathbf{Set}$ defined as follows:

   $F(C) = \prod_{i\in I} \operatorname{Hom}_{\mathcal{C}}(A_i,\, C)$[2],

   $F(u)\big((\eta_i)_{i\in I}\big) = \big(u \circ \eta_i\big)_{i\in I}$

[2] $\prod_{i\in I} \operatorname{Hom}_{\mathcal{C}}(A_i,\, C)$ denotes the product in the category **Set** of sets.

for all C, $D \in \mathrm{Ob}\,\mathcal{C}$, $u \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ and $\eta_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, C)$, $i \in I$. Prove that F is representable if and only if there exists the coproduct of the family $(A_i)_{i\in I}$ in the category $\mathcal{C}$.

7. Let $\mathcal{C}$ be a category. Prove that:

   - If $\mathcal{C}$ has a final object and pullbacks then $\mathcal{C}$ has binary products and equalizers;
   - If $\mathcal{C}$ has an initial object and pushouts then $\mathcal{C}$ has binary coproducts and coequalizers.

8. Let $\mathcal{C}$ be a category. Prove that:

   - If $\mathcal{C}$ has binary products and equalizers then $\mathcal{C}$ has pullbacks;
   - If $\mathcal{C}$ has binary coproducts and coequalizers then $\mathcal{C}$ has pushouts.

9. Let $\mathcal{C}$ be a category with equalizers and $F : \mathcal{C} \to \mathcal{D}$ a functor which preserves equalizers and reflects isomorphisms. Prove that F is faithful.

10. Let I be a small category, $\mathcal{C}$ a category with pullbacks (resp. pushouts) and F, $G : I \to \mathcal{C}$ two functors. Prove that a natural transformation $\alpha : F \to G$ is a monomorphism (resp. an epimorphism) in the functor category $\mathrm{Fun(I, \mathcal{C})}$ if and only if all its components are monomorphisms (resp. epimorphisms) in $\mathcal{C}$.

11. Let I and $\mathcal{C}$ be two categories with I small and $F : I \to \mathcal{C}$ a functor. Prove that the pair $\big(X, (\alpha_i)_{i\in \mathrm{Ob}\,I}\big)$, $\alpha_i \in \mathrm{Hom}_{\mathcal{C}}(X, F(i))$ for all $i \in \mathrm{Ob}\,I$, is a cone on F if and only if $\alpha : \Delta(X) \to F$, $\alpha = (\alpha_i)_{i\in \mathrm{Ob}\,I}$ is a natural transformation, where $\Delta : \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ is the diagonal functor.

12. Let $(X, \leqslant)$ be a pre-ordered set and $\mathrm{PO}(X, \leqslant)$ the corresponding category. Describe (co)limits in $\mathrm{PO}(X, \leqslant)$.

13. Let G be a group and $\mathcal{G}$ the corresponding category. Is $\mathcal{G}$ (co)complete?

14. Decide if the following categories are (co)complete: **Grp**, **Ab**, **Top**, **Ring**, ${}_R\mathcal{M}$, **Field**.

15. Let $\mathrm{PO}(\mathbb{Z}, \leqslant)$ be the category corresponding to the poset $(\mathbb{Z}, \leqslant)$, where $\leqslant$ is the usual ordering on the integers. Decide if the identity functor $\mathrm{Id} : \mathrm{PO}(\mathbb{Z}, \leqslant) \to \mathrm{PO}(\mathbb{Z}, \leqslant)$ has (co)limit.

16. Prove that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are (co)complete categories then the product category $\mathcal{C}_1 \times \mathcal{C}_2$ is also (co)complete.

17. Let I be a small category, $F : I \to \mathcal{C}$ and $\Delta : \mathcal{C} \to \mathrm{Fun(I, \mathcal{C})}$ the diagonal functor. Prove that:

   - The functor F has a limit if and only if the comma category $(\Delta \downarrow F)$ has a final object;

- The functor F has a colimit if and only if the comma category $(F \downarrow \Delta)$ has an initial object.

3

Adjoint functors

Adjoint functors were first defined by Kan in the 50's motivated by homological algebra. Nowadays they are present in most fields of mathematics. The terminology was inspired by adjoint operators whose definition is somewhat similar to the correspondence in Definition 3.1.1 between hom sets.

3.1 Definition and a generic example

Definition 3.1.1 *An adjunction consists of a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ and for any $X \in \operatorname{Ob}\mathcal{C}$, $Y \in \operatorname{Ob}\mathcal{D}$ a bijective map*

$$\theta_{X,\,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X),\,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,\,G(Y))$$

which is natural in both variables. In this case we say that F is left adjoint to G or equivalently that G is right adjoint to F. The notation $F \dashv G$ is used to designate a pair of adjoint functors.

Remark 3.1.2 *Unpacking the naturality assumption in the two variables comes down to the following: for any $X \in \operatorname{Ob}\mathcal{C}$, the functors $\mathit{Hom}_{\mathcal{D}}(F(X),\,-)$ and $\mathit{Hom}_{\mathcal{C}}(X,\,G(-))$ are naturally isomorphic and for any $Y \in \operatorname{Ob}\mathcal{D}$, the (contravariant) functors $\mathit{Hom}_{\mathcal{D}}(F(-),\,Y)$ and $\mathit{Hom}_{\mathcal{C}}(-,\,G(Y))$ are naturally isomorphic. More precisely, this amounts to the commutativity of the following diagrams for all $f \in \mathrm{Hom}_{\mathcal{C}}(X',\,X)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(Y,\,Y')$:*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X),\,Y) \arrow[r, "\theta_{X,\,Y}"] \arrow[d, "\mathit{Hom}_{\mathcal{D}}(F(f),Y)"'] & \mathrm{Hom}_{\mathcal{C}}(X,\,G(Y)) \arrow[d, "\mathit{Hom}_{\mathcal{C}}(f,G(Y))"] \\
\mathrm{Hom}_{\mathcal{D}}(F(X'),\,Y) \arrow[r, "\theta_{X',\,Y}"'] & \mathrm{Hom}_{\mathcal{C}}(X',\,G(Y))
\end{tikzcd} \tag{3.1}$$

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(F(X),\, Y) & \xrightarrow{\theta_{X,\,Y}} & \mathrm{Hom}_{\mathcal{C}}(X,\, G(Y)) \\
{\scriptstyle Hom_{\mathcal{D}}(F(X),g)}\downarrow & & \downarrow{\scriptstyle Hom_{\mathcal{C}}(X,G(g))} \\
\mathrm{Hom}_{\mathcal{D}}(F(X),\, Y') & \xrightarrow[\theta_{X,\,Y'}]{} & \mathrm{Hom}_{\mathcal{C}}(X,\, G(Y'))
\end{array}
\tag{3.2}
$$

Most of the categories we considered so far are categories of sets endowed with some extra structure (i.e. groups, rings, vector spaces, topological spaces etc.) which allow for a forgetful functor to **Set**. Whenever these categories have free objects the forgetful functor to **Set** has a left adjoint. We consider below the case of the category of groups. However, the same strategy works if we replace **Grp** by the category of monoids, modules, algebras etc.

Example 3.1.3 *Let U : **Grp** $\to$ **Set** be the forgetful functor. We will see that U has a left adjoint F : **Set** $\to$ **Grp** called the free functor. More precisely, F is constructed as follows:*

- *for any $X \in \mathrm{Ob}\,\textbf{\textit{Set}}$, define $F(X) = FX$ the free group on the set X;*
- *given $f \in \mathrm{Hom}_{\textbf{\textit{Set}}}(X,\, Y)$, define $F(f) : FX \to FY$ by $F(f) = \overline{f}$, where $\overline{f}$ is obtained from the universal property of the free group FX, i.e. $\overline{f} \in \mathrm{Hom}_{\textbf{\textit{Grp}}}(FX,\, FY)$ is the unique group homomorphism which makes the following diagram commute:*

$$
\begin{tikzcd}
X \arrow[rr, "i_X"] \arrow[dr, "f"'] & & FX \arrow[dd, "\overline{f}"] \\
& Y \arrow[dr, "i_Y"'] & \\
& & FY
\end{tikzcd}
\qquad \text{i.e. } \overline{f} \circ i_X = i_Y \circ f
\tag{3.3}
$$

where i_X and i_Y are the inclusion maps.

Let $X \in \mathrm{Ob}\,\textbf{\textit{Set}}$ and $G \in \mathrm{Ob}\,\textbf{\textit{Grp}}$. We will prove that there is a bijection:

$$
\theta_{X,G} : \mathrm{Hom}_{\textbf{\textit{Grp}}}(FX,\, G) \to \mathrm{Hom}_{\textbf{\textit{Set}}}(X,\, U(G)), \text{ given by } \theta_{X,G}(v) = v \circ i_X
$$

for any $v \in \mathrm{Hom}_{\textbf{\textit{Grp}}}(FX,\, G)$. The inverse of $\theta_{X,G}$, denoted by $\psi_{X,G}$, is defined as follows:

$$
\begin{aligned}
\psi_{X,G} &: \mathrm{Hom}_{\textbf{\textit{Set}}}(X,\, U(G)) \to \mathrm{Hom}_{\textbf{\textit{Grp}}}(FX,\, G), \\
\psi_{X,G}(u) &= \overline{u}, \text{ for all } u \in \mathrm{Hom}_{\textbf{\textit{Set}}}(X,\, U(G))
\end{aligned}
$$

where $\overline{u} \in \mathrm{Hom}_{\textbf{\textit{Grp}}}(FX,\, G)$ is the unique group homomorphism which makes the following diagram commute:

$$
\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[d, "u"'] & FX \arrow[dl, "\overline{u}"] \\
G &
\end{tikzcd}
\qquad \text{i.e. } \overline{u} \circ i_X = u
\tag{3.4}
$$

Indeed, for any $v \in \operatorname{Hom}_{\boldsymbol{Grp}}(FX, G)$ *we have:*

$$\psi_{X,G} \circ \theta_{X,G}(v) = \psi_{X,G}(v \circ i_X) = \overline{v \circ i_X}$$

where $\overline{v \circ i_X}$ *is the unique group homomorphism which makes the following diagram commute:*

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "v \circ i_X"'] & FX \arrow[dl, "\overline{v \circ i_X}"] \\ G & \end{tikzcd}$$

Since v *makes the above diagram commutative we get* $\psi_{X,G} \circ \theta_{X,G}(v) = v$*. On the other hand, if* $u \in \operatorname{Hom}_{\boldsymbol{Set}}(X, U(G))$*, we have:*

$$\theta_{X,G} \circ \psi_{X,G}(u) = \theta_{X,G}(\overline{u}) = \overline{u} \circ i_X$$

where $\overline{u}$ *is the unique group homomorphism which makes diagram (3.4) commute. Thus* $\overline{u} \circ i_X = u$ *and we obtain* $\theta_{X,G} \circ \psi_{X,G}(u) = u$ *as desired.*

Finally we check that the isomorphism θ *is natural in both variables. First, fix* $G \in \operatorname{Ob} \boldsymbol{Grp}$ *and consider* $f \in \operatorname{Hom}_{\boldsymbol{Set}}(X', X)$*. We need to prove the commutativity of the following diagram:*

$$\begin{tikzcd} \operatorname{Hom}_{\boldsymbol{Grp}}(FX, G) \arrow[r, "\theta_{X,G}"] \arrow[d, "Hom_{Grp}(F(f), G)"'] & \operatorname{Hom}_{\boldsymbol{Set}}(X, U(G)) \arrow[d, "Hom_{Set}(f, U(G))"] \\ \operatorname{Hom}_{\boldsymbol{Grp}}(FX', G) \arrow[r, "\theta_{X',G}"'] & \operatorname{Hom}_{\boldsymbol{Set}}(X', U(G)) \end{tikzcd}$$

i.e. $Hom_{Set}(f, U(G)) \circ \theta_{X,G} = \theta_{X',G} \circ Hom_{Grp}(F(f), G)$.

To this end, consider $r \in \operatorname{Hom}_{\boldsymbol{Grp}}(FX, G)$*; we have:*

$$\begin{aligned} Hom_{Set}(f, U(G)) \circ \theta_{X,G}(r) &= Hom_{Set}(f, U(G))(r \circ i_X) \\ &= r \circ \underline{i_X \circ f} \\ &\overset{(3.3)}{=} r \circ \overline{f} \circ i_{X'} = r \circ F(f) \circ i_{X'} \\ &= \theta_{X',G}(r \circ F(f)) = \theta_{X',G} \circ Hom_{Grp}(F(f), G)(r) \end{aligned}$$

Finally, fix $X \in \operatorname{Ob} \boldsymbol{Set}$ *and consider* $g \in \operatorname{Hom}_{\boldsymbol{Grp}}(G, G')$*. We need to prove that the following diagram is commutative:*

$$\begin{tikzcd} \operatorname{Hom}_{\boldsymbol{Grp}}(F(X), G) \arrow[r, "\theta_{X,G}"] \arrow[d, "Hom_{Grp}(F(X), g)"'] & \operatorname{Hom}_{\boldsymbol{Set}}(X, U(G)) \arrow[d, "Hom_{Set}(X, U(g))"] \\ \operatorname{Hom}_{\boldsymbol{Grp}}(F(X), G') \arrow[r, "\theta_{X,G'}"'] & \operatorname{Hom}_{\boldsymbol{Set}}(X, U(G')) \end{tikzcd}$$

$$\text{i.e. } \mathtt{Hom}_{\boldsymbol{Set}}(X,\, U(g)) \circ \theta_{X,G} = \theta_{X,G'} \circ \mathtt{Hom}_{\boldsymbol{Grp}}(F(X),\, g)$$

Let $t \in \mathrm{Hom}_{\boldsymbol{Grp}}(F(X),\, G)$*; we have:*

$$\begin{aligned}
\mathtt{Hom}_{\boldsymbol{Set}}(X,\, U(g)) \circ \theta_{X,G}(t) &= U(g) \circ t \circ i_X \\
&= g \circ t \circ i_X \\
&= \theta_{X,G'}(g \circ t) = \theta_{X,G'} \circ \mathtt{Hom}_{\boldsymbol{Grp}}(F(X),\, g)(t)
\end{aligned}$$

3.2 More examples and properties of adjoint functors

We start by presenting more examples of adjoint functors, spanning various fields:

Examples 3.2.1 *1) For any set* X*, the functor* $- \times X : \boldsymbol{Set} \to \boldsymbol{Set}$ *has a right adjoint given by* $\mathtt{Hom}_{\boldsymbol{Set}}(X,\, -) : \boldsymbol{Set} \to \boldsymbol{Set}$*;*

2) If $X \in \mathrm{Ob}\,{}_R\mathcal{M}$*, then the tensor product functor* $- \otimes_R X : {}_R\mathcal{M} \to {}_R\mathcal{M}$ *has a right adjoint given by* $\mathtt{Hom}_{{}_R\mathcal{M}}(X,\, -) : {}_R\mathcal{M} \to {}_R\mathcal{M}$*, where for any* $Y \in \mathrm{Ob}\,{}_R\mathcal{M}$ *the* R*-module structure on* $\mathrm{Hom}_{{}_R\mathcal{M}}(X,\, Y)$ *is given by scalar multiplication;*

3) If X *is a locally compact topological space then the functor* $- \times X : \boldsymbol{Top} \to \boldsymbol{Top}$ *has a right adjoint given by* $\mathtt{Hom}_{\boldsymbol{Top}}(X,\, -) : \boldsymbol{Top} \to \boldsymbol{Top}$*, where for any* $Y \in \mathrm{Ob}\,\boldsymbol{Top}$ *we consider on* $\mathrm{Hom}_{\boldsymbol{Top}}(X,\, Y)$ *the compact open topology;*

4) The inclusion functor $I : \boldsymbol{Ab} \to \boldsymbol{Grp}$ *has a left adjoint called abelianization which assigns to every group* G *the quotient group* $G/[G, G]$*, where* $[G, G]$ *denotes the commutator subgroup;*

5) If R *is a commutative ring, the forgetful functor* $F : \boldsymbol{Alg}_R \to {}_R\mathcal{M}$ *(forgetting the multiplicative structure) has a left adjoint called the tensor algebra;*

6) The inclusion functor $I : \boldsymbol{KHaus} \to \boldsymbol{Top}$ *has a left adjoint called the Stone-Čech compactification;*

7) The forgetful functor $U : \boldsymbol{Top} \to \boldsymbol{Set}$ *has both left and right adjoints. The left adjoint equips a set* X *with its discrete topology, while the right adjoint equips* X *with the indiscrete topology;*

8) The inclusion functor $I : \boldsymbol{Grp} \to \boldsymbol{Mon}$ *has a left adjoint, the so-called anvelopant group of a monoid (or the group completion of a monoid), and a right adjoint, which assigns to each monoid the group of its invertible elements;*

9) The forgetful functor $U : {}_R\mathcal{M} \to \boldsymbol{Ab}$ *has both a left and a right adjoint. The left adjoint sends any abelian group* A *to the tensor product* $R \otimes A$*, while the right adjoint sends* A *to the group of morphisms* $\mathrm{Hom}_{\mathbb{Z}}(U(R),\, A)$ *endowed with the following* R*-module structure:*

$$rf(x) = f(rx), \quad \text{for all } r \in R,\ x \in U(R),\ f \in \mathrm{Hom}_{\mathbb{Z}}(U(R),\, A).$$

10) Let $\mathbf{1}$ *be the category with only one object and one morphism (the identity on the unique object). For any category* $\mathcal{C}$ *we can define a unique functor* $T : \mathcal{C} \to \mathbf{1}$*. It can* *be easily seen that the functor T has a left (resp. right) adjoint if and only if $\mathcal{C}$ has an initial (resp. final) object.*

Next we look at compositions of adjoint functors.

