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# *Basic Category Theory and Topos Theory*

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# 1 Categories and Functors

## 1.1 Definitions and examples

A *category*  $\mathcal{C}$  is given by a collection  $\mathcal{C}_0$  of *objects* and a collection  $\mathcal{C}_1$  of *arrows* which have the following structure.

- Each arrow has a *domain* and a *codomain* which are objects; one writes  $f : X \rightarrow Y$  or  $X \xrightarrow{f} Y$  if  $X$  is the domain of the arrow  $f$ , and  $Y$  its codomain. One also writes  $X = \text{dom}(f)$  and  $Y = \text{cod}(f)$ ;
- Given two arrows  $f$  and  $g$  such that  $\text{cod}(f) = \text{dom}(g)$ , the *composition* of  $f$  and  $g$ , written  $gf$ , is defined and has domain  $\text{dom}(f)$  and codomain  $\text{cod}(g)$ :

$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \mapsto (X \xrightarrow{gf} Z)$$

- Composition is *associative*, that is: given  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  and  $h : Z \rightarrow W$ ,  $h(gf) = (hg)f$ ;
- For every object  $X$  there is an *identity* arrow  $\text{id}_X : X \rightarrow X$ , satisfying  $\text{id}_X g = g$  for every  $g : Y \rightarrow X$  and  $f \text{id}_X = f$  for every  $f : X \rightarrow Y$ .

**Exercise 1** Show that  $\text{id}_X$  is the *unique* arrow with domain  $X$  and codomain  $X$  with this property.

Instead of “arrow” we also use the terms “morphism” or “map”.

### Examples

- 1 is the category with one object  $*$  and one arrow,  $\text{id}_*$ ;
- 0 is the empty category. It has no objects and no arrows.
- A *preorder* is a set  $X$  together with a binary relation  $\leq$  which is reflexive (i.e.  $x \leq x$  for all  $x \in X$ ) and transitive (i.e.  $x \leq y$  and  $y \leq z$  imply  $x \leq z$  for all  $x, y, z \in X$ ). This can be viewed as a category, with set of objects  $X$  and for every pair of objects  $(x, y)$  such that  $x \leq y$ , exactly one arrow:  $x \rightarrow y$ .

**Exercise 2** Prove this. Prove also the converse: if  $\mathcal{C}$  is a category such that  $\mathcal{C}_0$  is a set, and such that for any two objects  $X, Y$  of  $\mathcal{C}$  there is at most one arrow:  $X \rightarrow Y$ , then  $\mathcal{C}_0$  is a preordered set.

- A *monoid* is a set  $X$  together with a binary operation, written like multiplication:  $xy$  for  $x, y \in X$ , which is associative and has a unit element  $e \in X$ , satisfying  $ex = xe = x$  for all  $x \in X$ . Such a monoid is a category with one object, and an arrow  $x$  for every  $x \in X$ .
- Set is the category which has the class of all sets as objects, and functions between sets as arrows.

Most basic categories have as objects certain mathematical structures, and the structure-preserving functions as morphisms. Examples:

- f) Top is the category of topological spaces and continuous functions.
- g) Grp is the category of groups and group homomorphisms.
- h) Rng is the category of rings and ring homomorphisms.
- i) Grph is the category of graphs and graph homomorphisms.
- j) Pos is the category of partially ordered sets and monotone functions.

Given two categories  $\mathcal{C}$  and  $\mathcal{D}$ , a *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  consists of operations  $F_0 : \mathcal{C}_0 \rightarrow \mathcal{D}_0$  and  $F_1 : \mathcal{C}_1 \rightarrow \mathcal{D}_1$ , such that for each  $f : X \rightarrow Y$ ,  $F_1(f) : F_0(X) \rightarrow F_0(Y)$  and:

- for  $X \xrightarrow{f} Y \xrightarrow{g} Z$ ,  $F_1(gf) = F_1(g)F_1(f)$ ;
- $F_1(\text{id}_X) = \text{id}_{F_0(X)}$  for each  $X \in \mathcal{C}_0$ .

But usually we write just  $F$  instead of  $F_0, F_1$ .

### Examples.

- a) There is a functor  $U : \text{Top} \rightarrow \text{Set}$  which assigns to any topological space  $X$  its underlying set. We call this functor “forgetful”: it “forgets” the mathematical structure. Similarly, there are forgetful functors  $\text{Grp} \rightarrow \text{Set}$ ,  $\text{Grph} \rightarrow \text{Set}$ ,  $\text{Rng} \rightarrow \text{Set}$ ,  $\text{Pos} \rightarrow \text{Set}$  etcetera;
- b) For every category  $\mathcal{C}$  there is a unique functor  $\mathcal{C} \rightarrow \mathbf{1}$  and a unique one  $\mathbf{0} \rightarrow \mathcal{C}$ ;
- c) Given two categories  $\mathcal{C}$  and  $\mathcal{D}$  we can define the *product category*  $\mathcal{C} \times \mathcal{D}$  which has as objects pairs  $(C, D) \in \mathcal{C}_0 \times \mathcal{D}_0$ , and as arrows:  $(C, D) \rightarrow (C', D')$  pairs  $(f, g)$  with  $f : C \rightarrow C'$  in  $\mathcal{C}$ , and  $g : D \rightarrow D'$  in  $\mathcal{D}$ . There are functors  $\pi_0 : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C}$  and  $\pi_1 : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$ ;
- d) Given two functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{E}$  one can define the composition  $GF : \mathcal{C} \rightarrow \mathcal{E}$ . This composition is of course associative and since we have, for any category  $\mathcal{C}$ , the *identity functor*  $\mathcal{C} \rightarrow \mathcal{C}$ , we have a category  $\text{Cat}$  which has categories as objects and functors as morphisms.
- e) Given a set  $A$ , consider the set  $\tilde{A}$  of strings  $a_1 \dots a_n$  on the alphabet  $A \cup A^{-1}$  ( $A^{-1}$  consists of elements  $a^{-1}$  for each element  $a$  of  $A$ ; the sets  $A$  and  $A^{-1}$  are disjoint and in 1-1 correspondence with each other), such that for no  $x \in A$ ,  $xx^{-1}$  or  $x^{-1}x$  is a substring of  $a_1 \dots a_n$ . Given two such strings  $\vec{a} = a_1 \dots a_n, \vec{b} = b_1 \dots b_m$ , let  $\vec{a} \star \vec{b}$  be the string formed by first taking  $a_1 \dots a_n b_1 \dots b_m$  and then removing from this string, successively, substrings of form  $xx^{-1}$  or  $x^{-1}x$ , until one has an element of  $\tilde{A}$ .

This defines a group structure on  $\tilde{A}$ .  $\tilde{A}$  is called the *free group* on the set  $A$ .

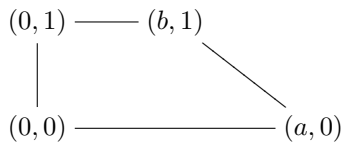
**Exercise 3** Prove this, and prove that the assignment  $A \mapsto \tilde{A}$  is part of a functor:  $\text{Set} \rightarrow \text{Grp}$ . This functor is called the *free functor*.

- f) Every directed graph can be made into a category as follows: the objects are the vertices of the graph and the arrows are paths in the graph. This defines a functor from the category  $\text{Dgrph}$  of directed graphs to  $\text{Cat}$ . The image of a directed graph  $D$  under this functor is called the category *generated* by the graph  $D$ .
- g) **Quotient categories.** Given a category  $\mathcal{C}$ , a *congruence relation* on  $\mathcal{C}$  specifies, for each pair of objects  $X, Y$ , an equivalence relation  $\sim_{X,Y}$  on the class of arrows  $\mathcal{C}(X, Y)$  which have domain  $X$  and codomain  $Y$ , such that
- for  $f, g : X \rightarrow Y$  and  $h : Y \rightarrow Z$ , if  $f \sim_{X,Y} g$  then  $hf \sim_{X,Z} hg$ ;
  - for  $f : X \rightarrow Y$  and  $g, h : Y \rightarrow Z$ , if  $g \sim_{Y,Z} h$  then  $gf \sim_{X,Z} hf$ .

Given such a congruence relation  $\sim$  on  $\mathcal{C}$ , one can form the quotient category  $\mathcal{C}/\sim$  which has the same objects as  $\mathcal{C}$ , and arrows  $X \rightarrow Y$  are  $\sim_{X,Y}$ -equivalence classes of arrows  $X \rightarrow Y$  in  $\mathcal{C}$ .

**Exercise 4** Show this and show that there is a functor  $\mathcal{C} \rightarrow \mathcal{C}/\sim$ , which takes each arrow of  $\mathcal{C}$  to its equivalence class.

- h) An example of this is the following (“homotopy”). Given a topological space  $X$  and points  $x, y \in X$ , a *path* from  $x$  to  $y$  is a continuous mapping  $f$  from some closed interval  $[0, a]$  to  $X$  with  $f(0) = x$  and  $f(a) = y$ . If  $f : [0, a] \rightarrow X$  is a path from  $x$  to  $y$  and  $g : [0, b] \rightarrow X$  is a path from  $y$  to  $z$  there is a path  $gf : [0, a+b] \rightarrow X$  (defined by  $gf(t) = \begin{cases} f(t) & t \leq a \\ g(t-a) & \text{else} \end{cases}$ ) from  $x$  to  $z$ . This makes  $X$  into a category, the *path category* of  $X$ , and of course this also defines a functor  $\text{Top} \rightarrow \text{Cat}$ . Now given paths  $f : [0, a] \rightarrow X$ ,  $g : [0, b] \rightarrow X$ , both from  $x$  to  $y$ , one can define  $f \sim_{x,y} g$  if there is a continuous map  $F : A \rightarrow X$  where  $A$  is the area:



in  $\mathbb{R}^2$ , such that

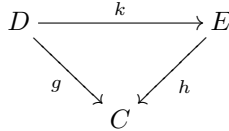
$$\begin{aligned} F(t, 0) &= f(t) \\ F(t, 1) &= g(t) \\ F(0, s) &= x \quad s \in [0, 1] \\ F(s, t) &= y \quad (s, t) \text{ on the segment } (b, 1) - (a, 0) \end{aligned}$$

One can easily show that this is a congruence relation. The quotient of the path category by this congruence relation is a category called the category of *homotopy classes* of paths in  $X$ .

- i) let  $\mathcal{C}$  be a category such that for every pair  $(X, Y)$  of objects the collection  $\mathcal{C}(X, Y)$  of arrows from  $X$  to  $Y$  is a set (such  $\mathcal{C}$  is called *locally small*).

For any object  $C$  of  $\mathcal{C}$  then, there is a functor  $y_C : \mathcal{C} \rightarrow \text{Set}$  which assigns to any object  $C'$  the set  $\mathcal{C}(C, C')$ . Any arrow  $f : C' \rightarrow C''$  gives by composition a function  $\mathcal{C}(C, C') \rightarrow \mathcal{C}(C, C'')$ , so we have a functor. A functor of this form is called a *representable functor*.

- j) Let  $\mathcal{C}$  be a category and  $C$  an object of  $\mathcal{C}$ . The *slice category*  $\mathcal{C}/C$  has as objects all arrows  $g$  which have codomain  $C$ . An arrow from  $g : D \rightarrow C$  to  $h : E \rightarrow C$  in  $\mathcal{C}/C$  is an arrow  $k : D \rightarrow E$  in  $\mathcal{C}$  such that  $hk = g$ . Draw like:



We say that *this diagram commutes* if we mean that  $hk = g$ .

**Exercise 5** Convince yourself that the assignment  $C \mapsto \mathcal{C}/C$  gives rise to a functor  $\mathcal{C} \rightarrow \text{Cat}$ .

- k) Recall that for every group  $(G, \cdot)$  we can form a group  $(G, \star)$  by putting  $f \star g = g \cdot f$ .

For categories the same construction is available: given  $\mathcal{C}$  we can form a category  $\mathcal{C}^{\text{op}}$  which has the same objects and arrows as  $\mathcal{C}$ , but with reversed direction; so if  $f : X \rightarrow Y$  in  $\mathcal{C}$  then  $\bar{f} : Y \rightarrow X$  in  $\mathcal{C}^{\text{op}}$ . To make it notationally clear, write  $\bar{f}$  for the arrow  $Y \rightarrow X$  corresponding to  $f : X \rightarrow Y$  in  $\mathcal{C}$ . Composition in  $\mathcal{C}^{\text{op}}$  is defined by:

$$\bar{f}\bar{g} = \overline{gf}$$

Often one reads the term “contravariant functor” in the literature. What I call functor, is then called “covariant functor”. A contravariant functor  $F$  from  $\mathcal{C}$  to  $\mathcal{D}$  inverts the direction of the arrows, so  $F_1(f) : F_0(\text{cod}(f)) \rightarrow F_0(\text{dom}(f))$  for arrows  $f$  in  $\mathcal{C}$ . In other words, a contravariant functor from  $\mathcal{C}$  to  $\mathcal{D}$  is a functor from  $\mathcal{C}^{\text{op}}$  to  $\mathcal{D}$  (equivalently, from  $\mathcal{C}$  to  $\mathcal{D}^{\text{op}}$ ).

Of course, any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  gives a functor  $F^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ . In fact, we have a functor  $(-)^{\text{op}} : \text{Cat} \rightarrow \text{Cat}$ .

**Exercise 6** Let  $\mathcal{C}$  be locally small. Show that there is a functor (the “Hom functor”)  $\mathcal{C}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set}$ , assigning to the pair  $(A, B)$  of objects of  $\mathcal{C}$ , the set  $\mathcal{C}(A, B)$ .

- l) Given a partially ordered set  $(X, \leq)$  we make a topological space by defining  $U \subseteq X$  to be open iff for all  $x, y \in X$ ,  $x \leq y$  and  $x \in U$  imply  $y \in U$  ( $U$  is “upwards closed”, or an “upper set”). This is a topology, called the *Alexandroff topology* w.r.t. the order  $\leq$ .



If  $(X, \leq)$  and  $(Y, \leq)$  are two partially ordered sets, a function  $f : X \rightarrow Y$  is monotone for the orderings if and only if  $f$  is continuous for the Alexandroff topologies. This gives an important functor:  $\text{Pos} \rightarrow \text{Top}$ .

**Exercise 7** Show that the construction of the quotient category in example g) generalizes that of a quotient group by a normal subgroup. That is, regard a group  $G$  as a category with one object; show that there is a bijection between congruence relations on  $G$  and normal subgroups of  $G$ , and that for a normal subgroup  $N$  of  $G$ , the quotient category by the congruence relation corresponding to  $N$ , is to the quotient group  $G/N$ .

- m) “Abelianization”. Let  $\text{Abgp}$  be the category of abelian groups and homomorphisms. For every group  $G$  the subgroup  $[G, G]$  generated by all elements of form  $aba^{-1}b^{-1}$  is a normal subgroup.  $G/[G, G]$  is abelian, and for every group homomorphism  $\phi : G \rightarrow H$  with  $H$  abelian, there is a unique homomorphism  $\bar{\phi} : G/[G, G] \rightarrow H$  such that the diagram

$$\begin{array}{ccc} & G & \\ p \swarrow & & \searrow \phi \\ G/[G, G] & \xrightarrow{\bar{\phi}} & H \end{array}$$

commutes. Show that this gives a functor:  $\text{Grp} \rightarrow \text{Abgp}$ .

- n) “Specialization ordering”. Given a topological space  $X$ , you can define a preorder  $\leq_s$  on  $X$  as follows: say  $x \leq_s y$  if for all open sets  $U$ , if  $x \in U$  then  $y \in U$ .  $\leq_s$  is a partial order iff  $X$  is a  $T_0$ -space.

For many spaces,  $\leq_s$  is trivial (in particular when  $X$  is  $T_1$ ) but in case  $X$  is for example the Alexandroff topology on a poset  $(X, \leq)$  as in l), then  $x \leq_s y$  iff  $x \leq y$ .

**Exercise 8** If  $f : X \rightarrow Y$  is a continuous map of topological spaces then  $f$  is monotone w.r.t. the specialization orderings  $\leq_s$ . This defines a functor  $\text{Top} \rightarrow \text{Preord}$ , where  $\text{Preord}$  is the category of preorders and monotone functions.

**Exercise 9** Let  $X$  be the category defined as follows: objects are pairs  $(I, x)$  where  $I$  is an open interval in  $\mathbb{R}$  and  $x \in I$ . Morphisms  $(I, x) \rightarrow (J, y)$  are differentiable functions  $f : I \rightarrow J$  such that  $f(x) = y$ .

Let  $Y$  be the (multiplicative) monoid  $\mathbb{R}$ , considered as a category. Show that the operation which sends an arrow  $f : (I, x) \rightarrow (J, y)$  to  $f'(x)$ , determines a functor  $X \rightarrow Y$ . On which basic fact of elementary Calculus does this rely?

## 1.2 Some special objects and arrows

We call an arrow  $f : A \rightarrow B$  *mono* (or a monomorphism, or monomorphic) in a category  $\mathcal{C}$ , if for any other object  $C$  and for any pair of arrows  $g, h : C \rightarrow A$ ,  $fg = fh$  implies  $g = h$ .

In  $\mathbf{Set}$ ,  $f$  is mono iff  $f$  is an injective function. The same is true for  $\mathbf{Grp}$ ,  $\mathbf{Grph}$ ,  $\mathbf{Rng}$ ,  $\mathbf{Preord}$ ,  $\mathbf{Pos}$ ,...

We call an arrow  $f : A \rightarrow B$  *epi* (epimorphism, epimorphic) if for any pair  $g, h : B \rightarrow C$ ,  $gf = hf$  implies  $g = h$ .

The definition of epi is “dual” to the definition of mono. That is,  $f$  is epi in the category  $\mathcal{C}$  if and only if  $f$  is mono in  $\mathcal{C}^{\text{op}}$ , and vice versa. In general, given a property  $P$  of an object, arrow, diagram,... we can associate with  $P$  the dual property  $P^{\text{op}}$ : the object or arrow has property  $P^{\text{op}}$  in  $\mathcal{C}$  iff it has  $P$  in  $\mathcal{C}^{\text{op}}$ .

The *duality principle*, a very important, albeit trivial, principle in category theory, says that any valid statement about categories, involving the properties  $P_1, \dots, P_n$  implies the “dualized” statement (where direction of arrows is reversed) with the  $P_i$  replaced by  $P_i^{\text{op}}$ .

**Example.** If  $gf$  is mono, then  $f$  is mono. From this, “by duality”, if  $fg$  is epi, then  $f$  is epi.

**Exercise 10** Prove these statements.

In  $\mathbf{Set}$ ,  $f$  is epi iff  $f$  is a surjective function. This holds (less trivially!) also for  $\mathbf{Grp}$ , but not for  $\mathbf{Mon}$ , the category of monoids and monoid homomorphisms:

**Example.** In  $\mathbf{Mon}$ , the embedding  $\mathbb{N} \rightarrow \mathbb{Z}$  is an epimorphism.

For, suppose  $\mathbb{Z} \xrightleftharpoons[g]{f} (M, e, \star)$  two monoid homomorphisms which agree on the nonnegative integers. Then

$$f(-1) = f(-1) \star g(1) \star g(-1) = f(-1) \star f(1) \star g(-1) = g(-1)$$

so  $f$  and  $g$  agree on the whole of  $\mathbb{Z}$ .

We say a functor  $F$  *preserves* a property  $P$  if whenever an object or arrow (or...) has  $P$ , its  $F$ -image does so.

Now a functor does not in general preserve monos or epis: the example of  $\mathbf{Mon}$  shows that the forgetful functor  $\mathbf{Mon} \rightarrow \mathbf{Set}$  does not preserve epis.

An epi  $f : A \rightarrow B$  is called *split* if there is  $g : B \rightarrow A$  such that  $fg = \text{id}_B$  (other names: in this case  $g$  is called a *section* of  $f$ , and  $f$  a *retraction* of  $g$ ).

**Exercise 11** By duality, define what a split mono is. Prove that every functor preserves split epi and monos.

A morphism  $f : A \rightarrow B$  is an *isomorphism* if there is  $g : B \rightarrow A$  such that  $fg = \text{id}_B$  and  $gf = \text{id}_A$ . We call  $g$  the *inverse* of  $f$  (and vice versa, of course); it is unique if it exists. We also write  $g = f^{-1}$ .

Every functor preserves isomorphisms.

**Exercise 12** In  $\mathbf{Set}$ , every arrow which is both epi and mono is an isomorphism. Not so in  $\mathbf{Mon}$ , as we have seen. Here’s another one: let  $\mathbf{CRng1}$  be the category of commutative rings with 1, and ring homomorphisms (preserving 1) as arrows. Show that the embedding  $\mathbb{Z} \rightarrow \mathbb{Q}$  is epi in  $\mathbf{CRng1}$ .

**Exercise 13** i) If two of  $f$ ,  $g$  and  $fg$  are iso, then so is the third;

ii) if  $f$  is epi and split mono, it is iso;

iii) if  $f$  is split epi and mono,  $f$  is iso.

A functor  $F$  *reflects* a property  $P$  if whenever the  $F$ -image of something (object, arrow, ...) has  $P$ , then that something has.

A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called *full* if for every two objects  $A, B$  of  $\mathcal{C}$ ,  $F : \mathcal{C}(A, B) \rightarrow \mathcal{D}(FA, FB)$  is a surjection.  $F$  is *faithful* if this map is always injective.

**Exercise 14** A faithful functor reflects epis and monos.

An object  $X$  is called *terminal* if for any object  $Y$  there is exactly one morphism  $Y \rightarrow X$  in the category. Dually,  $X$  is *initial* if for all  $Y$  there is exactly one  $X \rightarrow Y$ .

**Exercise 15** A full and faithful functor reflects the property of being a terminal (or initial) object.

**Exercise 16** If  $X$  and  $X'$  are two terminal objects, they are *isomorphic*, that is there exists an isomorphism between them. Same for initial objects.

**Exercise 17** Let  $\sim$  be a congruence on the category  $\mathcal{C}$ , as in example g). Show: if  $f$  and  $g$  are arrows  $X \rightarrow Y$  with inverses  $f^{-1}$  and  $g^{-1}$  respectively, then  $f \sim g$  iff  $f^{-1} \sim g^{-1}$ .

**Exercise 18** Recall that a topological space is *normal* if every one-point subset is closed and for every pair  $A, B$  of disjoint closed subsets, there exist disjoint open subsets  $U, V$  with  $A \subset U$ ,  $B \subset V$ . We denote by  $\mathcal{N}$  the full subcategory of Top on the normal topological spaces.

a) Characterise the epimorphisms in  $\mathcal{N}$ . Hint: you may find it useful to invoke Urysohn's Lemma.

b) Show that for two morphisms  $f, g : A \rightarrow B$  in  $\mathcal{N}$  we have:  $f = g$  if and only if for every morphism  $h : B \rightarrow \mathbb{R}$ ,  $hf = hg$  holds (this property of  $\mathbb{R}$  in  $\mathcal{N}$  is sometimes called a *coseparator*)

## 2 Natural transformations

### 2.1 The Yoneda lemma

A *natural transformation* between two functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  consists of a family of morphisms  $(\mu_C : FC \rightarrow GC)_{C \in \mathcal{C}_0}$  indexed by the collection of objects of  $\mathcal{C}$ , satisfying the following requirement: for every morphism  $f : C \rightarrow C'$  in  $\mathcal{C}$ , the diagram

$$\begin{array}{ccc} FC & \xrightarrow{\mu_C} & GC \\ Ff \downarrow & & \downarrow Gf \\ FC' & \xrightarrow{\mu_{C'}} & GC' \end{array}$$

commutes in  $\mathcal{D}$  (the diagram above is called the naturality square). We say  $\mu = (\mu_C)_{C \in \mathcal{C}_0} : F \Rightarrow G$  and we call  $\mu_C$  the *component at C* of the natural transformation  $\mu$ .

Given natural transformations  $\mu : F \Rightarrow G$  and  $\nu : G \Rightarrow H$  we have a natural transformation  $\nu\mu = (\nu_C\mu_C)_C : F \Rightarrow H$ , and with this composition there is a category  $\mathcal{D}^{\mathcal{C}}$  with functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  as objects, and natural transformations as arrows.

One of the points of the naturality square condition in the definition of a natural transformation is given by the following proposition. Compare with the situation in  $\mathbf{Set}$ : denoting the set of all functions from  $X$  to  $Y$  by  $Y^X$ , for any set  $Z$  there is a bijection between functions  $Z \rightarrow Y^X$  and functions  $Z \times X \rightarrow Y$  ( $\mathbf{Set}$  is *cartesian closed*: see chapter 7).

**Proposition 2.1** *For categories  $\mathcal{C}$ ,  $\mathcal{D}$  and  $\mathcal{E}$  there is a bijection:*

$$\mathbf{Cat}(\mathcal{E} \times \mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathbf{Cat}(\mathcal{E}, \mathcal{D}^{\mathcal{C}})$$

**Proof.** Given  $F : \mathcal{E} \times \mathcal{C} \rightarrow \mathcal{D}$  define for every object  $E$  of  $\mathcal{E}$  the functor  $F_E : \mathcal{C} \rightarrow \mathcal{D}$  by  $F_E(C) = F(E, C)$ ; for  $f : C \rightarrow C'$  let  $F_E(f) = F(\mathrm{id}_E, f) : F_E(C) \rightarrow F_E(C') = F(E, C') = F_E(C')$ .

Given  $g : E \rightarrow E'$  in  $\mathcal{E}$ , the family  $(F(g, \mathrm{id}_C) : F_E(C) \rightarrow F_{E'}(C))_{C \in \mathcal{C}_0}$  is a natural transformation:  $F_E \Rightarrow F_{E'}$ . So we have a functor  $F \mapsto F_{(-)} : \mathcal{E} \rightarrow \mathcal{D}^{\mathcal{C}}$ .

Conversely, given a functor  $G : \mathcal{E} \rightarrow \mathcal{D}^{\mathcal{C}}$  we define a functor  $\tilde{G} : \mathcal{E} \times \mathcal{C} \rightarrow \mathcal{D}$  on objects by  $\tilde{G}(E, C) = G(E)(C)$ , and on arrows by  $\tilde{G}(g, f) = G(g)_{C'}G(E)(f) = G(E')(f)G(g)_C$ :

$$\begin{array}{ccc} G(E)(C) = \tilde{G}(E, C) & \xrightarrow{G(g)_C} & \tilde{G}(E', C) = G(E')(C) \\ G(E)(f) \downarrow & & \downarrow G(E')(f) \\ \tilde{G}(E, C') & \xrightarrow{G(g)_{C'}} & \tilde{G}(E', C') = G(E')(C') \end{array}$$

**Exercise 19** Write out the details. Check that  $\tilde{G}$  as just defined, is a functor, and that the two operations

$$\text{Cat}(\mathcal{E} \times \mathcal{C}, \mathcal{D}) \xleftarrow{\quad} \text{Cat}(\mathcal{E}, \mathcal{D}^{\mathcal{C}})$$

are inverse to each other. ■

An important example of natural transformations arises from the functors  $y_C : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$  (see example i) in the preceding chapter); defined on objects by  $y_C(C') = \mathcal{C}(C', C)$  and on arrows  $f : C'' \rightarrow C'$  so that  $y_C(f)$  is composition with  $f$ :  $\mathcal{C}(C', C) \rightarrow \mathcal{C}(C'', C)$ .

Given  $g : C_1 \rightarrow C_2$  there is a natural transformation

$$y_g : y_{C_1} \Rightarrow y_{C_2}$$

whose components are composition with  $g$ .

**Exercise 20** Spell this out.

We have, in other words, a functor

$$Y = y_{(-)} : \mathcal{C} \rightarrow \text{Set}^{\mathcal{C}^{\text{op}}}$$

This functor is called the *Yoneda embedding*.

An embedding is a functor which is full and faithful and injective on objects. That  $Y$  is injective on objects is easy to see, because  $\text{id}_C \in y_C(C)$  for each object  $C$ , and  $\text{id}_C$  is in no other set  $y_D(E)$ ; that  $Y$  is full and faithful follows from the following

**Proposition 2.2 (Yoneda lemma)** *For every object  $F$  of  $\text{Set}^{\mathcal{C}^{\text{op}}}$  and every object  $C$  of  $\mathcal{C}$ , there is a bijection  $f_{C,F} : \text{Set}^{\mathcal{C}^{\text{op}}}(y_C, F) \rightarrow F(C)$ . Moreover, this bijection is natural in  $C$  and  $F$  in the following sense: given  $g : C' \rightarrow C$  in  $\mathcal{C}$  and  $\mu : F \Rightarrow F'$  in  $\text{Set}^{\mathcal{C}^{\text{op}}}$ , the diagram*

$$\begin{array}{ccc} \text{Set}^{\mathcal{C}^{\text{op}}}(y_C, F) & \xrightarrow{f_{C,F}} & F(C) \\ \text{Set}^{\mathcal{C}^{\text{op}}}(g, \mu) \downarrow & & \downarrow \mu_{C'} F(g) = F'(g) \mu_C \\ \text{Set}^{\mathcal{C}^{\text{op}}}(y_{C'}, F') & \xrightarrow{f_{C',F'}} & F'(C') \end{array}$$

*commutes in Set.*

**Proof.** For every object  $C'$  of  $\mathcal{C}$ , every element  $f$  of  $y_C(C') = \mathcal{C}(C', C)$  is equal to  $\text{id}_C f$  which is  $y_C(f)(\text{id}_C)$ .

If  $\kappa = (\kappa_{C'} | C' \in \mathcal{C}_0)$  is a natural transformation:  $y_C \Rightarrow F$  then,  $\kappa_{C'}(f)$  must be equal to  $F(f)(\kappa_C(\text{id}_C))$ . So  $\kappa$  is completely determined by  $\kappa_C(\text{id}_C) \in F(C)$

and conversely, any element of  $F(C)$  determines a natural transformation  $y_C \Rightarrow F$ .

Given  $g : C' \rightarrow C$  in  $\mathcal{C}$  and  $\mu : F \Rightarrow F'$  in  $\text{Set}^{\mathcal{C}^{\text{op}}}$ , the map  $\text{Set}^{\mathcal{C}^{\text{op}}}(g, \mu)$  sends the natural transformation  $\kappa = (\kappa_{C''} | C'' \in \mathcal{C}_0) : y_C \Rightarrow F$  to  $\lambda = (\lambda_{C''} | C'' \in \mathcal{C}_0)$  where  $\lambda_{C''} : y_{C'}(C'') \rightarrow F'(C'')$  is defined by

$$\lambda_{C''}(h : C'' \rightarrow C') = \mu_{C''}(\kappa_{C''}(gh))$$

Now

$$\begin{aligned} f_{C', F'}(\lambda) &= \lambda_{C'}(\text{id}_{C'}) \\ &= \mu_{C'}(\kappa_{C'}(g)) \\ &= \mu_{C'}(F(g)(\kappa_C(\text{id}_C))) \\ &= (\mu_{C'} F(g))(f_{C, F}(\kappa)) \end{aligned}$$

which proves the naturality statement. ■

**Corollary 2.3** *The functor  $Y : \mathcal{C} \rightarrow \text{Set}^{\mathcal{C}^{\text{op}}}$  is full and faithful.*

**Proof.** Immediate by the Yoneda lemma, since

$$\mathcal{C}(C, C') = y_{C'}(C) \cong \text{Set}^{\mathcal{C}^{\text{op}}}(y_C, y_{C'})$$

and this bijection is induced by  $Y$ . ■

The use of the Yoneda lemma is often the following. One wants to prove that objects  $A$  and  $B$  of  $\mathcal{C}$  are isomorphic. Suppose one can show that for every object  $X$  of  $\mathcal{C}$  there is a bijection  $f_X : \mathcal{C}(X, A) \rightarrow \mathcal{C}(X, B)$  which is natural in  $X$ ; i.e. given  $g : X' \rightarrow X$  in  $\mathcal{C}$  one has that

$$\begin{array}{ccc} \mathcal{C}(X, A) & \xrightarrow{f_X} & \mathcal{C}(X, B) \\ \downarrow \mathcal{C}(g, \text{id}_A) & & \downarrow \mathcal{C}(g, \text{id}_B) \\ \mathcal{C}(X', A) & \xrightarrow{f_{X'}} & \mathcal{C}(X', B) \end{array}$$

commutes.

Then one can conclude that  $A$  and  $B$  are isomorphic in  $\mathcal{C}$ ; for, from what one has just shown it follows that  $y_A$  and  $y_B$  are isomorphic objects in  $\text{Set}^{\mathcal{C}^{\text{op}}}$ ; that is,  $Y(A)$  and  $Y(B)$  are isomorphic. Since  $Y$  is full and faithful,  $A$  and  $B$  are isomorphic by the following exercise:

**Exercise 21** Check: if  $F : \mathcal{C} \rightarrow \mathcal{D}$  is full and faithful, and  $F(A)$  is isomorphic to  $F(B)$  in  $\mathcal{D}$ , then  $A$  is isomorphic to  $B$  in  $\mathcal{C}$ .

**Exercise 22** Suppose objects  $A$  and  $B$  are such that for every object  $X$  in  $\mathcal{C}$  there is a bijection  $f_X : \mathcal{C}(A, X) \rightarrow \mathcal{C}(B, X)$ , naturally in a sense you define yourself. Conclude that  $A$  and  $B$  are isomorphic (hint: duality + the previous).

This argument can be carried further. Suppose one wants to show that two functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  are isomorphic as objects of  $\mathcal{D}^{\mathcal{C}}$ . Let's first spell out what this means:

**Exercise 23** Show that  $F$  and  $G$  are isomorphic in  $\mathcal{D}^{\mathcal{C}}$  if and only if there is a natural transformation  $\mu : F \Rightarrow G$  such that all components  $\mu_C$  are isomorphisms (in particular, if  $\mu$  is such, the family  $((\mu_C)^{-1} | C \in \mathcal{C}_0)$  is also a natural transformation  $G \Rightarrow F$ ).

Now suppose one has for each  $C \in \mathcal{C}_0$  and  $D \in \mathcal{D}_0$  a bijection

$$\mathcal{D}(D, FC) \cong \mathcal{D}(D, GC)$$

natural in  $D$  and  $C$ . This means that the objects  $y_{FC}$  and  $y_{GC}$  of  $\text{Set}^{\mathcal{D}^{\text{op}}}$  are isomorphic, by isomorphisms which are natural in  $C$ . By full and faithfulness of  $Y$ ,  $FC$  and  $GC$  are isomorphic in  $\mathcal{D}$  by isomorphisms natural in  $C$ ; which says exactly that  $F$  and  $G$  are isomorphic as objects of  $\mathcal{D}^{\mathcal{C}}$ .

## 2.2 Examples of natural transformations

- a) Let  $M$  and  $N$  be two monoids, regarded as categories with one object as in chapter 1. A functor  $F : M \rightarrow N$  is then just the same as a homomorphism of monoids. Given two such, say  $F, G : M \rightarrow N$ , a natural transformation  $F \Rightarrow G$  is (given by) an element  $n$  of  $N$  such that  $nF(x) = G(x)n$  for all  $x \in M$ ;
- b) Let  $P$  and  $Q$  two preorders, regarded as categories. A functor  $P \rightarrow Q$  is a monotone function, and there exists a unique natural transformation between two such,  $F \Rightarrow G$ , exactly if  $F(x) \leq G(x)$  for all  $x \in P$ .

**Exercise 24** In fact, show that if  $\mathcal{D}$  is a preorder and the category  $\mathcal{C}$  is *small*, i.e. the classes  $\mathcal{C}_0$  and  $\mathcal{C}_1$  are sets, then the functor category  $\mathcal{D}^{\mathcal{C}}$  is a preorder.

- c) Let  $U : \text{Grp} \rightarrow \text{Set}$  denote the forgetful functor, and  $F : \text{Set} \rightarrow \text{Grp}$  the free functor (see chapter 1). There are natural transformations  $\varepsilon : FU \Rightarrow \text{id}_{\text{Grp}}$  and  $\eta : \text{id}_{\text{Set}} \Rightarrow UF$ .

Given a group  $G$ ,  $\varepsilon_G$  takes the string  $\sigma = g_1 \dots g_n$  to the product  $g_1 \dots g_n$  (here, the “formal inverses”  $g_i^{-1}$  are interpreted as the real inverses in  $G!$ ).

Given a set  $A$ ,  $\eta_A(a)$  is the singleton string  $a$ .

- d) Let  $i : \text{Abgp} \rightarrow \text{Grp}$  be the inclusion functor and  $r : \text{Grp} \rightarrow \text{Abgp}$  the abelianization functor defined in example m) in chapter 1. There is  $\varepsilon : ri \Rightarrow \text{id}_{\text{Abgp}}$  and  $\eta : \text{id}_{\text{Grp}} \Rightarrow ir$ .

The components  $\eta_G$  of  $\eta$  are the quotient maps  $G \rightarrow G/[G, G]$ ; the components of  $\varepsilon$  are isomorphisms.

- e) There are at least two ways to associate a category to a set  $X$ : let  $F(X)$  be the category with as objects the elements of  $X$ , and as only arrows identities (a category of the form  $F(X)$  is called *discrete*; and  $G(X)$  the category with the same objects but with exactly one arrow  $f_{x,y} : x \rightarrow y$  for each pair  $(x, y)$  of elements of  $X$  (We might call  $G(X)$  an *indiscrete category*).

**Exercise 25** Check that  $F$  and  $G$  can be extended to functors:  $\text{Set} \rightarrow \text{Cat}$  and describe the natural transformation  $\mu : F \Rightarrow G$  which has, at each component, the identity function on objects.

- f) Every class of arrows of a category  $\mathcal{C}$  can be viewed as a natural transformation. Suppose  $S \subseteq \mathcal{C}_1$ . Let  $F(S)$  be the discrete category on  $S$  as in the preceding example. There are the two functors  $\text{dom}, \text{cod} : F(S) \rightarrow \mathcal{C}$ , giving the domain and the codomain, respectively. For every  $f \in S$  we have  $f : \text{dom}(f) \rightarrow \text{cod}(f)$ , and the family  $(f | f \in S)$  defines a natural transformation:  $\text{dom} \Rightarrow \text{cod}$ .
- g) Let  $A$  and  $B$  be sets. There are functors  $(-) \times A : \text{Set} \rightarrow \text{Set}$  and  $(-) \times B : \text{Set} \rightarrow \text{Set}$ . Any function  $f : A \rightarrow B$  gives a natural transformation  $(-) \times A \Rightarrow (-) \times B$ .
- h) A category  $\mathcal{C}$  is called a *groupoid* if every arrow of  $\mathcal{C}$  is an isomorphism. Let  $\mathcal{C}$  be a groupoid, and suppose we are given, for each object  $X$  of  $\mathcal{C}$ , an arrow  $\mu_X$  in  $\mathcal{C}$  with domain  $X$ .

**Exercise 26** Show that there is a functor  $F : \mathcal{C} \rightarrow \mathcal{C}$  in this case, which acts on objects by  $F(X) = \text{cod}(\mu_X)$ , and that  $\mu = (\mu_X | X \in \mathcal{C}_0)$  is a natural transformation:  $\text{id}_{\mathcal{C}} \Rightarrow F$ .

- i) Given categories  $\mathcal{C}, \mathcal{D}$  and an object  $D$  of  $\mathcal{D}$ , there is the constant functor  $\Delta_D : \mathcal{C} \rightarrow \mathcal{D}$  which assigns  $D$  to every object of  $\mathcal{C}$  and  $\text{id}_D$  to every arrow of  $\mathcal{C}$ .

Every arrow  $f : D \rightarrow D'$  gives a natural transformation  $\Delta_f : \Delta_D \Rightarrow \Delta_{D'}$  defined by  $(\Delta_f)_C = f$  for each  $C \in \mathcal{C}_0$ .

- j) Let  $\mathcal{P}(X)$  denote the power set of a set  $X$ : the set of subsets of  $X$ . The powerset operation can be extended to a functor  $\mathcal{P} : \text{Set} \rightarrow \text{Set}$ . Given a function  $f : X \rightarrow Y$  define  $\mathcal{P}(f)$  by  $\mathcal{P}(f)(A) = f[A]$ , the image of  $A \subseteq X$  under  $f$ .

There is a natural transformation  $\eta : \text{id}_{\text{Set}} \Rightarrow \mathcal{P}$  such that  $\eta_X(x) = \{x\} \in \mathcal{P}(X)$  for each set  $X$ .

There is also a natural transformation  $\mu : \mathcal{P}\mathcal{P} \Rightarrow \mathcal{P}$ . Given  $A \in \mathcal{P}\mathcal{P}(X)$ , so  $A$  is a set of subsets of  $X$ , we take its union  $\bigcup(A)$  which is a subset of  $X$ . Put  $\mu_X(A) = \bigcup(A)$ .



## 2.3 Equivalence of categories; an example

As will become clear in the following chapters, equality between objects plays only a minor role in category theory. The most important categorical notions are only defined “up to isomorphism”. This is in accordance with mathematical practice and with common sense: just renaming all elements of a group does not really give you another group.

We have already seen one example of this: the property of being a terminal object defines an object up to isomorphism. That is, any two terminal objects are isomorphic. There is, in the language of categories, no way of distinguishing between two isomorphic objects, so this is as far as we can get.

However, once we also consider functor categories, it turns out that there is another relation of “sameness” between categories, weaker than isomorphism of categories, and yet preserving all “good” categorical properties. Isomorphism of categories  $\mathcal{C}$  and  $\mathcal{D}$  requires the existence of functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  such that  $FG = \text{id}_{\mathcal{D}}$  and  $GF = \text{id}_{\mathcal{C}}$ ; but bearing in mind that we can’t really say meaningful things about equality between objects, we may relax the requirement by just asking that  $FG$  is *isomorphic* to  $\text{id}_{\mathcal{D}}$  in the functor category  $\mathcal{D}^{\mathcal{D}}$  (and the same for  $GF$ ); doing this we arrive at the notion of *equivalence of categories*, which is generally regarded as the proper notion of sameness.

So two categories  $\mathcal{C}$  and  $\mathcal{D}$  are *equivalent* if there are functors  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $G : \mathcal{D} \rightarrow \mathcal{C}$  and natural transformations  $\mu : \text{id}_{\mathcal{C}} \Rightarrow GF$  and  $\nu : \text{id}_{\mathcal{D}} \Rightarrow FG$  whose components are all isomorphisms.  $F$  and  $G$  are called *pseudo inverses* of each other. A functor which has a pseudo inverse is also called an *equivalence of categories*.

As a simple example of an equivalence of categories, take a preorder  $P$ . Let  $Q$  be the quotient of  $P$  by the equivalence relation which contains the pair  $(x, y)$  iff both  $x \leq y$  and  $y \leq x$  in  $P$ . Let  $\pi : P \rightarrow Q$  be the quotient map. Regarding  $P$  and  $Q$  as categories,  $\pi$  is a functor, and in fact an equivalence of categories, though not in general an isomorphism.

**Exercise 27** Work this out.

**Exercise 28** Show that a category is equivalent to a discrete category if and only if it is a groupoid and a preorder.

In this section I want to give an important example of an equivalence of categories: the so-called “Lindenbaum-Tarski duality between Set and Complete Atomic Boolean Algebras”. A duality between categories  $\mathcal{C}$  and  $\mathcal{D}$  is an equivalence between  $\mathcal{C}^{\text{op}}$  and  $\mathcal{D}$  (equivalently, between  $\mathcal{C}$  and  $\mathcal{D}^{\text{op}}$ ).

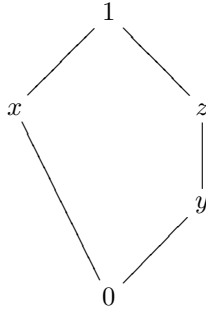
We need some definitions. A *lattice* is a partially ordered set in which every two elements  $x, y$  have a least upper bound (or join)  $x \vee y$  and a greatest lower bound (or meet)  $x \wedge y$ ; moreover, there exist a least element 0 and a greatest element 1.

Such a lattice is called a *Boolean algebra* if every element  $x$  has a *complement*  $\neg x$ , that is, satisfying  $x \vee \neg x = 1$  and  $x \wedge \neg x = 0$ ; and the lattice is *distributive*, which means that  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$  for all  $x, y, z$ .

In a Boolean algebra, complements are unique, for if both  $y$  and  $z$  are complements of  $x$ , then

$$y = y \wedge 1 = y \wedge (x \vee z) = (y \wedge x) \vee (y \wedge z) = 0 \vee (y \wedge z) = y \wedge z$$

so  $y \leq z$ ; similarly,  $z \leq y$  so  $y = z$ . This is a non-example:



It is a lattice, and every element has a complement, but it is not distributive (check!).

A Boolean algebra  $B$  is *complete* if every subset  $A$  of  $B$  has a least upper bound  $\bigvee A$  and a greatest lower bound  $\bigwedge A$ .

An *atom* in a Boolean algebra is an element  $x$  such that  $0 < x$  but for no  $y$  we have  $0 < y < x$ . A Boolean algebra is atomic if every  $x$  is the join of the atoms below it:

$$x = \bigvee \{a \mid a \leq x \text{ and } a \text{ is an atom}\}$$

The category **CABool** is defined as follows: the objects are complete atomic Boolean algebras, and the arrows are complete homomorphisms, that is:  $f : B \rightarrow C$  is a complete homomorphism if for every  $A \subseteq B$ ,

$$f(\bigvee A) = \bigvee \{f(a) \mid a \in A\} \text{ and } f(\bigwedge A) = \bigwedge \{f(a) \mid a \in A\}$$

**Exercise 29** Show that  $1 = \bigwedge \emptyset$  and  $0 = \bigvee \emptyset$ . Conclude that a complete homomorphism preserves 1, 0 and complements.

**Exercise 30** Show that  $\bigwedge A = \neg \bigvee \{\neg a \mid a \in A\}$  and conclude that if a function preserves all  $\bigvee$ 's, 1 and complements, it is a complete homomorphism.

**Theorem 2.4** *The category CABool is equivalent to  $\mathbf{Set}^{\text{op}}$ .*

**Proof.** For every set  $X$ ,  $\mathcal{P}(X)$  is a complete atomic Boolean algebra and if  $f : Y \rightarrow X$  is a function, then  $f^{-1} : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  which takes, for each subset

of  $X$ , its inverse image under  $f$ , is a complete homomorphism. So this defines a functor  $F : \text{Set}^{\text{op}} \rightarrow \text{CABool}$ .

Conversely, given a complete atomic Boolean algebra  $B$ , let  $G(B)$  be the set of atoms of  $B$ . Given a complete homomorphism  $g : B \rightarrow C$  we have a function  $G(g) : G(C) \rightarrow G(B)$  defined by:  $G(g)(c)$  is the unique  $b \in G(B)$  such that  $c \leq g(b)$ . This is well-defined: first, there is an atom  $b$  with  $c \leq g(b)$  because  $1_B = \bigvee G(B)$  ( $B$  is atomic), so  $1_C = g(1_B) = \bigvee \{g(b) | b \text{ is an atom}\}$  and:

**Exercise 31** Prove: if  $c$  is an atom and  $c \leq \bigvee A$ , then there is  $a \in A$  with  $c \leq a$  (hint: prove for all  $a, b$  that  $a \wedge b = 0 \Leftrightarrow a \leq \neg b$ , and prove for  $a, c$  with  $c$  atom:  $c \not\leq a \Leftrightarrow a \leq \neg c$ ).

Secondly, the atom  $b$  is unique since  $c \leq g(b)$  and  $c \leq g(b')$  means  $c \leq g(b) \wedge g(b') = g(b \wedge b') = g(0) = 0$ .

So we have a functor  $G : \text{CABool} \rightarrow \text{Set}^{\text{op}}$ .

Now the atoms of the Boolean algebra  $\mathcal{P}(X)$  are exactly the singleton subsets of  $X$ , so  $GF(X) = \{\{x\} | x \in X\}$  which is clearly isomorphic to  $X$ . On the other hand,  $FG(B) = \mathcal{P}(\{b \in B | b \text{ is an atom}\})$ . There is a map from  $FG(B)$  to  $B$  which sends each set of atoms to its least upper bound in  $B$ , and this map is an isomorphism in  $\text{CABool}$ . ■

**Exercise 32** Prove the last statement: that the map from  $FG(B)$  to  $B$ , defined in the last paragraph of the proof, is an isomorphism.

**Exercise 33** Prove that  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an equivalence of categories if and only if  $F$  is full and faithful, and *essentially surjective on objects*, that means: for any  $D \in \mathcal{D}_0$  there is  $C \in \mathcal{C}_0$  such that  $F(C)$  is isomorphic to  $D$ .

## 3 (Co)cones and (Co)limits

### 3.1 Limits

Given a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , a *cone* for  $F$  consists of an object  $D$  of  $\mathcal{D}$  together with a natural transformation  $\mu : \Delta_D \Rightarrow F$  ( $\Delta_D$  is the constant functor with value  $D$ ). In other words, we have a family  $(\mu_C : D \rightarrow F(C) | C \in \mathcal{C}_0)$ , and the naturality requirement in this case means that for every arrow  $f : C' \rightarrow C$  in  $\mathcal{C}$ ,

$$\begin{array}{ccc} & D & \\ \mu_C \swarrow & & \searrow \mu_{C'} \\ F(C) & \xrightarrow{F(f)} & F(C') \end{array}$$

commutes in  $\mathcal{D}$  (this diagram explains, I hope, the name “cone”). Let us denote the cone by  $(D, \mu)$ .  $D$  is called the *vertex* of the cone.

A *map of cones*  $(D, \mu) \rightarrow (D', \mu')$  is a map  $g : D \rightarrow D'$  such that  $\mu'_C g = \mu_C$  for all  $C \in \mathcal{C}_0$ .

Clearly, there is a category  $\text{Cone}(F)$  which has as objects the cones for  $F$  and as morphisms maps of cones.

A *limiting cone* for  $F$  is a terminal object in  $\text{Cone}(F)$ . Since terminal objects are unique up to isomorphism, as we have seen, any two limiting cones are isomorphic in  $\text{Cone}(F)$  and in particular, their vertices are isomorphic in  $\mathcal{D}$ .

A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is also called a *diagram* in  $\mathcal{D}$  of *type*  $\mathcal{C}$ , and  $\mathcal{C}$  is the *index category* of the diagram.

Let us see what it means to be a limiting cone, in some simple, important cases.

- i) A limiting cone for the unique functor  $! : \mathbf{0} \rightarrow \mathcal{D}$  ( $\mathbf{0}$  is the empty category) “is” a terminal object in  $\mathcal{D}$ . For every object  $D$  of  $\mathcal{D}$  determines, together with the empty family, a cone for  $!$ , and a map of cones is just an arrow in  $\mathcal{D}$ . So  $\text{Cone}(!)$  is isomorphic to  $\mathcal{D}$ .
- ii) Let  $\mathbf{2}$  be the discrete category with two objects  $x, y$ . A functor  $\mathbf{2} \rightarrow \mathcal{D}$  is just a pair  $\langle A, B \rangle$  of objects of  $\mathcal{D}$ , and a cone for this functor consists of

an object  $C$  of  $\mathcal{D}$  and two maps

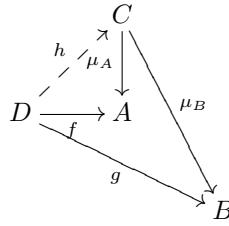
$$\begin{array}{ccc} C & \xrightarrow{\mu_A} & A \\ & \searrow \mu_B & \\ & & B \end{array}$$

since there are no nontrivial

arrows in  $\mathbf{2}$ .

$(C, (\mu_A, \mu_B))$  is a limiting cone for  $\langle A, B \rangle$  iff the following holds: for any object  $D$  and arrows  $f : D \rightarrow A$ ,  $g : D \rightarrow B$ , there is a unique arrow

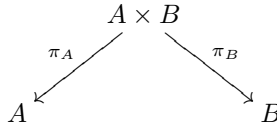
$h : D \rightarrow C$  such that



commutes. In other words, there is, for any  $D$ , a 1-1 correspondence

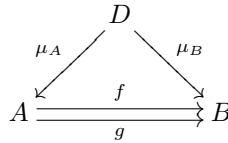
between maps  $D \rightarrow C$  and pairs of maps  $D \rightarrow A$  and  $D \rightarrow B$ . This is

the property of a *product*; a limiting cone for  $\langle A, B \rangle$  is therefore called a product cone, and usually denoted:



The arrows  $\pi_A$  and  $\pi_B$  are called *projections*.

- iii) Let  $\hat{\mathbf{2}}$  denote the category  $x \xrightarrow[a]{a} y$ . A functor  $\hat{\mathbf{2}} \rightarrow \mathcal{D}$  is the same thing as a parallel pair of arrows  $A \xrightarrow[f]{g} B$  in  $\mathcal{D}$ ; I write  $\langle f, g \rangle$  for this functor. A cone for  $\langle f, g \rangle$  is:



But  $\mu_B = f\mu_A = g\mu_A$  is already defined from  $\mu_A$ , so giving a cone is the same as giving a map  $\mu_A : D \rightarrow A$  such that  $f\mu_A = g\mu_A$ . Such a cone is limiting iff for any other map  $h : C \rightarrow A$  with  $fh = gh$ , there is a unique  $k : C \rightarrow D$  such that  $h = \mu_A k$ .

We call  $\mu_A$ , if it is limiting, an *equalizer* of the pair  $f, g$ , and the diagram

$$D \xrightarrow{\mu_A} A \xrightarrow[f]{g} B \text{ an equalizer diagram.}$$

In Sets, an equalizer of  $f, g$  is isomorphic (as a cone) to the inclusion of  $\{a \in A | f(a) = g(a)\}$  into  $A$ . In categorical interpretations of logical systems (see chapters 4 and 7), equalizers are used to interpret *equality between terms*.

**Exercise 34** Show that every equalizer is a monomorphism.

**Exercise 35** If  $E \xrightarrow{e} X \rightrightarrows^f_g Y$  is an equalizer diagram, show that  $e$  is an isomorphism if and only if  $f = g$ .

**Exercise 36** Show that in  $\mathbf{Set}$ , every monomorphism fits into an equalizer diagram.

iv) Let  $J$  denote the category

$$\begin{array}{ccc} & y & \\ & \downarrow b & \\ x & \xrightarrow{a} & z \end{array}$$

A functor  $F : J \rightarrow \mathcal{D}$  is specified

by giving two arrows in  $\mathcal{D}$  with the same codomain, say  $f : X \rightarrow Z$ ,  $g : Y \rightarrow Z$ . A limit for such a function is given by an object  $W$  together

with projections

$$\begin{array}{ccc} W & \xrightarrow{p_Y} & Y \\ p_X \downarrow & & \\ X & & \end{array}$$

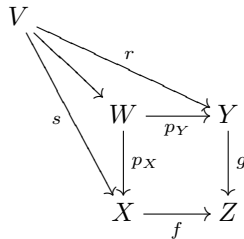
satisfying  $fp_X = gp_Y$ , and such that,

given any other pair of arrows:

$$\begin{array}{ccc} V & \xrightarrow{r} & Y \\ s \downarrow & & \\ X & & \end{array}$$

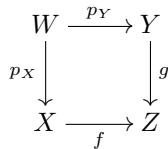
with  $gr = fs$ , there is a

unique arrow  $V \rightarrow W$  such that



commutes.

The diagram



is called a *pullback diagram*. In  $\mathbf{Set}$ , the pullback cone for  $f, g$  is isomorphic to

$$\{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

with the obvious projections.

We say that a category  $\mathcal{D}$  has *binary products* (equalizers, pullbacks) iff every functor  $\mathbf{2} \rightarrow \mathcal{D}$  ( $\hat{\mathbf{2}} \rightarrow \mathcal{D}$ ,  $J \rightarrow \mathcal{D}$ , respectively) has a limiting cone. Some dependencies hold in this context:

**Proposition 3.1** *If a category  $\mathcal{D}$  has a terminal object and pullbacks, it has binary products and equalizers.*

*If  $\mathcal{D}$  has binary products and equalizers, it has pullbacks.*

**Proof.** Let  $1$  be the terminal object in  $\mathcal{D}$ ; given objects  $X$  and  $Y$ , if

$$\begin{array}{ccc} C & \xrightarrow{p_X} & X \\ p_Y \downarrow & & \downarrow \\ Y & \longrightarrow & 1 \end{array}$$

is a pullback diagram, then

$$\begin{array}{ccc} C & \xrightarrow{p_X} & X \\ p_Y \downarrow & & \\ Y & & \end{array}$$

is a product cone.

Given a product cone

$$\begin{array}{ccc} A \times B & \xrightarrow{\pi_A} & A \\ \pi_B \downarrow & & \\ B & & \end{array}$$

and maps

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ g \downarrow & & \\ B & & \end{array}$$

we write

$X \xrightarrow{\langle f, g \rangle} A \times B$  for the unique factorization through the product. Write also  $\delta : Y \rightarrow Y \times Y$  for  $\langle \text{id}_Y, \text{id}_Y \rangle$ .

Now given  $f, g : X \rightarrow Y$ , if

$$\begin{array}{ccc} E & \xrightarrow{e} & X \\ \downarrow & & \downarrow \langle f, g \rangle \\ Y & \xrightarrow{\delta} & Y \times Y \end{array}$$

is a pullback diagram, then  $E \xrightarrow{e} X \xrightarrow[g]{f} Y$  is an equalizer diagram. This proves the first statement.

As for the second: given

$$\begin{array}{ccc} & X & \\ & \downarrow f & \\ Y & \xrightarrow{g} & Z \end{array}$$

let  $E \xrightarrow{e} X \times Y \xrightarrow[g\pi_Y]{f\pi_X} Z$  be an equalizer; then

$$\begin{array}{ccc} E & \xrightarrow{\pi_X e} & X \\ \pi_Y e \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

is a pullback diagram. ■

**Exercise 37** Let

$$\begin{array}{ccc} A & \xrightarrow{b} & B \\ a \downarrow & & \downarrow f \\ X & \xrightarrow{g} & Y \end{array}$$

a pullback diagram with  $f$  mono. Show that  $a$  is also mono. Also, if  $f$  is iso (an isomorphism), so is  $a$ .

**Exercise 38** Given two commuting squares:

$$\begin{array}{ccccc} A & \xrightarrow{b} & B & \xrightarrow{c} & C \\ a \downarrow & & \downarrow f & & \downarrow d \\ X & \xrightarrow{g} & Y & \xrightarrow{h} & Z \end{array}$$

- a) if both squares are pullback squares, then so is the composite square

$$\begin{array}{ccc} A & \xrightarrow{cb} & C \\ a \downarrow & & \downarrow d \\ X & \xrightarrow{hg} & Z \end{array}$$

- b) If the right hand square and the composite square are pullbacks, then so is the left hand square.

**Exercise 39**  $f : A \rightarrow B$  is a monomorphism if and only if

$$\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ \text{id}_A \downarrow & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback diagram.

A monomorphism  $f : A \rightarrow B$  which fits into an equalizer diagram

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{g} \\ \xleftarrow{h} \end{array} C$$

is called a *regular mono*.



**Exercise 40** If

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ b \downarrow & & \downarrow g \\ B & \xrightarrow{f} & Y \end{array}$$

is a pullback and  $g$  is regular mono, so is  $b$ .

**Exercise 41** If  $f$  is regular mono and epi,  $f$  is iso. Every split mono is regular.

**Exercise 42** Give an example of a category in which not every mono is regular.

**Exercise 43** In  $\mathbf{Grp}$ , every mono is regular [This is not so easy].

**Exercise 44** Characterize the regular monos in  $\mathbf{Pos}$ .

**Exercise 45** If a category  $\mathcal{D}$  has binary products and a terminal object, it has *all finite products*, i.e. limiting cones for every functor into  $\mathcal{D}$  from a finite discrete category.

**Exercise 46** Suppose  $\mathcal{C}$  has binary products and suppose for every ordered pair

$$(A, B) \text{ of objects of } \mathcal{C} \text{ a product cone } \begin{array}{ccc} A \times B & \xrightarrow{\pi_A} & A \\ \pi_B \downarrow & & \\ B & & \end{array} \text{ has been chosen.}$$

- Show that there is a functor:  $\mathcal{C} \times \mathcal{C} \xrightarrow{- \times -} \mathcal{C}$  (the product functor) which sends each pair  $(A, B)$  of objects to  $A \times B$  and each pair of arrows  $(f : A \rightarrow A', g : B \rightarrow B')$  to  $f \times g = \langle f\pi_A, g\pi_B \rangle$ .
- From a), there are functors:

$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \begin{array}{c} \xrightarrow{(- \times -) \times -} \\ \xrightarrow{- \times (- \times -)} \end{array} \mathcal{C}$$

sending  $(A, B, C)$  to  $\begin{array}{c} (A \times B) \times C \\ A \times (B \times C) \end{array}$  Show that there is a natural transformation  $a = (a_{A,B,C} | A, B, C \in \mathcal{C}_0)$  from  $(- \times -) \times -$  to  $- \times (- \times -)$

such that for any four objects  $A, B, C, D$  of  $\mathcal{C}$ :

$$\begin{array}{ccc}
 ((A \times B) \times C) \times D & \xrightarrow{a_{A \times B, C, D}} & (A \times B) \times (C \times D) \\
 a_{A, B, C} \times \text{id}_D \downarrow & & \downarrow a_{A, B, C \times D} \\
 (A \times (B \times C)) \times D & & A \times (B \times (C \times D)) \\
 & \searrow a_{A, B \times C, D} \quad \nearrow \text{id}_A \times a_{B, C, D} & \\
 & A \times ((B \times C) \times D) &
 \end{array}$$

commutes (This diagram is called “MacLane’s pentagon”).

A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is said to *preserve limits of type  $\mathcal{E}$*  if for all functors  $M : \mathcal{E} \rightarrow \mathcal{C}$ , if  $(D, \mu)$  is a limiting cone for  $M$  in  $\mathcal{C}$ , then  $(FD, F\mu = (F(\mu_E)|E \in \mathcal{E}_0))$  is a limiting cone for  $FM$  in  $\mathcal{D}$ .

So, a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  preserves binary products if for every product dia-

$$\begin{array}{ccccc}
 A \times B & \xrightarrow{\pi_B} & B & & F(A \times B) \xrightarrow{F(\pi_B)} F(B) \\
 \text{diagram } \pi_A \downarrow & & \text{its } F\text{-image} & F(\pi_A) \downarrow & \text{is again a product} \\
 A & & & F(A) &
 \end{array}$$

gram. Similarly for equalizers and pullbacks.

Some more terminology:  $F$  is said to *preserve all finite limits* if it preserves limits of type  $\mathcal{E}$  for every finite  $\mathcal{E}$ . A category which has all finite limits is called *lex* (left exact), *cartesian* or *finitely complete*.

**Exercise 47** If a category  $\mathcal{C}$  has equalizers, it has *all finite equalizers*: for every category  $\mathcal{E}$  of the form

$$\begin{array}{ccc}
 & \xrightarrow{f_1} & \\
 X & \begin{array}{c} \vdots \\ \vdots \end{array} & Y \\
 & \xrightarrow{f_n} &
 \end{array}$$

every functor  $\mathcal{E} \rightarrow \mathcal{C}$  has a limiting cone.

**Exercise 48** Suppose  $F : \mathcal{C} \rightarrow \mathcal{D}$  preserves equalizers (and  $\mathcal{C}$  has equalizers) and reflects isomorphisms. Then  $F$  is faithful.

**Exercise 49** Let  $\mathcal{C}$  be a category with finite limits. Show that for every object  $C$  of  $\mathcal{C}$ , the slice category  $\mathcal{C}/C$  (example j) of 1.1) has binary products: the vertex of a product diagram for two objects  $D \rightarrow C$ ,  $D' \rightarrow C$  is  $D'' \rightarrow C$  where

$$\begin{array}{ccc}
 D'' & \longrightarrow & D \\
 \downarrow & & \downarrow \\
 D' & \longrightarrow & C
 \end{array}$$

is a pullback square in  $\mathcal{C}$ .

## 3.2 Limits by products and equalizers

In  $\mathbf{Set}$ , every small diagram has a limit; given a functor  $F : \mathcal{E} \rightarrow \mathbf{Set}$  with  $\mathcal{E}$  small, there is a limiting cone for  $F$  in  $\mathbf{Set}$  with vertex

$$\{(x_E)_{E \in \mathcal{E}_0} \in \prod_{E \in \mathcal{E}_0} F(E) \mid \forall E \xrightarrow{f} E' \in \mathcal{E}_1 (F(f)(x_E) = x_{E'})\}$$

So in  $\mathbf{Set}$ , limits are equationally defined subsets of suitable products. This holds in any category:

**Proposition 3.2** *Suppose  $\mathcal{C}$  has all small products (including the empty product, i.e. a terminal object 1) and equalizers; then  $\mathcal{C}$  has all small limits.*

**Proof.** Given a set  $I$  and an  $I$ -indexed family of objects  $(A_i \mid i \in I)$  of  $\mathcal{C}$ , we denote the product by  $\prod_{i \in I} A_i$  and projections by  $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$ ; an arrow  $f : X \rightarrow \prod_{i \in I} A_i$  which is determined by the compositions  $f_i = \pi_i f : X \rightarrow A_i$ , is also denoted  $(f_i \mid i \in I)$ .

Now given  $\mathcal{E} \rightarrow \mathcal{C}$  with  $\mathcal{E}_0$  and  $\mathcal{E}_1$  sets, we construct

$$E \xrightarrow{e} \prod_{i \in \mathcal{E}_0} F(i) \xrightleftharpoons[(F(u)\pi_{\text{dom}(u)} \mid u \in \mathcal{E}_1)]{(\pi_{\text{cod}(u)} \mid u \in \mathcal{E}_1)} \prod_{u \in \mathcal{E}_1} F(\text{cod}(u))$$

in  $\mathcal{C}$  as an equalizer diagram. The family  $(\mu_i = \pi_i e : E \rightarrow F(i) \mid i \in \mathcal{E}_0)$  is a natural transformation  $\Delta_E \Rightarrow F$  because, given an arrow  $u \in \mathcal{E}_1$ , say  $u : i \rightarrow j$ , we have that

$$\begin{array}{ccc} & E & \\ \pi_i e \swarrow & & \searrow \pi_j e \\ F(i) & \xrightarrow{F(u)} & F(j) \end{array}$$

commutes since  $F(u)\pi_i e = F(u)\pi_{\text{dom}(u)} e = \pi_{\text{cod}(u)} e = \pi_j e$ .

So  $(E, \mu)$  is a cone for  $F$ , but every other cone  $(D, \nu)$  for  $F$  gives a map  $d : D \rightarrow \prod_{i \in \mathcal{E}_0} F(i)$  equalizing the two horizontal arrows; so factors uniquely through  $E$ . ■

**Exercise 50** Check that “small” in the statement of the proposition, can be replaced by “finite”: if  $\mathcal{C}$  has all finite products and equalizers,  $\mathcal{C}$  is finitely complete.

**Exercise 51** Show that if  $\mathcal{C}$  is complete, then  $F : \mathcal{C} \rightarrow \mathcal{D}$  preserves all limits if  $F$  preserves products and equalizers. This no longer holds if  $\mathcal{C}$  is not complete! That is,  $F$  may preserve all products and equalizers which exist in  $\mathcal{C}$ , yet not preserve all limits which exist in  $\mathcal{C}$ .

### 3.3 Complete Categories

A category is called *complete* if it has limits of type  $\mathcal{E}$  for all small  $\mathcal{E}$ .

In general, limits over large (i.e. not small) diagrams need not exist. For example in  $\mathbf{Set}$ , there is a limiting cone for the identity functor  $\mathbf{Set} \rightarrow \mathbf{Set}$  (its vertex is the empty set), but not for the constant functor  $\Delta_A : \mathcal{C} \rightarrow \mathbf{Set}$  if  $\mathcal{C}$  is a large discrete category and  $A$  has more than one element.

The categories  $\mathbf{Set}$ ,  $\mathbf{Top}$ ,  $\mathbf{Pos}$ ,  $\mathbf{Mon}$ ,  $\mathbf{Grp}$ ,  $\mathbf{Grph}$ ,  $\mathbf{Rng}$ , ... are all complete. For instance in  $\mathbf{Top}$ , the product of a set  $(X_i \mid i \in I)$  of topological spaces is the set  $\prod_{i \in I} X_i$  with the product topology; the equalizer of two continuous maps

$X \xrightleftharpoons[g]{f} Y$  is the inclusion  $X' \subseteq X$  where  $X' = \{x \in X \mid f(x) = g(x)\}$  with the subspace topology from  $X$ .

**Exercise 52** What are monomorphisms in  $\mathbf{Top}$ ? Is every mono regular in  $\mathbf{Top}$ ?

The category  $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$  is also complete, and limits are “computed pointwise”. That is, let  $F : \mathcal{D} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$  be a diagram in  $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ . For every  $C \in \mathcal{C}_0$  there is a functor  $F_C : \mathcal{D} \rightarrow \mathbf{Set}$ , given by  $F_C(D) = F(D)(C)$  and for  $f : D \rightarrow D'$  in  $\mathcal{D}$ ,  $F_C(f) = F(f)_C : F(D)(C) \rightarrow F(D')(C)$ .

Since  $\mathbf{Set}$  is complete, every  $F_C$  has a limiting cone  $(X_C, \mu_C)$  in  $\mathbf{Set}$ . Now if  $C' \xrightarrow{g} C$  is a morphism in  $\mathcal{C}$ , the collection of arrows

$$\{X_C \xrightarrow{(\mu_C)^D} F(D)(C) \xrightarrow{F(D)(g)} F(D)(C') = F_{C'}(D) \mid D \in \mathcal{D}_0\}$$

is a cone for  $F_{C'}$  with vertex  $X_C$ , since for any  $f : D \rightarrow D'$  we have  $F(f)_{C'} \circ F(D)(g) \circ (\mu_C)_D = F(D')(g) \circ F(f)_C \circ (\mu_C)_D$  (by naturality of  $F(f)$ )  $= F(D')(g) \circ (\mu_C)_{D'}$  (because  $(X_C, \mu_C)$  is a cone).

Because  $(X_{C'}, \mu_{C'})$  is a limiting cone for  $F_{C'}$ , there is a unique arrow  $X_g : X_C \rightarrow X_{C'}$  in  $\mathbf{Set}$  such that  $F(D)(g) \circ (\mu_C)_D = (\mu_{C'})_{D'} \circ X_g$  for all  $D \in \mathcal{D}_0$ . By the uniqueness of these arrows, we have an object  $X$  of  $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ , and arrows  $\nu_D : X \rightarrow F(D)$  for all  $D \in \mathcal{D}_0$ , and the pair  $(X, \nu)$  is a limiting cone for  $F$  in  $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ .

**Exercise 53** Check the remaining details.

It is a consequence of the Yoneda lemma that the Yoneda embedding  $Y : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$  preserves all limits which exist in  $\mathcal{C}$ . For, let  $F : \mathcal{D} \rightarrow \mathcal{C}$  be a diagram with limiting cone  $(E, \nu)$  and let  $(X, \delta)$  be a limiting cone for the composition  $Y \circ F : \mathcal{D} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ . By the Yoneda lemma,  $X(C)$  is in natural 1-1 correspondence with the set of arrows  $Y(C) \rightarrow X$  in  $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ ; which by the fact that  $(X, \delta)$  is a limiting cone, is in natural 1-1 correspondence with cones  $(Y(C), \mu)$  for  $Y \circ F$  with vertex  $Y(C)$ ; since  $Y$  is full and faithful every such cone comes from a unique cone  $(C, \mu')$  for  $F$  in  $\mathcal{C}$ , hence from a unique map  $C \rightarrow E$  in  $\mathcal{C}$ .

So,  $X(C)$  is naturally isomorphic to  $\mathcal{C}(C, E)$  whence  $X$  is isomorphic to  $Y(E)$ , by an isomorphism which transforms  $\delta$  into  $Y \circ \nu = (Y(\nu_C) \mid C \in \mathcal{C}_0)$ .

To finish this section a little theorem by Peter Freyd which says that every small, complete category is a complete preorder:

**Proposition 3.3** *Suppose  $\mathcal{C}$  is small and complete. Then  $\mathcal{C}$  is a preorder.*

**Proof.** If not, there are objects  $A, B$  in  $\mathcal{C}$  such that there are two distinct maps  $f, g : A \rightarrow B$ . Since  $\mathcal{C}_1$  is a set and  $\mathcal{C}$  complete, the product  $\prod_{h \in \mathcal{C}_1} B$  exists. Arrows  $k : A \rightarrow \prod_{h \in \mathcal{C}_1} B$  are in 1-1 correspondence with families of arrows  $(k_h : A \rightarrow B \mid h \in \mathcal{C}_1)$ . For every subset  $X \subseteq \mathcal{C}_1$  define such a family by:

$$k_h = \begin{cases} f & \text{if } h \in X \\ g & \text{else} \end{cases}$$

This gives an injective function from  $2^{\mathcal{C}_1}$  into  $\mathcal{C}(A, \prod_{h \in \mathcal{C}_1} B)$  hence into  $\mathcal{C}_1$ , contradicting Cantor's theorem in set theory. ■

### 3.4 Colimits

The dual notion of limit is colimit. Given a functor  $F : \mathcal{E} \rightarrow \mathcal{C}$  there is clearly a functor  $F^{\text{op}} : \mathcal{E}^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$  which does “the same” as  $F$ . We say that a *colimiting cocone* for  $F$  is a limiting cone for  $F^{\text{op}}$ .

So: a cocone for  $F : \mathcal{E} \rightarrow \mathcal{C}$  is a pair  $(\nu, D)$  where  $\nu : F \Rightarrow \Delta_D$  and a colimiting cocone is an initial object in the category  $\text{Cocone}(F)$ .

#### Examples

- i) a colimiting cocone for  $! : \mathbf{0} \rightarrow \mathcal{C}$  “is” an initial object of  $\mathcal{C}$
- ii) a colimiting cocone for  $\langle A, B \rangle : \mathbf{2} \rightarrow \mathcal{C}$  is a *coproduct* of  $A$  and  $B$  in  $\mathcal{C}$ : usually denoted  $A + B$  or  $A \sqcup B$ ; there are *coprojections* or *coproduct inclusions*

$$\begin{array}{ccc} A & & \\ & \searrow \nu_A & \\ B & \xrightarrow{\nu_B} & A \sqcup B \end{array}$$

with the property that, given any pair of arrows  $A \xrightarrow{f} C, B \xrightarrow{g} C$  there is a unique map  $\begin{bmatrix} f \\ g \end{bmatrix} : A \sqcup B \rightarrow C$  such that  $f = \begin{bmatrix} f \\ g \end{bmatrix} \nu_A$  and  $g = \begin{bmatrix} f \\ g \end{bmatrix} \nu_B$

- iii) a colimiting cocone for  $A \xrightarrow{f} B$  (as functor  $\hat{\mathbf{2}} \rightarrow \mathcal{C}$ ) is given by a map  $B \xrightarrow{c} C$  satisfying  $cf = cg$ , and such that for any  $B \xrightarrow{h} D$  with  $hf = hg$  there is a unique  $C \xrightarrow{h'} D$  with  $h = h'c$ .  $c$  is called a *coequalizer* for  $f$  and  $g$ ; the diagram  $A \rightrightarrows B \longrightarrow C$  a coequalizer diagram.

