Teach Yourself Logic 2017: A Study Guide

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Pass it on, ... That's the game I want you to learn. Pass it on.

Alan Bennett, *The History Boys*

The latest version of this Guide can always be downloaded from logicmatters.net/tyl/

URLs in blue are live links to external web-pages or PDF documents. Internal cross-references to sections etc. are also live. But to avoid the text being covered in ugly red rashes internal cross-references are also live links but are not coloured. If you are reading on a computer, your cursor should change as you hover over such live links: *try here*.

The Guide's layout is optimized for reading on-screen: try, for example, reading in iBooks on an iPad, or in double-page mode in Adobe Reader on a laptop.

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A very quick introduction

Before I retired from the University of Cambridge it was my greatest good fortune to have secure, decently paid, university posts for forty years in leisurely times, with almost total freedom to follow my interests wherever they led. Like many of my contemporaries, for most of that time I didn't really appreciate just how lucky I was. This Study Guide to logic textbooks is one very small attempt to give a little back by way of heartfelt thanks.

Please don't be scared off by the Guide's length! This is due to its coverage starting just one step above the level of what is often called 'baby logic' and then going quite a long way down the road towards pretty advanced stuff. Different readers will therefore want to jump on and off the bus at different stops. Simply choose the sections which are most relevant to your background and your interests.

- If you are impatient and want a **quick start** guide to the Guide, read §§1.3, 2.2 and 2.4. Then glance through §3.1 to see whether you already know enough about first-order logic to skip the rest of the chapter on FOL. If you don't have a solid background in FOL, continue from §3.2. Otherwise, start in earnest with Ch. 4.

- For a **slower introduction**, read through Chs. 1 and 2 over a cup of coffee to get a sense of the purpose, shape and coverage of this Guide, and also to get some advice about how to structure your logic reading. That should enable you to sensibly pick and choose your way through the remaining chapters.

- However, if you are hoping for help with **very elementary logic** (e.g. as typically encountered by philosophers in their first-year courses), then let me say straight away that this Guide *isn't* designed for you. The only section that directly pertains to this 'baby logic' is §1.5; all the rest is about rather more advanced – and eventually *very* much more advanced – material.

Chapter 1

Preliminaries

This chapter and the next describe the Guide's aims and structure – and also explain why it is so long. I imagine new readers to be mainly either philosophy or mathematics students at various levels (to be sure, computer scientists and some theoretical linguists can be very interested in logic too: but I haven't written with them in mind). I explain the role of the Guide for those two main constituencies in the opening two sections.

1.1 Why this Guide for philosophers?

It is an odd phenomenon, and a very depressing one too. Logic, beyond the most elementary introduction, is seemingly taught less and less, at least in UK philosophy departments. Fewer and fewer serious logicians get appointed to teaching posts. Yet logic itself is, of course, no less exciting and rewarding a subject than it ever was, and the amount of good formally-informed work in philosophy is ever greater as time goes on. Moreover, logic is far too important to be left entirely to the mercies of technicians from maths or computer science departments with different agendas (who often reveal an insouciant casualness about conceptual details that will matter to the philosophical reader).

So how is a competence in logic to be passed on if there are not enough courses, or are none at all?

It seems then that many beginning graduate students in philosophy – if they are not to be quite dismally uneducated in logic and therefore cut off from working in some of the most exciting areas of their discipline – will need to teach themselves from books, either solo or (better, but not always possible) by organizing their own study groups.

In a way, that's perhaps no real hardship, as there are some wonderful books written by great expositors out there. But *what* to read and work through? Logic books can have a very long shelf life, and you shouldn't at all dismiss older texts when starting out on some topic area: so there's more than a fifty year span of publications to select from. Without having tried very hard, I seem over the years to have accumulated on my own shelves perhaps three hundred formal logic books that might feature somewhere in a Guide such as this – and of course these are still only a selection of what's available.

Philosophy students evidently need a Study Guide if they are to find their way around the large literature old and new: this is my (on-going, still developing) attempt to provide one.

1.2 Why this Guide for mathematicians too?

The situation of logic teaching in mathematics departments can also be pretty dire. Students will of course pick up a passing acquaintance with some very basic notions about sets and some logical symbolism. But there are full university maths courses in good UK universities with precisely zero courses offered on logic, computability theory, or serious set theory. And I believe that the situation can be equally patchy elsewhere.

So if you want to teach yourself some logic or related areas, where should you start? What are the topics you might want to cover? What textbooks are likely to prove accessible and tolerably enjoyable and rewarding to work through? Again, this Guide – or at least, the sections on the core mathematical logic curriculum – will give you pointers.

True, it is written by someone who has, apart from a few guest mini-courses, taught in a philosophy department and who is no research mathematician. Which probably gives a rather distinctive tone to the Guide (and certainly explains why it occasionally ranges into areas of logic of special interest to philosophers). Still, mathematics remains my first love, and these days it is mathematicians whom I mostly get to hang out with. Most of the books I recommend are very definitely paradigm *mathematics* texts. So I won't be leading you astray!

I didn't want to try to write *two* overlapping Guides, one primarily for philosophers and one aimed primarily at mathematicians. This was not just to avoid multiplying work. It would also be difficult to decide what should go where. After all, a number of philosophers will develop serious interests in more mathematical corners of the broad field of logic; and a number of mathematicians will find themselves becoming interested in more foundational/conceptual issues. So rather than impose artificial divisions, I provide here a single but wide-ranging menu for everyone to choose from as their interests dictate.

1.3 A strategy for reading logic books (and why this Guide is so long)

We cover a lot of ground in this Guide then, which is one reason for its initially daunting length. But there is another reason which I want to highlight:

> I very strongly recommend tackling an area of logic by reading a series of books which *overlap* in level (with the next one covering some of the same ground and then pushing on from the previous one), rather than trying to proceed by big leaps.

In fact, I probably can't stress this advice too much (which applies equally to getting to grips with any new area of mathematics). This approach will really help to reinforce and deepen understanding as you re-encounter the same material at different levels, coming at it from different angles, with different emphases.

Exaggerating only a little, there are many instructors who say 'This is the textbook we are using/here is my set of notes: take it or leave it'. But you will always gain from looking at some overlapping texts. (When responding to student queries on a question-and-answer internet site, I'm repeatedly struck by how much puzzlement would be quickly resolved by taking the occasional look outside the course textbook/lecturer's notes.)

The multiple overlaps in coverage in the reading lists below, which make the Guide rather long, are therefore fully intended. They also mean that you should always be able to find options that suit your degree of mathematical competence.

To repeat: you will certainly miss a lot if you concentrate on just one text in a given area, especially at the outset. Yes, do very carefully read one or two central texts, at a level that appeals to you. But do also cultivate the crucial further habit of judiciously skipping and skimming through a number of other works so that you can build up a good overall picture of an area seen from various somewhat different angles of approach.

1.4 On the question of exercises

While we are talking about strategies for reading logic books, I should say something very briefly on the question of doing exercises.

Mathematics is, as they say, not merely a spectator sport: so you should try some of the exercises in the books as you read along to check and reinforce comprehension. On the other hand, don't obsess about doing exercises if you are a philosopher – understanding proof ideas is very much the crucial thing, not the ability to roll-your-own proofs. And even mathematicians shouldn't get too hung up on routine additional exercises beyond those needed to initially fix ideas (unless you have specific exams to prepare for!): concentrate on the exercises that look interesting and/or might deepen understanding.

Do note however that some authors have the irritating(?) habit of burying important results among the exercises, mixed in with routine homework. It is therefore always a good policy to skim through the exercises in a book even if you don't plan to work on answers to very many of them.

1.5 Assumed background

So what do you need to bring to the party, if you are going to tackle some of the books recommended in the body of this Guide?

If you are a mathematician, there is no specific assumed background you need before tackling the books recommended in chapters to come. The entry-level books mentioned in Chs 3 and 4 don't presuppose much 'mathematical maturity'. So you can really just dive in – and here in this Guide, now skip on to §1.7.

If, however, you are a philosopher without any mathematical background, then how to proceed will depend on how much logic you have already encountered. Let's distinguish three levels you might have reached:

L1. If you have only done an 'informal logic' or 'critical reasoning course', then you'll probably need to read a good introductory formal logic text before tackling the more advanced work covered in this Guide. See below.

L2. If you *have* taken a logic course with a formal element, but it was based on some really, *really*, elementary text book like Sam Guttenplan's *The Languages of Logic*, Howard Kahane's *Logic and Philosophy*, or Patrick Hurley's *Concise Introduction to Logic* (to mention some frequently used texts), then you *might* still struggle with the initial suggestions in this Guide – though this will of course vary a lot from person to person. So the best advice is probably just to make a start and see how you go. If you do struggle, one possibility would be to use *Intermediate Logic* by Bostock mentioned in §3.3 to bridge the gap between what you know and the beginnings of mathematical logic. Or again, try one of the books I'm about to mention, skipping quickly over what you already know.

L3. If you have taken an elementary logic course based on a substantial text like the ones mentioned in just a moment, then you should be well prepared.

Here then, for those that need them, are two initial suggestions of formal logic books that start from scratch and go far enough to provide a good foundation for further work – the core chapters of these cover the so-called 'baby logic' that it would be ideal for a non-mathematician to have under his or her belt:

1. My *Introduction to Formal Logic** (CUP 2003: a second edition is in preparation): for more details see the IFL pages, where there are also answers to the exercises). This is intended for beginners, and was the first year text in Cambridge for a decade. It was written as an accessible teach-yourself book, covering propositional and predicate logic 'by trees'. It in fact gets as far as a completeness proof for the tree system of predicate logic without identity, though for a beginner's book that's very much an optional extra.