Proposition 3.2.2 *Consider the functors $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$ such that $F \dashv G$ and $H : \mathcal{B} \to \mathcal{C}$, $K : \mathcal{C} \to \mathcal{B}$ such that $H \dashv K$. Then $HF \dashv GK$.*

Proof: We have the following natural isomorphism for all $A \in \operatorname{Ob}\mathcal{A}$ and $C \in \operatorname{Ob}\mathcal{C}$:

$$\operatorname{Hom}_{\mathcal{C}}(HF(A),\, C) \approx \operatorname{Hom}_{\mathcal{B}}(F(A),\, K(C)) \approx \operatorname{Hom}_{\mathcal{A}}(A,\, GK(C)).$$

□

The next result gives a necessary condition for the existence of adjoints: if a left (resp. right) adjoint exists, the functor has to preserve limits (resp. colimits).

Proposition 3.2.3 *Consider the functors $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$ such that $F \dashv G$. Then F preserves small colimits while G preserves small limits.*

Proof: We will only prove that G preserves all existing limits of $\mathcal{B}$. Consider $\theta : \mathtt{Hom}_{\mathcal{B}}(F(-),\, -) \to \mathtt{Hom}_{\mathcal{A}}(-,\, G(-))$ to be the natural isomorphism corresponding to the adjunction $F \dashv G$. Let I be a small category and $H : I \to \mathcal{B}$ a functor whose limit we denote by $\big(L,\, (p_i : L \to H(i))_{i \in \operatorname{Ob} I}\big)$. We will prove that $\big(G(L),\, (G(p_i) : G(L) \to GH(i))_{i \in \operatorname{Ob} I}\big)$ is the limit of $GH : I \to \mathcal{A}$. To start with, $\big(G(L),\, (G(p_i) : G(L) \to GH(i))_{i \in \operatorname{Ob} I}\big)$ is a cone on GH by Lemma 2.4.3.

Consider now $\big(A,\, (q_i : A \to GH(i))_{i \in \operatorname{Ob} I}\big)$ to be another cone on GH. Since the map $\theta_{A,\, H(i)} : \operatorname{Hom}_{\mathcal{B}}(F(A),\, H(i)) \to \operatorname{Hom}_{\mathcal{A}}(A,\, GH(i))$ is a bijection, there exists a unique morphism $r_i \in \operatorname{Hom}_{\mathcal{B}}(F(A),\, H(i))$ such that $\theta_{A,\, H(i)}(r_i) = q_i$. We will prove that $\big(F(A),\, (r_i : F(A) \to H(i))_{i \in \operatorname{Ob} I}\big)$ is a cone on H, i.e. for any $d \in \operatorname{Hom}_I(i,\, j)$ we have $H(d) \circ r_i = r_j$. To this end, the naturality of θ renders the following diagram commutative:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{B}}(F(A),\, H(i)) \arrow[r, "\theta_{A,\, H(i)}"] \arrow[d, "\mathtt{Hom}_{\mathcal{B}}(F(A), H(d))"'] & \operatorname{Hom}_{\mathcal{A}}(A,\, GH(i)) \arrow[d, "\mathtt{Hom}_{\mathcal{A}}(A, GH(d))"] \\
\operatorname{Hom}_{\mathcal{B}}(F(A),\, H(j)) \arrow[r, "\theta_{A,\, H(j)}"'] & \operatorname{Hom}_{\mathcal{A}}(A,\, GH(j))
\end{tikzcd}$$

$$\text{i.e.} \quad \mathtt{Hom}_{\mathcal{A}}(A,\, GH(d)) \circ \theta_{A,\, H(i)} = \theta_{A,\, H(j)} \circ \mathtt{Hom}_{\mathcal{B}}(F(A),\, H(d)) \tag{3.5}$$

Moreover, since $\big(A,\, (q_i : A \to GH(i))_{i \in \operatorname{Ob} I}\big)$ is a cone on GH the following diagram is commutative:

$$\begin{tikzcd}
 & A \arrow[dl, "q_i"'] \arrow[dr, "q_j"] & \\
GH(i) \arrow[rr, "GH(d)"'] & & GH(j)
\end{tikzcd} \qquad \text{i.e.} \quad GH(d) \circ q_i = q_j \tag{3.6}$$

Now by evaluating (3.5) at r_i we obtain:

$$\begin{aligned}
&\mathtt{Hom}_{\mathcal{A}}(A,\, GH(d)) \circ \theta_{A,\,H(i)}(r_i) = \theta_{A,\,H(j)} \circ \mathtt{Hom}_{\mathcal{B}}(F(A),\, H(d))(r_i)\\
&\Leftrightarrow \mathtt{Hom}_{\mathcal{A}}(A,\, GH(d))(q_i) = \theta_{A,\,H(j)}(H(d) \circ r_i)\\
&\Leftrightarrow \underline{GH(d) \circ q_i} = \theta_{A,\,H(j)}(H(d) \circ r_i)\\
&\overset{(3.6)}{\Leftrightarrow} q_j = \theta_{A,\,H(j)}(H(d) \circ r_i)\\
&\Leftrightarrow r_j = H(d) \circ r_i
\end{aligned}$$

Thus $\big(F(A),\, (r_i)_{i \in \operatorname{Ob} I}\big)$ is a cone on H. Therefore, we have a unique morphism $f \in \operatorname{Hom}_{\mathcal{B}}(F(A),\, L)$ such that the following diagram is commutative for all $i \in \operatorname{Ob} I$:

$$\begin{tikzcd}
L \arrow[r, "p_i"] & H(i) \\
F(A) \arrow[u, "f"] \arrow[ur, "r_i"'] &
\end{tikzcd} \qquad \text{i.e.} \quad p_i \circ f = r_i. \tag{3.7}$$

Denote $g = \theta_{A,L}(f) \in \operatorname{Hom}_{\mathcal{A}}(A,\, G(L))$. We are left to prove that the following diagram is commutative for all $i \in \operatorname{Ob} I$:

$$\begin{tikzcd}
G(L) \arrow[r, "G(p_i)"] & GH(i) \\
A \arrow[u, "g"] \arrow[ur, "q_i"'] &
\end{tikzcd} \qquad \text{i.e.} \quad G(p_i) \circ g = q_i. \tag{3.8}$$

Using again the naturality of the bijection θ we obtain the following commutative diagram for all $i \in \operatorname{Ob} I$:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{B}}(F(A),\, L) \arrow[r, "\theta_{A,\,L}"] \arrow[d, "\mathtt{Hom}_{\mathcal{B}}(F(A),p_i)"'] & \operatorname{Hom}_{\mathcal{A}}(A,\, G(L)) \arrow[d, "\mathtt{Hom}_{\mathcal{A}}(A,G(p_i))"] \\
\operatorname{Hom}_{\mathcal{B}}(F(A),\, H(i)) \arrow[r, "\theta_{A,\,H(i)}"'] & \operatorname{Hom}_{\mathcal{A}}(A,\, GH(i))
\end{tikzcd}$$

$$\text{i.e.} \quad \mathtt{Hom}_{\mathcal{A}}(A,\, G(p_i)) \circ \theta_{A,\,L} = \theta_{A,\,H(i)} \circ \mathtt{Hom}_{\mathcal{B}}(F(A),\, p_i) \tag{3.9}$$

By evaluating (3.9) at $f \in \operatorname{Hom}_{\mathcal{B}}(F(A),\, L)$ we obtain:

$$\begin{aligned}
&\mathtt{Hom}_{\mathcal{A}}(A,\, G(p_i)) \circ \theta_{A,\,L}(f) = \theta_{A,\,H(i)} \circ \mathtt{Hom}_{\mathcal{B}}(F(A),\, p_i)(f)\\
&\Leftrightarrow G(p_i) \circ g = \theta_{A,\,H(i)}(\underline{p_i \circ f})\\
&\overset{(3.7)}{\Leftrightarrow} G(p_i) \circ g = \theta_{A,\,H(i)}(r_i)\\
&\Leftrightarrow G(p_i) \circ g = q_i
\end{aligned}$$

for all $i \in \operatorname{Ob} I$. Hence the diagram (3.8) is indeed commutative.

Assume now that there exists another morphism $\overline{g} \in \mathrm{Hom}_{\mathcal{A}}(A, G(L))$ such that $G(p_i) \circ \overline{g} = q_i$ for all $i \in \mathrm{Ob}\, I$. Since $\theta_{A,L} : \mathrm{Hom}_{\mathcal{B}}(F(A), L) \to \mathrm{Hom}_{\mathcal{A}}(A, G(L))$ is bijective, there exists a unique morphism $\overline{f} \in \mathrm{Hom}_{\mathcal{B}}(F(A), L)$ such that $\theta_{A,L}(\overline{f}) = \overline{g}$. Now by evaluating (3.9) at $\overline{f}$ we arrive at:

$$\begin{aligned}
&\mathtt{Hom}_{\mathcal{A}}(A, G(p_i)) \circ \theta_{A,L}(\overline{f}) = \theta_{A,H(i)} \circ \mathtt{Hom}_{\mathcal{B}}(F(A), p_i)(\overline{f}) \\
&\Leftrightarrow G(p_i) \circ \overline{g} = \theta_{A,H(i)}(p_i \circ \overline{f}) \\
&q_i = \theta_{A,H(i)}(p_i \circ \overline{f})
\end{aligned}$$

for any $i \in \mathrm{Ob}\, I$. Therefore, we have $p_i \circ \overline{f} = r_i$ for all $i \in \mathrm{Ob}\, I$ which implies $f = \overline{f}$ and consequently $g = \overline{g}$, as desired. □

Examples 3.2.4 *1) The forgetful functor F : **Ab** $\to$ **Set** does not preserve coproducts. Therefore, by Proposition 3.2.3 it does not have a right adjoint.*
*2) Consider now the inclusion functor I : **Ring** $\to$ **Rng**. As $\mathbb{Z}$ is an initial object in **Ring** but not in **Rng** we can conclude by Proposition 3.2.3 that it does not admit a right adjoint.*
*3) The forgetful functor U : **Field** $\to$ **Set** does not have a left adjoint. Indeed, if F : **Set** $\to$ **Field** is a left adjoint to U then by Proposition 3.2.3, F needs to preserve colimits. In particular, $F(\varnothing)$ would be an initial object in **Field** which contradicts Example 1.2.6.*

3.3 The unit and counit of an adjunction

Our next result gives an important equivalent description of adjoint functors.

Theorem 3.3.1 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors. Then F is left adjoint to G if and only if there exist natural transformations:*

$$\eta : 1_{\mathcal{C}} \to GF, \qquad \varepsilon : FG \to 1_{\mathcal{D}}$$

such that for all $C \in Ob\,\mathcal{C}$ and $D \in \mathrm{Ob}\,\mathcal{D}$ we have:

$$1_{F(C)} = \varepsilon_{F(C)} \circ F\big(\eta_C\big) \tag{3.10}$$

$$1_{G(D)} = G\big(\varepsilon_D\big) \circ \eta_{G(D)} \tag{3.11}$$

In this case η and ε are called the unit, respectively the counit of the adjunction.

Proof: Suppose first that $F \dashv G$ and let $\theta : \mathtt{Hom}_{\mathcal{D}}(F(-), -) \to \mathtt{Hom}_{\mathcal{C}}(-, G(-))$ be the corresponding natural isomorphism. For each $C \in \mathrm{Ob}\mathcal{C}$ and $D \in \mathrm{Ob}\mathcal{D}$ we have the following bijective maps:

$$\begin{aligned}
\theta_{C,F(C)} &: \mathrm{Hom}_{\mathcal{D}}(F(C), F(C)) \to \mathrm{Hom}_{\mathcal{C}}(C, GF(C)) \\
\theta_{G(D),D} &: \mathrm{Hom}_{\mathcal{D}}(FG(D), D) \to \mathrm{Hom}_{\mathcal{C}}(G(D), G(D)).
\end{aligned}$$

Now define $\eta_C = \theta_{C,\,F(C)}\big(1_{F(C)}\big)\colon C \to GF(C)$ and $\varepsilon_D = \theta^{-1}_{G(D),\,D}\big(1_{G(D)}\big)\colon FG(D) \to D$. We are left to prove (3.10) and (3.11) as well as the naturality of η and ε. We start by proving (3.10); indeed, if we consider the commutative diagram (3.1) for $X = GF(C)$, $X' = C$, $Y = F(C)$ and $f = \eta_C \in \mathrm{Hom}_{\mathcal{C}}(C,\, GF(C))$ we get:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(FGF(C),\, F(C)) \arrow[r, "\theta_{GF(C),\,F(C)}"] \arrow[d, "\mathtt{Hom}_{\mathcal{D}}(F(\eta_C),F(C))"'] & \mathrm{Hom}_{\mathcal{C}}(GF(C),\, GF(C)) \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(\eta_C,GF(C))"] \\
\mathrm{Hom}_{\mathcal{D}}(F(C),\, F(C)) \arrow[r, "\theta_{C,\,F(C)}"'] & \mathrm{Hom}_{\mathcal{C}}(C,\, GF(C))
\end{tikzcd}$$

From the commutativity of the above diagram applied to $\varepsilon_{F(C)} \in \mathrm{Hom}_{\mathcal{D}}(FGF(C),\, F(C))$ we obtain:

$$\begin{aligned}
&\mathtt{Hom}_{\mathcal{C}}(\eta_C,\, GF(C)) \circ \theta_{GF(C),F(C)}(\varepsilon_{F(C)}) = \theta_{C,\,F(C)} \circ \mathtt{Hom}_{\mathcal{D}}(F(\eta_C),\, F(C))(\varepsilon_{F(C)}) \\
&\Leftrightarrow \underline{\theta_{GF(C),\,F(C)}(\varepsilon_{F(C)})} \circ \eta_C = \theta_{C,\,F(C)}\big(\varepsilon_{F(C)} \circ F(\eta_C)\big) \\
&\Leftrightarrow 1_{GF(C)} \circ \eta_C = \theta_{C,\,F(C)}\big(\varepsilon_{F(C)} \circ F(\eta_C)\big) \\
&\Leftrightarrow \underline{\theta^{-1}_{C,\,F(C)}(\eta_C)} = \varepsilon_{F(C)} \circ F(\eta_C) \\
&\Leftrightarrow 1_{F(C)} = \varepsilon_{F(C)} \circ F(\eta_C) \text{ i.e. (3.10) holds.}
\end{aligned}$$

Similarly, we consider the commutative diagram (3.2) for $X = G(D)$, $Y = FG(D)$, $Y' = D$ and $g = \varepsilon_D \in \mathrm{Hom}_{\mathcal{D}}(FG(D),\, D)$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(FG(D),\, FG(D)) \arrow[r, "\theta_{G(D),\,FG(D)}"] \arrow[d, "\mathtt{Hom}_{\mathcal{D}}(FG(D),\varepsilon_D)"'] & \mathrm{Hom}_{\mathcal{C}}(G(D),\, GFG(D)) \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(G(D),G(\varepsilon_D))"] \\
\mathrm{Hom}_{\mathcal{D}}(FG(D),\, D) \arrow[r, "\theta_{G(D),\,D}"'] & \mathrm{Hom}_{\mathcal{C}}(G(D),\, G(D))
\end{tikzcd}$$

The commutativity of the above diagram applied to $1_{FG(D)} \in \mathrm{Hom}_{\mathcal{D}}(FG(D),\, FG(D))$ yields:

$$\begin{aligned}
&\theta_{G(D),\,D} \circ \mathtt{Hom}_{\mathcal{D}}(FG(D),\, \varepsilon_D)(1_{FG(D)}) = \mathtt{Hom}_{\mathcal{C}}\big(G(D),\, G(\varepsilon_D)\big) \circ \underline{\theta_{G(D),\,FG(D)}(1_{FG(D)})} \\
&\Leftrightarrow \theta_{G(D),\,D} \circ \mathtt{Hom}_{\mathcal{D}}(FG(D),\, \varepsilon_D) \circ 1_{FG(D)} = G(\varepsilon_D) \circ \eta_{G(D)} \\
&\Leftrightarrow \underline{\theta_{G(D),\,D}(\varepsilon_D)} = G(\varepsilon_D) \circ \eta_{G(D)} \\
&\Leftrightarrow 1_{G(D)} = G(\varepsilon_D) \circ \eta_{G(D)} \text{ i.e. (3.11) holds.}
\end{aligned}$$

Finally, we move on to proving that η and ε are natural transformations. First we will collect some compatibilities using the commutativity of the diagrams (3.1) and (3.2). Setting $X = C$, $X' = C'$ and $Y = F(C)$ in (3.1), yields the following commutative diagram for all $f \in \mathrm{Hom}_{\mathcal{C}}(C', C)$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(C),\, F(C)) \arrow[r, "\theta_{C,\, F(C)}"] \arrow[d, "\mathtt{Hom}_{\mathcal{D}}(F(f), F(C))"'] & \mathrm{Hom}_{\mathcal{C}}(C,\, GF(C)) \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(f, GF(C))"] \\
\mathrm{Hom}_{\mathcal{D}}(F(C'),\, F(C)) \arrow[r, "\theta_{C',\, F(C)}"'] & \mathrm{Hom}_{\mathcal{C}}(C',\, GF(C))
\end{tikzcd}$$