**Exercise 54** Is the terminology “coproduct inclusions” correct? That is, it suggests they are monos. Is this always the case?

Formulate a condition on  $A$  and  $B$  which implies that  $\nu_A$  and  $\nu_B$  are monic.

In  $\mathbf{Set}$ , the coproduct of  $X$  and  $Y$  is the disjoint union  $(\{0\} \times X) \cup (\{1\} \times Y)$  of  $X$  and  $Y$ . The coequalizer of  $X \rightrightarrows Y$  is the quotient map  $Y \rightarrow Y/\sim$  where  $\sim$  is the equivalence relation generated by

$$y \sim y' \text{ iff there is } x \in X \text{ with } f(x) = y \text{ and } g(x) = y'$$

The dual notion of pullback is *pushout*. A pushout diagram is a colimiting cocone for a functor  $\Gamma \rightarrow \mathcal{C}$  where  $\Gamma$  is the category

$$\begin{array}{ccc} & x & \longrightarrow y \\ & \downarrow & \\ & z & \end{array}$$

is a square

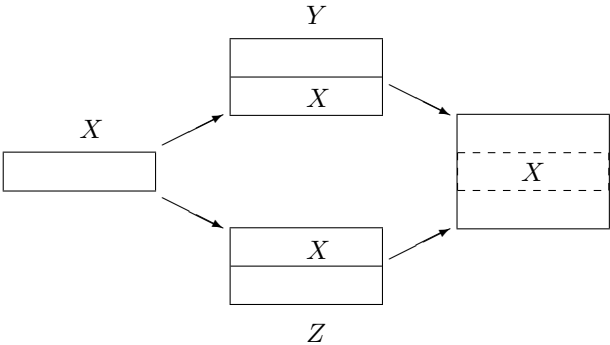
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow a \\ Z & \xrightarrow{b} & P \end{array}$$

which commutes and such that, given

$$\begin{array}{ccc} & Y & \\ & \searrow \alpha & \\ Z & \xrightarrow{\beta} & Q \end{array}$$

with  $\alpha f = \beta g$ , there is a

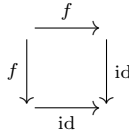
unique  $P \xrightarrow{p} Q$  with  $\alpha = pa$  and  $\beta = pb$ . In  $\mathbf{Set}$ , the pushout of  $X \xrightarrow{f} Y$  and  $X \xrightarrow{g} Z$  is the coproduct  $Y \sqcup Z$  where the two images of  $X$  are identified:



**Exercise 55** Give yourself, in terms of  $X \xrightarrow{f} Y$  and  $X \xrightarrow{g} Z$ , a formal definition of a relation  $R$  on  $Y \sqcup Z$  such that the pushout of  $f$  and  $g$  is  $Y \sqcup Z/\sim$ ,  $\sim$  being the equivalence relation generated by  $R$ .

One can now dualize every result and exercise from the section on limits:

**Exercise 56**  $f$  is epi if and only if



is a pushout diagram.

**Exercise 57** Every coequalizer is an epimorphism; if  $e$  is a coequalizer of  $f$  and  $g$ , then  $e$  is iso iff  $f = g$

**Exercise 58** If  $\mathcal{C}$  has an initial object and pushouts,  $\mathcal{C}$  has binary coproducts and coequalizers; if  $\mathcal{C}$  has binary coproducts and coequalizers,  $\mathcal{C}$  has pushouts.

**Exercise 59** If  $\begin{array}{ccc} & \xrightarrow{\quad} & \\ a \downarrow & & \downarrow f \\ & \xrightarrow{\quad} & \end{array}$  is a pushout diagram, then  $a$  epi implies  $f$  epi, and  $a$  regular epi (i.e. a coequalizer) implies  $f$  regular epi.

**Exercise 60** The composition of two pushout squares is a pushout; if both the first square and the composition are pushouts, the second square is.

**Exercise 61** If  $\mathcal{C}$  has all small (finite) coproducts and coequalizers,  $\mathcal{C}$  has all small (finite) colimits.

Some miscellaneous exercises:

**Exercise 62** Call an arrow  $f$  a *stably regular epi* if whenever  $\begin{array}{ccc} & \xrightarrow{\quad} & \\ a \downarrow & & \downarrow f \\ & \xrightarrow{\quad} & \end{array}$  is a pullback diagram, the arrow  $a$  is a regular epi. Show: in  $\text{Pos}$ ,  $X \xrightarrow{f} Y$  is a stably regular epi if and only if for all  $y, y'$  in  $Y$ :

$$y \leq y' \Leftrightarrow \exists x \in f^{-1}(y) \exists x' \in f^{-1}(y'). x \leq x'$$

Show by an example that not every epi is stably regular in  $\text{Pos}$ .

**Exercise 63** In  $\text{Grp}$ , every epi is regular.

**Exercise 64** Characterize coproducts in  $\text{Abgrp}$ .

## 4 A little piece of categorical logic

One of the major achievements of category theory in mathematical logic and in computer science, has been a unified treatment of semantics for all kinds of logical systems and term calculi which are the basis for programming languages.

One can say that mathematical logic, seen as the study of classical first order logic, first started to be a real subject with the discovery, by Gödel, of the *completeness theorem* for set-theoretic interpretations: a sentence  $\varphi$  is provable if and only if  $\varphi$  is true in all possible interpretations. This unites the two approaches to logic: proof theory and model theory, makes logic accessible for mathematical methods and enables one to give nice and elegant proofs of proof theoretical properties by model theory (for example, the Beth and Craig definability and interpolation theorems).

However the completeness theorem needs generalization once one considers logics, such as intuitionistic logic (which does not admit the principle of excluded middle), minimal logic (which has no negation) or modal logic (where the logic has an extra operator, expressing “necessarily true”), for which the set-theoretic interpretation is not complete. One therefore comes with a general definition of “interpretation” in a category  $\mathcal{C}$  of a logical system, which generalizes Tarski’s truth definition: this will then be the special case of classical logic and the category  $\mathbf{Set}$ .

In this chapter I treat, for reasons of space, only a fragment of first order logic: regular logic. On this fragment the valid statements of classical and intuitionistic logic coincide.

For an interpretation of a term calculus like the  $\lambda$ -calculus, which is of paramount importance in theoretical computer science, the reader is referred to chapter 7.

### 4.1 Regular categories and subobjects

**Definition 4.1** *A category  $\mathcal{C}$  is called regular if the following conditions hold:*

- a)  *$\mathcal{C}$  has all finite limits;*
- b) *For every arrow  $f$ , if*

$$\begin{array}{ccc} Z & \xrightarrow{p_0} & X \\ p_1 \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

*is a pullback (then  $Z \rightrightarrows X$  is called the kernel pair of  $f$ ), the coequalizer of  $p_0, p_1$  exists;*

- c) *Regular epimorphisms (coequalizers) are stable under pullback, that is: in*



a pullback square

$$\begin{array}{ccc} & \xrightarrow{\quad} & \\ a \downarrow & & \downarrow f \\ & \xrightarrow{\quad} & \end{array}$$

if  $f$  is regular epi, so is  $a$ .

**Examples.** In **Set**, as in **Grp**, **Top**, etc., the (underlying) set which is the vertex of the kernel pair of  $f : X \rightarrow Y$  is  $X_f = \{(x, x') \mid f(x) = f(x')\}$ . The coequalizer of  $X_f \xrightarrow[\pi_2]{\pi_1} X$  is (up to isomorphism) the map  $X \rightarrow \text{Im}(f)$  where  $\text{Im}(f)$  is the set-theoretic image of  $f$  as subset of  $Y$ .

These coequalizers exist in **Set**, **Top**, **Grp**, **Pos**. . . Moreover, in **Set** and **Grp** every epi is regular, and (since epis in **Set** and **Grp** are just surjective functions) stable under pullback; hence **Set** and **Grp** are examples of regular categories.

**Top** is not regular! It satisfies the first two requirements of the definition, but not the third. One can prove that the functor  $(-) \times S : \mathbf{Top} \rightarrow \mathbf{Top}$  preserves all quotient maps only if the space  $S$  is locally compact. Since every coequalizer is a quotient map, if  $S$  is not locally compact there will be pullbacks of form

$$\begin{array}{ccc} X \times S & \xrightarrow{f \times \text{id}_S} & Y \times S \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

with  $f$  regular epi, but  $f \times \text{id}_S$  not.

**Exercise 65** This exercise shows that **Pos** is not regular either. Let  $X$  and  $Y$  be the partial orders  $\{x \leq y, y' \leq z\}$  and  $\{a \leq b \leq c\}$  respectively.

- Prove that  $f(x) = a, f(y) = f(y') = b, f(z) = c$  defines a regular epimorphism:  $X \rightarrow Y$ .
- Let  $Z$  be  $\{a \leq c\} \subset Y$  and  $W = f^{-1}(Z) \subset X$ . Then

$$\begin{array}{ccc} W & \longrightarrow & Z \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback, but  $W \rightarrow Z$  is not the coequalizer of anything.

**Proposition 4.2** *In a regular category, every arrow  $f : X \rightarrow Y$  can be factored as  $f = me : X \xrightarrow{e} E \xrightarrow{m} Y$  where  $e$  is regular epi and  $m$  is mono; and this factorization is unique in the sense that if  $f$  is also  $m'e' : X \xrightarrow{e'} E' \xrightarrow{m'} Y$  with  $m'$  mono and  $e'$  regular epi, there is an isomorphism  $\sigma : E \rightarrow E'$  such that  $\sigma e = e'$  and  $m'\sigma = m$ .*

**Proof.** Given  $f : X \rightarrow Y$  we let  $X \xrightarrow{e} E$  be the coequalizer of the kernel pair  $Z \xrightleftharpoons[p_1]{p_0} X$  of  $f$ . Since  $fp_0 = fp_1$  there is a unique  $m : E \rightarrow Y$  such that  $f = me$ . By construction  $e$  is regular epi; we must show that  $m$  is mono, and the uniqueness of the factorization.

Suppose  $mg = mh$  for  $g, h : W \rightarrow E$ ; we prove that  $g = h$ . Let

$$\begin{array}{ccc} V & \xrightarrow{a} & W \\ \langle q_0, q_1 \rangle \downarrow & & \downarrow \langle g, h \rangle \\ X \times X & \xrightarrow{e \times e} & E \times E \end{array}$$

be a pullback square. Then

$$fq_0 = meq_0 = mga = mha = meq_1 = fq_1$$

so there is a unique arrow  $V \xrightarrow{b} Z$  such that  $\langle q_0, q_1 \rangle = \langle p_0, p_1 \rangle b : V \rightarrow X \times X$  (because of the kernel pair property). It follows that

$$ga = eq_0 = ep_0b = ep_1b = eq_1 = ha$$

I claim that  $a$  is epi, so it follows that  $g = h$ . It is here that we use the requirement that regular epis are stable under pullback. Now  $e \times e : X \times X \rightarrow E \times E$  is the composite

$$X \times X \xrightarrow{e \times \text{id}_X} E \times X \xrightarrow{\text{id}_E \times e} E \times E$$

and both maps are regular epis since both squares

$$\begin{array}{ccc} X \times X & \xrightarrow{e \times \text{id}_X} & E \times X \\ \pi_0 \downarrow & & \downarrow \pi_0 \\ X & \xrightarrow{e} & E \end{array} \quad \text{and} \quad \begin{array}{ccc} E \times X & \xrightarrow{\text{id}_E \times e} & E \times E \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ X & \xrightarrow{e} & E \end{array}$$

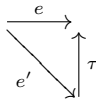
are pullbacks. The map  $a$ , being the pullback of a composite of regular epis, is then itself the composite of regular epis (check this!), so in particular epi.

This proves that  $m$  is mono, and we have our factorization.

As to uniqueness, suppose we had another factorization  $f = m'e'$  with  $m'$  mono and  $e'$  regular epi. Then  $m'e'p_0 = fp_0 = fp_1 = m'e'p_1$  so since  $m'$  mono,  $e'p_0 = e'p_1$ . Because  $e$  is the coequalizer of  $p_0$  and  $p_1$ , there is a unique  $\sigma :$

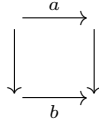
$$\begin{array}{ccc} & \xrightarrow{e} & \\ \swarrow e' & & \downarrow \sigma \\ & \searrow & \end{array} \quad \text{Then } m'\sigma e = m'e' = f = me \text{ so since } e \text{ epi, } m = m'\sigma.$$

Now  $e' : X \rightarrow E'$  is a coequalizer; say  $U \xrightleftharpoons[l]{k} X \xrightarrow{e'} E'$  is a coequalizer diagram. Then it follows that  $ek = el$  (since  $mek = m'e'k = m'e'l = mel$

and  $m$  mono) so there is a unique  $\tau$ :  Then  $m\tau\sigma e = m\tau e' = me$ ;

since  $m$  mono and  $e$  epi,  $\tau\sigma = \text{id}_E$ . Similarly,  $\sigma\tau = \text{id}_{E'}$ . So  $\sigma$  is the required isomorphism. ■

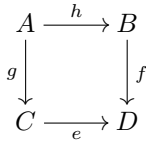
**Exercise 66** Check this detail: in a regular category  $\mathcal{C}$ , if



is a pullback diagram and  $b = c_1 c_2$  with  $c_1$  and  $c_2$  regular epis, then  $a = a_1 a_2$  for some regular epis  $a_1, a_2$ .

**Proposition 4.3** *Let  $\mathcal{C}$  be a regular category.*

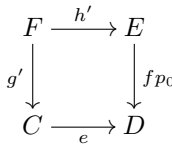
a) *Suppose that*



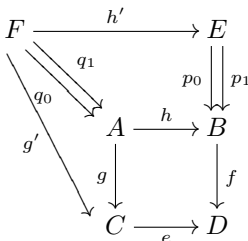
*is a pullback diagram in  $\mathcal{C}$  with  $e$  regular epi. Then if  $g$  is mono, so is  $f$ .*

b) *The composition of two regular epis in  $\mathcal{C}$  is again regular epi in  $\mathcal{C}$ .*

**Proof.** For a), suppose  $E \xrightarrow[p_1]{p_0} B$  is a parallel pair for which  $fp_0 = fp_1$ . Let



be a pullback. Then by the pullback property of the original diagram there are arrows  $q_0, q_1 : F \rightarrow A$  such that  $gq_0 = g', hq_0 = p_0 h'$  and  $gq_1 = g', hq_1 = p_1 h'$ :



From  $gq_0 = g' = gq_1$  and the assumption that  $g$  is mono, we get  $q_0 = q_1$ . Therefore  $p_0h' = hq_0 = hq_1 = p_1h'$ . Since  $h'$ , being a pullback of the regular epi  $e$ , is regular epi (hence epi), we find  $p_0 = p_1$ . We conclude that  $f$  is mono.

For b), suppose in  $A \xrightarrow{e_1} B \xrightarrow{e_2} C$  the arrows  $e_1, e_2$  are both regular epi. In order to show that the composite  $e_2e_1$  is regular epi, we factor this composite as  $me$  with  $m$  mono and  $e$  regular epi:

$$\begin{array}{ccccc} A & \xrightarrow{e_1} & B & \xrightarrow{e_2} & C \\ & \searrow e & & \nearrow m & \\ & & D & & \end{array}$$

If  $E \xrightarrow[p_1]{p_0} A$  is the kernel pair of  $e_1$  then  $me p_0 = e_2e_1p_0 = e_2e_1p_1 = me p_1$  so since  $m$  is mono,  $ep_0 = ep_1$ . Therefore, since  $e_1$  is the coequalizer of  $p_0, p_1$  we have a unique map  $n : B \rightarrow D$  satisfying  $ne_1 = e$ . Then we also have:  $mne_1 = me = e_2e_1$ , so since  $e_1$  is epi,  $mn = e_2$  and the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{e_1} & B & \xrightarrow{e_2} & C \\ & \searrow e & \downarrow n & \nearrow m & \\ & & D & & \end{array}$$

Repeating the argument for the kernel pair  $q_0, q_1$  of  $e_2$ , we get that  $nq_0 = nq_1$ ; so since  $e_2$  is the coequalizer of its kernel pair, we get a unique arrow  $k : C \rightarrow D$  such that  $ke_2 = n$ .

Then  $mke_2 = mn = e_2$  so since  $e_2$  is epi,  $mk = \text{id}_C$ ; and  $kme = ke_2e_1 = ne_1 = e$ , so since  $e$  is epi,  $km = \text{id}_D$ . We find that  $k$  is a two-sided inverse for  $m$ , which is therefore an isomorphism. We conclude that  $e_2e_1$  is regular epi. ■

**Subobjects.** In any category  $\mathcal{C}$  we define a *subobject* of an object  $X$  to be an equivalence class of monomorphisms  $Y \xrightarrow{m} X$ , where  $Y \xrightarrow{m} X$  is equivalent to  $Y' \xrightarrow{m'} X$  if there is an isomorphism  $\sigma : Y \rightarrow Y'$  with  $m'\sigma = m$  (then  $m\sigma^{-1} = m'$  follows). We say that  $Y \xrightarrow{m} X$  represents a *smaller* subobject than  $Y' \xrightarrow{m'} X$  if there is  $\sigma : Y \rightarrow Y'$  with  $m'\sigma = m$  ( $\sigma$  not necessarily iso; but check that  $\sigma$  is always mono).

The notion of subobject is the categorical version of the notion of subset in set theory. In  $\text{Set}$ , two injective functions represent the same subobject iff their images are the same; one can therefore identify subobjects with subsets. Note however, that in  $\text{Set}$  we have a “canonical” choice of representative for each subobject: the inclusion of the subset to which the subobject corresponds. This choice is not always available in general categories.

We have a partial order  $\text{Sub}(X)$  of subobjects of  $X$ , ordered by the smaller-than relation.

**Proposition 4.4** *In a category  $\mathcal{C}$  with finite limits, each pair of elements of  $\text{Sub}(X)$  has a greatest lower bound. Moreover,  $\text{Sub}(X)$  has a largest element.*

**Proof.** If  $Y \xrightarrow{m} X$  and  $Y' \xrightarrow{m'} X$  represent two subobjects of  $X$  and

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow m \\ Y' & \xrightarrow{m'} & X \end{array}$$

is a pullback, then  $Z \rightarrow X$  is mono, and represents the greatest lower bound (check!).

Of course, the identity  $X \xrightarrow{\text{id}_X} X$  represents the top element of  $\text{Sub}(X)$ . ■

Because the factorization of  $X \xrightarrow{f} Y$  as  $X \xrightarrow{e} E \xrightarrow{m} Y$  with  $e$  regular epi and  $m$  mono, in a regular category  $\mathcal{C}$ , is only defined up to isomorphism, it defines rather a subobject of  $Y$  than a mono into  $Y$ ; this defined subobject is called the *image* of  $f$  and denoted  $\text{Im}(f)$  (compare with the situation in  $\text{Set}$ ).

**Exercise 67**  $\text{Im}(f)$  is the smallest subobject of  $Y$  through which  $f$  factors: for a subobject represented by  $n : A \rightarrow Y$  we have that there is  $X \xrightarrow{a} A$  with  $f = na$ , if and only if  $\text{Im}(f)$  is smaller than the subobject represented by  $n$ .

Since monos and isos are stable under pullback, in any category  $\mathcal{C}$  with pullbacks, any arrow  $f : X \rightarrow Y$  determines an order preserving map  $f^* : \text{Sub}(Y) \rightarrow \text{Sub}(X)$  by pullback along  $f$ : if  $E \xrightarrow{m} Y$  represents the subobject  $M$  of  $Y$  and

$$\begin{array}{ccc} F & \longrightarrow & E \\ n \downarrow & & \downarrow m \\ X & \xrightarrow{f} & Y \end{array} \text{ is a pullback, } F \xrightarrow{n} X \text{ represents } f^*(M).$$

**Exercise 68** Check that  $f^*$  is well defined and order preserving.

**Proposition 4.5** In a regular category, each  $f^*$  preserves greatest lower bounds and images, that is: for  $f : X \rightarrow Y$ ,

i) for subobjects  $M, N$  of  $Y$ ,  $f^*(M \wedge N) = f^*(M) \wedge f^*(N)$ ;

ii) if  $\begin{array}{ccc} & \longrightarrow & \\ g' \downarrow & & \downarrow g \\ & \xrightarrow{f} & \end{array}$  is a pullback, then  $f^*(\text{Im}(g)) = \text{Im}(g')$ .

**Exercise 69** Prove proposition 4.5.

**Exercise 70** Suppose  $f : X \rightarrow Y$  is an arrow in a regular category. For a subobject  $M$  of  $X$ , represented by a mono  $E \xrightarrow{m} X$ , write  $\exists_f(M)$  for the subobject  $\text{Im}(fm)$  of  $Y$ .

- Show that  $\exists_f(M)$  is well-defined, that is: depends only on  $M$ , not on the representative  $m$ .
- Show that if  $M \in \text{Sub}(X)$  and  $N \in \text{Sub}(Y)$ , then  $\exists_f(M) \leq N$  if and only if  $M \leq f^*(N)$ .

## 4.2 The logic of regular categories

The fragment of first order logic we are going to interpret in regular categories is the so-called regular fragment.

The logical symbols are  $=$  (equality),  $\wedge$  (conjunction) and  $\exists$  (existential quantification). A *language* consists of a set of *sorts*  $S, T, \dots$ ; a denumerable collection of *variables*  $x_1^S, x_2^S, \dots$  of sort  $S$ , for each sort  $S$ ; a collection of *function symbols*  $(f : S_1, \dots, S_n \rightarrow S)$  and *relation symbols*  $(R \subseteq S_1, \dots, S_m)$ . The case  $n = 0$  is not excluded (one thinks of *constants* of a sort in case of 0-placed function symbols, and of *atomic propositions* in the case of 0-placed relation symbols), but not separately treated. We now define, inductively, terms of sort  $S$  and formulas.

**Definition 4.6** *Terms of sort  $S$  are defined by:*

- i)  $x^S$  is a term of sort  $S$  if  $x^S$  is a variable of sort  $S$ ;
- ii) if  $t_1, \dots, t_n$  are terms of sorts  $S_1, \dots, S_n$  respectively, and

$$(f : S_1, \dots, S_n \rightarrow S)$$

is a function symbol of the language, then  $f(t_1, \dots, t_n)$  is a term of sort  $S$ .

*Formulas are defined by:*

- i)  $\top$  is a formula (the formula “true”);
- ii) if  $t$  and  $s$  are terms of the same sort, then  $t = s$  is a formula;
- iii) if  $(R \subseteq S_1, \dots, S_m)$  is a relation symbol and  $t_1, \dots, t_m$  are terms of sorts  $S_1, \dots, S_m$  respectively, then  $R(t_1, \dots, t_m)$  is a formula;
- iv) if  $\varphi$  and  $\psi$  are formulas then  $(\varphi \wedge \psi)$  is a formula;
- v) if  $\varphi$  is a formula and  $x$  a variable of some sort, then  $\exists x\varphi$  is a formula.

An *interpretation* of such a language in a regular category  $\mathcal{C}$  is given by choosing for each sort  $S$  an object  $\llbracket S \rrbracket$  of  $\mathcal{C}$ , for each function symbol  $(f : S_1, \dots, S_n \rightarrow S)$  of the language, an arrow  $\llbracket f \rrbracket : \llbracket S_1 \rrbracket \times \dots \times \llbracket S_n \rrbracket \rightarrow \llbracket S \rrbracket$  in  $\mathcal{C}$ , and for each relation symbol  $(R \subseteq S_1, \dots, S_m)$  a subobject  $\llbracket R \rrbracket$  of  $\llbracket S_1 \rrbracket \times \dots \times \llbracket S_m \rrbracket$ .

Given this, we define interpretations  $\llbracket t \rrbracket$  for terms  $t$  and  $\llbracket \varphi \rrbracket$  for formulas  $\varphi$ , as follows.

Write  $FV(t)$  for the set of variables which occur in  $t$ , and  $FV(\varphi)$  for the set of free variables in  $\varphi$ .

We put  $\llbracket FV(t) \rrbracket = \llbracket S_1 \rrbracket \times \dots \times \llbracket S_n \rrbracket$  if  $FV(t) = \{x_1^{S_1}, \dots, x_n^{S_n}\}$ ; the same for  $\llbracket FV(\varphi) \rrbracket$ . Note: in the products  $\llbracket FV(t) \rrbracket$  and  $\llbracket FV(\varphi) \rrbracket$  we take a copy of  $\llbracket S \rrbracket$  for every variable of sort  $S$ ! Let me further emphasize that the empty product is 1, so if  $FV(t)$  (or  $FV(\varphi)$ ) is  $\emptyset$ ,  $\llbracket FV(t) \rrbracket$  (or  $\llbracket FV(\varphi) \rrbracket$ ) is the terminal object of the category.

**Definition 4.7** *The interpretation of a term  $t$  of sort  $S$  is a morphism  $\llbracket t \rrbracket : \llbracket FV(t) \rrbracket \rightarrow \llbracket S \rrbracket$  and is defined by the clauses:*

- i)  $\llbracket x^S \rrbracket$  is the identity on  $\llbracket S \rrbracket$ , if  $x^S$  is a variable of sort  $S$ ;
- ii) Given  $\llbracket t_i \rrbracket : \llbracket FV(t_i) \rrbracket \rightarrow \llbracket S_i \rrbracket$  for  $i = 1, \dots, n$  and a function symbol  $(f : S_1, \dots, S_n \rightarrow S)$  of the language,  $\llbracket f(t_1, \dots, t_n) \rrbracket$  is the map

$$\llbracket FV(f(t_1, \dots, t_n)) \rrbracket \xrightarrow{(\tilde{t}_i | i=1, \dots, n)} \prod_{i=1}^n \llbracket S_i \rrbracket \xrightarrow{\llbracket f \rrbracket} \llbracket S \rrbracket$$

where  $\tilde{t}_i$  is the composite

$$\llbracket FV(f(t_1, \dots, t_n)) \rrbracket \xrightarrow{\pi_i} \llbracket FV(t_i) \rrbracket \xrightarrow{\llbracket t_i \rrbracket} \llbracket S_i \rrbracket$$

in which  $\pi_i$  is the appropriate projection, corresponding to the inclusion  $FV(t_i) \subseteq FV(f(t_1, \dots, t_n))$ .

Finally, we interpret formulas  $\varphi$  as subobjects  $\llbracket \varphi \rrbracket$  of  $\llbracket FV(\varphi) \rrbracket$ . You should try to keep in mind the intuition that  $\llbracket \varphi(x_1, \dots, x_n) \rrbracket$  is the “subset”

$$\{(a_1, \dots, a_n) \in A_1 \times \dots \times A_n \mid \varphi[a_1, \dots, a_n]\}$$

**Definition 4.8** *The interpretation  $\llbracket \varphi \rrbracket$  as subobject of  $\llbracket FV(\varphi) \rrbracket$  is defined as follows:*

- i)  $\llbracket \top \rrbracket$  is the maximal subobject of  $\llbracket FV(\top) \rrbracket = 1$ ;
- ii)  $\llbracket t = s \rrbracket \rightarrow \llbracket FV(t = s) \rrbracket$  is the equalizer of

$$\llbracket FV(t = s) \rrbracket \rightrightarrows \begin{array}{c} \llbracket FV(t) \rrbracket \\ \llbracket FV(s) \rrbracket \end{array} \xrightarrow[\llbracket s \rrbracket]{\llbracket t \rrbracket} \llbracket T \rrbracket$$

if  $t$  and  $s$  are of sort  $T$ ; again, the left hand side maps are projections, corresponding to the inclusions of  $FV(t)$  and  $FV(s)$  into  $FV(t = s)$ ;

- iii) For  $(R \subseteq S_1, \dots, S_m)$  a relation symbol and terms  $t_1, \dots, t_m$  of sorts  $S_1, \dots, S_m$  respectively, let  $\bar{t} : \llbracket FV(R(t_1, \dots, t_m)) \rrbracket \rightarrow \prod_{i=1}^m \llbracket S_i \rrbracket$  be the composite map

$$\llbracket FV(R(t_1, \dots, t_m)) \rrbracket \rightarrow \prod_{i=1}^m \llbracket FV(t_i) \rrbracket \xrightarrow{\prod_{i=1}^m \llbracket t_i \rrbracket} \prod_{i=1}^m \llbracket S_i \rrbracket$$

Then  $\llbracket R(t_1, \dots, t_m) \rrbracket \rightarrow \llbracket FV(R(t_1, \dots, t_m)) \rrbracket$  is the subobject  $(\bar{t})^*(\llbracket R \rrbracket)$ , defined by pullback along  $\bar{t}$ .

iv) if  $\llbracket \varphi \rrbracket \rightarrow \llbracket FV(\varphi) \rrbracket$  and  $\llbracket \psi \rrbracket \rightarrow \llbracket FV(\psi) \rrbracket$  are given and

$$\begin{array}{ccc} \llbracket FV(\varphi \wedge \psi) \rrbracket & \xrightarrow{\pi_1} & \llbracket FV(\varphi) \rrbracket \\ & \searrow \pi_2 & \\ & & \llbracket FV(\psi) \rrbracket \end{array}$$

are again the suitable projections, then  $\llbracket (\varphi \wedge \psi) \rrbracket \rightarrow \llbracket FV(\varphi \wedge \psi) \rrbracket$  is the greatest lower bound in  $\text{Sub}(\llbracket FV(\varphi \wedge \psi) \rrbracket)$  of  $\pi_1^*(\llbracket \varphi \rrbracket)$  and  $\pi_2^*(\llbracket \psi \rrbracket)$ ;

v) if  $\llbracket \varphi \rrbracket \rightarrow \llbracket FV(\varphi) \rrbracket$  is given and  $\pi : \llbracket FV(\varphi) \rrbracket \rightarrow \llbracket FV(\exists x \varphi) \rrbracket$  the projection, let  $\llbracket FV'(\varphi) \rrbracket$  be the product of the interpretations of the sorts of the variables in  $FV(\varphi) \cup \{x\}$  (so  $\llbracket FV'(\varphi) \rrbracket = \llbracket FV(\varphi) \rrbracket$  if  $x$  occurs freely in  $\varphi$ ; and  $\llbracket FV'(\varphi) \rrbracket = \llbracket FV(\varphi) \rrbracket \times \llbracket S \rrbracket$  if  $x = x^S$  does not occur free in  $\varphi$ ). Write  $\pi' : \llbracket FV'(\varphi) \rrbracket \rightarrow \llbracket FV(\varphi) \rrbracket$ .

Now take  $\llbracket \exists x \varphi \rrbracket \rightarrow \llbracket FV(\exists x \varphi) \rrbracket$  to be the image of the composition:

$$(\pi')^*(\llbracket \varphi \rrbracket) \rightarrow \llbracket FV'(\varphi) \rrbracket \xrightarrow{\pi'} \llbracket FV(\exists x \varphi) \rrbracket$$

We have now given an interpretation of formulas. Basically, a formula  $\varphi$  is *true* under this interpretation if  $\llbracket \varphi \rrbracket \rightarrow \llbracket FV(\varphi) \rrbracket$  is the maximal subobject; but since we formulate the logic in terms of sequents we rather define when a sequent is true under the interpretation.

**Definition 4.9** A labelled sequent is an expression of the form  $\psi \vdash_\sigma \varphi$  or  $\vdash_\sigma \varphi$  where  $\psi$  and  $\varphi$  are the formulas of the sequent (but  $\psi$  may be absent), and  $\sigma$  is a finite set of variables which includes all the variables which occur free in a formula of the sequent.

Let  $\llbracket \sigma \rrbracket = \llbracket S_1 \rrbracket \times \cdots \times \llbracket S_n \rrbracket$  if  $\sigma = \{x_1^{S_1}, \dots, x_n^{S_n}\}$ ; there are projections  $\llbracket \sigma \rrbracket \xrightarrow{\pi_\varphi} \llbracket FV(\varphi) \rrbracket$  and (in case  $\psi$  is there)  $\llbracket \sigma \rrbracket \xrightarrow{\pi_\psi} \llbracket FV(\psi) \rrbracket$ ; we say that the sequent  $\psi \vdash_\sigma \varphi$  is true for the interpretation if  $(\pi_\psi)^*(\llbracket \psi \rrbracket) \leq (\pi_\varphi)^*(\llbracket \varphi \rrbracket)$  as subobjects of  $\llbracket \sigma \rrbracket$ , and  $\vdash_\sigma \varphi$  is true if  $(\pi_\varphi)^*(\llbracket \varphi \rrbracket)$  is the maximal subobject of  $\llbracket \sigma \rrbracket$ .

We also say that  $\varphi$  is true if  $\vdash_{FV(\varphi)} \varphi$  is true.

**Exercise 71** Show that the sequent  $\vdash \exists x^S (x^S = x^S)$  is true if and only if the unique map  $\llbracket S \rrbracket \rightarrow 1$  is a regular epimorphism. What about the sequent  $\vdash_S \top$ ?

We now turn to the logic. Instead of giving deduction rules and axioms, I formulate a list of closure conditions which determine what sets of labelled sequents will be called a *theory*. I write  $\vdash_x$  for  $\vdash_{\{x\}}$  and  $\vdash$  for  $\vdash_\emptyset$ .

**Definition 4.10** Given a language, a set  $T$  of labelled sequents of that language is called a *theory* iff the following conditions hold (the use of brackets around  $\psi$  caters in a, I hope, self-explanatory way for the case distinction as to whether  $\psi$  is or is not present):



- i)  $\vdash \top$  is in  $T$ ;  
 $\vdash_x x = x$  is in  $T$  for every variable  $x$ ;  
 $x = y \vdash_{\{x,y\}} y = x$  is in  $T$  for variables  $x, y$  of the same sort;  
 $x = y \wedge y = z \vdash_{\{x,y,z\}} x = z$  is in  $T$  for variables  $x, y, z$  of the same sort;  
 $R(x_1, \dots, x_m) \vdash_{\{x_1, \dots, x_m\}} R(x_1, \dots, x_m)$  is in  $T$ ;
- ii) if  $(\psi) \vdash_\sigma \varphi$  is in  $T$  then  $(\psi) \vdash_\tau \varphi$  is in  $T$  whenever  $\sigma \subseteq \tau$ ;
- iii) if  $(\psi) \vdash_\sigma \varphi$  is in  $T$  and  $FV(\chi) \subseteq \sigma$  then  $(\psi \wedge \chi) \vdash_\sigma \varphi$  and  $\chi \wedge (\psi) \vdash_\sigma \varphi$  are in  $T$ ;
- iv) if  $(\psi) \vdash_\sigma \varphi$  and  $(\psi) \vdash_\sigma \chi$  are in  $T$  then  $(\psi) \vdash_\sigma \varphi \wedge \chi$  and  $(\psi) \vdash_\sigma \chi \wedge \varphi$  are in  $T$ ;
- v) if  $(\chi \wedge) \psi \vdash_\sigma \varphi$  is in  $T$  and  $x$  is a variable not occurring in  $\varphi$  or  $\chi$  then  $(\chi \wedge) \exists x \psi \vdash_{\sigma \setminus \{x\}} \varphi$  is in  $T$ ;
- vi) if  $x$  occurs in  $\varphi$  and  $(\psi) \vdash_\sigma \varphi[t/x]$  is in  $T$  then  $(\psi) \vdash_\sigma \exists x \varphi$  is in  $T$ ;  
if  $x$  does not occur in  $\varphi$  and  $(\psi) \vdash_\sigma \varphi$  and  $(\psi) \vdash_\sigma \exists x(x = x)$  are in  $T$ ,  
then  $(\psi) \vdash_\sigma \exists x \varphi$  is in  $T$ ;
- vii) if  $(\psi) \vdash_\sigma \varphi$  is in  $T$  then  $(\psi[t/x]) \vdash_{\sigma \setminus \{x\} \cup FV(t)} \varphi[t/x]$  is in  $T$ ;
- viii) if  $(\psi) \vdash_\sigma \varphi[t/x]$  and  $(\psi) \vdash_\sigma t = s$  are in  $T$  then  $(\psi) \vdash_\sigma \varphi[s/x]$  is in  $T$ ;
- ix) if  $(\psi) \vdash_\sigma \varphi$  and  $\varphi \vdash_\sigma \chi$  are in  $T$  then  $(\psi) \vdash_\sigma \chi$  is in  $T$

**Exercise 72** Show that the sequent  $\varphi \vdash_{FV(\varphi)} \varphi$  is in every theory, for every formula  $\varphi$  of the language.

As said, the definition of a theory is a list of closure conditions: every item can be seen as a rule, and a theory is a set of sequents closed under every rule. Therefore, the intersection of any collection of theories is again a theory, and it makes sense to speak, given a set of sequents  $S$ , of the theory  $Cn(S)$  generated by  $S$ :

$$Cn(S) = \bigcap \{T \mid T \text{ is a theory and } S \subseteq T\}$$

We have the following theorem:

**Theorem 4.11 (Soundness theorem)** Suppose  $T = Cn(S)$  and all sequents of  $S$  are true under the interpretation in the category  $\mathcal{C}$ . Then all sequents of  $T$  are true under that interpretation.

Before embarking on the proof, first a lemma:

**Lemma 4.12** Suppose  $t$  is substitutable for  $x$  in  $\varphi$ . There is an obvious map

$$[t] : \llbracket FV(\varphi) \setminus \{x\} \cup FV(t) \rrbracket = \llbracket FV(\varphi[t/x]) \rrbracket \rightarrow \llbracket FV(\varphi) \rrbracket$$

induced by  $\llbracket t \rrbracket$ ; the components of  $\llbracket t \rrbracket$  are projections except for the factor of  $\llbracket \varphi \rrbracket$  corresponding to  $x$ , where it is

$$\llbracket FV(\varphi[t/x]) \rrbracket \rightarrow \llbracket FV(t) \rrbracket \xrightarrow{\llbracket t \rrbracket} \llbracket \{x\} \rrbracket$$

There is a pullback diagram:

$$\begin{array}{ccc} \llbracket \varphi[t/x] \rrbracket & \longrightarrow & \llbracket FV(\varphi[t/x]) \rrbracket \\ \downarrow & & \downarrow \llbracket t \rrbracket \\ \llbracket \varphi \rrbracket & \longrightarrow & \llbracket FV(\varphi) \rrbracket \end{array}$$

**Exercise 73** Prove this lemma [not trivial. Use induction on  $\varphi$  and proposition 4.5].

**Proof.** (of theorem 4.11) The proof checks that for every rule in the list of definition 4.10, if the premiss is true then the conclusion is true; in other words, that the set of true sequents is a theory.

i)  $\vdash \top$  is true by the definition  $\llbracket \top \rrbracket = 1$ ;

$\llbracket x^S = x^S \rrbracket$  is the equalizer of two maps which are both the identity on  $\llbracket S \rrbracket$ , so isomorphic to  $\llbracket S \rrbracket$ . For  $x = y \wedge y = z \vdash_{\{x,y,z\}} x = z$ , just observe that  $E_{12} \wedge E_{23} \leq E_{13}$  if  $E_{ij}$  is the equalizer of the two projections  $\pi_i, \pi_j : \llbracket S \rrbracket \times \llbracket S \rrbracket \rightarrow \llbracket S \rrbracket$ .

ii) This is because if  $\sigma \subseteq \tau$  and  $\rho : \llbracket \tau \rrbracket \rightarrow \llbracket \sigma \rrbracket$  is the projection,  $\rho^*$  is monotone.

iii)-iv) By the interpretation of  $\wedge$  as the greatest lower bound of two subobjects, and proposition 4.5.

v) Let

$$\begin{array}{ccccc} \llbracket \sigma \rrbracket & \xrightarrow{\pi} & \llbracket \sigma \setminus \{x\} \rrbracket & \xrightarrow{\rho} & \llbracket FV(\varphi) \rrbracket \\ \mu \downarrow & & \downarrow \nu & & \\ \llbracket FV(\psi) \rrbracket & & \llbracket FV(\exists x \psi) \rrbracket & & \end{array}$$

the projections. Since by assumption  $\mu^*(\llbracket \psi \rrbracket) \leq (\rho\pi)^*(\llbracket \varphi \rrbracket)$  there is a commutative diagram

$$\begin{array}{ccc} \mu^*(\llbracket \psi \rrbracket) & \longrightarrow & \llbracket \sigma \rrbracket \\ \downarrow & & \downarrow \pi \\ \rho^*(\llbracket \varphi \rrbracket) & \longrightarrow & \llbracket \sigma \setminus \{x\} \rrbracket \end{array}$$

By proposition 4.5,  $\nu^*(\llbracket \exists x \psi \rrbracket)$  is the image of the map  $\mu^*(\llbracket \psi \rrbracket) \rightarrow \llbracket \sigma \setminus \{x\} \rrbracket$ , so  $\nu^*(\llbracket \psi \rrbracket) \leq \rho^*(\llbracket \varphi \rrbracket)$  in  $\text{Sub}(\llbracket \sigma \setminus \{x\} \rrbracket)$ .

vi) Suppose  $x$  occurs free in  $\varphi$ . Consider the commutative diagram

$$\begin{array}{ccccccc} & & \llbracket \sigma \rrbracket & & & & \\ & \swarrow \pi & \downarrow \pi' & \searrow \pi'' & & & \\ \llbracket FV(\psi) \rrbracket & & \llbracket FV(\varphi[t/x]) \rrbracket & \xrightarrow{\llbracket t \rrbracket} & \llbracket FV(\varphi) \rrbracket & \xrightarrow{\rho} & \llbracket FV(\varphi) \setminus \{x\} \rrbracket \end{array}$$

with  $[t]$  as in lemma 4.12 and the other maps projections. Now  $\llbracket \varphi \rrbracket \leq \rho^*(\llbracket \exists x \varphi \rrbracket)$  because  $\llbracket \varphi \rrbracket \rightarrow \llbracket FV(\varphi) \rrbracket \xrightarrow{\rho} \llbracket FV(\varphi) \setminus \{x\} \rrbracket$  factors through  $\llbracket \exists x \varphi \rrbracket$  by definition; so if  $\pi^*(\llbracket \psi \rrbracket) \leq \pi'^*(\llbracket \varphi[t/x] \rrbracket)$  then with lemma 4.12,

$$\pi^*(\llbracket \psi \rrbracket) \leq \pi'^*(\llbracket \varphi[t/x] \rrbracket) = \pi'^*[t]^*(\llbracket \varphi \rrbracket) \leq \pi'^*[t]^* \rho^*(\llbracket \exists x \varphi \rrbracket) = \pi''^*(\llbracket \exists x \varphi \rrbracket)$$

in  $\text{Sub}(\llbracket \sigma \rrbracket)$  and we are done.

The case of  $x$  not occurring in  $\varphi$  is left to you.

vii) Direct application of lemma 4.12

viii-ix) Left to you. ■

**Exercise 74** Fill in the “left to you” gaps in the proof.

### 4.3 The language $\mathcal{L}(\mathcal{C})$ and theory $T(\mathcal{C})$ associated to a regular category $\mathcal{C}$

Given a regular category  $\mathcal{C}$  (which, to be precise, must be assumed to be small), we associate to  $\mathcal{C}$  the language which has a sort  $C$  for every object of  $\mathcal{C}$ , and a function symbol  $(f : C \rightarrow D)$  for every arrow  $f : C \rightarrow D$  of  $\mathcal{C}$ .

This language is called  $\mathcal{L}(\mathcal{C})$  and it has trivially an interpretation in  $\mathcal{C}$ .

The theory  $T(\mathcal{C})$  is the set of sequents of  $\mathcal{L}(\mathcal{C})$  which are true for this interpretation.

One of the points of categorical logic is now, that categorical statements about objects and arrows in  $\mathcal{C}$  can be reformulated as statements about the truth of certain sequents in  $\mathcal{L}(\mathcal{C})$ . You should read the relevant sequents as expressing that we can “do as if the category were Set”.

#### Examples

- a)  $C$  is a terminal object of  $\mathcal{C}$  if and only if the sequents  $\vdash_{x,y} x = y$  and  $\vdash \exists x(x = x)$  are valid, where  $x, y$  variables of sort  $C$ ;
- b) the arrow  $f : A \rightarrow B$  is mono in  $\mathcal{C}$  if and only if the sequent  $f(x) = f(y) \vdash_{x,y} x = y$  is true;
- c) The square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & \xrightarrow{d} & D \end{array}$$

is a pullback square in  $\mathcal{C}$  if and only if the sequents

$$h(x^B) = d(y^C) \vdash_{x,y} \exists z^A (f(z) = x \wedge g(z) = y)$$

and

$$f(z^A) = f(z'^A) \wedge g(z^A) = g(z'^A) \vdash_{z,z'} z = z'$$

are true;

- d) the fact that  $f : A \rightarrow B$  is epi can not similarly be expressed! But:  $f$  is regular epi if and only if

$$\vdash_{x^B} \exists y^A (f(y) = x)$$

is true;

- e)  $A \xrightarrow{f} B \xrightleftharpoons[h]{g} C$  is an equalizer diagram iff  $f$  is mono (see b) and the sequent

$$g(x^B) = h(x^B) \vdash_{x^B} \exists y^A (f(y) = x)$$

is true.

**Exercise 75** Check (a number of) these statements. Give the sequent(s) cor-

responding to the statement that  $\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \\ C & & \end{array}$  is a product diagram.

**Exercise 76** Check that the formulas  $\exists x\varphi$  and  $\exists x(x = x \wedge \varphi)$  are *equivalent*, that is, every theory contains the sequents

$$\exists x\varphi \vdash_{\sigma} \exists x(x = x \wedge \varphi)$$

and

$$\exists x(x = x \wedge \varphi) \vdash_{\sigma} \exists x\varphi$$

for any  $\sigma$  containing the free variables of  $\exists x\varphi$ .

**Exercise 77** Suppose

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & \xrightarrow{d} & D \end{array}$$

is a pullback diagram in a regular category  $\mathcal{C}$ , with  $d$  regular epi. Prove the result of Proposition 4.3 a) (that is, if  $g$  is mono, then so is  $h$ ), by using regular logic. Can you say anything about the relation between the two proofs?

**Exercise 78** Can you express:  $A \xrightarrow{f} B$  is regular mono? [Hint: don't waste too much time in trying!]

## 4.4 The category $\mathcal{C}(T)$ associated to a theory $T$ : Completeness Theorem

The counterpart of Theorem 4.11 (the Soundness Theorem) is of course a completeness theorem: suppose that the sequent  $(\psi) \vdash_{\sigma} \varphi$  is true in every interpretation which makes all sequents from  $T$  true. We want to conclude that  $(\psi) \vdash_{\sigma} \varphi$  is in  $T$ .

To do this, we build a category  $\mathcal{C}(T)$ , a so-called *syntactic category*, which will be regular, and which allows an interpretation  $\llbracket \cdot \rrbracket$  such that exactly the sequents in  $T$  will be true for  $\llbracket \cdot \rrbracket$ .

The construction of  $\mathcal{C}(T)$  is as follows. *Objects* are formulas of the language  $\mathcal{L}$ , up to renaming of free variables; so if  $\varphi$  is a formula in distinct free variables  $x_1, \dots, x_n$  and  $v_1, \dots, v_n$  are distinct variables of matching sorts, then  $\varphi$  and  $\varphi(v_1/x_1, \dots, v_n/x_n)$  are the same object. Because of this stipulation, if I define morphisms between  $\varphi(\vec{x})$  and  $\psi(\vec{y})$ , I can always assume that the collections of variables  $\vec{x}$  and  $\vec{y}$  are disjoint (even when treating a morphism from  $\varphi$  to itself, I take  $\varphi$  and a renaming  $\varphi(\vec{x}')$ ). Given  $\varphi(\vec{x})$  and  $\psi(\vec{y})$  (where  $\vec{x} = (x_1, \dots, x_n)$ ,  $\vec{y} = (y_1, \dots, y_m)$ ), a *functional relation* from  $\varphi$  to  $\psi$  is a formula  $\chi(\vec{x}, \vec{y})$  such that the sequents

$$\begin{aligned} \chi(\vec{x}, \vec{y}) &\vdash_{\{x_1, \dots, x_n, y_1, \dots, y_m\}} \varphi(\vec{x}) \wedge \psi(\vec{y}) \\ \chi(\vec{x}, \vec{y}) \wedge \chi(\vec{x}, \vec{y}') &\vdash_{\{x_1, \dots, x_n, y_1, \dots, y_m, y'_1, \dots, y'_m\}} y_1 = y'_1 \wedge \dots \wedge y_m = y'_m \\ \varphi(\vec{x}) &\vdash_{\{x_1, \dots, x_n\}} \exists y_1 \dots \exists y_m \chi(\vec{x}, \vec{y}) \end{aligned}$$

are all in  $T$ . If  $FV(\psi) = \emptyset$ , the second requirement is taken to be vacuous, i.e. trivially fulfilled.

A *morphism* from  $\varphi$  to  $\psi$  is an equivalence class  $[\chi]$  of functional relations from  $\varphi$  to  $\psi$ , where  $\chi_1$  and  $\chi_2$  are equivalent iff the sequents

$$\begin{aligned} \chi_1(\vec{x}, \vec{y}) &\vdash_{\{x_1, \dots, x_n, y_1, \dots, y_m\}} \chi_2(\vec{x}, \vec{y}) \\ \chi_2(\vec{x}, \vec{y}) &\vdash_{\{x_1, \dots, x_n, y_1, \dots, y_m\}} \chi_1(\vec{x}, \vec{y}) \end{aligned}$$

are in  $T$  (in fact, given that  $\chi_1$  and  $\chi_2$  are functional relations, one of these sequents is in  $T$  iff the other is).

*Composition* of morphisms is defined as follows: if  $\chi_1(\vec{x}, \vec{y})$  represents a morphism  $\varphi \rightarrow \psi$  and  $\chi_2(\vec{y}, \vec{z})$  a morphism  $\psi \rightarrow \omega$ , the composition  $[\chi_2] \circ [\chi_1] : \varphi \rightarrow \omega$  is represented by the functional relation

$$\chi_{21} \equiv \exists y_1 \dots \exists y_m (\chi_1(\vec{x}, \vec{y}) \wedge \chi_2(\vec{y}, \vec{z}))$$

**Exercise 79** Show:

- $\chi_{21}$  is a functional relation from  $\varphi$  to  $\omega$ ;
- the class of  $\chi_{21}$  does not depend on the choice of representatives  $\chi_1$  and  $\chi_2$ ;
- composition is associative.

The *identity* arrow from  $\varphi(\vec{x})$  to itself (i.e. to  $\varphi(\vec{x}')$ , given our renaming convention), is represented by the formula

$$\varphi(\vec{x}) \wedge x_1 = x'_1 \wedge \cdots \wedge x_n = x'_n$$

**Exercise 80** Show that this definition is correct: that this formula is a functional relation, and defines an identity arrow.

We have defined the category  $\mathcal{C}(T)$ .

**Theorem 4.13**  $\mathcal{C}(T)$  is a regular category.

**Proof.** The formula  $\top$  is a terminal object in  $\mathcal{C}(T)$ : for every formula  $\varphi$ ,  $\varphi$  itself represents the unique morphism  $\varphi \rightarrow \top$ . Given formulas  $\varphi(\vec{x})$  and  $\psi(\vec{y})$ , the formula  $\varphi \wedge \psi$  is a product, with projections  $\varphi \wedge \psi \rightarrow \varphi(\vec{x}')$  (renaming!) represented by the formula

$$\varphi(\vec{x}) \wedge \psi(\vec{y}) \wedge x_1 = x'_1 \wedge \cdots \wedge x_n = x'_n$$

and  $\varphi \wedge \psi$  to  $\psi(\vec{y}')$  similarly defined.

If  $\chi_1(\vec{x}, \vec{y})$  and  $\chi_2(\vec{x}, \vec{y})$  represent morphisms  $\varphi(\vec{x}) \rightarrow \psi(\vec{y})$  let  $\omega(\vec{x})$  be the formula

$$\exists y_1 \cdots \exists y_m (\chi_1(\vec{x}, \vec{y}) \wedge \chi_2(\vec{x}, \vec{y}))$$

Then  $\omega(\vec{x}) \wedge x_1 = x'_1 \wedge \cdots \wedge x_n = x'_n$  represents a morphism  $\omega \rightarrow \varphi(\vec{x}')$  which is the equalizer of  $[\chi_1]$  and  $[\chi_2]$ .

This takes care of finite limits.

Now if  $[\chi_1(\vec{x}, \vec{y})] : \varphi(\vec{x}) \rightarrow \psi_1(\vec{y})$  and  $[\chi_2(\vec{x}, \vec{y})] : \varphi(\vec{x}) \rightarrow \psi_2(\vec{y})$ , then you can check that  $[\chi_2]$  coequalizes the kernel pair of  $[\chi_1]$  if and only if the sequent

$$\exists \vec{y} (\chi_1(\vec{x}, \vec{y}) \wedge \chi_1(\vec{x}', \vec{y})) \vdash_{\{\vec{x}, \vec{x}'\}} \exists \vec{v} (\chi_2(\vec{x}, \vec{v}) \wedge \chi_2(\vec{x}', \vec{v}))$$

is in  $T$ . This is the case if and only if  $[\chi_2]$  factors through the obvious map from  $\varphi$  to  $\exists \vec{x} \chi_1(\vec{x}, \vec{y})$  which is therefore the image of  $[\chi_1]$ , i.e. the coequalizer of its kernel pair.

We see at once that  $[\chi] : \varphi \rightarrow \psi$  is regular epi iff the sequent

$$\psi(\vec{y}) \vdash_{\{\vec{y}\}} \exists \vec{x} \chi(\vec{x}, \vec{y})$$

is in  $T$ ; using the description of finite limits one easily checks that these are stable under pullback ■

We define  $\llbracket \cdot \rrbracket$ , the standard interpretation of  $\mathcal{L}$  in  $\mathcal{C}(T)$ , as follows:

- the interpretation  $\llbracket S \rrbracket$  of a sort  $S$  is the formula  $x = x$ , where  $x$  is a variable of sort  $S$ ;
- if  $(f : S_1, \dots, S_n \rightarrow S)$  is a function symbol of  $\mathcal{L}$ ,  $\llbracket f \rrbracket$  is the functional relation

$$f(x_1, \dots, x_n) = x$$

- if  $(R \subseteq S_1, \dots, S_n)$  is a relation symbol of  $\mathcal{L}$ , the subobject  $\llbracket R \rrbracket$  of  $\llbracket S_1 \rrbracket \times \dots \times \llbracket S_n \rrbracket$  is represented by the mono

$$R(x_1, \dots, x_n) \wedge x_1 = x'_1 \wedge \dots \wedge x_n = x'_n$$

I now state the important facts about  $\mathcal{C}(T)$  and  $\llbracket \cdot \rrbracket$  as nontrivial exercises. Here we say that, given the theory  $T$ , formulas  $\varphi$  and  $\psi$  (in the same free variables  $x_1, \dots, x_n$  are *equivalent in  $T$*  if the sequents  $\varphi \vdash_{\{x_1, \dots, x_n\}} \psi$  and  $\psi \vdash_{\{x_1, \dots, x_n\}} \varphi$  are both in  $T$ .

**Exercise 81** If  $t$  is a term of sort  $S$  in  $\mathcal{L}$ , in variables  $x_1^{S_1}, \dots, x_n^{S_n}$ , the functional relation from  $\llbracket S_1 \rrbracket \times \dots \times \llbracket S_n \rrbracket \equiv x_1 = x_1 \wedge \dots \wedge x_n = x_n$ , to  $\llbracket S \rrbracket \equiv x = x$ , representing  $\llbracket t \rrbracket$ , is equivalent in  $T$  to the formula

$$t(x_1, \dots, x_n) = x$$

**Exercise 82** If  $\varphi$  is a formula of  $\mathcal{L}$  in free variables  $x_1^{S_1}, \dots, x_n^{S_n}$ , the subobject  $\llbracket \varphi \rrbracket \rightarrow \llbracket S_1 \rrbracket \times \dots \times \llbracket S_n \rrbracket$  is represented by a functional relation  $\chi$  from a formula  $\psi$  to  $x_1 = x_1 \wedge \dots \wedge x_n = x_n$ , such that:

- $\psi$  is a formula in variables  $x'_1, \dots, x'_n$  and  $\psi(x'_1, \dots, x'_n)$  is equivalent in  $T$  to  $\varphi(x'_1, \dots, x'_n)$ ;
- $\chi(x'_1, \dots, x'_n, x_1, \dots, x_n)$  is equivalent in  $T$  to

$$\varphi(x'_1, \dots, x'_n) \wedge x'_1 = x_1 \wedge \dots \wedge x'_n = x_n$$

**Exercise 83** The sequent  $(\psi) \vdash_{\sigma} \varphi$  is true in the interpretation  $\llbracket \cdot \rrbracket$  if and only if  $(\psi) \vdash_{\sigma} \varphi$  is in  $T$ .

**Exercise 84** Let  $\mathcal{E}$  be a regular category and  $T$  a theory. Call a functor between regular categories *regular* if it preserves finite limits and regular epis.

Then every regular functor:  $\mathcal{C}(T) \rightarrow \mathcal{E}$  gives rise to an interpretation of the language of  $T$  in  $\mathcal{E}$ , which makes all sequents in  $T$  true.

Conversely, given such an interpretation of  $T$  in  $\mathcal{E}$ , there is, up to natural isomorphism, a unique regular functor:  $\mathcal{C}(T) \rightarrow \mathcal{E}$  with the property that it maps the standard interpretation of  $T$  in  $\mathcal{C}(T)$  to the given one in  $\mathcal{E}$ .

**Exercise 85** This exercise constructs the “free regular category on a given category  $\mathcal{C}$ ”. Given  $\mathcal{C}$ , which is not assumed to be regular (or to have finite limits), let  $\mathcal{L}$  be the language of  $\mathcal{C}$  as before: it has a sort  $C$  for every object  $C$  of  $\mathcal{C}$ , and a function symbol  $(f : C \rightarrow D)$  for every arrow of  $\mathcal{C}$ . Let  $T$  be the theory generated by the following set  $S$  of sequents: for every identity arrow  $i : C \rightarrow C$  in  $\mathcal{C}$ , the sequent  $\vdash_x x = i(x)$  is in  $S$ ; and for every composition  $f = gh$  of arrows in  $\mathcal{C}$ , the sequent  $\vdash_x f(x) = g(h(x))$  is in  $S$ .

- a) Show that interpretations of  $\mathcal{L}$  in a regular category  $\mathcal{E}$  which make all sequents of  $S$  true, correspond bijectively to functors from  $\mathcal{C}$  to  $\mathcal{E}$ .
- b) Show that there is a functor  $\eta : \mathcal{C} \rightarrow \mathcal{C}(T)$  such that for every functor  $F : \mathcal{C} \rightarrow \mathcal{E}$  with  $\mathcal{E}$  regular, there is, up to isomorphism, a unique regular functor  $\tilde{F} : \mathcal{C}(T) \rightarrow \mathcal{E}$  such that  $\tilde{F}\eta = F$ .  $\mathcal{C}(T)$  is the *free regular category* on  $\mathcal{C}$ .
- c) Show that if, in this situation,  $\mathcal{C}$  has finite limits,  $\eta$  does not preserve them! How would one construct the “free regular category on  $\mathcal{C}$ , preserving the limits which exist in  $\mathcal{C}$ ”?

## 4.5 Example of a regular category

In this section, I treat an example of a type of regular categories which are important in categorical logic. They are categories of  $\Omega$ -valued sets for some frame  $\Omega$ . Let's define some things.

**Definition 4.14** *A frame  $\Omega$  is a partially ordered set which has suprema (least upper bounds)  $\bigvee B$  of all subsets  $B$ , and infima (meets)  $\bigwedge A$  for finite subsets  $A$  (so, there is a top element  $\top$  and every pair of elements  $x, y$  has a meet  $x \wedge y$ ), and moreover,  $\wedge$  distributes over  $\bigvee$ , that is,*

$$x \wedge \bigvee B = \bigvee \{x \wedge b \mid b \in B\}$$

for  $x \in \Omega$ ,  $B \subseteq \Omega$ .

Given a frame  $\Omega$  we define the category  $\mathcal{C}_\Omega$  as follows:

Objects are functions  $X \xrightarrow{E_X} \Omega$ ,  $X$  a set;

Maps from  $(X, E_X)$  to  $(Y, E_Y)$  are functions  $X \xrightarrow{f} Y$  satisfying the requirement that  $E_X(x) \leq E_Y(f(x))$  for all  $x \in X$ .

It is easily seen that the identity function satisfies this requirement, and if two composable functions satisfy it, their composition does; so we have a category.

**Proposition 4.15**  *$\mathcal{C}_\Omega$  is a regular category.*

**Proof.** Let  $\{*\}$  be any one-element set, together with the function which sends  $*$  to the top element of  $\Omega$ . Then  $\{*\} \rightarrow \Omega$  is a terminal object of  $\mathcal{C}_\Omega$ .

Given  $(X, E_X)$  and  $(Y, E_Y)$ , a product of the two is the object  $(X \times Y, E_{X \times Y})$  where  $E_{X \times Y}(x, y)$  is defined as  $E_X(x) \wedge E_Y(y)$ .

Given two arrows  $f, g : (X, E_X) \rightarrow (Y, E_Y)$  their equalizer is  $(X', E_{X'})$  where  $X' \subseteq X$  is  $\{x \in X \mid f(x) = g(x)\}$  and  $E_{X'}$  is the restriction of  $E_X$  to  $X'$ .

This is easily checked, and  $\mathcal{C}_\Omega$  is a finitely complete category.

An arrow  $f : (X, E_X) \rightarrow (Y, E_Y)$  is regular epi if and only if the function  $f$  is surjective and for all  $y \in Y$ ,  $E_Y(y) = \bigvee \{E_X(x) \mid f(x) = y\}$ .

For suppose  $f$  is such, and  $g : (X, E_X) \rightarrow (Z, E_Z)$  coequalizes the kernel pair of  $f$ . Then  $g(x) = g(x')$  whenever  $f(x) = f(x')$ , and so for all  $y \in Y$ , since  $f(x) = y$  implies  $E_X(x) \leq E_Z(g(x))$ , we have

$$E_Y(y) = \bigvee \{E_X(x) \mid f(x) = y\} \leq E_Z(g(x))$$



so there is a unique map  $h : (Y, E_Y) \rightarrow (Z, E_Z)$  such that  $g = hf$ ; that is  $f$  is the coequalizer of its kernel pair.

The proof of the converse is left to you.

Finally we must show that regular epis are stable under pullback. This is an exercise. ■

**Exercise 86** Show that the pullback of  $\begin{array}{ccc} X & & \\ & \downarrow f & \\ Y & \xrightarrow{g} & Z \end{array}$  (I suppress reference to the  $E_X$  etc.) is (up to isomorphism) the set  $\{(x, y) | f(x) = g(y)\}$ , with  $E(x, y) = E_X(x) \wedge E_Y(y)$ ; and then, use the distributivity of  $\Omega$  to show that regular epis are stable under pullback.

**Exercise 87** Fill in the other gap in the proof: if  $f : (X, E_X) \rightarrow (Y, E_Y)$  is a regular epi, then  $f$  satisfies the condition given in the proof.

**Exercise 88** Given  $(X, E_X) \xrightarrow{f} (Y, E_Y)$ , give the interpretation of the formula  $\exists x(f(x) = y)$ , as subobject of  $(Y, E_Y)$ .

**Exercise 89** Characterize those objects  $(X, E_X)$  for which the unique map to the terminal object is a regular epimorphism.

**Exercise 90** Give a categorical proof of the statement: if  $f$  is the coequalizer of something, it is the coequalizer of its kernel pair.

**Exercise 91** Characterize the regular monos in  $\mathcal{C}_\Omega$ .

**Exercise 92** For every element  $u$  of  $\Omega$ , let  $1_u$  be the object of  $\mathcal{C}_\Omega$  which is the function from a one-element set into  $\Omega$ , with value  $u$ . Prove that every object of  $\mathcal{C}_\Omega$  is a coproduct of objects of form  $1_u$ .

**Exercise 93** Let  $\Omega$  be a frame. We consider the category  $\mathcal{C}_\Omega$  and the presheaf category  $\text{Set}^{\Omega^{\text{op}}}$ .

We have the Yoneda embedding  $y_{(-)} : \Omega \rightarrow \text{Set}^{\Omega^{\text{op}}}$  and we have a functor  $H : \Omega \rightarrow \mathcal{C}_\Omega$ , which sends  $u \in \Omega$  to the object  $1_u$  (in the notation of the previous exercise).

- a) Show that there is an essentially unique functor  $F : \mathcal{C}_\Omega \rightarrow \mathbf{Set}^{\Omega^{\text{op}}}$  which preserves all small coproducts and moreover makes the diagram

$$\begin{array}{ccc} \Omega & & \\ \downarrow H & \searrow y & \\ \mathcal{C}_\Omega & \xrightarrow{F} & \mathbf{Set}^{\Omega^{\text{op}}} \end{array}$$

commute. Give a concrete description of  $F(X, E_X)$  as a presheaf on  $\Omega$ .

- b) Suppose  $\Omega$  has a nonempty subset  $B$  with the property that  $\bigvee B \notin B$ . Show that the functor  $F$  does not preserve regular epis.

## 5 Adjunctions

The following kind of problem occurs quite regularly: suppose we have a functor  $\mathcal{C} \xrightarrow{G} \mathcal{D}$  and for a given object  $D$  of  $\mathcal{D}$ , we look for an object  $G(C)$  which “best approximates”  $D$ , in the sense that there is a map  $D \xrightarrow{\eta} G(C)$  with the property that any other map  $D \xrightarrow{g} G(C')$  factors uniquely as  $G(f)\eta$  for  $f : C \rightarrow C'$  in  $\mathcal{C}$ .

We have seen, that if  $G$  is the inclusion of  $\text{Abgp}$  into  $\text{Grp}$ , the abelianization of a group is an example. Another example is the Čech-Stone compactification in topology: for a completely regular topological space  $X$  one constructs a compact Hausdorff space  $\beta X$  out of it, and a map  $X \rightarrow \beta X$ , such that any continuous map from  $X$  to a compact Hausdorff space factors uniquely through this map.

Of course, what we described here is a sort of “right-sided” approximation; the reader can define for himself what the notion for a left-sided approximation would be.

The categorical description of this kind of phenomena goes via the concept of *adjunction*, which this chapter is about.

### 5.1 Adjoint functors

Let  $\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D}$  be a pair of functors between categories  $\mathcal{C}$  and  $\mathcal{D}$ .

We say that  $F$  is *left adjoint* to  $G$ , or  $G$  is *right adjoint* to  $F$ , notation:  $F \dashv G$ , if there is a natural bijection:

$$\mathcal{C}(FD, C) \xrightarrow{m_{D,C}} \mathcal{D}(D, GC)$$

for each pair of objects  $C \in \mathcal{C}_0, D \in \mathcal{D}_0$ . Two maps  $f : FD \rightarrow C$  in  $\mathcal{C}$  and  $g : D \rightarrow GC$  in  $\mathcal{D}$  which correspond to each other under this correspondence are called *transposes* of each other.

The naturality means that, given  $f : D \rightarrow D', g : C' \rightarrow C$  in  $\mathcal{D}$  and  $C$  respectively, the diagram

$$\begin{array}{ccc} \mathcal{C}(FD, C) & \xrightarrow{m_{D,C}} & \mathcal{D}(D, GC) \\ \mathcal{C}(Ff, g) \uparrow & & \uparrow \mathcal{D}(f, Gg) \\ \mathcal{C}(FD', C') & \xrightarrow{m_{D',C'}} & \mathcal{D}(D', GC') \end{array}$$

commutes in  $\text{Set}$ . Remind yourself that given  $\alpha : FD' \rightarrow C'$ ,  $\mathcal{C}(Ff, g)(\alpha) : FD \rightarrow C$  is the composite

$$FD \xrightarrow{Ff} FD' \xrightarrow{\alpha} C' \xrightarrow{g} C$$

Such a family  $m = (m_{D,C} | D \in \mathcal{D}_0, C \in \mathcal{C}_0)$  is then completely determined by the values it takes on identities; i.e. the values

$$m_{D,FD}(\text{id}_{FD}) : D \rightarrow GFD$$

For, given  $\alpha : FD \rightarrow C$ , since  $\alpha = \mathcal{C}(\text{id}_{FD}, \alpha)(\text{id}_{FD})$ ,

$$m_{D,C}(\alpha) = m_{D,C}(\mathcal{C}(\text{id}_{FD}, \alpha)(\text{id}_{FD})) = \mathcal{D}(\text{id}_D, G(\alpha))(m_{D,FD}(\text{id}_{FD}))$$

which is the composite

$$D \xrightarrow{m_{D,FD}(\text{id}_{FD})} GFD \xrightarrow{G(\alpha)} G(C)$$

The standard notation for  $m_{D,FD}(\text{id}_{FD})$  is  $\eta_D : D \rightarrow GF(D)$ .

**Exercise 94** Show that  $(\eta_D : D \in \mathcal{D}_0)$  is a natural transformation:

$$\text{id}_{\mathcal{D}} \Rightarrow GF$$

By the same reasoning, the natural family  $(m_{D,C}^{-1} | D \in \mathcal{D}_0, C \in \mathcal{C}_0)$  is completely determined by its actions on identities

$$m_{GC,C}^{-1}(\text{id}_{GC}) : FGC \rightarrow C$$

Again, the family  $(m_{GC,C}^{-1}(\text{id}_{GC}) | C \in \mathcal{C}_0)$  is a natural transformation:  $FG \Rightarrow \text{id}_C$ . We denote its components by  $\varepsilon_C$  and this is also standard notation.

We have that  $m_{D,C}^{-1}(\beta : D \rightarrow GC)$  is the composite

$$FD \xrightarrow{F\beta} FGC \xrightarrow{\varepsilon_C} C$$

Now making use of the fact that  $m_{D,C}$  and  $m_{D,C}^{-1}$  are each others inverse we get that for all  $\alpha : FD \rightarrow C$  and  $\beta : D \rightarrow GC$  the diagrams

$$\begin{array}{ccc} D & \xrightarrow{\beta} & GC \\ \eta_D \downarrow & & \uparrow G(\varepsilon_C) \\ GFD & \xrightarrow{GF(\beta)} & GFG(C) \end{array} \quad \text{and} \quad \begin{array}{ccc} FD & \xrightarrow{\alpha} & C \\ F(\eta_D) \downarrow & & \uparrow \varepsilon_C \\ FGFD & \xrightarrow{FG(\alpha)} & FGC \end{array}$$

commute; applying this to the identities on  $FD$  and  $GC$  we find that we have commuting diagrams of natural transformations:

$$\begin{array}{ccc} G & \xrightarrow{\eta \star G} & GFG \\ \text{id}_G \searrow & & \downarrow G \circ \varepsilon \\ & & G \end{array} \quad \begin{array}{ccc} F & \xrightarrow{F \circ \eta} & FGF \\ \text{id}_F \searrow & & \downarrow \varepsilon \star F \\ & & F \end{array}$$

Here  $\eta \star G$  denotes  $(\eta_{GC} | C \in \mathcal{C}_0)$  and  $G \circ \varepsilon$  denotes  $(G(\varepsilon_C) | C \in \mathcal{C}_0)$ .

Conversely, given  $\mathcal{C} \xleftarrow{F} \mathcal{D}$  with natural transformations  $\eta : \text{id}_{\mathcal{D}} \Rightarrow GF$  and  $\varepsilon : FG \Rightarrow \text{id}_{\mathcal{C}}$  which satisfy the above *triangle equalities*, we have that  $F \dashv G$ .

The tuple  $(F, G, \varepsilon, \eta)$  is called an *adjunction*.  $\eta$  is the *unit* of the adjunction,  $\varepsilon$  the *counit*.

**Exercise 95** Prove the statement above, that is: given  $\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D}$ ,  $\eta : \text{id}_{\mathcal{D}} \Rightarrow GF$  and  $\varepsilon : FG \Rightarrow \text{id}_{\mathcal{C}}$  satisfying  $(G \circ \varepsilon) \cdot (\eta \star G) = \text{id}_G$  and  $(\varepsilon \star F) \cdot (F \circ \eta) = \text{id}_F$ , we have  $F \dashv G$ .

**Exercise 96** Given  $\mathcal{C} \xrightleftharpoons[G_1]{F_1} \mathcal{D} \xrightleftharpoons[G_2]{F_2} \mathcal{E}$ , if  $F_1 \dashv G_1$  and  $F_2 \dashv G_2$  then  $F_1 F_2 \dashv G_2 G_1$ .

**Examples.** The world is full of examples of adjoint functors. We have met several:

- a) Consider the forgetful functor  $U : \text{Grp} \rightarrow \text{Set}$  and the free functor  $F : \text{Set} \rightarrow \text{Grp}$ . Given a function from a set  $A$  to a group  $G$  (which is an arrow  $A \rightarrow U(G)$  in  $\text{Set}$ ) we can uniquely extend it to a group homomorphism from  $(\bar{A}, \star)$  to  $G$  (see example e) of 1.1), i.e. an arrow  $F(A) \rightarrow G$  in  $\text{Grp}$ , and conversely. This is natural in  $A$  and  $G$ , so  $F \dashv U$ ;
- b) The functor  $\text{Dgrph} \rightarrow \text{Cat}$  given in example f) of 1.1 is left adjoint to the forgetful functor  $\text{Cat} \rightarrow \text{Dgrph}$ ;
- c) Given functors  $P \xrightleftharpoons[G]{F} Q$  between two preorders  $P$  and  $Q$ ,  $F \dashv G$  if and only if we have the equivalence

$$y \leq G(x) \Leftrightarrow F(y) \leq x$$

for  $x \in P, y \in Q$ ; if and only if we have  $FG(x) \leq x$  and  $y \leq GF(y)$  for all  $x, y$ ;

- d) In example m) of 1.1 we did “abelianization” of a group  $G$ . We made use of the fact that any homomorphism  $G \rightarrow H$  with  $H$  abelian, factors uniquely through  $G/[G, G]$ , giving a natural 1-1 correspondence

$$\text{Grp}(G, I(H)) \xrightarrow{\sim} \text{Abgp}(G/[G, G], H)$$

where  $I : \text{Abgp} \rightarrow \text{Grp}$  is the inclusion. So abelianization is left adjoint to the inclusion functor of abelian groups into groups;

- e) The *free monoid*  $F(A)$  on a set  $A$  is just the set of strings on the alphabet  $A$ .  $F : \text{Set} \rightarrow \text{Mon}$  is a functor left adjoint to the forgetful functor from  $\text{Mon}$  to  $\text{Set}$ ;
- f) Given a set  $X$  we have seen (example g) of 2.2) the product functor  $(-) \times X : \text{Set} \rightarrow \text{Set}$ , assigning the product  $Y \times X$  to a set  $Y$ .