2. Paul Teller's *A Modern Formal Logic Primer*** (Prentice Hall 1989) predates my book, is now out of print, but the scanned pages freely available online at the book's website, which makes it unbeatable value! The book (two slim volumes) is in many ways excellent, and had I known about it at the time (or listened to Paul's good advice, when I got to know him, about how long it takes to write an intro book), I'm not sure that I'd have written my own book, full of good things though it is! As well as introducing trees, Teller also covers a version of 'Fitch-style' natural deduction – regrettably, I didn't have the page allowance to do this in the first edition of my book. (Like me, he also goes beyond the really elementary by getting as far as a completeness proof.) Notably user-friendly. Answers to exercises are available at the author's website.

Of course, those are just two possibilities from very many.

This is not the place to discuss lots more options for elementary logic texts (indeed, I have not in recent years kept up with all of the seemingly never-ending flow of new alternatives). But despite that, I *will* mention here two other books:

3. I have been asked frequently about Dave Barker-Plummer, Jon Barwise and John Etchemendy's *Language, Proof and Logic* (CSLI Publications, 1999; 2nd edition 2011). The unique selling point for this widely used book is that it comes as part of a 'courseware package', which includes software such as

a famous program called ‘Tarski’s World’ in which you build model worlds and can query whether various first-order sentences are true of them. Some students really like it, but at least equally many don’t find this kind of thing particularly useful. There is an associated online course from Stanford, with video lectures by the authors which you can watch for free, though you have to buy the courseware package to complete the course with a ‘statement of accomplishment’. For more details, see the book’s website.

This is another book which is in many respects user-friendly, goes slowly, and does Fitch-style natural deduction. It is a very respectable option. But Teller is rather snappier, I think no less clear, and certainly wins on price!

4. Nicholas Smith’s recent *Logic: The Laws of Truth* (Princeton UP 2012) is very clearly written and seems to have many virtues (if you like your texts to go slowly and discursively). The first two parts of the book overlap very largely with mine (it too introduces logic by trees). But the third part ranges wider, including a brisk foray into natural deduction – indeed the logical coverage goes almost as far as Bostock’s book, mentioned below in §3.3, and are there some extras too. It is a particularly readable addition to the introductory literature. I have commented further here. Answers to exercises can be found at the book’s website.

1.6 Do you *really* need more logic?

This section is again for philosophers; mathematicians can skip on to the next section.

It is perhaps worth pausing to ask whether you, as a budding philosopher, really *do* want or need to pursue your logical studies much further if you have already worked through a book like mine or Paul Teller’s or Nick Smith’s. Far be it from me to put people off doing more logic: perish the thought! But for many philosophical purposes, you might well survive by just reading this:

> Eric Steinhart, *More Precisely: The Math You Need to Do Philosophy** (Broadview 2009) The author writes: ‘The topics presented …include: basic set theory; relations and functions; machines; probability; formal semantics; utilitarianism; and infinity. The chapters on sets, relations, and functions provide you with all you need to know to apply set theory in any branch of philosophy. The chapter of machines includes finite state machines, networks of machines, the game of life, and Turing machines. The chapter on formal semantics includes both extensional semantics, Kripkean

> possible worlds semantics, and Lewisian counterpart theory. The chapter on probability covers basic probability, conditional probability, Bayes theorem, and various applications of Bayes theorem in philosophy. The chapter on utilitarianism covers act utilitarianism, applications involving utility and probability (expected utility), and applications involving possible worlds and utility. The chapters on infinity cover recursive definitions, limits, countable infinity, Cantor's diagonal and power set arguments, uncountable infinities, the aleph and beth numbers, and definitions by transfinite recursion. *More Precisely* is designed both as a text book and reference book to meet the needs of upper level undergraduates and graduate students. It is also useful as a reference book for any philosopher working today.'

Steinhart's book is admirable, and will give many philosophers a handle on some technical ideas going well beyond 'baby logic' and which they really should know just a little about, without all the hard work of doing a full mathematical logic course. What's not to like? It could be enough for you. And then, if there indeed turns out to be some particular area (modal logic, for example) that seems especially germane to your particular philosophical interests, you always can go to the relevant section of this Guide for more.

1.7 How to prove it

Before getting down to the main business let me mention one other terrific book:

> Daniel J. Velleman, *How to Prove It: A Structured Approach** (CUP, 2nd edition, 2006). From the Preface: 'Students of mathematics ... often have trouble the first time that they're asked to work seriously with mathematical proofs, because they don't know 'the rules of the game'. What is expected of you if you are asked to prove something? What distinguishes a correct proof from an incorrect one? This book is intended to help students learn the answers to these questions by spelling out the underlying principles involved in the construction of proofs.'

There are chapters on the propositional connectives and quantifiers, and *informal* proof-strategies for using them, and chapters on relations and functions, a chapter on mathematical induction, and a final chapter on infinite sets (countable vs. uncountable sets). This truly excellent student text could certainly be of use both to many philosophers and perhaps also to some mathematicians reading this Guide.[1]

True, if you are a mathematician who has got to the point of embarking on an upper level undergraduate course in some area of mathematical logic, you *should* certainly have already mastered nearly all the content of Velleman's splendidly clear book. However, a few hours speed-reading through this text (except perhaps for the very final section), pausing over anything that doesn't look very comfortably familiar, could still be time extremely well spent.

What if you are a philosophy student who (as we are now assuming) has done some elementary logic? Well, experience shows that being able to handle e.g. natural deduction proofs in a formal system doesn't always translate into being able to construct good *informal* proofs. For example, one of the few meta-theoretic results that might be met in a first logic course is the expressive completeness of the set of formal connectives $\{\wedge, \vee, \neg\}$. The proof of this result is really easy, based on a simple proof-idea. But many students who can ace the part of the end-of-course exam asking for quite complex *formal* proofs inside a deductive system find themselves all at sea when asked to replicate this *informal* bookwork proof *about* a formal system.

Another example: it is only too familiar to find philosophy students introduced to set notation not even being able to make a start on a good informal proof that $\{\{a\}, \{a, b\}\} = \{\{a'\}, \{a', b'\}\}$ if and only if $a = a'$ and $b = b'$.

Well, if you *are* one of those students who jumped through the formal hoops but were unclear about how to set out elementary mathematical proofs (e.g. from the 'meta-theory' theory of baby logic, or from very introductory set theory), then again working through Velleman's book from the beginning could be just what you need to get you prepared for the serious study of logic. And even if you were one of those comfortable with the informal proofs, you will probably still profit from skipping and skimming through (perhaps paying especial attention to the chapter on mathematical induction).

[1] By the way, if you want to check your answers to exercises in the book, here's a long series of blog posts (in reverse order).

Chapter 2

How the Guide is structured

This chapter, still by way of introduction, explains how the field of logic is being carved up into subfields in this Guide. It also explains how many of these subfields are visited twice in the Guide, once to give entry-level readings, and then again later when we consider some more advanced texts.

2.1 Logic, in the broad sense

'Logic', in the broad sense, is a big field. Its technical development is of concern to philosophers and mathematicians, not to mention computer scientists (and others). Different constituencies will be particularly interested in different areas and give different emphases. Core 'classical first-order logic' is basic by anyone's lights. But after that, interests can diverge. For example, modal logic is of considerable interest to some philosophers, not so much to mathematicians, though parts of this sub-discipline are of concern to computer scientists. Set theory (which falls within the purview of mathematical logic, broadly understood) is an active area of research interest in mathematics, but – because of its (supposed) foundational status – even quite advanced results can be of interest to philosophers too. Type theory started off as a device of philosophy-minded logicians looking to avoid the paradoxes: it has become primarily the playground of computer scientists. The incompleteness theorems are relatively elementary results of the theory of computable functions, but are of particular conceptual interest to philosophers. Finite model theory is of interest to mathematicians and computer scientists, but perhaps not so much to philosophers. And so it goes.

In this Guide, I'm going to have to let the computer scientists and others largely look after themselves. Our main focus is going to be on the core mathematical logic curriculum of most concern to philosophers and mathematicians, together with some extras of particular interest to philosophers – if only because that's what I know a little about.

So what's the geography of these areas?

2.2 Mapping the field

Here's an overview map of the territory we cover in the Guide:

- **First-order logic (Ch. 3)**. The serious study of logic always starts with a reasonably rigorous treatment of quantification theory, covering both a proof-system for classical first-order logic (FOL), and the standard classical semantics, getting at least as far as a soundness and completeness proof for your favourite proof system. I both give the headline news about the topics that need to be covered and also suggest a number of different ways of covering them (I'm not suggesting you read all the texts mentioned!). This part of the Guide is therefore quite long even though it doesn't cover a lot of ground: it does, however, provide the essential foundation for ...

- **Continuing the basic 'Mathematical Logic' curriculum (Ch. 4)** Mathematical logic programmes typically comprise – in one order or another and in various proportions – three or perhaps four elements in addition to a serious treatment of FOL:

    1. **A little model theory**, i.e. a little more exploration of the fit between theories cast framed in formal languages and the structures they are supposed to be 'about'. This will start with the compactness theorem and Löwenheim-Skolem theorems (if these aren't already covered in your basic FOL reading), and then will push on just a bit further. You will need to know a very little set theory as background (mainly ideas about cardinality), so you might want to interweave beginning model theory with the very beginnings of your work on set theory if those cardinality ideas are new to you.
    2. **Computability and decidability**, and proofs of epochal results such as Gödel's incompleteness theorems. This is perhaps the most readily approachable area of mathematical logic.
    3. **Some introductory set theory.** Informal set theory, basic notions of cardinals and ordinals, constructions in set theory, the role of the axiom of choice, etc. The formal axiomatization of ZFC.

4. **Extras: variants of standard FOL** The additional material that you could (should?) meet in a first serious encounter with mathematical logic includes:

   (a) Second Order Logic (where we can quantify over properties as well as objects), and

   (b) Intuitionistic Logic (which drops the law of excluded middle, motivated by a non-classical understanding of the significance of the logical operators).