From the commutativity of the above diagram applied to $1_{F(C)}$ we get:

$$\begin{aligned}
&\mathtt{Hom}_{\mathcal{C}}(f,\, GF(C)) \circ \theta_{C,\, F(C)}(1_{F(C)}) = \theta_{C',\, F(C)} \circ \mathtt{Hom}_{\mathcal{D}}(F(f),\, F(C))(1_{F(C)}) \\
&\Leftrightarrow \mathtt{Hom}_{\mathcal{C}}(f,\, GF(C)) \circ \eta_C = \theta_{C',\, F(C)}\big(F(f)\big)
\end{aligned}$$

$$\text{i.e. } \eta_C \circ f = \theta_{C',\, F(C)}\big(F(f)\big) \tag{3.12}$$

On the other hand, setting $X = C'$, $Y = F(C')$ and $Y' = F(C)$ in (3.2) yields the following commutative diagram for all $g \in \mathrm{Hom}_{\mathcal{D}}(F(C'),\, F(C))$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(C'),\, F(C')) \arrow[r, "\theta_{C',\, F(C')}"] \arrow[d, "\mathtt{Hom}_{\mathcal{D}}(F(C'), g)"'] & \mathrm{Hom}_{\mathcal{C}}(C',\, GF(C')) \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(C', G(g))"] \\
\mathrm{Hom}_{\mathcal{D}}(F(C'),\, F(C)) \arrow[r, "\theta_{C',\, F(C)}"'] & \mathrm{Hom}_{\mathcal{C}}(C',\, GF(C))
\end{tikzcd}$$

By applying the commutativity of the above diagram to $1_{F(C')}$ we obtain:

$$\mathtt{Hom}_{\mathcal{C}}(C',\, G(g)) \circ \theta_{C',\, F(C')}(1_{F(C')}) = \theta_{C',\, F(C)} \circ \mathtt{Hom}_{\mathcal{D}}(F(C'),\, g)(1_{F(C')})$$

$$\text{i.e. } G(g) \circ \eta_{C'} = \theta_{C',\, F(C)}(g) \tag{3.13}$$

Next we use the commutativity of the diagram (3.1) for $X = G(D)$, $X' = G(D')$ and $Y = D$. It comes down to the following commutative diagram for all $f \in \mathrm{Hom}_{\mathcal{C}}(G(D'),\, G(D))$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(FG(D),\, D) \arrow[r, "\theta_{G(D),\, D}"] \arrow[d, "\mathtt{Hom}_{\mathcal{D}}(F(f), D)"'] & \mathrm{Hom}_{\mathcal{C}}(G(D),\, G(D)) \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(f, G(D))"] \\
\mathrm{Hom}_{\mathcal{D}}(FG(D'),\, D) \arrow[r, "\theta_{G(D'),\, D}"'] & \mathrm{Hom}_{\mathcal{C}}(G(D'),\, G(D))
\end{tikzcd}$$

From the commutativity of the above diagram applied to ε_D we get:

$$\begin{aligned}
&\theta_{G(D'),\, D} \circ \mathtt{Hom}_{\mathcal{D}}(F(f),\, D)(\varepsilon_D) = \mathtt{Hom}_{\mathcal{C}}(f,\, G(D)) \circ \theta_{G(D),\, D}(\varepsilon_D) \\
&\Leftrightarrow \theta_{G(D'),\, D}(\varepsilon_D \circ F(f)) = 1_{G(D)} \circ f
\end{aligned}$$

$$\text{i.e. } \varepsilon_D \circ F(f) = \theta^{-1}_{G(D'),\, D}(f) \tag{3.14}$$

Finally, we use the commutativity of the diagram (3.2) for $X = G(D')$, $Y = D'$ and $Y' = D$. It yields the following commutative diagram for all $g \in \mathrm{Hom}_{\mathcal{D}}(D', D)$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(FG(D'),\, D') \arrow[r, "\theta_{G(D'),\, D'}"] \arrow[d, "\mathtt{Hom}_{\mathcal{D}}(FG(D'),g)"'] & \mathrm{Hom}_{\mathcal{C}}(G(D'),\, G(D')) \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(G(D'),G(g))"] \\
\mathrm{Hom}_{\mathcal{D}}(FG(D'),\, D) \arrow[r, "\theta_{G(D'),\, D}"'] & \mathrm{Hom}_{\mathcal{C}}(G(D'),\, G(D))
\end{tikzcd}$$

The commutativity of the above diagram applied to $\varepsilon_{D'}$ gives:

$$\theta_{G(D'),\,D} \circ \mathtt{Hom}_{\mathcal{D}}(FG(D'),\, g)(\varepsilon_{D'}) = \mathtt{Hom}_{\mathcal{C}}(G(D'),\, G(g)) \circ \theta_{G(D'),\,D'}(\varepsilon_{D'})$$
$$\Leftrightarrow \theta_{G(D'),\,D}(g \circ \varepsilon_{D'}) = G(g) \circ 1_{G(D')}$$

$$\text{i.e. } g \circ \varepsilon_{D'} = \theta^{-1}_{G(D'),\,D}\big(G(g)\big) \tag{3.15}$$

We are now in a position to prove that η and ε are natural transformations. Indeed, the naturality of η comes down to proving the commutativity of the above diagram for all $h \in \mathrm{Hom}_{\mathcal{C}}(C', C)$:

$$\begin{tikzcd}
C' \arrow[r, "\eta_{C'}"] \arrow[d, "h"'] & GF(C') \arrow[d, "GF(h)"] \\
C \arrow[r, "\eta_C"'] & GF(C)
\end{tikzcd}$$

To this end we have:

$$\eta_C \circ h \overset{(3.12)}{=} \theta_{C',\,F(C)}\big(F(h)\big) \overset{(3.13)}{=} GF(h) \circ \eta_{C'}$$

where in the second equality we used (3.13) for $g = F(h)$. Thus η is a natural transformation.

The naturality of ε comes down to proving the commutativity of the following diagram for all $t \in \mathrm{Hom}_{\mathcal{D}}(D', D)$:

$$\begin{tikzcd}
FG(D') \arrow[r, "\varepsilon_{D'}"] \arrow[d, "FG(t)"'] & D' \arrow[d, "t"] \\
FG(D) \arrow[r, "\varepsilon_D"'] & D
\end{tikzcd}$$

which can be proved using (3.14) and respectively (3.15):

$$\varepsilon_D \circ FG(t) \overset{(3.14)}{=} \theta^{-1}_{G(D'),\,D}\big(G(t)\big) \overset{(3.15)}{=} g \circ \varepsilon_{D'}$$

Remark that the first equality follows by applying (3.14) for $f = G(t)$.

Assume now that there exist natural transformations $\eta : 1_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to 1_{\mathcal{D}}$ such that (3.10) and (3.11) are fulfilled for any $C \in \operatorname{Ob}\mathcal{C}$ and $D \in \operatorname{Ob}\mathcal{D}$. Define the following maps:

$$\theta_{C,D} : \operatorname{Hom}_{\mathcal{D}}(F(C), D) \to \operatorname{Hom}_{\mathcal{C}}(C, G(D)), \quad \theta_{C,D}(u) = G(u) \circ \eta_C$$
$$\varphi_{C,D} : \operatorname{Hom}_{\mathcal{C}}(C, G(D)) \to \operatorname{Hom}_{\mathcal{D}}(F(C), D), \quad \varphi_{C,D}(v) = \varepsilon_D \circ F(v)$$

for any $u \in \operatorname{Hom}_{\mathcal{D}}(F(C), D)$ and $v \in \operatorname{Hom}_{\mathcal{C}}(C, G(D))$. First we will prove that $\theta_{C,D}$ and $\varphi_{C,D}$ are inverses to each other for any $C \in \operatorname{Ob}\mathcal{C}$ and $D \in \operatorname{Ob}\mathcal{D}$. To start with, we note for further use that the naturality of η and ε imply the commutativity of the following diagrams for all $u \in \operatorname{Hom}_{\mathcal{D}}(F(C), D)$ and $v \in \operatorname{Hom}_{\mathcal{C}}(C, G(D))$:

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "v"'] & GF(C) \arrow[d, "GF(v)"] \\ G(D) \arrow[r, "\eta_{G(D)}"'] & GFG(D) \end{tikzcd} \qquad \text{i.e.} \quad GF(v) \circ \eta_C = \eta_{G(D)} \circ v \tag{3.16}$$

$$\begin{tikzcd} FGF(C) \arrow[r, "\varepsilon_{F(C)}"] \arrow[d, "FG(u)"'] & F(C) \arrow[d, "u"] \\ FG(D) \arrow[r, "\varepsilon_D"'] & D \end{tikzcd} \qquad \text{i.e.} \quad \varepsilon_D \circ FG(u) = u \circ \varepsilon_{F(C)} \tag{3.17}$$

Now, we have:

$$\begin{aligned}
\theta_{C,D} \circ \varphi_{C,D}(v) &= \theta_{C,D}\big(\varepsilon_D \circ F(v)\big) = G\big(\varepsilon_D \circ F(v)\big) \circ \eta_C \\
&= G(\varepsilon_D) \circ \underline{GF(v) \circ \eta_C} \\
&\overset{(3.16)}{=} \underline{G(\varepsilon_D) \circ \eta_{G(D)}} \circ v \\
&\overset{(3.11)}{=} v \\
\varphi_{C,D} \circ \theta_{C,D}(u) &= \varphi_{C,D}\big(G(u) \circ \eta_C\big) = \varepsilon_D \circ F\big(G(u) \circ \eta_C\big) \\
&= \underline{\varepsilon_D \circ FG(u)} \circ F(\eta_C) \\
&\overset{(3.17)}{=} u \circ \underline{\varepsilon_{F(C)} \circ F(\eta_C)} \\
&\overset{(3.10)}{=} u
\end{aligned}$$

Thus $\theta_{C,D}$ and $\varphi_{C,D}$ are inverses to each other for any $C \in \operatorname{Ob}\mathcal{C}$ and $D \in \operatorname{Ob}\mathcal{D}$. We are left to prove that θ is natural in both variables, i.e. the diagrams (3.1) and (3.2) are commutative. Indeed, let $f \in \operatorname{Hom}_{\mathcal{C}}(C', C)$ and $u \in \operatorname{Hom}_{\mathcal{D}}(F(C), D)$; we have:

$$\begin{aligned}
\mathtt{Hom}_{\mathcal{C}}(f, G(D)) \circ \theta_{C,D}(u) &= \mathtt{Hom}_{\mathcal{C}}(f, G(D)) \circ \big(G(u) \circ \eta_C\big) \\
&= G(u) \circ \underline{\eta_C \circ f} \\
&= G(u) \circ GF(f) \circ \eta_{C'} \\
&= \theta_{C',D}\big(u \circ F(f)\big) \\
&= \theta_{C',D} \circ \mathtt{Hom}_{\mathcal{D}}(F(f), D)(u)
\end{aligned}$$

where in the third equality we used the naturality of η applied to f. Thus, (3.1) holds. Consider now $g \in \mathrm{Hom}_{\mathcal{D}}(D, D')$ and $u \in \mathrm{Hom}_{\mathcal{D}}(F(C), D)$. Then:

$$\begin{aligned} \mathtt{Hom}_{\mathcal{C}}(C, G(g)) \circ \theta_{C, D}(u) &= G(g) \circ G(u) \circ \eta_C \\ &= G(g \circ u) \circ \eta_C \\ &= \theta_{C, D'} \circ \mathtt{Hom}_{\mathcal{D}}(F(C), g)(u) \end{aligned}$$

This proves that (3.2) also holds and the proof is now finished. □

Remark 3.3.2 *Let $F : \mathcal{C} \to \mathcal{D}$ be an isomorphism of categories with inverse $G : \mathcal{D} \to \mathcal{C}$. Then (F, G) is a pair of adjoint functors with unit and counit given by the identity natural transformations.*

Example 3.3.3 *Consider the pair of adjoint functors $F = - \otimes_R X$, $G = \mathit{Hom}_{R\mathcal{M}}(X, -) : {}_R\mathcal{M} \to {}_R\mathcal{M}$ from Example 3.2.1, 2). The unit and counit of the adjunction $F \dashv G$ are given as follows for any Y, $Z \in \mathrm{Ob}\, {}_R\mathcal{M}$:*

$$\eta \colon 1_{R\mathcal{M}} \to \mathit{Hom}_{R\mathcal{M}}(X, - \otimes_R X), \quad \varepsilon \colon \mathit{Hom}_{R\mathcal{M}}(X, -) \otimes_R X \to 1_{R\mathcal{M}}$$
$$\eta_Y \colon Y \to \mathit{Hom}_{R\mathcal{M}}(X, Y \otimes_R X), \ \ \eta_Y(y)(x) = y \otimes_R x, \ y \in Y, x \in X$$
$$\varepsilon_Z \colon \mathit{Hom}_{R\mathcal{M}}(X, Z) \otimes_R X \to Z, \ \ \varepsilon_Z(f \otimes_R x) = f(x), \ f \in \mathit{Hom}_{R\mathcal{M}}(X, Z), x \in X.$$

3.4 Another characterisation of adjoint functors

Theorem 3.4.1 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors. The following are equivalent:*

1) $F \dashv G$;

2) There exists a natural transformation $\eta : 1_{\mathcal{C}} \to GF$ such that for any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(C, G(D))$ there exists a unique morphism $g \in \mathrm{Hom}_{\mathcal{D}}(F(C), D)$ which makes the following diagram commutative:

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & GF(C) \arrow[d, "G(g)"] \\ & G(D) \end{tikzcd} \qquad \text{i.e.} \quad G(g) \circ \eta_C = f \tag{3.18}$$

3) There exists a natural transformation $\varepsilon : FG \to 1_{\mathcal{D}}$ such that for any morphism $f \in \mathrm{Hom}_{\mathcal{D}}(F(C), D)$ there exists a unique morphism $g \in \mathrm{Hom}_{\mathcal{C}}(C, G(D))$ which makes the following diagram commutative:

$$\begin{tikzcd} & F(C) \arrow[dl, "F(g)"'] \arrow[d, "f"] \\ FG(D) \arrow[r, "\varepsilon_D"'] & D \end{tikzcd} \qquad \text{i.e.} \quad \varepsilon_D \circ F(g) = f \tag{3.19}$$

Proof: We will only prove the equivalence between 1) and 2); the equivalence between 1) and 3) follows similarly.

Suppose first that $F \dashv G$ and let θ be the corresponding natural isomorphism. We define the natural transformation $\eta : 1_{\mathcal{C}} \to GF$ as in the proof of Theorem 3.3.1, namely by $\eta_C = \theta_{C,F(C)}(1_{F(C)})$ for any $C \in \operatorname{Ob}\mathcal{C}$. Let $f \in \operatorname{Hom}_{\mathcal{C}}(C, G(D))$; we will prove that $g = \theta^{-1}_{C,D}(f) \in \operatorname{Hom}_{\mathcal{D}}(F(C), D)$ is the unique morphism in $\mathcal{D}$ which makes the diagram (3.18) commute. Indeed, by setting $X = C$, $Y = F(C)$ and $Y' = D$ in (3.2) yields the following commutative diagram for all $u \in \operatorname{Hom}_{\mathcal{C}}(F(C), D)$:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{D}}(F(C),\, F(C)) \arrow[r, "\theta_{C,\,F(C)}"] \arrow[d, "\mathtt{Hom}_{\mathcal{D}}(F(C),u)"'] & \operatorname{Hom}_{\mathcal{C}}(C,\, GF(C)) \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(C,G(u))"] \\
\operatorname{Hom}_{\mathcal{D}}(F(C),\, D) \arrow[r, "\theta_{C,\,D}"'] & \operatorname{Hom}_{\mathcal{C}}(C,\, G(D))
\end{tikzcd}$$

By applying the commutativity of the above diagram to $1_{F(C)}$ we obtain:

$$\mathtt{Hom}_{\mathcal{C}}(C,\, G(u)) \circ \theta_{C,\,F(C)}(1_{F(C)}) = \theta_{C,\,D} \circ \mathtt{Hom}_{\mathcal{D}}(F(C),\, u)(1_{F(C)})$$

$$\text{i.e. } G(u) \circ \eta_C = \theta_{C,\,D}(u) \tag{3.20}$$

Thus, we have:

$$G(g) \circ \eta_C \overset{(3.20)}{=} \theta_{C,D}(g) = \theta_{C,D} \circ \theta^{-1}_{C,D}(f) = f$$

The morphism g is unique with the above property since θ is a bijection.