Since there is a natural bijection between functions  $Y \times X \rightarrow Z$  and functions  $Y \rightarrow Z^X$ , the functor  $(-)^X : \text{Set} \rightarrow \text{Set}$  is right adjoint to  $(-) \times X$ ;

- g) Example e) of 2.2 gives two functors  $F, G : \text{Set} \rightarrow \text{Cat}$ .  $F$  and  $G$  are respectively left and right adjoint to the functor  $\text{Cat} \xrightarrow{\text{Ob}} \text{Set}$  which assigns to a (small) category its set of objects (to be precise, for this example to work we have to take for  $\text{Cat}$  the category of *small* categories), and to a functor its action on objects.
- h) Given a regular category  $\mathcal{C}$  we saw in 4.1 that every arrow  $f : X \rightarrow Y$  can be factored as a regular epi followed by a monomorphism. In Exercise 70 you were asked to show that there is a function  $\exists_f : \text{SUB}(X) \rightarrow \text{Sub}(Y)$  such that the equivalence  $\exists_f(M) \leq N \Leftrightarrow M \leq f^*(N)$  holds, for arbitrary subobjects  $M$  and  $N$  of  $X$  and  $Y$ , respectively.

But this just means that  $\exists_f \dashv f^*$

We can also express this logically: in the logic developed in chapter 4, for any formulas  $M(x)$  and  $N(y)$ , the sequents

$$\exists x(f(x) = y \wedge M(x)) \vdash_y N(y)$$

and

$$M(x) \vdash_x N(f(x))$$

are equivalent.

One of the slogans of categorical logic is therefore, that “existential quantification is left adjoint to substitution”.

- i) Let  $\mathcal{C}$  be a category with finite products; for  $C \in \mathcal{C}_0$  consider the slice category  $\mathcal{C}/C$ . There is a functor  $C^* : \mathcal{C} \rightarrow \mathcal{C}/C$  which assigns to  $D$  the object  $C \times D \xrightarrow{\pi_C} C$  of  $\mathcal{C}/C$ , and to maps  $D \xrightarrow{f} D'$  the map  $\text{id}_C \times f$ .  $C^*$  has a left adjoint  $\Sigma_C$  which takes the domain:  $\Sigma_C(D \rightarrow C) = D$ .
- j) Let  $P : \text{Set}^{\text{op}} \rightarrow \text{Set}$  be the functor which takes the powerset on objects, and for  $X \xrightarrow{f} Y$ ,  $P(f) : P(Y) \rightarrow P(X)$  gives for each subset  $B$  of  $Y$  its inverse image under  $f$ .

Now  $P$  might as well be regarded as a functor  $\text{Set} \rightarrow \text{Set}^{\text{op}}$ ; let's write  $\bar{P}$  for that functor. Since there is a natural bijection:

$$\text{Set}(X, P(Y)) \xrightarrow{\sim} \text{Set}(Y, P(X)) = \text{Set}^{\text{op}}(\bar{P}(X), Y)$$

we have an adjunction  $\bar{P} \dashv P$ .

**Exercise 97** A general converse to the last example. Suppose that  $F : \text{Set}^{\text{op}} \rightarrow \text{Set}$  is a functor, such that for the corresponding functor  $\bar{F} : \text{Set} \rightarrow \text{Set}^{\text{op}}$  we have that  $\bar{F} \dashv F$ . Then there is a set  $A$  such that  $F$  is naturally isomorphic to  $\text{Set}(-, A)$ .

**Exercise 98** Suppose that  $\mathcal{C} \xleftarrow{F} \mathcal{D}$  is a functor and that for each object  $C$  of  $\mathcal{C}$  there is an object  $GC$  of  $\mathcal{D}$  and an arrow  $\varepsilon_C : FGC \rightarrow C$  with the property that for every object  $D$  of  $\mathcal{D}$  and any map  $FD \xrightarrow{f} C$ , there is a unique  $\tilde{f} : D \rightarrow GC$  such that

$$\begin{array}{ccc} FD & \xrightarrow{f} & C \\ & \searrow F\tilde{f} & \nearrow \varepsilon_C \\ & FGC & \end{array}$$

commutes.

Prove that  $G : \mathcal{C}_0 \rightarrow \mathcal{D}_0$  extends to a functor  $G : \mathcal{C} \rightarrow \mathcal{D}$  which is right adjoint to  $F$ , and that  $(\varepsilon_C : FGC \rightarrow C | C \in \mathcal{C}_0)$  is the counit of the adjunction.

Construct also the unit of the adjunction.

**Exercise 99** Given  $\mathcal{C} \xrightarrow{G} \mathcal{D}$ , for each object  $D$  of  $\mathcal{D}$  we let  $(D \downarrow G)$  denote the category which has as objects pairs  $(C, g)$  where  $C$  is an object in  $\mathcal{C}$  and  $g : D \rightarrow GC$  is an arrow in  $\mathcal{D}$ . An arrow  $(C, g) \rightarrow (C', g')$  in  $(D \downarrow G)$  is an arrow  $f : C \rightarrow C'$  in  $\mathcal{C}$  which makes

$$\begin{array}{ccc} & D & \\ g \swarrow & & \searrow g' \\ GC & \xrightarrow{Gf} & GC' \end{array}$$

commute.

Show that  $G$  has a left adjoint if and only if for each  $D$ , the category  $(D \downarrow G)$  has an initial object.

**Exercise 100** (Uniqueness of adjoints) Any two left (or right) adjoints of a given functor are isomorphic as objects of the appropriate functor category.

**Exercise 101**  $\mathcal{D} \rightarrow \mathbf{1}$  has a right adjoint iff  $\mathcal{D}$  has a terminal object, and a left adjoint iff  $\mathcal{D}$  has an initial object.

**Exercise 102** Suppose  $\mathcal{D}$  has both an initial and a terminal object; denote by  $L$  the functor  $\mathcal{D} \rightarrow \mathcal{D}$  which sends everything to the initial, and by  $R$  the one which sends everything to the terminal object. Prove that  $L \dashv R$ .

**Exercise 103** We are given an adjunction  $\mathcal{E} \xleftarrow[I]{R} \mathcal{S}$  with  $R \dashv I$ , unit  $\eta$  and counit  $\varepsilon$ .

- a) Prove:  $I$  is faithful if and only if every component of  $\varepsilon$  is epi; and  $I$  is full if and only if every component of  $\varepsilon$  is split mono. Hint: use the fact that for an arrow  $A \xrightarrow{f} B$  in  $\mathcal{E}$ , the composite arrow  $RIA \xrightarrow{\varepsilon_A} A \xrightarrow{f} B$  transposes under the adjunction to the arrow  $I(f) : I(A) \rightarrow I(B)$ .

- b) Now suppose  $I$  is full and faithful. Prove: if  $F : \mathcal{A} \rightarrow \mathcal{E}$  is a diagram and  $IF$  has a limit in  $\mathcal{S}$ , then  $F$  has a limit in  $\mathcal{E}$ .

**Exercise 104** Suppose that  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an equivalence of categories with pseudo inverse  $G : \mathcal{D} \rightarrow \mathcal{C}$ . Show that both  $F \dashv G$  and  $G \dashv F$  hold.

**Exercise 105** Consider the situation of Exercise 93 and in particular the diagram given in part a) of that exercise.

- Show that the functor  $F$  has a left adjoint  $L$ .
- Show that this functor  $L$  does not preserve equalizers.

**Exercise 106** An object  $M$  of a category  $\mathcal{C}$  is called *injective* if for any diagram

$$\begin{array}{ccc} & A & \\ m \downarrow & \searrow f & \\ & B & \rightarrow M \end{array}$$

with  $m$  a monomorphism, there exists an arrow  $g : B \rightarrow M$  satisfying  $gm = f$ .

- Let  $\mathcal{C}, \mathcal{D}$  be categories and  $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$  functors with  $F \dashv G$ . Prove: if  $F$  preserves monos, then  $G$  preserves injective objects.
- Formulate the statement dual to part a) (the dual notion of ‘injective’ is *projective*).
- Now assume that in  $\mathcal{C}$ , for any object  $X$  there is an injective object  $M$  and a monomorphism  $m : X \rightarrow M$  (one says: the category  $\mathcal{C}$  has enough injectives). Prove the converse of part a).

## 5.2 Expressing (co)completeness by existence of adjoints; preservation of (co)limits by adjoint functors

Given categories  $\mathcal{C}$  and  $\mathcal{D}$ , we defined for every functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  its *limit* (or limiting cone), if it existed, as a pair  $(D, \mu)$  with  $\mu : \Delta_D \Rightarrow F$ , and  $(D, \mu)$  terminal in the category of cones for  $F$ .

Any other natural transformation  $\mu' : \Delta_{D'} \Rightarrow F$  factors uniquely through  $(D, \mu)$  via an arrow  $D' \rightarrow D$  in  $\mathcal{D}$  and conversely, every arrow  $D' \rightarrow D$  gives rise to a natural transformation  $\mu' : \Delta_{D'} \Rightarrow F$ .

So there is a 1-1 correspondence between

$$\mathcal{D}(D', D) \text{ and } \mathcal{D}^{\mathcal{C}}(\Delta_{D'}, F)$$

which is natural in  $D'$ .

Since every arrow  $D' \rightarrow D''$  in  $\mathcal{D}$  gives a natural transformation  $\Delta_{D'} \Rightarrow \Delta_{D''}$  (example i) of 2.2), there is a functor  $\Delta_{(-)} : \mathcal{D} \rightarrow \mathcal{D}^{\mathcal{C}}$ .

The above equation now means that:



**Proposition 5.1**  $\mathcal{D}$  has all limits of type  $\mathcal{C}$  (i.e. every functor  $\mathcal{C} \xrightarrow{F} \mathcal{D}$  has a limiting cone in  $\mathcal{D}$ ) if and only if  $\Delta_{(-)} : \mathcal{D} \rightarrow \mathcal{D}^{\mathcal{C}}$  has a right adjoint.

**Exercise 107** Give an exact proof of this proposition.

**Exercise 108** Use duality to deduce the dual of the proposition:  $\mathcal{D}$  has all colimits of type  $\mathcal{C}$  if and only if  $\Delta_{(-)} : \mathcal{D} \rightarrow \mathcal{D}^{\mathcal{C}}$  has a left adjoint.

A very important aspect of adjoint functors is their behaviour with respect to limits and colimits.

**Theorem 5.2** Let  $\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D}$  such that  $F \dashv G$ . Then:

- a)  $G$  preserves all limits which exist in  $\mathcal{C}$ ;
- b)  $F$  preserves all colimits which exist in  $\mathcal{D}$ .

**Proof.** Suppose  $M : \mathcal{E} \rightarrow \mathcal{C}$  has a limiting cone  $(C, \mu)$  in  $\mathcal{C}$ . Now a cone  $(D, \nu)$  for  $GM$  is a natural family  $D \xrightarrow{\nu_E} GM(E)$ , i.e. such that

$$\begin{array}{ccc} D & \xrightarrow{\nu_E} & GM(E) \\ & \searrow \nu_{E'} & \downarrow GM(e) \\ & & GM(E') \end{array}$$

commutes for every  $E \xrightarrow{e} E'$  in  $\mathcal{E}$ .

This transposes under the adjunction to a family  $(FD \xrightarrow{\tilde{\nu}_E} ME | E \in \mathcal{E}_0)$  and the naturality requirement implies that

$$\begin{array}{ccc} FD & \xrightarrow{\tilde{\nu}_E} & ME \\ & \searrow \tilde{\nu}_{E'} & \downarrow M(e) \\ & & ME' \end{array}$$

commutes in  $\mathcal{C}$ , in other words, that  $(FD, \tilde{\nu})$  is a cone for  $M$  in  $\mathcal{C}$ . There is, therefore, a unique map of cones from  $(FD, \tilde{\nu})$  to  $(C, \mu)$ .

Transposing back again, we get a unique map of cones  $(D, \nu) \rightarrow (GC, G \circ \mu)$ . That is to say that  $(GC, G \circ \mu)$  is terminal in  $\text{Cone}(GM)$ , so a limiting cone for  $GM$ , which was to be proved.

The argument for the other statement is dual. ■

**Exercise 109** Given  $\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D}$ ,  $F \dashv G$  and  $M : \mathcal{E} \rightarrow \mathcal{C}$ . Show that the functor  $\text{Cone}(M) \rightarrow \text{Cone}(GM)$  induced by  $G$  has a left adjoint.

From the theorem on preservation of (co)limits by adjoint functors one can often conclude that certain functors cannot have a right or a left adjoint.

### Examples

- a) The forgetful functor  $\mathbf{Mon} \rightarrow \mathbf{Set}$  does not preserve epis, as we have seen in 1.2. In chapter 3 we've seen that  $f$  is epi iff  $f$  is a pushout; since left adjoints preserve identities and pushouts, they preserve epis; therefore the forgetful functor  $\mathbf{Mon} \rightarrow \mathbf{Set}$  does not have a right adjoint;
- b) The functor  $(-) \times X : \mathbf{Set} \rightarrow \mathbf{Set}$  does not preserve the terminal object unless  $X$  is itself terminal in  $\mathbf{Set}$ ; therefore, it does not have a left adjoint for non-terminal  $X$ .
- c) The forgetful functor  $\mathbf{Pos} \rightarrow \mathbf{Set}$  has a left adjoint, but it cannot have a right adjoint: it preserves all coproducts, including the initial object, but not all coequalizers.

**Exercise 110** Prove the last example. Hint: think of the coequalizer of the following two maps  $\mathbb{Q} \rightarrow \mathbb{R}$ : one is the inclusion, the other is the constant zero map.

Another use of the theorem has to do with the computation of limits. Many categories, as we have seen, have a forgetful functor to  $\mathbf{Set}$  which has a left adjoint. So the forgetful functor preserves limits, and since these can easily be computed in  $\mathbf{Set}$ , one already knows the “underlying set” of the vertex of the limiting cone one wants to compute.

Does a converse to the theorem hold? I.e. given  $G : \mathcal{C} \rightarrow \mathcal{D}$  which preserves all limits; does  $G$  have a left adjoint? In general *no*, unless  $\mathcal{C}$  is sufficiently complete, and a rather technical condition, the “solution set condition” holds. The *adjoint functor theorem* (Freyd) tells that in that case there is a converse:

**Definition 5.3 (Solution set condition)**  $G : \mathcal{C} \rightarrow \mathcal{D}$  satisfies the solution set condition (ssc) for an object  $D$  of  $\mathcal{D}$ , if there is a set  $X_D$  of objects of  $\mathcal{C}$ , such that every arrow  $D \rightarrow GC$  factors as

$$\begin{array}{ccc} D & \longrightarrow & GC \\ & \searrow & \uparrow G(f) \\ & & GC' \end{array}$$

for some  $C' \in X_D$ .

**Theorem 5.4 (Adjoint Functor Theorem)** Let  $\mathcal{C}$  be a locally small, complete category and  $G : \mathcal{C} \rightarrow \mathcal{D}$  a functor.  $G$  has a left adjoint if and only if  $G$  preserves all small limits and satisfies the ssc for every object  $D$  of  $\mathcal{D}$ .

**Proof.** I sketch the proof for the ‘if’ part; convince yourself that the ‘only if’ part is trivial.

For any object  $D$  of  $\mathcal{D}$  let  $D\downarrow G$  be the category defined in exercise 98. By that exercise, we are looking for an initial object of  $D\downarrow G$ .

The solution set condition means, that there is a set  $\mathcal{K}_0$  of objects of  $D\downarrow G$  such that for any object  $x$  of  $D\downarrow G$  there is an element  $k \in \mathcal{K}_0$  and an arrow  $k \rightarrow x$  in  $D\downarrow G$ .

The fact that  $\mathcal{C}$  is complete and that  $G$  preserves all small limits, entails that  $D\downarrow G$  is complete. Moreover,  $D\downarrow G$  is locally small as  $\mathcal{C}$  is. Now let  $\mathcal{K}$  be the full subcategory of  $D\downarrow G$  with set of objects  $\mathcal{K}_0$ . Then since  $D\downarrow G$  is locally small and  $\mathcal{K}_0$  a set,  $\mathcal{K}$  is small. Take, by completeness of  $\mathcal{C}$ , a vertex of a limiting cone for the inclusion:  $\mathcal{K} \rightarrow D\downarrow G$ . Call this vertex  $x_0$ .  $x_0$  may not yet be an initial object of  $D\downarrow G$ , but now let  $\mathcal{M}$  be the full subcategory of  $D\downarrow G$  on the single object  $x_0$  ( $M$  is a monoid), and let  $x$  be a vertex of a limiting cone for the inclusion  $\mathcal{M} \rightarrow D\downarrow G$ .  $x$  is the *joint equalizer* of all arrows  $f : x_0 \rightarrow x_0$  in  $D\downarrow G$ , and this will be an initial object in  $D\downarrow G$ . ■

Let me remark that in natural situations, the ssc is always satisfied. But then in those situations, one generally does not invoke the Adjoint Functor Theorem in order to conclude to the existence of a left adjoint. The value of this theorem is theoretical, rather than practical.

For small categories  $\mathcal{C}$ , the ssc is of course irrelevant. But categories which are small *and* complete are complete preorders, as we saw in chapter 3.

For preorders  $\mathcal{C}, \mathcal{D}$  we have: if  $\mathcal{C}$  is complete, then  $G : \mathcal{C} \rightarrow \mathcal{D}$  has a left adjoint if and only if  $G$  preserves all limits, that is: greatest lower bounds  $\bigwedge B$  for all  $B \subseteq \mathcal{C}$ . For, put

$$F(d) = \bigwedge \{c \mid d \leq G(c)\}$$

Then  $F(d) \leq c'$  implies (since  $G$  preserves  $\bigwedge$ )  $\bigwedge \{G(c) \mid d \leq G(c)\} \leq G(c')$  which implies  $d \leq G(c')$  since  $d \leq \bigwedge \{G(c) \mid d \leq G(c)\}$ ; conversely,  $d \leq G(c')$  implies  $c' \in \{c \mid d \leq G(c)\}$  so  $F(d) = \bigwedge \{c \mid d \leq G(c)\} \leq c'$ .

## 6 Monads and Algebras

Given an adjunction  $(F, G, \varepsilon, \eta) : \mathcal{C} \rightleftarrows \mathcal{D}$  let us look at the functor  $T = GF : \mathcal{D} \rightarrow \mathcal{D}$ .

We have a natural transformation  $\eta : \text{id}_{\mathcal{D}} \Rightarrow T$  and a natural transformation  $\mu : T^2 \Rightarrow T$ . The components  $\mu_D$  are

$$T^2(D) = GFGFD \xrightarrow{G(\varepsilon_{FD})} GFD = T(D)$$

Furthermore the equalities

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \mu T \downarrow & & \downarrow \mu \\ T^2 & \xrightarrow{\mu} & T \end{array} \quad \text{and} \quad \begin{array}{ccc} T & \xrightarrow{\eta T} & T^2 & \xleftarrow{T\eta} & T \\ & \searrow \text{id}_T & \downarrow \mu & \swarrow \text{id}_T & \\ & & T & & \end{array}$$

hold. Here  $(T\mu)_D = T(\mu_D) : T^3D \rightarrow TD$  and  $(\mu T)_D = \mu_{TD} : T^3D \rightarrow TD$  (Similar for  $\eta T$  and  $T\eta$ ).

**Exercise 111** Prove these equalities.

A triple  $(T, \mu, \eta)$  satisfying these identities is called a *monad*. Try to see the formal analogy between the defining equalities for a monad and the axioms for a monoid: writing  $m(e, f)$  for  $ef$  in a monoid, and  $\eta$  for the unit element, we have

$$\begin{aligned} m(e, m(g, h)) &= m(m(e, g), h) & (\text{associativity}) \\ m(\eta, e) &= m(e, \eta) = e & (\text{unit}) \end{aligned}$$

Following this one calls  $\mu$  the *multiplication* of the monad, and  $\eta$  its *unit*.

**Example.** The powerset functor  $\mathcal{P} : \text{Set} \rightarrow \text{Set}$  (example j) of 2.2, with  $\eta$  and  $\mu$  indicated there) is a monad (check).

Dually, there is the notion of a *comonad*  $(L, \delta, \varepsilon)$  on a category  $\mathcal{C}$ , with equalities

$$\begin{array}{ccc} L & \xrightarrow{\delta} & L^2 \\ \delta \downarrow & & \downarrow L\delta \\ L^2 & \xrightarrow{\delta L} & L^3 \end{array} \quad \begin{array}{ccccc} & & L & & \\ & \swarrow \text{id}_L & \downarrow \delta & \searrow \text{id}_L & \\ L & \xleftarrow{\varepsilon L} & L^2 & \xrightarrow{L\varepsilon} & L \end{array}$$

Given an adjunction  $(F, G, \varepsilon, \eta)$ ,  $(FG, \delta = F\eta G, \varepsilon)$  is a comonad on  $\mathcal{C}$ . We call  $\delta$  the *comultiplication* and  $\varepsilon$  the *counit* (this is in harmony with the unit-counit terminology for adjunctions).

Although, in many contexts, comonads and the notions derived from them are at least as important as monads, the treatment is dual so I concentrate on monads.

Every adjunction gives rise to a monad; conversely, every monad arises from an adjunction, but in more than one way. Essentially, there are a maximal and

a minimal solution to the problem of finding an adjunction from which a given monad arises.

**Example.** A monad on a poset  $P$  is a monotone function  $T : P \rightarrow P$  with the properties  $x \leq T(x)$  and  $T^2(x) \leq T(x)$  for all  $x \in P$ ; such an operation is also often called a *closure operation* on  $P$ . Note that  $T^2 = T$  because  $T$  is monotone.

In this situation, let  $Q \subseteq P$  be the image of  $T$ , with the ordering inherited from  $P$ . We have maps  $r : P \rightarrow Q$  and  $i : Q \rightarrow P$  such that  $ri$  is the identity on  $Q$  and  $ir = T : P \rightarrow P$ .

For  $x \in P$ ,  $y \in Q$  we have  $x \leq i(y) \Leftrightarrow r(x) \leq y$  (check); so  $r \dashv i$  and the operation  $T$  arises from this adjunction.

## 6.1 Algebras for a monad

Given a monad  $(T, \eta, \mu)$  on a category  $\mathcal{C}$ , we define the category  $T\text{-Alg}$  of *T-algebras* as follows:

- *Objects* are pairs  $(X, h)$  where  $X$  is an object of  $\mathcal{C}$  and  $h : T(X) \rightarrow X$  is an arrow in  $\mathcal{C}$  such that

$$\begin{array}{ccc} T^2(X) & \xrightarrow{T(h)} & T(X) \\ \mu_X \downarrow & & \downarrow h \\ T(X) & \xrightarrow{h} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\eta_X} & T(X) \\ & \searrow \text{id}_X & \downarrow h \\ & & X \end{array}$$

commute;

- *Morphisms:*  $(X, h) \rightarrow (Y, k)$  are morphisms  $X \xrightarrow{f} Y$  in  $\mathcal{C}$  for which

$$\begin{array}{ccc} T(X) & \xrightarrow{T(f)} & T(Y) \\ h \downarrow & & \downarrow k \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

**Theorem 6.1** *There is an adjunction between  $T\text{-Alg}$  and  $\mathcal{C}$  which brings about the given monad  $T$ .*

**Proof.** There is an obvious forgetful functor  $U^T : T\text{-Alg} \rightarrow \mathcal{C}$  which takes  $(X, h)$  to  $X$ . I claim that  $U^T$  has a left adjoint  $F^T$ :

$F^T$  assigns to an object  $X$  the  $T$ -algebra  $T^2(X) \xrightarrow{\mu_X} T(X)$ ; to  $X \xrightarrow{f} Y$  the map  $T(f)$ ; this is an algebra map because of the naturality of  $\mu$ . That  $T^2(X) \xrightarrow{\mu_X} T(X)$  is an algebra follows from the defining axioms for a monad  $T$ .

Now given any arrow  $g : X \rightarrow U^T(Y, h)$  we let  $\tilde{g} : (T(X), \mu_X) \rightarrow (Y, h)$  be the arrow  $T(X) \xrightarrow{T(g)} T(Y) \xrightarrow{h} Y$ . This is a map of algebras since

$$\begin{array}{ccccc} T^2(X) & \xrightarrow{T^2(g)} & T^2(Y) & \xrightarrow{T(h)} & T(Y) \\ \mu_X \downarrow & & \mu_Y \downarrow & & \downarrow h \\ T(X) & \xrightarrow{T(g)} & T(Y) & \xrightarrow{h} & Y \end{array}$$

commutes; the left hand square is the naturality of  $\mu$ ; the right hand square is because  $(Y, h)$  is a  $T$ -algebra.

Conversely, given a map of algebras  $f : (TX, \mu_X) \rightarrow (Y, h)$  we have an arrow  $\tilde{f} : X \rightarrow Y$  by taking the composite  $X \xrightarrow{\eta_X} TX \xrightarrow{f} Y$ .

Now  $\tilde{\tilde{f}} : TX \rightarrow Y$  is the composite

$$TX \xrightarrow{T(\eta_X)} T^2X \xrightarrow{T(f)} TY \xrightarrow{h} Y$$

Since  $f$  is a  $T$ -algebra map, this is

$$T(X) \xrightarrow{T(\eta_X)} T^2(X) \xrightarrow{\mu_X} T(X) \xrightarrow{f} Y$$

which is  $f$  by the monad laws.

Conversely,  $\tilde{\tilde{g}} : X \rightarrow Y$  is the composite

$$X \xrightarrow{\eta_X} TX \xrightarrow{T(g)} TY \xrightarrow{h} Y$$

By naturality of  $\eta$  and the fact that  $(Y, h)$  is a  $T$ -algebra, we conclude that  $\tilde{\tilde{g}} = g$ . So we have a natural 1-1 correspondence

$$\mathcal{C}(X, U^T(Y, h)) \simeq T\text{-Alg}(F^T(X), (Y, h))$$

and our adjunction.

Note that the composite  $U^T F^T$  is the functor  $T$ , and that the unit  $\eta$  of the adjunction is the unit of  $T$ ; the proof that for the counit  $\varepsilon$  of  $F^T \dashv U^T$  we have that

$$T^2 = U^T F^T U^T F^T \xrightarrow{U^T \varepsilon_{F^T}} U^T F^T = T$$

is the original multiplication  $\mu$ , is left to you. ■

**Exercise 112** Complete the proof.

**Example.** The *group monad*. Combining the forgetful functor  $U : \text{Grp} \rightarrow \text{Set}$  with the left adjoint, the free functor  $\text{Set} \rightarrow \text{Grp}$ , we get the following monad on  $\text{Set}$ :

$T(A)$  is the set of sequences on the alphabet  $A \sqcup A^{-1}$  ( $A^{-1}$  is the set  $\{a^{-1} | a \in A\}$  of formal inverses of elements of  $A$ , as in example e) of 1.1) in which no

$aa^{-1}$  or  $a^{-1}a$  occur. The unit  $A \xrightarrow{\eta_A} TA$  sends  $a \in A$  to the string  $\langle a \rangle$ . The multiplication  $\mu : T^2(A) \rightarrow T(A)$  works as follows. Define  $(-)^- : A \sqcup A^{-1} \rightarrow A \sqcup A^{-1}$  by  $a^- = a^{-1}$  and  $(a^{-1})^- = a$ . Define also  $(-)^-$  on strings by  $(a_1 \dots a_n)^- = a_n^- \dots a_1^-$ . Now for an element of  $TT(A)$ , which is a string on the alphabet  $T(A) \sqcup T(A)^{-1}$ , say  $\sigma_1 \dots \sigma_n$ , we let  $\mu_A(\sigma_1 \dots \sigma_n)$  be the concatenation of the strings  $\tilde{\sigma}_1, \dots, \tilde{\sigma}_n$  on the alphabet  $A \sqcup A^{-1}$ , where  $\tilde{\sigma}_i = \sigma_i$  if  $\sigma_i \in T(A)$ , and  $\tilde{\sigma}_i = (\sigma_i)^-$  if  $\sigma_i \in T(A)^{-1}$ . Of course we still have to remove possible substrings of the form  $aa^{-1}$  etc.

Now let us look at algebras for the group monad: maps  $T(A) \xrightarrow{h} A$  such that for a string of strings

$$\alpha = \sigma_1, \dots, \sigma_n = \langle \langle s_1^1, \dots, s_1^{k_1} \rangle, \dots, \langle s_n^1, \dots, s_n^{k_n} \rangle \rangle$$

we have that

$$h(\langle h\sigma_1, \dots, h\sigma_n \rangle) = h(\langle s_1^1, \dots, s_1^{k_1}, \dots, s_n^1, \dots, s_n^{k_n} \rangle)$$

and

$$h(\langle a \rangle) = a \text{ for } a \in A$$

I claim that this is the same thing as a group structure on  $A$ , with multiplication  $a \cdot b = h(\langle a, b \rangle)$ .

The unit element is given by  $h(\langle \rangle)$ ; the inverse of  $a \in A$  is  $h(\langle a^{-1} \rangle)$  since

$$\begin{aligned} h(\langle a, h(\langle a^{-1} \rangle) \rangle) &= h(\langle h(\langle a \rangle), h(\langle a^{-1} \rangle) \rangle) = \\ h(\langle a, a^{-1} \rangle) &= h(\langle \rangle), \text{ the unit element} \end{aligned}$$

Try to see for yourself how the associativity of the monad and its algebras transforms into associativity of the group law.

**Exercise 113** Finish the proof of the theorem: for the group monad  $T$ , there is an equivalence of categories between  $T\text{-Alg}$  and  $\text{Grp}$ .

This situation is very important and has its own name:

**Definition 6.2** Given an adjunction  $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$ ,  $F \dashv G$ , there is always a comparison functor  $K : \mathcal{C} \rightarrow T\text{-Alg}$  for  $T = GF$ , the monad induced by the adjunction.  $K$  sends an object  $C$  of  $\mathcal{C}$  to the  $T$ -algebra  $GFG(C) \xrightarrow{G(\varepsilon_C)} G(C)$ .

We say that the functor  $G : \mathcal{C} \rightarrow \mathcal{D}$  is monadic, or by abuse of language (if  $G$  is understood), that  $\mathcal{C}$  is monadic over  $\mathcal{D}$ , if  $K$  is an equivalence.

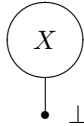
**Exercise 114** Check that  $K(C)$  is a  $T$ -algebra. Complete the definition of  $K$  as a functor. Check that in the example of the group monad, the functor  $T\text{-Alg} \rightarrow \text{Grp}$  defined there is a pseudo inverse to the comparison functor  $K$ .

In many cases however, the situation is not monadic. Take the forgetful functor  $U : \text{Pos} \rightarrow \text{Set}$ . It has a left adjoint  $F$  which sends a set  $X$  to the discrete ordering on  $X$  ( $x \leq y$  iff  $x = y$ ). Of course,  $UF$  is the identity on  $\text{Set}$  and the  $UF$ -algebras are just sets. The comparison functor  $K$  is the functor  $U$ , and this is not an equivalence.

**Exercise 115** Why not?

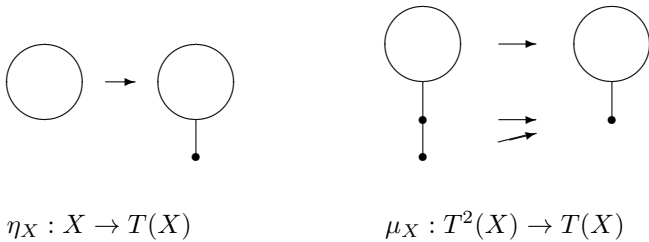
Another example of a monadic situation is of importance in domain theory. Let  $\text{Pos}_\perp$  be the category of partially ordered sets with a least element, and order preserving maps which also preserve the least element.

There is an obvious inclusion functor  $U : \text{Pos}_\perp \rightarrow \text{Pos}$ , and  $U$  has a left adjoint  $F$ . Given a poset  $X$ ,  $F(X)$  is  $X$  “with a bottom element added”:



Given  $f : X \rightarrow Y$  in  $\text{Pos}$ ,  $F(f)$  sends the new bottom element of  $X$  to the new bottom element of  $Y$ , and is just  $f$  on the rest. If  $f : X \rightarrow Y$  in  $\text{Pos}$  is a map and  $Y$  has a least element, we have  $F(X) \rightarrow Y$  in  $\text{Pos}_\perp$  by sending  $\perp$  to the least element of  $Y$ . So the adjunction is clear.

The monad  $UF : \text{Pos} \rightarrow \text{Pos}$ , just adding a least element, is called the *lifting monad*. Unit and multiplication are:



A  $T$ -algebra  $h : TX \rightarrow X$  is first of all a monotone map, but since  $h\eta_X = \text{id}_X$ ,  $h$  is epi in  $\text{Pos}$  so surjective. It follows that  $X$  must have a least element  $h(\perp)$ . From the axioms for an algebra one deduces that  $h$  must be the identity when restricted to  $X$ , and  $h(\perp)$  the least element of  $X$ .

**Exercise 116** Given  $\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D}$ ,  $F \dashv G$ ,  $T = GF$ . Prove that the comparison functor  $K : \mathcal{C} \rightarrow T\text{-Alg}$  satisfies  $U^T K = G$  and  $KF = F^T$  where  $T\text{-Alg} \xrightleftharpoons[U^T]{F^T} \mathcal{D}$  as in theorem 6.1.

Another poset example: algebras for the power set monad  $\mathcal{P}$  on  $\text{Set}$  (example j) of 2.2). Such an algebra  $h : \mathcal{P}(X) \rightarrow X$  must satisfy  $h(\{x\}) = x$  and for  $\alpha \subseteq \mathcal{P}(X)$ :

$$h(\{h(A) \mid A \in \alpha\}) = h(\{x \mid \exists A \in \alpha (x \in A)\}) = h\left(\bigcup \alpha\right)$$



Now we can, given an algebra structure on  $X$ , define a partial order on  $X$  by putting  $x \leq y$  iff  $h(\{x, y\}) = y$ .

Indeed, this is clearly reflexive and antisymmetric. As to transitivity, if  $x \leq y$  and  $y \leq z$  then

$$\begin{aligned} h(\{x, z\}) &= h(\{x, h(\{y, z\})\}) &= \\ h(\{h(\{x\}), h(\{y, z\})\}) &= h(\{x\} \cup \{y, z\}) &= \\ h(\{x, y\} \cup \{z\}) &= h(\{h(\{x, y\}), h(\{z\})\}) = \\ h(\{y, z\}) &= z \end{aligned}$$

so  $x \leq z$ .

Furthermore this partial order is *complete*: least upper bounds for arbitrary subsets exist. For  $\bigvee B = h(B)$  for  $B \subseteq X$ : for  $x \in B$  we have  $h(\{x, h(B)\}) = h(\{x\} \cup B) = h(B)$  so  $x \leq \bigvee B$ ; and if  $x \leq y$  for all  $x \in B$  then

$$\begin{aligned} h(\{h(B), y\}) &= h(B \cup \{y\}) &= \\ h(\bigcup_{x \in B} \{x, y\}) &= h(\{h(\{x, y\}) \mid x \in B\}) = \\ h(\{y\}) &= y \end{aligned}$$

so  $\bigvee B \leq y$ .

We can also check that a map of algebras is a  $\bigvee$ -preserving monotone function. Conversely, every  $\bigvee$ -preserving monotone function between complete posets determines a  $\mathcal{P}$ -algebra homomorphism.

We have an equivalence between the category of complete posets and  $\bigvee$ -preserving functions, and  $\mathcal{P}$ -algebras.

**Exercise 117** Let  $P : \text{Set}^{\text{op}} \rightarrow \text{Set}$  be the contravariant powerset functor, and  $\bar{P}$  its left adjoint, as in j) of 5.1. Let  $T : \text{Set} \rightarrow \text{Set}$  the induced monad.

- Describe unit and multiplication of this monad explicitly.
- Show that  $\text{Set}^{\text{op}}$  is equivalent to  $T\text{-Alg}$  [Hint: if this proves hard, have a look at VI.4.3 of Johnstone's "Stone Spaces"].
- Conclude that there is an adjunction

$$\text{CABool} \longleftrightarrow \text{Set}$$

which presents CABool as monadic over Set.

## 6.2 $T$ -Algebras at least as complete as $\mathcal{D}$

Let  $T$  be a monad on  $\mathcal{D}$ . The following exercise is meant to show that if  $\mathcal{D}$  has all limits of a certain type, so does  $T\text{-Alg}$ . In particular, if  $\mathcal{D}$  is complete, so is  $T\text{-Alg}$ ; this is often an important application of a monadic situation.

**Exercise 118** Let  $\mathcal{E}$  be a category such that every functor  $M : \mathcal{E} \rightarrow \mathcal{D}$  has a limiting cone. Now suppose  $M : \mathcal{E} \rightarrow T\text{-Alg}$ . For objects  $E$  of  $\mathcal{E}$ , let  $M(E)$  be the  $T$ -algebra  $T(m_E) \xrightarrow{h_E} m_E$ .

- a) Let  $(D, (\nu_E | E \in \mathcal{E}_0))$  be a limiting cone for  $U^T M : \mathcal{E} \rightarrow \mathcal{D}$ . Using the  $T$ -algebra structure on  $M(E)$  and the fact that  $U^T M(E) = m_E$ , show that there is also a cone  $(TD, (\pi_E | E \in \mathcal{E}_0))$  for  $U^T M$ ;
- b) Show that the unique map of cones:  $(TD, \pi) \rightarrow (D, \nu)$  induces a  $T$ -algebra structure  $TD \xrightarrow{h} D$  on  $D$ ;
- c) Show that  $TD \xrightarrow{h} D$  is the vertex of a limiting cone for  $M$  in  $T\text{-Alg}$ .

This exercise gives a situation which has its own name. For a functor  $G : \mathcal{C} \rightarrow \mathcal{D}$  we say that  $G$  *creates limits* of type  $\mathcal{E}$  if for every functor  $M : \mathcal{E} \rightarrow \mathcal{C}$  and every limiting cone  $(D, \mu)$  for  $GM$  in  $\mathcal{D}$ , there is a unique cone  $(C, \nu)$  for  $M$  in  $\mathcal{C}$  which is taken by  $G$  to  $(D, \mu)$ , and moreover this unique cone is limiting for  $M$  in  $\mathcal{C}$ .

Clearly, if  $G$  creates limits of type  $\mathcal{E}$  and  $\mathcal{D}$  has all limits of type  $\mathcal{E}$ , then  $\mathcal{C}$  has them, too. The exercise proves that the forgetful functor  $U^T : T\text{-Alg} \rightarrow \mathcal{D}$  creates limits of every type.

### 6.3 The Kleisli category of a monad

I said before that for a monad  $T$  on a category  $\mathcal{D}$ , there are a “maximal and a minimal solution” to the problem of finding an adjunction which induces the given monad.

We’ve seen the category  $T\text{-Alg}$ , which we now write as  $\mathcal{D}^T$ ; we also write  $G^T : T\text{-Alg} \rightarrow \mathcal{D}$  for the forgetful functor. In case  $T$  arises from an adjunction  $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$ , there was a comparison functor  $\mathcal{C} \xrightarrow{K} \mathcal{D}^T$ . In the diagram

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{K} & \mathcal{D}^T \\
 \swarrow F & & \searrow G^T \\
 & \mathcal{D} & \\
 \nwarrow G & & \nearrow F^T
 \end{array}$$

we have that  $KF = F^T$  and  $G^T K = G$ .

Moreover, the functor  $K$  is unique with this property.

This leads us to define a category  $T\text{-Adj}$  of adjunctions  $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$  such that  $GF = T$ . A *map* of such  $T$ -adjunctions from  $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$  to  $\mathcal{C}' \xrightleftharpoons[G']{F'} \mathcal{D}$  is a functor  $K : \mathcal{C} \rightarrow \mathcal{C}'$  satisfying  $KF = F'$  and  $G'K = G$ .

What we have proved about  $T\text{-Alg}$  can be summarized by saying that the adjunction  $\mathcal{D}^T \xrightleftharpoons[G^T]{F^T} \mathcal{D}$  is a *terminal object* in  $T\text{-Adj}$ . This was the “maximal” solution.

$T\text{-Adj}$  has also an initial object: the *Kleisli category* of  $T$ , called  $\mathcal{D}_T$ .  $\mathcal{D}_T$  has the same objects as  $\mathcal{D}$ , but a map in  $\mathcal{D}_T$  from  $X$  to  $Y$  is an arrow  $X \xrightarrow{f} T(Y)$

in  $\mathcal{D}$ . Composition is defined as follows: given  $X \xrightarrow{f} T(Y)$  and  $Y \xrightarrow{g} T(Z)$  in  $\mathcal{D}$ , considered as a composable pair of morphisms in  $\mathcal{D}_T$ , the composition  $gf$  in  $\mathcal{D}_T$  is the composite

$$X \xrightarrow{f} T(Y) \xrightarrow{T(g)} T^2(Z) \xrightarrow{\mu_Z} T(Z)$$

in  $\mathcal{D}$ .

**Exercise 119** Prove that composition is associative. What are the identities of  $\mathcal{D}_T$ ?

The adjunction  $\mathcal{D}_T \xrightleftharpoons[G_T]{F_T} \mathcal{D}$  is defined as follows: the functor  $G_T$  sends the object  $X$  to  $T(X)$  and  $f : X \rightarrow Y$  ( $f : X \rightarrow T(Y)$  in  $\mathcal{D}$ ) to

$$T(X) \xrightarrow{T(f)} T^2(Y) \xrightarrow{\mu_Y} T(Y)$$

The functor  $F_T$  is the identity on objects and sends  $X \xrightarrow{f} Y$  to  $X \xrightarrow{f} Y \xrightarrow{\eta_X} T(Y)$ , considered as  $X \rightarrow Y$  in  $\mathcal{D}_T$ .

**Exercise 120** Define unit and counit; check  $F_T \dashv G_T$ .

**Exercise 121** Let  $T$  be a monad on  $\mathcal{D}$ . Call an object of  $T\text{-Alg}$  *free* if it is in the image of  $F^T : \mathcal{D} \rightarrow T\text{-Alg}$ . Show that the Kleisli category  $\mathcal{D}_T$  is equivalent to the full subcategory of  $T\text{-Alg}$  on the free  $T$ -algebras.

Now for every adjunction  $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$  with  $GF = T$ , there is a unique comparison functor  $L : \mathcal{D}_T \rightarrow \mathcal{C}$  such that  $GL = G_T$  and  $LF_T = F$ .

$L$  sends the object  $X$  to  $F(X)$  and  $f : X \rightarrow Y$  (so  $f : X \rightarrow T(Y) = GF(Y)$  in  $\mathcal{D}$ ) to its transpose  $\tilde{f} : F(X) \rightarrow F(Y)$ .

**Exercise 122** Check the commutations. Prove the uniqueness of  $L$  w.r.t. these properties.

**Exercise 123** Let  $\text{Rng1}$  be the category of rings with unit and unitary ring homomorphisms. Since every ring with 1 is a (multiplicative) monoid, there is a forgetful functor  $G : \text{Rng1} \rightarrow \text{Mon}$ . For a monoid  $M$ , let  $Z[M]$  be the ring of formal expressions

$$n_1 c_1 + \cdots + n_k c_k$$

with  $k \geq 0$ ,  $n_1, \dots, n_k \in Z$  and  $c_1, \dots, c_k \in M$ . This is like a ring of polynomials, but multiplication uses the multiplication in  $M$ . Show that this defines a functor  $F : \text{Mon} \rightarrow \text{Rng1}$  which is left adjoint to  $G$ , and that  $G$  is monadic, i.e. the category of  $GF$ -algebras is equivalent to  $\text{Rng1}$ . [Hint: Proceed as in the example of the powerset monad. That is, let  $h : GF(M) \rightarrow M$  be a monoid homomorphism which gives  $M$  the structure of a  $GF$ -algebra. Find an abelian group structure on  $M$  such that  $M$  becomes a ring with unit]

**Exercise 124** What does the Kleisli category for the covariant powerset monad look like?

## 7 Cartesian closed categories and the $\lambda$ -calculus

Many set-theoretical constructions are completely determined (up to isomorphism, as always) by their categorical properties in **Set**. We are therefore tempted to generalize them to arbitrary categories, by taking the characteristic categorical property as a definition. Of course, this procedure is not really well-defined and it requires sometimes a real insight to pick the ‘right’ categorical generalization. For example, the category of sets has very special properties:

- $f : X \rightarrow Y$  is mono if and only if  $fg = fh$  implies  $g = h$  for any two maps  $g, h : 1 \rightarrow X$ , where  $1$  is a terminal object (we say  $1$  is *a generator*);
- objects  $X$  and  $Y$  are isomorphic if there exist monos  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  (the Cantor-Bernstein theorem);
- every mono  $X \xrightarrow{f} Y$  is part of a coproduct diagram

$$\begin{array}{ccc} X & & \\ & \searrow f & \\ Z & \xrightarrow{g} & Y \end{array}$$

And if you believe the axiom of choice, there is its categorical version:

- Every epi is split

None of these properties is generally valid, and categorical generalizations based on them are usually of limited value.

In this chapter we focus on a categorical generalization of a set-theoretical concept which has proved to have numerous applications: Cartesian closed categories as the generalization of “function space”.

In example f) of 5.1 we saw that the set of functions  $Z^X$  appears as the value at  $Z$  of the right adjoint to the product functor  $(-) \times X$ . A category is called *cartesian closed* if such right adjoints always exist. In such categories we may really think of this right adjoint as giving the “object of functions (or arrows)”, as the treatment of the  $\lambda$ -calculus will make clear.

### 7.1 Cartesian closed categories (ccc’s); examples and basic facts

**Definition 7.1** *A category  $\mathcal{C}$  is called cartesian closed or a ccc if it has finite products, and for every object  $X$  of  $\mathcal{C}$  the product functor  $(-) \times X$  has a right adjoint.*

Of course, “the” product functor only exists once we have chosen a product diagram for every pair of objects of  $\mathcal{C}$ . In this chapter we assume that we have such a choice, as well as a distinguished terminal object  $1$ ; and we assume

also that for each object  $X$  we have a *specified* right adjoint to the functor  $(-) \times X$ , which we write as  $(-)^X$  (Many authors write  $X \Rightarrow (-)$ , but I think that overloads the arrows notation too much). Objects of the form  $Z^X$  are called *exponents*.

We have the *unit*

$$Y \xrightarrow{\eta_{Y,X}} (Y \times X)^X$$

and *counit*

$$Y^X \times X \xrightarrow{\varepsilon_{Y,X}} Y$$

of the adjunction  $(-) \times X \dashv (-)^X$ . Anticipating the view of  $Y^X$  as the object of arrows  $X \rightarrow Y$ , we call  $\varepsilon$  *evaluation*.

## Examples

- a) A preorder (or partial order) is cartesian closed if it has a top element 1, binary meets  $x \wedge y$  and for any two elements  $x, y$  an element  $x \rightarrow y$  satisfying for each  $z$ :

$$z \leq x \rightarrow y \text{ iff } z \wedge x \leq y$$

- b) Set is cartesian closed; Cat is cartesian closed (2.1);
- c) Top is not cartesian closed. In chapter 4 it was remarked, that for non-locally compact spaces  $X$ , the functor  $X \times (-)$  will not preserve quotients (coequalizers); hence, it cannot have a right adjoint. There are various subcategories of Top which are cartesian closed, if one takes as exponent  $Y^X$  the set of continuous maps  $Y \rightarrow X$ , topologized with the compact-open topology.
- d) Pos is cartesian closed. The exponent  $Y^X$  is the set of all monotone maps  $X \rightarrow Y$ , ordered pointwise ( $f \leq g$  iff for all  $x \in X$ ,  $fx \leq gx$  in  $Y$ );
- e) Grp and Abgp are not cartesian closed. In both categories, the initial object is the one-element group. Since for non-initial groups  $G$ ,  $(-) \times G$  does not preserve the initial object, it cannot have a right adjoint (the same argument holds for Mon);
- f) **1** is cartesian closed; **0** isn't (why?);
- g)  $\text{Set}^{\text{cop}}$  is cartesian closed. Products and 1 are given “pointwise” (in fact all limits are), that is  $F \times G(C) = F(C) \times G(C)$  and  $1(C)$  is the terminal 1 in Set, for all  $C \in \mathcal{C}_0$ .

The construction of the exponent  $G^F$  is a nice application of the Yoneda lemma. Indeed, for  $G^F$  to be the right adjoint (at  $G$ ) of  $(-) \times F$ , we need for every object  $C$  of  $\mathcal{C}$ :

$$\text{Set}^{\text{cop}}(h_C \times F, G) \simeq \text{Set}^{\text{cop}}(h_C, G^F) \simeq G^F(C)$$

where the last isomorphism is by the Yoneda lemma.

Now the assignment  $C \mapsto \text{Set}^{\mathcal{C}^{\text{op}}}(h_C \times F, G)$  defines a functor  $\mathcal{C}^{\text{op}} \rightarrow \text{Set}$ , which we denote by  $G^F$ . At the same time, this construction defines a functor  $(-)^F : \text{Set}^{\mathcal{C}^{\text{op}}} \rightarrow \text{Set}^{\mathcal{C}^{\text{op}}}$ , which is right adjoint to  $(-) \times F$ . It is a nice exercise to prove this.

- h) A monoid is never cartesian closed unless it is trivial. However, if from the definition of ‘cartesian closed’ one would delete the requirement that it has a terminal object, an interesting class of ‘cartesian closed’ monoids exists: the *C-mnoids* in the book “Higher Order Categorical Logic” by J. Lambek and Ph. Scott.

**Exercise 125** Show that every Boolean algebra is cartesian closed (as a partial order).

**Exercise 126** Show that CABool is not cartesian closed [use 2.3].

**Exercise 127** Show that a complete partial order is cartesian closed if and only if it’s a frame [see section 4.5].

**Exercise 128** Let  $\Omega$  be a frame. By the preceding exercise, it is cartesian closed; denote by  $x \rightarrow y$  the exponent in  $\Omega$ . This exercise is meant to let you show that the category  $\mathcal{C}_\Omega$  from section 4.5 is cartesian closed.

- a) Show that  $\Omega$  also has greatest lower bounds  $\bigwedge B$  for all subsets  $B$ .
- b) Given objects  $(X, E_X)$  and  $(Y, E_Y)$ , define their exponent  $(Y, E_Y)^{(X, E_X)}$  as  $(Y^X, E)$  where  $Y^X$  is the set of all functions  $X \rightarrow Y$  in  $\text{Set}$ , and

$$E(f) = \bigwedge \{E_X(x) \rightarrow E_Y(f(x)) \mid x \in X\}$$

Show that this defines a right adjoint (at  $(Y, E_Y)$ ) of  $(-) \times (X, E_X)$ .

Some useful facts:

- $\mathcal{C}$  is cartesian closed if and only if it has finite products, and for each pair of objects  $X, Y$  there is an object  $Y^X$  and an arrow  $\varepsilon : Y^X \times X \rightarrow Y$  such that for every  $Z$  and map  $Z \times X \xrightarrow{f} Y$  there is a unique  $Z \xrightarrow{\tilde{f}} Y^X$  such that

$$\begin{array}{ccc} Z \times X & \xrightarrow{f} & Y \\ & \searrow \tilde{f} \times \text{id}_X & \nearrow \varepsilon \\ & Y^X \times X & \end{array}$$

commutes (use the result of exercise 98).

- In a ccc, there are natural isomorphisms  $1^X \simeq 1$ ;  $(Y \times Z)^X \simeq Y^X \times Z^X$ ;  $(Y^Z)^X \simeq Y^{Z \times X}$ .

- If a ccc has coproducts, we have  $X \times (Y + Z) \simeq (X \times Y) + (X \times Z)$  and  $Y^{Z+X} \simeq Y^Z \times Y^X$ .

**Exercise 129** Prove these facts.

Recall that two maps  $Z \times X \rightarrow Y$  and  $Z \rightarrow Y^X$  which correspond to each other under the adjunction are called each other's *transposes*.

**Exercise 130** In a ccc, prove that the transpose of a composite  $Z \xrightarrow{g} W \xrightarrow{f} Y^X$  is

$$Z \times X \xrightarrow{g \times \text{id}_X} W \times X \xrightarrow{\tilde{f}} Y$$

if  $\tilde{f}$  is the transpose of  $f$ .

**Lemma 7.2** In a ccc, given  $f : X' \rightarrow X$  let  $Y^f : Y^X \rightarrow Y^{X'}$  be the transpose of  $f$

$$Y^X \times X' \xrightarrow{\text{id} \times f} Y^X \times X \xrightarrow{\varepsilon} Y$$

Then for each  $f : X' \rightarrow X$  and  $g : Y \rightarrow Y'$  the diagram

$$\begin{array}{ccc} Y^X & \xrightarrow{g^X} & Y'^X \\ Y^f \downarrow & & \downarrow Y'^f \\ Y^{X'} & \xrightarrow{g^{X'}} & Y'^{X'} \end{array}$$

commutes.

**Proof.** By the exercise, the transposes of both composites are the top and bottom composites of the following diagram:

$$\begin{array}{ccccccc} & & Y'^X \times X' & \xrightarrow{\text{id} \times f} & Y'^X \times X & & \\ & \nearrow g^X \times \text{id} & & \nearrow g^X \times \text{id} & & \searrow \varepsilon & \\ Y^X \times X' & \xrightarrow{\text{id} \times f} & Y^X \times X & \xrightarrow{\varepsilon} & Y & \xrightarrow{g} & Y' \\ & \searrow Y^f \times \text{id} & & \searrow \varepsilon & & \nearrow \varepsilon & \\ & & Y^{X'} \times X' & \xrightarrow{g^{X'} \times \text{id}} & Y'^{X'} \times X' & & \end{array}$$

This diagram commutes because the right hand “squares” are naturality squares for  $\varepsilon$ , the lower left hand square commutes because both composites are the transpose of  $Y^f$ , and the upper left hand square commutes because both composites are  $g^X \times f$ . ■

**Proposition 7.3** For every ccc  $\mathcal{C}$  there is a functor  $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ , assigning  $Y^X$  to  $(X, Y)$ , and given  $g : Y \rightarrow Y'$  and  $f : X' \rightarrow X$ ,  $g^f : Y^X \rightarrow Y'^{X'}$  is either of the composites in the lemma.

**Exercise 131** Prove the proposition.

## 7.2 Typed $\lambda$ -calculus and cartesian closed categories

The  $\lambda$ -calculus is an extremely primitive formalism about functions. Basically, we can form functions (by  $\lambda$ -abstraction) and apply them to arguments; that's all. Here I treat briefly the *typed*  $\lambda$ -calculus.

We start with a set  $\mathcal{S}$  of *type symbols*  $S_1, S_2, \dots$

Out of  $\mathcal{S}$  we make the set of types as follows: every type symbol is a type, and if  $T_1$  and  $T_2$  are types then so is  $(T_1 \Rightarrow T_2)$ .

We have also *terms* of each type (we label the terms like  $t:T$  to indicate that  $t$  is a term of type  $T$ ):

- we may have constants  $c:T$  of type  $T$ ;
- for every type  $T$  we have a denumerable set of variables  $x_1:T, x_2:T, \dots$ ;
- given a term  $t:(T_1 \Rightarrow T_2)$  and a term  $s:T_1$ , there is a term  $(ts):T_2$ ;
- given  $t:T_2$  and a variable  $x:T_1$  there is a term  $\lambda x.t:T_1 \Rightarrow T_2$ .

Terms  $\lambda x.t$  are said to be formed by  *$\lambda$ -abstraction*. This procedure binds the variable  $x$  in  $t$ . Furthermore we have the usual notion of substitution for free variables in a term  $t$  (types have to match, of course). Terms of form  $(ts)$  are said to be formed by *application*.

In the  $\lambda$ -calculus, the only statements we can make are equality statements about terms. Again, I formulate the rules in terms of theories. First, let us say that a *language* consists of a set of type symbols and a set of constants, each of a type generated by the set of type symbols.

An *equality judgement* is an expression of the form  $\Gamma|t = s:T$  (to be read: “ $\Gamma$  thinks that  $s$  and  $t$  are equal terms of type  $T$ ”), where  $\Gamma$  is a finite set of variables which includes all the variables free in either  $t$  or  $s$ , and  $t$  and  $s$  are terms of type  $T$ .

A *theory* is then a set  $\mathcal{T}$  of equality judgements which is closed under the following rules:

- i)  $\Gamma|t = s:T$  in  $\mathcal{T}$  implies  $\Delta|t = s:T$  in  $\mathcal{T}$  for every superset  $\Delta$  of  $\Gamma$ ;
- ii)  $FV(t)|t = t:T$  is in  $\mathcal{T}$  for every term  $t:T$  of the language (again,  $FV(t)$  is the set of free variables of  $t$ );  
if  $\Gamma|t = s:T$  and  $\Gamma|s = u:T$  are in  $\mathcal{T}$  then so is  $\Gamma|t = u:T$ ;
- iii) if  $t(x_1, \dots, x_n):T$  is a term of the language, with free variables  $x_1:S_1, \dots, x_n:S_n$ , and  $\Gamma|s_1 = t_1:S_1, \dots, \Gamma|s_n = t_n:S_n$  are in  $\mathcal{T}$  then

$$\Gamma|t[s_1/x_1, \dots, s_n/x_n] = t[t_1/x_1, \dots, t_n/x_n]:T$$

is in  $\mathcal{T}$ ;

- iv) if  $t$  and  $s$  are terms of type  $(T_1 \Rightarrow T_2)$ ,  $x$  a variable of type  $T_1$  which does not occur in  $t$  or  $s$ , and  $\Gamma \cup \{x\}|(tx) = (sx):T_2$  is in  $\mathcal{T}$ , then  $\Gamma \setminus \{x\}|t = s:(T_1 \Rightarrow T_2)$  is in  $\mathcal{T}$ ;



v) if  $s:T_1$  and  $t:T_2$  are terms and  $x$  a variable of type  $T_2$ , then

$$FV(s) \setminus \{x\} \cup FV(t) | ((\lambda x.s)t) = s[t/x]:T_1$$

is in  $\mathcal{T}$ .

Given a language, an interpretation of it into a ccc  $\mathcal{C}$  starts by choosing objects  $\llbracket S \rrbracket$  of  $\mathcal{C}$  for every type symbol  $S$ . This then generates objects  $\llbracket T \rrbracket$  for every type  $T$  by the clause

$$\llbracket T_1 \Rightarrow T_2 \rrbracket = \llbracket T_2 \rrbracket^{\llbracket T_1 \rrbracket}$$

The interpretation is completed by choosing interpretations

$$1 \xrightarrow{\llbracket c \rrbracket} \llbracket T \rrbracket$$

for every constant  $c:T$  of the language.

Such an interpretation then generates, in much the same way as in chapter 4, interpretations of all terms. For a finite set  $\Gamma = \{x_1:T_1, \dots, x_n:T_n\}$  let's again write  $\llbracket \Gamma \rrbracket$  for the product  $\llbracket T_1 \rrbracket \times \dots \times \llbracket T_n \rrbracket$  (this is only defined modulo a permutation of the factors of the product, but that will cause us no trouble).

The interpretation of  $t:T$  will now be an arrow

$$\llbracket FV(t) \rrbracket \xrightarrow{\llbracket t \rrbracket} \llbracket T \rrbracket$$

defined as follows:

- $\llbracket x \rrbracket$  is the identity on  $\llbracket T \rrbracket$  for every variable  $x:T$ ;
- given  $\llbracket t \rrbracket : \llbracket FV(t) \rrbracket \rightarrow \llbracket T_2 \rrbracket^{\llbracket T_1 \rrbracket}$  and  $\llbracket s \rrbracket : \llbracket FV(s) \rrbracket \rightarrow \llbracket T_1 \rrbracket$  we let  $\llbracket (ts) \rrbracket : \llbracket FV((ts)) \rrbracket \rightarrow \llbracket T_2 \rrbracket$  be the composite

$$\llbracket FV((ts)) \rrbracket \xrightarrow{\langle \llbracket t \rrbracket \pi_t, \llbracket s \rrbracket \pi_s \rangle} \llbracket T_2 \rrbracket^{\llbracket T_1 \rrbracket} \times \llbracket T_1 \rrbracket \xrightarrow{\varepsilon} \llbracket T_2 \rrbracket$$

where  $\pi_t$  and  $\pi_s$  are the projections from  $\llbracket FV((ts)) \rrbracket$  to  $\llbracket FV(t) \rrbracket$  and  $\llbracket FV(s) \rrbracket$ , respectively;

- given  $\llbracket t \rrbracket : \llbracket FV(t) \rrbracket \rightarrow \llbracket T_2 \rrbracket$  and the variable  $x:T_1$  we let  $\llbracket \lambda x.t \rrbracket : \llbracket FV(t) \setminus \{x\} \rrbracket \rightarrow \llbracket T_2 \rrbracket^{\llbracket T_1 \rrbracket}$  be the transpose of

$$\llbracket FV(t) \setminus \{x\} \rrbracket \times \llbracket T_1 \rrbracket \xrightarrow{\tilde{t}} \llbracket T_2 \rrbracket$$

where, if  $x$  occurs free in  $t$  so  $\llbracket FV(t) \setminus \{x\} \rrbracket \times \llbracket T_1 \rrbracket \simeq \llbracket FV(t) \rrbracket$ ,  $\tilde{t}$  is just  $\llbracket t \rrbracket$ ; and if  $x$  doesn't occur in  $t$ ,  $\tilde{t}$  is  $\llbracket t \rrbracket$  composed with the obvious projection.

We now say that an equality judgement  $\Gamma|t = s:T$  is *true* in this interpretation, if the diagram

$$\begin{array}{ccc}
 & \llbracket FV(t) \rrbracket & \\
 \pi_t \nearrow & & \searrow \llbracket t \rrbracket \\
 \llbracket \Gamma \rrbracket & & \llbracket T \rrbracket \\
 \pi_s \searrow & & \nearrow \llbracket s \rrbracket \\
 & \llbracket FV(s) \rrbracket &
 \end{array}$$

commutes (again,  $\pi_s$  and  $\pi_t$  projections).

**Lemma 7.4** *Let  $t(x_1, \dots, x_n):T$  have free variables  $x_i:T_i$  and let  $t_i:T_i$  be terms. Let*

$$\tilde{t}_i : \llbracket FV(t[t_1/x_1, \dots, t_n/x_n]) \rrbracket \rightarrow \llbracket T_i \rrbracket$$

*be the obvious composite of projection and  $\llbracket t_i \rrbracket$ .*

*Then the composition*

$$\llbracket FV(t[t_1/x_1, \dots, t_n/x_n]) \rrbracket \xrightarrow{\langle \tilde{t}_i |_{i=1, \dots, n} \rangle} \prod_{i=1}^n \llbracket T_i \rrbracket = \llbracket FV(t) \rrbracket \xrightarrow{\llbracket t \rrbracket} \llbracket T \rrbracket$$

*is the interpretation  $\llbracket t[t_1/x_1, \dots, t_n/x_n] \rrbracket$ .*

**Exercise 132** Prove the lemma [take your time. This is not immediate].

**Theorem 7.5** *Let  $S$  be a set of equality judgements and  $\mathcal{T} = Cn(S)$  be the least theory containing  $S$ . If every judgement of  $S$  is true in the interpretation, so is every judgement in  $\mathcal{T}$ .*

**Proof.** Again, we show that the set of true judgements is a theory, i.e. closed under the rules in the definition of a theory.

i) and ii) are trivial;

iii) follows at once by lemma 7.4;

iv) Since  $\Gamma \cup \{x\} = (\Gamma \setminus \{x\}) \cup \{x\}$  and because of the inductive hypothesis, we have that

$$\begin{array}{ccccc}
 & \llbracket FV(s) \rrbracket \times \llbracket T_1 \rrbracket & \xrightarrow{\llbracket s \rrbracket \times \text{id}} & \llbracket T_1 \Rightarrow T_2 \rrbracket \times \llbracket T_1 \rrbracket & \\
 \pi_s \times \text{id} \nearrow & & & \searrow \varepsilon & \\
 \llbracket \Gamma \setminus \{x\} \cup \{x\} \rrbracket & & & & \llbracket T_2 \rrbracket \\
 \pi_t \times \text{id} \searrow & & & \nearrow \varepsilon & \\
 & \llbracket FV(t) \rrbracket \times \llbracket T_1 \rrbracket & \xrightarrow{\llbracket t \rrbracket \times \text{id}} & \llbracket T_1 \Rightarrow T_2 \rrbracket \times \llbracket T_1 \rrbracket &
 \end{array}$$

commutes. Taking the transposes of both maps, we get the equality we want.

v) According to lemma 7.4,  $\llbracket FV(s[t/x]) \rrbracket \llbracket s[t/x] \rrbracket \llbracket T_1 \rrbracket$  is

$$\llbracket FV(s[t/x]) \rrbracket \xrightarrow{\tilde{t}} \llbracket FV(s) \rrbracket \xrightarrow{\llbracket s \rrbracket} \llbracket T_1 \rrbracket$$

This is the same as

$$\llbracket FV(s[t/x]) \rrbracket \xrightarrow{\langle \pi, \llbracket t \rrbracket \rangle} \llbracket FV(s) \setminus \{x\} \rrbracket \times \llbracket T_2 \rrbracket \xrightarrow{\llbracket \lambda x.s \rrbracket \times \text{id}} \llbracket T_2 \Rightarrow T_1 \rrbracket \times \llbracket T_2 \rrbracket \xrightarrow{\epsilon} \llbracket T_1 \rrbracket$$

which is

$$\llbracket FV((\lambda x.s)t) \rrbracket \xrightarrow{\llbracket ((\lambda x.s)t) \rrbracket} \llbracket T_1 \rrbracket$$

■

There is also a *completeness theorem*: if a judgement  $\Gamma | t = s : T$  is true in all possible interpretations, then every theory (in a language this judgement is in) contains it.

The relevant construction is that of a syntactic cartesian closed category out of a theory, and an interpretation into it which makes exactly true the judgements in the theory. The curious reader can find the, somewhat laborious, treatment in Lambek & Scott's “Higher Order Categorical Logic”.

### 7.3 Representation of primitive recursive functions in ccc's with natural numbers object

Dedekind observed, that in  $\text{Set}$ , the set  $\omega$  is characterized by the following property: given any set  $X$ , any element  $x \in X$  and any function  $X \xrightarrow{f} X$ , there is a unique function  $F : \omega \rightarrow X$  such that  $F(0) = x$  and  $F(x+1) = f(F(x))$ .

Lawvere took this up, and proposed this *categorical* property as a definition (in a more general context) of a “natural numbers object” in a category.

**Definition 7.6** *In a category  $\mathcal{C}$  with terminal object 1, a natural numbers object is a triple  $(0, N, S)$  where  $N$  is an object of  $\mathcal{C}$  and  $1 \xrightarrow{0} N$ ,  $N \xrightarrow{S} N$  arrows in  $\mathcal{C}$ , such that for any other such diagram*

$$1 \xrightarrow{x} X \xrightarrow{f} X$$

*there is a unique map  $\phi : N \rightarrow X$  making*

$$\begin{array}{ccccc} 1 & \xrightarrow{0} & N & \xrightarrow{S} & N \\ & \searrow x & \downarrow \phi & & \downarrow \phi \\ & & X & \xrightarrow{f} & X \end{array}$$

*commute.*

Of course we think of 0 as the zero element, and of  $S$  as the successor map. The defining property of a natural numbers object enables one to “do recursion”, a weak version of which we show here: we show that every primitive recursive function can be represented in a ccc with natural numbers object.

**Definition 7.7** Let  $\mathcal{C}$  be a ccc with natural numbers object  $(0, N, S)$ . We say that a number-theoretic function  $F : \mathbb{N}^k \rightarrow \mathbb{N}$  is represented by an arrow  $f : N^k \rightarrow N$  if for any  $k$ -tuple of natural numbers  $n_1, \dots, n_k$ , the diagram

$$\begin{array}{ccccc} 1 & \xrightarrow{0} & N & \xrightarrow{\langle S^{n_1}, \dots, S^{n_k} \rangle} & N^k \\ & \searrow 0 & & & \downarrow f \\ & & N & \xrightarrow{S^{F(n_1, \dots, n_k)}} & N \end{array}$$

commutes.

Recall that the class of *primitive recursive* functions is given by the following clauses:

- The constant zero function  $\lambda \vec{x}. 0 : \mathbb{N}^k \rightarrow \mathbb{N}$ , the function  $\lambda x. x + 1 : \mathbb{N} \rightarrow \mathbb{N}$  and the projections  $\lambda \vec{x}. x_i : \mathbb{N} \rightarrow \mathbb{N}$  are primitive recursive;
- The primitive recursive functions are closed under composition: if  $F_1, \dots, F_k : \mathbb{N}^l \rightarrow \mathbb{N}$  and  $G : \mathbb{N}^k \rightarrow \mathbb{N}$  are primitive recursive, then so is  $G(\langle F_1, \dots, F_k \rangle) : \mathbb{N}^l \rightarrow \mathbb{N}$ ;
- The primitive recursive functions are closed under *definition by primitive recursion*: if  $G : \mathbb{N}^k \rightarrow \mathbb{N}$  and  $H : \mathbb{N}^{k+2} \rightarrow \mathbb{N}$  are primitive recursive, and  $F : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$  is defined by  $F(0, \vec{x}) = G(\vec{x})$  and  $F(n + 1, \vec{x}) = H(n, F(n, \vec{x}), \vec{x})$  then  $F$  is primitive recursive.

**Proposition 7.8** In a ccc  $\mathcal{C}$  with natural numbers object, every primitive recursive function is representable.

**Proof.** I do only the case for definition by primitive recursion. So by inductive hypothesis we have arrows  $G$  and  $H$  representing the homonymous functions. By interpretation of the  $\lambda$ -calculus, I use  $\lambda$ -terms: so there is an arrow

$$\lambda \vec{x}. G(\vec{x}) : 1 \rightarrow N^{(N^k)}$$

and an arrow

$$\lambda \vec{x}. H(n, \phi(\vec{x}), \vec{x}) : N^{(N^k)} \times N \rightarrow N^{(N^k)}$$

which is the interpretation of a term with free variables  $\phi : N^{(N^k)}$  and  $n : N$ ; this map is the exponential transpose of the map which intuitively sends  $(n, \phi, \vec{x})$  to  $(n, \phi(\vec{x}), \vec{x})$ . Now look at

$$1 \xrightarrow{\langle \lambda \vec{x}. G(\vec{x}), 0 \rangle} N^{(N^k)} \times N \xrightarrow{(\lambda \vec{x}. H(n, \phi(\vec{x}), \vec{x})) \times S} N^{(N^k)} \times N$$

By the natural numbers object property, there is now a unique map

$$\bar{F} = \langle \tilde{F}, \sigma \rangle : N \rightarrow N^{(N^k)} \times N$$

which makes the following diagram commute:

$$\begin{array}{ccccc}
 1 & \xrightarrow{0} & N & \xrightarrow{S} & N \\
 & \searrow \langle \lambda \vec{x}. G(\vec{x}), 0 \rangle & \downarrow \bar{F} & & \downarrow \bar{F} \\
 & & N^{(N^k)} \times N & \xrightarrow{(\lambda \vec{x}. H(n, \phi(\vec{x}), \vec{x})) \times S} & N^{(N^k)} \times N
 \end{array}$$

One verifies that  $\sigma$  is the identity, and that the composite

$$N^{k+1} \xrightarrow{\tilde{F} \times \text{id}} N^{(N^k)} \times N^k \xrightarrow{\varepsilon} N$$

represents  $F$ . ■

**Exercise 133** Make these verifications.

One could ask: what is the class of those numerical functions (that is, functions  $\mathbb{N}^k \rightarrow \mathbb{N}$ ) that are representable in every ccc with natural numbers object? It is not hard to see, that there are representable functions which are not primitive recursive (for example, the Ackermann function). On the other hand, Logic teaches us that every such representable function must be recursive, and that there are recursive, non-representable functions.

The answer is: the representable functions are precisely the so-called  $\varepsilon_0$ -recursive functions from Proof Theory; and this was essentially shown by Gödel in 1958.

## 8 Recursive Domain Equations

A *recursive domain equation* in a category  $\mathcal{C}$  is an “equation” of the form:

$$X \cong F(X, \dots, X)$$

where  $F$  is a functor:  $(\mathcal{C}^{\text{op}})^n \times \mathcal{C}^m \rightarrow \mathcal{C}$ .

Often, we are interested in not just any solution of such an equation, but in certain ‘universal’ solutions. Consider, as an example, the case  $\mathcal{C} = \text{Set}$ , and  $F(X) = 1 + (A \times X)$  for a fixed, nonempty set  $A$ . There are many solutions of  $X \cong F(X)$  but one stands out: it is the set of *finite sequences of elements of  $A$* .

In what sense is this a universal solution? Do such solutions always exist?

There is a piece of theory about this, which is by now a classic in theoretical Computer Science and was developed by Dana Scott around 1970; it is concerned with a certain subcategory of  $\text{Pos}$ . It is a nice application of the methods of category theory.

### 8.1 The category $\mathbb{CPO}$

Let  $(P, \leq)$  be a partially ordered set. A *downset* or *downwards closed subset* of  $P$  is a subset  $A \subseteq P$  such that if  $a \in A$  and  $p \leq a$ , then  $p \in A$ . The *downwards closure*  $\downarrow A$  of  $A \subseteq P$  is the least downset of  $P$  containing  $A$ :  $\downarrow A = \{p \in P \mid \exists a \in A. p \leq a\}$ . We write  $\downarrow p$  for  $\downarrow\{p\}$ .

An  $\omega$ -chain in  $P$  is a function  $f : \mathbb{N} \rightarrow P$  such that  $f(0) \leq f(1) \leq \dots$ .

$(P, \leq)$  is a *cpo* or  $\omega$ -complete partial order if every  $\omega$ -chain in  $P$  has a colimit (i.e., least upper bound). This colimit is denoted  $\bigsqcup_{n \in \mathbb{N}} f(n)$ .

A monotone function  $f : P \rightarrow Q$  between cpo’s is *continuous* if it preserves least upper bounds of  $\omega$ -chains.

**Exercise 134** Every cpo  $P$  can be regarded as a topological space, in the following way: open sets are those sets  $A \subseteq P$  which are *upwards closed* ( $a \in A \wedge a \leq b \Rightarrow b \in A$ ) and such that for any chain  $f : \mathbb{N} \rightarrow P$ , if  $\bigsqcup_{n \in \mathbb{N}} f(n) \in A$  then  $f(n) \in A$  for some  $n \in \mathbb{N}$ . Show that  $f : P \rightarrow Q$  is continuous if and only if  $f$  is continuous w.r.t. the topology just defined.