   These two variants are both of technical and philosophical interest.

- **Modal and other logics (Ch. 5)** In this chapter, we consider a number of logical topics of particular concern to philosophers (many mathematicians will want to skip – though perhaps you should at least know that there *is* a logical literature on these topics). These topics are actually quite approachable even if you know little other logic. So the fact that this material is mentioned after the heavier-duty mathematical topics in Ch. 4 doesn't mean that there is a jump in difficulty from Ch. 3. We look at:

  1. **Modal logic** Even before encountering a full-on treatment of first-order logic, philosophers are often introduced to modal logic (the logic of necessity and possibility) and to its 'possible world semantics'. You can indeed do an amount of propositional modal logic with no more than 'baby logic' as background.

  2. **Other classical variations** and extensions of standard logic which are of conceptual interest but still classical in spirit, e.g. free logic, plural logic.

  3. **Further non-classical variations** The most important non-classical logic – and the one of real interest to mathematicians too – is intuitionist logic which we've already mentioned. But here we might also consider e.g. relevant logics which drop the classical rule that a contradiction entails anything.

- **Exploring further in core mathematical logic (Ch. 6)** The introductory texts mentioned in Chapters 3 and 4 will already contain numerous pointers onwards to further books, more than enough to put you in a position to continue exploring solo. However, Chapter 6 offers my own suggestions for more advanced reading on model theory, computability, formal arithmetic, and on proof theory too. This chapter is for specialist graduate students (among philosophers) and for final year undergraduate/beginning graduate students (among mathematicians).

- **Serious set theory (Ch 7)** We continue the exploration beyond the basics of set theory covered in Ch. 4 with a chapter on more advanced set theory.

- **Other topics (Ch. 8)** The very brief final chapter mentions a number of additional topics, with a pointer to my web-page on category theory.

Don't be alarmed if (some of) the descriptions of topics above are at the moment opaque to you: we will be explaining things rather more as we go through the Guide.

2.3 Three comments on the Guide's structure

Three comments on all this:

1. The Guide divides up the broad field of logic into subfields in a pretty conventional way. But of course, even the 'horizontal' divisions into different areas can in places be a little arbitrary. And the 'vertical' division between the entry-level readings on mathematical logic in Chapter 4 and the further explorations in Chapter 6 is necessarily going to be a lot more arbitrary. I think that everyone will agree (at least in retrospect!) that e.g. the elementary theory of ordinals and cardinals belongs to the basics of set theory, while explorations of 'large cardinals' or independence proofs via forcing are decidedly advanced. But in most areas, there are far fewer natural demarcation lines between the entry-level basics and more advanced work. Still, it is surely very much better to have *some* such structuring than to heap everything together.

2. Within sections in the coming chapters, I have usually put the main recommendations into what strikes me as a sensible reading order of increasing difficulty (without of course supposing you will want to read everything – those with stronger mathematical backgrounds might sometimes want to try starting in the middle of a list). Some further books are listed in asides or postscripts.

3. The Guide used also to have a substantial Appendix considering some of 'The Big Books on mathematical logic' (meaning typically broader-focus

books that cover first-order logic together with one or more subfields from the further menu of mathematical logic). These books vary a lot in level and coverage, but can provide very useful consolidating/amplifying reading. This supplement to the main Guide is now available online as a separate Appendix which I hope to continue adding to. Alternatively, for individual webpages on those texts and a number of additional reviews, visit the Logic Matters Book Notes.

2.4 Choices, choices

(a) So what has guided my choices of what to recommend within the sections of this Guide?

Different people find different expository styles congenial. For example, what is agreeably discursive for one reader is irritatingly verbose and slow-moving for another. For myself, I do particularly like books that are good on conceptual details and good at explaining the motivation for the technicalities while avoiding needless complications, excessive hacking through routine detail, or misplaced 'rigour', though I do like elegance too. Given the choice, I tend to prefer a treatment that doesn't rush too fast to become too general, too abstract, and thereby obscures intuitive motivation: this is surely what we want in books to be used for self-study. (There's a certain tradition of masochism in older maths writing, of going for brusque formal abstraction from the outset with little by way of explanatory chat: this is quite unnecessary in other areas, and just because logic is all about formal theories, that doesn't make it any more necessary here.)

The selection of books in the following chapters no doubt reflects these tastes. But overall, I don't think that I have been downright idiosyncratic. Nearly all the books I recommend will very widely be agreed to have significant virtues (even if other logicians would have different preference-orderings).

(b) Most of the books mentioned here should be held by any university library which has been paying reasonable attention to maintaining core collections in mathematics and philosophy, and every book should be borrowable through your local inter-library loans system. (We must pass over in silence the question of using the well-known copyright-infringing file repositories.)

Since I'm not assuming you will be buying personal copies, I have *not* made cost or even being currently in print a significant consideration: indeed it has to be said that the list price of some of the books is just ridiculous (though second-hand copies of some books at better prices might well be available via

Amazon sellers or from abebooks.com). However, I have marked with one star* books that are available new at a reasonable price (or at least are unusually good value for the length and/or importance of the book). And I've marked with two stars** those books for which e-copies are freely and legally available, and links are provided. Most articles, encyclopaedia entries, etc., can also be downloaded, again with links supplied.

(c) And yes, the references here *are* very largely to published books and articles rather than to on-line lecture notes etc. Many such notes are excellent, but they do tend to be a pretty terse (as entirely befits material intended to support a lecture course) and so they are perhaps not as helpful as fully-worked-out book-length treatments for students needing to teach themselves. But I'm sure that there is an increasing number of excellent e-resources out there which do amount, more or less, to free stand-alone textbooks: I mention a couple of recent arrivals, and I'd be very happy indeed to get recommendations about others.

(d) Finally, the earliest versions of this Guide kept largely to positive recommendations: I didn't pause to explain the reasons for the then absence of some well-known books. This was partly due to considerations of length which have now quite gone by the wayside; but also I wanted to keep the tone enthusiastic, rather than to start criticizing or carping.

However, enough people kept asking what I think about an alternative X, or asking why the old warhorse Y wasn't mentioned, to change my mind. So I have occasionally added some reasons why I *don't* particularly recommended certain books.

Chapter 3

First order logic

So, at last, we get down to work! This chapter starts with a checklist of the topics we will be treating as belonging to the basics of first-order logic (predicate logic, quantificational logic, call it what you will: we'll use 'FOL' for short). There are then some recommendations for texts covering these topics.

3.1 FOL: the basic topics

For those who already have some significant background in formal logic, this checklist will enable you to determine whether you can skip forward in the Guide. If you find that you are indeed familiar with these topics, then go straight to the next chapter.

Alternatively, for those who know only (some fragments of) elementary logic, this list might provide some preliminary orientation – though don't worry, of course, if at present you don't fully grasp the import of every point: the next section gives recommended reading to help you reach enlightenment!

So, without further ado, here in headline terms is what you need to get to know about:

1. Starting with *syntax*, you need to know how first-order *languages* are constructed. And now you ideally should get to understand how to prove various things about such languages that might seem obvious and that you previously took for granted, e.g. that 'bracketing works' to avoid ambiguities, meaning that every well-formed formula has a unique parsing.

   By the way, it is worth remarking that introductory logic courses for philosophers very often ignore *functions*; but given that FOL is deployed to regiment everyday mathematical reasoning, and that functions are of course crucial to mathematics, function expressions now become centrally important (even if there are tricks that make them in principle eliminable).

2. On the *semantic* side, you need to understand the idea of a structure (a domain of objects equipped with some relations and/or functions, and perhaps having some particular objects especially picked out to be the denotations of constants in the language); and you need to grasp the idea of an interpretation of a language in such a structure. You'll also need to understand how an interpretation generates a unique assignment of truth-values to every sentence of the interpreted language – this means grasping a proper formal semantic story with the bells and whistles required to cope with quantifiers adequately.

   With these ideas to hand, you now can define the crucial relation of semantic entailment, where the set of sentences Γ semantically entails φ when no interpretation in any appropriate structure can make all the sentences among Γ true without making φ true too. You'll need to know some of the basic properties of this relation.

3. Back to syntax: you need to get to know a deductive proof-system for FOL reasonably well. But what sort of system should you explore first? It is surely natural to give centre stage, at least at the outset, to so-called *natural deduction* systems.

   The key feature of such systems is that they allow us to make temporary assumptions 'for the sake of argument' and then later discharge the temporary assumptions. This is, of course, exactly what we do all the time in everyday reasoning, mathematical or otherwise – as, for example, when we temporarily suppose A, show it leads to absurdity, and then drop the supposition and conclude $\neg A$. So surely it is indeed natural to seek to formalize these key informal ways of reasoning.

   Different formal natural deduction systems will offer different ways of handling temporary assumptions, keeping track of them while they stay 'live', and then showing where in the argument they get discharged. Let's give a couple of examples of familiar styles of layout.

   Suppose, then, we want to show that from $\neg(P \wedge \neg Q)$ we can infer $(P \to Q)$ (where '$\neg$', '$\wedge$' and '$\to$' are of course our symbols for, respectively, *not*, *and*, and [roughly] *implies*.) Then one way of laying out a natural deduction proof would be like this, where we have added informal commentary on the right:

$$
\begin{array}{lll}
1 & \neg(P \wedge \neg Q) & \text{premiss} \\
2 & \mid\ P & \text{supposition for the sake of argument} \\
3 & \mid\ \mid\ \neg Q & \text{supposition for the sake of argument} \\
4 & \mid\ \mid\ (P \wedge \neg Q) & \text{from 2, 3 by and-introduction} \\
5 & \mid\ \mid\ \bot & \text{1 and 4 give us a contradiction} \\
6 & \mid\ \neg\neg Q & \text{supp'n 3 must be false, by reductio} \\
7 & \mid\ Q & \text{from 6, eliminating the double negation} \\
8 & (P \to Q) & \text{if the supposition 2 is true, so is 7.}
\end{array}
$$

Here, the basic layout principle for such Fitch-style proofs[1] is that, whenever we make a new temporary assumption we indent the line of argument a column to the right (vertical bars marking the indented columns), and whenever we 'discharge' the assumption at the top of an indented column we move back to the left.