Assume now that 2) holds, i.e. for any $f \in \operatorname{Hom}_{\mathcal{C}}(C, G(D))$ there exists a unique morphism $g \in \operatorname{Hom}_{\mathcal{D}}(F(C), D)$ such that (3.18) is fulfilled. Given $C \in \operatorname{Ob}\mathcal{C}$ and $D \in \operatorname{Ob}\mathcal{D}$ we define the following map:

$$\theta_{C,\,D} : \operatorname{Hom}_{\mathcal{D}}(F(C),\, D) \to \operatorname{Hom}_{\mathcal{C}}(C,\, G(D)), \quad \theta_{C,\,D}(u) = G(u) \circ \eta_C$$

for any $u \in \operatorname{Hom}_{\mathcal{D}}(F(C), D)$. Obviously our assumption implies that $\theta_{C,\,D}$ is a set bijection for all $C \in \operatorname{Ob}\mathcal{C}$ and $D \in \operatorname{Ob}\mathcal{D}$. We are left to show that θ is natural in both variables, i.e. the diagrams (3.1) and (3.2) are commutative. Indeed, let $f \in \operatorname{Hom}_{\mathcal{C}}(C', C)$ and $u \in \operatorname{Hom}_{\mathcal{D}}(F(C), D)$; we have:

$$\begin{aligned}
\mathtt{Hom}_{\mathcal{C}}(f,\, G(D)) \circ \theta_{C,\,D}(u) &= \mathtt{Hom}_{\mathcal{C}}(f,\, G(D)) \circ \big(G(u) \circ \eta_C\big) \\
&= G(u) \circ \underline{\eta_C \circ f} \\
&= G(u) \circ GF(f) \circ \eta_{C'} \\
&= \theta_{C',\,D}\big(u \circ F(f)\big) \\
&= \theta_{C',\,D} \circ \mathtt{Hom}_{\mathcal{D}}(F(f),\, D)(u)
\end{aligned}$$

where in the third equality we used the naturality of η applied to f. Thus, (3.1) holds.

Consider now $g \in \mathrm{Hom}_{\mathcal{D}}(D,\, D')$ and $u \in \mathrm{Hom}_{\mathcal{D}}(F(C),\, D)$. Then:

$$\begin{aligned} \mathtt{Hom}_{\mathcal{C}}(C,\, G(g)) \circ \theta_{C,\,D}(u) &= G(g) \circ G(u) \circ \eta_C \\ &= G(g \circ u) \circ \eta_C \\ &= \theta_{C,\,D'} \circ \mathtt{Hom}_{\mathcal{D}}(F(C),\, g)(u) \end{aligned}$$

This proves that (3.2) also holds and the proof is now finished. □

Corollary 3.4.2 *Suppose $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are two functors such that $F \dashv G$ and consider $\eta : 1_{\mathcal{C}} \to GF$, $\varepsilon : FG \to 1_{\mathcal{D}}$ to be the unit, respectively the counit of the adjunction.*

1) If $g,\, g' \in \mathrm{Hom}_{\mathcal{D}}(F(C), D)$ such that $G(g) \circ \eta_C = G(g') \circ \eta_C$ then $g = g'$;

2) If $g,\, g' \in \mathrm{Hom}_{\mathcal{C}}(C, G(D))$ such that $\varepsilon_D \circ F(g) = \varepsilon_D \circ F(g')$ then $g = g'$.

Proof: 1) Follows trivially from Theorem 3.4.1, 2) by considering $f = G(g') \circ \eta_C$. Then both morphisms g and g' make diagram (3.18) commutative which implies $g = g'$. The second part follows in a similar manner by using Theorem 3.4.1, 3). □

Example 3.4.3 *The forgetful functor U : **Top** $\to$ **Set** has both a left and a right adjoint.*

*We start by constructing the left adjoint functor F : **Set** $\to$ **Top** which endows each $X \in \mathrm{Ob}\,$**Set** with the discrete topology (i.e. every subset of X is open). We define a natural transformation $\eta : 1_{\mathbf{Set}} \to UF$ by $\eta_X(x) = x$ for any $X \in \mathrm{Ob}\,$**Set** and $x \in X$. Consider now $f \in \mathrm{Hom}_{\mathbf{Set}}(X,\, U(Y))$. According to Theorem 3.4.1, in order to prove that $F \dashv U$ we need to find a unique morphism $g \in \mathrm{Hom}_{\mathbf{Set}}(F(X),\, Y)$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & F(X) \arrow[d, "g"] \\ & Y \end{tikzcd}$$

To this end, it is enough to consider $g = f$. Remark that f is obviously continous since we are dealing with the discrete topology.

*On the other hand, the right adjoint G endows each $X \in \mathrm{Ob}\,$**Set** with the indiscrete topology (i.e. the only open subsets are X and the empty set). We define a natural transformation $\eta : 1_{\mathbf{Set}} \to GU$ by $\eta_X(x) = x$ for any $X \in \mathrm{Ob}\,$**Set** and $x \in X$. Now since $\eta_X^{-1}(\emptyset) = \emptyset$ and $\eta_X^{-1}(G(X)) = G(X) = X$ we obtain that each η_X is continous. Consider now $f \in \mathrm{Hom}_{\mathbf{Top}}(X,\, G(Y))$. We need to find a unique morphism $g \in \mathrm{Hom}_{\mathbf{Set}}(U(X),\, Y)$ such that the following diagram is commutative:*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & G(X) \arrow[d, "G(g)"] \\ & G(Y) \end{tikzcd}$$

As before, we set $g = f$.

Theorem 3.4.4 *Any two left (right) adjoints of a given functor are naturally isomorphic.*

Proof: Assume $F, F' : \mathcal{C} \to \mathcal{D}$ are both left adjoint functors of $G : \mathcal{D} \to \mathcal{C}$. Then there exist natural transformations $\eta : 1_{\mathcal{C}} \to GF$ and $\eta' : 1_{\mathcal{C}} \to GF'$ satisfying the conditions in Theorem 3.4.1, 2). Given $C \in \operatorname{Ob}\mathcal{C}$, as $F' \dashv G$ and $\eta_C \in \operatorname{Hom}_{\mathcal{C}}(C, GF(C))$, there exists a unique morphism $\gamma_C \in \operatorname{Hom}_{\mathcal{D}}(F'(C), F(C))$ such that:

$$G(\gamma_C) \circ \eta'_C = \eta_C \tag{3.21}$$

Similarly, as $F \dashv G$ and $\eta'_C \in \operatorname{Hom}_{\mathcal{C}}(C, GF'(C))$, there exists a unique morphism $\gamma'_C \in \operatorname{Hom}_{\mathcal{D}}(F(C), F'(C))$ such that:

$$G(\gamma'_C) \circ \eta_C = \eta'_C. \tag{3.22}$$

We will see that each γ_C is an isomorphism with the inverse given precisely by γ'_C. Indeed, using (3.21) and (3.22) we can easily see that $G(\gamma'_C \circ \gamma_C) \circ \eta'_C = \eta'_C$ and since we obviously also have $G(1_{F'(C)}) \circ \eta'_C = \eta'_C$ it follows by Corollary 3.4.2 that $\gamma'_C \circ \gamma_C = 1_{F'(C)}$. Similarly, one can prove that $\gamma_C \circ \gamma'_C = 1_{F(C)}$.

We are left to prove that $\gamma : F' \to F$ is a natural transformation, i.e. for any $f \in \operatorname{Hom}_{\mathcal{C}}(C, C')$ the following diagram is commutative:

$$\begin{tikzcd} F'(C) \arrow[r, "\gamma_C"] \arrow[d, "F'(f)"'] & F(C) \arrow[d, "F(f)"] \\ F'(C') \arrow[r, "\gamma_{C'}"'] & F(C') \end{tikzcd} \qquad \text{i.e.} \quad F(f) \circ \gamma_C = \gamma_{C'} \circ F'(f)$$

Using Corollary 3.4.2 it is enough to prove the following:

$$G(F(f) \circ \gamma_C) \circ \eta'_C = G(\gamma_{C'} \circ F'(f)) \circ \eta'_C \tag{3.23}$$

To this end, we use the naturality of η and respectively η'; that is, the commutativity of the following diagrams:

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "f"'] & GF(C) \arrow[d, "GF(f)"] \\ C' \arrow[r, "\eta_{C'}"'] & GF(C') \end{tikzcd} \qquad \text{i.e.} \quad GF(f) \circ \eta_C = \eta_{C'} \circ f \tag{3.24}$$

$$\begin{tikzcd} C \arrow[r, "\eta'_C"] \arrow[d, "f"'] & GF'(C) \arrow[d, "GF'(f)"] \\ C' \arrow[r, "\eta'_{C'}"'] & GF'(C') \end{tikzcd} \qquad \text{i.e.} \quad GF'(f) \circ \eta'_C = \eta'_{C'} \circ f \tag{3.25}$$

Then we have:

$$
\begin{aligned}
GF(f)\circ \underline{G(\gamma_C)\circ\eta'_C} &\overset{(3.21)}{=} \underline{GF(f)\circ\eta_C}\\
&\overset{(3.24)}{=} \underline{\eta_{C'}}\circ f\\
&\overset{(3.21)}{=} G(\gamma_{C'})\circ \underline{\eta'_{C'}\circ f}\\
&\overset{(3.25)}{=} G(\gamma_{C'})\circ GF'(f)\circ\eta'_C
\end{aligned}
$$

Therefore, (3.23) holds and the proof is now complete. □

3.5 Equivalence of categories

When studying categories which are *essentially the same*, the first notion we usually encounter is that of an isomorphism of categories. However, this concept turns out to be too strict as there are many examples of categories with similar properties (such as completeness, cocompleteness etc.) which are not isomorphic. A much more natural notion than isomorphism for saying when two categories are *essentially the same* is the following:

Definition 3.5.1 *A functor $F:\mathcal{C}\to\mathcal{D}$ is called an equivalence of categories if there exists another functor $G:\mathcal{D}\to\mathcal{C}$ such that we have natural isomorphisms $GF\cong 1_{\mathcal{C}}$ and $FG\cong 1_{\mathcal{D}}$. A contravariant functor $F:\mathcal{C}\to\mathcal{D}$ such that $F:\mathcal{C}^{op}\to\mathcal{D}$ is an equivalence of categories is called a duality of categories.*

Example 3.5.2 *Let K be a field; the category $\boldsymbol{Mat}_K$ defined in Example 1.1.4, 18), is equivalent to the category of finite dimensional K vector spaces ${}_K\mathcal{M}^{fd}$. Indeed, the functor $F\colon {}_K\mathcal{M}^{fd}\to\boldsymbol{Mat}_K$ defined below is an equivalence of categories:*

$$
\begin{aligned}
&F(V)=\dim(V),\ \text{for all finite dimensional vector spaces } V,\\
&F(\alpha)=U_\alpha,\ \text{where } U_\alpha \text{ is the matrix of the linear map } \alpha\colon V\to W \text{ with respect to}\\
&\text{the chosen bases of V and W.}
\end{aligned}
$$

To this end, we define the functor $G\colon\boldsymbol{Mat}_K\to{}_K\mathcal{M}^{fd}$ as follows:

$$
\begin{aligned}
&G(n)=K^n,\ \text{the n} - \text{dimensional space of column vectors over K, for all } n\in\mathbb{N},\\
&G(A)=M_A,\ \text{for all morphisms } A\colon m\to n \text{ in } \boldsymbol{Mat}_K,\ \text{where } M_A\colon K^m\to K^n \text{ is}\\
&\text{given by } M_A(v)=Av \text{ for all } v\in K^m.
\end{aligned}
$$

Note that for each K^n the basis we consider will be the standard basis.

To start with, we show that $FG=1_{\boldsymbol{Mat}_K}$. For any $n\in\mathbb{N}$ we have:

$$FG(n)=F\big(K^n\big)=\dim\big(K^n\big)=n=1_{\boldsymbol{Mat}_K}(n).$$

Moreover, if $A\colon m \to n$ *is a morphism in* $\boldsymbol{Mat}_K$*, we have* $FG(A) = F(M_A) = U_{M_A}$*, where* U_{M_A} *is the matrix of the linear map* $M_A\colon K^m \to K^n$ *given by* $M_A(v) = Av$ *with respect to the standard bases* $\{e_1, e_2, \cdots, e_m\}$ *and* $\{f_1, f_2, \cdots, f_n\}$ *of* K^m *and* K^n *respectively. Having in mind that the element* e_i *(resp.* f_j*) of the standard basis is the column vector in* K^m *(resp. in* K^n*) having 1 on the* i*-th (resp.* j*-th) position and zeros elsewhere for all* $i = 1, 2, \cdots, m$ *(resp.* $j = 1, 2, \cdots, n$*) we obtain* $M_A(e_i) = Ae_i = \sum_{j=1}^{n} a_{ji} f_j$ *where we denote* $A = \left(a_{kl}\right)_{k=\overline{1,n},\, l=\overline{1,m}}$*. This proves that* $U_{M_A} = A$*, i.e.* $FG(A) = A$ *as desired. Hence, we proved that* $FG = 1_{\boldsymbol{Mat}_K}$*; in particular* FG *is naturally isomorphic to* $1_{\boldsymbol{Mat}_K}$*.*

We are left to show that GF *is naturally isomorphic to* $1_{{}_K\mathcal{M}^{fd}}$*. Consider* $\eta\colon 1_{{}_K\mathcal{M}^{fd}} \to GF$ *defined for any vector space* V *by* $\eta_V\colon V \to K^{\dim(V)}$*,* $\eta_V(v) = [v]$ *where we denote by* $[v]$ *the (column) coordinate vector of* v *with respect to the chosen basis of* V*. We claim that* η *is a natural isomorphism. Indeed, each* η_V *is obviously a linear bijection. We are left to check the naturality condition. Indeed, let* $\alpha\colon V \to W$ *be a morphism in* ${}_K\mathcal{M}^{fd}$*,* $m = \dim(V)$*,* $n = \dim(W)$ *and consider* $\{u_1, u_2, \cdots, u_m\}$ *and* $\{w_1, w_2, \cdots, w_n\}$ *bases in* V *and* W *respectively. The proof will be finished once we show that the following diagram is commutative:*

$$\begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[d, "\alpha"'] & K^m \arrow[d, "GF(\alpha)"] \\ W \arrow[r, "\eta_W"'] & K^n \end{tikzcd} \tag{3.26}$$

For any $v = \sum_{i=1}^{m} v_i u_i \in V$ *we have* $\alpha(v) = \sum_{i=1}^{m} v_i\, \alpha(u_i) = \sum_{i=1}^{m} v_i \left(\sum_{j=1}^{n} u_{ji}\, w_j\right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} v_i\, u_{ji}\right) w_j$ *and therefore the* j*-th component of* $[v]$ *is* $\sum_{i=1}^{m} v_i\, u_{ji}$ *for all* $j = 1, 2, \cdots, n$*, where* $U_\alpha = \left(u_{kl}\right)_{k=\overline{1,n},\, l=\overline{1,m}}$*. Moreover, a similar straightforward computation shows that the* j*-th component of* $U_\alpha[v]$ *is also* $\sum_{i=1}^{m} v_i\, u_{ji}$ *for all* $j = 1, 2, \cdots, n$*. Putting everything together we proved the following:*

$$[\alpha(v)] = U_\alpha[v]. \tag{3.27}$$

Therefore, for any $v \in V$ *we have:*

$$GF(\alpha) \circ \eta_V(v) = GF(\alpha)\big([v]\big) = M_{U_\alpha}\big([v]\big) = U_\alpha\,[v] \overset{(3.27)}{=} \big[\alpha(v)\big] = \eta_W \circ \alpha(v)$$

which proves the commutativity of diagram (3.26). The proof is now finished.

Remark 3.5.3 *The two categories* $\boldsymbol{Mat}_K$ *and* ${}_K\mathcal{M}^{fd}$ *from the previous example are equivalent but not isomorphic. Indeed, this follows easily by noticing that* $\boldsymbol{Mat}_K$ *is a small category while* ${}_K\mathcal{M}^{fd}$ *is not.*

We prove the following characterization of equivalences:

Theorem 3.5.4 *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor. The following are equivalent:*

1) F is an equivalence of categories;

2) F is faithful, full and essentially surjective;

3) There exists $G : \mathcal{D} \to \mathcal{C}$ a right adjoint of F such that the unit and counit of the adjunction are natural isomorphisms;

4) There exists $G : \mathcal{D} \to \mathcal{C}$ a left adjoint of F such that the unit and counit of the adjunction are natural isomorphisms.

Proof: 1) $\Rightarrow$ 2) Since F is an equivalence of categories, there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms $\eta : 1_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to 1_{\mathcal{D}}$. To start with, for any $D \in \operatorname{Ob}\mathcal{D}$ the morphism $\varepsilon_D \in \operatorname{Hom}_{\mathcal{D}}(FG(D), D)$ is an isomorphism and therefore F is essentially surjective. Next we prove that F is fully faithful. Let h_1, $h_2 \in \operatorname{Hom}_{\mathcal{C}}(C, C')$ such that $F(h_1) = F(h_2)$. Then we also have $GF(h_1) = GF(h_2)$. Moreover, the naturality of η renders the following diagrams commutative:

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "h_1"'] & GF(C) \arrow[d, "GF(h_1)"] \\ C' \arrow[r, "\eta_{C'}"'] & GF(C') \end{tikzcd} \qquad \text{i.e.} \quad GF(h_1) \circ \eta_C = \eta_{C'} \circ h_1 \tag{3.28}$$

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "h_2"'] & GF(C) \arrow[d, "GF(h_2)"] \\ C' \arrow[r, "\eta_{C'}"'] & GF(C') \end{tikzcd} \qquad \text{i.e.} \quad GF(h_2) \circ \eta_C = \eta_{C'} \circ h_2 \tag{3.29}$$

From (3.28) and (3.29) we obtain $\eta_{C'} \circ h_1 = \eta_{C'} \circ h_2$ and since $\eta_{C'}$ is an isomorphism we get $h_1 = h_2$ as desired. Similarly, using the naturality of ε it follows that G is faithful as well.