There is a category  $\mathbb{CPO}$  of cpo’s and continuous maps, and this category is our object of study for a while. Since every continuous function is monotone, there is a forgetful functor  $U : \mathbb{CPO} \rightarrow \text{Pos}$ .

**Theorem 8.1**  $U : \mathbb{CPO} \rightarrow \text{Pos}$  is *monadic*.

**Proof.** We have to show that  $U$  has a left adjoint  $F : \text{Pos} \rightarrow \mathbb{CPO}$  such that  $\mathbb{CPO}$  is equivalent to the category of  $UF$ -algebras on  $\text{Pos}$ .

Call a subset  $A$  of a poset  $P$  an  $\omega$ -ideal if there is an  $\omega$ -chain  $f : \mathbb{N} \rightarrow P$  such that  $A$  is the downwards closure of the image of  $f$ . Let  $\omega\text{-Idl}(P)$  the set of  $\omega$ -ideals of  $P$ , ordered by inclusion. If  $\varphi : P \rightarrow Q$  is a monotone map and

$A \subseteq P$  an  $\omega$ -ideal, then  $\downarrow\varphi[A] = \{q \in Q \mid \exists a \in A. q \leq \varphi(a)\}$  is also an  $\omega$ -ideal of  $Q$ , for if  $A = \downarrow\text{im}(f)$  for  $f : \mathbb{N} \rightarrow P$  then  $\downarrow\varphi[A] = \downarrow\text{im}(\varphi \circ f)$ .

If  $A_0 \subseteq A_1 \subseteq \dots$  is an  $\omega$ -chain of elements of  $\omega\text{-Idl}(P)$  then also  $\bigcup_{n \in \mathbb{N}} A_n \in \omega\text{-Idl}(P)$ , for, if  $A_i = \downarrow\text{im}(f_i)$  define  $f : \mathbb{N} \rightarrow P$  by:

$$f(n) = \begin{array}{l} f_n(m) \text{ where } m \text{ is minimal such that} \\ f_n(m) \text{ is an upper bound of} \\ \{f_i(k) \mid i, k \in \{0, \dots, n\}\} \cup \{f(k) \mid k < n\} \end{array}$$

Then  $f$  is a chain and  $\bigcup_{n \in \mathbb{N}} A_n = \downarrow\text{im}(f)$ .

So  $\omega\text{-Idl}(P)$  is a cpo; and since (for a monotone  $\varphi : P \rightarrow Q$ ) the map  $A \mapsto \downarrow\varphi[A]$  commutes with unions of  $\omega$ -chains, it is a continuous map:  $\omega\text{-Idl}(P) \rightarrow \omega\text{-Idl}(Q)$ . So we have a functor  $F : \text{Pos} \rightarrow \mathbb{CPO}$ :  $F(P) = \omega\text{-Idl}(P)$ , and for  $\varphi : P \rightarrow Q$  in  $\text{Pos}$ ,  $F(\varphi) : \omega\text{-Idl}(P) \rightarrow \omega\text{-Idl}(Q)$  is the map which sends  $A$  to  $\downarrow\varphi[A]$ .

Every monotone function  $f : P \rightarrow U(Q)$  where  $Q$  is a cpo, gives a continuous function  $\tilde{f} : \omega\text{-Idl}(P) \rightarrow Q$  defined as follows: given  $A \in \omega\text{-Idl}(P)$ , if  $A = \downarrow\text{im}(g)$  for a chain  $g : \mathbb{N} \rightarrow P$ , let  $\tilde{f}(A)$  be the least upper bound in  $Q$  of the chain  $f \circ g$ . This is independent of the choice of  $g$ , for if  $\downarrow\text{im}(g) = \downarrow\text{im}(g')$  then the chains  $f \circ g$  and  $f \circ g'$  have the same least upper bound in  $Q$ .

In the other direction, first let  $\eta_P : P \rightarrow \omega\text{-Idl}(P)$  be defined by  $\eta_P(p) = \downarrow p$ . Every continuous function  $g : \omega\text{-Idl}(P) \rightarrow Q$  gives a monotone  $\bar{g} : P \rightarrow U(Q)$  by composition with  $\eta_P$ .

**Exercise 135** Check that these two operations define a natural 1-1 correspondence between  $\text{Pos}(P, U(Q))$  and  $\mathbb{CPO}(\omega\text{-Idl}(P), Q)$  and therefore an adjunction  $F \dashv U$  of which  $\eta = (\eta_P)_P$  is the unit. What is the counit of this adjunction? Is it iso? Epi? What do you conclude about the functor  $U$ ?

The monad  $UF$  has  $\eta$  as unit, and as multiplication

$$\mu = \bigcup : \omega\text{-Idl}(\omega\text{-Idl}(P)) \rightarrow \omega\text{-Idl}(P)$$

taking the union of an  $\omega$ -ideal of  $\omega$ -ideals of  $P$ .

**Exercise 136** Check that  $\mu$  is well-defined. Prove that every  $UF$ -algebra is a cpo [Hint: compare with the proof that algebras for the powerset monad are equivalent to join-complete posets and join-preserving maps], so that  $\mathbb{CPO}$  is equivalent to  $UF\text{-Alg}$ .

■

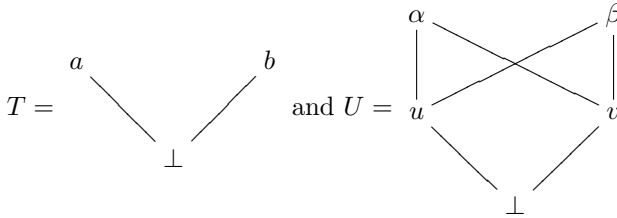
A corollary is now that  $\mathbb{CPO}$  has all the limits that  $\text{Pos}$  has (that is, all small limits), and that these are created by the forgetful functor  $U$ . So, the limit in  $\text{Pos}$  of a diagram of cpo's and continuous maps, is also a cpo.

We shall also consider the category  $\mathbb{CPO}_\perp$  of cpo's with a least element.  $\mathbb{CPO}_\perp$  is a full subcategory of  $\mathbb{CPO}$  (i.e., maps between objects of  $\mathbb{CPO}_\perp$  are continuous but don't have to preserve the least element).

It is, categorically speaking, bad practice to require properties of objects without requiring the maps to preserve them. This is borne out by the fact that  $\mathbf{CPO}_\perp$  loses the nice properties of  $\mathbf{CPO}$ :

**Fact.**  $\mathbf{CPO}_\perp$  is neither finitely complete nor finitely cocomplete.

For instance consider the cpo's:



Both  $T$  and  $U$  are objects of  $\mathbf{CPO}_\perp$ . Let  $f, g : T \rightarrow U$  be defined by:  $f(a) = g(a) = \alpha$ ,  $f(b) = g(b) = \beta$ ,  $f(\perp) = u$ ,  $g(\perp) = v$ .  $f$  and  $g$  are maps of  $\mathbf{CPO}_\perp$ , but cannot have an equalizer in  $\mathbf{CPO}_\perp$ .

**Exercise 137** Prove this. Prove also that the coproduct of two one-element cpo's cannot exist in  $\mathbf{CPO}_\perp$ .

A map of cpo's with least elements which preserves the least element is called *strict*. The category  $(\mathbf{CPO}_\perp)_s$  of cpo's with least element and strict continuous maps, is monadic over  $\mathbf{CPO}$  by the “lifting monad”: adding a least element (see chapter 6), and therefore complete.

**Lemma 8.2** Let  $P$  be a cpo and  $(x_{ij})_{i,j \in \mathbb{N}}$  be a doubly indexed set of elements of  $P$  such that  $i \leq i'$  and  $j \leq j'$  implies  $x_{ij} \leq x_{i'j'}$ . Then

$$\bigsqcup_{i \in \mathbb{N}} \bigsqcup_{j \in \mathbb{N}} x_{ij} = \bigsqcup_{j \in \mathbb{N}} \bigsqcup_{i \in \mathbb{N}} x_{ij} = \bigsqcup_{i \in \mathbb{N}} x_{ii}$$

**Exercise 138** Prove lemma 8.2.

**Theorem 8.3**  $\mathbf{CPO}$  is cartesian closed.

**Proof.** The exponent  $P^Q$  of two cpo's is the set of continuous maps from  $Q$  to  $P$ , ordered pointwise (i.e.  $f \leq g$  iff  $\forall q \in Q. f(q) \leq g(q)$ ). This is a cpo, because given a chain  $f_0 \leq f_1 \leq \dots$  of continuous maps, taking least upper bounds pointwise yields a continuous map:

$$f(q) = \bigsqcup_{i \in \mathbb{N}} f_i(q)$$

For, using lemma 8.2,  $f(\bigsqcup_j q_j) = \bigsqcup_i \bigsqcup_j f_i(q_j) = \bigsqcup_j \bigsqcup_i f_i(q_j) = \bigsqcup_j f(q_j)$  ■

**Theorem 8.4** Let  $P$  be a cpo with least element  $\perp$ . Then:



a) Every continuous map  $f : P \rightarrow P$  has a least fixed point  $\text{fix}(f)$  (i.e. a least  $x$  with  $f(x) = x$ );

b) The assignment  $f \mapsto \text{fix}(f)$  is a continuous function:  $P^P \rightarrow P$ .

**Proof.** Consider the chain  $\perp \leq f(\perp) \leq f^2(\perp) \leq \dots$ . It's a chain because  $f$  is monotone. Let  $a$  be its least upper bound in  $P$ . Since  $f$  is continuous,  $f(a) = f(\bigsqcup_{n \in \mathbb{N}} f^n(\perp)) = \bigsqcup_{n \in \mathbb{N}} f^{n+1}(\perp) = a$ , so  $a$  is a fixed point; if  $b$  is another fixed point of  $f$  then since  $\perp \leq b = f(b)$ ,  $f^n(\perp) \leq b$  for all  $n$ , hence  $a \leq b$ , so  $a$  is the least fixed point of  $f$ .

For the second statement, first notice that if  $f \leq g$  in  $P^P$  then  $f^n(\perp) \leq g^n(\perp)$  so  $\text{fix}(f) \leq \text{fix}(g)$ , so  $\text{fix}$  is monotone; and if  $f_1 \leq f_2 \leq \dots$  then  $\text{fix}(\bigsqcup_n f_n) = \bigsqcup_i (\bigsqcup_n f_n)^i(\perp) \geq \bigsqcup_i \bigsqcup_n f_n^i(\perp) = (\text{by lemma 8.2}) \bigsqcup_n \bigsqcup_i f_n^i(\perp) = \bigsqcup_n \text{fix}(f_n)$ . The other inequality follows from the monotonicity of  $\text{fix}$ . ■

For purposes of interpretation of recursion equations, it is convenient to have a notation for fixed points of functions of more than one variable. Let  $P$  and  $Q$  be cpo's with  $\perp$  and  $f : P \times Q \rightarrow P$  continuous; by cartesian closedness of  $\mathbb{CPO}$  we have  $\tilde{f} : Q \rightarrow P^P$  and we can consider the composite

$$Q \xrightarrow{\tilde{f}} P^P \xrightarrow{\text{fix}} P$$

which is a continuous map by theorem 8.4.  $\text{fix}(\tilde{f}(q))$  is the least fixed point of the function which sends  $p$  to  $f(p, q)$ ; we write  $\mu p.f(p, q)$  for this.

Békić's theorem<sup>1</sup> says that if we want to find a *simultaneous* fixed point  $(x, y) \in P \times Q$  of a function  $f : P \times Q \rightarrow P \times Q$ , we can do it in two steps, each involving a single fixed point calculation:

**Theorem 8.5 (Békić's simultaneous fixed point theorem)** *Let  $P$  and  $Q$  be cpo's with  $\perp$  and  $f : P \times Q \rightarrow P$ ,  $g : P \times Q \rightarrow Q$  be continuous maps. Then the least fixed point of the map  $\langle f, g \rangle : P \times Q \rightarrow P \times Q$  is the pair  $(\hat{x}, \hat{y}) \in P \times Q$ , where*

$$\begin{aligned}\hat{x} &= \mu x.f(x, \mu y.g(x, y)) \\ \hat{y} &= \mu y.g(\hat{x}, y)\end{aligned}$$

**Proof.** The least fixed point  $a$  of a function  $f$  has the property, that for any  $y$ , if  $f(y) \leq y$  then  $a \leq y$  (check this!), and moreover it is *characterized* by this property. Therefore the element  $\mu x.\Phi(x, \vec{y})$  satisfies the *rule*:

$$\Phi(x', \vec{y}) \leq x' \Rightarrow \mu x.\Phi(x, \vec{y}) \leq x'$$

and is characterized by it. Now suppose:

$$\begin{aligned}(1) \quad & f(a, b) \leq a \\ (2) \quad & g(a, b) \leq b\end{aligned}$$

From (2) and the rule we get  $\mu y.g(a, y) \leq b$ , hence by (1) and monotonicity of  $f$ ,  $f(a, \mu y.g(a, y)) \leq f(a, b) \leq a$ . Applying the rule again yields:

$$\hat{x} = \mu x.f(a, \mu y.g(x, y)) \leq a$$

---

<sup>1</sup>This theorem is known in recursion theory as *Smullyan's double recursion theorem*

so by (2) and monotonicity of  $g$ :  $g(\hat{x}, b) \leq g(a, b) \leq b$ , so by the rule,  $\mu y. g(\hat{x}, y) \leq b$ . We have derived that  $(\hat{x}, \hat{y}) \leq (a, b)$  from the assumption that  $\langle f, g \rangle(a, b) \leq (a, b)$ ; this characterizes the least fixed point of  $\langle f, g \rangle$ , which is therefore  $(\hat{x}, \hat{y})$ . ■

**Exercise 139** Generalize theorem 8.5 to 3 continuous functions with 3 variables, and, if you have the courage, to  $n$  continuous functions with  $n$  variables.

**Exercise 140** Suppose  $D, E$  are cpo's with  $\perp$  and  $f : D \rightarrow E$ ,  $g : E \rightarrow D$  continuous. Show that  $\mu d. gf(d) = g(\mu e. fg(e))$  [Hint: use the rule given in the proof of theorem 8.5]

## 8.2 The category of cpo's with $\perp$ and embedding-projection pairs; limit-colimit coincidence; recursive domain equations

So far, we have seen that the category  $\mathbb{CPO}$  is cartesian closed and complete. About  $\mathbb{CPO}_\perp$  we can say that:

- $\mathbb{CPO}_\perp$  has products and the inclusion  $\mathbb{CPO}_\perp \rightarrow \mathbb{CPO}$  preserves them;
- if  $Y$  has  $\perp$  then  $Y^X$  has  $\perp$ , for any  $X$ .

So, also  $\mathbb{CPO}_\perp$  is cartesian closed and supports therefore interpretation of simply typed  $\lambda$ -calculus (see chapter 7) and recursion (by the fixed point property). However, the structure of cpo's is much richer than that. First, we shall see that by restricting the morphisms of  $\mathbb{CPO}_\perp$  we get a “cpo of cpo's”. This will then, later, allow us to solve recursive domain equations like:

$$\begin{array}{ll} X \cong 1 + A \times X & \text{lists on alphabet } A \\ X \cong X^X & \text{untyped } \lambda\text{-calculus} \end{array}$$

First we have to go through some technique.

**Definition 8.6** Let  $P$  and  $Q$  posets. A pair  $P \xrightleftharpoons[i]{r} Q$  of monotone maps is called an embedding-projection pair (e-p pair for short), where  $i$  is the embedding,  $r$  the projection, if  $i \dashv r$  and  $i$  is full and faithful; equivalently:  $ri = \text{id}_P$  and  $ir \leq \text{id}_Q$ .

By uniqueness of adjoints, each member of an e-p pair determines the other. It is evident that  $\langle \text{id}_P, \text{id}_P \rangle$  is an e-p pair, and that e-p pairs compose. We can therefore define a category  $\mathbb{CPO}_\perp^{\text{EP}}$ : objects are cpo's with  $\perp$ , and morphisms

$P \rightarrow Q$  are e-p pairs  $P \xrightleftharpoons[i]{r} Q$  such that both  $i$  and  $r$  are continuous.

**Lemma 8.7** Let  $P \xrightleftharpoons[i]{r} Q$  be an e-p pair, where  $P$  and  $Q$  are cpo's with  $\perp$ . Then both  $i$  and  $r$  are strict, and  $i$  is continuous.

**Proof.** Being a left adjoint,  $i$  preserves colimits, so  $i$  is strict and continuous; since  $ri = \text{id}_P$  we also have  $r(\perp_Q) = ri(\perp_P) = \perp_P$ . ■

For the following theorem, recall that every diagram in  $\mathbb{CPO}_\perp$  with *strict* continuous maps will have a limit in  $\mathbb{CPO}_\perp$ , since this takes place in  $(\mathbb{CPO}_\perp)_s$  which is monadic over  $\mathbb{CPO}$ .

**Theorem 8.8** ( $\mathbb{CPO}_\perp^{\text{EP}}$  as “cpo of cpo’s”; limit-colimit coincidence)

a) Any chain of maps

$$P_1 \xleftarrow[i_1]{r_1} P_2 \xleftarrow[i_2]{r_2} P_3 \xleftarrow[i_3]{r_3} \dots$$

in  $\mathbb{CPO}_\perp^{\text{EP}}$  has a colimit in  $\mathbb{CPO}_\perp^{\text{EP}}$ ;

b) the vertex of this colimit,  $P$ , is the limit (in  $\mathbb{CPO}_\perp$ ) of the diagram

$$P_1 \xleftarrow{r_1} P_2 \xleftarrow{r_2} \dots$$

and  $P$  is also the colimit in  $\mathbb{CPO}_\perp$  of the diagram

$$P_1 \xrightarrow{i_1} P_2 \xrightarrow{i_2} \dots$$

**Proof.** We prove a) and b) simultaneously. Note that the limit of  $P_1 \xleftarrow{r_1} P_2 \xleftarrow{r_2} \dots$  exists in  $\mathbb{CPO}_\perp$  since all maps are strict by lemma 8.7; it is the object  $P = \{(p_1, p_2, \dots) \in \prod_{n \geq 1} P_n \mid \forall i \geq 1 \ r_i(p_{i+1}) = p_i\}$  with pointwise order.

For any  $k$  we have maps  $P_k \xleftarrow[e_k]{\pi_k} P$  where  $\pi_k$  is the  $k$ -th projection, and  $e_k$  is defined by:

$$(e_k(p))_j = \begin{cases} r_j r_{j+1} \dots r_{k-1}(p) & \text{if } j < k \\ p & \text{if } j = k \\ i_{j_1} \dots i_k(p) & \text{if } j > k \end{cases}$$

Now  $\pi_k e_k(p) = (e_k(p))_k = p$  and

$$\begin{aligned} e_k \pi_k(p_1, p_2, \dots) &= (p_1, p_2, \dots, p_k, i_k r_k(p_{k+1}), i_{k+1} i_k r_k r_{k+1}(p_{k+2}), \dots) \\ &\leq (p_1, p_2, \dots) \end{aligned}$$

So  $\langle e_k, \pi_k \rangle$  is an e-p pair. Since obviously  $r_i \pi_{i+1} = \pi_i$  for all  $i \in \mathbb{N}$ , also  $e_{k+1} i_k = e_k$  must hold (since one component of an e-p pair uniquely determines the other), hence

$$\{P_k \xrightarrow{\langle e_k, \pi_k \rangle} P \mid k \geq 1\}$$

is a cocone in  $\mathbb{CPO}_\perp^{\text{EP}}$  for the given chain.

Suppose now that  $\{P_k \xrightarrow{\langle j_k, s_k \rangle} Q \mid k \geq 1\}$  is another cocone for the chain in  $\mathbb{CPO}_\perp^{\text{EP}}$ . Immediately, we have (since  $P$  is the limit of  $P_1 \xleftarrow{r_1} P_2 \xleftarrow{r_2} \dots$ ) a unique

$\sigma : Q \rightarrow P$  such that  $s_k = \pi_k \sigma$  for all  $k$ ;  $\sigma(q) = (s_1(q), s_2(q), \dots)$ . Note that  $\sigma$  is continuous. Since we have a cocone, for any  $(p_1, p_2, \dots)$  in  $P$  we have that in  $Q$ :

$$j_k(p_k) = j_{k+1}i_k(p_k) = j_{k+1}i_k r_k(p_{k+1}) \leq j_{k+1}(p_{k+1})$$

so  $j_1(p_1) \leq j_2(p_2) \leq \dots$  and we define  $J : P \rightarrow Q$  by

$$J(p_1, p_2, \dots) = \bigsqcup_k j_k(p_k)$$

Then  $J(\sigma(q)) = \bigsqcup_k j_k s_k(q) \leq q$  because  $\langle j_k, s_k \rangle$  is an e-p pair, and by continuity of  $s_n$ ,

$$\begin{aligned} (\sigma J(p_1, p_2, \dots))_n &= s_n(\bigsqcup_k j_k(p_k)) = \\ s_n(\bigsqcup_{k \geq n} j_k(p_k)) &= \bigsqcup_{k \geq n} s_n j_k(p_k) \end{aligned}$$

For  $k \geq n$  write  $\langle i_{nk}, r_{nk} \rangle$  for the e-p pair  $P_n \xleftarrow{\quad} P_k$ . Using that  $r_{nk} s_k = s_n$ ,  $s_k j_k = \text{id}_{P_k}$ ,

$$\begin{aligned} \bigsqcup_{k \geq n} s_n j_k(p_k) &= \bigsqcup_{k \geq n} r_{nk} s_k j_k(p_k) = \\ \bigsqcup_{k \geq n} r_{nk}(p_k) &= \bigsqcup_{k \geq n} p_k = p_n \end{aligned}$$

So  $\sigma J = \text{id}_P$ ; i.e. the cocone with vertex  $Q$  factors uniquely through the one with vertex  $P$ ; hence the latter is colimiting.

The only thing which remains to be proven, is that  $\{P_k \xrightarrow{e_k} P \mid k \geq 1\}$  is also a colimiting cocone in  $\mathbb{CPO}_\perp$ . Firstly, from the definition of  $P_k \xleftarrow[e_k]{\pi_k} P$  and the already seen

$$e_k \pi_k(p_1, p_2, \dots) = (p_1, \dots, p_k, i_k r_k(p_{k+1}), i_{k+1} i_k r_k r_{k+1}(p_{k+2}), \dots)$$

it is immediate that, in  $P$ ,

$$(p_1, p_2, \dots) = \bigsqcup_{k \geq 1} e_k \pi_k(p_1, p_2, \dots) = \bigsqcup_{k \geq 1} e_k(p_k)$$

So if  $\{P_k \xrightarrow{f_k} Q \mid k \geq 1\}$  is another cocone in  $\mathbb{CPO}_\perp$  we can define a continuous factorization  $P \xrightarrow{f} Q$  by

$$f(p_1, p_2, \dots) = \bigsqcup_k f_k(p_k)$$

but in fact we have no other choice, hence the factorization is unique. ■

Define an  $\omega$ -category as a category where every chain of maps

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \rightarrow \dots$$

has a colimiting cocone; and call a functor between  $\omega$ -categories *continuous* if it preserves colimits of chains. Theorem 8.8 says that  $\mathbb{CPO}_\perp^{\text{EP}}$  is an  $\omega$ -category.

**Lemma 8.9** *Let  $\mathcal{A}$  be an  $\omega$ -category and  $F : \mathcal{A} \rightarrow \mathcal{A}$  continuous. If  $A \in \mathcal{A}_0$  and  $A \xrightarrow{f} F(A)$  a map, and the chain*

$$A \xrightarrow{f} F(A) \xrightarrow{F(f)} F^2(A) \xrightarrow{F^2(f)} F^3(A) \rightarrow \dots$$

*has colimit with vertex  $D$ , then  $D$  is isomorphic to  $F(D)$ .*

In particular, if  $\mathcal{A}$  has an initial object,  $F$  has an up to isomorphism fixed point.

**Exercise 141** Prove lemma 8.9. It generalizes the idea of the fixed point property for cpo's.

For any endofunctor  $F : \mathcal{A} \rightarrow \mathcal{A}$  we define the category  $F\text{-Alg}$  of  $F$ -algebras in a similar way as for a monad, but simpler (since the functor has less structure than the monad): objects are maps  $F X \xrightarrow{h} X$ , just like that, and maps  $(X, h) \rightarrow (Y, k)$  are maps  $f : X \rightarrow Y$  in  $\mathcal{A}$  such that

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ h \downarrow & & \downarrow k \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. We have:

**Lemma 8.10 (Lambek's Lemma)** *If  $F(X) \xrightarrow{h} X$  is an initial object of  $F\text{-Alg}$ , then  $h$  is an isomorphism in  $\mathcal{A}$ .*

**Proof.**  $F^2(X) \xrightarrow{F(h)} F(X)$  is also an  $F$ -algebra, so there is a unique  $k : X \rightarrow F(X)$  such that

$$\begin{array}{ccc} F(X) & \xrightarrow{F(k)} & F^2(X) \\ h \downarrow & & \downarrow F(h) \\ X & \xrightarrow{k} & F(X) \end{array}$$

commutes. Since also

$$\begin{array}{ccc} F^2(X) & \xrightarrow{F(h)} & F(X) \\ F(h) \downarrow & & \downarrow h \\ F(X) & \xrightarrow{h} & X \end{array}$$

commutes,  $h$  is a map of  $F$ -algebras:  $(F(X), F(h)) \rightarrow (X, h)$ . Therefore is  $hk : (X, h) \rightarrow (X, h)$  a map in  $F\text{-Alg}$  and since  $(X, h)$  is initial,  $hk = \text{id}_X$ . Then  $kh = F(h)F(k) = F(hk) = F(\text{id}_X) = \text{id}_{F(X)}$ , so  $h$  is iso with inverse  $k$ . ■

**Exercise 142** In the situation of lemma 8.9, i.e.  $\mathcal{A}$  an  $\omega$ -category,  $F : \mathcal{A} \rightarrow \mathcal{A}$  continuous, the colimit of

$$0 \xrightarrow{!} F(0) \xrightarrow{F(!)} F^2(0) \xrightarrow{F^2(!)} F^3(0) \rightarrow \dots$$

(where 0 is initial in  $\mathcal{A}$ , and  $!$  the unique map  $0 \rightarrow F(0)$ ) gives the initial  $F$ -algebra.

In view of Lambek's Lemma and other considerations (such as the desirability of induction principles for elements of recursively defined domains), we aim to solve an equation:

$$X \cong F(X)$$

as an initial  $F$ -algebra. So we have seen, that as long as  $F$  is a continuous functor, we do have initial  $F$ -algebras in  $\mathbf{CPO}_{\perp}^{\text{EP}}$ . But this in itself did not require the introduction of  $\mathbf{CPO}_{\perp}^{\text{EP}}$  for also  $\mathbf{CPO}$  is an  $\omega$ -category with an initial object (as is, by the way,  $\mathbf{Set}$ ). The force of the embedding-projection pairs resides in the possibilities of handling "mixed variance". Since the expression  $X^X$  does not define a functor but in general, the expression  $X^Y$  defines a functor  $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ , one says that in  $X^X$ , the variable  $X$  occurs both covariantly and contravariantly (or, positively and negatively). We shall see that functors of mixed variance on  $\mathbf{CPO}_{\perp}$ , that is: functors  $(\mathbf{CPO}_{\perp}^{\text{op}})^n \times (\mathbf{CPO}_{\perp})^m \rightarrow \mathbf{CPO}_{\perp}$ , can, under certain conditions, be transformed into continuous covariant functors:  $(\mathbf{CPO}_{\perp}^{\text{EP}})^{n+m} \rightarrow \mathbf{CPO}_{\perp}^{\text{EP}}$ . Composition with the diagonal functor  $\Delta : \mathbf{CPO}_{\perp}^{\text{EP}} \rightarrow (\mathbf{CPO}_{\perp}^{\text{EP}})^{n+m}$  gives a continuous endofunctor on  $\mathbf{CPO}_{\perp}^{\text{EP}}$  which has a fixed point (up to isomorphism).

The first ingredient we need is the notion of *local continuity*. Recall that in the proof that  $\mathbf{CPO}$  was a ccc, we have basically seen that for cpo's  $P$  and  $Q$  the set  $\mathbf{CPO}(P, Q)$  is itself a cpo. Of course, this holds for  $\mathbf{CPO}^{\text{op}}$  too, and also for products of copies of  $\mathbf{CPO}$  and  $\mathbf{CPO}^{\text{op}}$ :

$$(\mathbf{CPO}^{\text{op}})^n \times (\mathbf{CPO})^m ((A'_1, \dots, A'_n, B_1, \dots, B_m), (A_1, \dots, A_n, B'_1, \dots, B'_m))$$

is the cpo  $\mathbf{CPO}(A_1, A'_1) \times \dots \times \mathbf{CPO}(B_m, B'_m)$ .

**Definition 8.11** A functor  $F : (\mathbf{CPO}^{\text{op}})^n \times (\mathbf{CPO})^m \rightarrow \mathbf{CPO}$  is called *locally continuous* if its action on maps:

$$F_1 : (\mathbf{CPO}^{\text{op}})^n \times (\mathbf{CPO})^m ((\vec{A}', \vec{B}), (\vec{A}, \vec{B}')) \rightarrow \mathbf{CPO}(F(\vec{A}', \vec{B}), F(\vec{A}, \vec{B}'))$$

is a map of cpo's, that is: continuous. We have the same notion if we replace  $\mathbf{CPO}$  by  $\mathbf{CPO}_{\perp}$ .

**Example.** The product and coproduct functors:  $\mathbf{CPO} \times \mathbf{CPO} \rightarrow \mathbf{CPO}$ , and the exponent functor:  $\mathbf{CPO}^{\text{op}} \times \mathbf{CPO} \rightarrow \mathbf{CPO}$  are locally continuous.

**Theorem 8.12** Suppose  $F : \mathbf{CPO}_{\perp}^{\text{op}} \times \mathbf{CPO}_{\perp} \rightarrow \mathbf{CPO}_{\perp}$  is locally continuous. Then there is an  $\hat{F} : \mathbf{CPO}_{\perp}^{\text{EP}} \times \mathbf{CPO}_{\perp}^{\text{EP}} \rightarrow \mathbf{CPO}_{\perp}^{\text{EP}}$  which is continuous and has the same action on objects as  $F$ .

**Proof.** We put  $\hat{F}(P, Q) = F(P, Q)$ , so the last statement of the theorem has been taken care of. Given e-p pairs  $P \xrightarrow{i} P'$  and  $Q \xrightarrow{j} Q'$  we have an

e-p pair  $F(P, Q) \xrightarrow[F(r, j)]{F(i, s)} F(P', Q')$  since  $F(i, s) \circ F(r, j) = F(r \circ i, s \circ j)$  (recall that  $F$  is contravariant in its first argument!)  $= F(\text{id}, \text{id}) = \text{id}_{F(P, Q)}$ , and  $F(r, j) \circ F(i, s) = F(i \circ r, j \circ s) \leq F(\text{id}, \text{id}) = \text{id}_{F(P', Q')}$  (the last inequality is by local continuity of  $F$ ). So we let

$$\hat{F}(\langle i, r \rangle, \langle j, s \rangle) = \langle F(r, j), F(i, s) \rangle$$

Then clearly,  $\hat{F}$  is a functor. To see that  $\hat{F}$  is continuous, suppose we have a chain of maps in  $\mathbb{CPO}_{\perp}^{\text{EP}} \times \mathbb{CPO}_{\perp}^{\text{EP}}$ :  $(A_1, B_1) \rightarrow (A_2, B_2) \rightarrow \dots$  with colimit  $(D, E)$ . That means in  $\mathbb{CPO}_{\perp}^{\text{EP}}$  we have two chains, each with its colimit:

$$\begin{array}{ccc} A_1 & \longrightarrow & A_2 \longrightarrow \dots \\ & \searrow \langle i^1, r^1 \rangle & \downarrow \langle i^2, r^2 \rangle \dots \\ & & D \end{array} \quad \begin{array}{ccc} B_1 & \longrightarrow & B_2 \longrightarrow \dots \\ & \searrow \langle j^1, s^1 \rangle & \downarrow \langle j^2, s^2 \rangle \dots \\ & & E \end{array}$$

From the proof of theorem 8.8 we know that  $\text{id}_D = \bigsqcup_n i^n \circ r^n$  and  $\text{id}_E = \bigsqcup_n j^n \circ s^n$  so  $\text{id}_{(D, E)} = \bigsqcup (i^n \circ r^n, j^n \circ s^n)$ . From local continuity of  $F$  then,  $\text{id}_{\hat{F}(D, E)} = \hat{F}(\text{id}_{(D, E)}) = \langle F(\text{id}_D, \text{id}_E), F(\text{id}_D, \text{id}_E) \rangle = \bigsqcup_n \langle F(i^n \circ r^n, j^n \circ s^n), F(i^n \circ r^n, j^n \circ s^n) \rangle$ . But this characterizes the colimit of a chain in  $\mathbb{CPO}_{\perp}^{\text{EP}}$ , so  $\hat{F}(D, E) = F(D, E)$  is isomorphic to the vertex of the colimit in  $\mathbb{CPO}_{\perp}^{\text{EP}}$  of the chain:

$$\hat{F}(A_1, B_1) \rightarrow \hat{F}(A_2, B_2) \rightarrow \dots$$

■

**Example:** a model of the untyped  $\lambda$ -calculus. In the *untyped*  $\lambda$ -calculus, we have a similar formalism as the one given in Chapter 7, but now there are no types. That means, that variables denote functions and arguments at the same time!

In order to model this, we seek a nontrivial solution to:

$$X \cong X^X$$

(Since  $X = 1$  is always a solution, “nontrivial” means: not this one) According to theorem 8.12, the exponential functor  $\mathbb{CPO}_{\perp}^{\text{OP}} \times \mathbb{CPO}_{\perp} \rightarrow \mathbb{CPO}_{\perp}$ , which sends  $(X, Y)$  to  $Y^X$  and is locally continuous, gives rise to a continuous functor  $\mathbb{CPO}_{\perp}^{\text{EP}} \times \mathbb{CPO}_{\perp}^{\text{EP}} \xrightarrow{\text{Exp}} \mathbb{CPO}_{\perp}^{\text{EP}}$ . Let  $F : \mathbb{CPO}_{\perp}^{\text{EP}} \rightarrow \mathbb{CPO}_{\perp}^{\text{EP}}$  be the composite

$$\mathbb{CPO}_{\perp}^{\text{EP}} \xrightarrow{\Delta} \mathbb{CPO}_{\perp}^{\text{EP}} \times \mathbb{CPO}_{\perp}^{\text{EP}} \xrightarrow{\text{Exp}} \mathbb{CPO}_{\perp}^{\text{EP}}$$

$\Delta$  (the diagonal functor) is continuous, so  $F$  is, since continuous functors compose.

**Exercise 143** Show that the functor  $F$  works as follows: for an e-p pair  $P \xrightleftharpoons[i]{r} Q$ ,

$F(\langle i, r \rangle)$  is the e-p pair  $P^P \xrightleftharpoons[I]{R} Q^Q$  where  $I(f) = i \circ f \circ r$  and  $R(g) = r \circ g \circ i$ .

Let us try and apply lemma 8.9; we need a non-initial object  $P$  and a map  $P \xrightarrow{f} F(P)$  in  $\mathbb{CPO}_{\perp}^{\text{EP}}$ .

That is, an embedding-projection pair  $P \xrightleftharpoons{\quad} P^P$ . Well, it is readily checked that for *any* cpo  $P$  with  $\perp$  the pair  $\langle \iota, \rho \rangle$  is an e-p pair, where  $\iota : P \rightarrow P^P$  sends  $p$  to the function which has constant value  $p$ , and  $\rho : P^P \rightarrow P$  sends the function  $f$  to  $f(\perp)$ . We now summarize all our preparations into the final theorem.

**Theorem 8.13 (Scott)** *Let  $P$  be a cpo with  $\perp$ . Define a diagram*

$$D_0 \xrightleftharpoons[i_0]{r_0} D_1 \xrightleftharpoons[i_1]{r_1} D_2 \xrightleftharpoons[i_2]{r_2} \dots$$

*in  $\mathbb{CPO}$  as follows:*

- $D_0 = P$ ;  $D_{n+1} = D_n^{D_n}$ ;
- $i_0 : P \rightarrow P^P$  is  $\lambda p. \lambda q. p$ ;
- $r_0 : P^P \rightarrow P$  is  $\lambda f. f(\perp)$ ;
- $i_{n+1} : D_n^{D_n} = D_{n+1} \rightarrow D_{n+1}^{D_{n+1}}$  is  $\lambda f. i_n \circ f \circ r_n$ ;
- $r_{n+1} : D_{n+1}^{D_{n+1}} \rightarrow D_n^{D_n}$  is  $\lambda g. r_n \circ g \circ i_n$ .

*Let  $D_{\infty}$  be (vertex of) the limit in  $\mathbb{CPO}$  of*

$$D_0 \xleftarrow{r_0} D_1 \xleftarrow{r_1} D_2 \xleftarrow{r_2} \dots$$

*Then  $D_{\infty} \cong D_{\infty}^{D_{\infty}}$  in  $\mathbb{CPO}$ .*



## 9 Presheaves

We start by reviewing the category  $\text{Set}^{\mathcal{C}^{\text{op}}}$  of contravariant functors from  $\mathcal{C}$  to  $\text{Set}$ .  $\mathcal{C}$  is assumed to be a small category throughout. Objects of  $\text{Set}^{\mathcal{C}^{\text{op}}}$  are called *presheaves on  $\mathcal{C}$* .

We have the Yoneda embedding  $y : \mathcal{C} \rightarrow \text{Set}^{\mathcal{C}^{\text{op}}}$ ; we write its effect on objects  $C$  and arrows  $f$  as  $y_C, y_f$  respectively. So for  $f : C \rightarrow D$  we have  $y_f : y_C \rightarrow y_D$ . Recall:  $y_C(C') = \mathcal{C}(C', C)$ , the set of arrows  $C' \rightarrow C$  in  $\mathcal{C}$ ; for  $\alpha : C'' \rightarrow C'$  we have  $y_C(\alpha) : y_C(C') \rightarrow y_C(C'')$  which is defined by composition with  $\alpha$ , so  $y_C(\alpha)(g) = g\alpha$  for  $g : C' \rightarrow C$ . For  $f : C \rightarrow D$  we have  $y_f : y_C \rightarrow y_D$  which is a natural transformation with components

$$(y_f)_{C'} : y_C(C') \rightarrow y_D(C')$$

given by  $(y_f)_{C'}(g) = fg$ . Note, that the naturality of  $y_f$  is just the associativity of composition in  $\mathcal{C}$ .

Presheaves of the form  $y_C$  are called *representable*.

The *Yoneda Lemma* (2.2) says that there is a 1-1 correspondence between elements of  $X(C)$  and arrows in  $\text{Set}^{\mathcal{C}^{\text{op}}}$  from  $y_C$  to  $X$ , for presheaves  $X$  and objects  $C$  of  $\mathcal{C}$ , and this correspondence is natural in both  $X$  and  $C$ . To every element  $x \in X(C)$  corresponds a natural transformation  $\mu : y_C \rightarrow X$  such that  $(\mu)_C(\text{id}_C) = x$ ; and natural transformations from  $y_C$  are completely determined by their effect on  $\text{id}_C$ . An important consequence of the Yoneda lemma is that the Yoneda embedding is actually an embedding, that is: full and faithful, and injective on objects.

### Examples of presheaf categories

1. A first example is the category of presheaves on a monoid (a one-object category)  $M$ . Such a presheaf is nothing but a set  $X$  together with a right  $M$ -action, that is: we have a map  $X \times M \rightarrow X$ , written  $x, f \mapsto xf$ , satisfying  $xe = x$  (for the unit  $e$  of the monoid), and  $(xf)g = x(fg)$ . There is only one representable presheaf.
2. If the category  $\mathcal{C}$  is a poset  $(P, \leq)$ , for  $p \in P$  we have the representable  $y_p$  with  $y_p(q) = \{*\}$  if  $q \leq p$ , and  $\emptyset$  otherwise. So we can identify the representable  $y_p$  with the *downset*  $\downarrow(p) = \{q \mid q \leq p\}$ .
3. The category of directed graphs and graph morphisms is a presheaf category: it is the category of presheaves on the category with two objects  $e$  and  $v$ , and two non-identity arrows  $\sigma, \tau : v \rightarrow e$ . For a presheaf  $X$  on this category,  $X(v)$  can be seen as the set of vertices,  $X(e)$  the set of edges, and  $X(\sigma), X(\tau) : X(e) \rightarrow X(v)$  as the source and target maps.
4. A *tree* is a partially ordered set  $T$  with a least element, such that for any  $x \in T$ , the set  $\downarrow(x) = \{y \in T \mid y \leq x\}$  is a finite linearly ordered subset of  $T$ . A *morphism* of trees  $f : T \rightarrow S$  is an order-preserving function with the property that for any element  $x \in T$ , the restriction of  $f$  to  $\downarrow(x)$  is a

bijection from  $\downarrow(x)$  to  $\downarrow(f(x))$ . A *forest* is a set of trees; a map of forests  $X \rightarrow Y$  is a function  $\phi : X \rightarrow Y$  together with an  $X$ -indexed collection  $(f_x \mid x \in X)$  of morphisms of trees such that  $f_x : x \rightarrow \phi(x)$ . The category of forests and their maps is just the category of presheaves on  $\omega$ , the first infinite ordinal.

Recall the definition of the category  $y\downarrow X$  (an example of a ‘comma category’ construction): objects are pairs  $(C, \mu)$  with  $C$  an object of  $\mathcal{C}$  and  $\mu : y_C \rightarrow X$  an arrow in  $\text{Set}^{\mathcal{C}^{\text{op}}}$ . A morphism  $(C, \mu) \rightarrow (C', \nu)$  is an arrow  $f : C \rightarrow C'$  in  $\mathcal{C}$  such that the triangle

$$\begin{array}{ccc} y_C & \xrightarrow{y_f} & y_{C'} \\ & \searrow \mu & \swarrow \nu \\ & X & \end{array}$$

commutes.

Note that if this is the case and  $\mu : y_C \rightarrow X$  corresponds to  $\xi \in X(C)$  and  $\nu : y_{C'} \rightarrow X$  corresponds to  $\eta \in X(C')$ , then  $\xi = X(f)(\eta)$ .

There is a functor  $U_X : y\downarrow X \rightarrow \mathcal{C}$  (the forgetful functor) which sends  $(C, \mu)$  to  $C$  and  $f$  to itself; by composition with  $y$  we get a diagram

$$y \circ U_X : y\downarrow X \rightarrow \text{Set}^{\mathcal{C}^{\text{op}}}$$

Clearly, there is a natural transformation  $\rho$  from  $y \circ U_X$  to the constant functor  $\Delta_X$  from  $y\downarrow X$  to  $\text{Set}^{\mathcal{C}^{\text{op}}}$  with value  $X$ : let  $\rho_{(C, \mu)} = \mu : y_C \rightarrow X$ . So there is a cocone in  $\text{Set}^{\mathcal{C}^{\text{op}}}$  for  $y \circ U_X$  with vertex  $X$ .

**Proposition 9.1** *The cocone  $\rho : y \circ U_X \Rightarrow \Delta_X$  is colimiting.*

**Proof.** Suppose  $\lambda : y \circ U_X \Rightarrow \Delta_Y$  is another cocone. Define  $\nu : X \rightarrow Y$  by  $\nu_C(\xi) = (\lambda_{(C, \mu)})_C(\text{id}_C)$ , where  $\mu : y_C \rightarrow X$  corresponds to  $\xi$  in the Yoneda Lemma.

Then  $\nu$  is natural: if  $f : C' \rightarrow C$  in  $\mathcal{C}$  and  $\mu' : y_{C'} \rightarrow X$  corresponds to  $X(f)(\xi)$ , the diagram

$$\begin{array}{ccc} y_{C'} & \xrightarrow{y_f} & y_C \\ & \searrow \mu' & \swarrow \mu \\ & X & \end{array}$$

commutes, so  $f$  is an arrow  $(C', \mu') \rightarrow (C, \mu)$  in  $y\downarrow X$ . Since  $\lambda$  is a cocone, we have that

$$\begin{array}{ccc} y_{C'} & \xrightarrow{y_f} & y_C \\ & \searrow \lambda_{(C', \mu')} & \swarrow \lambda_{(C, \mu)} \\ & Y & \end{array}$$

commutes; so

$$\begin{aligned}
\nu_{C'}(X(f)(\xi)) &= (\lambda_{(C', \mu')})_{C'}(\text{id}_{C'}) = \\
(\lambda_{(C, \mu)})_{C'}((y_f)_{C'}(\text{id}_{C'})) &= (\lambda_{(C, \mu)})_{C'}(f) = \\
Y(f)((\lambda_{(C, \mu)})_C(\text{id}_C)) &= Y(f)(\nu_C(\xi))
\end{aligned}$$

It is easy to see that  $\lambda : y \circ U_X \Rightarrow \Delta_Y$  factors through  $\rho$  via  $\nu$ , and that the factorization is unique. ■

Proposition 9.1 is often referred to by saying that “every presheaf is a colimit of representables”.

Let us note that the category  $\text{Set}^{C^{\text{op}}}$  is complete and cocomplete, and that limits and colimits are calculated ‘pointwise’: if  $I$  is a small category and  $F : I \rightarrow \text{Set}^{C^{\text{op}}}$  is a diagram, then for every object  $C$  of  $\mathcal{C}$  we have a diagram  $F_C : I \rightarrow \text{Set}$  by  $F_C(i) = F(i)(C)$ ; if  $X_C$  is a colimit for this diagram in  $\text{Set}$ , there is a unique presheaf structure on the collection  $(X_C \mid C \in \mathcal{C}_0)$  making it into the vertex of a colimit for  $F$ . The same holds for limits. Some immediate consequences of this are:

- i) an arrow  $\mu : X \rightarrow Y$  in  $\text{Set}^{C^{\text{op}}}$  is mono (resp. epi) if and only if every component  $\mu_C$  is an injective (resp. surjective) function of sets;
- ii) the category  $\text{Set}^{C^{\text{op}}}$  is regular, and every epimorphism is a regular epi;
- iii) the initial object of  $\text{Set}^{C^{\text{op}}}$  is the constant presheaf with value  $\emptyset$ ;
- iv)  $X$  is terminal in  $\text{Set}^{C^{\text{op}}}$  if and only if every set  $X(C)$  is a singleton;
- v) for every presheaf  $X$ , the functor  $(-) \times X : \text{Set}^{C^{\text{op}}} \rightarrow \text{Set}^{C^{\text{op}}}$  preserves colimits.

Furthermore we note the following fact: the Yoneda embedding  $\mathcal{C} \rightarrow \text{Set}^{C^{\text{op}}}$  is the ‘free colimit completion’ of  $\mathcal{C}$ . That is: for any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  where  $\mathcal{D}$  is a cocomplete category, there is, up to isomorphism, exactly one *colimit preserving* functor  $\tilde{F} : \text{Set}^{C^{\text{op}}} \rightarrow \mathcal{D}$  such that the diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
& \searrow y & \nearrow \tilde{F} \\
& \text{Set}^{C^{\text{op}}} &
\end{array}$$

commutes.  $\tilde{F}(X)$  is computed as the colimit in  $\mathcal{D}$  of the diagram

$$y \downarrow X \xrightarrow{U_X} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

The functor  $\tilde{F}$  is also called the ‘left Kan extension of  $F$  along  $y$ ’.

We shall now calculate explicitly some structure of  $\text{Set}^{C^{\text{op}}}$ . Exponentials can be calculated using the Yoneda Lemma and proposition 9.1. For  $Y^X$ , we need a natural 1-1 correspondence

$$\text{Set}^{C^{\text{op}}}(Z, Y^X) \simeq \text{Set}^{C^{\text{op}}}(Z \times X, Y)$$

In particular this should hold for representable presheaves  $y_C$ ; so, by the Yoneda Lemma, we should have a 1-1 correspondence

$$Y^X(C) \simeq \text{Set}^{\text{cop}}(y_C \times X, Y)$$

which is natural in  $C$ . This leads us to *define* a presheaf  $Y^X$  by:  $Y^X(C) = \text{Set}^{\text{cop}}(y_C \times X, Y)$ , and for  $f : C' \rightarrow C$  we let  $Y^X(f) : Y^X(C) \rightarrow Y^X(C')$  be defined by composition with  $y_f \times \text{id}_X : y_{C'} \times X \rightarrow y_C \times X$ . Then certainly,  $Y^X$  is a well-defined presheaf and for representable presheaves we have the natural bijection  $\text{Set}^{\text{cop}}(y_C, Y^X) \simeq \text{Set}^{\text{cop}}(y_C \times X, Y)$  we want. In order to show that it holds for arbitrary presheaves  $Z$  we use proposition 9.1. Given  $Z$ , we have the diagram  $y \circ U_Z : y \downarrow Z \rightarrow \mathcal{C} \rightarrow \text{Set}^{\text{cop}}$  of which  $Z$  is a colimit. Therefore arrows  $Z \rightarrow Y^X$  correspond to cocones on  $y \circ U_Z$  with vertex  $Y^X$ . Since we have our correspondence for representables  $y_C$ , such cocones correspond to cocones on the diagram

$$y \downarrow Z \xrightarrow{U_Z} \mathcal{C} \xrightarrow{y} \text{Set}^{\text{cop}} \xrightarrow{(-) \times X} \text{Set}^{\text{cop}}$$

with vertex  $Y$ . Because, as already noted, the functor  $(-) \times X$  preserves colimits, these correspond to arrows  $Z \times X \rightarrow Y$ , as desired.

It is easy to see that the construction of  $Y^X$  gives a functor  $(-)^X : \text{Set}^{\text{cop}} \rightarrow \text{Set}^{\text{cop}}$  which is right adjoint to  $(-) \times X$ , thus establishing that  $\text{Set}^{\text{cop}}$  is cartesian closed. The *evaluation map*  $\text{ev}_{X,Y} : Y^X \times X \rightarrow Y$  is given by

$$(\phi, x) \mapsto \phi_C(\text{id}_C, x)$$

**Exercise 144** Show that the map  $\text{ev}_{X,Y}$ , thus defined, is indeed a natural transformation.

**Exercise 145** Prove that  $y : \mathcal{C} \rightarrow \text{Set}^{\text{cop}}$  preserves all limits which exist in  $\mathcal{C}$ . Prove also, that if  $\mathcal{C}$  is cartesian closed,  $y$  preserves exponents.

Another piece of structure we shall need is that of a *subobject classifier*.

Suppose  $\mathcal{E}$  is a category with finite limits. A subobject classifier is a monomorphism  $t : T \rightarrow \Omega$  with the property that for any monomorphism  $m : A \rightarrow B$  in  $\mathcal{E}$  there is a *unique* arrow  $\phi : B \rightarrow \Omega$  such that there is a pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & T \\ m \downarrow & & \downarrow t \\ B & \xrightarrow{\phi} & \Omega \end{array}$$

We say that the unique arrow  $\phi$  *classifies*  $m$  or rather, the subobject represented by  $m$  (if  $m$  and  $m'$  represent the same subobject, they have the same classifying arrow). In  $\text{Set}$ , any two element set  $\{a, b\}$  together with a specific choice of one of them, say  $b$  (considered as arrow  $1 \rightarrow \{a, b\}$ ) acts as a subobject classifier: for  $A \subset B$  we have the unique *characteristic function*  $\phi_A : B \rightarrow \{a, b\}$  defined by  $\phi_A(x) = b$  if  $x \in A$ , and  $\phi_A(x) = a$  otherwise.

It is no coincidence that in  $\mathbf{Set}$ , the domain of  $t : T \rightarrow \Omega$  is a terminal object:  $T$  is always terminal. For, for any object  $A$  the arrow  $\phi : A \rightarrow \Omega$  which classifies the identity on  $A$  factors as  $tn$  for some  $n : A \rightarrow T$ . On the other hand, if  $k : A \rightarrow T$  is any arrow, then we have pullback diagrams

$$\begin{array}{ccccc} A & \xrightarrow{k} & T & \xrightarrow{\text{id}_T} & T \\ \text{id}_A \downarrow & & \downarrow \text{id}_T & & \downarrow t \\ A & \xrightarrow{k} & T & \xrightarrow{t} & \Omega \end{array}$$

so  $tk$  classifies  $\text{id}_A$ . By uniqueness of the classifying map,  $tn = tk$ ; since  $t$  is mono,  $n = k$ . So  $T$  is terminal. Henceforth we shall write  $1 \xrightarrow{t} \Omega$  for the subobject classifier, or, by abuse of language, just  $\Omega$ .

Note: if  $1 \xrightarrow{t} \Omega$  is a subobject classifier in  $\mathcal{E}$  then we have a 1-1 correspondence between arrows  $A \xrightarrow{\phi} \Omega$  and *subobjects* of  $A$ . This correspondence is natural in the following sense: given  $f : B \rightarrow A$  and a subobject  $U$  of  $A$ ; by  $f^\#(U)$  we denote the subobject of  $B$  obtained by pulling back  $U$  along  $f$ . Then if  $\phi$  classifies  $U$ ,  $\phi f$  classifies  $f^\#(U)$ .

First a remark about subobjects in  $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ . A subobject of  $X$  can be identified with a *subpresheaf* of  $X$ : that is, a presheaf  $Y$  such that  $Y(C) \subseteq X(C)$  for each  $C$ , and  $Y(f)$  is the restriction of  $X(f)$  to  $Y(\text{cod}(f))$ . This follows easily from epi-mono factorizations pointwise, and the corresponding fact in  $\mathbf{Set}$ .

Again, we use the Yoneda Lemma to compute the subobject classifier in  $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ . We need a presheaf  $\Omega$  such that at least for each representable presheaf  $y_C$ ,  $\Omega(C)$  is in 1-1 correspondence with the set of subobjects (in  $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ ) of  $y_C$ . So we define  $\Omega$  such that  $\Omega(C)$  is the set of subpresheaves of  $y_C$ ; for  $f : C' \rightarrow C$  we have  $\Omega(f)$  defined by the action of pulling back along  $y_f$ .

What do subpresheaves of  $y_C$  look like? If  $R$  is a subpresheaf of  $y_C$  then  $R$  can be seen as a set of arrows with codomain  $C$  such that if  $f : C' \rightarrow C$  is in  $R$  and  $g : C'' \rightarrow C'$  is arbitrary, then  $fg$  is in  $R$  (for,  $fg = y_C(g)(f)$ ). Such a set of arrows is called a *sieve* on  $C$ .

Under the correspondence between subobjects of  $y_C$  and sieves on  $C$ , the operation of pulling back a subobject along a map  $y_f$  (for  $f : C' \rightarrow C$ ) sends a sieve  $R$  on  $C$  to the sieve  $f^*(R)$  on  $C'$  defined by

$$f^*(R) = \{g : D \rightarrow C' \mid fg \in R\}$$

So  $\Omega$  can be defined as follows:  $\Omega(C)$  is the set of sieves on  $C$ , and  $\Omega(f)(R) = f^*(R)$ . The map  $t : 1 \rightarrow \Omega$  sends, for each  $C$ , the unique element of  $1(C)$  to the maximal sieve on  $C$  (i.e., the unique sieve which contains  $\text{id}_C$ ).

**Exercise 146** Suppose  $\mathcal{C}$  is a preorder  $(P, \leq)$ . For  $p \in P$  we let  $\downarrow(p) = \{q \in P \mid q \leq p\}$ . Show that sieves on  $p$  can be identified with downwards closed subsets of  $\downarrow(p)$ . If we denote the unique arrow  $q \rightarrow p$  by  $qp$  and  $U$  is a downwards closed subset of  $\downarrow(p)$ , what is  $(qp)^*(U)$ ?

Let us now prove that  $t : 1 \rightarrow \Omega$ , thus defined, is a subobject classifier in  $\text{Set}^{C^{\text{op}}}$ . Let  $Y$  be a subpresheaf of  $X$ . Then for any  $C$  and any  $x \in X(C)$ , the set

$$\phi_C(x) = \{f : D \rightarrow C \mid X(f)(x) \in Y(D)\}$$

is a sieve on  $C$ , and defining  $\phi : X \rightarrow \Omega$  in this way gives a natural transformation: for  $f : C' \rightarrow C$  we have

$$\begin{aligned} \phi_{C'}(X(f)(x)) &= \{g : D \rightarrow C' \mid X(g)(X(f)(x)) \in Y(D)\} \\ &= \{g : D \rightarrow C' \mid X(gf)(x) \in Y(D)\} \\ &= \{g : D \rightarrow C' \mid fg \in \phi_C(x)\} \\ &= f^*(\phi_C(x)) \\ &= \Omega(f)(\phi_C(x)) \end{aligned}$$

Moreover, if we take the pullback of  $t$  along  $\phi$ , we get the subpresheaf of  $X$  consisting of (at each object  $C$ ) of those elements  $x$  for which  $\text{id}_C \in \phi_C(x)$ ; that is, we get  $Y$ . So  $\phi$  classifies the subpresheaf  $Y$ .

On the other hand, if  $\phi : X \rightarrow \Omega$  is any natural transformation such that pulling back  $t$  along  $\phi$  gives  $Y$ , then for every  $x \in X(C)$  we have that  $x \in Y(C)$  if and only if  $\text{id}_C \in \phi_C(x)$ . But then by naturality we get for any  $f : C' \rightarrow C$  that

$$X(f)(x) \in Y(C') \Leftrightarrow \text{id}_{C'} \in f^*(\phi_C(x)) \Leftrightarrow f \in \phi_C(x)$$

which shows that the classifying map  $\phi$  is unique.

Combining the subobject classifier with the cartesian closed structure, we obtain *power objects*. In a category  $\mathcal{E}$  with finite products, we call an object  $A$  a *power object* of the object  $X$ , if there is a natural 1-1 correspondence

$$\mathcal{E}(Y, A) \simeq \text{Sub}_{\mathcal{E}}(Y \times X)$$

The naturality means that if  $f : Y \rightarrow A$  and  $g : Z \rightarrow Y$  are arrows in  $\mathcal{E}$  and  $f$  corresponds to the subobject  $U$  of  $Y \times X$ , then  $fg : Z \rightarrow A$  corresponds to the subobject  $(g \times \text{id}_X)^{\sharp}(U)$  of  $Z \times X$ .

Power objects are unique up to isomorphism; the power object of  $X$ , if it exists, is usually denoted  $\mathcal{P}(X)$ . Note the following consequence of the definition: to the identity map on  $\mathcal{P}(X)$  corresponds a subobject of  $\mathcal{P}(X) \times X$  which we call the “element relation”  $\in_X$ ; it has the property that whenever  $f : Y \rightarrow \mathcal{P}(X)$  corresponds to the subobject  $U$  of  $Y \times X$ , then  $U = (f \times \text{id}_X)^{\sharp}(\in_X)$ .

Convince yourself that power objects in the category  $\text{Set}$  are just the familiar power sets.

In a cartesian closed category with subobject classifier  $\Omega$ , power objects exist: let  $\mathcal{P}(X) = \Omega^X$ . Clearly, the defining 1-1 correspondence is there.

$$\mathcal{P}(X)(C) = \text{Sub}(y_C \times X)$$

with action  $\mathcal{P}(X)(f)(U) = (y_f \times \text{id}_X)^{\sharp}(U)$ .

**Exercise 147** Show that  $\mathcal{P}(X)(C) = \text{Sub}(y_C \times X)$  and that, for  $f : C' \rightarrow C$ ,  $\mathcal{P}(X)(f)(U) = (y_f \times \text{id}_X)^\#(U)$ . Prove also, that the element relation, as a subpresheaf  $\in_X$  of  $\mathcal{P}(X) \times X$ , is given by

$$(\in_X)(C) = \{(U, x) \in \text{Sub}(y_C \times X) \times X(C) \mid (\text{id}_C, x) \in U(C)\}$$

**Definition 9.2** A *topos* is a category with finite limits, which is cartesian closed and has a subobject classifier.

**Exercise 148** Let  $\mathcal{C}$  be a small category; we work in the category  $\text{Set}^{\mathcal{C}^{\text{op}}}$  of presheaves on  $\mathcal{C}$ . Let  $P$  be such a presheaf. We define a presheaf  $\tilde{P}$  as follows: for an object  $C$  of  $\mathcal{C}$ ,  $\tilde{P}(C)$  consists of those subobjects  $\alpha$  of  $y_C \times P$  which satisfy the following condition: for all arrows  $f : D \rightarrow C$ , the set

$$\{y \in P(D) \mid (f, y) \in \alpha(D)\}$$

has at most one element.

- Complete the definition of  $\tilde{P}$  as a presheaf.
- Show that there is a monic map  $\eta_P : P \rightarrow \tilde{P}$  with the following property: for every diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & P \\ m \downarrow & & \\ B & & \end{array}$$

with  $m$  mono, there is a unique map  $\tilde{g} : B \rightarrow \tilde{P}$  such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & P \\ m \downarrow & & \downarrow \eta_P \\ B & \xrightarrow{\tilde{g}} & \tilde{P} \end{array}$$

is a pullback square. The arrow  $P \xrightarrow{\eta_P} \tilde{P}$  is called a *partial map classifier* for  $P$ .

- Show that the assignment  $P \mapsto \tilde{P}$  is part of a functor  $(\tilde{\cdot})$  in such a way that the maps  $\eta_P$  form a natural transformation from the identity functor to  $(\tilde{\cdot})$ , and all naturality squares for  $\eta$  are pullbacks.

**Exercise 149** Let  $\mathcal{E}$  be a topos with subobject classifier  $1 \xrightarrow{t} \Omega$ .

- Prove that  $\Omega$  is injective.
- Prove that every object of the form  $\Omega^X$  is injective.
- Conclude that  $\mathcal{E}$  has enough injectives.

## 9.1 Recovering the category from its presheaves?

In this short section we shall see to what extent the category  $\text{Set}^{\mathcal{C}^{\text{op}}}$  determines  $\mathcal{C}$ . In other words, suppose  $\text{Set}^{\mathcal{C}^{\text{op}}}$  and  $\text{Set}^{\mathcal{D}^{\text{op}}}$  are equivalent categories; what can we say about  $\mathcal{C}$  and  $\mathcal{D}$ ?

**Definition 9.3** In a regular category an object  $P$  is called (regular) *projective* if for every regular epi  $f : A \rightarrow B$ , any arrow  $P \rightarrow B$  factors through  $f$ . Equivalently, every regular epi with codomain  $P$  has a section.

**Exercise 150** Prove the equivalence claimed in definition 9.3.

**Definition 9.4** An object  $X$  is called *indecomposable* if whenever  $X$  is a coproduct  $\coprod_i U_i$ , then for *exactly* one  $i$  the object  $U_i$  is not initial.

Note, that an initial object is not indecomposable, just as 1 is not a prime number.

In  $\text{Set}^{\mathcal{C}^{\text{op}}}$ , coproducts are *stable*, which means that they are preserved by pull-back functors; this is easy to check. Another triviality is that the initial object is *strict*: the only maps into it are isomorphisms.

**Proposition 9.5** In  $\text{Set}^{\mathcal{C}^{\text{op}}}$ , a presheaf  $X$  is indecomposable and projective if and only if it is a retract of a representable presheaf: there is a diagram  $X \xrightarrow{i} y_{\mathcal{C}} \xrightarrow{r} X$  with  $ri = \text{id}_X$ .

**Proof.** Check yourself that every retract of a projective object is again projective. Similarly, a retract of an indecomposable object is indecomposable: if  $X \xrightarrow{i} Y \xrightarrow{r} X$  is such that  $ri = \text{id}_X$  and  $Y$  is indecomposable, any presentation of  $X$  as a coproduct  $\coprod_i U_i$  can be pulled back along  $r$  to produce, by stability of coproducts, a presentation of  $Y$  as coproduct  $\coprod_i V_i$  such that

$$\begin{array}{ccc} V_i & \longrightarrow & Y \\ \downarrow & & \downarrow r \\ U_i & \longrightarrow & X \end{array}$$

is a pullback; for exactly one  $i$  then,  $V_i$  is non-initial; hence since  $r$  is epi and the initial object is strict, for exactly one  $i$  we have that  $U_i$  is non-initial. We see that the property of being projective and indecomposable is inherited by retracts. Moreover, every representable is indecomposable and projective, as we leave for you to check.

Conversely, assume  $X$  is indecomposable and projective. By proposition 9.1 and the standard construction of colimits from coproducts and coequalizers, there is an epi  $\coprod_i y_{\mathcal{C}_i} \rightarrow X$  from a coproduct of representables. Since  $X$  is projective, this epi has a section  $\iota$ . Pulling back along  $\iota$  we get a presentation



of  $X$  as a coproduct  $\coprod_i V_i$  such that

$$\begin{array}{ccc} V_i & \longrightarrow & X \\ \downarrow & & \downarrow \iota \\ y_{C_i} & \longrightarrow & \coprod_i y_{C_i} \end{array}$$

is a pullback diagram.  $X$  was assumed indecomposable, so exactly one  $V_i$  is non-initial. But this means that  $X$  is a retract of  $y_{C_i}$ .  $\blacksquare$

If  $X$  is a retract of  $y_C$ , say  $X \xrightarrow{\mu} y_C \xrightarrow{\nu} X$  with  $\nu\mu = \text{id}_X$ , consider  $\mu\nu : y_C \rightarrow y_C$ . This arrow is *idempotent*:  $(\mu\nu)(\mu\nu) = \mu(\nu\mu)\nu = \mu\nu$ , and since the Yoneda embedding is full and faithful,  $\mu\nu = y_e$  for an idempotent  $e : C \rightarrow C$  in  $\mathcal{C}$ .

A category  $\mathcal{C}$  is said to be *Cauchy complete* if for every idempotent  $e : C \rightarrow C$  there is a diagram  $D \xrightarrow{i} C \xrightarrow{r} D$  with  $ri = \text{id}_D$  and  $ir = e$ . One also says: “idempotents split”. In the situation above (where  $X$  is a retract of  $y_C$ ) we see that  $X$  must then be isomorphic to  $y_D$  for a retract  $D$  of  $C$  in  $\mathcal{C}$ . We conclude:

**Theorem 9.6** *If  $\mathcal{C}$  is Cauchy complete,  $\mathcal{C}$  is equivalent to the full subcategory of  $\text{Set}^{\mathcal{C}^{\text{op}}}$  on the indecomposable projectives. Hence if  $\mathcal{C}$  and  $\mathcal{D}$  are Cauchy complete and  $\text{Set}^{\mathcal{C}^{\text{op}}}$  and  $\text{Set}^{\mathcal{D}^{\text{op}}}$  are equivalent, so are  $\mathcal{C}$  and  $\mathcal{D}$ .*

**Exercise 151** Show that if  $\mathcal{C}$  has equalizers,  $\mathcal{C}$  is Cauchy complete.

## 9.2 The Logic of Presheaves

**Definition 9.7** A *Heyting algebra*  $H$  is a lattice  $(\perp, \top, \vee, \wedge)$  together with a binary operation  $\rightarrow$  (called *Heyting implication*), which satisfies the following equivalence for all  $a, b, c \in H$ :

$$a \wedge b \leq c \Leftrightarrow a \leq b \rightarrow c$$

**Exercise 152** Prove that every Heyting algebra, as a lattice, is *distributive*:  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$  and  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$  hold, for all  $x, y, z \in H$ .

For a presheaf  $X$  we shall write  $\text{Sub}(X)$  for the set of subpresheaves of  $X$ . So if  $\phi : X \rightarrow Y$  in  $\text{Set}^{\mathcal{C}^{\text{op}}}$  we gave  $\phi^\sharp : \text{Sub}(Y) \rightarrow \text{Sub}(X)$  by pulling back.

**Theorem 9.8** *Every poset  $\text{Sub}(X)$  is a Heyting algebra. For every  $\phi : Y \rightarrow X$ , the map  $\phi^\sharp$  commutes with the Heyting structure  $(\perp, \top, \wedge, \vee, \rightarrow)$ . Moreover,  $\phi^\sharp$  has both a right and a left adjoint, denoted  $\forall_\phi$  and  $\exists_\phi$  respectively.*

**Proof.** Since limits and colimits are computed pointwise,  $\wedge$  and  $\vee$  (between subpresheaves) are given by pointwise intersection and union, respectively. The empty subpresheaf is, of course, the bottom element of  $\text{Sub}(X)$ , and  $X$  itself is

the top element. Heyting implication is *not* done pointwise, since if  $A$  and  $B$  are subpresheaves of  $X$ , setting

$$(A \rightarrow B)(C) = \{x \in X(C) \mid x \in A(C) \Rightarrow x \in B(C)\}$$

does not necessarily define a subpresheaf of  $X$  (check this!). Therefore we put

$$(A \rightarrow B)(C) = \{x \in X(C) \mid \forall f : C' \rightarrow C (X(f)(x) \in A(C') \Rightarrow X(f)(x) \in B(C'))\}$$

Then  $(A \rightarrow B)$  is a subpresheaf of  $X$ . It is easy to verify that if  $D$  is another subpresheaf of  $X$  then  $D$  is a subpresheaf of  $(A \rightarrow B)$  if and only if  $D \cap A$  is a subpresheaf of  $B$ .

Let us check that  $\phi^\sharp$  preserves Heyting implication (the rest of the structure is left to you):

$$\begin{aligned} (\phi^\sharp(A) \rightarrow \phi^\sharp(B))(C) &= \{y \in Y(C) \mid \forall f : C' \rightarrow C (Y(f)(y) \in \phi^\sharp(A)(C') \Rightarrow \\ &\quad Y(f)(y) \in \phi^\sharp(B)(C'))\} \\ &= \{y \in Y(C) \mid \forall f : C' \rightarrow C (\phi_{C'}(Y(f)(y)) \in A(C') \Rightarrow \\ &\quad \phi_{C'}(Y(f)(y)) \in B(C'))\} \\ &= \{y \in Y(C) \mid \forall f : C' \rightarrow C (X(f)(\phi_C(y)) \in A(C') \Rightarrow \\ &\quad X(f)(\phi_C(y)) \in B(C'))\} \\ &= \{y \in Y(C) \mid \phi_C(y) \in (A \rightarrow B)(C)\} \\ &= \phi^\sharp(A \rightarrow B)(C) \end{aligned}$$

The left adjoint  $\exists_\phi(A)$  (where now  $A$  is a subpresheaf of  $Y$ ) is, just as in the case of regular categories, given by the *image* of  $A$  under  $\phi$ , and this is done pointwise. So,

$$\exists_\phi(A)(C) = \{x \in X(C) \mid \exists y \in A(C)(x = \phi_C(y))\}$$

Clearly then, if  $B$  is a subpresheaf of  $X$ , we have  $\exists_\phi(A) \leq B$  in  $\text{Sub}(X)$  if and only if  $A \leq \phi^\sharp(B)$  in  $\text{Sub}(Y)$ .

The right adjoint  $\forall_\phi(A)$  is given by

$$\forall_\phi(A)(C) = \{x \in X(C) \mid \forall f : D \rightarrow C \forall y \in Y(D)(\phi_D(y) = x \Rightarrow y \in A(D))\}$$

Check for yourself that then,  $B \leq \forall_\phi(A)$  in  $\text{Sub}(X)$  if and only if  $\phi^\sharp(B) \leq A$  in  $\text{Sub}(Y)$ . ■

**Exercise 153** Prove that for  $\phi : Y \rightarrow X$ ,  $A$  a subpresheaf of  $X$  and  $B$  a subpresheaf of  $Y$ ,

$$\exists_\phi(\phi^\sharp(A) \wedge B) = A \wedge \exists_\phi(B)$$

Which property of the map  $\phi^\sharp$  do you need?

**Exercise 154** Suppose  $X$  is a presheaf on  $\mathcal{C}$ . Let  $\mathcal{S}$  be the set of all those  $\mathcal{C}_0$ -indexed collections of sets  $A = (A_C \mid c \in \mathcal{C}_0)$  for which  $A_C$  is a subset of  $X(C)$  for each  $C$ .  $\mathcal{S}$  is ordered pointwise:  $A \leq B$  iff for each  $C$ ,  $A_C \subseteq B_C$ .

Let  $\iota : \text{Sub}(X) \rightarrow \mathcal{S}$  be the inclusion. Show that  $\iota$  has both a left and a right adjoint.

### 9.2.1 First-order structures in categories of presheaves

In any regular category which satisfies the properties of theorem 9.8 (such a category is often called a ‘Heyting category’), one can extend the interpretation of ‘regular logic’ in regular categories to full first-order logic. We shall retain as much as possible the notation from chapter 4.

We have a language  $\mathcal{L}$ , which consists of a collection of *sorts*  $S, T, \dots$ , possibly *constants*  $c^S$  of sort  $S$ , *function symbols*  $f : S_1, \dots, S_n \rightarrow S$ , and *relation symbols*  $R \subseteq S_1, \dots, S_n$ . The definition of formula is extended with the clauses:

- i) If  $\varphi$  and  $\psi$  are formulas then  $(\varphi \vee \psi)$ ,  $(\varphi \rightarrow \psi)$  and  $\neg\varphi$  are formulas;
- ii) if  $\varphi$  is a formula and  $x^S$  a variable of sort  $S$  then  $\forall x^S \varphi$  is a formula.

For the notations  $FV(t)$  and  $FV(\varphi)$  we refer to the mentioned chapter 4. Again, an interpretation assigns objects  $\llbracket S \rrbracket$  to the sorts  $S$ , arrows to the function symbols and subobjects to relation symbols. This then leads to the definition of the interpretation of a formula  $\varphi$  as a subobject  $\llbracket \varphi \rrbracket$  of  $\llbracket FV(\varphi) \rrbracket$ , which is a chosen product of the interpretations of all the sorts of the free variables of  $\varphi$ : if  $FV(\varphi) = \{x_1^{S_1}, \dots, x_n^{S_n}\}$  then  $\llbracket FV(\varphi) \rrbracket = \llbracket S_1 \rrbracket \times \dots \times \llbracket S_n \rrbracket$ .