An alternative layout (going back to Genzten) would display the same proof like this, where the guiding idea is (roughly speaking) that a wff below an inference line follows from what is immediately above it, or by discharge of some earlier assumption:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P]^{(2)} \qquad [\neg Q]^{(1)}}{(P \wedge \neg Q)} \qquad \neg(P \wedge \neg Q)}{\bot}}{\neg\neg Q}\,(1)}{Q}}{(P \to Q)}\,(2)
$$

Here numerical labels are used to indicate where a supposition gets discharged, and the supposition to be discharged gets bracketed off (or struck through) and also given the corresponding label.

Fitch-style proofs are perhaps easier to use for beginners (indeed, we might say, especially natural, by virtue of more closely follow the basically

[1]See the introductory texts by Teller or by Barker-Plummer, Barwise and Etchemendy mentioned in §1.5 for more examples of natural deduction done in this style.

linear style of ordinary reasoning). But Gentzen-style proofs are usually preferred for more advanced work, and that's what we find in the natural deduction texts that I'll be mentioning below (as contrasted with more elementary texts like Teller's).

4. Next – and for philosophers this is likely to be the key move beyond even a substantial first logic course, and for mathematicians this is probably the point at which things actually start getting really interesting – you need to know how to prove a *soundness* and a *completeness* theorem for your favourite deductive system for first-order logic. That is to say, you need to be able to show that there's a deduction in your chosen system of a conclusion from given premisses only if the premisses do indeed semantically entail the conclusion (the system doesn't give false positives), and whenever an inference is semantically valid there's a formal deduction of the conclusion from the premisses (the system captures all the semantical entailments).

5. As an optional bonus at this stage, depending on what text you read, you might catch a first glimpse of e.g. the downward Löwenheim-Skolem theorem (roughly, assuming we are dealing with an ordinary sort of first-order language L, if there is any infinite model that makes a given set of L-sentences Γ all true, then there is a model which does this whose domain is the natural numbers), and the compactness theorem (if Γ semantically entails φ then there is some finite subset of Γ which already semantically entails φ). These are initial results of model theory which flow quickly from the proof of the completeness theorem.

   However, we will be returning to these results in §4.1 when we consider model theory proper, so it *won't* be assumed that your knowledge of basic FOL covers them.

6. Of course you don't want to get confused at the very outset by running together variant versions of standard FOL. However, you should eventually become aware of some of the alternative ways of doing things, so you don't become fazed when you encounter different approaches in different more advanced texts. In particular, you will need to get to know a little about old-school Hilbert-style *axiomatic* linear proof-systems. A standard such system for e.g. the propositional calculus has a single rule of inference (modus ponens) and a bunch of axioms which can be called on at any stage in a proof. A Hilbert-style proof is just a linear sequence of wffs, each one of which is a given premiss, or a logical axiom, or follows from earlier wffs

in the sequence by modus ponens. Such systems in their unadorned form are pretty horrible to work inside (proofs can be long and very unobvious), even though their Bauhaus simplicity makes them easy to theorize about from the outside. It does strike me as potentially off-putting, even a bit masochistic, to concentrate entirely on axiomatic systems when beginning serious logic – Wilfrid Hodges rightly calls them 'barbarously unintuitive'. But for various reasons they are often used in more advanced texts, so you certainly need to get to know about them sooner or later.

3.2 The main recommendations on FOL

There is, unsurprisingly, a long list of possible texts. But here are four of the best (and in this and the next couple of chapters we'll use display boxes to mark off main recommendations like these):

Let's start with a couple of stand-out books which, taken together, make an excellent introduction to the serious study of FOL.

1. Ian Chiswell and Wilfrid Hodges, *Mathematical Logic* (OUP 2007). This very nicely written text is only one notch up in actual difficulty from 'baby logic' texts like mine or Paul Teller's or Nick Smith's: but – as its title might suggest – it does have a notably more mathematical 'look and feel' (being indeed written by mathematicians). Despite that, it remains particularly friendly and approachable and should be entirely manageable for self study by philosophers and mathematicians alike. It is also pleasingly short. Indeed, I'm rather tempted to say that if you *don't* like this lovely book then serious logic might not be for you!

   The briefest headline news is that authors explore a (Gentzen-style) natural deduction system. But by building things up in three stages – so after propositional logic, they consider an interesting fragment of first-order logic before turning to the full-strength version – they make proofs of e.g. the completeness theorem for first-order logic quite unusually comprehensible. For a more detailed description see my book note on C&H.

   Very warmly recommended, then. For the moment, you only *need* read up to and including §7.7; but having got that far, you might as well read the final couple of sections and the Postlude too! (The book has brisk solutions to some exercises. A demerit mark to OUP for not publishing C&H more cheaply.)

Next, complement C&H by reading the first half of

2. Christopher Leary and Lars Kristiansen's *A Friendly Introduction to Mathematical Logic** (1st edn by Leary alone, Prentice Hall 2000; inexpensive 2nd edn Milne Library 2015; for the first three chapters, either edition is fine). There is a great deal to like about this book. Chs. 1–3 do make a very friendly and helpful introduction to first-order logic, this time done in axiomatic style. At this stage you could stop reading after §3.2, which means you will be reading just over 100 pages. Unusually, L&K dive straight into a full treatment of FOL without spending an introductory chapter or two on propositional logic: but that happily means (in the present context) that you won't feel that you are labouring through the very beginnings of logic one more time than is really necessary – so this book dovetails very nicely with C&H. The book *is* again written by mathematicians for a mostly mathematical audience so some illustrations of ideas can presuppose a smattering of elementary background mathematical knowledge; but you will miss very little if you occasionally have to skip an example (and curious philosophers can always resort to Wikipedia, which is quite reliable in this area, for explanations of mathematical terms). I like the tone very much indeed, and say more about this admirable book in another book note.

Now here's an alternative to the C&H/L&K pairing which is also wonderfully approachable and can be warmly recommended:

3. Derek Goldrei's *Propositional and Predicate Calculus: A Model of Argument** (Springer, 2005) is explicitly designed for self-study. Read Chs. 1 to 5 (you could skip §§4.4 and 4.5, leaving them until you turn to elementary model theory). While C&H and the first half of L&K together cover overlapping material twice, Goldrei – in much the same total number of pages – covers very similar ground once. So this is a somewhat more gently-paced book, allowing Goldrei to be more expansive about fundamentals, and to give a lot of examples and exercises to test comprehension along the way. A very great deal of thought has gone into making

this text as helpful as possible. So if you struggle slightly with the alternative reading, or just want a comfortingly manageable additional text, you should find this exceptionally accessible and useful. Or you might just warm more to Goldrei's style anyway.

Like L&K, Goldrei uses an axiomatic system (which is one reason why, on balance, I still recommend starting with C&H instead: you'll need to get to know about natural deduction at some point). As with C&H and L&K, I like the tone and approach a great deal.

Fourthly, even though it is giving a second bite to an author we've already met, I must mention

4. Wilfrid Hodges's 'Elementary Predicate Logic', in the *Handbook of Philosophical Logic*, Vol. 1, ed. by D. Gabbay and F. Guenthner, (Kluwer 2nd edition 2001). This is a slightly expanded version of the essay in the first edition of the *Handbook* (read that earlier version if this one isn't available), and is written with Hodges's usual enviable clarity and verve. As befits an essay aimed at philosophically minded logicians, it is full of conceptual insights, historical asides, comparisons of different ways of doing things, etc., so it very nicely complements the more conventional textbook presentations of C&H and L&K (or of Goldrei). Read at this stage the first twenty sections (70 pp.): they are wonderfully illuminating.

Note to philosophers: if you have carefully read and mastered a book covering baby-logic-plus-a-little, you could well already know about quite a lot of the material covered in the reading mentioned so far, except perhaps for the completeness theorems. However, the big change is that you will now have begun to see the familiar old material being re-presented in the sort of mathematical style and with the sort of rigorous detail that you will necessarily encounter more and more as you progress in logic. You very much need to start feeling entirely comfortable with this mode of presentation at an early stage.

3.3 Some parallel reading

The material covered in the last section is so very fundamental, and the alternative options so very many, that I really do need to say at least something about a few other books and note some different approaches.

So in this section I will list – in rough order of difficulty/sophistication – a small handful of further texts which could well make for useful parallel reading at different levels. In the following section I'll mention just three books which push on the discussion of FOL in different ways. In the final section of this chapter of the Guide, I will then mention some other books I've been asked about but don't particularly recommend for one reason or another.

If you've read C&H and L&K you will know about both natural deduction and axiomatic approaches to logic. If you are a philosopher, you may also have already encountered a (downward-branching) 'tree' or 'tableau' system of logic which is often used in introductory logic courses. To illustrate, a tableau proof to warrant the same illustrative inference from $\neg(P \land \neg Q)$ to $(P \to Q)$ runs as follows:

$$\begin{array}{c} \neg(P \land \neg Q) \\ \neg(P \to Q) \\ P \\ \neg Q \end{array}$$

$$\begin{array}{cc} \neg P & \neg\neg Q \\ \star & \star \end{array}$$

What we are doing here (if you haven't seen this sort of thing before) is starting with the given premiss and the *negation* of the target conclusion; we then proceed to unpack the implications of these assumptions for their component wffs. In this case, for the second (negated-conditional) premiss to be true, it must have a true antecedent and false consequent: while for the first premiss to be true we have to consider cases (a false conjunction must have at least one false conjunct but we don't know in advance which to blame) – which is why the tree splits. But as we pursue either branch we immediately find a contradiction (conventionally marked with a star in tableau systems), showing that we can't after all consistently have the premiss true and the *negation* of the conclusion true too, so the inference is indeed valid.