Consider now C, $C' \in \operatorname{Ob}\mathcal{C}$ and $g \in \operatorname{Hom}_{\mathcal{D}}(F(C), F(C'))$. Now define:

$$f = \eta_{C'}^{-1} \circ G(g) \circ \eta_C \in \operatorname{Hom}_{\mathcal{C}}(C, C') \tag{3.30}$$

We will prove that $F(f) = g$. Indeed, using the naturality of η we obtain:

$$\eta_{C'}^{-1} \circ GF(f) \circ \eta_C \overset{(3.28)}{=} f \overset{(3.30)}{=} \eta_{C'}^{-1} \circ G(g) \circ \eta_C$$

Since η_C and $\eta_{C'}$ are isomorphisms we get $GF(f) = G(g)$. As G is faithful the above equality comes down to $F(f) = g$.

2) $\Rightarrow$ 1) Assume now that F is fully faithful and essentially surjective. First remark that since F is fully faithful it reflects isomorphisms (see Proposition 1.4.17), thus two objects C and C' are isomorphic in $\mathcal{C}$ if and only if $F(C)$ and $F(C')$ are isomorphic in $\mathcal{D}$. As F is essentially surjective, for any $D \in \operatorname{Ob}\mathcal{D}$ there exists a unique (up to isomorphism)

$C \in \operatorname{Ob}\mathcal{C}$ such that $F(C) \simeq D$. Thus, for any $D \in \operatorname{Ob}\mathcal{D}$ we choose[1] an object $G(D) \in \operatorname{Ob}\mathcal{C}$ and an isomorphism $\varepsilon_D : FG(D) \to D$ in $\mathcal{D}$. Now if $g \in \operatorname{Hom}_{\mathcal{D}}(D, D')$ we have the following morphism in $\mathcal{D}$:

$$\varepsilon_{D'}^{-1} \circ g \circ \varepsilon_D : FG(D) \to FG(D')$$

Since F is fully faithful, there exists a unique morphism $G(g) \in \operatorname{Hom}_{\mathcal{C}}(G(D), G(D'))$ such that $FG(g) = \varepsilon_{D'}^{-1} \circ g \circ \varepsilon_D$. The last equality implies the commutativity of the following diagram:

$$\begin{tikzcd} FG(D) \arrow[r, "\varepsilon_D"] \arrow[d, "FG(g)"'] & D \arrow[d, "g"] \\ FG(D') \arrow[r, "\varepsilon_{D'}"'] & D' \end{tikzcd} \tag{3.31}$$

We will prove that G defined above is in fact a functor and $\varepsilon : FG \to 1_{\mathcal{D}}$ is a natural transformation. Indeed, setting $D = D'$ and $g = 1_D$ yields:

$$\varepsilon_D^{-1} \circ 1_D \circ \varepsilon_D = 1_{FG(D)} : FG(D) \to FG(D)$$

and there exists a unique morphism $G(1_D) \in \operatorname{Hom}_{\mathcal{C}}(G(D), G(D))$ such that $FG(1_D) = 1_{FG(D)} = F(1_{G(D)})$. Since F is faithful we get $G(1_D) = 1_{G(D)}$. Consider now $g \in \operatorname{Hom}_{\mathcal{D}}(D, D')$, $g' \in \operatorname{Hom}_{\mathcal{D}}(D', D'')$ and the unique morphisms $G(g) \in \operatorname{Hom}_{\mathcal{C}}(G(D), G(D'))$, respectively $G(g') \in \operatorname{Hom}_{\mathcal{C}}(G(D'), G(D''))$ such that:

$$FG(g) = \varepsilon_{D'}^{-1} \circ g \circ \varepsilon_D \quad \text{and} \quad FG(g') = \varepsilon_{D''}^{-1} \circ g' \circ \varepsilon_{D'}.$$

This yields:

$$F\big(G(g') \circ G(g)\big) = \varepsilon_{D''}^{-1} \circ g' \circ g \circ \varepsilon_D \tag{3.32}$$

Now having in mind that there exists a unique morphism $G(g' \circ g) \in \operatorname{Hom}_{\mathcal{C}}(G(D), G(D''))$ such that $FG(g' \circ g) = \varepsilon_{D''}^{-1} \circ (g' \circ g) \circ \varepsilon_D$, it follows from (3.32) that $G(g' \circ g) = G(g') \circ G(g)$. Therefore, G is indeed a functor. Now the commutativity of diagram (3.31) implies that ε is a natural transformation. Recall that every ε_D is an isomorphism and thus ε is in fact a natural isomorphism. We are left to construct a natural isomorphism $\eta : 1_{\mathcal{C}} \to GF$. For any $C \in \operatorname{Ob}\mathcal{C}$ we have $\varepsilon_{F(C)}^{-1} \in \operatorname{Hom}_{\mathcal{D}}(F(C), FGF(C))$ an isomorphism in $\mathcal{D}$. Since F is fully faithful, there exists a unique $\eta_C \in \operatorname{Hom}_{\mathcal{C}}(C, GF(C))$ such that $F(\eta_C) = \varepsilon_{F(C)}^{-1}$. Obviously, η_C is an isomorphism for all $C \in \operatorname{Ob}\mathcal{C}$ since F reflects isomorphisms. We prove now that η is a natural transformation. To this end, let $f \in \operatorname{Hom}_{\mathcal{C}}(C, C')$; we need to prove the commutativity of the following diagram:

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "f"'] & GF(C) \arrow[d, "GF(f)"] \\ C' \arrow[r, "\eta_{C'}"'] & C' \end{tikzcd} \qquad \text{i.e.} \quad \eta_{C'} \circ f = GF(f) \circ \eta_C \tag{3.33}$$

[1] We assume that the axiom of choice holds.

By naturality of ε applied to $F(f)$ we have the following commutative diagram:

$$\begin{tikzcd}
FGF(C) \arrow[r, "\varepsilon_{F(C)}"] \arrow[d, "FGF(f)"'] & F(C) \arrow[d, "F(f)"] \\
FGF(C') \arrow[r, "\varepsilon_{F(C')}"'] & F(C')
\end{tikzcd}$$

$$\begin{aligned}
\text{i.e.} \qquad & \varepsilon_{F(C')} \circ FGF(f) = F(f) \circ \varepsilon_{F(C)} \\
& \Leftrightarrow FGF(f) \circ \varepsilon^{-1}_{F(C)} = \varepsilon^{-1}_{F(C')} \circ F(f) \\
& \Leftrightarrow FGF(f) \circ F(\eta_C) = F(\eta_{C'}) \circ F(f) \\
& \Leftrightarrow F\big(GF(f) \circ \eta_C\big) = F(\eta_{C'} \circ f)
\end{aligned}$$

Since F is faithful we get $GF(f) \circ \eta_C = \eta_{C'} \circ f$, i.e. (3.33) is commutative, as desired.

3) $\Rightarrow$ 1) Obvious.

1) $\Rightarrow$ 3) Suppose F is an equivalence of categories and let $G : \mathcal{D} \to \mathcal{C}$ such that $GF \cong 1_{\mathcal{C}}$ and $FG \cong 1_{\mathcal{D}}$. We will prove that G is right adjoint to F. Denote by $\varepsilon : FG \to 1_{\mathcal{D}}$ the natural isomorphism arising from the above equivalence. Thus, for any $C \in \operatorname{Ob}\mathcal{C}$ the morphism $\varepsilon_{F(C)} \in \operatorname{Hom}_{\mathcal{D}}(FGF(C), F(C))$ is an isomorphism. Therefore, $\varepsilon^{-1}_{F(C)} \in \operatorname{Hom}_{\mathcal{D}}(F(C), FGF(C))$ and since F is fully faithful (see 1) $\Rightarrow$ 2)) there exists a unique morphism $\eta_C \in \operatorname{Hom}_{\mathcal{C}}(C, GF(C))$ such that $F(\eta_C) = \varepsilon^{-1}_{F(C)}$. Since F is fully faithful, it preserves and reflects isomorphisms; thus η_C is also an isomorphism. Furthermore, one can show exactly as in the proof of 2) $\Rightarrow$ 1) that η is also a natural transformation. In light of Theorem 3.3.1, the proof will be finished once we show that (3.10) and (3.11) hold. To start with, for all $C \in \operatorname{Ob}\mathcal{C}$ we have:

$$\varepsilon_{F(C)} \circ F(\eta_C) = \varepsilon_{F(C)} \circ \varepsilon^{-1}_{F(C)} = 1_{F(C)}, \quad \text{i.e. (3.10) is fulfilled.}$$

Consider now $D \in \operatorname{Ob}\mathcal{D}$ and $\varepsilon^{-1}_D : D \to FG(D)$. From the naturality of ε applied to the morphism ε^{-1}_D we obtain the following commutative diagram:

$$\begin{tikzcd}
FG(D) \arrow[r, "\varepsilon_D"] \arrow[d, "FG(\varepsilon_D^{-1})"'] & D \arrow[d, "\varepsilon_D^{-1}"] \\
FGFG(D) \arrow[r, "\varepsilon_{FG(D)}"'] & FG(D)
\end{tikzcd} \qquad \text{i.e.} \quad \varepsilon_{FG(D)} \circ FG(\varepsilon_D^{-1}) = 1_{FG(D)}$$

Therefore, since F is faithful, we have:

$$\begin{aligned}
& FG(\varepsilon_D) \circ \varepsilon^{-1}_{FG(D)} = 1_{FG(D)} \\
& \Leftrightarrow FG(\varepsilon_D) \circ F(\eta_{G(D)}) = 1_{FG(D)} \\
& \Leftrightarrow F(G(\varepsilon_D) \circ \eta_{G(D)}) = F\big(1_{G(D)}\big) \\
& \Leftrightarrow G(\varepsilon_D) \circ \eta_{G(D)} = 1_{G(D)} \quad \text{i.e. (3.11) holds as well.}
\end{aligned}$$

$\square$

Example 3.5.5 *A skeleton of a category $\mathcal{C}$ is a full subcategory $\mathcal{A}$ of $\mathcal{C}$ such that each object of $\mathcal{C}$ is isomorphic to exactly one object of $\mathcal{A}$*[2]*. Using Theorem 3.5.4, 2), we can easily conclude that the inclusion $I : \mathcal{A} \to \mathcal{C}$ is an equivalence of categories, i.e. a category is equivalent to any of its skeletons. Loosely speaking, two equivalent categories are identical except that they might have different numbers of isomorphic copies of the same object.*

Corollary 3.5.6 *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. Then:*

1) $\mathcal{C}$ is complete if and only if $\mathcal{D}$ is complete;

2) $\mathcal{C}$ is cocomplete if and only if $\mathcal{D}$ is cocomplete.

Proof: By Theorem 3.5.4, F admits both a left a right adjoint. Now the desired conclusion follows by Proposition 3.2.3.

3.6 (Co)reflective subcategories

Definition 3.6.1 *A full subcategory $\mathcal{A}$ of $\mathcal{B}$ is called reflective if the inclusion functor $I : \mathcal{A} \to \mathcal{B}$ admits a left adjoint $R : \mathcal{B} \to \mathcal{A}$ called reflector. Dually, a full subcategory $\mathcal{A}$ of $\mathcal{B}$ is called coreflective if the inclusion functor admits a right adjoint called coreflector.*

Examples 3.6.2 *1) **Ab** is a reflective subcategory of **Grp**;*

*2) **Grp** is both a reflective and a coreflective subcategory of **Mon**;*

*3) **KHaus** is a reflective subcategory of **Top**. The Stone-Čech compactification provides the reflector.*

Proposition 3.6.3 1) *A reflective subcategory of a complete category is itself complete;* 2) *A coreflective subcategory of a cocomplete category is itself cocomplete.*

Proof: We will only prove the first assertion; the second one follows in a similar manner. Let $\mathcal{B}$ be a complete category, $\mathcal{A}$ a reflective subcategory of $\mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ the reflector of the inclusion functor $I : \mathcal{A} \to \mathcal{B}$, i.e. $R \dashv I$. Consider J to be a small category and $F : J \to \mathcal{A}$ a functor. Since $\mathcal{B}$ is complete, the functor $IF : J \to \mathcal{B}$ has a limit which we denote by $\big(L, (p_j : L \to IF(j))_{j \in \mathrm{Ob}\, J}\big)$. Since $\big(L, (p_j : L \to IF(j))_{j \in \mathrm{Ob}\, J}\big)$ is in particular a cone on IF, the following diagram is commutative for any $d \in \mathrm{Hom}_J(j, j')$:

$$\begin{tikzcd} & L \arrow[dl, "p_j"'] \arrow[dr, "p_{j'}"] & \\ IF(j) \arrow[rr, "IF(d)"'] & & IF(j') \end{tikzcd} \qquad \text{i.e. } IF(d) \circ p_j = p_{j'}. \tag{3.34}$$

[2] The existence of a skeleton for a given category is ensured by the axiom of choice for classes.

Let $\eta : 1_{\mathcal{B}} \to IR$ be the unit of the adjunction $R \dashv I$ (see Theorem 3.3.1). By Theorem 3.4.1, for any $p_j \in \operatorname{Hom}_{\mathcal{B}}(L, IF(j))$ there exists an unique morphism $q_j \in \operatorname{Hom}_{\mathcal{A}}(R(L), F(j))$ such that the following diagram commutes for all $j \in \operatorname{Ob} J$:

$$\begin{tikzcd} L \arrow[r, "\eta_L"] \arrow[dr, "p_j"'] & IR(L) \arrow[d, "I(q_j)"] \\ & IF(j) \end{tikzcd} \qquad \text{i.e.} \quad I(q_j) \circ \eta_L = p_j \tag{3.35}$$

We will prove that $\big(R(L), (q_j : R(L) \to F(j))_{j \in \operatorname{Ob} J}\big)$ is a cone on F. Indeed, for any $d \in \operatorname{Hom}_J(j, j')$ we have:

$$I\big(F(d) \circ q_j\big) \circ \eta_L = IF(d) \circ \underline{I\big(q_j\big) \circ \eta_L} \overset{(3.35)}{=} IF(d) \circ p_j \overset{(3.34)}{=} p_{j'} \overset{(3.35)}{=} I\big(q_{j'}\big) \circ \eta_L$$

Now by Corollary 3.4.2 we get $F(d) \circ q_j = q_{j'}$, i.e. $\big(R(L), (q_j : R(L) \to F(j))_{j \in \operatorname{Ob} J}\big)$ is a cone on F. We will see that $\big(R(L), (q_j)_{j \in \operatorname{Ob} J}\big)$ is in fact the limit of F. We start by showing that $\eta_L : L \to IR(L)$ is an isomorphism. Indeed, since $\big(IR(L), (I(q_j) : IR(L) \to IF(j))_{j \in \operatorname{Ob} J}\big)$ is a cone on IF by Lemma 2.4.3 and $\big(L, (p_j : L \to IF(j))_{j \in \operatorname{Ob} J}\big)$ is its limit, there exists a unique morphism $f \in \operatorname{Hom}_{\mathcal{B}}(IR(L), L)$ such that the following diagram is commutative for all $j \in \operatorname{Ob} J$:

$$\begin{tikzcd} L \arrow[r, "p_j"] & IF(j) \\ IR(L) \arrow[u, "f"] \arrow[ur, "I(q_j)"'] & \end{tikzcd} \qquad \text{i.e.} \quad p_j \circ f = I(q_j). \tag{3.36}$$

Thus, for any $j \in \operatorname{Ob} J$ we have $\underline{p_j \circ f} \circ \eta_L \overset{(3.36)}{=} I(q_j) \circ \eta_L \overset{(3.35)}{=} p_j = p_j \circ 1_L$ and using Proposition 2.2.5 we obtain:

$$f \circ \eta_L = 1_L \tag{3.37}$$

On the other hand $\eta_L \circ f \in \operatorname{Hom}_{\mathcal{B}}(IR(L), IR(L))$ and since $I : \mathcal{A} \to \mathcal{B}$ is fully faithful, there exists a unique morphism $t \in \operatorname{Hom}_{\mathcal{A}}(R(L), R(L))$ such that:

$$\eta_L \circ f = I(t) \tag{3.38}$$

Moreover, we have:

$$\underline{I(t)} \circ \eta_L \overset{(3.38)}{=} \eta_L \circ \underline{f \circ \eta_L} \overset{(3.37)}{=} \eta_L = I(1_{R(L)}) \circ \eta_L$$

Using again Corollary 3.4.2 we get $t = 1_{R(L)}$ and hence $\eta_L \circ f = 1_{IR(L)}$ so η_L is an isomorphism.