The definition of  $\llbracket \varphi \rrbracket$  of the mentioned chapter 4 is now extended by the clauses:

- i) If  $\llbracket \varphi \rrbracket \rightarrow \llbracket FV(\varphi) \rrbracket$  and  $\llbracket \psi \rrbracket \rightarrow \llbracket FV(\psi) \rrbracket$  are given and

$$\begin{array}{ccc} \llbracket FV(\varphi \wedge \psi) \rrbracket & \xrightarrow{\pi_1} & \llbracket FV(\varphi) \rrbracket \\ & \searrow \pi_2 & \\ & & \llbracket FV(\psi) \rrbracket \end{array}$$

are the projections, then

$$\begin{array}{lll} \llbracket \varphi \vee \psi \rrbracket & = & (\pi_1)^\sharp(\llbracket \varphi \rrbracket) \vee (\pi_2)^\sharp(\llbracket \psi \rrbracket) & \text{in } \text{Sub}(\llbracket FV(\varphi \wedge \psi) \rrbracket) \\ \llbracket \varphi \rightarrow \psi \rrbracket & = & (\pi_1)^\sharp(\llbracket \varphi \rrbracket) \rightarrow (\pi_2)^\sharp(\llbracket \psi \rrbracket) & \text{in } \text{Sub}(\llbracket FV(\varphi \wedge \psi) \rrbracket) \\ \llbracket \neg\varphi \rrbracket & = & \llbracket \varphi \rrbracket \rightarrow \perp & \text{in } \text{Sub}(\llbracket FV(\varphi) \rrbracket) \end{array}$$

(Note that  $FV(\varphi \wedge \psi) = FV(\varphi \vee \psi) = FV(\varphi \rightarrow \psi)$ )

- ii) if  $\llbracket \varphi \rrbracket \rightarrow \llbracket FV(\varphi) \rrbracket$  is given and  $\pi : \llbracket FV(\varphi) \rrbracket \rightarrow \llbracket FV(\exists x \varphi) \rrbracket$  is the projection, let  $FV'(\varphi) = FV(\varphi \wedge x = x)$  and  $\pi' : \llbracket FV'(\varphi) \rrbracket \rightarrow \llbracket FV(\varphi) \rrbracket$  the projection. Then

$$\llbracket \forall x \varphi \rrbracket = \forall_{\pi \pi'} ((\pi')^\sharp(\llbracket \varphi \rrbracket))$$

We shall now write out what this means, concretely, in  $\text{Set}^{c^{\text{op}}}$ . For a formula  $\varphi$ , we have  $\llbracket \varphi \rrbracket$  as a subobject of  $\llbracket FV(\varphi) \rrbracket$ , hence we have a classifying map  $\{\varphi\} : \llbracket FV(\varphi) \rrbracket \rightarrow \Omega$  with components  $\{\varphi\}_C : \llbracket FV(\varphi) \rrbracket(C) \rightarrow \Omega(C)$ ; for  $(a_1, \dots, a_n) \in \llbracket FV(\varphi) \rrbracket(C)$ ,  $\{\varphi\}_C(a_1, \dots, a_n)$  is a sieve on  $C$ .

**Definition 9.9** For  $\varphi$  a formula with free variables  $x_1, \dots, x_n$ ,  $C$  an object of  $\mathcal{C}$  and  $(a_1, \dots, a_n) \in \llbracket FV(\varphi) \rrbracket(C)$ , the notation  $C \Vdash \varphi(a_1, \dots, a_n)$  means that  $\text{id}_C \in \{\varphi\}_C(a_1, \dots, a_n)$ .

The pronunciation of “ $\Vdash$ ” is ‘forces’.

**Notation.** For  $\varphi$  a formula with free variables  $x_1^{S_1}, \dots, x_n^{S_n}$ ,  $C$  an object of  $\mathcal{C}$  and  $(a_1, \dots, a_n) \in \llbracket FV(\varphi) \rrbracket(C)$  as above, so  $a_i \in \llbracket S_i \rrbracket(C)$ , if  $f : C' \rightarrow C$  is an arrow in  $\mathcal{C}$  we shall write  $a_i f$  for  $\llbracket S_i \rrbracket(f)(a_i)$ .

Note: with this notation and  $\varphi$ ,  $C$ ,  $a_1, \dots, a_n$ ,  $f : C' \rightarrow C$  as above, we have  $f \in \{\varphi\}_C(a_1, \dots, a_n)$  if and only if  $C' \Vdash \varphi(a_1 f, \dots, a_n f)$ .

Using the characterization of the Heyting structure of  $\text{Set}^{\text{cop}}$  given in the proof of theorem 9.8, we can easily write down an inductive definition for the notion  $C \Vdash \varphi(a_1, \dots, a_n)$ :

- $C \Vdash (t = s)(a_1, \dots, a_n)$  if and only if  $\llbracket t \rrbracket_C(a_1, \dots, a_n) = \llbracket s \rrbracket_C(a_1, \dots, a_n)$
- $C \Vdash R(t_1, \dots, t_k)(a_1, \dots, a_n)$  if and only if

$$(\llbracket t_1 \rrbracket_C(a_1, \dots, a_n), \dots, \llbracket t_k \rrbracket_C(a_1, \dots, a_n)) \in \llbracket R \rrbracket(C)$$

- $C \Vdash (\varphi \wedge \psi)(a_1, \dots, a_n)$  if and only if

$$C \Vdash \varphi(a_1, \dots, a_n) \text{ and } C \Vdash \psi(a_1, \dots, a_n)$$

- $C \Vdash (\varphi \vee \psi)(a_1, \dots, a_n)$  if and only if

$$C \Vdash \varphi(a_1, \dots, a_n) \text{ or } C \Vdash \psi(a_1, \dots, a_n)$$

- $C \Vdash (\varphi \rightarrow \psi)(a_1, \dots, a_n)$  if and only if for every arrow  $f : C' \rightarrow C$ ,

$$\text{if } C' \Vdash \varphi(a_1 f, \dots, a_n f) \text{ then } C' \Vdash \psi(a_1 f, \dots, a_n f)$$

- $C \Vdash \neg \varphi(a_1, \dots, a_n)$  if and only if for no arrow  $f : C' \rightarrow C$ ,  $C' \Vdash \varphi(a_1 f, \dots, a_n f)$
- $C \Vdash \exists x^S \varphi(a_1, \dots, a_n)$  if and only if for some  $a \in \llbracket S \rrbracket(C)$ ,  $C \Vdash \varphi(a, a_1, \dots, a_n)$
- $C \Vdash \forall x^S \varphi(a_1, \dots, a_n)$  if and only if for every arrow  $f : C' \rightarrow C$  and every  $a \in \llbracket S \rrbracket(C')$ ,

$$C' \Vdash \varphi(a, a_1 f, \dots, a_n f)$$

**Exercise 155** Prove: if  $C \Vdash \varphi(a_1, \dots, a_n)$  and  $f : C' \rightarrow C$  is an arrow, then  $C' \Vdash \varphi(a_1 f, \dots, a_n f)$ .

Now let  $\phi$  be a sentence of the language, so  $\llbracket \phi \rrbracket$  is a subobject of 1 in  $\text{Set}^{\text{cop}}$ . Note: a subobject of 1 is ‘the same thing’ as a collection  $X$  of objects of  $\mathcal{C}$  such that whenever  $C \in X$  and  $f : C' \rightarrow C$  is arbitrary, then  $C' \in X$  also. The following theorem is straightforward.

**Theorem 9.10** For a language  $\mathcal{L}$  and interpretation  $\llbracket \cdot \rrbracket$  of  $\mathcal{L}$  in  $\text{Set}^{\mathcal{C}^{\text{op}}}$ , we have that for every  $\mathcal{L}$ -sentence  $\phi$ ,  $\llbracket \phi \rrbracket = \{C \in \mathcal{C}_0 \mid C \Vdash \phi\}$ . Hence,  $\phi$  is true for the interpretation in  $\text{Set}^{\mathcal{C}^{\text{op}}}$  if and only if for every  $C$ ,  $C \Vdash \phi$ .

If  $\Gamma$  is a set of  $\mathcal{L}$ -sentences and  $\phi$  an  $\mathcal{L}$ -sentence, we write  $\Gamma \Vdash \phi$  to mean: in every interpretation in a presheaf category such that every sentence of  $\Gamma$  is true,  $\phi$  is true.

We mention without proof:

**Theorem 9.11 (Soundness and Completeness)** If  $\Gamma$  is a set of  $\mathcal{L}$ -sentences and  $\phi$  an  $\mathcal{L}$ -sentence, we have  $\Gamma \Vdash \phi$  if and only if  $\phi$  is provable from  $\Gamma$  in intuitionistic predicate calculus.

Intuitionistic predicate calculus is what one gets from classical logic by deleting the rule which infers  $\phi$  from a proof that  $\neg\phi$  implies absurdity. In a Gentzen calculus, this means that one restricts attention to those sequents  $\Gamma \Rightarrow \Delta$  for which  $\Delta$  consists of at most one formula.

**Exercise 156** Let  $N$  denote the constant presheaf with value  $\mathbb{N}$ .

- i) Show that there are maps  $0 : 1 \rightarrow N$  and  $S : N \rightarrow N$  which make  $N$  into a natural numbers object in  $\text{Set}^{\mathcal{C}^{\text{op}}}$ .
- ii) Accordingly, there is an interpretation of the language of first-order arithmetic in  $\text{Set}^{\mathcal{C}^{\text{op}}}$ , where the unique sort is interpreted by  $N$ . Prove, that for this interpretation, a sentence in the language of arithmetic is true if and only if it is true classically in the standard model  $\mathbb{N}$ .

**Exercise 157** Prove that for every object  $C$  of  $\mathcal{C}$ , the set  $\Omega(C)$  of sieves on  $C$  is a Heyting algebra, and that for every map  $f : C' \rightarrow C$  in  $\mathcal{C}$ ,  $\Omega(f) : \Omega(C) \rightarrow \Omega(C')$  preserves the Heyting structure. Write out explicitly the Heyting implication  $(R \rightarrow S)$  of two sieves.

## 9.3 Two examples and applications

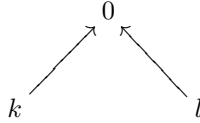
### 9.3.1 Kripke semantics

*Kripke semantics* is a special kind of presheaf semantics:  $\mathcal{C}$  is taken to be a poset, and the sorts are interpreted by presheaves  $X$  such that for every  $q \leq p$  the map  $X(q \leq p) : X(p) \rightarrow X(q)$  is an inclusion of sets. Let us call these presheaves *Kripke presheaves*.

The soundness and completeness theorem 9.11 already holds for Kripke semantics. This raises the question whether the greater generality of presheaves achieves anything new. In this example, we shall see that general presheaves are richer than Kripke models if one considers *intermediate logics*: logics stronger than intuitionistic logic but weaker than classical logic.

In order to warm up, let us look at Kripke models for *propositional logic*. The propositional variables are interpreted as subobjects of  $1$  in  $\text{Set}^{\mathcal{K}^{\text{op}}}$  (for a

poset  $(\mathcal{K}, \leq)$ ; that means, as downwards closed subsets of  $\mathcal{K}$  (see the remark just before theorem 9.10). Let, for example,  $\mathcal{K}$  be the poset:



and let  $\llbracket p \rrbracket = \{k\}$ . Then  $0 \not\models p$ ,  $0 \not\models \neg p$  (since  $k \leq 0$  and  $k \Vdash p$ ) and  $0 \not\models \neg\neg p$  (since  $l \leq 0$  and  $l \not\Vdash p$ ). So  $p \vee \neg p \vee \neg\neg p$  is not true for this interpretation. Even simpler, if  $\mathcal{K} = \{0 \leq 1\}$  and  $\llbracket p \rrbracket = \{0\}$ , then  $1 \not\models p \vee \neg p$ . However, if  $\mathcal{K}$  is a linear order, then  $(p \rightarrow q) \vee (q \rightarrow p)$  is always true on  $\mathcal{K}$ , since if  $\mathcal{K}$  is linear, then so is the poset of its downwards closed subsets. From this one can conclude that if one adds to intuitionistic propositional logic the axiom scheme

$$(\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$$

one gets a logic which is strictly between intuitionistic and classical logic.

**Exercise 158** Prove that  $(p \rightarrow q) \vee (q \rightarrow p)$  is always true on  $\mathcal{K}$  if and only if  $\mathcal{K}$  has the property that for every  $x \in \mathcal{K}$ , the set  $\downarrow x = \{y \in \mathcal{K} \mid y \leq x\}$  is linearly ordered.

Prove also, that  $\neg p \vee \neg\neg p$  is always true on  $\mathcal{K}$  if and only if  $\mathcal{K}$  has the following property: whenever two elements have an upper bound, they also have a lower bound.

Not only certain properties of posets can be characterized by the propositional logic they satisfy in the sense of exercise 158, also properties of presheaves.

**Exercise 159** Let  $X$  be a Kripke presheaf on a poset  $\mathcal{K}$ . Show that the following axiom scheme of predicate logic:

$$\text{D} \quad \forall x(A(x) \vee B) \rightarrow (\forall x A(x) \vee B)$$

(where  $A$  and  $B$  may contain additional variables, but the variable  $x$  is not allowed to occur in  $B$ ) is always true in  $X$ , if and only if for every  $k' \leq k$  in  $\mathcal{K}$ , the map  $X(k) \rightarrow X(k')$  is the identity.

Suppose now one considers the logic D-J, which is intuitionistic logic extended with the axiom schemes  $\neg\phi \vee \neg\neg\phi$  and the axiom scheme D from exercise 159. One might expect (in view of exercises 158 and 159) that this logic is complete with respect to constant presheaves on posets  $\mathcal{K}$  which have the property that whenever two elements have an upper bound, they also have a lower bound. However, this is not the case!

**Proposition 9.12** *Suppose  $X$  is a constant presheaf on a poset  $\mathcal{K}$  which has the property that whenever two elements have an upper bound, they also have a lower bound. Then the following axiom scheme is always true on  $X$ :*

$$\begin{array}{c} \forall x[(R \rightarrow (S \vee A(x))) \vee (S \rightarrow (R \vee A(x)))] \wedge \neg\forall x A(x) \\ \rightarrow \\ [(R \rightarrow S) \vee (S \rightarrow R)] \end{array}$$

**Exercise 160** Prove proposition 9.12.

However, the axiom scheme in proposition 9.12 is not a consequence of the logic D-J, which fact can be shown using presheaves. This was also shown by Ghilardi. We give the relevant statements without proof; the interested reader is referred to *Arch.Math.Logic* **29** (1989), 125–136.

- Proposition 9.13** *i) The axiom scheme  $\neg\phi \vee \neg\neg\phi$  is true in every interpretation in  $\text{Set}^{\mathcal{C}^{\text{op}}}$  if and only if the category  $\mathcal{C}$  has the property that every pair of arrows with common codomain fits into a commutative square.*
- ii) Let  $X$  be a presheaf on a category  $\mathcal{C}$ . Suppose  $X$  has the property that for all  $f : C' \rightarrow C$  in  $\mathcal{C}$ , all  $n \geq 0$ , all  $x_1, \dots, x_n \in X(C)$  and all  $y \in X(C')$  there is  $f' : C' \rightarrow C$  and  $x \in X(C)$  such that  $xf = y$  and  $x_1f = x_1f', \dots, x_nf = x_nf'$ . Then for every interpretation on  $X$  the axiom scheme D of exercise 159 is true.*
- iii) There exist a category  $\mathcal{C}$  satisfying the property of i), and a presheaf  $X$  on  $\mathcal{C}$  satisfying the property of ii), and an interpretation on  $X$  for which an instance of the axiom scheme of proposition 9.12 is not true.*

### 9.3.2 Failure of the Axiom of Choice

In this example, due to M. Fourman and A. Scedrov (*Manuscr. Math.* **38** (1982), 325–332), we explore a bit the higher-order structure of a presheaf category. Recall that the Axiom of Choice says: if  $X$  is a set consisting of nonempty sets, there is a function  $F : X \rightarrow \bigcup X$  such that  $F(x) \in x$  for every  $x \in X$ . This axiom is not provable in Zermelo-Fraenkel set theory, but it is classically totally unproblematic for *finite*  $X$  (induction on the cardinality of  $X$ ).

We exhibit here a category  $\mathcal{C}$ , a presheaf  $Y$  on  $\mathcal{C}$ , and a subpresheaf  $X$  of the power object  $\mathcal{P}(Y)$  such that the following statements are true in  $\text{Set}^{\mathcal{C}^{\text{op}}}$ :

$\forall \alpha \beta \in X (\alpha = \beta)$  (“ $X$  has at most one element”)

$\forall \alpha \in X \exists xy \in Y (x \neq y \wedge \forall z \in Y (z \in \alpha \leftrightarrow z = x \vee z = y))$  (“every element of  $X$  has exactly two elements”)

There is no arrow  $X \rightarrow \bigcup X$  (this is stronger than:  $X$  has no choice function).

Consider the category  $\mathcal{C}$  with two objects and two non-identity arrows:

$$\beta \begin{array}{c} \curvearrowright \\ \hookrightarrow \end{array} D \xrightarrow{\alpha} E$$

subject to the equations  $\beta^2 = \text{id}_D$  and  $\alpha\beta = \alpha$ .

We calculate the representables  $y_D$  and  $y_E$ , and the map  $y_\alpha : y_D \rightarrow y_E$ :

$$\begin{array}{ll} y_D(E) = \emptyset & (y_\alpha)_D(\text{id}_D) = \alpha \\ y_D(D) = \{\text{id}_D, \beta\} & (y_\alpha)_D(\beta) = \alpha \\ y_D(\alpha) \text{ is the empty function} & (y_\alpha)_E \text{ is the empty function} \\ y_D(\beta)(\text{id}_D) = \beta & y_D(\beta)(\beta) = \text{id}_D \end{array}$$

Since  $E$  is terminal in  $\mathcal{C}$ ,  $y_E$  is a terminal object in  $\text{Set}^{\mathcal{C}^{\text{op}}}$ :

$$y_E(E) = \{\text{id}_E\}, \quad y_E(D) = \{\alpha\}, \quad y_E(\alpha)(\text{id}_E) = \alpha, \quad y_E(\beta)(\alpha) = \alpha$$

Now let us calculate the power object  $\mathcal{P}(y_D)$ . According to the explicit construction of power objects in presheaf categories, we have

$$\begin{aligned} \mathcal{P}(y_D)(E) &= \text{Sub}(y_E \times y_D) \\ \mathcal{P}(y_D)(D) &= \text{Sub}(y_D \times y_D) \end{aligned}$$

$(y_E \times y_D)(D)$  is the two-element set  $\{(\alpha, \text{id}_D), (\alpha, \beta)\}$  which are permuted by the action of  $\beta$ , and  $(y_E \times y_D)(E) = \emptyset$ . So we see that  $\text{Sub}(y_E \times y_D)$  has two elements:  $\emptyset$  (the empty presheaf) and  $y_E \times y_D$  itself.  $(y_D \times y_D)(D)$  has 4 elements:  $(\text{id}_D, \beta), (\beta, \text{id}_D), (\beta, \beta), (\text{id}_D, \text{id}_D)$  and we have:  $(\text{id}_D, \beta)\beta = (\beta, \text{id}_D)$  and  $(\beta, \beta)\beta = (\text{id}_D, \text{id}_D)$ .

So  $\text{Sub}(y_D \times y_D)$  has 4 elements:  $\emptyset, y_D \times y_D, A, B$  where  $A$  and  $B$  are such that

$$\begin{aligned} A(E) &= \emptyset & A(D) &= \{(\text{id}_D, \beta), (\beta, \text{id}_D)\} \\ B(E) &= \emptyset & B(D) &= \{(\beta, \beta), (\text{id}_D, \text{id}_D)\} \end{aligned}$$

Summarizing: we have  $\mathcal{P}(y_D)(E) = \{\emptyset, y_E \times y_D\}$ ,  $\mathcal{P}(y_D)(D) = \{\emptyset, y_D \times y_D, A, B\}$ . The map  $\mathcal{P}(y_D)(\alpha)$  is given by pullback along  $y_\alpha \times \text{id}_{y_D}$  and sends therefore  $\emptyset$  to  $\emptyset$  and  $y_E \times y_D$  to  $y_D \times y_D$ .  $\mathcal{P}(y_D)(\beta)$  is by pullback along  $y_\beta \times \text{id}_{y_D}$  and sends  $\emptyset$  to  $\emptyset$ ,  $y_D \times y_D$  to  $y_D \times y_D$ , and permutes  $A$  and  $B$ .

Now let  $X$  be the subpresheaf of  $\mathcal{P}(y_D)$  given by:

$$X(E) = \emptyset \quad X(D) = \{y_D \times y_D\}$$

Then  $X$  is a ‘set of sets’ (a subobject of a power object), and clearly, in  $X$ , the sentence  $\forall xy(x = y)$  is true. So  $X$  ‘has at most one element’. We have the element relation  $\in_{y_D}$  as a subobject of  $\mathcal{P}(y_D) \times y_D$ , and its restriction to a subobject of  $X \times y_D$ . This is the presheaf  $Z$  with  $Z(E) = \emptyset$  and  $Z(D) = \{(y_D \times y_D, \text{id}_D), (y_D \times y_D, \beta)\}$ . So we see that the sentence expressing ‘every element of  $X$  has exactly two elements’ is true. The presheaf  $\bigcup X$  of ‘elements of elements of  $X$ ’ is the presheaf  $(\bigcup X)(E) = \emptyset$ ,  $(\bigcup X)(D) = \{\text{id}_D, \beta\}$  as subpresheaf of  $y_D$ . Now there cannot be any arrow in  $\text{Set}^{\mathcal{C}^{\text{op}}}$  from  $X$  to  $\bigcup X$ , because, in  $X(D)$ , the unique element is fixed by the action of  $\beta$ ; however, in  $(\bigcup X)(D)$  there is no fixed point for the action of  $\beta$ . Hence there is no ‘choice function’.



# 10 Sheaves

In this chapter we shall generalize the notion of a ‘sheaf on a topological space’ to arbitrary categories.

First, let us recall what a sheaf on a topological space is. Let  $X$  be a space with set of opens  $\mathcal{O}(X)$ , considered as a poset with the inclusion order.

A *presheaf*  $F$  on  $X$  is simply a presheaf on  $\mathcal{O}(X)$ . So for two opens  $U \subseteq V$  in  $X$  we have  $F(U \subseteq V) : F(V) \rightarrow F(U)$ , with the usual conditions.  $F$  is called *separated* if for any two elements  $x, y$  of  $F(U)$  and any open cover  $U = \bigcup_i U_i$  of  $U$ , if  $F(U_i \subseteq U)(x) = F(U_i \subseteq U)(y)$  for all  $i$ , then  $x = y$ .

$F$  is called a *sheaf*, if for every system of elements  $x_i \in F(U_i)$ , indexed by an open cover  $U = \bigcup_i U_i$  of  $U$ , such that for every pair  $i, j$  we have  $F(U_i \cap U_j \subseteq U_i)(x_i) = F(U_i \cap U_j \subseteq U_j)(x_j)$ , there is a *unique* element  $x \in F(U)$  such that  $x_i = F(U_i \subseteq U)(x)$  for each  $i$ .

Such a system of elements  $x_i$  is called a *compatible family*, and  $x$  is called an *amalgamation* of it. The most common examples of sheaves on  $X$  are sheaves of partial functions:  $F(U)$  is a set of functions  $U \rightarrow Y$  (for example, continuous functions to a space  $Y$ ), and  $F(U \subseteq V)(f)$  is the restriction of  $f$  to  $U$ .

**Example.** Let  $X$  be  $\mathbb{R}$  with the discrete topology; for  $U \subseteq \mathbb{R}$  let  $F(U)$  be the set of *injective* functions from  $U$  to  $\mathbb{N}$ .  $F$  is separated, but not a sheaf (check!).

In generalizing from  $\mathcal{O}(X)$  to an arbitrary category  $\mathcal{C}$ , we see that what we lack is the notion of a ‘cover’. Because  $\mathcal{C}$  is in general not a preorder, it will not do to define a ‘cover of an object  $C$ ’ as a collection of *objects* (as in the case of  $\mathcal{O}(X)$ ); rather, a cover of  $C$  will be a *sieve* on  $C$ .

We shall denote the maximal sieve on  $C$  by  $\max(C)$ .

**Definition 10.1** Let  $\mathcal{C}$  be a category. A *Grothendieck topology* on  $\mathcal{C}$  specifies, for every object  $C$  of  $\mathcal{C}$ , a family  $\text{Cov}(C)$  of ‘covering sieves’ on  $C$ , in such a way that the following conditions are satisfied:

- i)  $\max(C) \in \text{Cov}(C)$
- ii) If  $R \in \text{Cov}(C)$  then for every  $f : C' \rightarrow C$ ,  $f^*(R) \in \text{Cov}(C')$
- iii) If  $R$  is a sieve on  $C$  and  $S$  is a covering sieve on  $C$ , such that for every arrow  $f : C' \rightarrow C$  from  $S$  we have  $f^*(R) \in \text{Cov}(C')$ , then  $R \in \text{Cov}(C)$

We note an immediate consequence of the definition:

**Proposition 10.2** a) If  $R \in \text{Cov}(C)$ ,  $S$  a sieve on  $C$  and  $R \subseteq S$ , then  $S \in \text{Cov}(C)$ ;

b) If  $R, S \in \text{Cov}(C)$  then  $R \cap S \in \text{Cov}(C)$

**Proof.** For a), just observe that for every  $f \in R$ ,  $f^*(S) = \max(C')$ ; apply i) and iii) of 10.1. For b), note that if  $f \in R$  then  $f^*(S) = f^*(R \cap S)$ , and apply ii) and iii). ■

**Definition 10.3** A *universal closure operation* on  $\text{Set}^{\text{C}^{\text{op}}}$  assigns to every presheaf  $X$  an operation  $(\bar{\cdot}) : \text{Sub}(X) \rightarrow \text{Sub}(X)$  such that the following hold:

- i)  $A \leq \bar{A}$
- ii)  $\bar{A} = \bar{\bar{A}}$
- iii)  $A \leq B \Rightarrow \bar{A} \leq \bar{B}$
- iv) For  $\phi : Y \rightarrow X$  and  $A \in \text{Sub}(X)$ ,  $\phi^\sharp(\bar{A}) = \overline{\phi^\sharp(A)}$

**Definition 10.4** A *Lawvere-Tierney topology* on  $\text{Set}^{\text{C}^{\text{op}}}$  is an arrow  $J : \Omega \rightarrow \Omega$  (where, as usual,  $\Omega$  denotes the subobject classifier of  $\text{Set}^{\text{C}^{\text{op}}}$ ), such that the following hold:

- i)  $R \subseteq J_C(R)$  for every sieve  $R$  on  $C$
- ii)  $J_C(R \cap S) = J_C(R) \cap J_C(S)$
- iii)  $J_C(J_C(R)) = J_C(R)$

**Theorem 10.5** *The following notions are equivalent (that is, each of them determines the others uniquely):*

- 1) A Grothendieck topology on  $\mathcal{C}$
- 2) A Lawvere-Tierney topology on  $\text{Set}^{\text{C}^{\text{op}}}$
- 3) A universal closure operation on  $\text{Set}^{\text{C}^{\text{op}}}$
- 4) A full subcategory  $\mathcal{E}$  of  $\text{Set}^{\text{C}^{\text{op}}}$ , such that the inclusion  $\mathcal{E} \rightarrow \text{Set}^{\text{C}^{\text{op}}}$  has a left adjoint which preserves finite limits

**Proof.** We first prove the equivalence of the first three notions; the equivalence of these with notion 4 requires more work and is relegated to a separate proof.

1)  $\Rightarrow$  2). Given a Grothendieck topology  $\text{Cov}$  on  $\mathcal{C}$ , define  $J : \Omega \rightarrow \Omega$  by

$$J_C(R) = \{h : C' \rightarrow C \mid h^*(R) \in \text{Cov}(C')\}$$

If  $h : C' \rightarrow C$  is an element of  $J_C(R)$  and  $g : C'' \rightarrow C'$  arbitrary, then  $(hg)^*(R) = g^*(h^*(R)) \in \text{Cov}(C'')$  so  $hg \in J_C(R)$ . Hence  $J_C(R)$  is a sieve on  $C$ . Similarly,  $J$  is a natural transformation: for  $f : C' \rightarrow C$  we have

$$\begin{aligned} J_{C'}(f^*(R)) &= \{h : C'' \rightarrow C' \mid h^*(f^*(R)) \in \text{Cov}(C'')\} \\ &= \{h : C'' \rightarrow C' \mid (fh)^*(R) \in \text{Cov}(C'')\} \\ &= \{h : C'' \rightarrow C' \mid fh \in J_C(R)\} \\ &= f^*(J_C(R)) \end{aligned}$$

To prove  $R \subseteq J_C(R)$  we just apply condition i) of 10.1, since  $h^*(R) = \max(C')$  for  $h \in R$ .

By 10.2 we have  $R \cap S \in \text{Cov}(C)$  if and only if both  $R \in \text{Cov}(C)$  and  $S \in \text{Cov}(C)$ , and together with the equation  $h^*(R \cap S) = h^*(R) \cap h^*(S)$  this implies that  $J_C(R \cap S) = J_C(R) \cap J_C(S)$ .

Finally, since  $J$  preserves  $\cap$ , it preserves  $\subseteq$  ( $A \subseteq B$  iff  $A \cap B = A$ ). We have proved  $R \subseteq J_C(R)$ , so  $J_C(R) \subseteq J_C(J_C(R))$  follows. For the converse, suppose  $h \in J_C(J_C(R))$  so  $h^*(J_C(R)) \in \text{Cov}(C')$ . We need to prove  $h^*(R) \in \text{Cov}(C')$ . Now for any  $g \in h^*(J_C(R))$  we have  $(hg)^*(R) \in \text{Cov}(C'')$  so  $g^*(h^*(R)) \in \text{Cov}(C'')$ . Hence by condition iii) of 10.1,  $h^*(R) \in \text{Cov}(C')$ . This completes the proof that  $J$  is a Lawvere-Tierney topology.

2)  $\Rightarrow$  3). Suppose we are given a Lawvere-Tierney topology  $J$ . Define the operation  $(\bar{\cdot}) : \text{Sub}(X) \rightarrow \text{Sub}(X)$  as follows: if  $A \in \text{Sub}(X)$  is classified by  $\phi : X \rightarrow \Omega$  then  $\bar{A}$  is classified by  $J\phi$ . So

$$\bar{A}(C) = \{x \in X(C) \mid J_C(\phi_C(x)) = \max(C)\}$$

Then if  $f : Y \rightarrow X$  is a map of presheaves and  $A \in \text{Sub}(X)$ , both subobjects  $f^\sharp(\bar{A})$  and  $\bar{f^\sharp(A)}$  are classified by  $J\phi f : Y \rightarrow \Omega$ , hence they are equal. This proves iv) of definition 10.3. i) follows from condition i) of 10.4 and ii) from iii) of 10.4; finally, that  $(\bar{\cdot})$  is order-preserving follows from the fact that  $J_C$  preserves  $\subseteq$ .

3)  $\Rightarrow$  1). Given a universal closure operation  $(\bar{\cdot})$  on  $\text{Set}^{C^{\text{op}}}$ , we define  $\text{Cov}(C) = \{R \in \Omega(C) \mid \bar{R} = y_C \text{ in } \text{Sub}(y_C)\}$ . Under the identification of sieves on  $C$  with subobjects of  $y_C$ ,  $f^*(R)$  corresponds to  $(y_f)^\sharp(R)$ . So from condition iv) in 10.3 it follows that if  $R \in \text{Cov}(C)$  and  $f : C' \rightarrow C$ ,  $f^*(R) \in \text{Cov}(C')$ . Condition i) ( $\max(C) \in \text{Cov}(C)$ ) follows from i) of 10.3.

To prove iii) of 10.1, suppose  $R \in \Omega(C)$ ,  $S \in \text{Cov}(C)$  and for every  $f : C' \rightarrow C$  in  $S$  we have  $f^*(R) \in \text{Cov}(C')$ . So  $\bar{S} = y_C$  and for all  $f \in S$ ,  $y_{C'} = f^*(\bar{R}) = (y_f)^\sharp(\bar{R}) = f^*(\bar{R})$ . But that means that for all  $f \in S$ ,  $f \in \bar{R}$ . So  $S \subseteq \bar{R}$ ; hence by iii) of 10.3,  $y_C = \bar{S} = \bar{\bar{R}}$ ; but  $\bar{\bar{R}} = \bar{R}$  by ii) of 10.3, so  $y_C = \bar{R}$ , so  $R \in \text{Cov}(C)$ , as desired. ■

As said in the beginning of this proof, the equivalence of 4) with the other notions requires more work. We start with some definitions.

**Definition 10.6** Let  $\text{Cov}$  be a Grothendieck topology on  $\mathcal{C}$ , and  $(\bar{\cdot})$  the associated universal closure operation on  $\text{Set}^{C^{\text{op}}}$ .

A presheaf  $F$  is *separated* for  $\text{Cov}$  if for each  $C \in \mathcal{C}_0$  and  $x, y \in F(C)$ , if the sieve  $\{f : C' \rightarrow C \mid F(f)(x) = F(f)(y)\}$  covers  $C$ , then  $x = y$ .

A subpresheaf  $G$  of  $F$  is *closed* if  $\bar{G} = G$  in  $\text{Sub}(F)$ .

A subpresheaf  $G$  of  $F$  is *dense* if  $\bar{G} = F$  in  $\text{Sub}(F)$ .

**Exercise 161** i) A subpresheaf  $G$  of  $F$  is closed if and only if for each  $x \in F(C)$ : if  $\{f : C' \rightarrow C \mid F(f)(x) \in G(C')\}$  covers  $C$ , then  $x \in G(C)$

ii)  $F$  is separated if and only if the diagonal:  $F \rightarrow F \times F$  is a closed subobject (this explains the term ‘separated’: in French, the word ‘séparé’ is synonymous with ‘Hausdorff’)

- iii) A subpresheaf  $G$  of  $F$  is dense if and only if for each  $x \in F(C)$ , the sieve  $\{f : C' \rightarrow C \mid F(f)(x) \in G(C')\}$  is covering
- iv) A sieve  $R$  on  $C$ , considered as subobject of  $y_C$ , is dense if and only if it is covering

**Definition 10.7** Let  $F$  be a presheaf,  $C$  an object of  $\mathcal{C}$ . A *compatible family* in  $F$  at  $C$  is a family  $(x_f \mid f \in R)$  indexed by a sieve  $R$  on  $C$ , of elements  $x_f \in F(\text{dom}(f))$ , such that for  $f : C' \rightarrow C$  in  $R$  and  $g : C'' \rightarrow C'$  arbitrary,  $x_{fg} = F(g)(x_f)$ . In other words, a compatible family is an arrow  $R \rightarrow F$  in  $\text{Set}^{C^{\text{op}}}$ . An *amalgamation* of such a compatible family is an element  $x$  of  $F(C)$  such that  $x_f = F(f)(x)$  for all  $f \in R$ . In other words, an amalgamation is an extension of the map  $R \rightarrow F$  to a map  $y_C \rightarrow F$ .

**Exercise 162**  $F$  is separated if and only if each compatible family in  $F$ , indexed by a covering sieve, has *at most one* amalgamation.

**Definition 10.8**  $F$  is a *sheaf* if every compatible family in  $F$ , indexed by a covering sieve, has *exactly one* amalgamation.

**Exercise 163** Suppose  $G$  is a subpresheaf of  $F$ . If  $G$  is a sheaf, then  $G$  is closed in  $\text{Sub}(F)$ . Conversely, every closed subpresheaf of a sheaf is a sheaf.

**Example.** Let  $Y$  be a presheaf. Define a presheaf  $Z$  as follows:  $Z(C)$  consists of all pairs  $(R, \phi)$  such that  $R \in \text{Cov}(C)$  and  $\phi : R \rightarrow Y$  is an arrow in  $\text{Set}^{C^{\text{op}}}$ . If  $f : C' \rightarrow C$  then  $Z(f)(R, \phi) = (f^*(R), \phi f')$  where  $f'$  is such that

$$\begin{array}{ccc} f^*(R) & \xrightarrow{f'} & R \\ \downarrow & & \downarrow \\ y_{C'} & \xrightarrow{y_f} & y_C \end{array}$$

is a pullback.

Suppose we have a compatible family in  $Z$ , indexed by a covering sieve  $S$  on  $C$ . So for each  $f \in S$ ,  $f : C' \rightarrow C$  there is  $R_f \in \text{Cov}(C')$ ,  $\phi_f : R_f \rightarrow Y$ , such that for  $g : C'' \rightarrow C'$  we have that  $R_{fg} = g^*(R_f)$  and  $\phi_{fg} : R_{fg} \rightarrow Y$  is  $\phi_f g'$  where  $g' : R_{fg} \rightarrow R_f$  is the pullback of  $y_g : y_{C''} \rightarrow y_{C'}$ .

Then this family has an amalgamation in  $Z$ : define  $T \in \text{Cov}(C)$  by  $T = \{fg \mid f \in S, g \in R_f\}$ .  $T$  is covering since for every  $f \in S$  we have  $R_f \subseteq f^*(T)$ . We can define  $\chi : T \rightarrow Y$  by  $\chi(fg) = \phi_f(g)$ . So the presheaf  $Z$  satisfies the ‘existence’ part of the amalgamation condition for a sheaf. It does not in general satisfy the uniqueness part.

**Exercise 164** Prove that  $F$  is a sheaf if and only if for every presheaf  $X$  and every dense subpresheaf  $A$  of  $X$ , any arrow  $A \rightarrow F$  has a unique extension to an arrow  $X \rightarrow F$ .

**The ‘plus’ construction.** We define, for every presheaf  $X$  on  $\mathcal{C}$ , a presheaf  $X^+$  as follows.  $X^+(C)$  is the set of *equivalence classes* of pairs  $(R, \phi)$  with  $R \in \text{Cov}(C)$  and  $\phi : R \rightarrow X$ , where  $(R, \phi) \sim (S, \psi)$  holds if and only if there is a covering sieve  $T$  on  $C$ , such that  $T \subseteq R \cap S$  and  $\phi$  and  $\psi$  agree on  $T$ . Since  $\text{Cov}(C)$  is closed under intersections, this is evidently an equivalence relation.

The construction  $(\cdot)^+$  is a functor  $\text{Set}^{\mathcal{C}^{\text{op}}} \rightarrow \text{Set}^{\mathcal{C}^{\text{op}}}$ : for  $f : X \rightarrow Y$  define  $(f)^+ : X^+ \rightarrow Y^+$  by  $(f)_C^+(R, \phi) = (R, f\phi)$ . This is well-defined on equivalence classes.

We have a natural transformation  $\zeta$  from the identity on  $\text{Set}^{\mathcal{C}^{\text{op}}}$  to  $(\cdot)^+$ :  $(\zeta_X)_C(x) = (y_C, \phi)$  where  $\phi : y_C \rightarrow X$  corresponds to  $x$  in the Yoneda Lemma  $(\phi_C(\text{id}_C) = x)$ .

The following lemma is ‘by definition’.

**Lemma 10.9** *Let  $X$  be a presheaf on  $\mathcal{C}$ .*

- i)  $X$  is separated if and only if  $\zeta_X$  is mono.
- ii)  $X$  is a sheaf if and only if  $\zeta_X$  is an isomorphism.

**Lemma 10.10** *Let  $X$  be a presheaf,  $F$  a sheaf,  $g : X \rightarrow F$  an arrow in  $\text{Set}^{\mathcal{C}^{\text{op}}}$ . Then  $g$  factors through  $\zeta_X : X \rightarrow X^+$  via a unique  $\tilde{g} : X^+ \rightarrow F$ :*

$$\begin{array}{ccc} X & \xrightarrow{g} & F \\ & \searrow \zeta_X & \nearrow \tilde{g} \\ & X^+ & \end{array}$$

**Proof.** For  $[(R, \phi)] \in X^+(C)$ , define  $\tilde{g}_C([(R, \phi)])$  to be the unique amalgamation in  $F(C)$  of the composite  $g\phi : R \rightarrow F$ . This is well-defined, for if  $(R, \phi) \sim (S, \psi)$  then for some covering sieve  $T \subseteq R \cap S$  we have that  $g\phi$  and  $g\psi$  agree on  $T$ ; hence they have the same amalgamation. Convince yourself that  $\tilde{g}$  is natural. By inspection, the diagram commutes, and  $\tilde{g}$  is the unique arrow with this property. ■

**Lemma 10.11** *For every presheaf  $X$ ,  $X^+$  is separated.*

**Proof.** For, suppose  $(R, \phi)$  and  $(S, \psi)$  are representatives of elements of  $X^+(C)$  such that, for some covering sieve  $T$  of  $C$  it holds that for each  $f \in T$  there is a cover  $T_f \subseteq F^*(R) \cap F^*(S)$  such that  $\phi f'$  and  $\psi f''$  agree on  $T_f$ , where  $f'$  and  $f''$  are as in the pullback diagrams

$$\begin{array}{ccc} f^*(R) & \xrightarrow{f'} & R \\ \downarrow & & \downarrow \\ y_{C'} & \xrightarrow{f} & y_C \end{array} \quad \begin{array}{ccc} f^*(S) & \xrightarrow{f''} & S \\ \downarrow & & \downarrow \\ y_{C''} & \xrightarrow{f} & y_C \end{array}$$

Let  $U = \{fg \mid f \in T, g \in T_f\}$ . Then  $U$  is a covering sieve on  $C$ ,  $U \subseteq R \cap S$ , and  $\phi$  and  $\psi$  agree on  $U$ . Hence  $(R, \phi) \sim (S, \psi)$ , so they represent the same element of  $X^+(C)$ . ■

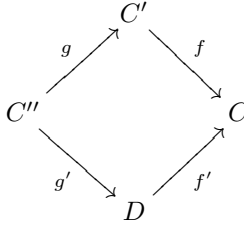
**Lemma 10.12** *If  $X$  is separated,  $X^+$  is a sheaf.*

**Proof.** Suppose we have a compatible family in  $X^+$ , indexed by a covering sieve  $R$  on  $C$ . So for each  $f : C' \rightarrow C$  in  $R$  we have  $(R_f, \phi_f)$ ,  $\phi_f : R \rightarrow X$ .

In order to find an amalgamation, we define a sieve  $S$  and a map  $\psi : S \rightarrow X$  by:

$$\begin{aligned} S &= \{fg \mid f \in R, g \in R_f\} \\ \psi(fg) &= \phi_f(g) \end{aligned}$$

Certainly  $S$  is a covering sieve on  $C$ , but it is not a priori clear that  $\psi$  is well-defined. For it may be the case that for  $f, f' \in R$ ,  $g \in R_f$  and  $g' \in R_{f'}$ ,  $fg = f'g'$ :



We need to show that in this case,  $\phi_f(g) = \phi_{f'}(g')$ .

The fact that we have a compatible family means that  $(g^*(R_f), \phi_f h) \sim ((g')^*(R_{f'}), \phi_{f'} h')$  in the equivalence relation defining  $X^+(C'')$ , where  $h$  and  $h'$  are as in the pullback diagrams

$$\begin{array}{ccc} g^*(R_f) & \xrightarrow{h} & R_f \\ \downarrow & & \downarrow \\ y_{C''} & \xrightarrow{y_g} & y_{C'} \end{array} \quad \begin{array}{ccc} (g')^*(R_{f'}) & \xrightarrow{h'} & R_{f'} \\ \downarrow & & \downarrow \\ y_{C''} & \xrightarrow{y_{g'}} & y_D \end{array}$$

That means that there is a covering sieve  $T$  on  $C''$  such that  $T \subseteq g^*(R_f) \cap (g')^*(R_{f'})$  on which  $\phi_f h$  and  $\phi_{f'} h'$  coincide; hence, for all  $k \in T$  we have that  $X(k)(\phi_f(g)) = X(k)(\phi_{f'}(g'))$ . Since  $X$  is separated by assumption,  $\phi_f(g) = \phi_{f'}(g')$  as desired.

We have obtained an amalgamation. It is unique because  $X^+$  is separated by lemma 10.11. ■

The functor  $\mathbf{a} : \text{Set}^{\text{cop}} \rightarrow \text{Set}^{\text{cop}}$  is defined by applying  $(\cdot)^+$  twice:  $\mathbf{a}(X) = X^{++}$ . By lemmas 10.11 and 10.12,  $\mathbf{a}(X)$  is always a sheaf. There is a natural transformation  $\eta$  from the identity to  $\mathbf{a}$  obtained by the composition  $X \xrightarrow{\zeta} X^+ \xrightarrow{\zeta} X^{++} = \mathbf{a}(X)$ ; by twice applying lemma 10.10 one sees that every arrow from a presheaf  $X$  to a sheaf  $F$  factors uniquely through  $\eta$ . This exhibits  $\eta$  as the unit of an adjunction between  $\text{Set}^{\text{cop}}$  and its full subcategory of sheaves. If we denote the latter by  $\text{Sh}(\mathcal{C}, \text{Cov})$  and regard  $\mathbf{a}$  as a functor  $\text{Set}^{\text{cop}} \rightarrow \text{Sh}(\mathcal{C}, \text{Cov})$ , then  $\mathbf{a}$  is left adjoint to the inclusion functor.

The functor  $\mathbf{a}$  is usually called *sheafification*, or the *associated sheaf* functor.

**Lemma 10.13** *The sheafification functor preserves finite limits.*

**Proof.** It is enough to show that  $(\cdot)^+$  preserves the terminal object and pullbacks. That  $(\cdot)^+$  preserves 1 is obvious (1 is always a sheaf). Suppose

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ v \downarrow & & \downarrow f \\ Z & \xrightarrow{g} & V \end{array}$$

is a pullback diagram. In order to show that also its  $(\cdot)^+$ -image is, suppose  $(R, \phi)$  and  $(S, \psi)$  represent elements of  $Y^+(C)$ ,  $Z^+(C)$ , respectively, such that  $(R, f\phi) \sim (S, g\psi)$  in the equivalence relation defining  $V^+(C)$ . Then there is a covering sieve  $T \subseteq R \cap S$  such that the square

$$\begin{array}{ccc} T & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ Z & \xrightarrow{g} & V \end{array}$$

commutes. By the pullback property, there is a unique factoring map  $\chi : T \rightarrow X$ ; that is, an element  $(T, \chi)$  of  $X^+(C)$  such that  $[(R, \phi)] = u^+([(T, \chi)])$  and  $[(S, \psi)] = v^+([(T, \chi)])$ , which shows that the  $(\cdot)^+$ -image of the given diagram is a pullback. ■

We now wrap up to finish the proof of Theorem 10.5: every Grothendieck topology gives a category of sheaves, which is a full subcategory of  $\text{Set}^{C^{\text{op}}}$  such that the inclusion has a left adjoint **a** which preserves finite limits by 10.13.

Conversely, suppose such a full subcategory  $\mathcal{E}$  of  $\text{Set}^{C^{\text{op}}}$  is given, with inclusion  $i : \mathcal{E} \rightarrow \text{Set}^{C^{\text{op}}}$  and left adjoint  $r : \text{Set}^{C^{\text{op}}} \rightarrow \mathcal{E}$ ,  $r$  preserving finite limits. This then determines a universal closure operation as follows. Given a subpresheaf  $A$  of  $X$ , define  $\bar{A}$  as given by the pullback:

$$\begin{array}{ccc} \bar{A} & \longrightarrow & ir(A) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\eta} & ir(X) \end{array}$$

where  $\eta : X \rightarrow ir(X)$  is the unit of the adjunction  $r \dashv i$ . Since  $r$  preserves finite limits,  $ir$  preserves monos, so the map  $\bar{A} \rightarrow X$  is monic. Writing down the naturality square for  $\eta$  for the inclusion  $A \rightarrow X$  we see that this inclusion factors through  $\bar{A} \rightarrow X$ , so  $A \leq \bar{A}$ . It is immediate that  $(\bar{\cdot})$  is order-preserving; and because  $i$  is full and faithful so  $ri$  is naturally isomorphic to the identity on  $\mathcal{E}$ ,  $ir$  is (up to isomorphism) idempotent, from which it is easy to deduce that  $\bar{\bar{A}} = \bar{A}$ . The final property of a universal closure operation, stability under pullback, follows again from the fact that  $r$ , and hence  $ir$ , preserves pullbacks. ■

## 10.1 Examples of Grothendieck topologies

1. As always, there are the two trivial extremes. The smallest Grothendieck topology (corresponding to the maximal subcategory of sheaves) has  $\text{Cov}(C)$  equal to  $\{\max(C)\}$  for all  $C$ . The only dense subpresheaves are the maximal ones; every presheaf is a sheaf.
2. The other extreme is the biggest Grothendieck topology:  $\text{Cov}(C) = \Omega(C)$ . Every subpresheaf is dense; the only sheaf is the terminal object 1.
3. Let  $X$  be a topological space with set of opens  $\mathcal{O}(X)$ , regarded as a category: a poset under the inclusion order. A sieve on an open set  $U$  can be identified with a downwards closed collection  $R$  of open subsets of  $U$ . The standard Grothendieck topology has  $R \in \text{Cov}(U)$  iff  $\bigcup R = U$ . Sheaves for this Grothendieck topology coincide with the familiar sheaves on the space  $X$ .
4. The *dense* or  $\neg\neg$ -topology is defined by:

$$\text{Cov}(C) = \{R \in \Omega(C) \mid \forall f : C' \rightarrow C \exists g : C'' \rightarrow C' (fg \in R)\}$$

This topology corresponds to the Lawvere-Tierney topology  $J : \Omega \rightarrow \Omega$  defined by

$$J_C(R) = \{h : C' \rightarrow C \mid \forall f : C'' \rightarrow C' \exists g : C''' \rightarrow C'' (hfg \in R)\}$$

This topology has the property that for every sheaf  $F$ , the collection of subsheaves of  $F$  forms a Boolean algebra.

5. For this example we assume that in the category  $\mathcal{C}$ , every pair of arrows with common codomain fits into a commutative square. Then the *atomic* topology takes all *nonempty* sieves as covers. This corresponds to the Lawvere-Tierney topology

$$J_C(R) = \{h : C' \rightarrow C \mid \exists f : C'' \rightarrow C' (hf \in R)\}$$

This topology has the property that for every sheaf  $F$ , the collection of subsheaves of  $F$  forms an *atomic* Boolean algebra: an *atom* in a Boolean algebra is a minimal non-bottom element. An atomic Boolean algebra is such that for every non-bottom  $x$ , there is an atom which is  $\leq x$ .

6. Let  $U$  be a subpresheaf of the terminal presheaf 1. With  $U$  we can associate a set of objects  $\tilde{U}$  of  $\mathcal{C}$  such that whenever  $f : C' \rightarrow C$  is an arrow and  $C \in \tilde{U}$ , then  $C' \in \tilde{U}$ . Namely,  $\tilde{U} = \{C \mid U(C) \neq \emptyset\}$ . To such  $U$  corresponds a Grothendieck topology, the *open topology* determined by  $U$ , given by

$$\text{Cov}(C) = \{R \in \Omega(C) \mid \forall f : C' \rightarrow C (C' \in \tilde{U} \Rightarrow fg \in R)\}$$



and associated Lawvere-Tierney topology

$$J_C(R) = \{h : C' \rightarrow C \mid \forall f : C'' \rightarrow C' (C'' \in \tilde{U} \Rightarrow hf \in R)\}$$

Let  $\mathcal{D}$  be the full subcategory of  $\mathcal{C}$  on the objects in  $\tilde{U}$ . Then there is an equivalence of categories between  $\text{Sh}(\mathcal{C}, \text{Cov})$  and  $\text{Set}^{\mathcal{D}^{\text{op}}}$ .

7. For  $U$  and  $\tilde{U}$  as in the previous example, there is also the *closed* topology determined by  $U$ , given by

$$\text{Cov}(C) = \{R \in \Omega(C) \mid C \in \tilde{U} \text{ or } R = \max(C)\}$$

There is an equivalence between  $\text{Sh}(\mathcal{C}, \text{Cov})$  and the category of presheaves on the full subcategory of  $\mathcal{C}$  on the objects *not* in  $\tilde{U}$ .

**Exercise 165** We consider the category  $\mathcal{C}$  whose objects are subsets of  $\mathbb{N}$ , and arrows  $A \rightarrow B$  are *finite-to-one* functions, i.e. functions  $f$  satisfying the requirement that for every  $b \in B$ , the set  $\{a \in A \mid f(a) = b\}$  is finite.

- a) Show that  $\mathcal{C}$  has pullbacks.
- b) Define for every object  $A$  of  $\mathcal{C}$  a set  $\text{Cov}(A)$  of sieves on  $A$  as follows:  $R \in \text{Cov}(A)$  if and only if  $R$  contains a finite family  $\{f_1, \dots, f_n\}$  of functions into  $A$ , which is *jointly almost surjective*, that is: the set

$$A - \bigcup_{i=1}^n \text{Im}(f_i)$$

is finite.

Show that  $\text{Cov}$  is a Grothendieck topology.

- c) Show that if  $R \in \text{Cov}(A)$ , then  $R$  contains a family  $\{f_1, \dots, f_n\}$  which is jointly almost surjective and moreover, every  $f_i$  is injective.
- d) Given a nonempty set  $X$  and an object  $A$  of  $\mathcal{C}$ , we define  $F_X(A)$  as the set of equivalence classes of functions  $\xi : A \rightarrow X$ , where  $\xi \sim \eta$  if  $\xi(n) = \eta(n)$  for all but finitely many  $n \in A$ .

Show that this definition can be extended to the definition of a presheaf  $F_X$  on  $\mathcal{C}$ .

- e) Show that  $F_X$  is a sheaf for  $\text{Cov}$ .

## 10.2 Structure of the category of sheaves

In this section we shall see, among other things, that also the category  $\text{Sh}(\mathcal{C}, \text{Cov})$  is a topos.

**Proposition 10.14**  $\text{Sh}(\mathcal{C}, \text{Cov})$  is closed under arbitrary limits in  $\text{Set}^{\mathcal{C}^{\text{op}}}$ .

**Proof.** This is rather immediate from the defining property of sheaves and the way (point-wise) limits are calculated in  $\text{Set}^{\text{C}^{\text{op}}}$ . Suppose  $F : I \rightarrow \text{Set}^{\text{C}^{\text{op}}}$  is a diagram of sheaves with limiting cone  $(X, (\mu_i : X \rightarrow F(i)))$  in  $\text{Set}^{\text{C}^{\text{op}}}$ . We show that  $X$  is a sheaf.

Suppose  $R \in \text{Cov}(C)$  and  $\phi : R \rightarrow X$  is a map in  $\text{Set}^{\text{C}^{\text{op}}}$ . Since every  $F(i)$  is a sheaf, every composite  $\mu_i \phi : R \rightarrow F(i)$  has a unique amalgamation  $y_i \in F(i)(C)$ , and by uniqueness these satisfy, for every map  $k : i \rightarrow j$  in the index category  $I$ , the equality  $(F(k))_C(y_i) = y_j$ . Since  $X(C)$  is the vertex of a limiting cone for the diagram  $F(\cdot)(C) : I \rightarrow \text{Set}$ , there is a unique  $x \in X(C)$  such that  $(\mu_i)_C(x) = y_i$  for each  $i$ . But this means that  $x$  is an amalgamation (and the unique such) for  $R \xrightarrow{\phi} X$ . ■

**Proposition 10.15** *Let  $X$  be a presheaf,  $Y$  a sheaf. Then  $Y^X$  is a sheaf.*

**Proof.** Suppose  $A \rightarrow Z$  is a dense subobject, and  $A \xrightarrow{\phi} Y^X$  a map. By exercise 164 we have to see that  $\phi$  has a unique extension to a map  $Z \rightarrow Y^X$ . Now  $\phi$  transposes to a map  $\tilde{\phi} : A \times X \rightarrow Y$ . By stability of the closure operation, if  $A \rightarrow Z$  is dense then so is  $A \times X \rightarrow Z \times X$ . Since  $Y$  is a sheaf,  $\tilde{\phi}$  has a unique extension to  $\psi : Z \times X \rightarrow Y$ . Transposing back gives  $\bar{\psi} : Z \rightarrow Y^X$ , which is the required extension of  $\phi$ . ■

**Corollary 10.16** *The category  $\text{Sh}(\mathcal{C}, \text{Cov})$  is cartesian closed.*

Now we turn to the subobject classifier in  $\text{Sh}(\mathcal{C}, \text{Cov})$ . Let  $J : \Omega \rightarrow \Omega$  be the associated Lawvere-Tierney topology. Sieves on  $C$  which are in the image of  $J_C$  are called *closed*. This is good terminology, since a closed sieve on  $C$  is the same thing as a closed subpresheaf of  $y_C$ .

By exercise 163 we know that subsheaves of a sheaf are the closed subpresheaves, and from exercise 161i) we know that a subpresheaf is closed if and only if its classifying map takes values in the image of  $J$ . This is a subobject of  $\Omega$ ; let us call it  $\Omega_J$ . So subobjects in  $\text{Sh}(\mathcal{C}, \text{Cov})$  admit unique classifying maps into  $\Omega_J$ ; note that the map  $1 \xrightarrow{t} \Omega$ , which picks out the maximal sieve on any  $C$ , factors through  $\Omega_J$  since every maximal sieve is closed. So  $1 \xrightarrow{t} \Omega_J$  is a subobject classifier in  $\text{Sh}(\mathcal{C}, \text{Cov})$  provided we can show that it is a map between sheaves. It is easy to see (and a special case of 10.14) that  $1$  is a sheaf. For  $\Omega_J$  this requires a little argument.

**Proposition 10.17** *The presheaf  $\Omega_J$  is a sheaf.*

**Proof.** We have seen that the arrow  $1 \xrightarrow{t} \Omega_J$  classifies closed subobjects. Therefore, in order to show that  $\Omega_J$  has the unique-extension property w.r.t. dense inclusions, it is enough to see that whenever  $X$  is a dense subpresheaf of  $Y$  there is a bijection between the closed subpresheaves of  $X$  and the closed subpresheaves of  $Y$ .

For a closed subpresheaf  $A$  of  $X$  let  $k(A)$  be the closure of  $A$  in  $\text{Sub}(Y)$ . For a closed subpresheaf  $B$  of  $Y$  let  $l(B) = B \cap X$ ; this is a closed subpresheaf of  $X$ .

Now  $kl(B) = k(B \cap X) = \overline{B \cap X} = \bar{B} \cap \bar{X} = \bar{B} = B$  since  $X$  is dense and  $B$  closed. Conversely,  $lk(A) = \bar{A} \cap X$  which is (by stability of closure) the closure of  $A$  in  $X$ . But  $A$  was closed, so this is  $A$ . Hence the maps  $k$  and  $l$  are inverse to each other, which finishes the proof. ■

**Corollary 10.18** *The category  $\text{Sh}(\mathcal{C}, \text{Cov})$  is a topos.*

**Definition 10.19** A pair  $(\mathcal{C}, \text{Cov})$  of a small category and a Grothendieck topology on it is called a *site*. For a sheaf on  $\mathcal{C}$  for  $\text{Cov}$ , we also say that it is a *sheaf on the site*  $(\mathcal{C}, \text{Cov})$ . A *Grothendieck topos* is a category of sheaves on a site.

Not every topos is a Grothendieck topos. For the moment, there is only one simple example to give of a topos that is not Grothendieck: the category of finite sets. It is not a Grothendieck topos, for example because it does not have all small limits.

**Exercise 166** The terminal category **1** is a topos. Is it a Grothendieck topos?

Let us say something about power objects and the natural numbers in  $\text{Sh}(\mathcal{C}, \text{Cov})$ .

For power objects there is not much more to say than this: for a sheaf  $X$ , its power object in  $\text{Sh}(\mathcal{C}, \text{Cov})$  is  $\Omega_J^X$ ; we shall also write  $\mathcal{P}_J(X)$ . By the Yoneda Lemma we have a natural 1-1 correspondence between  $\mathcal{P}_J(X)(C)$  and the set of closed subpresheaves of  $y_C \times X$ ; for  $f : C' \rightarrow C$  and  $A$  a closed subpresheaf of  $y_C \times X$ ,  $\mathcal{P}_J(X)(f)(A)$  is given by  $(y_f \times \text{id}_X)^\sharp(A)$ .

Next, let us discuss natural numbers. We use exercise 156 which says that the constant presheaf with value  $\mathbb{N}$  is a natural numbers object in  $\text{Set}^{\mathcal{C}^{\text{op}}}$ , and we also use the following result:

**Exercise 167** Suppose  $\mathcal{E}$  has a natural numbers object and  $F : \mathcal{E} \rightarrow \mathcal{F}$  is a functor which has a right adjoint and preserves the terminal object. Then  $F$  preserves the natural numbers object.

So the natural numbers object in  $\text{Sh}(\mathcal{C}, \text{Cov})$  is  $N^{++}$ , where  $N$  is the constant presheaf with value  $\mathbb{N}$ . In fact, we don't have to apply the 'plus' construction twice, because  $N$  is 'almost' separated: clearly, if  $n, m$  are two distinct natural numbers and  $R \in \text{Cov}(C)$  is such that for all  $f \in R$  we have  $nf = mf$ , then  $R = \emptyset$ . So the only way that  $N$  can fail to be separated is that for some objects  $C$  we have  $\emptyset \in \text{Cov}(C)$ . Now define the presheaf  $N'$  as follows:

$$N'(C) = \begin{cases} \mathbb{N} & \text{if } \emptyset \notin \text{Cov}(C) \\ \{*\} & \text{if } \emptyset \in \text{Cov}(C) \end{cases}$$

**Exercise 168** Prove:

- a)  $N'$  is separated
- b)  $\zeta_N : N \rightarrow N^+$  factors through  $N'$
- c)  $N^{++} \simeq (N')^+$

Colimits in  $\text{Sh}(\mathcal{C}, \text{Cov})$  are calculated as follows: take the colimit in  $\text{Set}^{\text{C}^{\text{op}}}$ , then apply the associated sheaf functor. For coproducts of sheaves, we have a simplification comparable to that of  $N$ . We write  $\bigsqcup$  for the coproduct in  $\text{Set}^{\text{C}^{\text{op}}}$  and  $\bigsqcup_J$  for the coproduct in  $\text{Sh}(\mathcal{C}, \text{Cov})$ . So  $\bigsqcup_J F_i = \mathbf{a}(\bigsqcup F_i)$ , but if we define  $\bigsqcup' F_i$  by

$$(\bigsqcup' F_i)(C) = \begin{cases} \bigsqcup F_i(C) & \text{if } \emptyset \notin \text{Cov}(C) \\ \{*\} & \text{if } \emptyset \in \text{Cov}(C) \end{cases}$$

then it is not too hard to show that  $\bigsqcup_J F_i \simeq (\bigsqcup' F_i)^+$ . Concretely, a compatible family in  $\bigsqcup' F_i$  indexed by a covering sieve  $R$  on  $C$ , i.e. a map  $\phi : R \rightarrow \bigsqcup' F_i$ , gives for each  $i$  a sub-sieve  $R_i$  and a map  $\phi_i : R_i \rightarrow F_i$ . The system of subsieves  $R_i$  has the property that if  $h : C' \rightarrow C$  is an element of  $R_i \cap R_j$  and  $i \neq j$ , then  $\emptyset \in \text{Cov}(C')$ . Of course, such compatible families are still subject to the equivalence relation defining  $(\bigsqcup' F_i)^+$ .

**Exercise 169** Prove:

- i) Coproducts are stable in  $\text{Sh}(\mathcal{C}, \text{Cov})$
- ii) For any sheaf  $F$ ,  $F^{N_J} \simeq \prod_{n \in \mathbb{N}} F$

Images in  $\text{Sh}(\mathcal{C}, \text{Cov})$ : given a map  $\phi : F \rightarrow G$  between sheaves, the image of  $\phi$  (as subsheaf of  $G$ ) is the closure of the image in  $\text{Set}^{\text{C}^{\text{op}}}$  of the same map. The arrow  $\phi$  is an epimorphism in  $\text{Sh}(\mathcal{C}, \text{Cov})$  if and only if for each  $C$  and each  $x \in G(C)$ , the sieve  $\{f : C' \rightarrow C \mid \exists y \in F(C')(\phi_{C'}(y) = xf)\}$  covers  $C$ .

**Exercise 170** Prove this characterization of epis in  $\text{Sh}(\mathcal{C}, \text{Cov})$ . Prove also that in  $\text{Sh}(\mathcal{C}, \text{Cov})$ , an arrow which is both mono and epi is an isomorphism.

Regarding the structure of the lattice of subobjects in  $\text{Sh}(\mathcal{C}, \text{Cov})$  of a sheaf  $F$ , we know that these are the closed subpresheaves, so the fixed points of the closure operation. That the subobjects again form a Heyting algebra is then a consequence of the following exercise.

**Exercise 171** Suppose  $H$  is a Heyting algebra with operations  $\perp, \top, \wedge, \vee, \rightarrow$  and let  $j : H \rightarrow H$  be order-preserving, idempotent, inflationary (that is:  $x \leq j(x)$  for all  $x \in H$ ), and such that  $j(x \wedge y) = j(x) \wedge j(y)$ . Let  $H_j$  be the set of fixed points of  $j$ . Then  $H_j$  is a Heyting algebra with operations:

$$\begin{aligned} \top_j &= \top & \perp_j &= j(\perp) \\ x \wedge_j y &= x \wedge y & x \vee_j y &= j(x \vee y) \\ x \rightarrow_j y &= x \rightarrow y \end{aligned}$$

**Exercise 172** If  $H$  is a Heyting algebra, show that the map  $\neg : x \mapsto (x \rightarrow \perp) \rightarrow \perp$  satisfies the requirements of the map  $j$  in exercise 171. Show also that  $H_{\neg}$  is a Boolean algebra.

**Exercise 173** Let  $J$  be the Lawvere-Tierney topology corresponding to the dense topology (see section 10.1). Show that in the Heyting algebra  $\Omega(C)$ ,  $J_C$  is the map  $\neg \neg$  of exercise 172.

As for presheaves, we can express the interpretation of first-order languages in  $\text{Sh}(\mathcal{C}, \text{Cov})$  in terms of a ‘forcing’ definition. The basic setup is the same; only now, of course, we take sheaves as interpretation of the sorts, and closed subpresheaves (subsheaves) as interpretation of the relation symbols. We then define  $\llbracket \varphi \rrbracket$  as a subsheaf of  $\llbracket FV(\varphi) \rrbracket$  and let  $\{\varphi\} : \llbracket FV(\varphi) \rrbracket \rightarrow \Omega_J$  be its classifying map. The notation  $C \Vdash_J \varphi(a_1, \dots, a_n)$  again means that  $\{\varphi\}_C(a_1, \dots, a_n)$  is the maximal sieve on  $C$ . This relation then again admits a definition by recursion on the formula  $\varphi$ . The inductive clauses of the definition of  $\Vdash_J$  are the same for  $\Vdash$  for the cases: atomic formula,  $\wedge$ ,  $\rightarrow$  and  $\forall$ , and we put:

- $C \Vdash_J \neg\varphi(a_1, \dots, a_n)$  if and only if for every arrow  $g : D \rightarrow C$  in  $\mathcal{C}$  we have: if  $D \Vdash_J \varphi(a_1g, \dots, a_ng)$  then  $\emptyset$  covers  $D$ ;
- $C \Vdash_J (\varphi \vee \psi)(a_1, \dots, a_n)$  if and only if the sieve  $\{g : C' \rightarrow C \mid C' \Vdash_J \varphi(a_1g, \dots, a_ng) \text{ or } C' \Vdash_J \psi(a_1g, \dots, a_ng)\}$  covers  $C$ ;
- $C \Vdash_J \exists x\varphi(x, a_1, \dots, a_n)$  if and only if the sieve  $\{g : C' \rightarrow C \mid \exists x \in F(C') C' \Vdash_J \varphi(x, a_1g, \dots, a_ng)\}$  covers  $C$  (where  $F$  is the interpretation of the sort of  $x$ ).

That this works should be no surprise in view of our characterisation of images in  $\text{Sh}(\mathcal{C}, \text{Cov})$  and our treatment of the Heyting structure on the subsheaves of a sheaf. We have the following properties of the relation  $\Vdash_J$ :

**Theorem 10.20** *i) If  $C \Vdash_J \varphi(a_1, \dots, a_n)$  then for each arrow  $f : C' \rightarrow C$ ,  $C' \Vdash_J \varphi(a_1f, \dots, a_nf)$ ;*  
*ii) if  $R$  is a covering sieve on  $C$  and for every arrow  $f : C' \rightarrow C$  in  $R$  we have  $C' \Vdash_J \varphi(a_1f, \dots, a_nf)$ , then  $C \Vdash_J \varphi(a_1, \dots, a_n)$ .*

**Exercise 174** Let  $N_J$  be the natural numbers object in  $\text{Sh}(\mathcal{C}, \text{Cov})$ . Prove the same result as we had in exercise 156, that is: for the standard interpretation of the language of arithmetic in  $N_J$ , a sentence is true if and only if it is true in the (classical) standard model of natural numbers.

**Exercise 175** We assume that we have a site  $(\mathcal{C}, \text{Cov})$  and an object  $I$  of  $\mathcal{C}$  satisfying the following conditions:

- i)  $\emptyset \notin \text{Cov}(I)$
- ii) If there is no arrow  $I \rightarrow A$  then  $\emptyset \in \text{Cov}(A)$
- iii) If there is an arrow  $I \rightarrow A$  then every arrow  $A \rightarrow I$  is split epi

We call a sheaf  $F$  in  $\text{Sh}(\mathcal{C}, \text{Cov})$   $\neg\neg$ -separated if for every object  $A$  of  $\mathcal{C}$  and all  $x, y \in F(A)$ ,

$$A \Vdash_J \neg\neg(x = y) \rightarrow x = y$$

Prove that the following two assertions are equivalent, for a sheaf  $F$ :

- a)  $F$  is  $\neg\neg$ -separated
- b) For every object  $A$  of  $\mathcal{C}$  and all  $x, y \in F(A)$  the following holds: if for every arrow  $\phi : I \rightarrow A$  we have  $x\phi = y\phi$  in  $F(I)$ , then  $x = y$

### 10.3 Application: a model for the independence of the Axiom of Choice

In this section we treat a model, due to P. Freyd, which shows that in toposes where *classical* logic always holds, the axiom of choice need not be valid. Specifically, we construct a topos  $\mathcal{F} = \text{Sh}(F, \text{Cov})$  and in  $\mathcal{F}$  a subobject  $E$  of  $N_J \times \mathcal{P}_J(N_J)$  with the properties:

- i)  $\mathcal{F}$  is *Boolean*, that is: every subobject lattice is a Boolean algebra;
- ii)  $\Vdash_J \forall n \exists \alpha ((n, \alpha) \in E)$
- iii)  $\Vdash \neg \exists f \in \mathcal{P}_J(N_J)^{N_J} \forall n ((n, f(n)) \in E)$

So,  $E$  is an  $N_J$ -indexed collection of nonempty (in a strong sense) subsets of  $\mathcal{P}_J(N_J)$ , but admits no choice function.

Let  $\mathbb{F}$  be the following category: it has objects  $\bar{n}$  for each natural number  $n$ , and an arrow  $f : \bar{m} \rightarrow \bar{n}$  is a function  $\{0, \dots, m\} \rightarrow \{0, \dots, n\}$  such that  $f(i) = i$  for every  $i$  with  $0 \leq i \leq n$ . It is understood that there are no morphisms  $\bar{m} \rightarrow \bar{n}$  for  $m < n$ . Note, that  $\bar{0}$  is a terminal object in this category.

On  $\mathbb{F}$  we let  $\text{Cov}$  be the dense topology, so a sieve  $R$  on  $\bar{m}$  covers  $\bar{m}$  if and only if for every arrow  $g : \bar{n} \rightarrow \bar{m}$  there is an arrow  $h : \bar{k} \rightarrow \bar{n}$  such that  $gh \in R$ . We shall work in the topos  $\mathcal{F} = \text{Sh}(\mathbb{F}, \text{Cov})$ , the *Freyd topos*. Let  $E_n$  be the object  $\mathbf{a}(y_{\bar{n}})$ , the sheafification of the representable presheaf on  $\bar{n}$ .

**Lemma 10.21** *Cov has the following properties:*

- a) *Every covering sieve is nonempty*
- b) *Every nonempty sieve on  $\bar{0}$  is a cover*
- c) *Every representable presheaf is separated*
- d)  *$y_{\bar{0}}$  has only two closed subobjects*

**Proof.** For a), apply the definition of ‘ $R$  covers  $\bar{m}$ ’ to the identity on  $\bar{m}$ ; it follows that there is an arrow  $h : \bar{k} \rightarrow \bar{m}$  such that  $h \in R$ .

For b), suppose  $S$  is a sieve on  $\bar{0}$  and  $\bar{k} \xrightarrow{f} \bar{0}$  is in  $S$ . Since  $\bar{0}$  is terminal, for any  $\bar{m} \xrightarrow{g} \bar{0}$  and any maps  $\bar{m} + \bar{k} \rightarrow \bar{k}$ ,  $\bar{m} + \bar{k} \rightarrow \bar{m}$ , the square

$$\begin{array}{ccc} \overline{m + k} & \longrightarrow & \bar{k} \\ \downarrow & & \downarrow f \\ \bar{m} & \xrightarrow{g} & \bar{0} \end{array}$$

commutes, so for any such  $g$  there is an  $h$  with  $gh \in R$ , hence  $R$  covers  $\bar{0}$ .

For c), suppose  $g, g' : \bar{k} \rightarrow \bar{n}$  are such that for a cover  $R$  of  $\bar{k}$  we have  $gf = g'f$  for all  $f \in R$ . We need to see that  $g = g'$ . Pick  $i \leq k$ . Let  $h : \bar{k} + 1 \rightarrow \bar{k}$  be such

that  $h(k+1) = i$ . Since  $R$  covers  $\bar{k}$  there is  $u : \bar{l} \rightarrow \overline{k+1}$  such that  $hu \in R$ . Then  $ghu = g'hu$ , which means that  $g(i) = ghu(k+1) = g'hu(k+1) = g'(i)$ . So  $g = g'$ , as desired.

Finally, d) follows directly from b): suppose  $R$  is a closed sieve on  $\bar{0}$ . If  $R \neq \emptyset$ , then  $R$  is covering by b), hence (being also closed) equal to  $\max(\bar{0})$ . Hence the only closed sieves are  $\emptyset$  and  $\max(\bar{0})$ . ■

**Proposition 10.22** *The unique map  $E_n \rightarrow 1$  is an epimorphism.*

**Proof.** By lemma 10.21d),  $1 = \mathbf{a}(y_{\bar{0}})$  has only two subobjects and  $y_{\bar{n}}$  is nonempty, so the image of  $E_n \rightarrow 1$  is 1. ■

**Proposition 10.23** *If  $n > m$  then  $E_n(\bar{m}) = \emptyset$ .*

**Proof.** Since  $y_{\bar{n}}$  is separated by 10.21c),  $E_n = (y_{\bar{n}})^+$ , so  $E_n(\bar{m})$  is an equivalence class of morphisms  $\tau : S \rightarrow y_{\bar{n}}$  in  $\text{Set}^{\mathbb{E}^{\text{op}}}$ , for a cover  $S$  of  $\bar{m}$ . We claim that such  $\tau$  don't exist.