If this is new to you, it is worth catching up. Unsurprisingly(!), I still think the best way to do this might be quickly to read the gentle treatment in

1. Peter Smith *Introduction to Formal Logic** (CUP 2003). Chs 16–18 cover trees for propositional logic and then Chs. 25, 29 are on on quantifier trees. (Chs 19 and 30 give soundness and completeness proofs for the tree system, and could also be illuminating.).

Or you could jump straight to the next recommendation which covers trees but also a considerable amount more.

This is a text by a philosopher, aimed at philosophers, though mathematicians could still profit from a quicker browse given the excellent coverage:

2. David Bostock's *Intermediate Logic* (OUP 1997) ranges more widely but not as deeply as Goldrei, for example, and in a more discursive style. From the preface: 'The book is confined to . . . what is called first-order predicate logic, but it aims to treat this subject in very much more detail than a standard introductory text. In particular, whereas an introductory text will pursue just one style of semantics, just one method of proof, and so on, this book aims to create a wider and a deeper understanding by showing how several alternative approaches are possible, and by introducing comparisons between them.' So Bostock does indeed usefully introduce you to tableaux (trees) *and* an Hilbert-style axiomatic proof system *and* natural deduction *and* even a so-called sequent calculus as well (as noted, it is important eventually to understand what is going on in these different kinds of proof-system). Anyone could profit from at least a quick browse of his Part II to pick up the headline news about the various approaches.

   Bostock eventually touches on issues of philosophical interest such as free logic which are not often dealt with in other books at this level. Still, the discussions mostly remain at much the same level of conceptual/mathematical difficulty as the later parts of Teller's book and my own. In particular, he proves completeness for tableaux in particular, which I always think makes the needed construction seem particularly natural. *Intermediate Logic* should therefore be, as intended, particularly accessible to philosophers who haven't done much formal logic before and should, if required, help ease the transition to work in the more mathematical style of the books mentioned in the last section.

Next let me mention a freely available alternative presentation of logic via natural deduction:

3. Neil Tennant, *Natural Logic*** (Edinburgh UP 1978, 1990). Now out of print, but downloadable as a scanned PDF.

   All credit to the author for writing the first textbook aimed at an introductory level which does Gentzen-style natural deduction. Tennant thinks that this approach to logic is philosophically highly significant, and in various ways this shows through in his textbook. Although not as conventionally mathematical in look-and-feel as some alternatives, it is in fact *very* careful about important details.

   This is not always an easy read, however, despite its being intended as a first logic text for philosophers, which is why I didn't mention it in the last section. However the book is there to freely sample, and some may well find it highly illuminating parallel reading on natural deduction.

Now, I recommended L&K's *A Friendly Introduction* as a follow-up to C&H: but the first edition might not be in every library (though do ensure your library gets the inexpensive new edition). As a possible alternative, here is an older and much used text which should certainly be very widely available:

4. Herbert Enderton's *A Mathematical Introduction to Logic* (Academic Press 1972, 2002) also focuses on a Hilbert-style axiomatic system, and is often regarded as a classic of exposition. However, it does strike me as a little more difficult than Leary, so I'm not surprised that some students report finding it a bit challenging *if used by itself as a first text*. Still, it is an admirable and very reliable piece of work which you should be able to cope with as a second text, e.g. after you have tackled C&H. Read up to and including §2.5 or §2.6 at this stage. Later, you can finish the rest of that chapter to take you a bit further into model theory. For more about this classic, see this book note.

Lastly in this section I'll mention – though this time with some hesitation – another much used text. This has gone through multiple editions and should also be in any library, making it a useful natural-deduction based alternative to C&H if the latter isn't available. Later chapters of this book are also mentioned below in the Guide as possible reading for more advanced work, so it could be worth making early acquaintance with ...

5. Dirk van Dalen, *Logic and Structure** (Springer, 1980; 5th edition 2012). The chapters up to and including §3.2 provide an introduction to FOL via natural-deduction. The treatment *is* often approachable and written with a relatively light touch. However – and this explains my hesitation – it has to be said that the book isn't without its quirks and flaws and inconsistencies of presentation (though perhaps you have to be an alert and pernickety reader to notice and be bothered by them). Still, the coverage and general approach is good.

   Mathematicians should be able to cope readily. I suspect, however, that the book would occasionally be tougher going for philosophers if taken from a standing start – which is another reason why I have recommended beginning with C&H instead. (See my more extended review of the whole book.)

An aside: The treatments of FOL I have mentioned so far here and in the last section include exercises, of course; but for many more exercises and this time with extensive worked solutions you could also look at the ends of Chs. 1, 3 and 4 of René Cori and Daniel Lascar, *Mathematical Logic: A Course with Exercises* (OUP, 2000). I can't, however, particularly recommend the main bodies of those chapters.

3.4 Further into FOL

In this section, which you can skip, I mention three books, very different from each other, which push on the story about FOL in various ways.

1. Raymond Smullyan, *First-Order Logic** (Springer 1968, Dover Publications 1995) is a classic, absolutely packed with good things. This is the tersest, most sophisticated, book I'm mentioning in this chapter. But enthusiasts can certainly try reading Parts I and II, just a hundred pages, after C&H; and those with a taste for mathematical neatness should be able to cope with these chapters and will appreciate their great elegance. This beautiful little book is the source and inspiration of many modern treatments of logic based on tree/tableau systems, such as my own.

   Not always easy, especially as the book progresses, but a delight for the mathematically minded.

2. Jan von Plato's *Elements of Logical Reasoning** (CUP, 2014) is based on the author's introductory lectures. A lot of material is touched on in a relatively short compass as von Plato talks about a range of different natural deduction and sequent calculi. So I suspect that, without any classroom work to round things out, this might not be easy as a first introduction to logic. But suppose you have already met one system of natural deduction (e.g., as in C&H), and now want to know more about 'proof-theoretic' aspects of this and related systems. Suppose, for example, that you want to know about variant ways of setting up ND systems, about proof-search, about the relation with so-called sequent calculi, etc. Then this is a very clear, approachable and interesting book. Experts will see that there are some novel twists, with deductive systems tweaked to have some very nice features: beginners will be put on the road towards understanding some of the initial concerns and issues in proof theory.

3. Don't be put off by the title of Melvin Fitting's *First-Order Logic and Automated Theorem Proving* (Springer, 1990, 2nd end. 1996). This is a

wonderfully lucid book by a terrific expositor. Yes, at various places in the book there are illustrations of how to implement various algorithms in Prolog. But either you can easily pick up the very small amount of background knowledge about Prolog that's needed to follow everything that is going on (and that's quite a fun thing to do anyway) or you can just skip those implementation episodes.

As anyone who has tried to work inside an axiomatic system knows, proof-discovery for such systems is often hard. Which axiom schema should we instantiate with which wffs at any given stage of a proof? Fitch-style natural deduction systems are nicer: but since we can make any new temporary assumption at any stage in a proof, again we still need to keep our wits about us if we are to avoid going off on useless diversions. By contrast, tableau proofs (a.k.a. tree proofs, as in my book) can pretty much write themselves even for quite complex FOL arguments, which is why I used to introduce formal proofs to students that way (in teaching tableaux, we can largely separate the business of getting across the idea of formality from the task of teaching heuristics of proof-discovery). And because tableau proofs very often write themselves, they are also good for automated theorem proving. Fitting explores both the tableau method and the related so-called resolution method in this exceptionally clearly written book.

This book's emphasis is, then, rather different from most of the other recommended books. So I initially hesitated to mention it here in this Guide. However, I think that the fresh light thrown on first-order logic makes the detour through this book *vaut le voyage*, as the Michelin guides say. (If you don't want to take the full tour, however, there's a nice introduction to proofs by resolution in Shawn Hedman, *A First Course in Logic* (OUP 2004): §1.8, §§3.4–3.5.)

3.5 Other treatments?

Obviously, I have still only touched on a very small proportion of books that cover first-order logic. The Appendix covers another handful. But I end this chapter responding to some Frequently Asked Questions, mostly questions raised in response to earlier versions of the Guide.

A blast from the past: What about Mendelson? Somewhat to my surprise, perhaps the most frequent question I used to get asked in response to earlier versions of the Guide is 'But what about Mendelson, Chs. 1 and 2'? Well, Elliot

Mendelson's *Introduction to Mathematical Logic* (Chapman and Hall/CRC 6th edn 2015) was first published in 1964 when I was a student and the world was a great deal younger. The book was I think the first modern textbook at its level (so immense credit to Mendelson for that), and I no doubt owe my career to it – it got me through tripos! And it seems that some who learnt using the book are in their turn still using it to teach from.

But let's not get sentimental! It has to be said that the book in its first incarnation was often brisk to the point of unfriendliness, and the basic look-and-feel of the book hasn't changed a great deal as it has run through successive editions. Mendelson's presentation of axiomatic systems of logic are quite tough going, and as the book progresses in later chapters through formal number theory and set theory, things if anything get somewhat less reader-friendly. Which certainly doesn't mean the book won't repay battling with. But unsurprisingly, fifty years on, there are many rather more accessible and more amiable alternatives for beginning serious logic. Mendelson's book is a landmark worth visiting one day, but I can't recommend starting there. For a little more about it, see here.