Consider now $\big(L', (t_j : L' \to F(j))_{j \in \operatorname{Ob} J}\big)$ another cone on F. Then $\big(I(L'), (I(t_j)_{j \in \operatorname{Ob} J}\big)$ is a cone on IF. Therefore, there exists a unique morphism $g \in \operatorname{Hom}_{\mathcal{B}}(I(L'), L)$ such that the following diagram is commutative for all $j \in \operatorname{Ob} J$:

$$\begin{tikzcd} IR(L) \arrow[dr, "I(q_j)"] & \\ L \arrow[u, "\eta_L"] \arrow[r, "p_j"] & IF(j) \\ I(L') \arrow[u, "g"] \arrow[ur, "I(t_j)"'] & \end{tikzcd} \qquad \text{i.e.} \quad p_j \circ g = I(t_j) \tag{3.39}$$

Since we also have $p_j = I(q_j) \circ \eta_L$ for all $j \in \operatorname{Ob} J$ we obtain:

$$I(q_j) \circ \eta_L \circ g = I(t_j) \tag{3.40}$$

As $\eta_L \circ g \in \operatorname{Hom}_{\mathcal{B}}(I(L'), IR(L))$ and I is fully faithful, there exists a unique morphism $h \in \operatorname{Hom}_{\mathcal{A}}(L', R(L))$ such that $I(h) = \eta_L \circ g$. Then (3.40) becomes $I(q_j \circ h) = I(t_j)$ and since I is fully faithful we get $q_j \circ h = t_j$ for all $j \in \operatorname{Ob} J$, i.e. the following diagram is commutative:

$$\begin{tikzcd} R(L) \arrow[r, "q_j"] & F(j) \\ L' \arrow[u, "h"] \arrow[ur, "t_j"'] & \end{tikzcd}$$

The proof will be finished once we show that h is the unique morphism which makes the above diagram commutative. Indeed, suppose there exists $\overline{h} \in \operatorname{Hom}_{\mathcal{A}}(L', R(L))$ such that $q_j \circ \overline{h} = t_j$ for all $j \in \operatorname{Ob} J$. Then we also have $I(q_j) \circ I(\overline{h}) = I(t_j)$ and using (3.35) and respectively (3.39) we get:

$$p_j \circ \eta_L^{-1} \circ I(\overline{h}) = p_j \circ g$$

for all $j \in \operatorname{Ob} J$. Proposition 2.2.5 implies $\eta_L^{-1} \circ I(\overline{h}) = g$ and thus $I(\overline{h}) = \eta_L \circ g$. Since h is the unique morphism such that $I(h) = \eta_L \circ g$ we get $\overline{h} = h$ as desired. $\square$

In light of Proposition 3.6.3 the next natural question we are led to concerns the co-completeness of reflective subcategories (and, dually, the completeness of coreflective subcategories). First we prove the following useful lemmas:

Lemma 3.6.4 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors such that $F \dashv G$.*

1) *If G is fully faithful then the counit of the adjunction $\varepsilon : FG \to 1_{\mathcal{D}}$ is a natural isomorphism;*

2) *If F is fully faithful then the unit of the adjunction $\eta : 1_{\mathcal{C}} \to GF$ is a natural isomorphism.*

Proof: We will only prove the first assertion; the second one follows in a similar manner. For any $D \in \operatorname{Ob}\mathcal{D}$ we have $\eta_{G(D)} \in \operatorname{Hom}_{\mathcal{C}}(G(D),\, GFG(D))$ and since G is fully faithful there exists a unique morphism $\alpha_D \in \operatorname{Hom}_{\mathcal{D}}(D,\, FG(D))$ such that $G(\alpha_D) = \eta_{G(D)}$. This yields:

$$G(\varepsilon_D \circ \alpha_D) = G(\varepsilon_D) \circ G(\alpha_D) = G(\varepsilon_D) \circ \eta_{G(D)} \overset{(3.11)}{=} 1_{G(D)} = G(1_D)$$

As G is faithful we arrive at $\varepsilon_D \circ \alpha_D = 1_D$. Moreover, we also have:

$$\begin{aligned} G(\alpha_D \circ \varepsilon_D) \circ \eta_{G(D)} &= G(\alpha_D) \circ G(\varepsilon_D) \circ \eta_{G(D)} \\ &= \eta_{G(D)} \circ \underline{G(\varepsilon_D) \circ \eta_{G(D)}} \\ &\overset{(3.11)}{=} \eta_{G(D)} = G(1_{FG(D)}) \circ \eta_{G(D)} \end{aligned}$$

Now Corollary 3.4.2, 1), implies $\alpha_D \circ \varepsilon_D = 1_{FG(D)}$ which finishes the proof. □

Lemma 3.6.5 *Let $F,\, G : J \to \mathcal{C}$ be two functors with J a small category and $\alpha : F \to G$ a natural isomorphism.*

1) *If $\big(L,\, (p_j : L \to F(j))_{j \in \operatorname{Ob} J}\big)$ is the limit of F then $\big(L,\, (\alpha_j \circ p_j : L \to G(j))_{j \in \operatorname{Ob} J}\big)$ is the limit of G;*

2) *If $\big(C,\, (q_j : F(j) \to C)_{j \in \operatorname{Ob} J}\big)$ is the colimit of F then $\big(C,\, (q_j \circ \alpha_j^{-1} : G(j) \to C)_{j \in \operatorname{Ob} J}\big)$ is the colimit of G.*

Proof: We will only prove the second assertion; the first one follows in a similar manner. To start with, we show that $\big(C,\, (q_j \circ \alpha_j^{-1} : G(j) \to C)_{j \in \operatorname{Ob} J}\big)$ is a cocone on G. To this end, consider $d \in \operatorname{Hom}_J(i,\, i')$; we need to prove the commutativity of the following diagram:

$$\begin{tikzcd} & C & \\ G(i) \arrow[ur, "q_i \circ \alpha_i^{-1}"] \arrow[rr, "G(d)"'] & & G(i') \arrow[ul, "q_{i'} \circ \alpha_{i'}^{-1}"'] \end{tikzcd}$$

i.e. $q_{i'} \circ \alpha_{i'}^{-1} \circ G(d) = q_i \circ \alpha_i^{-1}$. (3.41)

Since $\big(C,\, (q_j : F(j) \to C)_{j \in \operatorname{Ob} J}\big)$ is in particular a cocone on F the following diagram is commutative:

$$\begin{tikzcd} & C & \\ F(i) \arrow[ur, "q_i"] \arrow[rr, "F(d)"'] & & F(i') \arrow[ul, "q_{i'}"'] \end{tikzcd}$$

i.e. $q_{i'} \circ F(d) = q_i$. (3.42)

Furthermore, as α is in particular a natural transformation, the following diagram is commutative as well:

$$\begin{tikzcd} F(i) \arrow[r, "\alpha_i"] \arrow[d, "F(d)"'] & G(i) \arrow[d, "G(d)"] \\ F(i') \arrow[r, "\alpha_{i'}"'] & G(i') \end{tikzcd}$$

i.e. $G(d) \circ \alpha_i = \alpha_{i'} \circ F(d)$. (3.43)

Putting all the above together yields:

$$q_{i'} \circ \underline{\alpha_{i'}^{-1} \circ G(d)} \overset{(3.43)}{=} \underline{q_{i'} \circ F(d)} \circ \alpha_i^{-1} \overset{(3.42)}{=} q_i \circ \alpha_i^{-1}$$

which proves that (3.41) holds and therefore $\big(C,\, (q_j \circ \alpha_j^{-1} : G(j) \to C)_{j \in \mathrm{Ob}\, J}\big)$ is indeed a cocone on G. Consider now $\big(C',\, (t_j : G(j) \to C')_{j \in \mathrm{Ob}\, J}\big)$ another cocone on G. Then $\big(C',\, (t_j \circ \alpha_j : F(j) \to C')_{j \in \mathrm{Ob}\, J}\big)$ is a cocone on F. Indeed, for any $d \in \mathrm{Hom}_J(i,\, i')$ we have:

$$t_{i'} \circ \underline{\alpha_{i'} \circ F(d)} \overset{(3.43)}{=} \underline{t_{i'} \circ G(d)} \circ \alpha_i = t_i \circ \alpha_i$$

where in the last equality we used the fact that $\big(C',\, (t_j : G(j) \to C')_{j \in \mathrm{Ob}\, J}\big)$ is a cocone on G.

Now since $\big(C,\, (q_j : F(j) \to C)_{j \in \mathrm{Ob}\, J}\big)$ is the colimit of F, there exists a unique $f \in \mathrm{Hom}_{\mathcal{C}}(C,\, C')$ such that the following diagram is commutative for all $j \in \mathrm{Ob}\, J$:

$$\begin{tikzcd} F(j) \arrow[r, "q_j"] \arrow[dr, "t_j \circ \alpha_j"'] & C \arrow[d, "f"] \\ & C' \end{tikzcd} \qquad \text{i.e. } f \circ q_j = t_j \circ \alpha_j.$$

Thus $f \in \mathrm{Hom}_{\mathcal{C}}(C,\, C')$ is the unique morphism such that $f \circ (q_j \circ \alpha_j^{-1}) = t_j$, i.e. the unique morphism which makes the following diagram commutative for all $j \in \mathrm{Ob}\, J$:

$$\begin{tikzcd} G(j) \arrow[r, "q_j \circ \alpha_j^{-1}"] \arrow[dr, "t_j"'] & C \arrow[d, "f"] \\ & C' \end{tikzcd}$$

This proves that $\big(C,\, (q_j \circ \alpha_j^{-1} : G(j) \to C)_{j \in \mathrm{Ob}\, J}\big)$ is the initial object in the category of cocones on G and the proof is now finished. $\square$

Proposition 3.6.6 1) *If $\mathcal{A}$ is a reflective subcategory of a cocomplete category $\mathcal{B}$ then $\mathcal{A}$ is also cocomplete;*
2) *If $\mathcal{A}$ is a coreflective subcategory of a complete category $\mathcal{B}$ then $\mathcal{A}$ is also complete.*

Proof: We will only prove the first assertion; the second one follows in a similar manner. Consider $I : \mathcal{A} \to \mathcal{B}$ to be the inclusion functor and $R : \mathcal{B} \to \mathcal{A}$ the reflector. Let $F : J \to \mathcal{A}$ be a functor where J is a small category. Since $\mathcal{B}$ is cocomplete, the functor $IF : J \to \mathcal{B}$ has a colimit which we denote by $\big(C,\, (q_j : IF(j) \to C)_{j \in \mathrm{Ob}\, J}\big)$. R is left adjoint to I and by Proposition 3.2.3 it preserves colimits so $\big(R(C),\, (R(q_j) : RIF(j) \to R(C))_{j \in \mathrm{Ob}\, J}\big)$ is the colimit of the functor $RIF : J \to \mathcal{A}$. By Lemma 3.6.4, 1), we know that the counit $\varepsilon : RI \to 1_{\mathcal{A}}$ of the adjunction $R \dashv I$ is a natural isomorphism. Therefore, the natural transformation $\varepsilon_F : RIF \to F$ defined by:

$$(\varepsilon_F)_j = \varepsilon_{F(j)} \text{ for all } j \in \mathrm{Ob}\, J$$

is also a natural isomorphism (Example 1.5.2, 4)). Now in light of Lemma 3.6.5, 2) we can conclude that $\big(R(C),\, (R(q_j) \circ \varepsilon_{F(j)}^{-1} : F(j) \to R(C))_{j \in \mathrm{Ob}\, J}\big)$ is the colimit of F. $\square$

3.7 Localization

Consider $\mathcal{C}$ to be a category and let S be a class of morphisms in $\mathcal{C}$. The purpose of localization is to construct a new category $\mathcal{C}_S$ in which all morphisms in S became invertible, while approximating the original category as closely as possible. The idea of formally adjoining inverses exists for many algebraic structures such as rings or modules.

Definition 3.7.1 *A localization of a category $\mathcal{C}$ by a class of morphisms S of $\mathcal{C}$ is a category $\mathcal{C}_S$ together with a functor $F : \mathcal{C} \to \mathcal{C}_S$ such that:*

1) *for any $s \in S$, $F(s)$ is an isomorphism;*

2) *if $G : \mathcal{C} \to \mathcal{D}$ is a functor such that for all $s \in S$, $G(s)$ is an isomorphism, there exists a unique functor $H : \mathcal{C}_S \to \mathcal{D}$ such that the following diagram is commutative:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{C}_S \arrow[d, "H"] \\ & \mathcal{D} \end{tikzcd} \qquad \text{i.e.} \quad H \circ F = G \tag{3.44}$$

Theorem 3.7.2 *Let $\mathcal{C}$ be a category. Then there exists a localization of $\mathcal{C}$ by any set of morphisms S of $\mathcal{C}$.*

Proof: In order to construct the localization of $\mathcal{C}$ by the set S we start by defining an oriented graph Γ as follows:

- the vertices of Γ are the objects of $\mathcal{C}$;
- the edges of Γ are the morphisms of $\mathcal{C}$ (any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is seen as an oriented edge $X \xrightarrow{f} Y$) together with the set $\{x_s \mid s \in S\}$ where x_s is an edge having the same vertices as s but the opposite orientation (i.e. if $s \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ then $Y \xrightarrow{x_s} X$).

Two paths in the above graph will be called *equivalent* if one can be transformed into the other by applying the following elementary operations a finite number of times:

- if $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ then the path $X \xrightarrow{f} Y \xrightarrow{g} Z$ can be replaced by the composition path $X \xrightarrow{g \circ f} Z$;
- if $s \in S$, $s \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ then the path $X \xrightarrow{s} Y \xrightarrow{x_s} X$ can be replaced by the trivial path $X \xrightarrow{1_X} X$; similarly, the path $Y \xrightarrow{x_s} X \xrightarrow{s} Y$ can be replaced by the trivial path $Y \xrightarrow{1_Y} Y$.

Obviously, this is an equivalence relation on the paths of Γ. We denote by $\widehat{\gamma}$ the equivalence class of the path γ. The localization category $\mathcal{C}_S$ is defined as follows:

$$\mathrm{Ob}\,\mathcal{C}_S = \mathrm{Ob}\,\mathcal{C};$$

$$\mathrm{Hom}_{\mathcal{C}_S}(X, Y) = \{\widehat{\gamma} \mid \gamma \text{ is a path in } \Gamma \text{ from } X \text{ to } Y\}^3, \text{ for any } X, Y \in \mathrm{Ob}\mathcal{C}$$

with the composition of morphisms in $\mathcal{C}_S$ induced by the concatenation of paths and the identity maps given by the trivial paths. The functor $F : \mathcal{C} \to \mathcal{C}_S$ is defined as follows:

$$F(X) = X, \text{ for all } X \in \mathrm{Ob}\,\mathcal{C};$$

$$F(f) = \widehat{f}, \text{ for all } f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

Remark that if $s \in S$, $s \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, then $F(s) = \widehat{s}$ has an inverse in $\mathcal{C}_S$, namely $\widehat{x_s}$, where $Y \xrightarrow{x_s} X$. We are left to show that the pair $(\mathcal{C}_S, F)$ satisfies the second condition in Definition 3.7.1 as well. To this end, let $\mathcal{D}$ be a category and $G : \mathcal{C} \to \mathcal{D}$ a functor such that $G(s)$ is an isomorphism for any $s \in S$. Consider the functor $H : \mathcal{C}_S \to \mathcal{D}$ defined as follows:

$$H(X) = G(X), \text{ for all } X \in \mathrm{Ob}\,\mathcal{C}_s = \mathrm{Ob}\,\mathcal{C};$$

$$H(\widehat{f}) = G(f), \text{ for all } f \in \mathrm{Hom}_{\mathcal{C}}(X, Y);$$

$$H\big(\, W \xrightarrow{\widehat{x_s}} Z \,\big) = G(s)^{-1}, \text{ for all } s \in S,\ s \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$$

The way we defined the functor above ensures the commutativity of diagram (3.44) as well as the uniqueness of H with this property. We are left to prove that H is well-defined. To this end, consider two paths u and v in Γ such that $\widehat{u} = \widehat{v}$. Since the paths u and v are equivalent, we can turn u into v after a finite number of elementary operations. Thus the only thing left to prove is that by applying H to each of these elementary operations we obtain equalities in $\mathcal{D}$. Indeed, whenever $X \xrightarrow{\widehat{f}} Y \xrightarrow{\widehat{g}} Z = X \xrightarrow{\widehat{g \circ f}} Z$ in $\mathcal{C}_S$ we obviously have $H(\widehat{g \circ f}) = G(g \circ f) = G(g) \circ G(f) = H(\widehat{g}) \circ H(\widehat{f}) = H(\widehat{g} \circ \widehat{f})$. Analogously, whenever $X \xrightarrow{\widehat{s}} Y \xrightarrow{\widehat{x_s}} X = X \xrightarrow{\widehat{1_X}} X$ in $\mathcal{C}_S$ it follows that $H(\widehat{x_s} \circ \widehat{s}) = H(\widehat{x_s}) \circ H(\widehat{s}) = G(s)^{-1} \circ G(s) = 1_{G(X)} = G(1_X) = H(\widehat{1_X})$. Therefore H is well-defined and the proof is now finished. $\square$

Proposition 3.7.3 *When it exists, the localization of a category $\mathcal{C}$ by a class of morphisms S of $\mathcal{C}$ is unique up to isomorphism.*

Proof: Suppose $(\mathcal{C}_S, F)$ and $(\overline{\mathcal{C}_S}, \overline{F})$ are two localizations of $\mathcal{C}$ by S. Thus, there exists a unique functor $G : \mathcal{C}_S \to \overline{\mathcal{C}_S}$ such that:

$$G \circ F = \overline{F} \tag{3.45}$$

[3] This is obviously a set as a consequence of S being a set.