For, since such  $S$  is nonempty (10.21a)), pick  $s : \bar{k} \rightarrow \bar{m}$  in  $S$  and let  $f = \tau_{\bar{k}}(s)$ , so  $f : \bar{k} \rightarrow \bar{n}$ . Let  $g, h : \overline{k+1} \rightarrow \bar{k}$  be such that  $g(k+1) = n$ , and  $h(k+1) = s(n) \leq m < n$ . Then  $sg = sh$  (check!). So

$$fg = \tau_{\bar{k}}(s)g = \tau_{\overline{k+1}}(sg) = \tau_{\overline{k+1}}(sh) = \tau_{\bar{k}}(s)h = fh$$

However,  $fg(k+1) = f(n) = n$ , whereas  $fh(k+1) = f(s(n+1)) = s(n)$ . Contradiction. ■

**Corollary 10.24** *The product sheaf  $\prod_{n \in \mathbb{N}} E_n$  is empty.*

**Proof.** For, if  $(\prod_n E_n)(\bar{m}) \neq \emptyset$  then by applying the projection  $\prod_n E_n \rightarrow E_{m+1}$  we would have  $E_{m+1}(\bar{m}) \neq \emptyset$ , contradicting 10.23. ■

**Proposition 10.25** *For each  $n$  there is a monomorphism  $E_n \rightarrow \mathcal{P}_J(N_J)$ .*

**Proof.** Since  $E_n = \mathbf{a}(y_{\bar{n}})$  and  $\mathcal{P}_J(N_J)$  is a sheaf, it is enough to construct a monomorphism  $y_{\bar{n}} \rightarrow \mathcal{P}_J(N_J)$ , which gives then a unique extension to a map from  $E_n$ ; since  $\mathbf{a}$  preserves monos, the extension will be mono if the given map is.

Fix  $n$  for the rest of the proof. Let  $(g_k)_{k \in \mathbb{N}}$  be a 1-1 enumeration of all the arrows in  $\mathbb{F}$  with codomain  $\bar{n}$ . For each  $g_i$ , let  $C_i$  be the smallest closed sieve on  $\bar{n}$  containing  $g_i$  (i.e.,  $C_i$  is the  $J_{\bar{n}}$ -image of the sieve generated by  $g_i$ ).

$\mathcal{P}_J(N_J)(\bar{m})$  is the set of closed subpresheaves of  $y_{\bar{m}} \times N_J$ . Elements of  $(y_{\bar{m}} \times N_J)(\bar{k})$  are pairs  $(h, (S_i)_{i \in \mathbb{N}})$  where  $h : \bar{k} \rightarrow \bar{m}$  and  $(S_i)_i$  is an  $\mathbb{N}$ -indexed collection of sieves on  $\bar{k}$ , such that  $S_i \cap S_j = \emptyset$  for  $i \neq j$ , and  $\bigcup_i S_i$  covers  $\bar{k}$ .

Define  $\mu_{\bar{m}} : y_{\bar{n}}(\bar{m}) \rightarrow \mathcal{P}_J(N_J)(\bar{m})$  as follows. For  $f : \bar{m} \rightarrow \bar{n}$ ,  $\mu_{\bar{m}}(f)$  is the subpresheaf of  $y_{\bar{m}} \times N_J$  given by:  $(h, (S_i)_i) \in \mu_{\bar{m}}(f)(\bar{k})$  iff for each  $i$ ,  $S_i \subseteq (fh)^*(C_i)$ . It is easily seen that  $\mu_{\bar{m}}(f)$  is a closed subpresheaf of  $y_{\bar{m}} \times N_J$ .

Let us first see that  $\mu$  is a natural transformation. Suppose  $g : \bar{l} \rightarrow \bar{m}$ . For  $h' : \bar{k} \rightarrow \bar{l}$  we have:

$$\begin{aligned} & (h', (S_i)_i) \in (y_g \times \text{id}_{N_J})^\#(\mu_{\bar{m}}(f))(\bar{k}) \\ \text{iff } & (gh', (S_i)_i) \in \mu_{\bar{m}}(f)(\bar{k}) \\ \text{iff } & \forall i (S_i \subseteq (fgh')^*(C_i)) \\ \text{iff } & (h', (S_i)_i) \in \mu_{\bar{l}}(fg)(\bar{k}) \end{aligned}$$

Next, let us prove that  $\mu$  is mono. Suppose  $\mu_{\bar{m}}(f) = \mu_{\bar{m}}(f')$  for  $f, f' : \bar{m} \rightarrow \bar{n}$ . Let  $j$  and  $j'$  be such that in our enumeration,  $f = g_j$  and  $f' = g_{j'}$ . Now consider the pair  $\xi = (\text{id}_{\bar{m}}, (S_i)_i)$ , where  $S_i$  is the empty sieve if  $i \neq j$ , and  $S_j = \max(\bar{m})$ . Then  $\xi$  is easily seen to be an element of  $\mu_{\bar{m}}(f)(\bar{m})$ , so it must also be an element of  $\mu_{\bar{m}}(f')(\bar{m})$ , which means that  $f' \in C_j$ . So  $C_j \cap C_{j'} \neq \emptyset$ . But this means that we must have a commutative square in  $\mathbb{F}$ :

$$\begin{array}{ccc} \bar{l} & \longrightarrow & \bar{m} \\ \downarrow & & \downarrow f' \\ \bar{m} & \xrightarrow{f} & \bar{n} \end{array}$$

It is easy to conclude from this that  $f = f'$ . ■

## 10.4 Application: a model for “every function from reals to reals is continuous”

In 1924, L.E.J. Brouwer published a paper: *Beweis, dass jede volle Funktion gleichmässig stetig ist* (Proof, that every total function is uniformly continuous), *Nederl. Akad. Wetensch. Proc.* **27**, pp.189–193. His lucubrations on intuitionistic mathematics had led him to the conclusion that every function from  $\mathbb{R}$  to  $\mathbb{R}$  must be continuous. Among present-day researchers of constructive mathematics, this statement is known as *Brouwer’s Principle* (although die-hard intuitionists still refer to it as Brouwer’s Theorem).

The principle can be made plausible in a number of ways; one is, to look at the reals from a computational point of view. If a computer, which can only deal with finite approximations of reals, computes a function, then for every required precision for  $f(x)$  it must be able to approximate  $x$  closely enough and from there calculate  $f(x)$  within the prescribed precision; this just means that  $f$  must be continuous.

In this section we shall show that the principle is *consistent* with higher-order intuitionistic type theory, by exhibiting a topos in which it holds, for the standard real numbers. In order to do this, we have of course to say what the “object of real numbers” in a topos is. That will be done in the course of the construction.

We shall work with a full subcategory  $\mathbb{T}$  of the category  $\text{Top}$  of topological spaces and continuous functions. It doesn’t really matter so much what  $\mathbb{T}$  exactly is, but we require that:



- $\mathbb{T}$  is closed under finite products and open subspaces
- $\mathbb{T}$  contains the space  $\mathbb{R}$  (with the euclidean topology)

We specify a Grothendieck topology on  $\mathbb{T}$  by defining, for an object  $T$  of  $\mathbb{T}$ , that a sieve  $R$  on  $T$  covers  $T$ , if the set of open subsets  $U$  of  $T$  for which the inclusion  $U \rightarrow T$  is in  $R$ , forms an open covering of  $T$ . It is easy to verify that this is a Grothendieck topology.

The first thing to note is that for this topology (we call it  $\text{Cov}$ ), every representable presheaf is a sheaf, because it is a presheaf of (continuous) functions: given a compatible family  $R \rightarrow y_T$  for  $R$  a covering sieve on  $X$ , this family contains maps  $f_U : U \rightarrow T$  for every open  $U$  contained in a covering of  $X$ ; and these maps agree on intersections, because we have a sieve. So they have a unique amalgamation to a continuous map  $f : X \rightarrow T$ , i.e. an element of  $y_T(X)$ .

Also for spaces  $S$  not necessarily in the category  $\mathbb{T}$  we have sheaves  $y_S = \text{Cts}(-, S)$ .

Recall that the Yoneda embedding preserves existing exponents in  $\mathbb{T}$ . This also extends to exponents which exist in  $\text{Top}$  but are not in  $\mathbb{T}$ . If  $T$  is a locally compact space, then for any space  $X$  we have an exponent  $X^T$  in  $\text{Top}$ : it is the set of continuous functions  $T \rightarrow X$ , equipped with the compact-open topology (a subbase for this topology is given by the sets  $\mathcal{C}(C, U)$  of those continuous functions that map  $C$  into  $U$ , for a compact subset  $C$  of  $T$  and an open subset  $U$  of  $X$ ). Thus, even if  $X$  is not an object of  $\mathbb{T}$ , we still have in  $\text{Sh}(\mathbb{T}, \text{Cov})$ :

$$y_{X^T} \simeq (y_X)^{(Y_T)}$$

**Exercise 176** Prove this fact.

From now on, we shall denote the category  $\text{Sh}(\mathbb{T}, \text{Cov})$  by  $\mathcal{T}$ .

**Notation:** in this section we shall dispense with all subscripts  $(\cdot)_J$ , since we shall only work in  $\mathcal{T}$ . So,  $N$  denotes the *sheaf* of natural numbers,  $\mathcal{P}(X)$  is the power *sheaf* of  $X$ ,  $\Vdash$  refers to forcing in sheaves, etc.

The natural numbers are given by the constant sheaf  $N$ , the  $\mathbb{N}$ -fold coproduct of copies of 1. The rational numbers are formed as a quotient of  $N \times N$  by an equivalence relation which can be defined in a quantifier-free way, and hence is also a constant sheaf; therefore the object of rational numbers  $Q$  is the constant sheaf on the classical rational numbers  $\mathbb{Q}$ , and therefore the  $\mathbb{Q}$ -fold coproduct of copies of 1.

**Proposition 10.26** *In  $\mathcal{T}$ ,  $N$  and  $Q$  are isomorphic to the representable sheaves  $y_{\mathbb{N}}$ ,  $y_{\mathbb{Q}}$  respectively, where  $\mathbb{N}$  and  $\mathbb{Q}$  are endowed with the discrete topology.*

**Proof.** We shall do this for  $N$ ; the proof for  $Q$  is similar. An element of  $y_{\mathbb{N}}(X)$  is a continuous function from  $X$  to the discrete space  $\mathbb{N}$ ; this is the same thing as an open covering  $\{U_n \mid n \in \mathbb{N}\}$  of pairwise disjoint sets; which in turn is the same thing as an (equivalence class of an)  $\mathbb{N}$ -indexed collection  $\{R_n \mid n \in \mathbb{N}\}$  of sieves on  $X$  such that whenever for  $n \neq m$ ,  $f : Y \rightarrow X$  is in  $R_n \cap R_m$ ,  $Y = \emptyset$ ;

and moreover the sieve  $\bigcup_n R_n$  covers  $X$ . But that last thing is just an element of  $(\bigcup_n 1)(X)$ .  $\blacksquare$

Under this isomorphism, the order on  $N$  and  $Q$  corresponds to the pointwise ordering on functions.

**Exercise 177** Show that in  $\mathcal{T}$ , the objects  $N$  and  $Q$  are linearly ordered, that is: for every space  $X$  in  $\mathbb{T}$ ,  $X \Vdash \forall rs \in Q (r < s \vee r = r \vee s < r)$ .

We now construct the *object of Dedekind reals*  $R_d$ . Just as in the classical definition, a real number is a Dedekind cut of rational numbers, that is: a pair  $(L, R)$  of subsets of  $Q$  satisfying:

- i)  $\forall q \in Q \neg (q \in L \wedge q \in R)$
- ii)  $\exists q (q \in L) \wedge \exists r (r \in R)$
- iii)  $\forall qr (q < r \wedge r \in L \rightarrow q \in L) \wedge \forall st (s < t \wedge s \in R \rightarrow t \in R)$
- iv)  $\forall q \in L \exists r (q < r \wedge r \in L) \wedge \forall s \in R \exists t (t < s \wedge t \in R)$
- v)  $\forall qr (q < r \rightarrow q \in L \vee r \in R)$

Write  $\text{Cut}(L, R)$  for the conjunction of these formulas. So the object of reals  $R_d$  is the subsheaf of  $\mathcal{P}(Q) \times \mathcal{P}(Q)$  given by:

$$R_d(X) = \{(L, R) \in (\mathcal{P}(Q) \times \mathcal{P}(Q))(X) \mid X \Vdash \text{Cut}(L, R)\}$$

This is always a sheaf, by theorem 10.20ii).

**Proposition 10.27** *The sheaf  $R_d$  is isomorphic to the representable sheaf  $y_{\mathbb{R}}$ .*

**Proof.** Let  $W$  be an object of  $\mathbb{T}$  and  $(L, R) \in R_d(W)$ . Then  $L$  and  $R$  are subsheaves of  $y_W \times Q$ , which is isomorphic to  $y_{W \times \mathbb{Q}}$ . So both  $L$  and  $R$  consist of pairs of maps  $(\alpha, p)$  with  $\alpha : Y \rightarrow W$ ,  $p : Y \rightarrow \mathbb{Q}$  continuous. Since  $L$  and  $R$  are subsheaves we have: if  $(\alpha, p) \in L(Y)$  then for any  $f : V \rightarrow Y$ ,  $(\alpha f, p f) \in L(V)$ , and if  $(\alpha \upharpoonright V_i, p \upharpoonright V_i) \in L(V_i)$  for an open cover  $\{V_i\}_i$  of  $Y$ , then  $(\alpha, p) \in L(Y)$  (and similar for  $R$ , of course).

Now for such  $(L, R) \in \mathcal{P}(\mathbb{Q})(W) \times \mathcal{P}(\mathbb{Q})(W)$  we have  $(L, R) \in R_d(W)$  if and only if  $W \Vdash \text{Cut}(L, R)$ . We are now going to spell out what this means, and see that such  $(L, R)$  uniquely determine a continuous function  $W \rightarrow \mathbb{R}$ .

- i)' For  $\beta : W' \rightarrow W$  and  $q : W' \rightarrow \mathbb{Q}$ , not both  $(\beta, q) \in L(W')$  and  $(\beta, q) \in R(W')$
- ii)' There is an open covering  $\{W_i\}$  of  $W$  such that for each  $i$  there are  $W_i \xrightarrow{l_i} \mathbb{Q}$  and  $W_i \xrightarrow{r_i} \mathbb{Q}$  with  $(W_i \rightarrow W, W_i \xrightarrow{l_i} \mathbb{Q}) \in L(W_i)$ , and  $(W_i \rightarrow W, W_i \xrightarrow{r_i} \mathbb{Q}) \in R(W_i)$
- iii)' For any map  $\beta : W' \rightarrow W$  and any  $q, r : W' \rightarrow \mathbb{Q}$ : if  $(\beta, r) \in L(W)$  and  $q(x) < r(x)$  for all  $x \in W'$ , then  $(\beta, q) \in L(W')$ , and similar for  $R$

- iv)' For any  $\beta : W' \rightarrow W$  and  $q : W' \rightarrow \mathbb{Q}$ : if  $(\beta, q) \in L(W')$  there is an open covering  $\{W'_i\}$  of  $W'$ , and maps  $r_i : W'_i \rightarrow \mathbb{Q}$  such that  $(\beta \upharpoonright W'_i, r_i) \in L(W'_i)$ , and  $r_i(x) > q(x)$  for all  $x \in W'_i$ . And similar for  $R$
- v)' For any  $\beta : W' \rightarrow W$  and  $q, r : W' \rightarrow \mathbb{Q}$  satisfying  $q(x) < r(x)$  for all  $x \in W'$ , there is an open covering  $\{W'_i\}$  of  $W'$  such that for each  $i$ , either  $(\beta \upharpoonright W'_i, q \upharpoonright W'_i) \in L(W'_i)$  or  $(\beta \upharpoonright W'_i, q \upharpoonright W'_i) \in R(W'_i)$ .

Let  $\hat{q} : W \rightarrow \mathbb{Q}$  be the constant function with value  $q$ . For every  $x \in W$  we define:

$$\begin{aligned} L_x &= \{q \in \mathbb{Q} \mid \exists \text{open } V \subseteq W (x \in V \wedge (V \rightarrow W, \hat{q} \upharpoonright V) \in L(V))\} \\ R_x &= \{q \in \mathbb{Q} \mid \exists \text{open } V \subseteq W (x \in V \wedge (V \rightarrow W, \hat{q} \upharpoonright V) \in R(V))\} \end{aligned}$$

Then you should verify that  $(L_x, R_x)$  form a Dedekind cut in  $\text{Set}$ , hence determine a real number  $f_{L,R}(x)$ .

By definition of  $L_x$  and  $R_x$ , if  $q, r$  are rational numbers then  $q < f_{L,R}(x) < r$  holds if and only if  $q \in L_x$  and  $r \in R_x$ ; so the preimage of the open interval  $(q, r)$  under  $f_{L,R}$  is open; that is,  $f_{L,R}$  is continuous. We have therefore defined a map  $(L, R) \mapsto f_{L,R} : R_d(W) \rightarrow y_{\mathbb{R}}(W)$ . It is easy to verify that this gives a map of sheaves:  $R_d \rightarrow y_{\mathbb{R}}$ .

For the other direction, if  $f : W \rightarrow \mathbb{R}$  is continuous, one defines subsheaves  $L_f, R_f$  of  $y_{W \times \mathbb{Q}}$  as follows: for  $\beta : W' \rightarrow W, p : W' \rightarrow \mathbb{Q}$  put

$$\begin{aligned} (\beta, p) \in L_f(W') &\quad \text{iff} \quad \forall x \in W' (p(x) < f(\beta(x))) \\ (\beta, p) \in R_f(W') &\quad \text{iff} \quad \forall x \in W' (p(x) > f(\beta(x))) \end{aligned}$$

We leave it to you to verify that then  $W \Vdash \text{Cut}(L_f, R_f)$  and that the two given operations between  $y_{\mathbb{R}}(W)$  and  $R_d(W)$  are inverse to each other. You should observe that every continuous function  $f : W \rightarrow \mathbb{Q}$  is locally constant, as  $\mathbb{Q}$  is discrete. ■

**Corollary 10.28** *The exponential  $(R_d)^{R_d}$  is isomorphic to  $y_{\mathbb{R}^{\mathbb{R}}}$ , where  $\mathbb{R}^{\mathbb{R}}$  is the set of continuous maps  $\mathbb{R} \rightarrow \mathbb{R}$  with the compact-open topology.*

**Proof.** This follows at once from proposition 10.27, the observation that  $y$  preserves exponents, and the fact that  $\mathbb{R}$  is locally compact. ■

From the corollary we see at once that arrows  $R_d \rightarrow R_d$  in  $\mathcal{T}$  correspond bijectively to continuous functions  $\mathbb{R} \rightarrow \mathbb{R}$ , but this is not yet quite Brouwer's statement that all functions (defined, possibly, with extra parameters) from  $R_d$  to  $R_d$  are continuous. So we prove that now.

**Theorem 10.29**  $\mathcal{T} \Vdash$  “All functions  $R_d \rightarrow R_d$  are continuous”

**Proof.** . In other words, we have to prove that the sentence

$$\begin{aligned} \forall f \in (R_d)^{R_d} \forall x \in R_d \forall \epsilon \in R_d (\epsilon > 0 \rightarrow \exists \delta \in R_d (\delta > 0 \wedge \\ \forall y \in R_d (x - \delta < y < x + \delta \rightarrow f(x) - \epsilon < f(y) < f(x) + \epsilon))) \end{aligned}$$

is true in  $\mathcal{T}$ .

We can work in  $y_{\mathbb{R}}$  for  $(R_d)^{R_d}$ , so  $(R_d)^{R_d}(W) = \text{Cts}(W \times \mathbb{R}, \mathbb{R})$ . Take  $f \in (R_d)^{R_d}(W)$  and  $a, \epsilon \in R_d(W)$  such that  $W \Vdash \epsilon > 0$ . So  $f : W \times \mathbb{R} \rightarrow \mathbb{R}$ , and  $a, \epsilon : W \rightarrow \mathbb{R}$ ,  $\epsilon(x) > 0$  for all  $x \in W$ . We have to show:

$$(*) \quad W \Vdash \exists \delta \in R_d(\delta > 0 \wedge \forall y \in R_d(a - y < \delta < a + \delta \rightarrow f(a) - \epsilon < f(y) < f(a) + \epsilon))$$

Now  $f$  and  $\epsilon$  are continuous, so for each  $x \in W$  there is an open neighborhood  $W_x \subseteq W$  of  $x$ , and a  $\delta_x > 0$  such that for each  $\xi \in W_x$  and  $t \in (a(x) - \delta_x, a(x) + \delta_x)$ :

$$(1) \quad |a(\xi) - a(x)| < \frac{1}{2}\delta_x$$

$$(2) \quad |f(\xi, t) - f(\xi, a(x))| < \frac{1}{2}\epsilon(\xi)$$

We claim:

$$W_x \Vdash \forall y(a - \frac{1}{2}\delta_x < y < a + \frac{1}{2}\delta_x \rightarrow f(a) - \epsilon < f(y) < f(a) + \epsilon)$$

Note that this establishes what we want to prove.

To prove the claim, choose  $\beta : V \rightarrow W_x$ ,  $b : V \rightarrow \mathbb{R}$  such that

$$V \Vdash a\beta - \frac{1}{2}\delta_x < b < a\beta + \frac{1}{2}\delta_x$$

Then for all  $\zeta \in V$ ,  $|a\beta(\zeta) - b(\zeta)| < \frac{1}{2}\delta_x$ , so by (1),

$$|a(x) - b(\zeta)| < \delta_x$$

Therefore we can substitute  $\beta\zeta$  for  $\xi$ , and  $b(\zeta)$  for  $t$  in (2) to obtain

$$\begin{aligned} |f(\beta(\zeta), b(\zeta)) - f(x, a(x))| &< \frac{1}{2}\epsilon\beta(\zeta) \\ \text{and} \\ |f(\beta(\zeta), a\beta(\zeta)) - f(x, a(x))| &< \frac{1}{2}\epsilon\beta(\zeta) \end{aligned}$$

We conclude that  $|f(\beta(\zeta), b(\zeta)) - f(\beta(\zeta), a\beta(\zeta))| < \epsilon\beta(\zeta)$ . Hence,

$$V \Vdash (f\beta)(a\beta) - \epsilon\beta < (f\beta)(b) < (f\beta)(a\beta) + \epsilon\beta$$

which proves the claim and we are done. ■

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# Topos Theory

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## Preface

These lecture notes were written during a *Mastermath* (Dutch national programme for master-level courses in mathematics) course, taught in the fall of 2018.

The main sources I used are:

- 1) My course notes *Basic Category Theory and Topos Theory* ([8]), material for the lecture course to which the present course is a sequel. Referred to in the text as the *Basic Course*.
- 2) MacLane’s *Categories for the Working Mathematician* ([5]). Referred to as “MacLane”.
- 3) Peter Johnstone’s *Topos Theory* ([3]). This is referred to in the text by **PTJ**.
- 4) MacLane and Moerdijk’s *Sheaves in Geometry and Logic* ([6]). Referred to by **MM**.
- 5) Francis Borceux’s *Handbook of Categorical Algebra* ([1]).
- 6) Peter Johnstone’s *Sketches of an Elephant* ([4]). Referred to by **Elephant**.
- 7) Moerdijk’s *Classifying Spaces and Classifying Topoi* ([7]).
- 8) Olivia Caramello’s *Theories, Sites, Toposes* ([2]).
- 9) Jaap van Oosten’s *Realizability: an Introduction to its Categorical Side* ([9]).

There is no original material in the text, except for a few exercises and some proofs.

Conventions: in a categorical product, the projections are usually denoted by  $p_0$  and  $p_1$ , so  $p_1$  is the *second* projection.

### 0.1 The plural of the word “topos”

Everyone knows the quip at the end of the Introduction of [3], which asks *those toposophers who persist in talking about topoi whether, when they go out for a ramble on a cold day, they carry supplies of hot tea with them in thermoi*. Since then, everyone has to declare what, in his or her view, is the plural of “topos”. The form “topoi”, of course, is the plural of the

ancient Greek word for “place”. However, Topology is not the science of places, and the name Topology is what inspired Grothendieck to introduce the word Topos.

Someone (I forget who) proposed: the word “topos” is *French*, and its plural is “topos”. True, but English has adopted many French words, which are then treated as English words. The French plural of “bus” is “bus”, but in English it is “buses”.

I stick with “toposes”.

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# Introduction

We start by recalling some basic definitions from the course *Category Theory and Topos Theory*, which is a prerequisite for this course. For motivation, we start by exhibiting the elementary notions at work in the example of sheaves on a topological space. Later in this introductory chapter, we review a few definitions and results from Category Theory.

**Definition 0.1** A *topos* is a category with finite limits which is cartesian closed and has a *subobject classifier*. A subobject classifier is an arrow  $t : 1 \rightarrow \Omega$  such that every monomorphism is a pullback of  $t$  in a unique way: for every mono  $m : X \rightarrow Y$  there is a unique arrow  $\chi_m : Y \rightarrow \Omega$  (the *classifying map*, or *characteristic map* of  $m$ ) such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & 1 \\ m \downarrow & & \downarrow t \\ Y & \xrightarrow{\chi_m} & \Omega \end{array}$$

is a pullback.

In the introductory course we have seen that the category  $\widehat{\mathcal{C}}$  of presheaves on  $\mathcal{C}$  (for a small category  $\mathcal{C}$ ) is a topos, and if  $J$  is a Grothendieck topology on  $\mathcal{C}$  then the full subcategory of  $\widehat{\mathcal{C}}$  on the sheaves for  $J$ , usually denoted  $\text{Sh}(\mathcal{C}, J)$ , is also a topos. The pair  $(\mathcal{C}, J)$  is called a *site*; and a topos of the form  $\text{Sh}(\mathcal{C}, J)$  is called a *Grothendieck topos*.

Let us see a concrete example, in order to illustrate some of the themes which are important in Topos Theory.

## 0.2 Sheaves on Spaces

Given a topological space  $X$  with set of opens  $\mathcal{O}_X$ , we view  $\mathcal{O}_X$  as a (posetal) category, and form the topos  $\widehat{\mathcal{O}_X}$  of *presheaves on  $X$*  (as it is usually called). For an open  $U \subseteq X$ , a sieve on  $U$  can be identified with a set  $S$  of open subsets of  $U$  which is *downwards closed*: if  $V \subseteq W \subseteq U$  and  $W \in S$ , then also  $V \in S$ . There is a very natural and straightforward Grothendieck topology on  $\mathcal{O}_X$ : declare a sieve  $S$  on  $U$  to be covering if  $\bigcup S = U$ . The category of sheaves on  $\mathcal{O}_X$  for this Grothendieck topology is simply called the category of *sheaves on  $X$* , and denoted  $\text{Sh}(X)$ .

Let  $F$  be a presheaf on  $X$ ; an element  $s \in F(U)$  is called a *local section of  $F$  at  $U$* . For the action of  $F$  on local sections, that is:  $F(V \subseteq U)(s) \in F(V)$

(where  $V$  is a subset of  $U$  and the unique morphism from  $V$  to  $U$  is denoted by the inclusion), we write  $s|V$ .

Now let  $x$  be a point of the space  $X$ . We consider an equivalence relation on the set  $\{(s, U) \mid x \in U, s \in F(U)\}$  of local sections defined at  $x$ , by stipulating:  $(s, U) \sim_x (t, V)$  iff there is some neighbourhood  $W$  of  $x$  such that  $W \subseteq U \cap V$  and  $s|W = t|W$ . An equivalence class  $[(s, U)]$  is called a *germ at  $x$*  and is denoted  $s_x$ ; the set of all germs at  $x$  is  $G_x$ , the *stalk of  $x$* .

Define a topology on the disjoint union  $\coprod_{x \in X} G_x$  of all the stalks: a basic open set is of the form

$$\mathcal{O}_s^U = \{(y, s_y) \mid y \in U\}$$

for  $U \in \mathcal{O}_X$  and  $s \in F(U)$ . This is indeed a basis: suppose  $(x, g) \in \mathcal{O}_s^U \cap \mathcal{O}_t^V$ . then  $g = s_x = t_x$ , so there is a neighbourhood  $W$  of  $x$  such that  $W \subseteq U \cap V$  and  $s|W = t|W$ . We see that

$$(x, g) \in \mathcal{O}_{s|W}^W \subseteq \mathcal{O}_s^U \cap \mathcal{O}_t^V.$$

We have a map  $\pi : \coprod_{x \in X} G_x \rightarrow X$ , sending  $(x, g)$  to  $x$ . If  $U \in \mathcal{O}_X$  and  $(x, g) = (x, s_x) \in \pi^{-1}(U)$  then  $(x, s_x) \in \mathcal{O}_s^U \subseteq \pi^{-1}(U)$ , so  $\pi$  is continuous. Moreover,  $\pi(\mathcal{O}_s^U) = U$ , so  $\pi$  is an open map.

The map  $\pi$  has another important property. Let  $(x, g) = (x, s_x) \in \coprod_{x \in X} G_x$ . Fix some  $U$  such that  $x \in U$  and  $s \in F(U)$ . The restriction of the map  $\pi$  to  $\mathcal{O}_s^U$  gives a bijection from  $\mathcal{O}_s^U$  to  $U$ . Since this bijection is also continuous and open, it is a homeomorphism. We conclude that every element of  $\coprod_{x \in X} G_x$  has a neighbourhood such that the restriction of the map  $\pi$  to that neighbourhood is a homeomorphism. Such maps of topological spaces are called *local homeomorphisms*, or *étale maps*.

Let  $\mathbf{Top}$  denote the category of topological spaces and continuous functions. For a space  $X$  let  $\mathbf{Top}/X$  be the slice category of maps into  $X$ , and let  $\mathbf{Et}(X)$  be the full subcategory of  $\mathbf{Top}/X$  on the local homeomorphisms into  $X$ . We have the following theorem in sheaf theory:

**Theorem 0.2** *The categories  $\mathbf{Et}(X)$  and  $\mathbf{Sh}(X)$  are equivalent.*

**Proof.** [Outline] For an étale map  $p : Y \rightarrow X$ , define a presheaf  $\mathcal{F}$  on  $X$  by putting:

$$\mathcal{F}(U) = \{s : U \rightarrow Y \mid s \text{ continuous and } ps = \text{id}_U\}.$$

This explains the terminology *local sections*. Then  $\mathcal{F}$  is a sheaf on  $X$ . Conversely, given a sheaf  $F$  on  $X$ , define the corresponding étale map as the map  $\pi : \coprod_{x \in X} G_x \rightarrow X$  constructed above. These two operations are, up to isomorphism in the respective categories, each other's inverse. ■

**Exercise 1** Show that for a presheaf  $F$  and the associated local homeomorphism  $\pi : \coprod_{x \in X} G_x \rightarrow X$  that we have constructed, the following holds: every morphism of presheaves  $F \rightarrow H$ , where  $H$  is a sheaf, factors uniquely through the sheaf corresponding to  $\pi : \coprod_{x \in X} G_x \rightarrow X$ . Conclude that  $\pi : \coprod_{x \in X} G_x \rightarrow X$  is the associated sheaf of  $F$ .

Next, let us consider the effect of continuous maps on categories of sheaves. First of all, given a continuous map  $\phi : Y \rightarrow X$  we have the inverse image map  $\phi^{-1} : \mathcal{O}_X \rightarrow \mathcal{O}_Y$  and hence a functor

$$\phi_* = \text{Set}^{(\phi^{-1})^{\text{op}}} : \widehat{\mathcal{O}_Y} \rightarrow \widehat{\mathcal{O}_X}$$

and the functor  $\phi_*$  restricts to a functor  $\text{Sh}(Y) \rightarrow \text{Sh}(X)$ .

There is also a functor in the other direction: given a sheaf  $F$  on  $X$ , let  $\mathcal{F} \rightarrow X$  be the corresponding étale map. It is easy to verify that étale maps are stable under pullback, so if

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\phi} & X \end{array}$$

is a pullback diagram in  $\text{Top}$ , let  $\phi^*(F)$  be the sheaf on  $Y$  which corresponds to the local homeomorphism  $\mathcal{G} \rightarrow Y$ . This defines a functor  $\text{Sh}(X) \rightarrow \text{Sh}(Y)$ .

**Proposition 0.3** *We have an adjunction  $\phi^* \dashv \phi_*$ ; moreover, the left adjoint  $\phi^*$  preserves finite limits.*

**Definition 0.4** Let  $\mathcal{E}$  and  $\mathcal{F}$  be toposes. A *geometric morphism*:  $\mathcal{F} \rightarrow \mathcal{E}$  consists of functors  $f_* : \mathcal{F} \rightarrow \mathcal{E}$  and  $f^* : \mathcal{E} \rightarrow \mathcal{F}$  satisfying:  $f^* \dashv f_*$  and  $f^*$  preserves finite limits. The functor  $f_*$  is called the *direct image functor* of the geometric morphism, and  $f^*$  the *inverse image functor*.

It is clear that Definition 0.4 gives us a category  $\mathcal{Top}$  of toposes and geometric morphisms, and the treatment of categories of sheaves on spaces shows that we have a functor  $\text{Top} \rightarrow \mathcal{Top}$  from topological spaces to toposes. This functor allows us to relate topological properties of a space to category-theoretic properties of its associated topos of sheaves.

Another example of geometric morphism that we have seen, is the one  $\text{Sh}(\mathcal{C}, J) \rightarrow \widehat{\mathcal{C}}$ , where the direct image is inclusion as subcategory, and the inverse image is sheafification.

Other examples of geometric morphisms we shall meet during this course, are:

- i) Any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between small categories gives rise to a geometric morphism  $\widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$ .
- ii) If  $\mathcal{E}$  is a topos and  $X$  is an object of  $\mathcal{E}$ , then the slice category  $\mathcal{E}/X$  is a topos; and if  $f : X \rightarrow Y$  is an arrow in  $\mathcal{E}$  then we will have a geometric morphism  $\mathcal{E}/X \rightarrow \mathcal{E}/Y$ .
- iii) If  $\mathcal{E}$  is a topos and  $H : \mathcal{E} \rightarrow \mathcal{E}$  is a finite-limit preserving comonad on  $\mathcal{E}$ , then the category  $\mathcal{E}_H$  of coalgebras for  $H$  in  $\mathcal{E}$  is a topos, and there is a geometric morphism  $\mathcal{E} \rightarrow \mathcal{E}_H$ .

There is another important notion of “morphism between toposes”: logical functors.

**Definition 0.5** A *logical functor* between toposes is a functor which preserves the topos structure, that is: finite limits, exponentials and the sub-object classifier.

### 0.3 Notions from Category Theory

First, let us deal with a subtlety we overlooked in the *Category Theory and Topos Theory* course. There, we said (following MacLane) that a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  *creates limits of type  $J$*  if for every diagram  $M : J \rightarrow \mathcal{C}$  and every limiting cone  $(D, \mu)$  for  $FM$  in  $\mathcal{D}$ , there is a *unique* cone  $(C, \nu)$  for  $M$  in  $\mathcal{C}$  which is mapped by  $F$  to  $(D, \mu)$ , and moreover the cone  $(C, \nu)$  is a limiting cone for  $M$ .

For an adjunction  $F \dashv G : \mathcal{C} \rightarrow \mathcal{D}$  (so  $G : \mathcal{C} \rightarrow \mathcal{D}, F : \mathcal{D} \rightarrow \mathcal{C}$ ) we have a comparison functor  $K : \mathcal{C} \rightarrow \mathcal{D}^{GF}$ , where  $\mathcal{D}^{GF}$  is the category of algebras for the monad  $GF$  on  $\mathcal{D}$ . MacLane, consistently, defines the functor  $G$  to be *monadic* if  $K$  is an isomorphism of categories. It follows that every monadic functor creates limits.

However, we defined the functor  $G$  to be monadic if  $K$  is an *equivalence*. And whilst the forgetful functor  $U^T : \mathcal{C}^T \rightarrow \mathcal{C}$  always creates limits (here  $\mathcal{C}^T$  denotes the category of algebras for a monad  $T$ ), with the strict definition we gave this is no longer guaranteed if  $U^T$  is composed with an equivalence of categories. Yet, there are good reasons to consider “monadic” functors where the comparison is only an equivalence, and we would like to have a “creation of limits” definition which is stable under equivalence. For example, the “Crude Tripleability Theorem” (0.10) below only ensures an equivalence with the category of algebras.

**Definition 0.6 (Creation of Limits)** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  creates limits of type  $J$  if for any diagram  $M : J \rightarrow \mathcal{C}$  and any limiting cone  $(X, \mu)$  for  $FM$  in  $\mathcal{D}$  the following hold:

- i) There exists a cone  $(Y, \nu)$  for  $M$  in  $\mathcal{C}$  such that its  $F$ -image is isomorphic to  $(X, \mu)$  (in the category of cones for  $M$ ).
- ii) Any cone  $(Y, \nu)$  for  $M$  which is mapped by  $F$  to a cone isomorphic to  $(X, \mu)$ , is limiting.

We say that the functor  $F$  *creates limits* if  $F$  creates limits of every small type  $J$ .

For the record:

**Theorem 0.7** *Let  $\mathcal{C} \xrightarrow{G} \mathcal{D}$  monadic. Then  $G$  creates limits.*

The following remark appears on the first pages of Johnstone's *Sketches of an Elephant*, and is very useful.

**Remark 0.8 (Elephant A1.1.1)** Let  $A \xrightleftharpoons[U]{F} C$  be an adjunction with  $F \dashv U$ . If there is a natural isomorphism between  $FU$  and the identity on  $A$ , then the counit is a natural isomorphism. Of course, by duality a similar statement holds for units.

**Definition 0.9** A parallel pair of arrows  $X \xrightleftharpoons[g]{f} Y$  is a *reflexive pair* if  $f$  and  $g$  have a common section: a morphism  $s : Y \rightarrow X$  for which  $fs = gs = \text{id}_Y$ . A category is said to have *coequalizers of reflexive pairs* if for every reflexive pair the coequalizer exists.

**Theorem 0.10 (Beck's "Crude Tripleability Theorem")** *Let*

$$A \xrightleftharpoons[U]{F} C$$

*be an adjunction with  $F \dashv U$ ; let  $T = UF$  be the induced monad on  $C$ . Suppose that  $A$  has coequalizers of reflexive pairs, that  $U$  preserves them, and moreover that  $U$  reflects isomorphisms. Then the functor  $U$  is monadic.*

**Proof.** We start by constructing a left adjoint  $L$  to the functor  $K$ . Recall that  $K : A \rightarrow C^T$  sends an object  $Y$  of  $A$  to the  $T$ -algebra  $UFUY \xrightarrow{U(\varepsilon_Y)} UY$ .



Let  $UF X \xrightarrow{h} X$  be a  $T$ -algebra. We have that  $\eta_X$  is a section of  $h$  by the axioms for an algebra, and  $F(\eta_X)$  is a section of  $\varepsilon_{FX}$  by the triangular identities for an adjunction. So the parallel pair

$$FUF X \xrightleftharpoons[\varepsilon_{FX}]{F(h)} FX$$

is reflexive with common section  $F(\eta_X)$ ; let  $FX \xrightarrow{e} E$  be its coequalizer. We define  $L(h)$  to be the object  $E$ . Clearly, this is functorial in  $h$ .

Let us prove that  $KL(h)$  is isomorphic to  $h$ . Note that the underlying object of the  $T$ -algebra  $KL(h)$  is  $UE$ . By construction of  $L(h)$  and the assumptions on  $U$ , the diagram

$$UFUF X \xrightleftharpoons[U(\varepsilon_{FX})]{UF(h)} UFX \xrightarrow{U(e)} UE$$

is a coequalizer. By the associativity of the algebra  $h$ , the map  $h$  coequalizes the pair  $(UF(h), U(\varepsilon_{FX}))$ ; so we have a unique  $\xi : UE \rightarrow X$  satisfying

$$\xi \circ U(e) = h.$$

We also have the map  $U(e) \circ \eta_X : X \rightarrow UE$ . It is routine to check that these maps are each other's inverse, as well as that  $\xi$  is in fact an algebra map. This shows that  $KL(h)$  is naturally isomorphic to  $h$ .

Let us show that  $L \dashv K$ . Maps in  $A$  from  $E = L(h)$  to an object  $Y$  correspond, by the coequalizer property of  $E$ , to arrows  $f : FX \rightarrow Y$  satisfying  $f \circ F(h) = f \circ \varepsilon_{FX}$ . Transposing along the adjunction  $F \dashv U$ , these correspond to maps  $\bar{f} : X \rightarrow UY$  satisfying  $\bar{f} \circ h = U(\varepsilon_Y) \circ UF(\bar{f})$ ; that is, to  $T$ -algebra maps from  $h$  to  $K(Y)$ . This establishes the adjunction and applying Johnstone's remark 0.8 we conclude that the unit of the adjunction is an isomorphism.

In order to show that also the counit of  $L \dashv K$  is an isomorphism, we recall that for an object  $Y$  of  $A$ ,  $LK(Y)$  is the vertex of the coequalizer diagram

$$FUFUY \xrightleftharpoons[\varepsilon_{FUY}]{FU(\varepsilon_Y)} FUY \xrightarrow{w} W$$

Since also  $\varepsilon_Y$  coequalizes the parallel pair, we have a unique map  $W \xrightarrow{v} Y$  satisfying  $vw = \varepsilon_Y$ . It is now not too hard to prove that  $U(v)$  is an isomorphism; since  $U$  reflects isomorphisms,  $v$  is an isomorphism, and we are done. ■

The following theorem is called “Adjoint lifting theorem”.

**Theorem 0.11 (Adjoint Lifting Theorem; PTJ 0.15)** *Let  $T$  and  $S$  be monads on categories  $\mathcal{C}$  and  $\mathcal{D}$  respectively. Suppose we have a commutative diagram of functors*

$$\begin{array}{ccc} \mathcal{C}^T & \xrightarrow{\bar{F}} & \mathcal{D}^S \\ U^T \downarrow & & \downarrow U^S \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

where  $U^T, U^S$  are the forgetful functors. Suppose  $F$  has a left adjoint  $L$ . Moreover, assume that the category  $\mathcal{C}^T$  has coequalizers of reflexive pairs. Then the functor  $\bar{F}$  also has a left adjoint.

**Proof.** [Sketch] Let  $(T, \eta, \mu)$  and  $(S, \iota, \nu)$  be the respective monad structures on  $T$  and  $S$ . Our first remark is that every  $S$ -algebra is a coequalizer of a reflexive pair of arrows between free  $S$ -algebras. For an  $S$ -algebra  $SX \xrightarrow{h} X$ , consider the parallel pair

$$S^2X \xrightleftharpoons[\nu_X]{Sh} SX$$

This is a diagram of algebra maps  $F^S(SX) \rightarrow F^S(X)$ :  $\nu_X S^2h = Sh\nu_{SX}$  by naturality of  $\nu$ , and  $\nu_X \nu_{SX} = \nu_X S(\nu_X)$  by associativity of  $\nu$ . The two arrows have a common splitting  $S(\iota_X)$  which is also an algebra map since it is  $F^S(\iota_X)$ . That is: we have a reflexive pair in  $S\text{-Alg}$ . It is easy to see that  $h : SX \rightarrow X$  coequalizes this pair: this is the associativity of  $h$  as an algebra. If  $a : F^S(X) \rightarrow (\xi : SY \rightarrow Y)$  is an algebra map which coequalizes our reflexive pair then  $a$  factors through  $h : F^S(X) \rightarrow (h : SX \rightarrow X)$  by  $a\iota_X : (SX \xrightarrow{h} X) \rightarrow (SY \xrightarrow{\xi} Y)$  and the factorization is unique because the arrow  $h$  is split epi in  $\mathcal{C}$ .

This construction is functorial. Given an  $S$ -algebra map  $f : (SX \xrightarrow{h} X) \rightarrow (SY \xrightarrow{k} Y)$  the diagram

$$\begin{array}{ccc} S^2X & \xrightleftharpoons[\nu_X]{Sh} & SX \\ S^2f \downarrow & & \downarrow Sf \\ S^2Y & \xrightleftharpoons[\nu_Y]{Sk} & SY \end{array}$$

commutes serially (i.e.,  $Sf\nu_X = \nu_Y S^2f$  and  $SfSh = SkS^2f$ ). So, we have a functor  $R$  from  $S\text{-Alg}$  to the category of diagrams of shape  $\circ \rightrightarrows \circ$  in  $S\text{-Alg}$ , with the properties:

- i) The vertices of  $R(h)$  are free algebras.
- ii)  $R(h)$  is always a reflexive pair.
- iii) The colimit of  $R(h)$  is  $h$ .

Our second remark is that since  $\bar{F}$  is a lifting of  $F$  ( $U^S \bar{F} = F U^T$ ) there is a natural transformation  $\lambda : SF \rightarrow FT$  constructed as follows. Consider  $F(\eta) : F \rightarrow FT = F U^T F^T = U^S \bar{F} F^T$  and let  $\tilde{\lambda} : F^S F \rightarrow \bar{F} F^T$  be its transpose along  $F^S \dashv U^S$ . Define  $\lambda$  as the composite

$$SF = U^S F^S F \xrightarrow{U^S \tilde{\lambda}} U^S \bar{F} F^T = F U^T F^T = FT.$$

*Claim:* The natural transformation  $\lambda$  makes the following diagram commute:

$$\begin{array}{ccccc}
 F & \xrightarrow{\iota_F} & SF & \xleftarrow{\nu_F} & S^2 F \\
 & \searrow F(\eta) & \downarrow \lambda & & \downarrow S\lambda \\
 & & FT & & SFT \\
 & & & \swarrow F\mu & \downarrow \lambda T \\
 & & & & FT^2
 \end{array}$$

Now we are ready for the definition of  $\bar{L}$  on objects: if  $\bar{L}$  is going to be left adjoint to  $\bar{F}$  then, by uniqueness of adjoints and the fact that adjoints compose,  $\bar{L}F^S = F^T L$ , so we know what  $\bar{L}$  should do on free  $S$ -algebras  $F^S Y$ . Now every  $S$ -algebra  $\xi : SY \rightarrow Y$  is coequalizer of a reflexive pair of arrows between free  $S$ -algebras, and as a left adjoint,  $\bar{L}$  should preserve coequalizers. Therefore we expect  $\bar{L}(\xi)$  to be coequalizer of a reflexive pair

$$F^T LSY = \bar{L}F^S(SY) \xrightleftharpoons[g_\xi]{f_\xi} \bar{L}F^S(Y) = F^T LY$$

between free  $T$ -algebras. It is now our task to determine  $f_\xi$  and  $g_\xi$ .

By our first remark we have a coequalizer

$$F^S(SY) \xrightleftharpoons[\nu_Y]{S\xi} F^S Y \xrightarrow{\xi} (\xi)$$

and the topmost arrow of the reflexive pair is in the image of the functor  $F^S$ , so we can take  $F^T L(\xi)$  for  $f_\xi$ . The other map  $-\nu-$  is not in the image of  $F^S$  and needs a bit of doctoring using the adjunction  $L \dashv F$  and the natural

transformation  $\lambda$  we constructed. Let  $\alpha$  be the unit of the adjunction  $L \dashv F$ . Consider the arrow

$$SY \xrightarrow{S(\alpha_Y)} SFL(Y) \xrightarrow{\lambda_{L(Y)}} FTL(Y)$$

This transposes under  $L \dashv F$  to a map  $LS(Y) \rightarrow TL(Y) = U^T F^T L(Y)$ , and this in turn transposes under  $F^T \dashv U^T$  to a map

$$F^T LS(Y) \rightarrow F^T L(Y)$$

which we take as our  $g_\xi$ .

Note that the construction is natural in  $\xi$ , so if  $k : \xi \rightarrow \zeta$  is a map of  $S$ -algebras, we obtain a natural transformation from the diagram of parallel arrows  $f_\xi, g_\xi$  to the diagram with parallel arrows  $f_\zeta, g_\zeta$ . Hence we also get a map from the coequalizer of the first diagram, which is  $\bar{L}(\xi)$ , to the coequalizer of the second one, which is  $\bar{L}(\zeta)$ . And this map between coequalizers will be  $\bar{L}(k)$ .

There is still a lot to check. This is meticulously done in Volume 2 of Borceux's *Handbook of Categorical Algebra*, section 4.5. There the proof takes 10 pages. ■

**Remark 0.12** There is a better theorem than the one we just partially proved: the *Adjoint Triangle Theorem*. It says that whenever we have functors  $\mathcal{B} \xrightarrow{R} \mathcal{C} \xrightarrow{U} \mathcal{D}$  such that  $\mathcal{B}$  has reflexive coequalizers and  $U$  is of *descent type* (that is:  $U$  has a left adjoint  $J$  and the comparison functor  $K : \mathcal{C} \rightarrow UJ\text{-Alg}$  is full and faithful), then  $UR$  has a left adjoint if and only if  $R$  has one.

Note, that given the diagram of Theorem 0.11, the diagram

$$\mathcal{C}^T \xrightarrow{\bar{F}} \mathcal{D}^S \xrightarrow{U^S} \mathcal{D}$$

satisfies the conditions of the Adjoint Triangle Theorem. Since the composition  $U^S \bar{F}$ , which is  $F^T L$ , has a left adjoint, we conclude that  $\bar{F}$  has a left adjoint. Note in particular that we do not use that  $\mathcal{C}^T$  is monadic.

**Definition 0.13** A diagram  $a \xrightleftharpoons[g]{f} b \xrightarrow{h} c$  in a category is called a *split fork* if  $hf = hg$  and there exist maps

$$a \xleftarrow{t} b \xleftarrow{s} c$$

such that  $hs = \text{id}_c$ ,  $ft = \text{id}_b$  and  $gt = sh$ .

**Exercise 2** Show that every split fork is a coequalizer diagram, and moreover a coequalizer which is preserved by any functor (this is called an *absolute coequalizer*).

**Exercise 3** Suppose  $D_1$  is the diagram  $a \xrightarrow[f]{g} b \xrightarrow{h} c$  in a category  $\mathcal{C}$ ,

and  $D_2$  is the diagram  $a' \xrightarrow[g']{f'} b' \xrightarrow{h'} c'$  in  $\mathcal{C}$ . Assume that  $D_2$  is a retract of  $D_1$  in the category of diagrams in  $\mathcal{C}$  of type  $\bullet \rightrightarrows \bullet \longrightarrow \bullet$ . Prove that if  $D_1$  is a split fork, then so is  $D_2$ .

**Definition 0.14** In a category, a family of arrows  $\{f_i : A_i \rightarrow B \mid i \in I\}$  is called *epimorphic* if for every parallel pair of arrows  $u, v : B \rightarrow C$  the following holds: if  $uf_i = vf_i$  for all  $i \in I$ , then  $u = v$ .

**Exercise 4** If the ambient category has  $I$ -indexed coproducts, a family  $\{f_i : A_i \rightarrow B \mid i \in I\}$  is epimorphic if and only if the induced arrow from the coproduct  $\sum_{i \in I} A_i$  to  $B$  is an epimorphism.

We shall also have to deal with *comonads*; a comonad on a category  $\mathcal{C}$  is a monad on  $\mathcal{C}^{\text{op}}$ . Explicitly, we have a functor  $G : \mathcal{C} \rightarrow \mathcal{C}$  with natural transformations  $\varepsilon : G \Rightarrow \text{id}_{\mathcal{C}}$  (the “counit”) and  $\delta : G \Rightarrow G^2$  (the “comultiplication”) which make the following (coassociativity and counitarity) diagrams commute:

$$\begin{array}{ccc} G & \xrightarrow{\delta} & G^2 \\ \delta \downarrow & & \downarrow \delta G \\ G^2 & \xrightarrow{G\delta} & G^3 \end{array} \quad \begin{array}{ccccc} & & G & & \\ & \text{id} \nearrow & \downarrow \delta & \nwarrow \text{id} & \\ G & \xleftarrow{\varepsilon G} & G^2 & \xrightarrow{G\varepsilon} & G \end{array}$$

Dual to the treatment for monads, we have the category  $G\text{-Coalg}$  of  $G$ -coalgebras, the notion of a functor being “comonadic”, etcetera. We have the forgetful functor  $V : G\text{-Coalg} \rightarrow \mathcal{C}$  which has a *right* adjoint  $C : \mathcal{C} \rightarrow G\text{-Coalg}$ , the “cofree coalgebra functor”. Without proof we record the following theorem:

**Theorem 0.15 (Eilenberg-Moore; MM V.8.1-2; PTJ 0.14)** *Suppose  $T$  is a monad on a category  $\mathcal{C}$ , such that the functor  $T$  has a right adjoint  $G$ . Then there is a unique comonad structure  $(\varepsilon, \delta)$  on  $G$  such that the categories*

*T-Alg and G-Coalg are isomorphic by an isomorphism which commutes with the forgetful functors:*

$$\begin{array}{ccc} T\text{-Alg} & \xrightarrow{L} & G\text{-Coalg} \\ & \searrow U \quad \swarrow V & \\ & \mathcal{C} & \end{array}$$

**Corollary 0.16** *If  $(T, \eta, \mu)$  is a monad on  $\mathcal{C}$  and the functor  $T$  has a right adjoint  $G$ , then the forgetful functor  $T\text{-Alg} \rightarrow \mathcal{C}$  has both a left and a right adjoint.*

# 1 Elementary Toposes

In this chapter we are going through Chapter 1 of P.T. Johnstone's *Topos Theory*, expanding the proofs a bit when necessary.

**Example 1.1** Let  $\mathcal{G}$  be a group. In the topos  $\widehat{\mathcal{G}}$  of right  $\mathcal{G}$ -sets (sets  $X$  with  $\mathcal{G}$ -action  $X \times \mathcal{G} \rightarrow X$ , written  $(x, g) \mapsto x \cdot g$ ) we have:

- i) the subobject classifier  $1 \xrightarrow{t} \Omega$  is the map from  $\{*\}$  to  $\{0, 1\}$  which sends  $*$  to 1; here  $\{0, 1\}$  has the trivial  $\mathcal{G}$ -action.
- ii) The exponent  $Y^X$  of two  $\mathcal{G}$ -sets  $X$  and  $Y$  is the set of *all* functions  $X \xrightarrow{\phi} Y$ , with  $\mathcal{G}$ -action:

$$(\phi \cdot g)(x) = (\phi(x \cdot g^{-1})) \cdot g$$

We see at once that the forgetful functor  $\widehat{\mathcal{G}} \rightarrow \mathbf{Set}$  is logical, as is the functor  $\mathbf{Set} \rightarrow \widehat{\mathcal{G}}$  which sends a set  $X$  to the set  $X$  with trivial  $\mathcal{G}$ -action.

We can also consider the category  $\mathbf{Set}_f^{\mathcal{G}^{\text{op}}}$  of *finite*  $\mathcal{G}$ -sets; and we see that this is also a topos (even if  $\mathcal{G}$  itself is not finite); the inclusion functor  $\mathbf{Set}_f^{\mathcal{G}^{\text{op}}} \rightarrow \widehat{\mathcal{G}}$  is logical.

**Lemma 1.2 (PTJ 1.21)** *In a topos, every mono is regular.*

**Proof.** Every mono is a pullback of  $1 \xrightarrow{t} \Omega$ , and  $t$  is split mono, so regular. ■

**Corollary 1.3 (PTJ 1.22)** *Every map in a topos which is both epi and mono is an isomorphism (one says that a topos is balanced).*

**Definition 1.4** In a category with finite limits, an *equivalence relation on an object*  $X$  is a subobject  $R$  of  $X \times X$  for which the following statements hold:

- i) The diagonal embedding  $X \rightarrow X \times X$  factors through  $R$ .
- ii) The composition  $R \rightarrow X \times X \xrightarrow{\text{tw}} X \times X$  factors through  $R$ , where  $\text{tw}$  denotes the twist map

$$\langle p_1, p_0 \rangle : X \times X \rightarrow X \times X.$$

(Here  $p_0, p_1 : X \times X \rightarrow X$  are the projections)

- iii) The map  $\langle p_0s, p_1t \rangle : R' \rightarrow X \times X$  factors through  $R$ , where we assume that the subobject  $R$  is represented by the arrow  $\langle r_0, r_1 \rangle : R \rightarrow X \times X$ , and the maps  $s$  and  $t$  are defined by the pullback diagram

$$\begin{array}{ccc} R' & \xrightarrow{t} & R \\ s \downarrow & & \downarrow r_0 \\ R & \xrightarrow{r_1} & X \times X \end{array} .$$

The subobject  $R'$  is the “object of  $R$ -related triples”.

Equivalently, a subobject  $R$  of  $X \times X$  is an equivalence relation on  $X$  if and only if for every object  $Y$ , the relation

$$\{(f, g) \mid \langle f, g \rangle : Y \rightarrow X \times X \text{ factors through } R\}$$

is an equivalence relation on the set of arrows  $Y \rightarrow X$ .

Clearly, for every arrow  $f : X \rightarrow Y$ , the kernel pair of  $f$ , seen as a subobject of  $X \times X$ , is an equivalence relation on  $X$ . Equivalence relations which are kernel pairs are called *effective* (don’t ask me why).

**Proposition 1.5 (PTJ 1.23)** *In a topos, every equivalence relation is effective, i.e. a kernel pair.*

**Proof.** Let  $\phi : X \times X \rightarrow \Omega$  classify the subobject  $\langle r_0, r_1 \rangle : R \rightarrow X \times X$ , and let  $\bar{\phi} : X \rightarrow \Omega^X$  be its exponential transpose (in  $\mathbf{Set}$ ,  $\bar{\phi}(x)$  will be the  $R$ -equivalence class of  $x$ ). We claim that the square

$$\begin{array}{ccc} R & \xrightarrow{r_1} & X \\ r_0 \downarrow & & \downarrow \bar{\phi} \\ X & \xrightarrow{\bar{\phi}} & \Omega^X \end{array}$$

is a pullback, so that  $R$  is the kernel pair of  $\bar{\phi}$ . To see that it commutes, we look at the transposes of the compositions  $\bar{\phi}r_i$ , which are maps

$$R \times X \xrightarrow{r_i \times \text{id}} X \times X \xrightarrow{\phi} \Omega$$

Both these maps classify the object  $R'$  of  $R$ -related triples, seen as subobject of  $R \times X$ , so they are equal. To see that the given diagram is a pullback, suppose we have maps  $f, g : U \rightarrow X$  satisfying  $\bar{\phi}f = \bar{\phi}g$ . Then  $\phi(f \times \text{id}_X) =$



$\phi(g \times \text{id}_X) : U \times X \rightarrow \Omega$ . Composing with the map  $\langle \text{id}_U, g \rangle : U \rightarrow U \times X$  we get that the square

$$\begin{array}{ccc} U & \xrightarrow{\langle f, g \rangle} & X \times X \\ \langle g, g \rangle \downarrow & & \downarrow \phi \\ X \times X & \xrightarrow{\phi} & \Omega \end{array}$$

commutes. Now  $\phi$  classifies  $R$  and by reflexivity of  $R$  the map  $\langle g, g \rangle$  factors through  $R$ , so  $\phi \langle g, g \rangle$  is the composite map  $U \xrightarrow{!} 1 \xrightarrow{t} \Omega$ ; so this also holds for the other composite and therefore also  $\langle f, g \rangle$  must factor through  $R$ , which says that the given diagram is indeed a pullback. ■

The least equivalence relation on an object  $X$  is the *diagonal*  $\delta = \langle \text{id}_X, \text{id}_X \rangle : X \rightarrow X \times X$ . It is classified by some  $\Delta : X \times X \rightarrow \Omega$ ; let  $\{\cdot\} : X \rightarrow \Omega^X$  be the exponential transpose of  $\Delta$ . The map  $\{\cdot\}$  is of course thought of as the *singleton map* from  $X$  to its power object.

**Exercise 5** Prove that the map  $\{\cdot\}$  is monic. [Hint: for an arrow  $f : Y \rightarrow X$ , the composite  $\{\cdot\} \circ f$  transposes to a map which classifies the graph of  $f$  (as subobject of  $Y \times X$ ).]

**Definition 1.6** A *partial map* from  $X$  to  $Y$  is an arrow from a subobject of  $X$  to  $Y$ . More precisely, it is a diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & Y \\ m \downarrow & & \\ X & & \end{array}$$

with  $m$  mono.

We write  $f : X \rightarrow Y$  to emphasize that the map is partial.

We say that *partial maps are representable* if for each object  $Y$  there is a monomorphism  $\eta_Y : Y \rightarrow \tilde{Y}$  with the property that for every partial map  $X \xleftarrow{m} U \xrightarrow{f} Y$  from  $X$  to  $Y$  there is a *unique* arrow  $\tilde{f} : X \rightarrow \tilde{Y}$ , making the square

$$\begin{array}{ccc} U & \xrightarrow{f} & Y \\ m \downarrow & & \downarrow \eta_Y \\ X & \xrightarrow{\tilde{f}} & \tilde{Y} \end{array}$$

a pullback.

Let us spell out what this means for  $Y = 1$ : we have an arrow  $\eta : 1 \rightarrow \tilde{1}$  such that for every mono  $m : U \rightarrow X$  there is a unique map  $X \rightarrow \tilde{1}$  making the square

$$\begin{array}{ccc} U & \longrightarrow & 1 \\ m \downarrow & & \downarrow \eta \\ X & \longrightarrow & \tilde{1} \end{array}$$

a pullback. But this is just the definition of a subobject classifier; we conclude that  $1 \xrightarrow{\eta} \tilde{1}$  is  $1 \xrightarrow{t} \Omega$ .

**Theorem 1.7 (PTJ 1.26)** *In a topos, partial maps are representable.*

**Proof.** Let  $\phi : \Omega^Y \times Y \rightarrow \Omega$  classify the graph of the singleton map:

$$\begin{array}{ccc} Y & \xrightarrow{\langle \{\cdot\}, \text{id} \rangle} & \Omega^Y \times Y \\ \downarrow & & \downarrow \phi \\ 1 & \xrightarrow{t} & \Omega \end{array}$$

and let  $\bar{\phi} : \Omega^Y \rightarrow \Omega^Y$  be its exponential transpose.

Let

$$E \xrightarrow{e} \Omega^Y \xrightarrow[\text{id}]{\bar{\phi}} \Omega^Y$$

be an equalizer. We shall show that we can take  $E$  for  $\tilde{Y}$ . Think of  $E$  as the “set”

$$\{\alpha \subseteq Y \mid \forall y(y \in \alpha \leftrightarrow \alpha = \{y\})\},$$

that is: the set of subsets of  $Y$  having at most one element. We consider the pullback diagram

$$\begin{array}{ccc} Y & \xrightarrow{\text{id}} & Y \\ \delta \downarrow & & \downarrow \langle \{\cdot\}, \text{id} \rangle \\ Y \times Y & \xrightarrow{\{\cdot\} \times \text{id}_Y} & \Omega^Y \times Y \end{array}$$

Composing this with the diagram defining  $\phi$ , we obtain pullbacks

$$\begin{array}{ccc}
Y & \xrightarrow{\delta} & Y \times Y \\
\text{id} \downarrow & & \downarrow \{\cdot\} \times \text{id}_Y \\
Y & \xrightarrow{\langle \{\cdot\}, \text{id}_Y \rangle} & \Omega^Y \times Y \\
\downarrow & & \downarrow \phi \\
1 & \xrightarrow{t} & \Omega
\end{array}$$

from which we conclude that  $\phi(\{\cdot\} \times \text{id}_Y)$  classifies the diagonal map on  $Y$ ; hence its exponential transpose, which is  $\bar{\phi} \circ \{\cdot\} : Y \rightarrow \Omega^Y$ , is equal to  $\{\cdot\}$ . Therefore the map  $\{\cdot\} : Y \rightarrow \Omega^Y$  factors through the equalizer  $E$  above; so we have the required map  $Y \rightarrow E = \tilde{Y}$  (which is monic since  $\{\cdot\}$  is).

In order to show that the constructed mono  $Y \rightarrow \tilde{Y}$  indeed represents partial maps into  $Y$ , let

$$\begin{array}{ccc}
U & \xrightarrow{f} & Y \\
m \downarrow & & \\
X & & 
\end{array}$$

be a partial map  $X \rightarrow Y$ , so  $m$  is monic. Consider the graph of  $f: U \xrightarrow{\langle m, f \rangle} X \times Y$ . It is classified by a map  $\psi : X \times Y \rightarrow \Omega$ ; let  $\bar{\psi} : X \rightarrow \Omega^Y$  be the exponential transpose of  $\psi$ . We have a commutative diagram

$$(*) \quad \begin{array}{ccc}
U & \xrightarrow{f} & Y \\
\langle m, f \rangle \downarrow & & \downarrow \langle \{\cdot\}, \text{id} \rangle \\
X \times Y & \xrightarrow{\bar{\psi} \times \text{id}} & \Omega^Y \times Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{\bar{\psi}} & \Omega^Y
\end{array}$$

The lower square is a pullback, so the outer square is a pullback if and only if the upper square is. We prove that the outer square is a pullback. Suppose  $V \xrightarrow{a} X$ ,  $V \xrightarrow{b} Y$  are maps such that  $\{\cdot\}b = \bar{\psi}a$ . Then by transposing, the square

$$\begin{array}{ccc}
V \times Y & \xrightarrow{b \times \text{id}} & Y \times Y \\
a \times \text{id} \downarrow & & \downarrow \Delta \\
X \times Y & \xrightarrow{\psi} & \Omega
\end{array}$$

commutes (recall that  $\Delta$  classifies the diagonal  $Y \rightarrow Y \times Y$ ). Composing with the map  $V \xrightarrow{\langle \text{id}, b \rangle} V \times Y$  gives

$$\begin{aligned} \psi \circ \langle a, b \rangle &= \Delta \circ \langle b, b \rangle &&= \text{(by definition of } \Delta) \\ &= V \rightarrow 1 \xrightarrow{t} \Omega \end{aligned}$$

So  $\psi \circ \langle a, b \rangle$  factors through  $t$ , and since  $\psi$  classifies the graph of  $f$ , the map  $V \xrightarrow{\langle a, b \rangle} X \times Y$  factors through  $U$ ; we conclude that the outer square of  $(*)$  is indeed a pullback. Hence the upper square of  $(*)$  is a pullback.

Now since

$$\begin{array}{ccc} \Omega^Y \times Y & \xrightarrow{\phi} & \Omega \\ \langle \{\cdot\}, \text{id} \rangle \uparrow & & \uparrow \\ Y & \longrightarrow & 1 \end{array}$$

is a pullback by definition of  $\phi$ , composing with the upper square of  $(*)$  yields pullbacks

$$\begin{array}{ccccc} X \times Y & \xrightarrow{\bar{\psi} \times \text{id}} & \Omega^Y \times Y & \xrightarrow{\phi} & \Omega \\ \langle m, f \rangle \uparrow & & \uparrow \langle \{\cdot\}, \text{id} \rangle & & \uparrow t \\ U & \longrightarrow & Y & \longrightarrow & 1 \end{array}$$

So the graph of  $f$  is classified by  $\phi \circ (\bar{\psi} \times \text{id})$ . It follows that  $\phi \circ (\bar{\psi} \times \text{id}) = \psi$ , and by transposing we get  $\bar{\phi} \bar{\psi} = \bar{\psi} : X \rightarrow \Omega^Y$ . So  $\bar{\psi} : X \rightarrow \Omega^Y$  factors through  $\tilde{Y} \rightarrow \Omega^Y$  by a map  $\tilde{f} : X \rightarrow \tilde{Y}$ . The factorization is unique since  $\tilde{Y} \rightarrow \Omega^Y$  is monic. Summarizing, we have

$$\begin{array}{ccc} U & \xrightarrow{f} & Y \\ \downarrow m & & \downarrow \eta_Y \\ & \nearrow \tilde{f} & \tilde{Y} \\ X & \xrightarrow{\bar{\psi}} & \Omega^Y \\ & & \downarrow \\ & & \tilde{Y} \end{array}$$

where the outer square is a pullback (it is the outer square of  $(*)$ ), and since  $\tilde{Y} \rightarrow \Omega^Y$  is monic the upper square is a pullback too.

From the uniqueness of  $\tilde{f}$  we can prove that the assignment  $Y \Rightarrow \tilde{Y}$ , together with the maps  $\eta_Y : Y \rightarrow \tilde{Y}$ , gives a functor  $\mathcal{E} \rightarrow \mathcal{E}$  (where  $\mathcal{E}$

denotes the ambient topos): given a map  $f : X \rightarrow Y$ , let  $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$  represent the partial map

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \tilde{X} \\ f \downarrow & & \\ Y & & \end{array}$$

By uniqueness we see that  $\tilde{g}\tilde{f} = \widetilde{gf}$ . We also see that  $\eta$  is a natural transformation  $\text{id}_{\mathcal{E}} \Rightarrow \widetilde{(\cdot)}$ . It has the special property that all naturality squares are pullbacks. ■

An object of the form  $\tilde{X}$  is called a *partial map classifier*.

**Proposition 1.8 (PTJ 1.27)** *The partial map classifiers  $\tilde{Z}$  are injective.*

**Proof.** Given a diagram

$$\begin{array}{ccc} X' & & \\ m \uparrow & & \\ X & \xrightarrow{f} & \tilde{Z} \end{array}$$

with  $m$  mono, we need to find a map  $X' \rightarrow \tilde{Z}$  making the triangle commute. To this end, form the pullback

$$\begin{array}{ccc} X & \xrightarrow{f} & \tilde{Z} \\ n \uparrow & & \uparrow \eta_Z \\ Y & \xrightarrow{g} & Z \end{array}$$

Let the partial map  $X' \rightarrow Z$  given by  $X' \xleftarrow{mn} Y \xrightarrow{g} Z$  be represented by  $\tilde{g} : X' \rightarrow \tilde{Z}$ . It is left to you to verify that the square

$$\begin{array}{ccc} X & \xrightarrow{\tilde{g}m} & \tilde{Z} \\ n \uparrow & & \uparrow \eta_Z \\ Y & \xrightarrow{g} & Z \end{array}$$

is a pullback. We see that the arrows  $f$  and  $\tilde{g}m$  represent the same partial map, hence the triangle commutes. ■

**Corollary 1.9 (PTJ 1.28)** *Suppose we are given a pushout square*

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ f \downarrow & & \downarrow g \\ Z & \longrightarrow & T \end{array}$$

*with  $f$  mono. Then  $g$  is also mono, and the square is also a pullback.*

**Proof.** Consider the partial map  $Z \rightharpoonup Y$  given by the diagram  $Z \xleftarrow{f} X \xrightarrow{m} Y$  let it be represented by a map  $h : Z \rightarrow \tilde{Y}$ . Since the original square is a pushout, we have a unique map  $T \rightarrow \tilde{Y}$  making the diagram

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ f \downarrow & & \downarrow g \\ Z & \longrightarrow & T \end{array} \quad \begin{array}{c} \nearrow \eta_Y \\ \searrow h \\ \searrow \end{array} \quad \begin{array}{c} \\ \\ \tilde{Y} \end{array}$$

commute. Then  $g$  is mono because  $\eta_Y$  is mono, and the outer square is a pullback, so the inner square is a pullback too. ■

**Remark 1.10** Proposition 1.8 shows, in particular, that a topos *has enough injectives*: that is, for every object  $X$  there is a mono from  $X$  into an injective object. The following exercise elaborates on this.

- Exercise 6** a) Show that, in a topos, an object is injective if and only if it is a retract of  $\Omega^Y$  for some  $Y$ .
- b) Suppose  $\mathcal{A} \xrightarrow{G} \mathcal{B}$  be a functor with left adjoint  $\mathcal{B} \xrightarrow{F} \mathcal{A}$ . Show that if  $F$  preserves monos,  $G$  preserves injectives; and that the converse holds if  $\mathcal{A}$  has enough injectives.

In the following, let  $\mathcal{E}$  be a topos. We start by considering the category  $\mathcal{E}^{\text{op}}$ . We have a functor  $P : \mathcal{E}^{\text{op}} \rightarrow \mathcal{E}$ : on objects,  $PX = \Omega^X$  and for maps  $X \xrightarrow{f} Y$  we have  $Pf : \Omega^Y \rightarrow \Omega^X$ , the map which is the exponential transpose of the composition  $\Omega^Y \times X \xrightarrow{\text{id} \times f} \Omega^Y \times Y \xrightarrow{\text{ev}} \Omega$ .

Note that the same data define a functor  $P^* : \mathcal{E} \rightarrow \mathcal{E}^{\text{op}}$ , and we have:

**Lemma 1.11**  $P^* \dashv P$ .

**Proof.** We have natural bijections

$$\begin{aligned} \mathcal{E}^{\text{op}}(P^*X, Y) &= \mathcal{E}^{\text{op}}(\omega^X, Y) = \mathcal{E}(Y, \Omega^X) \simeq \mathcal{E}(Y \times X, \Omega) \\ &\simeq \mathcal{E}(X \times Y, \Omega) \simeq \mathcal{E}(X, \Omega^Y) = \mathcal{E}(X, PY) \end{aligned}$$

■

Hence, we have a monad  $T = PP^*$  on  $\mathcal{E}$ , and thus a comparison functor  $K : \mathcal{E}^{\text{op}} \rightarrow \mathcal{E}^T$ .

For a mono  $g : W \rightarrow Z$  we also have a map  $\exists g : \Omega^W \rightarrow \Omega^Z$ : it is the transpose of the map  $\widetilde{\exists}g : \Omega^W \times Z \rightarrow \Omega$  which classifies the mono

$$\in_W \longrightarrow \Omega^W \times W \xrightarrow{\text{id} \times g} \Omega^W \times Z$$

(here  $\in_W$  is the subobject of  $\Omega^W \times W$  classified by the evaluation map  $\text{ev}_W : \Omega^W \times W \rightarrow \Omega$ ).

We have that the square

$$\begin{array}{ccc} \in_W & \longrightarrow & \Omega^W \times W \xrightarrow{\text{id} \times g} \Omega^W \times Z \\ \downarrow & & \downarrow \widetilde{\exists}g \\ 1 & \xrightarrow{\quad t \quad} & \Omega \end{array}$$

is a pullback; hence, since  $g$  is mono, also the square

$$\begin{array}{ccc} \in_W & \longrightarrow & \Omega^W \times W \\ \downarrow & & \downarrow \text{id} \times g \\ & & \Omega^W \times Z \\ & & \downarrow \widetilde{\exists}g \\ 1 & \xrightarrow{\quad t \quad} & \Omega \end{array}$$

is a pullback. We see that  $\widetilde{\exists}g \circ (\text{id} \times g)$  classifies the mono  $\in_W \rightarrow \Omega^W \times W$ , and we conclude that  $\widetilde{\exists}g \circ (\text{id} \times g) = \text{ev}_W$ .