As an aside, if you do really want an old-school introduction from roughly the same era, I'd recommend instead Geoffrey Hunter, *Metalogic** (Macmillan 1971, University of California Press 1992). This is not groundbreaking in the way e.g. Smullyan's *First-Order Logic* is, nor is it as comprehensive as Mendelson: but it is still an exceptionally good student textbook from a time when there were few to choose from, and I still regard it with admiration. Read Parts One to Three at this stage. And if you are enjoying it, then do eventually finish the book: it goes on to consider formal arithmetic and proves the undecidability of first-order logic, topics we revisit in §4.2. Unfortunately, the typography – from pre-LaTeX days – isn't at all pretty to look at: this can make the book's pages appear rather unappealing. But in fact the treatment of an axiomatic system of logic is extremely clear and accessible. It might be worth blowing the dust off your library's copy!

The latest thing: What about the Open Logic Text? This is a collaborative, open-source, enterprise, and very much work in progress. Although this is referred to as a textbook, in fact it is a collection of slightly souped-up lecture hand-outs by various hands, written at various degrees of sophistication. Some parts would make for useful revision material. The chapters on FOL, in the version I last looked at, are respectable if a bit idiosyncratic (why start with the LK system of sequent calculus without explaining how it relates to more familiar deductive systems?), and could work if accompanied by a lot of lecture-room chat around and about. But I suspect that at the moment they don't amount to something that is likely to work as a stand-alone text for self-study. For more, see this book note.

Puzzles galore: What about some of Smullyan's other books? I have already warmly recommended Smullyan's terse 1968 classic *First-Order Logic*. He went on to write some classic texts on Gödel's theorem and on recursive functions, which we'll be mentioning later. But as well as these, Smullyan has written many 'puzzle' based-books aimed at a wider audience, including the justly famous 1981 *What is the Name of This Book?** (Dover Publications reprint, 2011).

More recently, he has written *Logical Labyrinths* (A. K. Peters, 2009). From the blurb: "This book features a unique approach to the teaching of mathematical logic by putting it in the context of the puzzles and paradoxes of common language and rational thought. It serves as a bridge from the author's puzzle books to his technical writing in the fascinating field of mathematical logic. Using the logic of lying and truth-telling, the author introduces the readers to informal reasoning preparing them for the formal study of symbolic logic, from propositional logic to first-order logic, ... The book includes a journey through the amazing labyrinths of infinity, which have stirred the imagination of mankind as much, if not more, than any other subject."

Smullyan starts, then, with puzzles of the kind where you are visiting an island where there are Knights (truth-tellers) and Knaves (persistent liars) and then in various scenarios you have to work out what's true from what the inhabitants say about each other and the world. And, without too many big leaps, he ends with first-order logic (using tableaux), completeness, compactness and more. This is no substitute for standard texts, but – for those with a taste for being led up to the serious stuff via sequences of puzzles – an entertaining and illuminating supplement.

Smullyan's later *A Beginner's Guide to Mathematical Logic** (Dover Publications, 2014) is more conventional. The first 170 pages are relevant to FOL. A rather uneven read, it seems to me, but again perhaps an illuminating supplement to the texts recommended above.

Designed for philosophers: What about Sider? Theodore Sider – a very well-known philosopher – has written a text called *Logic for Philosophy** (OUP, 2010) aimed at philosophers, which I've been asked to comment on. The book in fact falls into two halves. The second half (about 130 pages) is on modal logic, and I will return to that in §5.1. The first half of the book (almost exactly the same length) is on propositional and first-order logic, together with some variant logics, so is very much on the topic of this chapter. But while the coverage of modal logic is quite good, I can't at all recommend the first half of this book: I explain why here.

True, a potentially attractive additional feature of this part of Sider's book is that it does contain brief discussions about e.g. some non-classical propositional logics, and about descriptions and free logic. But remember all this is being done in 130 pages, which means that things are whizzing by very fast, so the breadth of Sider's coverage here goes with far too much superficiality. If you want some breadth, Bostock is still much to be preferred, plus perhaps some reading from §5.3 below.

Mostly for philosophers again: What about Bell, DeVidi and Solomon? If you concentrated at the outset on a one-proof-style book, you would do well to widen your focus at an early stage to look at other logical options. And one good thing about Bostock's book is that it tells you about different styles of proof-system. A potential alternative to Bostock at about the same level, and which initially looks promising, is John L. Bell, David DeVidi and Graham Solomon's *Logical Options: An Introduction to Classical and Alternative Logics* (Broadview Press 2001). This book covers a lot pretty snappily – for the moment, just Chapters 1 and 2 are relevant – and a few years ago I used it as a text for second-year seminar for undergraduates who had used my own tree-based book for their first year course. But many students found it quite hard going, as the exposition is terse, and I found myself having to write very extensive seminar notes. For example, see my notes on Types of Proof System, which gives a brisk overview of some different proof-styles (written for those who had first done logic using by tableau-based introductory book). If you want some breadth, you'd again do better sticking with Bostock.

Chapter 4

Continuing Mathematical Logic

We now press on from an initial look at first-order logic to consider other core elements of mathematical logic.

Recall, the three main topic-areas which we need to cover after the basics of FOL are:

- Some elements of the model theory for first-order theories.
- Formal arithmetic, theory of computation, Gödel's incompleteness theorems.
- Elements of set theory.

But at some point we'll also need to touch briefly on two standard 'extras'

- Second-order logic and second-order theories; intuitionistic logic

Recall: as I explained in §1.3, I do very warmly recommend reading a series of books on a topic which overlap in coverage and difficulty, rather than leaping immediately from an 'entry level' text to a really advanced one. Of course, you don't have to follow this excellent advice! But I mention it again here to remind you of one reason why the list of recommendations in most sections is quite extensive and why the increments in coverage/difficulty between successive recommendations are often quite small. So let me stress that *this level of logic really isn't as daunting as the overall length of this chapter might superficially suggest.* Promise!

4.1 From first-order logic to elementary model theory

The completeness theorem is the first high point – the first mathematically serious result – in a course in first-order logic; and some elementary treatments more or less stop there. Many introductory texts, however, continue just a little further with some first steps into model theory. It is clear enough what needs to come next: discussions of the so-called compactness theorem (also called the 'finiteness theorem'), of the downward and upward Löwenheim-Skolem theorems, and of their implications. There's less consensus about what other introductory model theoretic topics you should tackle at an early stage.

As you'll see, you very quickly meet claims that involve infinite cardinalities and also occasional references to the axiom of choice. Now in fact, even if you haven't yet done an official set theory course, you may well have picked up all you need to know in order to begin model theory. If you have met Cantor's proof that infinite collections come in different sizes, and if you have been warned to take note when a proof involves making an infinite series of choices, you will probably know enough. And Goldrei's chapter recommended in a moment in fact has a brisk section on the 'Set theory background' needed at this stage. (If that's *too* brisk, then perhaps do a skim read of e.g. Paul Halmos's very short *Naive Set Theory**, or one of the other books mentioned at the beginning of §4.3 below.)

What to read on model theory, then? Let me start by mentioning an old book that aims precisely to bridge the gap between an introductory study of FOL and more advanced work in model theory:

1. The very first volume in the prestigious and immensely useful Oxford Logic Guides series is Jane Bridge's very compact *Beginning Model Theory: The Completeness Theorem and Some Consequences* (Clarendon Press, 1977) which neatly takes the story onwards just a few steps from the reading on FOL mentioned in our §3.2 above. You can look at the opening chapter to remind yourself about the notion of a relational structure, then start reading Ch. 3 for a very clear account of what it takes to prove the completeness theorem that a consistent set of sentences has a model for different sizes of sets of sentences. Though now pay special attention to the compactness theorem, and §3.5 on 'Applications of the compactness theorem'.

   Then in the forty-page Ch. 4, Bridge proves the Löwenheim-Skolem theorems, which tell us that consistent first order theories that have an infinite model at all will have models of all different infinite sizes. But that still leaves open the possibility that a theory's models *of a particular size* are all isomorphic, all 'look the same'. So Bridge now explores some of the possibilities here, how they relate to a theory's being complete (i.e. settling whether φ or $\neg\varphi$ for any relevant sentence φ), and some connected issues.

The coverage strikes me as exemplary for a short first introduction, and the writing is pretty clear though rather terse. But – and this is a huge 'but' – very sadly, the book was printed in that brief period when publishers thought it a bright idea to save money by photographically printing work produced on electric typewriters. Accustomed as we now are to mathematical texts being beautifully LaTeXed, the *look* of Bridge's book is really very off-putting, and probably most will find that the book's real virtues simply do not outweigh that sad handicap.

Still, even setting aside the aesthetics of the book, and despite its excellent focus, I am not sure that – almost forty years on from publication – this is still the best place for beginners to start. But what are the alternatives?

Two of the introductions to FOL that I mentioned in $3.3 have treatments of some elementary model theory. Thus there are fragments of model theory in §2.6 of Herbert Enderton's *A Mathematical Introduction to Logic* (Academic Press 1972, 2002), followed by a discussion in §2.8 of non-standard analysis: but this, for our purposes here, is perhaps too little done too fast. Dirk van Dalen's *Logic and Structure** (Springer, 1980; 5th edition 2012) covers rather more model-theoretic material in more detail in his Ch. 3. You could read the first section for revision on the completeness theorem, then §3.2 on compactness, the Löwenheim-Skolem theorems and their implications, before moving on to the action-packed §3.3 which covers more model theory including non-standard analysis again, and indeed touches on slightly more advanced topics like 'quantifier elimination'. However, my top votes for a modern treatment of some first steps in elementary model theory go elsewhere:

2. I have already sung the praises of Derek Goldrei's *Propositional and Predicate Calculus: A Model of Argument** (Springer, 2005) for the accessibility of its treatment of FOL in the first five chapters. You can now read his §§4.4 and 4.5 (which I previously said you could skip) and then Ch. 6 on 'Some uses of compactness' to get a very clear introduction to some model theoretic ideas.