Similarly, as $(\overline{\mathcal{C}_S},\, \overline{F})$ is also a localization of $\mathcal{C}$ by S, there exists a unique functor $G' : \overline{\mathcal{C}_S} \to \mathcal{C}_S$ such that:

$$G' \circ \overline{F} = F \tag{3.46}$$

By putting all together we obtain:

$$F \overset{(3.46)}{=} G' \circ \overline{F} \overset{(3.45)}{=} G' \circ G \circ F = (G' \circ G) \circ F \tag{3.47}$$

Applying Definition 3.7.1 to the pair $(\mathcal{C}_S,\, F)$ seen both as a localization and as the other pair, yields a unique functor $H : \mathcal{C}_S \to \mathcal{C}_S$ such that $H \circ F = F$. By the uniqueness of H we must have $H = 1_{\mathcal{C}_S}$. Moreover, since by (3.47) the functor $G' \circ G$ makes the same diagram commutative we obtain $G' \circ G = 1_{\mathcal{C}_S}$.

$$\begin{tikzcd}
 & \mathcal{C}_S \arrow[d, "G"'] \arrow[dd, bend left=60, "1_{\mathcal{C}_S}"] \\
\mathcal{C} \arrow[ur, "F"] \arrow[r, "\overline{F}"] \arrow[dr, "F"'] & \overline{\mathcal{C}_S} \arrow[d, "G'"'] \\
 & \mathcal{C}_S
\end{tikzcd} \tag{3.48}$$

Similarly one can prove that $G \circ G' = 1_{\overline{\mathcal{C}_S}}$ and therefore the categories $\mathcal{C}_S$ and $\overline{\mathcal{C}_S}$ are isomorphic, as desired. The proof is now finished. $\square$

Our next result highlights the connection between reflective subcategories and localizations.

Theorem 3.7.4 *Let $I : \mathcal{A} \to \mathcal{B}$ be a reflective subcategory inclusion with reflector $R : \mathcal{B} \to \mathcal{A}$ and denote by S the class of all morphisms s of $\mathcal{B}$ such that $R(s)$ is an isomorphism. Then the localization of $\mathcal{B}$ by S is equivalent to $\mathcal{A}$.*

Proof: Consider $\eta : 1_{\mathcal{B}} \to IR$ and $\varepsilon : RI \to 1_{\mathcal{A}}$ the unit, respectively the counit of the adjunction $R \dashv I$. To start with, recall that by Lemma 3.6.4 the counit $\varepsilon : RI \to 1_{\mathcal{A}}$ is a natural isomorphism. In particular, each morphism $\varepsilon_A : RI(A) \to A$ is an isomorphism in $\mathcal{A}$. Now for any $B \in \mathrm{Ob}\mathcal{B}$ we have $1_{R(B)} \overset{(3.10)}{=} \varepsilon_{R(B)} \circ R(\eta_B)$ and given that $\varepsilon_{R(B)}$ is an isomorphism it follows that $R(\eta_B) : R(B) \to RIR(B)$ is an isomorphism in $\mathcal{A}$ as well. Therefore, $\eta_B \in S$ for any $B \in \mathrm{Ob}\,\mathcal{B}$.

Define a category $\mathcal{B}_S$ as follows:

$\mathrm{Ob}\,\mathcal{B}_S = \mathrm{Ob}\,\mathcal{B}$;

$\mathrm{Hom}_{\mathcal{B}_S}(B,\, B') = \mathrm{Hom}_{\mathcal{A}}(R(B),\, R(B'))$ for all $B,\, B' \in \mathrm{Ob}\,\mathcal{B}_S$,

with the composition of morphisms given by that of $\mathcal{A}$.

First we prove that the above category $\mathcal{B}_S$ is equivalent to $\mathcal{A}$. Indeed, consider the functor $T : \mathcal{A} \to \mathcal{B}_S$ defined as follows:

$T(A) = I(A)$, for all $A \in \operatorname{Ob}\mathcal{A}$;

$T(f) = RI(f)$, for all $f \in \operatorname{Hom}_{\mathcal{A}}(A,\, A')$.

T is full and faithful as RI is naturally isomorphic to $1_{\mathcal{A}}$ via ε. Indeed, let $h_1,\, h_2 \in \operatorname{Hom}_{\mathcal{A}}(A,\, A')$ such that $RI(h_1) = RI(h_2)$. The naturality of ε renders the following diagrams commutative for $i = 1,\, 2$:

$$\begin{tikzcd} RI(A) \arrow[r, "\varepsilon_A"] \arrow[d, "RI(h_i)"'] & A \arrow[d, "h_i"] \\ RI(A') \arrow[r, "\varepsilon_{A'}"'] & A' \end{tikzcd} \qquad \text{i.e.} \quad h_i \circ \varepsilon_A = \varepsilon_{A'} \circ RI(h_i) \tag{3.49}$$

Hence we obtain $h_1 \circ \varepsilon_A = h_2 \circ \varepsilon_A$ and since ε_A is an isomorphism we get $h_1 = h_2$ as desired. This shows that T is faithful.

Consider now $A,\, A' \in \operatorname{Ob}\mathcal{A}$, $v \in \operatorname{Hom}_{\mathcal{A}}(RI(A),\, RI(A'))$ and define:

$$u = \varepsilon_{A'} \circ v \circ \varepsilon_A^{-1} \in \operatorname{Hom}_{\mathcal{C}}(A,\, A') \tag{3.50}$$

We will prove that $RI(u) = v$. Indeed, using again the naturality of ε we obtain:

$$\varepsilon_{A'} \circ RI(u) \circ \varepsilon_A^{-1} \overset{(3.49)}{=} u \overset{(3.50)}{=} \varepsilon_{A'} \circ v \circ \varepsilon_A^{-1}$$

Since $\varepsilon_{A'}$ and ε_A are isomorphisms we get $RI(u) = v$ and we proved that T is full.

Moreover, for any $B \in \operatorname{Ob}\mathcal{B}_S$ there exists an isomorphism $R(\eta_B) \in \operatorname{Hom}_{\mathcal{A}}(R(B),\, RIR(B)) = \operatorname{Hom}_{\mathcal{B}_S}(B,\, IR(B)) = \operatorname{Hom}_{\mathcal{B}_S}(B,\, TR(B))$ that is T is essentially surjective as well. Therefore, by Theorem 3.5.4, T is an equivalence of categories.

Next we will show that $(\mathcal{B}_S, F)$ is in fact the localization of $\mathcal{B}$ with respect to S, where $F : \mathcal{B} \to \mathcal{B}_S$ is the functor defined as follows:

$F(B) = B$, for all $B \in \operatorname{Ob}\mathcal{B}$;

$F(f) = R(f)$, for all $f \in \operatorname{Hom}_{\mathcal{B}}(B,\, B')$.

Recall that S is the class of all morphisms s of $\mathcal{B}$ such that $R(s)$ is an isomorphism and therefore $F(s)$ is obviously an isomorphism for any $s \in S$.

Consider now another functor $G : \mathcal{B} \to \mathcal{D}$ such that $G(s)$ is an isomorphism for all $s \in S$. We need to find a functor $H : \mathcal{B}_S \to \mathcal{D}$ which makes the above diagram commutative:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{B}_S \arrow[d, "H"] \\ & \mathcal{D} \end{tikzcd} \qquad \text{i.e.} \quad H \circ F = G \tag{3.51}$$

Having in mind that $\eta_B \in S$ for any $B \in \mathrm{Ob}\mathcal{B}$, it can be easily seen that a functor H which makes the above diagram commute has the following property for all $B \in \mathrm{Ob}\mathcal{B}$:

$$H(\varepsilon_{R(B)}) \overset{(3.10)}{=} H\big(R(\eta_B)^{-1}\big) = \Big(H\big(R(\eta_B)\big)\Big)^{-1} = \Big(H\big(F(\eta_B)\big)\Big)^{-1} \overset{(3.51)}{=} G(\eta_B)^{-1} \tag{3.52}$$

Furthermore, for any morphism $f \in \mathrm{Hom}_{\mathcal{B}_S}(B, B') = \mathrm{Hom}_{\mathcal{A}}(R(B), R(B'))$, the naturality of $\varepsilon : RI \to 1_{\mathcal{A}}$ renders the following diagram commutative:

$$\begin{tikzcd} RIR(B) \arrow[r, "\varepsilon_{R(B)}"] \arrow[d, "RI(f)"'] & R(B) \arrow[d, "f"] \\ RIR(B') \arrow[r, "\varepsilon_{R(B')}"'] & R(B') \end{tikzcd} \quad \text{i.e.} \quad f \circ \varepsilon_{R(B)} = \varepsilon_{R(B')} \circ RI(f) \tag{3.53}$$

Therefore, for any $f \in \mathrm{Hom}_{\mathcal{B}_S}(B, B') = \mathrm{Hom}_{\mathcal{A}}(R(B), R(B'))$ we have:

$$\begin{aligned} H(f) &\overset{(3.10)}{=} H\big(\underline{f \circ \varepsilon_{R(B)}} \circ R(\eta_B)\big) \\ &\overset{(3.53)}{=} H\big(\varepsilon_{R(B')} \circ RI(f) \circ R(\eta_B)\big) \\ &= \underline{H\big(\varepsilon_{R(B')}\big)} \circ HRI(f) \circ HR(\eta_B) \\ &\overset{(3.52)}{=} G(\eta_{B'})^{-1} \circ \underline{HF}I(f) \circ \underline{HF}(\eta_B) \\ &\overset{(3.51)}{=} G(\eta_{B'})^{-1} \circ GI(f) \circ G(\eta_B) \end{aligned}$$

We define the functor $H : \mathcal{B}_S \to \mathcal{D}$ as follows:

$H(B) = G(B)$, for all $B \in \mathrm{Ob}\,\mathcal{B}_S$;

$H(f) = G(\eta_{B'})^{-1} \circ GI(f) \circ G(\eta_B)$, for all $f \in \mathrm{Hom}_{\mathcal{B}_S}(B, B')$.

The above discussion proves that H is the unique functor which might render diagram (3.51) commutative. We are left to prove that indeed H makes diagram (3.51) commute. To this end we will use the naturality of η, i.e. the commutativity of the above diagram for any $g \in \mathrm{Hom}_{\mathcal{B}}(B, B')$:

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[d, "g"'] & IR(B) \arrow[d, "IR(g)"] \\ B' \arrow[r, "\eta_{B'}"'] & IR(B') \end{tikzcd} \quad \text{i.e.} \quad IR(g) \circ \eta_B = \eta_{B'} \circ g \tag{3.54}$$

Obviously, for any $B \in \mathrm{Ob}\,\mathcal{B}_S$ we have $H \circ F(B) = H(B) = G(B)$. Moreover, for any $g \in \mathrm{Hom}_{\mathcal{B}}(B, B')$ we have:

$$\begin{aligned} H \circ F(g) &= H(R(g)) = G(\eta_{B'})^{-1} \circ GIR(g) \circ G(\eta_B) \\ &= G(\eta_{B'})^{-1} \circ G(\underline{IR(g) \circ \eta_B}) \\ &\overset{(3.54)}{=} G(\eta_{B'})^{-1} \circ G(\eta_{B'} \circ g) = G(g) \end{aligned}$$

$\square$

3.8 Freyd's adjoint functor theorem

We have seen in Proposition 3.2.3 that right (left) adjoints preserve all existing small limits (colimits). However, limit preservation alone does not guarantee the existence of a left adjoint as it can be easily seen from Example 3.2.1, 10). Indeed, the functor $T : \mathcal{C} \to \mathbf{1}$ defined in the aforementioned example trivially preserves limits while the existence of a left adjoint is conditioned by the existence of an initial object in the category $\mathcal{C}$. In this section we will prove that, in fact, limit preservation is part of a necessary and sufficient condition which needs to be fulfilled by a functor in order to admit a left adjoint. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor, $X \in \operatorname{Ob}\mathcal{C}$ and consider $(X \downarrow G)$ to be the comma-category defined in Example 1.4.6, 2). We have an obvious forgetful functor $U : (X \downarrow G) \to \mathcal{D}$ defined for any $(f, Y) \in \operatorname{Ob}\big(X \downarrow G\big)$ and any morphism h in $(X \downarrow G)$ as follows:

$$U(f, Y) = Y, \quad U(h) = h.$$

Lemma 3.8.1 *Let $\mathcal{D}$ be a complete category and $G : \mathcal{D} \to \mathcal{C}$ a functor which preserves small limits. Then, for any $X \in \operatorname{Ob}\mathcal{C}$, the category $(X \downarrow G)$ is complete.*

Proof: Consider $X \in \operatorname{Ob}\mathcal{C}$, J a small category and let $F : J \to (X \downarrow G)$ be a functor. We denote $F(j) = \big(f_j, UF(j)\big)$ where $f_j \in \operatorname{Hom}_{\mathcal{C}}(X, GUF(j))$ for all $j \in \operatorname{Ob} J$. Then $\big(X, (f_j)_{j \in \operatorname{Ob} J}\big)$ is a cone on $GUF : J \to \mathcal{C}$. Indeed, consider $t \in \operatorname{Hom}_J(i, k)$; since $F(t)$ is a morphism in $(X \downarrow G)$ the following diagram is commutative:

$$\begin{tikzcd} & X \arrow[dl, "f_i"'] \arrow[dr, "f_k"] & \\ GUF(i) \arrow[rr, "GUF(t)"'] & & GUF(k) \end{tikzcd} \qquad \text{i.e. } GUF(t) \circ f_i = f_k.$$

which obviously implies that $\big(X, (f_j)_{j \in \operatorname{Ob} J}\big)$ is a cone on $GUF : J \to \mathcal{C}$.

Since $\mathcal{D}$ is a complete category, we can consider $\big(L, (\lambda_j : L \to UF(j))_{j \in \operatorname{Ob} J}\big)$ to be the limit of the functor $UF : J \to \mathcal{D}$. Moreover, as G is a limit preserving functor we obtain that $\big(G(L), (G(\lambda_j) : G(L) \to GUF(j))_{j \in \operatorname{Ob} J}\big)$ is the limit of $GUF : J \to \mathcal{C}$. Thus, there exists a unique morphism $f \in \operatorname{Hom}_{\mathcal{C}}(X, G(L))$ such that the following diagram commutes for all $j \in \operatorname{Ob} J$:

$$\begin{tikzcd} G(L) \arrow[rr, "G(\lambda_j)"] & & GUF(j) \\ & X \arrow[ul, "f"] \arrow[ur, "f_j"'] & \end{tikzcd} \qquad \text{i.e.} \quad G(\lambda_j) \circ f = f_j. \tag{3.55}$$

In particular, the commutativity of the above diagram implies that the λ_j's are actually morphisms in $(X \downarrow G)$ from (f, L) to $F(j)$. Moreover, $\big((f, L), (\lambda_j : (f, L) \to$

$F(j))_{j \in \mathrm{Ob}\, J}\big)$ is a cone on $F : J \to (X \downarrow G)$. Indeed, since $\big(L, (\lambda_j : L \to UF(j))_{j \in \mathrm{Ob}\, J}\big)$ is in particular a cone on the functor $UF : J \to \mathcal{D}$, for any $t \in \mathrm{Hom}_J(i, k)$ we have

$$UF(t) \circ \lambda_i = \lambda_k. \tag{3.56}$$

As the λ_j's are morphisms in $(X \downarrow G)$, (3.56) is equivalent to $UF(t) \circ U(\lambda_i) = U(\lambda_k)$ and the faithfulness of the forgetful functor U leads to $F(t) \circ \lambda_i = \lambda_k$ as desired. We will prove that $\big((f, L), (\lambda_j : (f, L) \to F(j))_{j \in \mathrm{Ob}\, J}\big)$ is in fact the limit of the functor $F : J \to (X \downarrow G)$. To this end, consider $\big((g, D), (\mu_j : (g, D) \to F(j))_{j \in \mathrm{Ob}\, J}\big)$ another cone on $F : J \to (X \downarrow G)$. In particular, $\mu_j : (g, D) \to F(j)$ is a morphism in $(X \downarrow G)$ and thus the following diagram is commutative:

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, "f_j"] & \\ G(D) \arrow[rr, "G(\mu_j)"'] & & GUF(j) \end{tikzcd} \qquad \text{i.e. } G(\mu_j) \circ g = f_j. \tag{3.57}$$

By Lemma 2.4.3, applying U to the above cone on F yields a cone $\big(D, (\mu_j : D \to UF(j))_{j \in \mathrm{Ob}\, J}\big)$ on $UF : J \to \mathcal{D}$. Since $\big(L, (\lambda_j : L \to UF(j))_{j \in \mathrm{Ob}\, J}\big)$ is the limit of $UF : J \to \mathcal{D}$, there exists a unique morphism $h \in \mathrm{Hom}_{\mathcal{D}}(D, L)$ such that the following diagram is commutative for all $j \in \mathrm{Ob}\, J$:

$$\begin{tikzcd} L \arrow[r, "\lambda_j"] & UF(j) \\ D \arrow[u, "h"] \arrow[ur, "\mu_j"'] & \end{tikzcd} \qquad \text{i.e. } \lambda_j \circ h = \mu_j. \tag{3.58}$$