**Lemma 1.12 (PTJ 1.32; “Beck Condition”)** *Suppose the square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow h \\ Z & \xrightarrow{k} & T \end{array}$$

is a pullback with the arrows  $g$  and  $h$  monic. Then the following square commutes:

$$\begin{array}{ccc} \Omega^Y & \xrightarrow{Pf} & \Omega^X \\ \exists h \downarrow & & \downarrow \exists g \\ \Omega^T & \xrightarrow{Ph} & \Omega^Z \end{array}$$

**Proof.** We look at the exponential transposes of the two compositions. For the clockwise composition  $\exists g \circ Pf : \Omega^Y \rightarrow \Omega^Z$ , its transpose is the top row of

$$\begin{array}{ccccc} \Omega^Y \times Z & \xrightarrow{Pf \times \text{id}} & \Omega^X \times Z & \xrightarrow{\widetilde{\exists g}} & \Omega \\ \text{id} \times g \uparrow & & \uparrow \text{id} \times g & & \uparrow t \\ \Omega^Y \times X & \xrightarrow{Pf \times \text{id}} & \Omega^X \times X & & \\ \uparrow & & \uparrow & & \\ E & \longrightarrow & \in_X & \longrightarrow & 1 \end{array} \quad .$$

We see that this top row classifies the subobject  $E \rightarrow \Omega^Y \times X \xrightarrow{\text{id} \times g} \Omega^Y \times Z$ .

Since  $\widetilde{\exists g} \circ (\text{id} \times g) = \text{ev}_X$  as we noted just before the statement of the lemma, the subobject  $E \rightarrow \Omega^Y \times X$  is classified by the composition  $\Omega^Y \times X \xrightarrow{Pf \times \text{id}} \Omega^X \times X \xrightarrow{\text{ev}_X} \Omega$ , which equals the composition  $\Omega^Y \times X \xrightarrow{\text{id} \times f} \Omega^Y \times Y \xrightarrow{\text{ev}_Y} \Omega$  since both compositions are transposes of  $Pf$ . Therefore we have a pullback diagram

$$\begin{array}{ccc} E & \longrightarrow & \Omega^Y \times X \\ \downarrow & & \downarrow \text{id} \times f \\ \in_Y & \longrightarrow & \Omega^Y \times Y \end{array}$$

For the counterclockwise composition  $Pk \circ \exists h$ , its transpose is  $\Omega^Y \times Z \xrightarrow{\exists h \times \text{id}} \Omega^T \times Z \xrightarrow{\text{id} \times k} \Omega^T \times T \xrightarrow{\text{ev}_T} \Omega$  which equals  $\Omega^Y \times Z \xrightarrow{\text{id} \times k} \Omega^Y \times T \xrightarrow{\exists h \times \text{id}} \Omega^T \times T \xrightarrow{\text{ev}_T} \Omega$ .

Now  $\text{ev}_{T \circ (\exists h \times \text{id})}$  and  $\widetilde{\exists h} : \Omega^Y \times T \rightarrow \Omega$  both transpose to  $\exists h$ , so these maps are equal. We conclude that  $Pk \circ \exists h$  transposes to the composition



$\Omega^Y \times Z \xrightarrow{\text{id} \times k} \Omega^Y \times T \xrightarrow{\widetilde{\exists} h} \Omega$ , and we consider pullbacks

$$\begin{array}{ccccc}
E & \longrightarrow & \Omega^Y \times X & \xrightarrow{\text{id} \times g} & \Omega^Y \times Z \\
\downarrow & & \downarrow \text{id} \times f & & \downarrow \text{id} \times k \\
\in_Y & \longrightarrow & \Omega^Y \times Y & \xrightarrow{\text{id} \times h} & \Omega^Y \times T \\
\downarrow & & & & \downarrow \widetilde{\exists} h \\
1 & \longrightarrow & & & \Omega
\end{array}$$

Again, we have  $\widetilde{\exists} h \circ (\text{id} \times h) = \text{ev}_Y : \Omega^Y \times Y \rightarrow \Omega$  and we see that the counterclockwise composition transposes to a map which classifies the same subobject  $E \rightarrow \Omega^Y \times X \xrightarrow{\text{id} \times g} \Omega^Y \times Z$  as we saw for the clockwise composition.

Therefore the two compositions are equal, and the given diagram commutes. ■

**Corollary 1.13 (PTJ 1.33)** *If  $f : X \rightarrow Y$  is mono then  $Pf \circ \exists f = \text{id}_{\Omega^X}$ .*

**Proof.** Apply 1.12 to the pullback diagram

$$\begin{array}{ccc}
X & \xrightarrow{\text{id}} & X \\
\text{id} \downarrow & & \downarrow f \\
X & \xrightarrow{f} & Y
\end{array}$$

**Theorem 1.14 (PTJ 1.34)** *The functor  $P : \mathcal{E}^{\text{op}} \rightarrow \mathcal{E}$  is monadic.*

**Proof.** We use the Crude Tripleability Theorem (0.10). We need to verify its conditions:

- 1)  $\mathcal{E}^{\text{op}}$  has coequalizers of reflexive pairs.
- 2)  $P$  preserves coequalizers of reflexive pairs.
- 3)  $P$  reflects isomorphisms.

Verification of 1) is trivial, since coequalizers in  $\mathcal{E}^{\text{op}}$  are equalizers in  $\mathcal{E}$ , and  $\mathcal{E}$  has finite limits.

For 2), let  $X \xrightarrow{f} Y \xrightleftharpoons[h]{g} Z$  be a diagram in  $\mathcal{E}$  which is a coequalizer of a reflexive pair in  $\mathcal{E}^{\text{op}}$ . Since the pair  $(g, h)$  is reflexive in  $\mathcal{E}^{\text{op}}$  we have

an arrow  $Z \xrightarrow{d} Y$  satisfying  $dg = dh = \text{id}_Y$ . This means that  $g$  and  $h$  are monos, and the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{h} & Z \end{array}$$

is a pullback. We see that also  $f$  is mono, and applying 1.12 we find that  $\exists f \circ Pf = Ph \circ \exists g$ . Moreover by 1.13 we have the equalities  $Pf \circ \exists f = \text{id}_{\Omega^X}$ ,  $Pg \circ \exists g = \text{id}_{\Omega^Y}$ . Using these equalities we see that the  $P$ -image of the original coequalizer diagram:

$$\Omega^Z \xrightleftharpoons[P_h]{Pg} \Omega^Y \xrightarrow{Pf} \Omega^X$$

is a split fork in  $\mathcal{E}$ , with splittings  $\exists g : \Omega^Y \rightarrow \Omega^Z$ ,  $\exists f : \Omega^X \rightarrow \Omega^Y$ . In particular it is a coequalizer in  $\mathcal{E}$ .

For 3), we observe that for any morphism  $f : X \rightarrow Y$  in  $\mathcal{E}$ , the map  $Y \xrightarrow{\{\cdot\}} \Omega^Y \xrightarrow{Pf} \Omega^X$  transposes to the map  $Y \times X \rightarrow \Omega$  which classifies the graph of  $f$ , i.e. the subobject represented by  $\langle f, \text{id} \rangle : X \rightarrow Y \times X$ . Note that if the graphs of  $f$  and  $g : X \rightarrow Y$  coincide then  $f = g$ . Therefore,  $Pf = Pg$  implies  $f = g$  and  $P$  is faithful, hence reflects both monos and epis. By Corollary 1.3,  $P$  reflects isomorphisms. ■

**Corollary 1.15 (PTJ 1.36)** *A topos has finite colimits.*

**Proof.** For a finite diagram  $M : I \rightarrow \mathcal{E}$  consider  $M^{\text{op}} : I^{\text{op}} \rightarrow \mathcal{E}^{\text{op}}$  and compose with  $P : \mathcal{E}^{\text{op}} \rightarrow \mathcal{E}$ . The diagram  $P \circ M^{\text{op}}$  has a limit in  $\mathcal{E}$  since  $\mathcal{E}$  has finite limits. But  $P$ , being monadic, creates limits so  $M^{\text{op}}$  has a limit in  $\mathcal{E}^{\text{op}}$ ; that is,  $M$  has a limit in  $\mathcal{E}$ . ■

**Corollary 1.16 (PTJ 1.37)** *Let  $T : \mathcal{E} \rightarrow \mathcal{F}$  be a logical functor between toposes. Then the following hold:*

- i)  $T$  preserves finite limits.
- ii) If  $T$  has a left adjoint, it also has a right adjoint.

**Proof.** i) Since  $T$  is logical, the diagram

$$\begin{array}{ccc} \mathcal{E}^{\text{op}} & \xrightarrow{T^{\text{op}}} & \mathcal{F}^{\text{op}} \\ P \downarrow & & \downarrow P \\ \mathcal{E} & \xrightarrow{T} & \mathcal{F} \end{array}$$

commutes up to isomorphism. Proving that  $T$  preserves finite colimits amounts to proving that  $T^{\text{op}}$  preserves finite limits. So let  $M : I \rightarrow \mathcal{E}^{\text{op}}$  be a finite diagram, with limiting cone  $(D, \mu)$  in  $\mathcal{E}^{\text{op}}$ . Now  $T$  and  $P$  preserve finite limits, so  $TP(D, \mu)$  is a limiting cone for  $TPM$ ; hence  $PT^{\text{op}}(D, \mu)$  is a limiting cone for  $PT^{\text{op}}M$  by commutativity of the diagram. Since  $P$  creates limits,  $T^{\text{op}}(D, \mu)$  is a limiting cone for  $T^{\text{op}}M$ . We conclude that  $T^{\text{op}}$  preserves finite limits.

For ii), we employ the Adjoint Lifting Theorem (0.11) to the same diagram. The assumptions are readily verified, and we conclude that  $T^{\text{op}}$  has a left adjoint. But this means that  $T$  has a right adjoint. ■

We now discuss slice categories of toposes. In any category  $\mathcal{E}$  with finite limits, for each object  $X$  we have the category  $\mathcal{E}/X$  whose objects are arrows into  $X$  and whose arrows:  $(Y \xrightarrow{f} X) \rightarrow (Z \xrightarrow{g} X)$  are arrows  $Y \xrightarrow{h} Z$  such that  $f = gh$ . Given an arrow  $f : Y \rightarrow X$  we have a pullback functor  $f^* : \mathcal{E}/X \rightarrow \mathcal{E}/Y$ , which has a left adjoint  $\sum_f$ ;  $\sum_f(Z \xrightarrow{g} Y) = (Z \xrightarrow{fg} X)$ . In the case of the unique arrow  $X \rightarrow 1$  we write  $X^* : \mathcal{E} \cong \mathcal{E}/1 \rightarrow \mathcal{E}/X$  for the pullback functor. Note that  $X^*(Y)$  is the projection  $Y \times X \rightarrow X$ . Note also that  $X \xrightarrow{\text{id}} X$  is a terminal object of  $\mathcal{E}/X$ .

The following theorem was dubbed the “Fundamental Theorem of Topos Theory” by Peter Freyd.

**Theorem 1.17 (PTJ 1.42)** *Let  $\mathcal{E}$  be a topos and  $X$  an object of  $\mathcal{E}$ . Then  $\mathcal{E}/X$  is a topos, and the functor  $X^* : \mathcal{E} \rightarrow \mathcal{E}/X$  is logical.*

**Proof.** In the case  $\mathcal{E} = \text{Set}$ , it is useful to view objects of  $\mathcal{E}/X$  as “ $X$ -indexed families of sets” rather than as functions into  $X$ . This intuition will also guide us in the general case.

Binary products in  $\mathcal{E}/X$  are pullbacks over  $X$ : if we adopt the notation  $Y \times_X Z$  for the vertex of the pullback diagram

$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Z \\ \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

then in  $\mathcal{E}/X$ , the product  $f \times g$  is the arrow  $Y \times_X Z \rightarrow X$ . Equalizers in  $\mathcal{E}/X$  are just equalizers in  $\mathcal{E}$ . So  $\mathcal{E}/X$  has finite limits, and the functor  $X^*$  preserves finite limits since it has a left adjoint  $\sum_f$  as we remarked.

Monos in  $\mathcal{E}/X$  are monos in  $\mathcal{E}$ , and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\langle t, \text{id} \rangle} & \Omega \times X \\ & \searrow \text{id} & \downarrow \pi \\ & & X, \end{array}$$

seen as an arrow in  $\mathcal{E}/X$ , is a subobject classifier in  $\mathcal{E}/X$ . Note, that this map is  $X^*(1 \xrightarrow{t} \Omega)$ , so  $X^*$  preserves subobject classifiers.

In order to prove cartesian closure, first observe that for  $\mathcal{E} = \text{Set}$ , the exponent  $(Z \xrightarrow{g} X)^{(Y \xrightarrow{f} X)}$  is the  $X$ -indexed family  $(g^{-1}(x)^{f^{-1}(x)})_{x \in X}$ , or the projection function from the set  $\{(h, x) \mid h : f^{-1}(x) \rightarrow g^{-1}(x)\}$  to  $X$ .

We first construct the exponential  $(Z \xrightarrow{g} X)^{(Y \xrightarrow{f} X)}$ , then explain its meaning in intuitive terms (as if  $\mathcal{E}$  were the topos  $\text{Set}$ ); then we prove that it has the required universal property.

Let  $\theta : X \times Y \rightarrow \tilde{X}$  represent the partial map  $X \xleftarrow{f} Y \xrightarrow{\langle f, \text{id} \rangle} X \times Y$ , i.e. let

$$\begin{array}{ccc} Y & \xrightarrow{\langle f, \text{id} \rangle} & X \times Y \\ f \downarrow & & \downarrow \theta \\ X & \xrightarrow{\eta_X} & \tilde{X} \end{array}$$

be a pullback. Let  $\bar{\theta} : X \rightarrow \tilde{X}^Y$  be the exponential transpose of  $\theta$ . Finally, let

$$\begin{array}{ccc} E & \xrightarrow{q} & \tilde{Z}^Y \\ p \downarrow & & \downarrow \tilde{g}^Y \\ X & \xrightarrow{\bar{\theta}} & \tilde{X}^Y \end{array}$$

be a pullback; the claim is that  $E \xrightarrow{p} X$  is the required exponential.

Intuitive explanation: think of  $\tilde{X}$  as the set of subsets of  $X$  having at most one element. So  $\theta(x, y) = \{x \mid f(y) = x\}$ . The function  $\tilde{g} : \tilde{Z} \rightarrow \tilde{X}$  sends subset  $\alpha$  of  $Z$  to  $\{g(z) \mid z \in \alpha\}$ . Then, the function  $\tilde{g}^Y : \tilde{Z}^Y \rightarrow \tilde{X}^Y$  sends a function  $h : Y \rightarrow \tilde{Z}$  to the function  $y \mapsto \{g(z) \mid z \in h(y)\}$ . We have  $\bar{\theta}(x)(y) = \{x \mid f(y) = x\}$ . So the object  $E$  can be identified with the set of pairs  $(x, h)$  satisfying:

$$\begin{aligned} & x \in X, h : Y \rightarrow Z \\ & \text{dom}(h) = f^{-1}(x) \\ & \text{for all } y \in f^{-1}(x), h(y) \in g^{-1}(x). \end{aligned}$$

That is,  $E$  is isomorphic to  $\{(h, x) \mid h : f^{-1}(x) \rightarrow g^{-1}(x)\}$ .

Now we prove that the constructed  $E \xrightarrow{p} X$  has the property of the exponential  $(Z \xrightarrow{g} X)^{(Y \xrightarrow{f} X)}$ .

For an arbitrary object  $(T \xrightarrow{k} X)$  of  $\mathcal{E}/X$ , maps in  $\mathcal{E}/X$  from  $k$  to  $p$  correspond bijectively to maps  $T \xrightarrow{l} \tilde{Z}^Y$  satisfying  $\tilde{g}^Y l = \bar{\theta} k$ . These correspond to maps  $T \times Y \xrightarrow{\bar{l}} \tilde{Z}$  satisfying  $\theta(k \times \text{id}_Y) = \tilde{g} \bar{l}$ . These in turn, correspond to maps  $T \times_X Y \xrightarrow{\bar{l}} Z$  which satisfy that  $\tilde{g} \bar{l}$  is the composite  $T \times_X Y \rightarrow T \times Y \rightarrow Y \xrightarrow{f} X$ ; that is, to maps from  $T \times_X Y$  to  $Z$  making the triangle

$$\begin{array}{ccc} T \times_X Y & \xrightarrow{\quad} & Z \\ & \searrow & \swarrow g \\ & X & \end{array}$$

commute; that is, maps from  $k \times f$  to  $g$  in  $\mathcal{E}/X$ .

For the third correspondence in the chain above, suppose we have  $T \times Y \xrightarrow{\bar{l}} \tilde{Z}$  satisfying  $\theta(k \times \text{id}_Y) = \tilde{g} \bar{l}$ . Let

$$\begin{array}{ccc} W & \xrightarrow{\quad} & T \times Y \\ \downarrow & & \downarrow \bar{l} \\ Z & \xrightarrow{\eta_Z} & \tilde{Z} \end{array}$$

be a pullback. We then have a commutative diagram

$$\begin{array}{ccccccc} T & \xleftarrow{\pi} & T \times Y & \xleftarrow{\quad} & W & & \\ \downarrow k & & \downarrow k \times \text{id} & \searrow \bar{l} & \downarrow \eta_Z & \searrow & \\ & & & \tilde{Z} & \xleftarrow{\quad} & Z & \\ & & & \downarrow & \downarrow & \downarrow g & \\ X & \xleftarrow{\pi} & X \times Y & \xrightarrow{\tilde{g}} & Y & & \\ & & \searrow \theta & \downarrow & \searrow f & & \\ & & & \tilde{X} & \xleftarrow{\eta_X} & X & \end{array}$$

The front face and the top face of the cube are pullbacks, as is the bottom. Hence the back face is a pullback too. Composing the back face with the left hand square reveals that  $W$  is  $T \times_X Y$ .

The fact that  $X^*$  preserves exponentials is left as an exercise. ■

**Exercise 7** Show that  $X^*$  preserves exponentials.

**Corollary 1.18 (PTJ 1.43)** *For any arrow  $f : X \rightarrow Y$  in  $\mathcal{E}$  the pullback functor  $f^* : \mathcal{E}/Y \rightarrow \mathcal{E}/X$  is logical, and has a right adjoint  $\prod_f$ .*

**Proof.** We now know that  $\mathcal{E}/Y$  is a topos, so we can apply Theorem 1.17 with  $\mathcal{E}/Y$  in the role of  $\mathcal{E}$  and  $f$  in the role of  $X$ . We see that  $f^*$  is logical. By Corollary 1.16,  $f^*$  has a right adjoint, since it has a left adjoint  $\sum_f$ .

However, we can also exhibit the right adjoint  $\prod_f$  directly: we do this for the case  $Y = 1$ . Given an object  $(Y \xrightarrow{f} X)$  of  $\mathcal{E}/X$  let  $\lceil \text{id} \rceil : 1 \rightarrow X^X$  denote the exponential transpose of the identity arrow on  $X$ , and let

$$Z \longrightarrow Y^X \xrightleftharpoons[\lceil \text{id} \rceil \circ !]{f^X} X^X$$

be an equalizer diagram. Think of  $Z$  as the object of sections of  $f$ . Now for any object  $W$  of  $\mathcal{E}$ , arrows  $g : X^*(W) \rightarrow f$ :

$$\begin{array}{ccc} W \times X & \xrightarrow{g} & Y \\ & \searrow & \swarrow f \\ & X & \end{array}$$

correspond, via the exponential adjunction, to arrows  $\tilde{g} : W \rightarrow Y^X$  such that  $f^X \circ \tilde{g}$  factors through  $\lceil \text{id} \rceil$ ; that is to arrows  $W \rightarrow Z$ . Therefore  $Z$  is  $\prod_X(f)$ . ■

**Example 1.19** Consider the subobject classifier  $1 \xrightarrow{t} \Omega$ ; let us calculate  $\prod_t : \mathcal{E} \rightarrow \mathcal{E}/\Omega$ . For an object  $X$  of  $\mathcal{E}$  and an arrow  $Y \xrightarrow{m} \Omega$  we have that maps from  $m$  to  $\prod_t(X)$  in  $\mathcal{E}/\Omega$  correspond to maps from  $Y'$  to  $X$ , where  $Y'$  is the subobject of  $Y$  classified by  $m$ . That is, to maps  $g : Y \rightarrow \tilde{X}$  for which the domain (i.e. the map  $g^*(\eta_X) : Y' \rightarrow Y$ ) is the subobject of  $Y$  classified by  $m$ . But these correspond to maps in  $\mathcal{E}/\Omega$  from  $m$  to the arrow  $s : \tilde{X} \rightarrow \Omega$  which classifies the mono  $X \xrightarrow{\eta_X} \tilde{X}$ .

**Corollary 1.20 (PTJ 1.46)** *Every arrow  $f : X \rightarrow Y$  in  $\mathcal{E}$  induces a geometric morphism*

$$f : \mathcal{E}/X \xleftarrow[\prod_f]{f^*} \mathcal{E}/Y .$$

*This geometric morphism has the special features that the inverse image functor  $f^*$  is logical and has a left adjoint.*

**Definition 1.21** A geometric morphism  $f$  for which the inverse image functor  $f^*$  has a left adjoint is called *essential*.

Without proof, we mention the following partial converse to corollary 1.20.

**Theorem 1.22 (PTJ 1.47)** *Let  $f : \mathcal{F} \rightarrow \mathcal{E}$  be an essential geometric morphism such that  $f^*$  is logical and its left adjoint  $f_!$  preserves equalizers. Then there is an object  $X$  of  $\mathcal{E}$ , unique up to isomorphism, such that  $\mathcal{F}$  is equivalent to  $\mathcal{E}/X$  and, modulo this equivalence, the geometric morphism  $f$  is isomorphic to the geometric morphism  $(X^* \dashv \prod_X)$  of Corollary 1.20.*

We recall from the *Category Theory and Topos Theory* course that a *regular category* is a category with finite limits, which has coequalizers of kernel pairs, and in which regular epimorphisms are stable under pullback. Recall that in such a category, every arrow factors, essentially uniquely, as a regular epimorphism followed by a monomorphism. The construction is as follows: given  $f : X \rightarrow Y$ , let  $X \xrightarrow{e} E$  be the coequalizer of the kernel pair of  $f$ , and let  $m : E \rightarrow Y$  be the unique factorization of  $f$  through this coequalizer.

Since pullback functors have right adjoints, they preserve regular epimorphisms, so every topos is a regular category.

**Lemma 1.23 (PTJ 1.53)** *In a topos, every epi is regular.*

**Proof.** Given an epi  $f : X \rightarrow Y$ , let  $X \xrightarrow{e} E \xrightarrow{m} Y$  be its regular epi-mono factorization. Since  $f$  is epi,  $m$  must be epi; by 1.3,  $m$  is an isomorphism. So  $f$  is regular epi. ■

**Definition 1.24** An *exact* category is a regular category in which every equivalence relation is effective.

By 1.5 we have:

**Proposition 1.25** *Every topos is an exact category.*

**Proposition 1.26 (PTJ 1.56)** *In a topos the initial object  $0$  is strict; that is, every arrow into  $0$  is an isomorphism.*

**Proof.** Given  $X \xrightarrow{i} 0$ , we have a pullback

$$\begin{array}{ccc} X & \xrightarrow{i} & 0 \\ \text{id}_X \downarrow & & \downarrow \text{id}_0 \\ X & \xrightarrow{i} & 0 \end{array}$$

so  $\text{id}_X = i^*(\text{id}_0)$ . Now  $\text{id}_0$  is initial in  $\mathcal{E}/0$ , so  $\text{id}_X$  is initial in  $\mathcal{E}/X$  (since  $i^*$ , having a right adjoint, preserves initial objects). But that means that  $X$  is initial in  $\mathcal{E}$ , since for any object  $Y$  of  $\mathcal{E}$  there is a bijection between arrows  $X \rightarrow Y$  in  $\mathcal{E}$ , and arrows  $\text{id}_X \rightarrow X^*(Y)$  in  $\mathcal{E}/X$ . ■

**Corollary 1.27 (PTJ 1.57)** *In a topos, every coprojection  $X \rightarrow X + Y$  is monic. Moreover, “coproducts are disjoint”: that is, the square*

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X + Y \end{array}$$

*is a pullback.*

**Proof.** From Proposition 1.26 it follows easily that every map  $0 \rightarrow X$  is monic. Since the given square is always a pushout, the statement follows at once from Corollary 1.9. ■

**Exercise 8** Prove that for a topos  $\mathcal{E}$  and objects  $X, Y$  of  $\mathcal{E}$  the categories  $\mathcal{E}/(X + Y)$  and  $\mathcal{E}/X \times \mathcal{E}/Y$  are equivalent.

As a consequence of regularity (and existence of coproducts) we can form *unions of subobjects*: given subobjects  $M, N$  of  $X$ , represented by monos  $M \xrightarrow{m} X, N \xrightarrow{n} X$ , its union  $M \cup N$  (least upper bound in the poset  $\text{Sub}(X)$ ) is defined by the regular epi-mono factorization

$$M + N \rightarrow M \cup N \rightarrow X$$

of the map  $\begin{bmatrix} m \\ n \end{bmatrix} : M + N \rightarrow X$ . We have:

**Proposition 1.28** *In a topos, for any object  $X$  the poset  $\text{Sub}(X)$  of subobjects of  $X$  is a distributive lattice. Moreover, for any arrow  $X \xrightarrow{f} Y$  the pullback functor  $f^* : \text{Sub}(Y) \rightarrow \text{Sub}(X)$  between subobject lattices has both adjoints  $\exists_f$  and  $\forall_f$ .*

**Proof.** Finite meets in  $\text{Sub}(X)$  (from now on called “intersections” of subobjects) are given by pullbacks, and unions by the construction above. Distributivity follows from the fact that pullback functors preserve coproducts and regular epimorphisms. The left adjoint  $\exists_f$  is constructed using regular epi-mono factorization as in the course *Category Theory and Topos Theory*. The right adjoint  $\forall_f$  is just the restriction of  $\prod_f$  to subobjects:  $\prod_f$  preserves monos. ■

The following fact will be important later on.



**Proposition 1.29 (Elephant, A1.4.3)** *Let  $M \xrightarrow{m} X, N \xrightarrow{n} X$  be monos into  $X$  (we also write  $M, N$  for the subobjects represented by  $m$  and  $n$ ). Let the intersection and union of  $M$  and  $N$  be represented by arrows  $M \cap N \rightarrow X, M \cup N \rightarrow X$ , respectively. Then the diagram*

$$\begin{array}{ccc} M \cap N & \longrightarrow & M \\ \downarrow & & \downarrow \\ N & \longrightarrow & M \cup N \end{array}$$

*is both a pullback and a pushout in  $\mathcal{E}$ .*

**Proof.** This proof is not the proof given in **Elephant**.

The partial order  $\text{Sub}(X)$  is, as a category, equivalent to the full subcategory  $\text{Mon}/X$  of the slice  $\mathcal{E}/X$  on the monomorphisms into  $X$ . Since the given square is a pullback in  $\text{Sub}(X)$  hence in  $\text{Mon}/X$ , and the domain functor  $\text{Mon}/X \rightarrow \mathcal{E}$  preserves pullbacks, the square is a pullback in  $\mathcal{E}$ .

Let us define  $\text{Sub}_{\leq 1}(X)$  as the set of those subobjects  $M \xrightarrow{m} X$  for which the unique map  $M \rightarrow 1$  is a monomorphism. Note that there is a natural bijection between  $\text{Sub}_{\leq 1}(X)$  and  $\mathcal{E}(1, \tilde{X})$ , where  $\tilde{X}$  is the partial map classifier of  $X$ . Writing  $M$  both for a subobject of  $X$  and for the corresponding map  $1 \rightarrow \tilde{X}$ , we define the subobject  $\text{dom}(M)$  of  $1$  by the pullback

$$\begin{array}{ccc} \text{dom}(M) & \longrightarrow & 1 \\ \downarrow & & \downarrow M \\ X & \xrightarrow{\eta_X} & \tilde{X} \end{array}$$

Note, that  $\text{dom}(M)$  is also the image of the map  $M \rightarrow 1$ . For a subobject  $c$  of  $1$ , we define  $M \upharpoonright c$  by the pullback

$$\begin{array}{ccc} M \upharpoonright c & \longrightarrow & M \\ \downarrow & & \downarrow \\ c & \longrightarrow & 1 \end{array}$$

We have the following lemma.

**Lemma 1.30** *Let  $M, N \in \text{Sub}_{\leq 1}(X)$ , with  $\text{dom}(M) = c, \text{dom}(N) = d$ . If  $M \upharpoonright (c \cap d) = N \upharpoonright (c \cap d)$  as subobjects of  $X$ , then  $M \cup N \in \text{Sub}_{\leq 1}(X)$ .*

**Proof.** We must prove that the map  $\phi : M \cup N \rightarrow 1$  is monic. Clearly, this map factors through  $c \cup d$ , so it is enough to prove that  $(c \cup d)^*(\phi)$  is monic in  $\mathcal{E}/(c \cup d)$ .

We have  $c^*(M \cup N) = c^*(M) \cup c^*(N)$ . Since  $c^*(N)$  has domain  $c^*(d) = c \cap d$  and  $M$  and  $N$  agree on  $c \cap d$ , we have  $c^*(N) \leq c^*(M)$ , so  $c^*(M \cup N) = c^*(M)$  and  $c^*(\phi)$  is monic. In a symmetric way,  $d^*(M \cup N) = d^*(N)$  and  $d^*(\phi)$  is monic.

The topos  $\mathcal{E}/(c + d)$  is isomorphic to  $\mathcal{E}/c \times \mathcal{E}/d$  by Exercise 8, so we see that  $(c + d)^*(\phi)$  is monic. Now  $c + d \rightarrow c \cup d$  is epi, so the pullback functor  $\mathcal{E}/(c \cup d) \rightarrow \mathcal{E}/(c + d)$  reflects monomorphisms. We conclude that  $(c \cup d)^*(\phi)$  is monic, as required. This proves the lemma. ■

Continuing the proof of Proposition 1.29: as usual, we may do as if  $X = 1$ . So we have subobjects  $c, d$  of 1 and we wish to prove that the square

$$\begin{array}{ccc} c \cap d & \longrightarrow & c \\ \downarrow & & \downarrow \\ d & \longrightarrow & c \cup d \end{array}$$

is a pushout. Let  $M : c \rightarrow X$ ,  $N : d \rightarrow X$  be maps which agree on  $c \cap d$ . Then  $M$  and  $N$  define elements of  $\text{Sub}_{\leq 1}(X)$  for which the hypothesis of Lemma 1.30 holds. Therefore, the map  $c \cup d \rightarrow X$  which names the subobject  $M \cup N$  is a mediating map, which is unique because the maps  $\{c \rightarrow c \cup d, d \rightarrow c \cup d\}$  form an epimorphic family. ■

## 2 Geometric Morphisms

This section contains material from the book *Sheaves in Geometry and Logic* by MacLane and Moerdijk; hereafter referred to by “MM”.

We recall that a *geometric morphism*  $\mathcal{F} \rightarrow \mathcal{E}$  between toposes is an adjoint pair  $f^* \dashv f_*$  with  $f^* : \mathcal{E} \rightarrow \mathcal{F}$  (the *inverse image functor*),  $f_* : \mathcal{F} \rightarrow \mathcal{E}$  (the *direct image functor*), with the additional property that  $f^*$  preserves finite limits.

A geometric morphism  $\text{Set} \rightarrow \mathcal{E}$  is called a *point* of  $\mathcal{E}$ .

**Examples 2.1** 1) In the Introduction we have seen that every continuous function of topological spaces  $f : X \rightarrow Y$  determines a geometric morphism  $\text{Sh}(X) \rightarrow \text{Sh}(Y)$ . If the space  $Y$  is sufficiently separated (here we shall assume that  $Y$  is Hausdorff, although the weaker condition of *sober* suffices) then there is a converse to this: every geometric morphism  $\text{Sh}(X) \rightarrow \text{Sh}(Y)$  is induced by a unique continuous function. Indeed, let  $f$  be such a geometric morphism. In  $\text{Sh}(Y)$ , the lattice of subobjects of 1 (the terminal object) is in 1-1, order-preserving, bijection with  $\mathcal{O}(Y)$ , the set of open subsets of  $Y$ . The same for  $X$ , of course. Now the inverse image  $f^*$ , preserving finite limits, preserves subobjects of 1 and therefore induces a function  $f^- : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ . Since  $f^*$  preserves colimits and finite limits, the function  $f^-$  preserves the top element ( $f^-(Y) = X$ ), finite intersections and arbitrary unions (in particular,  $f^-(\emptyset) = \emptyset$ ).

Define a relation  $R$  from  $X$  to  $Y$  as follows:  $R(x, y)$  holds if and only if  $x \in f^-(V)$  for every open neighbourhood  $V$  of  $y$ . We shall show that  $R$  is in fact a function  $X \rightarrow Y$ , leaving the remaining details as an exercise.

- i) Assume  $R(x, y)$  and  $R(x, y')$  both hold, and  $y \neq y'$ . By the Hausdorff property,  $y$  and  $y'$  have disjoint open neighbourhoods  $V_y$  and  $V_{y'}$ . By assumption and the preservation properties of  $f^-$  we have:

$$x \in f^-(V_y) \cap f^-(V_{y'}) = f^-(V_y \cap V_{y'}) = f^-(\emptyset) = \emptyset$$

a clear contradiction. So the relation  $R$  is single-valued.

- ii) Suppose for  $x \in X$  there is no  $y \in Y$  satisfying  $R(x, y)$ . Then for every  $y$  there is a neighbourhood  $V_y$  such that  $x \notin f^-(V_y)$ . Then we have

$$x \notin \bigcup_{y \in Y} f^-(V_y) = f^-(\bigcup_{y \in Y} V_y) = f^-(Y) = X$$

also a clear contradiction. So the relation  $R$  is total, and therefore a function.

**Exercise 9** Show that the function  $R$  just constructed is continuous, and that it induces the given geometric morphism  $f$ .

- 2) Consider, for a group  $G$ , the category  $\widehat{G}$  of right  $G$ -sets. Let  $\Delta : \text{Set} \rightarrow \widehat{G}$  be the functor which sends a set  $X$  to the trivial  $G$ -set  $X$  (i.e. the  $G$ -action is the identity). Note that  $\Delta$  preserves finite limits. The functor  $\Delta$  has a right adjoint  $\Gamma$ , which sends a  $G$ -set  $X$  to its subset of  $G$ -invariant elements, i.e. to the set

$$\{x \in X \mid xg = x \text{ for all } g \in G\}$$

Note that  $\widehat{G}(\Delta(Y), X)$  is naturally isomorphic to  $\text{Set}(Y, \Gamma(X))$ , so we have a geometric morphism  $\widehat{G} \rightarrow \text{Set}$ . Actually, this geometric morphism is essential, because  $\Delta$  also has a left adjoint: we have that  $\widehat{G}(X, \Delta(Y))$  is naturally isomorphic to  $\text{Set}(\text{Orb}(X), Y)$ , where  $\text{Orb}(X)$  denotes the set of orbits of  $X$  under the  $G$ -action.

**Exercise 10** Prove that the functor  $\text{Orb}$  does not preserve equalizers (Hint: you can do this directly (think of two maps  $G \rightarrow G$ ), or apply Theorem 1.22).

This example can be generalized in two directions, as the following items show.

- 3) Let  $\mathcal{E}$  be a cocomplete topos. Then there is exactly one geometric morphism  $\mathcal{E} \rightarrow \text{Set}$ , up to natural isomorphism. For, a geometric morphism is determined by its inverse image functor, which must preserve 1 and coproducts; and since, in  $\text{Set}$ , every object  $X$  is the coproduct of  $X$  copies of 1, for  $f : \mathcal{E} \rightarrow \text{Set}$  we must have  $f^*(X) = \sum_{x \in X} 1$ . For a function  $\phi : X \rightarrow Y$  we have  $[\mu_{\phi(x)}]_{x \in X} : \sum_{x \in X} 1 \rightarrow \sum_{y \in Y} 1$  (where  $\mu_i$  sends 1 to the  $i$ 'th cofactor of the coproduct  $\sum_{y \in Y} 1$ ) which is  $f^*(\phi) : f^*(X) \rightarrow f^*(Y)$ . This defines  $f^* : \text{Set} \rightarrow \mathcal{E}$ .

**Exercise 11** Show that the functor  $f^*$  preserves finite limits.

The functor  $f^*$  has a right adjoint: for a set  $X$  and object  $Y$  of  $\mathcal{E}$  we have

$$\mathcal{E}(f^*(X), Y) \simeq \mathcal{E}\left(\sum_{x \in X} 1, Y\right) \simeq \prod_{x \in X} \mathcal{E}(1, Y) \simeq \text{Set}(X, \mathcal{E}(1, Y))$$

so the functor which sends  $Y$  to its set of *global sections* (arrows  $1 \rightarrow Y$ ) is right adjoint to  $f^*$ . The “global sections functor” is usually denoted by the letter  $\Gamma$ ; its left adjoint by  $\Delta$ .

- 4) Consider presheaf categories  $\widehat{\mathcal{C}}, \widehat{\mathcal{D}}$ , and let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. We have a geometric morphism  $\widehat{F} : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$  constructed as follows. We have a functor  $\widehat{F}^* : \widehat{\mathcal{D}} \rightarrow \widehat{\mathcal{C}}$  which sends a presheaf  $X : \mathcal{D}^{\text{op}} \rightarrow \text{Set}$  to  $X \circ F^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ . In other words,

$$\widehat{F}^*(X)(C) = X(F(C))$$

**Exercise 12** Prove that the functor  $\widehat{F}^*$  preserves all small limits.

A right adjoint  $\widehat{F}_*$  for  $\widehat{F}^*$  may be constructed using the Yoneda Lemma. Indeed, for  $\widehat{F}_*$  to exist, it should satisfy:

$$\widehat{F}_*(Y)(D) \simeq \widehat{\mathcal{D}}(y_D, \widehat{F}_*(Y)) \simeq \widehat{\mathcal{C}}(\widehat{F}^*(y_D), Y)$$

so we just *define*  $\widehat{F}_*$  on objects by putting  $\widehat{F}_*(Y)(D) = \widehat{\mathcal{C}}(\widehat{F}^*(y_D), Y)$ .

**Exercise 13** Complete the definition of  $\widehat{F}_*$  as a functor, and show that it is indeed a right adjoint for  $\widehat{F}^*$ .

The functor  $\widehat{F}^* : \widehat{\mathcal{D}} \rightarrow \widehat{\mathcal{C}}$  has also a left adjoint (so the geometric morphism  $\widehat{F}$  is essential). Recall that for a presheaf  $X$  on  $\mathcal{C}$  we have the *category of elements of  $X$* , denoted  $\text{Els}(X)$ : objects are pairs  $(x, C)$  with  $x \in X(C)$ , and arrows  $(x, C) \rightarrow (x', C')$  are arrows  $f : C \rightarrow C'$  in  $\mathcal{C}$  satisfying  $X(f)(x') = x$ . We have the projection functor  $\pi : \text{Els}(X) \rightarrow \mathcal{C}$ . Define the functor  $\widehat{F}_! : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$  as follows: for  $X \in \widehat{\mathcal{C}}$ ,  $\widehat{F}_!(X)$  is the colimit in  $\widehat{\mathcal{D}}$  of the diagram

$$\text{Els}(X) \xrightarrow{\pi} \mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{y} \widehat{\mathcal{D}}$$

We shall shortly see a more concrete presentation of functors of such “left Kan extensions”.

- 5) In the course *Basic Category Theory and Topos Theory* we have seen that if  $\text{Cov}$  is a Grothendieck topology on a small category  $\mathcal{C}$ , then the category  $\text{Sh}(\mathcal{C}, \text{Cov})$  of sheaves for  $\text{Cov}$  is a topos, and the inclusion functor  $\text{Sh}(\mathcal{C}, \text{Cov}) \rightarrow \widehat{\mathcal{C}}$  has a left adjoint (*sheafification*) which preserves finite limits; so this is also an example of a geometric morphism. Henceforth we shall denote a Grothendieck topology by  $J$  instead of  $\text{Cov}$ .

## 2.1 Points of $\widehat{\mathcal{C}}$

We recall from the course *Basic Category Theory and Topos Theory* that the functor  $y : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$  is the “free cocompletion of  $\mathcal{C}$ ”. That means the following: given an arbitrary functor  $F$  from  $\mathcal{C}$  to a cocomplete category  $\mathcal{E}$  there is a unique (up to natural isomorphism) colimit-preserving functor  $\tilde{F} : \widehat{\mathcal{C}} \rightarrow \mathcal{E}$  such that the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ & \searrow y & \uparrow \tilde{F} \\ & & \widehat{\mathcal{C}} \end{array}$$

commutes up to isomorphism. The functor  $\tilde{F}$  is called the “left Kan extension of  $F$  along  $y$ ”.

Of course,  $\tilde{F}(X)$  can be defined as the colimit in  $\mathcal{E}$  of the diagram  $\text{Elts}(X) \xrightarrow{\pi} \mathcal{C} \xrightarrow{F} \mathcal{E}$ . We wish to present this colimit as a form of “tensor product”. Let us review the definition from Commutative Algebra.

If  $R$  is a commutative ring and  $M, N$  are  $R$ -modules, the set  $\text{Hom}_R(M, N)$  of  $R$ -module homomorphisms from  $M$  to  $N$  is also an  $R$ -module (with point-wise operations), and the functor  $\text{Hom}_R(M, -) : R\text{-Mod} \rightarrow R\text{-Mod}$  has a left adjoint  $(-) \otimes_R M$ . For an  $R$ -module  $L$  we define an equivalence relation  $\sim$  on the set  $L \times M$ : it is the least equivalence relation satisfying

$$(x, y \cdot r) \sim (x \cdot r, y)$$

for all  $x \in L, y \in M, r \in R$ . The equivalence class of  $(x, y)$  is denoted  $x \otimes y$ , and  $L \otimes M$  is the  $R$ -module generated by all such elements  $x \otimes y$ , subject to the relations

$$((x + x') \otimes y) = x \otimes y + x' \otimes y \quad x \otimes (y + y') = x \otimes y + x \otimes y'$$

and with  $R$ -action  $(x \otimes y)r = (xr \otimes y) = (x \otimes ry)$ . In fact, one has a coequalizer diagram of abelian groups:

$$L \times R \times M \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\psi} \end{array} L \times M \longrightarrow L \otimes M$$

where  $\phi(x, r, y) = (xr, y)$  and  $\psi(x, r, y) = (x, ry)$ . The  $R$ -module  $M$  is called *flat* if the functor  $(-) \otimes M$  preserves exact sequences; given that this functor is a left adjoint, this is equivalent to saying that it preserves finite limits.

Something similar happens if we have a functor  $A : \mathcal{C} \rightarrow \mathbf{Set}$  and a presheaf  $X$  on  $\mathcal{C}$  and we wish to calculate the value of the left Kan extension  $\tilde{A}$  on  $X$ . Let  $\mathcal{C}_1$  be the set of arrows of  $\mathcal{C}$ . On  $\mathbb{A} = \sum_{C \in \mathcal{C}} A(C)$  there is a (partial) “left  $\mathcal{C}_1$ -action”  $x \mapsto f \cdot x = A(f)(x)$ , for  $x \in A(C)$  and  $f : C \rightarrow C'$ . Similarly, on  $\mathbb{X} = \sum_{C \in \mathcal{C}} X(C)$  there is a partial “right  $\mathcal{C}_1$ -action”  $x \mapsto x \cdot f = X(f)(x)$ , for  $x \in X(C')$  and  $f : C \rightarrow C'$ . We can now represent the set  $\tilde{A}(X)$  as a coequalizer of sets

$$\sum_{C, C' \in \mathcal{C}} X(C') \times \mathcal{C}(C, C') \times A(C) \xrightleftharpoons[\psi]{\phi} \sum_{C, C' \in \mathcal{C}} X(C) \times A(C) \longrightarrow \tilde{A}(X)$$

where  $\phi(x, f, a) = (x \cdot f, a)$  and  $\psi(x, f, a) = (x, f \cdot a)$ . Therefore we write, from now on,  $X \otimes_{\mathcal{C}} A$  for  $\tilde{A}(X)$ .

**Theorem 2.2 (MM VII.2.2)** *Let  $A : \mathcal{C} \rightarrow \mathbf{Set}$  be a functor. Then we have an adjunction*

$$\mathbf{Set} \xrightleftharpoons[R]{L} \hat{\mathcal{C}}$$

with  $L \dashv R$ ,  $R(Y)(C) = \mathbf{Set}(A(C), Y)$  and  $L(X) = X \otimes_{\mathcal{C}} A$ .

Now for geometric morphisms  $\mathbf{Set} \rightarrow \hat{\mathcal{C}}$  we need the left adjoint  $(-) \otimes_{\mathcal{C}} A$  to preserve finite limits.

**Definition 2.3 (MM VII.5.1)** A functor  $A : \mathcal{C} \rightarrow \mathbf{Set}$  is called *flat* if the functor  $(-) \otimes_{\mathcal{C}} A$  preserves finite limits.

The following theorem summarizes our remarks so far.

**Theorem 2.4 (MM VII.5.2)** *Points of the presheaf topos  $\hat{\mathcal{C}}$  correspond to flat functors  $\mathcal{C} \rightarrow \mathbf{Set}$ .*

**Definition 2.5** A category  $I$  is called *filtering* if the following conditions are satisfied:

- i)  $I$  is nonempty.
- ii) For each pair of objects  $(i, j)$  of  $I$  there is a diagram  $i \leftarrow k \rightarrow j$  in  $I$ .
- iii) For each parallel pair  $i \xrightarrow[a]{b} j$  there is an arrow  $k \xrightarrow{c} i$  which equalizes the pair.

Now let  $A : \mathcal{C} \rightarrow \mathbf{Set}$ . We have the category  $\mathbf{Elts}(A)$ : objects are pairs  $(x, C)$  with  $x \in A(C)$ ; an arrow  $(x, C) \rightarrow (x', C')$  is a morphism  $f : C \rightarrow C'$  in  $\mathcal{C}$  such that  $A(f)(x) = x'$ .

**Definition 2.6** A functor  $A : \mathcal{C} \rightarrow \mathbf{Set}$  is called *filtering* if the category  $\mathbf{Elts}(A)$  is filtering.

**Exercise 14** Let  $P$  be a poset and  $A : P \rightarrow \mathbf{Set}$  a filtering functor. Show that the category  $\mathbf{Elts}(A)$  is isomorphic to a *filter* in  $P$ , that is: a nonempty subset  $F \subseteq P$  with the following properties:

- i) The set  $F$  is upwards closed: if  $p \leq q$  and  $p \in F$ , then  $q \in F$ .
- ii) Any two elements of  $F$  have a common lower bound in  $F$ .

The following theorem provides a concrete handle on flat functors.

**Theorem 2.7 (MM VII.6.3)** A functor  $A : \mathcal{C} \rightarrow \mathbf{Set}$  is flat if and only if  $A$  is filtering.

**Proof.** Assume that  $A : \mathcal{C} \rightarrow \mathbf{Set}$  is flat. By definition, the following diagram commutes up to isomorphism:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{A} & \mathbf{Set} \\ & \searrow y_C & \nearrow (-) \otimes_{\mathcal{C}} A \\ & \hat{\mathcal{C}} & \end{array}$$

So,  $y_C \otimes_{\mathcal{C}} A \simeq A(C)$ , for objects  $C$  of  $\mathcal{C}$ . We check the conditions for a filtering category.

- i) Since  $(-) \otimes_{\mathcal{C}} A$  preserves terminal objects,  $1 \otimes_{\mathcal{C}} A$  is a one-point set. This shows that  $A$  is nonempty.
- ii) Since  $(-) \otimes_{\mathcal{C}} A$  preserves binary products, we have that the map

$$(y_C \times y_D) \otimes_{\mathcal{C}} A \rightarrow A(C) \times A(D)((B \xrightarrow{u} C, B \xrightarrow{v} D), a) \mapsto (u \cdot a, v \cdot a)$$

must be an isomorphism; in particular it is surjective. That is condition ii) of the definition of a filtering functor.



iii) Finally, consider a parallel pair  $C \xrightarrow[u]{u} D$  in  $\mathcal{C}$  and an element  $a \in A(C)$  such that  $u \cdot a = v \cdot a$  (that is, a parallel pair in  $\text{Els}(A)$ ). Let

$$P \longrightarrow y_C \xrightarrow[y_v]{y_u} y_D$$

be an equalizer diagram in  $\widehat{\mathcal{C}}$ . Since  $(-) \otimes_{\mathcal{C}} A$  preserves equalizers, we have an equalizer diagram

$$P \otimes_{\mathcal{C}} A \xrightarrow{i} A(C) \xrightarrow[A(v)]{A(u)} A(D)$$

in  $\text{Set}$ . Here, for  $w \in P(B), b \in A(B)$ ,  $i(w \otimes b) = w \cdot b \in A(C)$ . Since  $u \cdot a = v \cdot a$ , there must be some pair  $(w, b)$  for which  $i(w \otimes b) = a$ . This gives condition iii) of the definition of a filtering functor.

For the converse, only a sketch: suppose  $A$  is filtering. Now for  $R \in \widehat{\mathcal{C}}$ , the set  $R \otimes_{\mathcal{C}} A$  is a quotient of the sum  $\sum_{C \in \mathcal{C}} R(C) \times A(C)$  by the equivalence relation  $\sim$  generated by the set of equivalent pairs  $((r \cdot g, a), (r, g \cdot a))$  for  $r \in R(C), a \in A(C')$  and  $g : C' \rightarrow C$ . However, given that  $A$  is filtering this can be simplified. We have:  $(r, a) \in R(C) \times A(C)$  is equivalent to  $(r', a') \in R(C') \times A(C')$  if and only if there is a diagram  $C \xleftarrow{u} D \xrightarrow{v} C'$  in  $\mathcal{C}$  and an element  $b \in A(D)$  such that the equations

$$u \cdot b = a \quad v \cdot b = a' \quad r \cdot u = r \cdot v$$

hold. From this definition, it is straightforward to prove that  $(-) \otimes_{\mathcal{C}} A$  preserves finite limits. ■

**Corollary 2.8 (MM VII.6.4)** *Suppose  $\mathcal{C}$  is a category with finite limits. Then a functor  $A : \mathcal{C} \rightarrow \text{Set}$  is flat if and only if it preserves finite limits.*

**Proof.** Again we use that the composite functor  $((-) \otimes_{\mathcal{C}} A) \circ y : \mathcal{C} \rightarrow \text{Set}$  is naturally isomorphic to  $A$ . If  $A$  is flat, then  $(-) \otimes_{\mathcal{C}} A$  preserves finite limits and  $y$  always preserves existing finite limits, so then  $A$  preserves all finite limits. Note, that this direction does not require  $\mathcal{C}$  to have all finite limits.

Conversely, suppose  $\mathcal{C}$  has finite limits and  $A$  preserves them. Then  $A$  is filtering:

i)  $A(1) = 1$ , so  $A$  is nonempty.

- ii) We have  $A(C) \times A(D) \simeq A(C \times D)$  so in condition ii) of Definition 2.5 we can take the projections  $C \xleftarrow{\pi_C} C \times D \xrightarrow{\pi_D} D$  and appropriate element of  $A(C \times D)$ .
- iii) By a similar argument, now involving an equalizer in  $\mathcal{C}$ .

■

**Corollary 2.9 (MM VII.6.5)** *Let  $\mathcal{D}$  be a small category. Then the colimit functor  $\text{Set}^{\mathcal{D}} \rightarrow \text{Set}$  preserves finite limits if and only if  $\mathcal{D}^{\text{op}}$  is filtering.*

**Remark 2.10** In standard text books in category theory, for example MacLane, one finds a dual definition of “filtering” (i.e., a category is “filtering” in MacLane’s sense if its opposite category is filtering in our sense). For this notion of filtering, part of Corollary 2.9 is contained in the slogan that “filtered colimits commute with finite limits in  $\text{Set}$ ”.

**Exercise 15** Deduce Corollary 2.9.

## 2.2 Geometric Morphisms $\mathcal{E} \rightarrow \widehat{\mathcal{C}}$ for cocomplete $\mathcal{E}$

The universal property of the Yoneda embedding  $y : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$  ( $\widehat{\mathcal{C}}$  being the free cocompletion of  $\mathcal{C}$ ) holds with respect to all cocomplete categories, not just  $\text{Set}$ . Therefore, every geometric morphism  $f : \mathcal{E} \rightarrow \widehat{\mathcal{C}}$  is determined by the composite functor  $f^* \circ y : \mathcal{C} \rightarrow \mathcal{E}$ . Again, we have a suitably defined “tensor product”  $X \otimes_{\mathcal{C}} A$  (when  $A : \mathcal{C} \rightarrow \mathcal{E}$  is a functor and  $X \in \widehat{\mathcal{C}}$ ), which is now defined as a colimit in  $\mathcal{E}$  rather than in  $\text{Set}$ .

We cannot write down exactly the same formula for what will be the functor  $(-) \otimes_{\mathcal{C}} A$  as we did for the case of  $\text{Set}$ , as something like “ $X(C') \times \mathcal{C}(C, C') \times A(C)$ ” is not meaningful:  $X(C')$  and  $\mathcal{C}(C, C')$  are sets but  $A(C)$  is an object of  $\mathcal{E}$ . However, using the cocompleteness of  $\mathcal{E}$  we have the expression  $\sum_{x \in X(C'), f: C \rightarrow C'} A(C')$  which, in the case of  $\mathcal{E} = \text{Set}$ , is the same thing. Let, for a coproduct  $\sum_{i \in I} X_i$ ,  $\mu_i : X_i \rightarrow \sum_{i \in I} X_i$  denote the  $i$ ’th coprojection. Then we define  $X \otimes_{\mathcal{C}} A$  as the coequalizer

$$\sum_{C \in \mathcal{C}, x \in X(C), f: C' \rightarrow C} A(C') \xrightarrow[\tau]{\theta} \sum_{C \in \mathcal{C}, x \in X(C)} A(C) \longrightarrow X \otimes_{\mathcal{C}} A$$

where  $\theta = [\theta_{C,x,f}]_{C \in \mathcal{C}, x \in X(C), f: C' \rightarrow C}$ ; and  $\theta_{C,x,f}$  is defined to be the composite

$$A(C') \xrightarrow{A(f)} A(C) \xrightarrow{\mu_{C,x}} \sum_{C \in \mathcal{C}, x \in X(C)} A(C).$$

Likewise,  $\tau = [\tau_{C,x,f}]_{C \in \mathcal{C}, x \in X(C), f: C' \rightarrow C}$  where  $\tau_{C,x,f}$  is the map

$$A(C) \xrightarrow{\mu_{C',x,f}} \sum_{C \in \mathcal{C}, x \in X(C)} A(C).$$

Again, we define the functor  $A : \mathcal{C} \rightarrow \mathcal{E}$  to be *flat* if the functor  $(-) \otimes_{\mathcal{C}} A : \widehat{\mathcal{C}} \rightarrow \mathcal{E}$  preserves finite limits. And we have a similar notion of filtering as in 2.6:

**Definition 2.11 (MM VII.8.1)** A functor  $A : \mathcal{C} \rightarrow \mathcal{E}$  is *filtering* if the following conditions hold:

- i) The family of all maps  $A(C) \rightarrow 1$  is epimorphic.
- ii) For objects  $C, D$  of  $\mathcal{C}$ , the family of maps

$$\{\langle A(u), A(v) \rangle : A(B) \rightarrow A(C) \times A(D) \mid u : B \rightarrow C, v : B \rightarrow D\}$$

is epimorphic.

- iii) For any parallel pair of arrows  $u, v : C \rightarrow D$  in  $\mathcal{C}$  and equalizer diagram

$$E_{u,v} \xrightarrow{e} A(C) \xrightleftharpoons[A(v)]{A(u)} A(D)$$

in  $\mathcal{E}$ , the family of all arrows

$$\{A(B) \xrightarrow{f} E_{u,v} \mid \text{for some } w : B \rightarrow C \text{ in } \mathcal{C} \text{ with } uw = vw, ef = A(w)\}$$

is epimorphic.

Without proof, we record:

**Theorem 2.12 (MM VII.9.1)** *Let  $\mathcal{E}$  be a cocomplete topos, and  $\mathcal{C}$  a small category. Then a functor  $A : \mathcal{C} \rightarrow \mathcal{E}$  is flat if and only if it is filtering.*

We see that geometric morphisms  $\mathcal{E} \rightarrow \widehat{\mathcal{C}}$  correspond to filtering functors  $\mathcal{C} \rightarrow \mathcal{E}$ , for cocomplete  $\mathcal{E}$ .

### 2.3 Geometric morphisms to $\mathcal{E} \rightarrow \text{Sh}(\mathcal{C}, J)$ for cocomplete $\mathcal{E}$

Recall that we use the letter  $J$  to denote a general Grothendieck topology; so  $J(C)$  is a collection of covering sieves on  $C$  (where  $C$  is an object of  $\mathcal{C}$ ). Also recall that a sieve on  $C$  can be regarded as a subobject of the representable presheaf  $y_C$ . Finally, we established in the basic course that an object  $X$  of  $\widehat{\mathcal{C}}$  is a sheaf for  $J$ , if and only if for every object  $C$  of  $\mathcal{C}$  and every  $J$ -covering sieve  $R$  on  $C$ , any diagram

$$\begin{array}{ccc} R & \longrightarrow & y_C \\ \downarrow & & \\ X & & \end{array}$$

has a *unique* filler: an arrow  $y_C \rightarrow X$  making the triangle commute. For the remainder of this section,  $\mathcal{E}$  will always be a *cocomplete* topos.

**Exercise 16** Let  $i : \text{Sh}(\mathcal{C}, J) \rightarrow \widehat{\mathcal{C}}$  the geometric morphism where  $i_*$  is the inclusion and  $i^*$  is sheafification. Suppose  $p : \mathcal{E} \rightarrow \widehat{\mathcal{C}}$  is a geometric morphism such that the direct image  $p_*$  factors through  $i_*$  by a functor  $q : \mathcal{E} \rightarrow \text{Sh}(\mathcal{C}, J)$ . Show that the composite  $p^*i_*$  is left adjoint to  $q$  and conclude that the inverse image  $p^*$  is isomorphic to a functor which factors through  $\text{Sh}(\mathcal{C}, J)$ .

Exercise 16 tells us that a geometric morphism  $p : \mathcal{E} \rightarrow \widehat{\mathcal{C}}$  factors through  $\text{Sh}(\mathcal{C}, J)$  if and only if every object  $p_*(E)$  is a sheaf for  $J$ . The following exercise gives us a criterion for when this is the case.

**Exercise 17** Let  $p : \mathcal{E} \rightarrow \widehat{\mathcal{C}}$  be a geometric morphism, and let  $J$  be a Grothendieck topology on  $\mathcal{C}$ . Then the following two statements are equivalent:

- i) For every object  $E$  of  $\mathcal{E}$ ,  $p_*E$  is a sheaf for  $J$ .
- ii) For every  $J$ -covering sieve  $R$  on  $C$ ,  $p^*$  sends the inclusion  $R \rightarrow y_C$  to an isomorphism in  $\mathcal{E}$ .

Now we characterized geometric morphisms  $\mathcal{E} \rightarrow \widehat{\mathcal{C}}$  by flat functors  $\mathcal{C} \rightarrow \mathcal{E}$ ; so we would like to characterize also geometric morphisms  $p : \mathcal{E} \rightarrow \text{Sh}(\mathcal{C}, J)$  in terms of such functors. Every such geometric morphism determines a geometric morphism into  $\widehat{\mathcal{C}}$ , hence a flat functor  $A : \mathcal{C} \rightarrow \mathcal{E}$ ; we need to see which flat functors give rise to geometric morphisms which factor through  $\text{Sh}(\mathcal{C}, J)$ . It should not be a surprise that we can characterize these functors by their behaviour on covering sieves, now seen as diagrams in  $\mathcal{C}$ : every sieve on  $C$  is a diagram of arrows with codomain  $C$ .

**Lemma 2.13 (MM VII.7.3)** *Let  $J$  be a Grothendieck topology on a small category  $\mathcal{C}$ , and let  $f : \mathcal{E} \rightarrow \widehat{\mathcal{C}}$  be a geometric morphism. Then the following statements are equivalent:*

- i) *The geometric morphism  $f$  factors through  $\text{Sh}(\mathcal{C}, J)$ .*
- ii) *The composite  $f^* \circ y : \mathcal{C} \rightarrow \mathcal{E}$  sends  $J$ -covering sieves to colimiting cocones in  $\mathcal{E}$ .*
- iii) *The composite  $f^* \circ y$  sends  $J$ -covering sieves to epimorphic families in  $\mathcal{E}$ .*

**Definition 2.14** A functor  $A : \mathcal{C} \rightarrow \mathcal{E}$  is called *continuous* if it has the properties of the composite  $f^* \circ y$  in Lemma 2.13.

We can now state:

**Theorem 2.15 (MM, Corollary VII.7.4)** *There is an equivalence of categories between*

$$\text{Top}(\mathcal{E}, \text{Sh}(\mathcal{C}, J))$$

*and the category of flat and continuous functors  $\mathcal{C} \rightarrow \mathcal{E}$ .*

## 2.4 Surjections and the Topos of Coalgebras

**Theorem 2.16 (MM V.8.4; PTJ 2.32)** *Let  $(G, \delta, \varepsilon)$  be a comonad on a topos  $\mathcal{E}$  such that the functor  $G$  preserves finite limits. Then the category  $\mathcal{E}_G$  of  $G$ -coalgebras is a topos, and there is a geometric morphism*

$$\mathcal{E} \xrightleftharpoons[f_*]{f^*} \mathcal{E}_G$$

*where  $f^*$  is the forgetful functor and  $f_*$  the cofree coalgebra functor.*

**Proof.** Finite limits are created by  $V$  the forgetful functor  $\mathcal{E}_G \rightarrow \mathcal{E}$ , since  $G$  preserves finite limits; so  $\mathcal{E}_G$  has finite limits.

Let  $R : \mathcal{E} \rightarrow \mathcal{E}_G$  be the cofree coalgebra functor:  $RX = GX \xrightarrow{\delta_X} G^2X$ . For coalgebras  $(A, s), (B, t), (C, u)$  we have:

$$\mathcal{E}(A \times B, C) \simeq \mathcal{E}(A, C^B) \simeq \mathcal{E}_G((A, s), R(C^B))$$

where  $f : A \times B \rightarrow C$  corresponds to  $\tilde{f} : A \rightarrow C^B$  and to  $f' = G(\tilde{f}) \circ s : A \rightarrow G(C^B)$ . Note that  $f = \text{ev} \circ (\tilde{f} \times \text{id})$ .

Now  $f : A \times B \rightarrow C$  is a coalgebra map if and only if the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow{s \times t} & GA \times GB \xrightarrow{\sim} G(A \times B) \\ f \downarrow & & \downarrow G(f) \\ C & \xrightarrow{u} & GC \end{array}$$

We consider the exponential transposes of both compositions in this diagram. The clockwise composition transposes to

$$(*) \quad A \xrightarrow{f'} G(C^B) \xrightarrow{\rho} GC^{GB} \xrightarrow{GC^t} GC^B$$

where  $\rho$  is the transpose of the map  $G(C^B) \times GB \xrightarrow{\sim} G(C^B \times B) \xrightarrow{G(\text{ev})} GC$ .

The counterclockwise composition transposes to

$$(**) \quad A \xrightarrow{\tilde{f}} C^B \xrightarrow{u^B} GC^B$$

We wish to describe those maps  $f : A \times B \rightarrow C$  which make these two transposes equal. Let  $V : \mathcal{E}_G \rightarrow \mathcal{E}$  be the forgetful functor and  $C$  the cofree coalgebra functor; we have  $V \dashv C$  and  $VC = G$ . Under this adjunction, the map  $(*)$  corresponds to the composite

$$A \xrightarrow{f'} G(C^B) \xrightarrow{\delta} G^2(C^B) \xrightarrow{G\rho} G(GC^{GB}) \xrightarrow{G(GC^t)} G(GC^B)$$

and the map  $(**)$  corresponds to the composite

$$A \xrightarrow{f'} G(C^B) \xrightarrow{G(u^B)} G(GC^B)$$

Note that both these composites are maps of coalgebras. So, the maps  $f : A \times B \rightarrow C$  we are looking for, correspond to maps  $\tilde{f} : A \rightarrow E$ , where

$$\begin{array}{ccccc} & & G^2(C^B) & \xrightarrow{G\rho} & G(GC^{GB}) \\ & \nearrow \delta & & & \searrow G(GC^t) \\ E & \longrightarrow & G(C^B) & \xrightarrow{G(u^B)} & G(GC^B) \end{array}$$

is an equalizer in  $\mathcal{E}_G$  (equalizer of two maps between cofree coalgebras). So  $E$  is the exponent  $(C, u)^{(B, t)}$  in  $\mathcal{E}_G$ .

It remains to show that  $\mathcal{E}_G$  has a subobject classifier. To this end we have a look at subobjects of  $(A, s)$  in  $\mathcal{E}_G$ . Our first remark is that if  $m : D \rightarrow A$  is a subobject of  $A$  in  $\mathcal{E}$ , there is at most one coalgebra structure  $d : D \rightarrow GD$  on  $D$  such that  $m$  is a coalgebra map. Indeed, for  $m$  to be a coalgebra map we should have  $G(m)d = sm$ ; now  $G(m)$  is mono, so there is at most one such  $d$ .

On the other hand, if  $m : D \rightarrow A$  is a subobject and  $d : D \rightarrow GD$  is any map such that  $G(m)d = sm$ , then  $(D, d)$  is a  $G$ -coalgebra and the square

$$\begin{array}{ccc} D & \xrightarrow{d} & GD \\ m \downarrow & & \downarrow Gm \\ A & \xrightarrow{s} & GA \end{array}$$

is a pullback in  $\mathcal{E}$ . To see this, consider

$$\begin{array}{ccccc} D & \xrightarrow{\quad d \quad} & GD & & \\ \downarrow m & & \swarrow Gm & & \downarrow Gd \\ & A \xrightarrow{s} GA & & & \\ \downarrow s & & \downarrow Gs & & \\ & GA \xrightarrow{\delta_A} G^2A & & & \\ \swarrow Gm & & \nwarrow G^2m & & \\ GD & \xrightarrow{\quad \delta_D \quad} & G^2D & & \end{array}$$

The inner square commutes since  $(A, s)$  is a coalgebra. The three upper squares commute because of the assumption  $G(m)d = sm$ , and the lower square is a naturality square for  $\delta$ . Hence the outer square commutes, which says that the map  $d$  is coassociative. To see that  $d$  is also counitary, consider the diagram

$$\begin{array}{ccccc} D & \xrightarrow{d} & GD & \xrightarrow{\varepsilon_D} & D \\ m \downarrow & & \downarrow Gm & & \downarrow m \\ A & \xrightarrow{s} & GA & \xrightarrow{\varepsilon_A} & A \end{array}$$

Since  $m(\varepsilon_D d) = m$  and  $m$  is mono,  $\varepsilon_D d = \text{id}_D$ . Moreover, one sees that the left hand square is a pullback.

Now suppose  $m : (D, d) \rightarrow (A, s)$  is the inclusion of a subobject in  $\mathcal{E}_G$ . Let  $\tau : G(\Omega) \rightarrow \Omega$  be the classifying map of the mono  $1 \simeq G(1) \xrightarrow{G(t)} G(\Omega)$ .

Let  $h : A \rightarrow \Omega$  be the classifying map of  $m$ . In the diagram

$$\begin{array}{ccccccc}
D & \xrightarrow{d} & GD & \longrightarrow & 1 & \longrightarrow & 1 \\
\downarrow m & & \downarrow Gm & & \downarrow G(t) & & \downarrow t \\
A & \xrightarrow{s} & GA & \xrightarrow{G(h)} & G(\Omega) & \xrightarrow{\tau} & \Omega
\end{array}$$

all three squares are pullbacks (check!), and therefore  $\tau G(h)s = h$  by uniqueness of the classifying map. Moreover, since  $(A, s)$  is a coalgebra we have  $G(h)s = G(\tau)\delta_\Omega G(h)s$ , so if we form an equalizer

$$\Omega_G \xrightarrow{e} G(\Omega) \xrightleftharpoons[\text{id}]{G(\tau)\delta_\Omega} G(\Omega)$$

(equalizer taken in  $\mathcal{E}_G$ , the two maps seen as maps between cofree coalgebras), then we see that the map  $G(h)s$  factors through  $\Omega_G$ . Also the map  $G(t) : 1 \rightarrow G(\Omega)$  factors through this equalizer by a map  $e : 1 \rightarrow \Omega_G$ , which is the subobject classifier of  $\mathcal{E}_G$ . ■

**Corollary 2.17 (MM V.7.7)** *If  $(T, \eta, \mu)$  is a monad on a topos  $\mathcal{E}$  and the functor  $T$  has a right adjoint, then the category of  $T$ -algebras is again a topos.*

**Proof.** Combine Theorems 0.15 and 2.16. ■

To give an example, consider a monoid  $M$ : a set with an associative multiplication, for which it has a two-sided unit element. The functor  $(-) \times M : \text{Set} \rightarrow \text{Set}$  has the structure of a monad (using the multiplication and the unit element of  $M$ ). The category of algebras for this monad is the category of right  $M$ -sets, i.e. the category  $\widehat{M}$ . Note that the functor  $(-) \times M$  has a right adjoint  $(-)^M$ , so we have another proof that  $\widehat{M}$  is a topos.

The construction of the topos  $\mathcal{E}_G$  and its accompanying geometric morphism  $\mathcal{E} \rightarrow \mathcal{E}_G$  (the inverse image part of which is the forgetful functor, which is faithful) motivates the following definition.

**Definition 2.18** A geometric morphism  $f : \mathcal{F} \rightarrow \mathcal{E}$  is called a *surjection* if the inverse image functor  $f^*$  is faithful.

**Lemma 2.19 (MM VII.4.3)** *For a geometric morphism  $f : \mathcal{F} \rightarrow \mathcal{E}$  the following are equivalent:*

- i) *The inverse image  $f^*$  is faithful.*



- ii) Every component of the unit  $\eta$  of the adjunction  $f^* \dashv f_*$  is a monomorphism.
- iii) The functor  $f^*$  reflects isomorphisms.
- iv) The functor  $f^*$  induces an injective homomorphism of lattices  $\text{Sub}_{\mathcal{E}}(E) \rightarrow \text{Sub}_{\mathcal{F}}(f^*E)$ .
- v) The functor  $f^*$  reflects the order on subobjects: for  $A, B \in \text{Sub}_{\mathcal{E}}(E)$ ,  $f^*A \leq f^*B$  if and only if  $A \leq B$ .

**Proof.** The equivalence (i) $\Leftrightarrow$ (ii) is basic Category Theory.

For (i) $\Rightarrow$ (iii): a faithful functor reflects monos and epis, and a topos is balanced (1.3).

For (iii) $\Rightarrow$ (iv): Since  $f^*$  preserves monos, it induces a map on subobjects. Furthermore  $f^*$  preserves images and coproducts, hence unions of subobjects; also,  $f^*$  preserves intersections. So  $f^*$  induces a lattice homomorphism. Since  $f^*$  reflects isomorphisms, it is injective.

For (iv) $\Rightarrow$ (v): If  $f^*A \leq f^*B$  then  $f^*A = f^*A \cap f^*B = f^*(A \cap B)$  because  $f^*$  is a lattice homomorphism. Hence  $A = A \cap B$  since  $f^*$  is injective; so  $A \leq B$ .

For (v) $\Rightarrow$ (i): if  $X \xrightarrow[u]{u} Y$  is a parallel pair with equalizer  $E \xrightarrow{e} X$ , then  $f^*(u) = f^*(v)$  entails (since  $f^*$  preserves equalizers) that  $f^*(E)$  is the maximal subobject of  $f^*X$ . By (v), this entails that  $E$  is the maximal subobject of  $X$ ; in other words,  $u = v$ . So  $f^*$  is faithful.  $\blacksquare$

**Proposition 2.20 (MM VII.4.4)** *A geometric morphism  $f : \mathcal{F} \rightarrow \mathcal{E}$  is a surjection if and only if  $\mathcal{E}$  is equivalent to the topos of coalgebras for a finite limit preserving comonad on  $\mathcal{F}$  and  $f$  is, modulo this equivalence, the cofree-forgetful geometric morphism.*

**Proof.** One direction is clear, since the forgetful functor is always faithful. For the other, suppose  $f$  is a surjection and consider the comonad  $f^*f_*$  on  $\mathcal{F}$ . Let us spell out the dual version of Beck's Crude Tripleability Theorem (0.10):

CTT<sup>op</sup> Let  $A \xleftarrow[F]{U} C$  be an adjunction with  $F \dashv U$ . Suppose  $C$  has equalizers of coreflexive pairs,  $F$  preserves them and  $F$  reflects isomorphisms. Then the functor  $F$  is comonadic.

It is clear that for a surjection  $f$ , the conditions are satisfied. The conclusion follows.  $\blacksquare$

- Examples 2.21** 1) For a continuous map  $f$  of topological  $T_1$ -spaces, the induced geometric morphism is a surjection if and only if the map  $f$  is surjective (**MM**, start of §VII.4).
- 2) For a morphism  $f : A \rightarrow B$  in a topos  $\mathcal{E}$ , the induced geometric morphism  $\mathcal{E}/A \rightarrow \mathcal{E}/B$  is a surjection if and only if  $f$  is an epimorphism.
- 3) For a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between small categories, the induced geometric morphism  $\widehat{F} : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$  of Example 2.1 4) is a surjection if and only if every object of  $\mathcal{D}$  is a retract of an object in the image of  $F$  (**Elephant**, A4.2.7).

## 2.5 Embeddings and Sheaf Subtoposes

In this section we work again in an arbitrary (not necessarily cocomplete) topos  $\mathcal{E}$ . First we establish an internalization of the intersection  $(\cap)$  operation on subobjects.