   In a little more detail, §4.4 introduces some axiom systems describing various mathematical structures (partial orderings, groups, rings, etc.): this section could be particularly useful to philosophers who haven't really met the notions before. Then §4.5 introduces the notions of substructures and structure-preserving mappings. After proving the compactness theorem in §6.1 (as a corollary of his completeness proof), Goldrei proceeds to use it in §§6.2 and 6.3 to show various theories can't be finitely axiomatized, or can't be nicely axiomatized at all. §6.4 introduces the Löwenheim-Skolem theorems and some consequences, and the following section introduces the notion of diagrams and puts it to work. The final section, §6.6 considers issues about categoricity, completeness and decidability. All this is done with the same admirable clarity as marked out Goldrei's earlier chapters.

If you then want to go just a bit further, what you probably need next is the rather more expansive

3. Maria Manzano, *Model Theory*, Oxford Logic Guides 37 (OUP, 1999). This book aims to be an introduction at the kind of intermediate level we are currently concerned with. And standing back from the details, I do very much like the way that Manzano structures her book. The sequencing of chapters makes for a very natural path through her material, and the coverage seems very appropriate for a book at her intended level. After chapters about structures (and mappings between them) and about first-order languages, she proves the completeness theorem again, and then has a sequence of chapters on core model-theoretic notions and proofs. This is all done tolerably accessibly (just half a step up from Goldrei, perhaps).

   True, the discussions at some points would have benefitted from rather more informal commentary, motivating various choices. And there are some infelicities. But overall, Manzano's text should work well and there is no evident competitor book at this level. See this Book Note on Manzano for more details.

And this might already be about as far as philosophers, at least, may want or need to go. Many mathematicians, however, will be eager to take up the story about model theory again in §6.2.

Postscript Thanks to the efforts of the respective authors to be accessible, the path through Chiswell & Hodges → Leary & Kristiansen → Goldrei → Manzano is not at all a hard road to follow, yet we end up at least in the foothills of model theory. We can climb up to the same foothills by routes involving rather tougher scrambles, taking in some additional side-paths and new views along the way. Here are two suggestions for the more mathematical reader:

4. Shawn Hedman's *A First Course in Logic* (OUP, 2004) covers a surprising amount of model theory. Ch. 2 tells you about structures and relations between structures. Ch. 4 starts with a nice presentation of a Henkin completeness proof, and then pauses (as Goldrei does) to fill in some background about infinite cardinals etc., before going on to prove the Löwenheim-Skolem theorems and compactness theorems. Then the rest of Ch. 4 and the next chapter covers more introductory model theory, already touching on some topics beyond the scope of Manzano's book, and Hedman so far could serve as a rather tougher alternative to her treatment. (Then Ch. 6 takes the story on a lot further, quite a way beyond what I'd regard as 'entry level' model theory.) For more, see this Book Note on Hedman.

5. Peter Hinman's weighty *Fundamentals of Mathematical Logic* (A. K. Peters, 2005) is not for the faint-hearted, and I wouldn't recommend using this book as your guide in your first outing into this territory. But if you are mathematically minded and have already made a first foray along a gentler route, you could now try reading Ch. 1 – skipping material that is familiar – and then carefully working through Ch. 2 and Ch. 3 (leaving the last two sections, along with a further chapter on model theory, for later). This should significantly deepen your knowledge of FOL, or at least of its semantic features, and of the beginnings of model theory. For more, see this Book Note on Hinman.

4.2 Computability and Gödelian incompleteness

The standard mathematical logic curriculum, as well as looking at some elementary general results about formalized theories and their models, looks at two particular instances of non-trivial, rigorously formalized, axiomatic systems – arithmetic (a paradigm theory about finite whatnots) and set theory (a paradigm theory about infinite whatnots). We'll take arithmetic first.

In more detail, there are three inter-related topics here: (a) the elementary (informal) theory of arithmetic computations and of *computability* more generally, (b) an introduction to *formal* theories of arithmetic, leading up to (c) Gödel's epoch-making proof of the *incompleteness* of any sufficiently nice formal theory that can 'do' enough arithmetical computations (a result of profound interest to philosophers).

Now, Gödel's 1931 proof of his incompleteness theorem uses facts in particular about so-called *primitive recursive* functions: these functions are a subclass (but only a subclass) of the computable numerical functions, i.e. a subclass of the functions which a suitably programmed computer could evaluate (abstracting from practical considerations of time and available memory). A more general treatment of the effectively computable functions (arguably capturing *all* of them) was developed a few years later, and this in turn throws more light on the incompleteness phenomenon.

So there's a choice to be made. Do you look at things in roughly the historical order, first introducing just the primitive recursive functions and theories of formal arithmetic and learning how to prove initial versions of Gödel's incompleteness theorem before moving on to look at the general treatment of computable functions? Or do you do some of the general theory of computation first, turning to the incompleteness theorems later?

Here then to begin with are a couple of introductory books, one taking the first route, one the other route:

1. Peter Smith, *An Introduction to Gödel's Theorems** (CUP 2007, 2nd edition 2013) takes things in something like the historical order. Mathematicians: don't be put off by the series title 'Cambridge Introductions to Philosophy' – putting it in that series was the price I happily paid for cheap paperback publication. This is still quite a meaty logic book, with a lot of theorems and a lot of proofs, but I hope rendered very accessibly. The book's website is at http://godelbook.net, where there are supplementary materials of various kinds, including a freely available cut-down version of a large part of the book, *Gödel Without (Too Many) Tears.*
2. Richard Epstein and Walter Carnielli, *Computability: Computable Functions, Logic, and the Foundations of Mathematics* (Wadsworth 2nd edn. 2000: Advanced Reasoning Forum 3rd edn. 2008) does computability theory first. This is a very nicely introductory book on the standard basics, particularly clearly and attractively done, with lots of interesting and illuminating historical information too in Epstein's 28 page timeline on 'Computability and Undecidability' at the end of the book.

Those first two books should be very accessible to those without much mathematical background: but even more experienced mathematicians should appreciate the careful introductory orientation which they provide. And as you'll immediately see, this really is a delightful topic area. Elementary computability theory is conceptually very neat and natural, and the early Big Results are proved in quite remarkably straightforward ways (just get the hang of the basic 'diagonalization' construction, the idea of Gödel-style coding and one or two other tricks, and off you go . . .).

Here next are two more suggestions of excellent books which take us only a small step up in mathematical sophistication:

3. Herbert E. Enderton, *Computability Theory: An Introduction to Recursion Theory* (Associated Press, 2011). This short book completes Enderton's trilogy covering the basics of the mathematical logic curriculum – we've already mentioned his *A Mathematical Introduction to Logic* (1972) on first-order logic and a little model theory, and we will later meet his *The Elements of Set Theory* (1972) (see §§3.3 and 4.3).

   Enderton writes here with attractive zip and lightness of touch (this is a more relaxed book than his *Logic*); it makes a nice alternative to Epstein and Carnielli, though perhaps just a step more challenging/sophisticated/abstract. The first chapter is on the informal Computability Concept. There are then chapters on general recursive functions and on register machines (showing that the register-computable functions are exactly the recursive ones), and a chapter on recursive enumerability. Chapter 5 makes 'Connections to Logic' (including proving Tarski's theorem on the undefinability of arithmetical truth and a semantic incompleteness theorem). The final two chapters push on to say something about 'Degrees of Unsolvability' and 'Polynomial-time Computability'. This is all very nicely done, and in under 150 pages too. Enderton's book could particularly appeal to, and be very manageable by, philosophers.

4. George Boolos, John Burgess, Richard Jeffrey, *Computability and Logic* (CUP 5th edn. 2007). This, twice the length of Enderton's book, is the latest edition of a much admired classic. The first version – just by Boolos and Jeffrey – was published in 1974; and there's in fact something to be said for regarding their 1990 third edition as being the best (I know I am not the only reader who thinks this). The last two versions have been done by Burgess and have grown considerably and perhaps in the process lost just some of elegance and individuality.

   But still, whichever edition you get hold of, this is great stuff. Taking the divisions in the last two editions, you will want to read the first two parts of the book at this early stage, perhaps being more selective when it comes to the last part, 'Further Topics'.

5. I have already warmly recommended Christopher Leary and Lars Kristiansen's *A Friendly Introduction to Mathematical Logic** (Milne Library, 2015) for its coverage of first-order logic. Chs. 4 to 7 now give a very illuminating double treatment of matters related to incompleteness (you don't have to have read the previous chapters, other than noting the arithmetical system N introduced in their §2.8). In headline terms that you'll only come fully to understand in in retrospect:

   (a) L&K's first approach doesn't go overtly via computability. Instead of showing that certain syntactic properties are primitive recursive and showing that all primitive recursive properties can be 'represented' in theories like N (as I do in *IGT*), L&K first rely on more directly showing that some key syntactic properties can be represented. This representation result then leads to, inter alia, the incompleteness theorem.

   (b) L&K follow this, however, with a general discussion of computability, and then use the introductory results they obtain to prove various further theorems, including incompleteness again.

   This is all done with the same admirable clarity as the first part of the book on FOL.

If you twist my arm and tell me that I have to narrow the choices down further, I'd perhaps say you should ideally read *IGT* (well, I would, wouldn't I?), and then Boolos et al.

Postscript There are many other introductory treatments covering aspects of computability and/or incompleteness; and indeed there is no sharp boundary to be drawn between the entry-level accounts mentioned in this section and some of the more sophisticated books on computability and Gödelian incompleteness discussed in Ch. 6.