We need to prove that h is in fact a morphism in $(X \downarrow G)$, i.e. the following diagram is commutative:

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, "f"] & \\ G(D) \arrow[rr, "G(h)"'] & & G(L) \end{tikzcd} \qquad \text{i.e. } G(h) \circ g = f.$$

To this end, we will show that the morphism $G(h) \circ g$ makes diagram (3.55) commutative; as f is the unique morphism with this property we arrive at the desired conclusion. Indeed, for all $j \in \mathrm{Ob}\, J$ we have:

$$G(\lambda_j) \circ G(h) \circ g = G(\underline{\lambda_j \circ h}) \circ g \overset{(3.58)}{=} \underline{G(\mu_j) \circ g} \overset{(3.57)}{=} f_j.$$

Thus $G(h) \circ g = f$ and the proof is now complete. $\square$

Lemma 3.8.2 *A functor $G : \mathcal{D} \to \mathcal{C}$ admits a left adjoint if and only if for all $X \in \mathrm{Ob}\, \mathcal{C}$ the comma category $(X \downarrow G)$ has an initial object .*

Proof: Suppose first that G has a left adjoint $F : \mathcal{C} \to \mathcal{D}$ and let $\theta : \mathtt{Hom}_{\mathcal{D}}(F(-), -) \to \mathtt{Hom}_{\mathcal{C}}(-, G(-))$ be the natural isomorphism corresponding to the adjunction $F \dashv G$. Now let $X \in \operatorname{Ob}\mathcal{C}$ and consider $\eta : 1_{\mathcal{C}} \to GF$ to be the unit of the adjunction. We will prove that $\big(\eta_X, F(X)\big)$ is the initial object of the category $(X \downarrow G)$. Let (v, W) be another object in $(X \downarrow G)$, i.e. $W \in \operatorname{Ob}\mathcal{D}$ and $v \in \operatorname{Hom}_{\mathcal{C}}(X, G(W))$. To this end, we need to find a unique morphism $f : \big(\eta_X, F(X)\big) \to (v, W)$ in $(X \downarrow G)$, i.e. a morphism $f \in \operatorname{Hom}_{\mathcal{D}}(F(X), W)$ such that the following diagram is commutative:

$$\begin{tikzcd} & X \arrow[dl, "\eta_X"'] \arrow[dr, "v"] & \\ GF(X) \arrow[rr, "G(f)"'] & & G(W) \end{tikzcd} \qquad \text{i.e. } G(f) \circ \eta_X = v. \tag{3.59}$$

Consider $f = \theta^{-1}_{X, W}(v)$. By setting $Y = F(X)$ and $Y' = W$ in (3.2) we obtain the following commutative diagram:

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{D}}(F(X), F(X)) \arrow[r, "\theta_{X, F(X)}"] \arrow[d, "\mathtt{Hom}_{\mathcal{D}}(F(X), f)"'] & \operatorname{Hom}_{\mathcal{C}}(X, GF(X)) \arrow[d, "\mathtt{Hom}_{\mathcal{C}}(X, G(f))"] \\ \operatorname{Hom}_{\mathcal{D}}(F(X), W) \arrow[r, "\theta_{X, W}"'] & \operatorname{Hom}_{\mathcal{C}}(X, G(W)) \end{tikzcd}$$

Now since $\eta_X = \theta_{X, F(X)}(1_{F(X)})$ (see the proof of Theorem 3.3.1), the commutativity of the above diagram applied to $1_{F(X)} \in \operatorname{Hom}_{\mathcal{D}}(F(X), F(X))$ comes down to:

$$G(f) \circ \eta_X = \theta_{X, W}(f).$$

Therefore, we obtain $G(f) \circ \eta_X = \theta_{X, W}(f) = v$ as desired. The uniqueness of f with the above property follows from Corollary 3.4.2.

Suppose now that for each $X \in \operatorname{Ob}\mathcal{C}$ the comma-category $(X \downarrow G)$ has an initial object which we denote by (u_X, V_X), where $V_X \in \operatorname{Ob}\mathcal{D}$ and $u_X \in \operatorname{Hom}_{\mathcal{C}}(X, G(V_X))$. Hence, for any $(f, Y) \in \operatorname{Ob}(X \downarrow G)$ there exists a unique morphism $h : (u_X, V_X) \to (f, Y)$ in $(X \downarrow G)$; in other words, for any $f \in \operatorname{Hom}_{\mathcal{C}}(X, G(Y))$ there exists a unique morphism $h \in \operatorname{Hom}_{\mathcal{D}}(V_X, Y)$ making the following diagram commute:

$$\begin{tikzcd} & X \arrow[dl, "u_X"'] \arrow[dr, "f"] & \\ G(V_X) \arrow[rr, "G(h)"'] & & G(Y) \end{tikzcd} \qquad \text{i.e. } G(h) \circ u_X = f. \tag{3.60}$$

We define a functor $F : \mathcal{C} \to \mathcal{D}$ on objects by $F(X) = V_X$ for all $X \in \operatorname{Ob}\mathcal{C}$. Consider now $f \in \operatorname{Hom}_{\mathcal{C}}(X, X')$; then $u_{X'} \circ f \in \operatorname{Hom}_{\mathcal{C}}(X, G(F(X')))$ and, using (3.60), we define $F(f) \in \operatorname{Hom}_{\mathcal{D}}(F(X), F(X'))$ to be the unique morphism such that:

$$GF(f) \circ u_X = u_{X'} \circ f. \tag{3.61}$$

Obviously $F(1_X) = 1_{F(X)}$ for all $X \in \operatorname{Ob}\mathcal{C}$. Moreover, if $f \in \operatorname{Hom}_{\mathcal{C}}(X, X')$ and $f' \in \operatorname{Hom}_{\mathcal{C}}(X', X'')$ then $F(f' \circ f)$ and $F(f') \circ F(f)$ are both morphisms in the comma-category $(X \downarrow G)$ from $(u_X, F(X))$ to $(u_{X''} \circ f' \circ f, F(X''))$ so they must be equal as $(u_X, F(X))$ is the initial object of $(X \downarrow G)$. Hence F is a functor and furthermore, according to (3.61), $u : 1_{\mathcal{C}} \to GF$ is a natural transformation. To summarize, we proved that there exists a natural transformation $u : 1_{\mathcal{C}} \to GF$ such that for any $f \in \operatorname{Hom}_{\mathcal{C}}(X, G(Y))$ there exists a unique morphism $h \in \operatorname{Hom}_{\mathcal{D}}(F(X), Y)$ satisfying $G(h) \circ u_X = f$. Now Theorem 3.4.1 implies that F is left adjoint to G, as desired. $\square$

Definition 3.8.3 *Given a category $\mathcal{C}$, an I-indexed family $(K_i)_{i \in I}$ of objects of $\mathcal{C}$, where I is a set, is called a weakly initial set if for any $C \in \operatorname{Ob}\mathcal{C}$ there exists some $j \in I$ and a morphism $t_C^j \in \operatorname{Hom}_{\mathcal{C}}(K_j, C)$.*

Lemma 3.8.4 *If $\mathcal{C}$ is a complete category then $\mathcal{C}$ has an initial object if and only if $\mathcal{C}$ has a weakly initial set.*

Proof: Suppose $\mathcal{C}$ has an initial object I; then $\{I\}$ is obviously a weakly initial set.

Conversely, let $(K_i)_{i \in I}$ be a weakly initial set. As $\mathcal{C}$ is complete and I is a set we can consider $\big(P, (\pi_i : P \to K_i)_{i \in I}\big)$ to be the product of the family of objects $(K_i)_{i \in I}$. Notice that for each $C \in \operatorname{Ob}\mathcal{C}$ there exists at least one morphism $u_C \in \operatorname{Hom}_{\mathcal{C}}(P, C)$ given by the composition $P \xrightarrow{\pi_j} K_j \xrightarrow{t_C^j} C$ for some $j \in I$.

Consider now J to be the category with $\operatorname{Ob} J = \{P\}$ and $\operatorname{Hom}_J(P, P) = \operatorname{Hom}_{\mathcal{C}}(P, P)$ and let $(L, q : L \to P)$ be the limit of the inclusion functor $F : J \to \mathcal{C}$.

Now for any $C \in \operatorname{Ob}\mathcal{C}$ there exists at least one morphism in $\operatorname{Hom}_{\mathcal{C}}(L, C)$ given by the composition: $L \xrightarrow{q} P \xrightarrow{u_C} C$. We will prove that L is the initial object of the category $\mathcal{C}$. Indeed, suppose we have two such morphisms $f, g \in \operatorname{Hom}_{\mathcal{C}}(L, C)$ and consider $(E, e : E \to L)$ to be the equalizer of (f, g). Since $E \in \operatorname{Ob}\mathcal{C}$ there exists a morphism $u_E \in \operatorname{Hom}_{\mathcal{C}}(P, E)$ given by the composition $P \xrightarrow{\pi_j} K_j \xrightarrow{t_E^j} E$ for some $j \in I$. Thus $q \circ e \circ u_E \in \operatorname{Hom}_{\mathcal{C}}(P, P)$ and since $(L, q : L \to P)$ is in particular a cone on F, the following diagram is commutative:

$$\begin{tikzcd} & L \arrow[dl, "q"'] \arrow[dr, "q"] & \\ P \arrow[rr, "q \circ e \circ u_E"'] & & P \end{tikzcd}$$

Thus we have $q \circ e \circ u_E \circ q = q = q \circ 1_L$ and by Proposition 2.2.5 we get $e \circ u_E \circ q = 1_L$. This yields:

$$f = f \circ 1_L = f \circ e \circ u_E \circ q = g \circ e \circ u_E \circ q = g \circ 1_L = g$$

where in the third equality we used the fact that $(E, e : E \to L)$ is the equalizer of (f, g). We obtained $f = g$ and hence L is an initial object of $\mathcal{C}$. $\square$

Theorem 3.8.5 (Freyd's adjoint functor theorem) *Let $\mathcal{D}$ be a complete category. Then a functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint if and only if G preserves all small limits and for each $X \in \operatorname{Ob}\mathcal{C}$ the comma-category $(X \downarrow G)$ has a weakly initial set.*

Proof: Suppose G has a left adjoint F. Then G is a right adjoint to F and by Proposition 3.2.3 it preserves limits. Moreover, by (the proof of) Lemma 3.8.2, for any $X \in \operatorname{Ob}\mathcal{C}$, the pair $\big(\eta_X,\, F(X)\big)$ is an initial object in $(X \downarrow G)$, where $\eta : 1_{\mathcal{C}} \to GF$ is the unit of the adjunction (F, G).

Assume now that $G : \mathcal{D} \to \mathcal{C}$ preserves small limits and for each $X \in \operatorname{Ob}\mathcal{C}$ the comma-category $(X \downarrow G)$ has a weakly initial set. By Lemma 3.8.1, the category $(X \downarrow G)$ is complete. Thus, from Lemma 3.8.4 we obtain that $(X \downarrow G)$ has an initial object. The conclusion now follows by Lemma 3.8.2. $\square$

3.9 Exercises

1. Prove that the forgetful functor $U : \mathbf{Field} \to \mathbf{Ring}$ does not admit a right nor a left adjoint.

2. Decide if the forgetful functor $U : \mathcal{A} \to \text{Set}$ admits a right adjoint, where $\mathcal{A}$ is **Grp**, **Ring** or ${}_R\mathcal{M}$.

3. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors such that $F \dashv G$. Prove that F preserves epimorphisms and G preserves monomorphisms.

4. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors such that $F \dashv G$.

   - If GF is faithful then F is faithful;
   - If GF is full and G is faithful then F is full.

5. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{A}$ be functors such that $F \dashv G$, and let $\varepsilon : FG \to 1_{\mathcal{C}}$ be the counit of this adjunction. Prove that for every $X \in \operatorname{Ob}\mathcal{C}$ the pair $(G(X), \varepsilon_X)$ is a final object in the comma category $(F \downarrow X)$. [4]

6. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors such that $F \dashv G$, and consider $\eta : 1_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to 1_{\mathcal{D}}$ to be the unit and respectively the counit of this adjunction. Then the following are equivalent:

   - $F(\eta_C)$ is an isomorphism for all $C \in \operatorname{Ob}\mathcal{C}$;
   - $GF(\eta_C) = \eta_{GF(C)}$ for all $C \in \operatorname{Ob}\mathcal{C}$;
   - $\epsilon_{F(C)}$ is an isomorphism for all $C \in \operatorname{Ob}\mathcal{C}$;
   - $G(\epsilon_{F(C)})$ is an isomorphism for all $C \in \operatorname{Ob}\mathcal{C}$.

7. Let $H : \mathcal{C} \to \mathcal{D}$ and $F, G : \mathcal{D} \to \mathcal{C}$ be functors such that $F \dashv H$ and $H \dashv G$. If $\eta : 1_{\mathcal{D}} \to HF$ is the unit of the adjunction $F \dashv H$ and $\varepsilon : HG \to 1_{\mathcal{D}}$ is the counit of the adjunction $H \dashv G$ then $\eta_D : D \to HF(D)$ is an epimorphism for every $D \in \operatorname{Ob}\mathcal{D}$ if and only if $\varepsilon_D : HG(D) \to D$ is a monomorphism for every $D \in \operatorname{Ob}\mathcal{D}$.

8. Let I and $\mathcal{C}$ be two categories with I small. Prove that:

   - All functors $F : I \to \mathcal{C}$ have limit if and only if the diagonal functor $\Delta : \mathcal{C} \to \operatorname{Fun}(I, \mathcal{C})$ has a right adjoint;
   - All functors $F : I \to \mathcal{C}$ have colimit if and only if the diagonal functor $\Delta : \mathcal{C} \to \operatorname{Fun}(I, \mathcal{C})$ has a left adjoint.

9. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors such that $F \dashv G$. Prove that if G is faithful and X is a generator in $\mathcal{C}$ then $F(X)$ is a generator in $\mathcal{D}$.

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[4] See Lemma 3.8.2 for the proof of the dual statement.

10. Let $F,\, F' : \mathcal{C} \to \mathcal{D}$, $G,\, G' : \mathcal{D} \to \mathcal{C}$ be functors such that $F \dashv G$ and $F' \dashv G'$ and consider I to be a small category.

    - If $T : I \to \mathcal{C}$ is a functor which admits a limit and $\beta : G \to G'$ is a natural transformation such that $\beta_{T(i)}$ is an isomorphism in $\mathcal{C}$ for every $i \in \operatorname{Ob} I$, then $\beta_{\lim T}$ is also an isomorphism in $\mathcal{C}$;
    - If $H : I \to \mathcal{C}$ is a functor which admits a colimit and $\alpha : F \to F'$ is a natural transformation such that $\alpha_{H(i)}$ is an isomorphism in $\mathcal{D}$ for every $i \in \operatorname{Ob} I$, then $\alpha_{\operatorname{colim} H}$ is also an isomorphism in $\mathcal{D}$.

11. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors such that $F \dashv G$.

    - If the functor $F' : \mathcal{C} \to \mathcal{D}$ is naturally isomorphic to F then $F' \dashv G$;
    - If the functor $G' : \mathcal{D} \to \mathcal{C}$ is naturally isomorphic to G then $F \dashv G'$.

12. Prove that equivalence of categories is an equivalence relation.

13. Show that two categories are equivalent if and only if their skeletons are isomorphic.

14. Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. Prove that:

    - $f \in \operatorname{Hom}_{\mathcal{C}}(C,\, C')$ is a monomorphism if and only if $F(f)$ is a monomorphism;
    - $f \in \operatorname{Hom}_{\mathcal{C}}(C,\, C')$ is an epimorphism if and only if $F(f)$ is an epimorphism;
    - $f \in \operatorname{Hom}_{\mathcal{C}}(C,\, C')$ is an isomorphism if and only if $F(f)$ is an isomorphism.

15. Let $R,\, S$ be two rings. Show that the product category ${}_R\mathcal{M} \times {}_S\mathcal{M}$ is equivalent to the category ${}_{R\times S}\mathcal{M}$.

16. An object I of a category $\mathcal{C}$ is called *injective* if for any $u \in \operatorname{Hom}_{\mathcal{C}}(A,\, I)$ and any monomorphism $m \in \operatorname{Hom}_{\mathcal{C}}(A,\, B)$ there exists a morphism $v \in \operatorname{Hom}_{\mathcal{C}}(B,\, I)$ such that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[d, "m"'] \arrow[dr, "u"] & \\ B \arrow[r, "v"'] & I \end{tikzcd} \qquad \text{i.e. } v \circ m = u.$$

    Dually, an object P of $\mathcal{C}$ is called *projective* if for any $u \in \operatorname{Hom}_{\mathcal{C}}(P,\, B)$ and any epimorphism $e \in \operatorname{Hom}_{\mathcal{C}}(A,\, B)$ there exists a morphism $v \in \operatorname{Hom}_{\mathcal{C}}(P,\, A)$ such that the following diagram is commutative:

$$\begin{tikzcd} B & \\ A \arrow[u, "e"] & P \arrow[l, "v"] \arrow[ul, "u"'] \end{tikzcd} \qquad \text{i.e. } e \circ v = u.$$

    Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors such that $F \dashv G$. Prove that:

- if F preserves monomorphisms then G preserves injective objects;
- if G preserves epimorphisms then F preserves projective objects.

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