**Proposition 2.22** *Let  $1 \xrightarrow{t} \Omega$  be a subobject classifier and denote by  $\wedge : \Omega \times \Omega \rightarrow \Omega$  the classifying map of the monomorphism  $1 \xrightarrow{\langle t, t \rangle} \Omega \times \Omega$ . Then for subobjects  $M, N$  of  $X$  we have: if  $M$  is classified by  $\phi : X \rightarrow \Omega$  and  $N$  by  $\psi : X \rightarrow \Omega$  then the intersection  $M \cap N$  is classified by the composite*

$$X \xrightarrow{\langle \phi, \psi \rangle} \Omega \times \Omega \xrightarrow{\wedge} \Omega.$$

**Proof.** Consider maps  $f : Y \rightarrow X$ . If  $\langle \phi, \psi \rangle \circ f : Y \rightarrow \Omega \times \Omega$  is equal to  $\langle to!, to! \rangle : Y \rightarrow \Omega \times \Omega$ , then  $\phi f = t!$  and  $\psi f = t!$ , so  $f$  factors both through  $M$  and through  $N$ , hence  $f$  factors through the intersection  $M \cap N$ . We conclude that the diagram

$$\begin{array}{ccc} M \wedge N & \longrightarrow & X \\ \downarrow & & \downarrow \langle \phi, \psi \rangle \\ 1 & \xrightarrow{\langle t, t \rangle} & \Omega \times \Omega \end{array}$$

is a pullback, and the statement follows.  $\blacksquare$

**Definition 2.23** A *Lawvere-Tierney topology* (**MM**) or simply *topology* (**PTJ**) in a topos  $\mathcal{E}$  is an arrow  $j : \Omega \rightarrow \Omega$  with the following properties:

$$\text{i) } jt = t : \quad \begin{array}{ccc} 1 & \xrightarrow{t} & \Omega \\ & \searrow t & \downarrow j \\ & & \Omega \end{array}$$

$$\text{ii) } jj = j : \quad \begin{array}{ccc} \Omega & \xrightarrow{j} & \Omega \\ & \searrow j & \downarrow j \\ & & \Omega \end{array}$$

$$\text{iii) } j \circ \wedge = \wedge \circ (j \times j) : \quad \begin{array}{ccccc} \Omega \times \Omega & \xrightarrow{\wedge} & \Omega & & \\ j \times j \downarrow & & \downarrow j & & \\ \Omega \times \Omega & \xrightarrow{\wedge} & \Omega & & \end{array}$$

In the course *Basic Category Theory and Topos Theory* we have seen that for  $\mathcal{E} = \widehat{\mathcal{C}}$ , Lawvere-Tierney topologies correspond to Grothendieck topologies on  $\mathcal{C}$ .

**Definition 2.24 (PTJ 3.13)** A *universal closure operation* on a topos  $\mathcal{E}$  is given by, for each object  $X$ , a map  $c_X : \text{Sub}(X) \rightarrow \text{Sub}(X)$ , which system has the following properties:

- i)  $M \leq c_X(M)$  for every subobject  $M$  of  $X$  (the operation is *inflationary*).
- ii)  $M \leq N$  implies  $c_X(M) \leq c_X(N)$  for  $M, N \in \text{Sub}(X)$  (the operation is order-preserving).
- iii)  $c_X(c_X(M)) = c_X(M)$  for each  $M \in \text{Sub}(X)$  (the operation is idempotent).
- iv) For every arrow  $f : Y \rightarrow X$  and every  $M \in \text{Sub}(X)$  we have

$$c_Y(f^*(M)) = f^*(c_X(M))$$

(the operation is *stable*).

Instead of  $c_X(M)$  we shall also sometimes write  $\overline{M}$ , if the subobject lattice in which we work is clear.

**Exercise 18** Use the stability (requirement iv) of 2.24) to deduce that a closure operation commutes with finite intersections:  $\overline{M \cap N} = \overline{M} \cap \overline{N}$ .

Note that the result of Exercise 18 means that a universal closure operation is different from “closure” in Topology, where closure commutes with *union*, not with intersection of subsets.

**Proposition 2.25 (MM V.1.1; PTJ 3.14)** *There is a bijection between universal closure operations and Lawvere-Tierney topologies.*

**Proof.** If  $j$  is a Lawvere-Tierney topology, define for  $M \in \text{Sub}(X)$ , classified by  $\phi : X \rightarrow \Omega$ ,  $\overline{M}$  as the subobject of  $X$  classified by  $j\phi$ . We use the letter  $J$  to denote the subobject of  $\Omega$  classified by  $j$ :

$$\begin{array}{ccc} J & \longrightarrow & \Omega \\ \downarrow & & \downarrow j \\ 1 & \xrightarrow{t} & \Omega \end{array}$$

We see that  $J$  is the closure of the subobject  $(1 \xrightarrow{t} \Omega)$ . We have:  $\overline{M}$  is the vertex of the pullback

$$\begin{array}{ccc} \overline{M} & \longrightarrow & X \\ \downarrow & & \downarrow \phi \\ J & \longrightarrow & \Omega \end{array}$$

and we conclude that  $M \leq \overline{M}$ . The other properties of the universal closure operation are straightforward and left to you.

In the other direction, given a universal closure operation  $c_X(-)$ , let  $j$  be the classifying map of  $c_\Omega(1 \xrightarrow{t} \Omega)$ . The verification of the properties of a Lawvere-Tierney topology, as well as that the two described operations are inverse to each other, is again left to you. ■

**Definition 2.26** Given a Lawvere-Tierney topology  $j$  with associated closure operation  $c_X(-)$  (or  $\overline{(-)}$ ), we call a subobject  $M$  of  $X$ :

*dense* if  $\overline{M} = X$

*closed* if  $\overline{M} = M$ .

**Definition 2.27** Consider, for an object  $X$ , partial maps into  $X$  with domain a dense subobject:

$$\begin{array}{ccc} M' & \xrightarrow{m} & M \\ & \searrow & \\ & & X \end{array}$$

with  $m : M' \rightarrow M$  a dense mono (i.e., the subobject represented by the mono  $m$  is dense).

The object  $X$  is called *separated for  $j$*  if any such partial map has *at most one* extension to a map  $M \rightarrow X$ .

The object  $X$  is called a *sheaf for  $j$*  (or a  *$j$ -sheaf*) if any such partial map has *exactly one* extension to a map  $M \rightarrow X$ .

We write  $\text{Sh}_j(\mathcal{E})$  for the full subcategory of  $\mathcal{E}$  on the sheaves for  $j$ .

**Theorem 2.28 (MM V.2.5; PTJ §3.2)** *For any topos  $\mathcal{E}$  with Lawvere-Tierney topology  $j$ , the category  $\text{Sh}_j(\mathcal{E})$  is a topos. The inclusion functor  $\text{Sh}_j(\mathcal{E}) \rightarrow \mathcal{E}$  preserves finite limits and exponentials, and  $\text{Sh}_j(\mathcal{E})$  is closed under finite limits in  $\mathcal{E}$ .*

**Proof.** Suppose  $\mathcal{I}$  is a finite category and  $X : \mathcal{I} \rightarrow \text{Sh}_j(\mathcal{E})$  a functor with limiting cone  $(N, \mu)$  in  $\mathcal{E}$ . Given a diagram

$$\begin{array}{ccc} M' & \xrightarrow{m} & M \\ & \searrow & \\ & & N \end{array}$$

we have partial maps  $M \rightarrow X(i)$  for all objects  $i$  of  $\mathcal{I}$ , and these partial maps have unique extensions  $M \rightarrow X(i)$  since the  $X(i)$  are sheaves. Therefore we have a cone for  $X$  with vertex  $M$  and hence a unique map of cones  $M \rightarrow N$ , which is also the unique extension of the given partial map. Therefore,  $N$  is a sheaf and we see that  $\text{Sh}_j(\mathcal{E})$  is closed under the finite limits of  $\mathcal{E}$ , that it has finite limits and that the inclusion preserves them.

Secondly, if  $F$  is a sheaf, then the exponential  $F^Y$  is a sheaf, for any object  $Y$ . For, given a partial map

$$\begin{array}{ccc} M' & \xrightarrow{m} & M \\ & \searrow & \\ & & F^Y \end{array}$$

with  $m$  dense, this diagram transposes under the exponential adjunction to a partial map

$$\begin{array}{ccc} M' \times Y & \xrightarrow{m \times \text{id}} & M \times Y \\ & \searrow & \\ & & F \end{array}$$

Now by the stability of the closure operation, the subobject  $M' \times Y \xrightarrow{m \times \text{id}} M \times Y$  is dense. Since  $F$  is a sheaf we have a unique extension  $M \times Y \rightarrow F$ , which transposes back to give a unique extension for the original diagram. We conclude that  $\text{Sh}_j(\mathcal{E})$  is cartesian closed and that the inclusion into  $\mathcal{E}$  preserves exponentials.

For the subobject classifier of  $\text{Sh}_j(\mathcal{E})$  we need an intermediate result, which we have already seen in the case  $\mathcal{E} = \widehat{\mathcal{C}}$ .

**Lemma 2.29** *Let  $M$  be a sheaf and  $M'$  a subobject of  $M$ . Then  $M'$  is a sheaf if and only if  $M'$  is closed in  $M$ .*

**Proof.** Suppose  $M'$  is closed in  $M$  and  $M' \xleftarrow{f} N' \longrightarrow N$  is a partial map with  $N'$  dense in  $N$ . Let  $i : M' \rightarrow M$  be the inclusion. Now  $i \circ f$  has a unique extension  $g : N \rightarrow M$ . Let

$$\begin{array}{ccc} L & \longrightarrow & N \\ \downarrow & & \downarrow g \\ M' & \longrightarrow & M \end{array}$$

be a pullback. Then  $f : N' \rightarrow M'$  factors through  $L \rightarrow M'$ , so  $N' \leq L$  as subobjects of  $N$ , but  $L$  is closed (since it is a pullback of  $M' \rightarrow M$ ) and  $N'$  is dense. We see that  $N = \overline{N'} \leq \overline{L} = L$ , so  $L \rightarrow N$  is an isomorphism and we have  $g : N \rightarrow M'$ . So  $M'$  is a sheaf.

Conversely if  $M' \in \text{Sub}(M)$  is a sheaf, consider the partial map

$$\begin{array}{ccc} M' & \longrightarrow & \overline{M'} \\ \text{id} \downarrow & & \\ M' & & \end{array}$$

Since  $M' \rightarrow \overline{M'}$  is dense, there is a unique extension  $\overline{M'} \rightarrow M'$ . It follows that  $M' = \overline{M'}$ , so  $M'$  is closed in  $M$ . ■

Returning to the proof of 2.28: closed subobjects of  $X$  are classified by maps of the form  $j\phi$ , hence their classifying maps land in the image of  $j$ , which is (by the idempotence of  $j$ ) the equalizer

$$\Omega_j \longrightarrow \Omega \rightrightarrows \Omega$$

$\text{id} \quad j$

Hence,  $\Omega_j$  is a subobject classifier for  $\text{Sh}_j(\mathcal{E})$  provided we can show that it is a sheaf.

Now partial maps  $\Omega_j \longleftarrow M' \longrightarrow M$  correspond to closed subobjects of  $M'$ . But given that  $M'$  is dense in  $M$ , there is an order-preserving bijection between the closed subobjects of  $M'$  and of  $M$ , given as follows: for  $A$  closed in  $M$ , we have  $A \cap M'$  closed in  $M'$  and for  $B$  closed in  $M'$  we have  $c_M(B)$  closed in  $M$ . To see that these operations are each other's inverse, observe that for  $A$  closed in  $M$ :

$$c_M(A \cap M') = c_M(A) \cap c_M(M') = c_M(A) = A$$

and for  $B$  closed in  $M'$  we have

$$c_M(B) \cap M' = c_{M'}(B) = B$$

The given partial map has therefore a unique extension  $M \rightarrow \Omega_j$  (the classifier of the closed subobject of  $M$  corresponding to the closed subobject of  $M'$  classified by the partial map); and  $\Omega_j$  is a sheaf, as desired.  $\blacksquare$

**Proposition 2.30** *For an object  $X$  of  $\mathcal{E}$  the following are equivalent:*

- i)  $X$  is  $j$ -separated.
- ii)  $X$  is a subobject of a  $j$ -sheaf.
- iii)  $X$  is a subobject of a sheaf of the form  $\Omega_j^E$ .
- iv) The diagonal  $\delta : X \rightarrow X \times X$  is a  $j$ -closed subobject of  $X \times X$ .

**Proof.** We prove i) $\Rightarrow$ iv) $\Rightarrow$ iii) $\Rightarrow$ ii) $\Rightarrow$ i).

For i) $\Rightarrow$ iv): let  $X$  be separated and let  $\bar{\delta}$  be the closure of  $\delta$  as subobject of  $X \times X$ . Consider the partial map

$$\begin{array}{ccc} X & \longrightarrow & \bar{\delta} \\ \text{id} \downarrow & & \\ X & & \end{array}$$

If  $i : \bar{\delta} \rightarrow X \times X$  is the inclusion and  $p_1, p_2 : X \times X \rightarrow X$  are the projections, then both  $p_1 i$  and  $p_2 i$  are fillers for this diagram, so since  $X$  is separated,  $p_1 i = p_2 i$ . This means that  $i : \bar{\delta} \rightarrow X \times X$  factors through the equalizer of  $p_1$  and  $p_2$ , which is  $\delta$ . So  $\bar{\delta} = \delta$  as subobjects of  $X \times X$ .

For iv) $\Rightarrow$ iii): Let  $\Delta : X \times X \rightarrow \Omega$  classify the diagonal  $\delta$ , and  $\{\cdot\} : X \rightarrow \Omega^X$  its exponential transpose, which is a monomorphism. Since  $\delta$  is closed in  $X \times X$ ,  $\Delta$  factors through  $\Omega_j$ , and therefore  $\{\cdot\}$  factors through  $\Omega_j^X$ . So  $X$  is a subobject of  $\Omega_j^X$ , which is a  $j$ -sheaf.

The implication iii)  $\Rightarrow$  ii) is trivial.

For ii)  $\Rightarrow$  i): Let  $X \xrightarrow{i} F$  be mono, with  $F$  a  $j$ -sheaf. Suppose that

$$\begin{array}{ccc} M' & \xrightarrow{m} & M \\ \downarrow k & & \\ X & & \end{array}$$

is a partial map with  $m$  a dense mono. If both of  $f, g : M \rightarrow X$  are fillers for this diagram then  $if = ig$  since  $F$  is a sheaf; hence  $f = g$  since  $i$  is mono. So  $X$  is  $j$ -separated.  $\blacksquare$

**Lemma 2.31** *Let  $j$  be a Lawvere-Tierney topology in a topos  $\mathcal{E}$ , and let  $X$  be an object of  $\mathcal{E}$ . As usual, we denote the diagonal subobject of  $X \times X$  by  $\delta$  and its closure by  $\bar{\delta}$ .*

- a) *If  $f, g : Z \rightarrow X$  is a parallel pair of arrows into  $X$ , then the morphism  $\langle f, g \rangle : Z \rightarrow X \times X$  factors through  $\bar{\delta}$  if and only if the equalizer of  $f$  and  $g$  is a  $j$ -dense subobject of  $Z$ .*
- b) *The subobject  $\bar{\delta}$  of  $X \times X$  is an equivalence relation on  $X$ .*
- c) *Let  $X \rightarrow MX$  be the coequalizer of the pair  $\bar{\delta} \rightrightarrows X$ . Then any map  $X \rightarrow L$ , for a  $j$ -separated object  $L$  of  $\mathcal{E}$ , factors uniquely through  $X \rightarrow MX$ . Hence the assignment  $X \mapsto MX$  induces a functor which is left adjoint to the inclusion  $\text{sep}_j(\mathcal{E}) \rightarrow \mathcal{E}$ , where  $\text{sep}_j(\mathcal{E})$  denotes the full subcategory of  $\mathcal{E}$  on the  $j$ -separated objects.*

**Proof.** a) Let  $E_{fg} \rightarrow Z$  denote the equalizer of  $f, g$ . Consider the diagram:

$$\begin{array}{ccccc} & & E' & & \\ & \nearrow & \downarrow & \nwarrow & \\ E_{fg} & \xrightarrow{\quad} & Z & & \\ \downarrow & & \downarrow \bar{\delta} & & \downarrow \langle f, g \rangle \\ X & \xrightarrow{\quad \delta \quad} & X \times X & & \end{array}$$

where all the squares are pullbacks. We see that  $E'$  is the closure of  $E_{fg}$ , and we see that the map  $\langle f, g \rangle$  factors through  $\bar{\delta}$  if and only if  $E' \rightarrow Z$  is an isomorphism, which holds if and only if  $E_{fg}$  is a dense subobject of  $Z$ .



b) We prove that for an arbitrary object  $Z$  of  $\mathcal{E}$ , the set of ordered pairs

$$\{(f, g) \in \mathcal{E}(Z, X)^2 \mid \langle f, g \rangle \text{ factors through } \bar{\delta}\}$$

is an equivalence relation on  $\mathcal{E}(Z, X)$ . Now reflexivity and symmetry are obvious, and using the notation above for equalizers we easily see that  $E_{fg} \wedge E_{gh} \leq E_{fh}$ . Since the meet of two dense subobjects is dense, we see that the relation is transitive.

c) We have to prove that any map  $f : X \rightarrow L$  with  $L$  separated, co-equalizes the parallel pair  $r_0, r_1 : \bar{\delta} \rightarrow X$  which is the equivalence relation from part b). Now clearly for  $f \times f : X \times X \rightarrow L \times L$ , the composite  $(f \times f) \circ \delta$  factors through the diagonal subobject  $L \xrightarrow{\delta_L} L \times L$ , so the composite  $(f \times f) \circ \langle r_0, r_1 \rangle$  factors through the closure of  $\delta_L$ . But  $\delta_L$  is closed by Proposition 2.30iv), so  $fr_0 = fr_1$  and  $f$  factors uniquely through  $X \rightarrow MX$ . The adjointness is also clear, provided we can show that  $MX$  is separated. Now  $\delta$  is classified by  $\Delta : X \times X \rightarrow \Omega$ , which has as exponential transpose the map  $\{\cdot\} : X \rightarrow \Omega^X$ . So,  $\delta$  is the kernel pair of  $\{\cdot\}$ . Now  $\bar{\delta}$  is classified by  $j \circ \Delta$ , the exponential transpose of which is  $j^X \circ \{\cdot\} : X \rightarrow \Omega_j^X$ . And  $\bar{\delta}$  is the kernel pair of  $j^X \circ \{\cdot\}$ . We see that, by the construction of epi-mono factorizations in a regular category,  $X \rightarrow MX \rightarrow \Omega_j^X$  is an epi-mono factorization. So  $MX$  is a subobject of a sheaf, and therefore separated by 2.30. ■

**Lemma 2.32** *Suppose we have an operation which, to any object  $X$  of  $\mathcal{E}$ , assigns a sheaf  $LX$  and a dense inclusion  $MX \xrightarrow{i_X} LX$ . Then this extends to a unique functor  $L : \mathcal{E} \rightarrow \mathcal{E}$ . Moreover, this functor has the property that for every  $X$ , every map from  $X$  to a sheaf factors uniquely through the composite  $X \rightarrow MX \xrightarrow{i_X} LX$ , so  $L : \mathcal{E} \rightarrow \text{Sh}_j(\mathcal{E})$  is left adjoint to the inclusion of sheaves.*

**Proof.** For  $f : X \rightarrow X'$ , define  $Lf : LX \rightarrow LX'$  as the unique filler for the partial map

$$\begin{array}{ccc} MX & \xrightarrow{i_X} & LX \\ i_{X'} \circ Mf \downarrow & & \\ LX' & & . \end{array}$$

The functoriality and the adjointness follow at once. ■

**Theorem 2.33** *The inclusion functor  $\text{Sh}_j(\mathcal{E}) \rightarrow \mathcal{E}$  has a left adjoint which preserves finite limits. Hence, we have a geometric morphism  $i : \text{Sh}_j(\mathcal{E}) \rightarrow \mathcal{E}$ .*

**Proof.** Let, as before,  $\Delta : X \times X \rightarrow \Omega$  classify the diagonal  $\delta : X \rightarrow X \times X$ . Then  $j \circ \Delta : X \times X \rightarrow \Omega_j$  classifies the closure  $\bar{\delta}$ ; let  $\overline{\{\cdot\}} : X \rightarrow \Omega_j^X$  be its exponential transpose. One can easily verify that the kernel pair of  $\overline{\{\cdot\}}$  is  $\bar{\delta}$ , so  $\overline{\{\cdot\}}$  factors as  $X \rightarrow MX \rightarrow \Omega_j^X$  which, since a topos is regular, is the epi-mono factorization of  $\overline{\{\cdot\}}$ . Let  $LX$  be the closure of the subobject  $MX$  of  $\Omega_j^X$ . Then we have the assumptions of Lemma 2.32 verified, so  $L$  is a functor left adjoint to the inclusion  $\text{Sh}_j(\mathcal{E}) \rightarrow \mathcal{E}$ . We need to prove that  $L$  preserves finite limits. The following proof is taken from **Elephant**, A4.4.7.

First of all, we have seen in the proof of Theorem 2.28 that  $\text{sh}_j(\mathcal{E})$  is an exponential ideal in  $\mathcal{E}$  (for a sheaf  $F$  and an arbitrary  $X$ ,  $F^X$  is a sheaf). From this, it follows easily that  $L$  preserves finite products: for objects  $A$  and  $B$  of  $\mathcal{E}$  and a sheaf  $F$ , we have the following natural bijections:

$$\begin{aligned} \mathcal{E}(L(A \times B), F) &\simeq \mathcal{E}(A \times B, F) \simeq \mathcal{E}(A, F^B) \simeq \mathcal{E}(LA, F^B) \simeq \\ &\mathcal{E}(B, F^{LA}) \simeq \mathcal{E}(LB, F^{LA}) \simeq \mathcal{E}(LA \times LB, F) \end{aligned}$$

so  $L(A \times B) \simeq LA \times LB$ .

Furthermore, by Exercise 6, an object in  $\text{sh}_j(\mathcal{E})$  is injective if and only if it is a retract of some  $\Omega_j^X$ ; since the inclusion  $\text{sh}_j(\mathcal{E}) \rightarrow \mathcal{E}$  preserves exponentials and since  $\Omega_j$  is a retract of  $\Omega$  (hence  $\Omega_j^X$  is a retract of  $\Omega^X$ ), we see that the inclusion preserves injective objects. Given that  $\text{sh}_j(\mathcal{E})$  has enough injectives, by the same exercise we have that  $L$  preserves monos.

Now we wish to show that  $L$  preserves “coreflexive equalizers”. A *coreflexive pair* is a parallel pair  $X \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} Y$  with common retraction  $Y \xrightarrow{h} X$ :  $hf = hg = \text{id}_X$ . A coreflexive equalizer is an equalizer of a coreflexive pair.

In a category with finite products, every equalizer appears also as coreflexive equalizer: the arrow  $E \xrightarrow{e} X$  is an equalizer of  $f, g : X \rightarrow Y$  if and only if  $e$  is an equalizer of the coreflexive pair  $\langle \text{id}_X, f \rangle, \langle \text{id}_X, g \rangle : X \rightarrow X \times Y$  (which has as common retraction the projection  $X \times Y \rightarrow X$ ). Therefore, if coreflexive equalizers exist, all equalizers exist and if coreflexive equalizers are preserved (by a product-preserving functor) then all equalizers are preserved.

Let  $f, g : X \rightarrow Y$  be a coreflexive pair. You should check that  $e : E \rightarrow X$  is an equalizer of  $f, g$  if and only if the square

$$\begin{array}{ccc} E & \xrightarrow{e} & X \\ e \downarrow & & \downarrow f \\ X & \xrightarrow{g} & Y \end{array}$$

is a pullback. Therefore, if  $e$  is an equalizer of  $f, g$  then  $E \rightarrow X$  is the meet (intersection) in  $\text{Sub}(Y)$  of the subobjects represented by  $f$  and  $g$ . We wish therefore to show that  $L$  preserves meets of subobjects.

To this end, let  $M \xrightarrow{m} X, N \xrightarrow{n} X$  be monos representing subobjects  $M$  and  $N$ , and let  $M \cap N, M \cup N$  be their intersection and union. The square

$$\begin{array}{ccc} M \cap N & \longrightarrow & M \\ \downarrow & & \downarrow \\ N & \longrightarrow & M \cup N \end{array}$$

is a pushout in  $\mathcal{E}$  by Proposition 1.29. Since  $L$  is a left adjoint, the square

$$\begin{array}{ccc} L(M \cap N) & \longrightarrow & LM \\ \downarrow & & \downarrow \\ LN & \longrightarrow & L(M \cup N) \end{array}$$

is a pushout in  $\text{sh}_j(\mathcal{E})$ . We know that  $L$  preserves monos, so  $L(M \cap N) \rightarrow LN$  is mono; so Corollary 1.9 applies and the square is also a pullback. Since also  $L(M \cup N) \rightarrow LX$  is mono, also the square

$$\begin{array}{ccc} L(M \cap N) & \longrightarrow & LM \\ \downarrow & & \downarrow \\ LN & \longrightarrow & X \end{array}$$

is a pullback. We conclude that  $L(M \cap N) = LM \cap LN$  so  $L$  indeed preserves meets of subobjects. ■

**Definition 2.34** A geometric morphism  $f : \mathcal{F} \rightarrow \mathcal{E}$  is called an *embedding* if the direct image functor  $f_*$  is full and faithful.

The geometric morphism of Theorem 2.33 is an embedding. Moreover we shall see that every embedding is of this form (Proposition 2.37).

**Examples 2.35** For our usual examples of geometric morphisms, we have:

- 1) Given a continuous map of topological spaces  $f : X \rightarrow Y$ , the associated geometric morphism  $\text{Sh}(X) \rightarrow \text{Sh}(Y)$  is an embedding if and only if  $f$  is an embedding of topological spaces (i.e.  $X$  is a subspace of  $Y$ ).

- 2) For a morphism  $u : X \rightarrow Y$  in a topos  $\mathcal{E}$ , the geometric morphism  $\mathcal{E}/X \rightarrow \mathcal{E}/Y$  is an embedding if and only if  $u$  is mono.
- 3) For a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between small categories, the geometric morphism  $\widehat{F} : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$  is an embedding if and only if  $F$  is full and faithful.

## 2.6 The Factorization Theorem

**Theorem 2.36 (MM VII.4.6)** *Let  $f : \mathcal{F} \rightarrow \mathcal{E}$  be a geometric morphism. There exists a Lawvere-Tierney topology  $j$  in  $\mathcal{E}$  such that  $f$  factors as*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{f} & \mathcal{E} \\ & \searrow p \quad \nearrow i & \\ & \text{Sh}_j(\mathcal{E}) & \end{array}$$

where  $p$  is a surjection and  $i$  is the geometric morphism from Theorem 2.33. Moreover, given another factorization  $\mathcal{F} \xrightarrow{q} \mathcal{G} \xrightarrow{k} \mathcal{E}$  of  $f$  with  $q$  a surjection and  $k$  an embedding, there is an equivalence  $\mathcal{G} \rightarrow \text{Sh}_j(\mathcal{E})$  which makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{f} & \mathcal{E} \\ & \searrow p \quad \nearrow i & \\ & \text{Sh}_j(\mathcal{E}) & \\ & \uparrow & \\ \mathcal{G} & \xleftarrow{k} & \mathcal{E} \end{array} \quad \text{with } q : \mathcal{F} \rightarrow \mathcal{G} \text{ and } k : \mathcal{G} \rightarrow \mathcal{E}.$$

**Proof.** Consider the closure operation  $c_{(-)}$  on  $\mathcal{E}$  defined as follows: for a subobject  $U \xrightarrow{u} X$ ,  $c_X(u)$  is the subobject of  $X$  given by the following pullback:

$$\begin{array}{ccc} c_X(u) & \longrightarrow & f_* f^* U \\ \downarrow & & \downarrow f_* f^* u \\ X & \xrightarrow{\eta_X} & f_* f^* X \end{array}$$

where  $\eta$  is the unit of the adjunction  $f^* \dashv f_*$ .

**Exercise 19** Check yourself that this defines a universal closure operation.

We claim that for arbitrary subobjects  $U, V$  of  $X$  the following holds:  $V \leq c_X(U)$  if and only if  $f^*V \leq f^*U$ . Indeed, consider the commuting diagram:

$$\begin{array}{ccc}
V & \xrightarrow{\eta} & f_*f^*V \\
\downarrow & & \downarrow \\
X & \xrightarrow{\eta} & f_*f^*X \\
\uparrow & & \uparrow \\
c_X(U) & \longrightarrow & f_*f^*U
\end{array}$$

where  $\eta$  is the unit of the adjunction  $f^* \dashv f_*$ . If  $f^*V \leq f^*U$  then  $f_*f^*V \leq f_*f^*U$  so, since the lower square is a pullback, the arrow  $V \rightarrow X$  factors through  $c_X(U)$ ; i.e.,  $V \leq c_X(U)$ .

Conversely, if  $V \leq c_X(U)$ , we obtain an arrow  $V \xrightarrow{\mu} f_*f^*U$  such that the following diagram commutes:

$$\begin{array}{ccccc}
V & \xrightarrow{\mu} & f_*f^*U & \longrightarrow & f_*f^*X \\
& \searrow & & \nearrow \eta & \\
& & X & & 
\end{array}$$

Transposing along  $f^* \dashv f_*$  we get

$$\begin{array}{ccccc}
f^*V & \xrightarrow{\hat{\mu}} & f^*U & \longrightarrow & f^*X \\
& \searrow & & \nearrow \text{id} & \\
& & f^*X & & 
\end{array}$$

and, since  $f^*V \rightarrow f^*X$  is mono, also  $\hat{\mu}$  is mono, and  $f^*V \leq f^*U$ .

The following exercise is very similar to Exercise 17b):

**Exercise 20** Suppose  $\mathcal{F} \xrightarrow{f} \mathcal{E}$  is a geometric morphism and  $j$  is a Lawvere-Tierney topology in  $\mathcal{E}$ . Then  $f_*$  factors through the inclusion  $\text{sh}_j(\mathcal{E}) \rightarrow \mathcal{E}$  if and only if  $f^*$  maps  $j$ -dense monos to isomorphisms in  $\mathcal{F}$ .

Now if  $U \xrightarrow{u} X$  is a mono which is dense for (the topology associated to) the closure operator  $c_X$ , then  $X \leq c_X(U)$ , so  $f^*X \leq f^*U$  and  $f_*u$  is an isomorphism. By the exercise, we conclude that  $f_*$  factors through  $\text{sh}_j(\mathcal{E})$ . And by reasoning as in Exercise 16, we obtain a factorization of geometric

morphisms:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{f} & \mathcal{E} \\ & \searrow p & \nearrow i \\ & \text{sh}_j(\mathcal{E}) & \end{array} .$$

Remains to see that  $p$  is a surjection. Consider subobjects  $U \leq V$  of  $X$  in  $\text{sh}_j(\mathcal{E})$ ; suppose  $p^*U \simeq p^*V$ . Then  $f^*i^*U \simeq f^*i^*V$  so, since  $U$  and  $V$  are *closed* subobjects of  $X$ , we have  $i_*U \simeq i_*V$ . Since  $i_*$  is full and faithful,  $U \simeq V$  follows. We conclude that  $p^*$  reflects isomorphisms of subobjects; by Lemma 2.19,  $p$  is a surjection as claimed.

For the essential uniqueness of the decomposition, I refer to **MM**, Theorem VII.4.8. ■

We can now give the promised characterization of embeddings:

**Proposition 2.37** *For a geometric morphism  $f : \mathcal{F} \rightarrow \mathcal{E}$  the following statements are equivalent:*

- i)  $f$  is an embedding (i.e.,  $f_*$  is full and faithful).
- ii) The counit  $\varepsilon : f^*f_* \Rightarrow \text{id}_{\mathcal{F}}$  is an isomorphism.
- iii) There is a Lawvere-Tierney topology  $j$  in  $\mathcal{E}$  and an equivalence  $e : \mathcal{F} \rightarrow \text{sh}_j(\mathcal{E})$  such that the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{f} & \mathcal{E} \\ & \searrow e & \uparrow \\ & \text{sh}_j(\mathcal{E}) & \end{array}$$

*commutes up to isomorphism.*

**Proof.** The equivalence between i) and ii) is standard Category Theory, and the implication iii) $\Rightarrow$ i) is clear. For the converse, assume  $f$  is an embedding. By Theorem 2.36, there is a factorization

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{f} & \mathcal{E} \\ & \searrow p & \uparrow i \\ & \text{sh}_j(\mathcal{E}) & \end{array}$$

with  $p$  a surjection. Since  $i_*$  and  $f_*$  are full and faithful, so is  $p_*$  (check!). Therefore the counit  $\varepsilon$  for  $p^* \dashv p_*$  is an isomorphism. Consider the “triangular identity” from basic Category Theory for arbitrary  $E \in \mathcal{E}$ :

$$\begin{array}{ccc} p^*E & \xrightarrow{p^*\eta_E} & p^*p_*p^*E \\ & \searrow \text{id} & \downarrow \varepsilon_{p^*E} \\ & & p^*E \end{array}$$

Since  $\varepsilon$  is an isomorphism, we see that  $p^*(\eta_E)$  is an isomorphism. But  $p$  is a surjection, so  $\eta_E$  is an isomorphism. We see that both  $\varepsilon$  and  $\eta$  are isomorphisms, so  $p$  is an equivalence. ■

**Examples 2.38** Let us see how standard geometric morphisms decompose:

- 1) Every continuous map  $f : X \rightarrow Y$  of topological spaces factors as  $X \rightarrow Z \rightarrow Y$ , where  $Z$  is the image of  $X$ , topologized as a subspace of  $Y$ . The map  $X \rightarrow Z$  is surjective, the map  $Z \rightarrow Y$  is an embedding. Hence the geometric morphism  $\text{Sh}(X) \rightarrow \text{Sh}(Z)$  is a surjection and  $\text{Sh}(Z) \rightarrow \text{Sh}(Y)$  is an embedding.
- 2) Every morphism in a topos has an epi-mono factorization, as we have seen. This gives at once a surjection-embedding factorization of the geometric morphism between the slice toposes.
- 3) For a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between small categories, let  $\mathcal{B}$  be the full subcategory of  $\mathcal{D}$  on objects in the image of  $F$ ; and let  $\mathcal{C} \xrightarrow{G} \mathcal{B} \xrightarrow{H} \mathcal{D}$  be the evident factorization. Then  $G$  is surjective on objects and  $H$  is full and faithful; so  $\widehat{\mathcal{C}} \xrightarrow{\widehat{G}} \widehat{\mathcal{B}} \xrightarrow{\widehat{H}} \widehat{\mathcal{D}}$  is a surjection-embedding factorization of  $\widehat{F}$ .

### 3 Logic in Toposes

In 2.22 we have seen the map  $\wedge : \Omega \times \Omega \rightarrow \Omega$ , which classifies the subobject  $1 \xrightarrow{\langle t, t \rangle} \Omega \times \Omega$ ; whenever subobjects  $M, N$  of  $X$  are classified by  $\phi, \psi$  respectively, the intersection (meet)  $M \cap N$  is classified by the composite

$$X \xrightarrow{\langle \phi, \psi \rangle} \Omega \times \Omega \xrightarrow{\wedge} \Omega.$$

We have also seen that  $\text{Sub}(X)$  has unions (joins): for subobjects  $M \xrightarrow{m} X$ ,  $N \xrightarrow{n} X$ ,  $M \cup N \rightarrow X$  is the mono-part of the epi-mono factorization of  $\begin{bmatrix} m \\ n \end{bmatrix} : M + N \rightarrow X$ . Equivalently, by Proposition 1.29,  $M \cup N$  may be constructed as the pushout:

$$\begin{array}{ccc} M \cap N & \longrightarrow & M \\ \downarrow & & \downarrow \\ N & \longrightarrow & M \cup N \end{array}$$

Let  $\vee : \Omega \times \Omega \rightarrow \Omega$  classify the union of the subobjects  $\Omega \xrightarrow{\langle \text{id}, t \rangle} \Omega \times \Omega$  and  $\Omega \xrightarrow{\langle t, \text{id} \rangle} \Omega \times \Omega$ .

**Exercise 21** If the subobjects  $M, N$  of  $X$  are classified by  $\phi, \psi : X \rightarrow \Omega$  respectively, then  $M \cup N$  is classified by the composite

$$X \xrightarrow{\langle \phi, \psi \rangle} \Omega \times \Omega \xrightarrow{\vee} \Omega.$$

We define also the mono  $\Omega_1 \rightarrow \Omega \times \Omega$  as the equalizer of  $\pi_0, \wedge : \Omega \times \Omega \rightarrow \Omega$  (where  $\pi_0$  is the first projection).

**Exercise 22** For subobjects  $M, N$  of  $X$ , classified by  $\phi, \psi$  we have:  $M \leq N$  if and only if the map  $\langle \phi, \psi : X \rightarrow \Omega \times \Omega$  factors through  $\Omega_1$ .

Furthermore, we have the arrow  $f : 1 \rightarrow \Omega$ , which classifies the mono  $0 \rightarrow 1$ .

**Definition 3.1** In a category  $\mathcal{C}$  with finite limits, an *internal lattice* is an object  $L$  together with morphisms  $\top, \perp : 1 \rightarrow L$  and  $\sqcup, \sqcap : L \times L \rightarrow L$  such that the diagrams expressing the following equations commute:

|   |   |
|---|---|
| 1 $x \sqcap \top = x$                             | $x \sqcup \top = \top$                          |
| 2 $x \sqcap \perp = \perp$                        | $x \sqcup \perp = x$                            |
| 3 $x \sqcap y = y \sqcap x$                       | $x \sqcup y = y \sqcup x$                       |
| 4 $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$ | $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ |
| 5 $x \sqcap (x \sqcup y) = x$                     | $y \sqcup (x \sqcap y) = y$                     |
| 6 $x \sqcap x = x$                                | $x \sqcup x = x$                                |



Note that in a lattice  $(L, \perp, \top, \sqcap, \sqcup)$  a partial ordering can be defined as the equalizer of  $\sqcap$  and the first projection (equivalently, the equalizer of  $\sqcup$  and the second projection). If, in the context of an internal lattice, we use the symbol  $\leq$ , it refers to this ordering.

Such a lattice is *distributive* if moreover  $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$  holds.

**Exercise 23** Show that, with the operations  $\wedge$  and  $\vee$  and the elements  $t, f : 1 \rightarrow \Omega$ ,  $\Omega$  is a distributive internal lattice.

But  $\Omega$  has more structure. A *Heyting algebra* is a lattice which is cartesian closed as a poset: for elements  $x$  and  $y$  there is the exponent  $x \Rightarrow y$  which has the universal property that  $z \leq (x \Rightarrow y)$  if and only if  $z \sqcap x \leq y$ . This property can also be given equationally, so we can define the notion of *internal Heyting algebra*:

**Definition 3.2** In a category  $\mathcal{C}$  with finite limits, a *internal Heyting algebra* is an internal lattice  $(L, \top, \perp, \sqcap, \sqcup)$  and an operation  $\Rightarrow : L \times L \rightarrow L$  such that the diagram expressing the following equations commute:

$$\begin{array}{ll} (x \Rightarrow x) = \top & (x \Rightarrow (y \sqcap z)) = (x \Rightarrow y) \sqcap (x \Rightarrow z) \\ x \sqcap (x \Rightarrow y) = x \sqcap y & y \sqcap (x \Rightarrow y) = y \end{array}$$

For  $\Omega$ , we have a map  $\Rightarrow : \Omega \times \Omega \rightarrow \Omega$  which classifies the mono  $\Omega_1 \rightarrow \Omega \times \Omega$ .

**Exercise 24** Prove that  $(\Omega, t, f, \wedge, \vee, \Rightarrow)$  is an internal Heyting algebra. If  $M, N$  are subobjects of  $X$  classified by  $\phi, \psi$  respectively, then the composite

$$X \xrightarrow{\langle \phi, \psi \rangle} \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$$

classifies the largest subobject  $K$  of  $X$  which satisfies  $K \cap M \leq N$ .

In any Heyting algebra, we have the operation  $\neg$  (*pseudocomplement*):  $\neg x = x \Rightarrow \perp$ . A Heyting algebra is a *Boolean algebra* if  $x \sqcup \neg x = \top$  for all  $x$ .

**Definition 3.3** A topos  $\mathcal{E}$  is called *Boolean* if  $\Omega$  is an internal Boolean algebra.

**Proposition 3.4** For a topos  $\mathcal{E}$ , the following statements are equivalent:

- i)  $\mathcal{E}$  is Boolean.
- ii) The map  $\neg \neg$  is the identity map on  $\Omega$ .

- iii) Every subobject  $M$  of an object  $X$  has a complement, that is: a subobject  $N$  of  $X$  satisfying  $M \cap N = 0$  and  $M \cup N = X$ .
- iv) The map  $\begin{bmatrix} t \\ f \end{bmatrix} : 1 + 1 \rightarrow \Omega$  is an isomorphism.

Obviously, if  $f : \mathcal{F} \rightarrow \mathcal{E}$  is a logical functor and  $\mathcal{F}$  is Boolean, then so is  $\mathcal{E}$ .

**Theorem 3.5 (PTJ 5.17)** *For every topos  $\mathcal{E}$ ,  $\neg\neg$  is a Lawvere-Tierney topology in  $\mathcal{E}$ , and  $\text{sh}_{\neg\neg}(\mathcal{E})$  is Boolean.*

**Proof.** The map  $\neg$  is order-reversing, and for subobjects  $M, M'$  of  $X$   $M \leq \neg M'$  if and only if  $M' \leq \neg M$ . Hence (taking  $\neg M$  for  $M'$ )  $M \leq \neg\neg M$  (i.e.,  $\neg\neg$  is inflationary) and  $\neg M = \neg\neg\neg M$ ; so  $\neg\neg$  is idempotent. Also,  $\neg\neg X = X$  and  $\neg\neg$  commutes with meets:  $\neg\neg M \wedge \neg\neg M' = \neg\neg(M \wedge M')$ .

The subobject classifier of  $\text{sh}_{\neg\neg}(\mathcal{E})$  is  $\Omega_{\neg\neg}$ , which is an internal Boolean algebra in  $\mathcal{E}$ , hence also an internal Boolean algebra in  $\text{sh}_{\neg\neg}(\mathcal{E})$ . ■

For the following proposition, we need the pointwise ordering on maps into  $\Omega$ : for  $f, g : X \rightarrow \Omega$  we set  $f \leq g$  if and only if the map  $\langle f, g \rangle : X \rightarrow \Omega \times \Omega$  factors through  $\Omega_1$ .

**Proposition 3.6** *The map  $\neg\neg$  is the largest topology for which the inclusion  $0 \rightarrow 1$  is closed (equivalently, for which the object  $0$  is a sheaf).*

**Proof.** Clearly,  $\neg\neg f = f$  (since  $\neg f = t$ ), so  $0 \rightarrow 1$  is closed.

Conversely, let  $j$  be a topology for which  $0 \rightarrow 1$  is closed. Let  $X' \xrightarrow{\sigma} X$  be a  $j$ -dense mono. Let  $X'' = \neg X'$  (in  $\text{Sub}(X)$ ). We have a pullback

$$\begin{array}{ccc} 0 & \longrightarrow & X' \\ \downarrow & & \downarrow \sigma \\ X'' & \longrightarrow & X \end{array}$$

so  $0 \rightarrow X''$  is  $j$ -dense. But  $0 \rightarrow X''$  is also  $j$ -closed, since also the square

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ X'' & \longrightarrow & 1 \end{array}$$

is a pullback. So,  $0 \rightarrow X''$  is an isomorphism, and  $X = \neg X'' = \neg\neg X'$ ; so  $\sigma$  is  $\neg\neg$ -dense. It follows that  $j \leq \neg\neg$ . ■

**Definition 3.7** Let  $\mathcal{E}$  be a topos.

- 1) We say that  $\mathcal{E}$  *supports split* in  $\mathcal{E}$  (or, that  $\mathcal{E}$  *satisfies SS*) if, whenever

$$\begin{array}{ccc} X & \xrightarrow{\quad} & 1 \\ & \searrow \quad \nearrow & \\ & \sigma_1(X) & \end{array}$$

is an epi-mono factorization, the epi  $X \rightarrow \sigma_1(X)$  is split.

- 2) We say that  $\mathcal{E}$  *satisfies the Axiom of Choice (AC)* if every epi in  $\mathcal{E}$  is split.

**Exercise 25** Prove that  $\mathcal{E}$  satisfies SS if and only if every subobject of 1 is projective in  $\mathcal{E}$ ; and that  $\mathcal{E}$  satisfies AC if and only if every object is projective in  $\mathcal{E}$ .

The following is a classical result, but we shall later see a stronger statement, so the proof is deferred.

**Theorem 3.8 (Diaconescu; PTJ 5.23)** *If a topos satisfies AC, it is Boolean.*

**Exercise 26** For a group  $\mathcal{G}$ , show that  $\widehat{\mathcal{G}}$  is Boolean but does not satisfy AC.

**Definition 3.9** An object  $X$  of a topos  $\mathcal{E}$  is called *internally projective* if the functor  $(-)^X$  preserves epimorphisms.

An epimorphism  $f : X \rightarrow Y$  in  $\mathcal{E}$  is called *locally split* if there is an object  $V$  of  $\mathcal{E}$  such that  $V \rightarrow 1$  is epi and  $V^*(f)$  is split epi in  $\mathcal{E}/V$ .

**Exercise 27** Show that an epi  $f : X \rightarrow Y$  is locally split if and only if there is an epi  $h : Z \rightarrow Y$  such that  $h^*(f)$  is split epi in  $\mathcal{E}$ .

**Proposition 3.10 (PTJ 5.25)** *The following statements are equivalent for a topos  $\mathcal{E}$ :*

- i) *Every object of  $\mathcal{E}$  is internally projective.*
- ii) *Every epi in  $\mathcal{E}$  is locally split.*
- iii) *If  $X \xrightarrow{f} Y$  is epi, then  $\prod_Y(f) \rightarrow 1$  is epi.*

**Proof.** Recall from the proof of Corollary 1.18 that

$$\prod_Y(f) \longrightarrow Y^X \xrightarrow[\ulcorner \text{id} \urcorner \circ !]{f^X} X^X$$

is an equalizer diagram, where  $\ulcorner \text{id} \urcorner : 1 \rightarrow X^X$  denotes the exponential transpose of the identity arrow on  $X$ . ■

## 4 Classifying Toposes

### 4.1 Examples

**Example 4.1 (Torsors)** Let  $G$  be a group and suppose  $\gamma : \mathcal{E} \rightarrow \mathbf{Set}$  is a geometric morphism (we speak of a “topos over  $\mathbf{Set}$ ”, i.e. a topos with a geometric morphism to  $\mathbf{Set}$ ). Then  $\gamma^*(G)$  is a group object in  $\mathcal{E}$ . A  $G$ -torsor over  $\mathcal{E}$  is an object  $T$  of  $\mathcal{E}$  equipped with a left group action

$$\mu : \gamma^*(G) \times T \rightarrow T$$

which, apart from the axioms for a group action, satisfies the following conditions:

- i)  $T \rightarrow 1$  is an epimorphism.
- ii) The action  $\mu$  induces an isomorphism

$$\langle \mu, p_1 \rangle : \gamma^*(G) \times T \rightarrow T \times T$$

(recall that  $p_1$  denotes the projection on the *second* coordinate)

In the topos  $\widehat{G}$  of right  $G$ -sets, we have a torsor whose underlying set is  $G$  itself, with its canonical action on the left (note that the actions on the left and on the right commute with each other, so the left action is a map of  $G$ -sets). We call this  $G$ -torsor  $U_G$ .

The  $G$ -torsors in  $\mathcal{E}$  form a category  $\mathrm{Tor}(\mathcal{E}, G)$ , whose objects are  $G$ -torsors over  $\mathcal{E}$  and whose morphisms are morphisms of left  $G$ -sets in  $\mathcal{E}$ . Since for cocomplete toposes, the geometric morphism to  $\mathbf{Set}$  is essentially unique, we have, for a geometric morphism  $f : \mathcal{E} \rightarrow \widehat{G}$ , a diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \widehat{G} \\ & \searrow \gamma & \swarrow g \\ & \mathbf{Set} & \end{array}$$

which commutes up to isomorphism (where  $g$  is the geometric morphism we have already seen).

Clearly, the structures of a  $G$ -torsor and of a map between  $G$ -torsors are preserved by inverse images of geometric morphisms, so any geometric morphism  $f : \mathcal{F} \rightarrow \mathcal{E}$  gives rise to a functor  $f^* : \mathrm{Tor}(\mathcal{E}, G) \rightarrow \mathrm{Tor}(\mathcal{F}, G)$ .

For the following theorem we should state the 2-dimensional character of the category  $\mathcal{Top}$ : for two geometric morphisms  $f, g : \mathcal{F} \rightarrow \mathcal{E}$  we can also consider natural transformations  $f^* \rightarrow g^*$ . In this way we have, for any two toposes  $\mathcal{F}, \mathcal{G}$  a category  $\mathcal{Top}(\mathcal{F}, \mathcal{E})$ .

**Theorem 4.2 (MM VIII.2.7)** *For a topos  $\mathcal{E}$  over  $\mathbf{Set}$  there is an equivalence of categories*

$$\mathcal{Top}(\mathcal{E}, \widehat{G}) \simeq \mathbf{Tor}(\mathcal{E}, G).$$

*This equivalence is, on objects, induced by the operation which sends the geometric morphism  $g : \mathcal{E} \rightarrow \widehat{G}$  to the  $G$ -torsor  $g^*(U_G)$  and is therefore natural in  $\mathcal{E}$ .*

This example is an instance of a general phenomenon. We consider, for a topos  $\mathcal{E}$ , the category  $\mathcal{E}_T$  of “structures of a type  $T$ ” in  $\mathcal{E}$ . For the moment, let us not worry about what these structures are or what the morphisms could be, except that we suppose that when  $M$  is such a structure in  $\mathcal{E}$  and  $f : \mathcal{F} \rightarrow \mathcal{E}$  is a geometric morphism, then  $f^*M$  is such a structure in  $\mathcal{F}$ ; and similarly, if we have an arrow  $\mu : M \rightarrow N$  in  $\mathcal{E}_T$  then  $f^*(\mu)$  is an arrow  $f^*M \rightarrow f^*N$  in  $\mathcal{F}_T$ , so that we have a functor  $f^* : \mathcal{E}_T \rightarrow \mathcal{F}_T$ .

**Definition 4.3** A *classifying topos* for structures of type  $T$  is a topos  $\mathcal{B}(T)$  over  $\mathbf{Set}$ , for which there is a natural equivalence of categories

$$\mathcal{Top}(\mathcal{E}, \mathcal{B}(T)) \rightarrow \mathcal{E}_T$$

Applying the equivalence to the identity geometric morphism on  $\mathcal{B}(T)$  and reasoning like in the Yoneda Lemma, we see that there is a structure  $U_T$  of type  $T$  in  $\mathcal{B}(T)$  (the *universal  $T$ -structure*), such that the equivalence of Definition 4.3 is given by:  $f \mapsto f^*(U_T)$ .

We shall later specify what “structures of type  $T$ ” will be (models of a certain logical theory); for now, we continue with some more examples.

**Example 4.4 (Objects)** The simplest “structure of type  $T$ ” is: just an object. If  $\mathcal{B}$  is a classifying topos for objects, we have an equivalence of categories

$$\mathcal{Top}(\mathcal{E}, \mathcal{B}) \rightarrow \mathcal{E}$$

given by  $f \mapsto f^*(U)$  for some “universal object”  $U$  of  $\mathcal{B}$ .

**Lemma 4.5 (MM VIII.4.1)** *Let  $\mathbf{Set}_f$  be the category of finite sets. Then  $\mathbf{Set}_f$  is the free category with finite colimits generated by one object.*

**Proof.** The statement of the lemma means: there is a finite set  $X$  such that for every category  $\mathcal{C}$  with finite colimits and every object  $C$  of  $\mathcal{C}$ , there is an essentially unique functor  $F_C : \mathbf{Set}_f \rightarrow \mathcal{C}$  which preserves finite colimits and sends  $X$  to  $C$ . Indeed, let  $X$  be a one-element set. For an arbitrary finite set  $E$ , let

$$F_C(E) = \sum_{e \in E} C$$

Clearly,  $F_C(X) = C$ . Moreover,  $F_C$  preserves all finite colimits (see MM VIII.4.1 for details). ■

Dual to Lemma 4.5 we have:

**Lemma 4.6 (MM VIII.4.2)** *The category  $\text{Set}_f^{\text{op}}$  is the free category with finite limits, generated by one object.*

Now we have a chain of equivalences:

$$\begin{array}{rcl} \text{Geometric morphisms } \mathcal{E} \rightarrow \text{Set}^{\text{Set}_f} & \simeq & \\ \text{Flat functors } \text{Set}_f^{\text{op}} \rightarrow \mathcal{E} & \simeq & \\ \text{Finite limit preserving functors } \text{Set}_f^{\text{op}} \rightarrow \mathcal{E} & \simeq & \mathcal{E} \end{array}$$

So, the classifying topos for objects is  $\text{Set}^{\text{Set}_f}$ .

**Exercise 28** What is the “universal object” in  $\text{Set}^{\text{Set}_f}$ ?

**Example 4.7 (Rings)** Our next example concerns commutative rings, here just called rings. In a category  $\mathcal{C}$  with finite limits, a *ring object* is a diagram

$$1 \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{1} \end{array} R \begin{array}{c} \xleftarrow{+} \\ \xleftarrow{\cdot} \end{array} R \times R$$

for which the axioms for rings (expressed by commuting diagrams) hold. We have an obvious definition of homomorphism of ring objects in  $\mathcal{C}$ , and hence a category  $\text{ring}(\mathcal{C})$ . Any finite limit preserving functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  induces a functor  $\text{ring}(\mathcal{C}) \rightarrow \text{ring}(\mathcal{D})$ .

**Definition 4.8** *A ring is finitely presented if it is isomorphic to*

$$\mathbb{Z}[X_1, \dots, X_n]/I$$

where  $\mathbb{Z}[X_1, \dots, X_n]$  is the ring of polynomials in  $n$  variables with integer coefficients, and  $I$  is an ideal. Since  $\mathbb{Z}[X_1, \dots, X_n]$  is Noetherian, the ideal  $I$  can be written as  $(P_1, \dots, P_k)$  for elements  $P_1, \dots, P_k$  of  $\mathbb{Z}[X_1, \dots, X_n]$ .

Let **fp-rings** be the full subcategory of the category of rings on the finitely presented rings. A morphism

$$\alpha : \mathbb{Z}[X_1, \dots, X_n]/(P_1, \dots, P_k) \rightarrow \mathbb{Z}[Y_1, \dots, Y_m]/(Q_1, \dots, Q_l)$$

is given by an  $n$ -tuple  $(\alpha(X_1), \dots, \alpha(X_n))$  of polynomials in  $Y_1, \dots, Y_m$ , such that the polynomials

$$P_j(\alpha(X_1), \dots, \alpha(X_n))$$

are elements of the ideal  $(Q_1, \dots, Q_l)$ .

The category **fp-rings** has finite coproducts: the initial object is  $\mathbb{Z}$ , and the sum

$$\mathbb{Z}[X_1, \dots, X_n]/(P_1, \dots, P_k) + \mathbb{Z}[Y_1, \dots, Y_m]/(Q_1, \dots, Q_l)$$

(where we assume that the strings of variables  $\vec{X}$  and  $\vec{Y}$  are disjoint) is the ring

$$\mathbb{Z}[X_1, \dots, X_n, Y_1, \dots, Y_m]/(P_1, \dots, P_k, Q_1, \dots, Q_l)$$

Moreover, the category **fp-rings** has coequalizers: given a parallel pair of arrows

$$\mathbb{Z}[\vec{X}]/(\vec{P}) \xrightleftharpoons[\beta]{\alpha} \mathbb{Z}[\vec{Y}]/(\vec{Q})$$

its coequalizer is the quotient ring

$$\mathbb{Z}[\vec{Y}]/(\vec{Q}, \alpha(X_1) - \beta(X_1), \dots, \alpha(X_n) - \beta(X_n))$$

with the evident quotient map.

Now, we consider **fp-rings**<sup>op</sup>. This is a category with finite limits. Note that  $\mathbb{Z}$  is terminal in **fp-rings**<sup>op</sup>. A ring object in **fp-rings**<sup>op</sup> is a diagram

$$\mathbb{Z} \xleftarrow[1]{0} R \rightrightarrows R + R$$

in **fp-rings**, subject to the duals of the axioms for rings. An example of such a structure in **fp-rings** is the ring  $\mathbb{Z}[X]$  with maps  $0, 1 : \mathbb{Z}[X] \rightarrow \mathbb{Z}$  sending a polynomial  $P$  to  $P(0)$  and to  $P(1)$  respectively; and  $+, \cdot : \mathbb{Z}[X] \rightarrow \mathbb{Z}[X, Y]$  (note that  $\mathbb{Z}[X, Y] = \mathbb{Z}[X] + \mathbb{Z}[X]$  in **fp-rings**, sending  $P(X)$  to  $P(X + Y)$  and to  $P(XY)$  respectively).

**Lemma 4.9 (MM VIII.5.1)** *The category **fp-rings**<sup>op</sup>, together with the ring object  $\mathbb{Z}[X]$  as just described, is the free category with finite limits and a ring object.*

The statement of the lemma means: for any category  $\mathcal{C}$  with finite limits and ring object  $R$ , there is an essentially unique finite limit preserving functor from **fp-rings**<sup>op</sup> to  $\mathcal{C}$  which sends  $\mathbb{Z}[X]$  to  $R$ .

We can now argue in exactly the same way as in the two previous examples: ring objects in a topos  $\mathcal{E}$  correspond to flat, that is: finite limit preserving, functors from **fp-rings**<sup>op</sup> to  $\mathcal{E}$ , which correspond to geometric morphisms from  $\mathcal{E}$  to  $\mathbf{Set}^{\mathbf{fp-rings}}$ ; the latter therefore being the “classifying topos for rings”.

In each of the three examples we have just seen, the classifying topos was a presheaf topos. That is because of the “algebraic character” of the type of structures we considered: the structure is given by a number of operations and the axioms are equations. Not every structure which admits a classifying topos is of such a simple kind. But let us now define what kind of structures we have in mind: structures for *geometric logic*.

## 4.2 Geometric Logic

We consider a *multi-sorted language*. That is: we have a set of *sorts*, a stock of variables for each sort (we write  $x^S$  in order to indicate that the variable  $x$  has sort  $S$ ), and constants, function symbols and relation symbols with also specified sorts. We write:

$c^S$  to indicate that the constant  $c$  is of sort  $S$ ;

$f : S_1, \dots, S_n \rightarrow T$  to indicate that the function symbol  $f$  takes arguments of sorts  $S_1, \dots, S_n$ , and then yields something of sort  $T$ ;

$R \subseteq S_1, \dots, S_n$  to indicate that the relation symbol  $R$  takes arguments of sorts  $S_1, \dots, S_n$ .

All terms of the language have a specified sort: for a variable  $x^S$  of sort  $S$ ,  $x^S$  is a term of sort  $S$ . Every constant of sort  $S$  is a term of sort  $S$ . If  $f : S_1, \dots, S_n \rightarrow T$  is a function symbol and  $t_1, \dots, t_n$  are terms of sorts  $S_1, \dots, S_n$  respectively, then  $f(t_1, \dots, t_n)$  is a term of sort  $T$ .

An *atomic formula* is an expression of one of three forms: it is the symbol  $\top$  (for “true”), it is an equation  $t = s$  where  $t$  and  $s$  are terms of the same sort, or it is an expression  $R(t_1, \dots, t_n)$ , where  $R \subseteq S_1, \dots, S_n$  is a relation symbol and  $t_i$  is a term of sort  $S_i$  for  $i = 1, \dots, n$ .

The class of *geometric formulas* (for a given language) is defined as follows:

Every atomic formula is a geometric formula;

If  $\phi$  and  $\psi$  are geometric formulas, then  $\phi \wedge \psi$  is a geometric formula;

If  $\phi$  is a geometric formula and  $x^S$  is a variable, then  $\exists x^S \phi$  is a geometric formula;

If  $X$  is a set of geometric formulas and  $X$  contains only finitely many free variables, then  $\bigvee X$  is a geometric formula.



If  $\mathcal{E}$  is a cocomplete topos, then there is a straightforward definition of what a *structure* for a language in  $\mathcal{E}$  should be: for every sort  $S$ , we have an object  $\llbracket S \rrbracket$  of  $\mathcal{E}$ ; for every function symbol  $f : S_1, \dots, S_n \rightarrow T$  we have a morphism  $\llbracket f \rrbracket : \llbracket S_1 \rrbracket \times \dots \times \llbracket S_n \rrbracket \rightarrow \llbracket T \rrbracket$  in  $\mathcal{E}$ ; for every relation symbol  $R \subseteq S_1, \dots, S_n$  we have a subobject  $\llbracket R \rrbracket$  of  $\llbracket S_1 \rrbracket \times \dots \times \llbracket S_n \rrbracket$ .

Just as straightforwardly, one now obtains, for any formula  $\phi$  with free variables  $x_1^{S_1}, \dots, x_n^{S_n}$ , a subobject  $\llbracket \phi \rrbracket$  of  $\llbracket S_1 \rrbracket \times \dots \times \llbracket S_n \rrbracket$ . For the case when  $\phi$  is of the form  $\bigvee X$ , we use of course the cocompleteness of  $\mathcal{E}$ , which implies that subobject lattices are complete (have arbitrary joins).

A *geometric sequent* is an expression of the form  $\phi \vdash_{\vec{x}} \psi$ , where  $\phi$  and  $\psi$  are geometric formulas, and  $\vec{x}$  is a finite list of variables which contains every variable which appears freely in  $\phi$  or  $\psi$ .

If a structure for the language is given, let us write  $\llbracket \vec{x} \rrbracket$  for the product  $\prod_{i=1}^n \llbracket S_i \rrbracket$  if  $\vec{x} = (x_1^{S_1}, \dots, x_n^{S_n})$ . If  $\vec{y}_\phi$  is the list of variables appearing freely in  $\phi$  and  $\vec{y}_\psi$  the list of those in  $\psi$ , then we have evident projections  $p_\phi : \llbracket \vec{x} \rrbracket \rightarrow \llbracket \vec{y}_\phi \rrbracket$  and  $p_\psi : \llbracket \vec{x} \rrbracket \rightarrow \llbracket \vec{y}_\psi \rrbracket$ , and hence subobjects  $\llbracket \phi \rrbracket_{\vec{x}} = p_\phi^*(\llbracket \phi \rrbracket)$  and  $\llbracket \psi \rrbracket_{\vec{x}} = p_\psi^*(\llbracket \psi \rrbracket)$  of  $\llbracket \vec{x} \rrbracket$ .

We say that the sequent  $\phi \vdash_{\vec{x}} \psi$  is *true* in the given structure, if  $\llbracket \phi \rrbracket_{\vec{x}} \leq \llbracket \psi \rrbracket_{\vec{x}}$  in  $\text{Sub}(\llbracket \vec{x} \rrbracket)$ . We think of the sequent  $\phi \vdash_{\vec{x}} \psi$  as of the “formula”

$$\forall \vec{x} (\phi \Rightarrow \psi)$$

For instance, if for one of the variables  $x^S$  in  $\vec{x}$  we have that the object  $\llbracket S \rrbracket$  is initial, then the sequent  $\phi \vdash_{\vec{x}} \psi$  is always true.

Let us denote a structure for a given language by  $\mathcal{M}$ . So we have the interpretation  $\llbracket \cdot \rrbracket^{\mathcal{M}}$  of the sorts, function symbols, constants and relation symbols in some topos  $\mathcal{E}$ . If  $f : \mathcal{F} \rightarrow \mathcal{E}$  is a geometric morphism, we have a structure  $f^*\mathcal{M}$  in  $\mathcal{F}$  by applying the inverse image functor  $f^*$  to all the data of  $\mathcal{M}$ . We now have interpretations  $\llbracket \phi \rrbracket^{\mathcal{M}}$  in  $\mathcal{E}$  and  $\llbracket \phi \rrbracket^{f^*\mathcal{M}}$  in  $\mathcal{F}$ .

**Proposition 4.10** *Let  $\mathcal{M}$  be a structure for a language in a topos  $\mathcal{E}$ , and suppose  $f : \mathcal{F} \rightarrow \mathcal{E}$  is a geometric morphism. Then we have:*

- a) *For any formula  $\phi$  of the language,  $\llbracket \phi \rrbracket^{f^*\mathcal{M}} = f^*(\llbracket \phi \rrbracket^{\mathcal{M}})$ .*
- b) *If the sequent  $\phi \vdash_{\vec{x}} \psi$  is true with respect to the structure  $\mathcal{M}$ , then it is also true with respect to  $f^*\mathcal{M}$ .*
- c) *If the geometric morphism  $f$  is a surjection, then the converse of b) holds: if  $\phi \vdash_{\vec{x}} \psi$  is true with respect to the structure  $f^*\mathcal{M}$  then it is true with respect to  $\mathcal{M}$ .*

A *geometric theory* in a given language is a set of geometric sequents in that language. If  $\mathcal{M}$  is a structure in which every sequent of a theory is true, then  $\mathcal{M}$  is called a *model* of the theory.

Now we can be more precise about the “structures of a type  $T$ ” mentioned in Definition 4.3: they are, in fact, models of a geometric theory. One advantage of making this notion precise is, that we can investigate geometric theories also syntactically, and, much as in classical Model Theory, study relations between syntactic properties of theories and topos-theoretic properties of their classifying toposes.

For example, in the examples we have discussed so far, the classifying toposes were presheaf toposes (as we already remarked). This is connected to the fact that the respective theories are all *universal*: no  $\bigvee$  and no existential quantifier (you might object by saying that in the theory of rings we need to express that every element has an additive inverse, and that we need an existential quantifier for this; however, since the additive inverse is unique this existential quantifier is not essential and we could expand the language with an extra function symbol).

**Example 4.11 (Flat functors)** Let us now consider a theory where the use of existential quantifiers and (possibly infinite) disjunctions is necessary: the *theory of flat functors* form a small category  $\mathcal{C}$ .

Given a small category  $\mathcal{C}$ , let  $\mathcal{L}_{\mathcal{C}}$  be the language which has:

for every object  $C$  of  $\mathcal{C}$  a sort  $C$ ;

for every arrow  $f : C \rightarrow D$  in  $\mathcal{C}$ , a function symbol  $f : C \rightarrow D$ .

The geometric theory  $\text{Flat}(\mathcal{C})$  has the following sequents:

- 1) For every commutative triangle

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ & \searrow h & \downarrow g \\ & & E \end{array}$$

a sequent  $\top \vdash_{x^C} h(x) = g(f(x))$ .

- 2) A sequent

$$\top \vdash \bigvee_{C \in \mathcal{C}_0} \exists x^C (x = x).$$

3) A sequent

$$\top \vdash_{x^C, y^D} \bigvee_{f:E \rightarrow C, g:E \rightarrow D} \exists z^E (f(z) = x \wedge g(z) = y)$$

4) A sequent

$$f(x) = g(x) \vdash_{x^C} \bigvee_{h:D \rightarrow C, fh=gh} \exists y^D (h(y) = x)$$

**Exercise 29** Show that for a topos  $\mathcal{E}$ , a model of  $\text{Flat}(\mathcal{C})$  in  $\mathcal{E}$  is nothing but a flat functor  $\mathcal{C} \rightarrow \mathcal{E}$ ; and hence, that the topos  $\widehat{\mathcal{C}}$  classifies models of  $\text{Flat}(\mathcal{C})$ .

Admittedly, in this example the classifying topos is still a presheaf topos. However, this changes if we extend the theory  $\text{Flat}(\mathcal{C})$  according to section 2.3.

**Definition 4.12** Let  $(\mathcal{C}, J)$  be a site. The theory  $\text{FlatCont}(\mathcal{C}, J)$  of flat and  $J$ -continuous functors from  $\mathcal{C}$ , is an extension of the theory  $\text{Flat}(\mathcal{C})$  by the following axioms: for every object  $C$  of  $\mathcal{C}$  and every covering sieve  $R \in J(C)$  we have the axiom

$$\top \vdash_{x^C} \bigvee_{f:D \rightarrow C, f \in R} \exists y^D (f(y) = x)$$

Theorem 2.15 now implies:

**Proposition 4.13** *A model of  $\text{FlatCont}(\mathcal{C}, J)$  in a topos  $\mathcal{E}$  is nothing but a flat and  $J$ -continuous functor from  $\mathcal{C}$  to  $\mathcal{E}$ . Therefore, the topos  $\text{Sh}(\mathcal{C}, J)$  classifies models of  $\text{FlatCont}(\mathcal{C}, J)$ .*

And we conclude:

**Theorem 4.14 (Classifying Topos Theorem, part I)** *Every Grothendieck topos is the classifying topos of some geometric theory.*

The geometric theory which a Grothendieck topos classifies is by no means unique, as the following example shows.

**Example 4.15 (MM, §VIII.8)** Let  $\Delta$  be the category of nonempty finite ordinals and order-preserving (i.e.,  $\leq$ -preserving) functions. The presheaf category  $\widehat{\Delta}$  is of paramount importance in algebraic topology and higher

category theory; it is the *category of simplicial sets*. In the indicated section of their book, MacLane and Moerdijk give a detailed proof of the fact that  $\widehat{\Delta}$  classifies the theory of linear orders with distinct top and bottom elements, and order-preserving maps which also preserve top and bottom.

This looks rather different from the category  $\text{Flat}(\Delta)$ !

If geometric theories  $T$  and  $T'$  have equivalent classifying toposes, we call them *Morita equivalent*. In a picture strongly advocated by Olivia Caramello, the classifying topos forms a “bridge” between the theories  $T$  and  $T'$ .

### 4.3 Syntactic categories

In section 4.2 we have already seen (in the notations  $\phi \vdash_{\vec{x}} \psi$  and  $\llbracket \phi \rrbracket_{\vec{x}}$ ) that it is useful to consider so-called *formulas in context*: a formula in context is a pair  $[\vec{x}.\phi]$  where  $\phi$  is a geometric formula and  $\vec{x}$  a finite list of variables which contains all variables which appear freely in  $\phi$ .

Given a geometric theory  $T$  and a geometric sequent  $\phi \vdash_{\vec{x}} \psi$ , we write  $T \models (\phi \vdash_{\vec{x}} \psi)$  to mean that  $\phi \vdash_{\vec{x}} \psi$  is true in every model of  $T$  in every topos.

There is a deduction system for geometric logic, giving a notion  $T \vdash (\phi \vdash_{\vec{x}} \psi)$ , which is described in **Elephant**, §D1.3. We have a *Completeness Theorem*, which says that the notions  $T \models (\phi \vdash_{\vec{x}} \psi)$  and  $T \vdash (\phi \vdash_{\vec{x}} \psi)$  are equivalent; this theorem is outside the scope of these lecture notes. We shall only use the  $\models$ -notion.

We construct for any geometric theory  $T$  a so-called *syntactic category*  $\text{Syn}(T)$ , as follows.

Call two geometric formulas in context  $[\vec{x}.\phi]$  and  $[\vec{y}.\psi]$  equivalent if  $[\vec{y}.\psi]$  is obtained from  $[\vec{x}.\phi]$  by a renaming of variables (both free and bound). An object of  $\text{Syn}(T)$  is an equivalence class of such formulas in context. We shall just write  $[\vec{x}.\phi]$  for its equivalence class.

When discussing arrows from  $[\vec{x}.\phi]$  to  $[\vec{y}.\psi]$  we may, by our convention on equivalence, assume that the contexts  $\vec{x}$  and  $\vec{y}$  are disjoint.

A morphism  $[\vec{x}.\phi] \rightarrow [\vec{y}.\psi]$  in  $\text{Syn}(T)$  is an equivalence class of formulas in context  $[\vec{x}, \vec{y}.\theta]$  which satisfy:

- i)  $T \models (\theta(\vec{x}, \vec{y}) \vdash_{\vec{x}, \vec{y}} \phi(\vec{x}) \wedge \psi(\vec{y}))$ .
- ii)  $T \models (\phi(\vec{x}) \vdash_{\vec{x}} \exists \vec{y} \theta(\vec{x}, \vec{y}))$ .
- iii)  $T \models (\theta(\vec{x}, \vec{y}) \wedge \theta(\vec{x}, \vec{y}') \vdash_{\vec{x}, \vec{y}, \vec{y}'} \vec{y} = \vec{y}')$ .

where in the last clause, for  $\vec{y} = y_1, \dots, y_n$  and  $\vec{y}' = y'_1, \dots, y'_n$ ,  $\vec{y} = \vec{y}'$  abbreviates the formula  $y_1 = y'_1 \wedge \dots \wedge y_n = y'_n$ .

Two such  $\theta(\vec{x}, \vec{y})$  and  $\theta'(\vec{x}, \vec{y})$  represent the same morphism if they are equivalent modulo  $T$ .

Given morphisms  $\theta(\vec{x}, \vec{y}) : [\vec{x}.\phi] \rightarrow [\vec{y}.\psi]$  and  $\xi : [\vec{y}.\psi] \rightarrow [\vec{z}.\chi]$ , the composition  $\xi \circ \theta : [\vec{x}.\phi] \rightarrow [\vec{z}.\chi]$  is represented by the formula  $\exists \vec{y}(\theta(\vec{x}, \vec{y}) \wedge \xi(\vec{y}, \vec{z}))$ . For any object  $[\vec{x}.\phi]$ , the identity arrow  $[\vec{x}.\phi] \rightarrow [\vec{y}.\phi]$  (recall our convention about equivalent formulas in context) is the formula  $x_1 = y_1 \wedge \dots \wedge x_n = y_n$ .

**Exercise 30** Prove that  $\text{Syn}(T)$  is a category.

**Definition 4.16** A *geometric category* is a regular category in which subobject lattices have arbitrary joins, and these joins are stable under pullback.

**Exercise 31** i) Characterize the monomorphisms in the category  $\text{Syn}(T)$ .

ii) Show that  $\text{Syn}(T)$  is a regular category.

iii) Show that  $\text{Syn}(T)$  is a geometric category.

The category  $\text{Syn}(T)$  has a tautological model of  $T$ : for any sort  $S$ ,  $\llbracket S \rrbracket$  is the formula in context  $[x^S.x = x]$ ; for any function symbol  $f : S_1, \dots, S_n \rightarrow T$ , the arrow  $\llbracket f \rrbracket$  is the formula

$$f(x_1^{S_1}, \dots, x_n^{S_n}) = y^T$$

and for any relation symbol  $R \subseteq S_1, \dots, S_n$ , the subobject  $\llbracket R \rrbracket$  is represented by the evident monomorphism with domain  $R(x_1^{S_1}, \dots, x_n^{S_n})$ .

For every geometric category  $\mathcal{C}$ , there is a *geometric topology* on  $\mathcal{C}$ : the covering sieves are those families  $\{f_i : D_i \rightarrow C\}_{i \in I}$  for which the subobject

$$\bigvee_{i \in I} \text{im}(f_i)$$

is the maximal subobject of  $C$  (here  $\text{im}(f_i)$  denotes the image of  $f_i$  as subobject of  $C$ ).

Without proof, we state:

**Theorem 4.17** *Let  $T$  be a geometric theory. For any cocomplete topos  $\mathcal{E}$  (or, for any geometric category  $\mathcal{E}$ ), the category of models of  $T$  in  $\mathcal{E}$  is equivalent to the category of flat and continuous functors from  $\text{Syn}(T)$  to  $\mathcal{E}$ .*

Therefore we have:

**Theorem 4.18 (Classifying Topos Theorem, part II)** *The category  $\text{Sh}(\text{Syn}(T), J)$ , where  $J$  is the geometric topology on  $\text{Syn}(T)$ , is a classifying topos for  $T$ . Hence every geometric theory has a classifying topos.*

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