I'll only mention a very few more texts here. To begin, there are a couple of short books in the American Mathematical Society's 'Student Mathematical Library'. One is Rebecca Weber's *Computability Theory* (AMA, 2012) which strikes me as rather too uneven. It has some extremely elementary throat-clearing at the beginning, but then rushes on quite sketchily at a quick pace, getting as far as e.g. so-called priority arguments (an advanced topic). That could, I suppose, be useful supplementary reading; but for present purposes I much prefer

5. A. Shen and N. K. Vereshchagin, *Computable Functions*, (AMA, 2003). This is a lovely, elegant, little book, whose opening chapters can be recommended for giving a differently-structured quick tour through some of the Big Ideas, and hinting at ideas to come.

Then various of the Big Books on mathematical logic have more or less brisk treatments of computability and incompleteness. I've already mentioned the *Friendly Introduction*: and at a fairly similar level to that, though rather less friendly, there is

6. Herbert Enderton's *A Mathematical Introduction to Logic* (Academic Press 1972, 2002), Ch. 3 is a very good short treatment of different strengths of formal theories of arithmetic, and then proves the incompleteness theorem first for a formal arithmetic with exponentiation and then – after touching on other issues – shows how to use the β-function trick to extend the theorem to apply to Robinson arithmetic without exponentiation. Well worth reading after e.g. my book for consolidation.

   (By the way, here you will encounter one of those annoying terminological variations that can cause trouble for those new to the area. Some (most) define recursive relations and functions arithmetically, and then prove they can be represented in certain formal theories: others, like Enderton initially, go the other way about and define recursive relations and functions as those that can be represented in certain formal theories, and have to prove the arithmetic properties of the recursive functions.)

Finally, if only because I've been asked about it a good number of times, I suppose I should also mention

7. Douglas Hofstadter, *Gödel, Escher, Bach* (Penguin, first published 1979). When students enquire about this, I helpfully say that it is the sort of book that you might well like if you like that kind of book, and you won't if you don't. It is, to say the least, quirky and distinctive. As I far as I recall, though, the parts of the book which touch on techie logical things are pretty reliable and won't lead you astray. Which is a great deal more than can be said about many popularizing treatments of Gödel's theorems.

4.3 Beginning set theory

Let's say that the *elements of set theory* – the beginnings that any logician really ought to know about – will comprise enough to explain how numbers (natural, rational, real) are constructed in set theory (so enough to give us a glimmer of understanding about why it is said that set theory provides a foundation for mathematics). The elements also include the development of ordinal numbers and transfinite induction over ordinals, ordinal arithmetic, and something about the role of the axiom(s) of choice and its role in the arithmetic of cardinals. These initial ideas and constructions can (and perhaps should) be presented fairly informally: but something else that also belongs here at the beginning is an account of the development of ZFC as the now standard way of formally encapsulating and regimenting the key principles involved in the informal development of set theory.

Going beyond these elements we then have e.g. the exploration of 'large cardinals', proofs of the consistency and independence of e.g. the Continuum Hypothesis, and a lot more besides. But readings on these further delights are for Ch. 7: this present section is, as advertised, about the first steps for beginners in set theory. Even here, however, there are many books to choose from, so an annotated Guide should be particularly welcome.

I'll start by mentioning again a famous 'bare minimum' book (only 104 pp. long), which could well be very useful for someone making a start on exploring basic set-theoretic notation and some fundamental concepts.

1. Paul Halmos, *Naive Set Theory** (Originally published 1960, and now available inexpensively from Martino Fine Books). Informally written in an unusually conversational style for a maths book – though that won't be a recommendation for everyone! And mostly exceptionally clear, though sometimes (e.g. on Zorn's Lemma) there are perhaps lapses.

However, Halmos doesn't cover even all of what I just called the elements of set theory, and most readers will want to look at one or more of the following equally admirable 'entry level' treatments which cover a little more in a bit more depth but still very accessibly:

2. Herbert B. Enderton, *The Elements of Set Theory* (Academic Press, 1977) has exactly the right coverage. But more than that, it is particularly clear in marking off the informal development of the theory of sets, cardinals, ordinals etc. (guided by the conception of sets as constructed

in a cumulative hierarchy) and the formal axiomatization of ZFC. It is also particularly good and non-confusing about what is involved in (apparent) talk of classes which are too big to be sets – something that can mystify beginners. It is written with a certain lightness of touch and proofs are often presented in particularly well-signposted stages. The last couple of chapters or so perhaps do get a bit tougher, but overall this really is quite exemplary exposition.

3. Derek Goldrei, *Classic Set Theory* (Chapman & Hall/CRC 1996) has the subtitle 'For guided independent study'. It is as you might expect – especially if you looked at Goldrei's FOL text mentioned in §3.2 – extremely clear, and is indeed very well-structured for independent reading. And moreover, it is fairly attractively written (as set theory books go!). The coverage is very similar to Enderton's, and either book makes a fine introduction (for what little it is worth, I slightly prefer Enderton).

Still starting from scratch, and initially also only half a notch or so up in sophistication from Enderton and Goldrei, we find two more really nice books:

4. Karel Hrbacek and Thomas Jech, *Introduction to Set Theory* (Marcel Dekker, 3rd edition 1999). This eventually goes a bit further than Enderton or Goldrei (more so in the 3rd edition than earlier ones), and you could – on a first reading – skip some of the later material. Though do look at the final chapter which gives a remarkably accessible glimpse ahead towards large cardinal axioms and independence proofs. Again this is a very nicely put together book, and recommended if you want to consolidate your understanding by reading a second presentation of the basics and want then to push on just a bit. (Jech is of course a major author on set theory, and Hrbacek once won a AMA prize for maths writing.)
5. Yiannis Moschovakis, *Notes on Set Theory* (Springer, 2nd edition 2006). A slightly more individual path through the material than the previously books mentioned, again with glimpses ahead and again attractively written.

I'd strongly advise reading one of the first pair and then one of the second pair.

I will add two more firm recommendations at this level. The first might come as a bit of surprise, as it is something of a 'blast from the past'. But we shouldn't ignore old classics – they can have a lot to teach us even if we have read the modern books.

6. Abraham Fraenkel, Yehoshua Bar-Hillel and Azriel Levy, *Foundations of Set-Theory* (North-Holland, 2nd edition 1973). Both philosophers and mathematicians should appreciate the way this puts the development of our canonical ZFC set theory into some context, and also discusses alternative approaches. It really is attractively readable, and should be very largely accessible at this early stage. I'm not an enthusiast for history for history's sake: but it is very much worth knowing the stories that unfold here.

One intriguing feature of that last book is that it *doesn't* emphasize the 'cumulative hierarchy' – the picture of the universe of sets as built up in a hierarchy of stages or levels, each level containing all the sets at previous levels plus new ones (so the levels are cumulative). This picture is nowadays familiar to every beginner: you will find it e.g. in the opening pages of Joseph Shoenfield 'The axioms of set theory', *Handbook of mathematical logic*, ed. J. Barwise, (North-Holland, 1977) pp. 321–344: it is significant that the picture wasn't firmly in place from the beginning.

The hierarchical conception of the universe of sets is brought to the foreground again in

7. Michael Potter, *Set Theory and Its Philosophy* (OUP, 2004). For philosophers (and for mathematicians concerned with foundational issues) this surely is a 'must read', a unique blend of mathematical exposition (mostly about the level of Enderton, with a few glimpses beyond) and extensive conceptual commentary. Potter is presenting not straight ZFC but a very attractive variant due to Dana Scott whose axioms more directly encapsulate the idea of the cumulative hierarchy of sets. However, it has to be said that there are passages which are harder going, sometimes because of the philosophical ideas involved, but sometimes because of unnecessary expositional compression. In particular, at the key point at p. 41 where a trick is used to avoid treating the notion of a level (i.e. a level in the hierarchy) as a primitive, the definitions are presented too quickly, and I know that some relative beginners can get lost. However, if you have already read one or two set theory books from earlier in the list, you will be able to work out what is going on and read on past this stumbling block.

Mathematicians who get intrigued by set theory done for its own sake will want to continue the story in Ch. 6. But it is a nice question how much more technical knowledge of results in set theory a philosophy student interested in logic and the philosophy of maths *needs* (if she is not specializing in the technical philosophy of set theory). But getting this far will certainly be a useful start for both mathematicians and philosophers, so let's pause here.

Postscript Books by Ciesielski and by Hajnal and Hamburger, although in the LMS Student Text series and starting from scratch, are not really suitable for the present list (they go too far and probably too fast). But the following four(!) still-introductory books, listed in order of publication, each have things to recommend them for beginners. One is freely available online, and good libraries should have the others: so browse through and see which might suit your interests and mathematical level.

8. D. van Dalen, H.C. Doets and H. de Swart, *Sets: Naive, Axiomatic and Applied* (Pergamon, 1978). The first chapter covers the sort of elementary (semi)-naive set theory that any mathematician needs to know, up to an account of cardinal numbers, and then takes a first look at the paradox-avoiding ZF axiomatization. This is very attractively and illuminatingly done (or at least, the conceptual presentation is attractive – sadly, and a sign of its time of publication, the book seems to have been photo-typeset from original pages produced on electric typewriter, and the result is visually not attractive at all).

   The second chapter carries on the presentation axiomatic set theory, with a lot about ordinals, and getting as far as talking about higher infinities, measurable cardinals and the like. The final chapter considers some applications of various set theoretic notions and principles. Well worth seeking out, if you don't find the typography off-putting.

9. Keith Devlin, *The Joy of Sets* (Springer, 1979: 2nd edn. 1993). The opening chapters of this book are remarkably lucid and attractively written. The opening chapter explores 'naive' ideas about sets and some set-theoretic constructions, and the next chapter introducing axioms for ZFC pretty gently (indeed, non-mathematicians could particularly like Chs 1 and 2, omitting §2.6). Things then speed up a bit, and by the end of Ch. 3 – some 100 pages into the book – we are pretty much up to the coverage of Goldrei's much longer first six chapters, though Goldrei says more about (re)constructing classical maths in set theory. Some will prefer Devlin's fast-track version. (The rest of the book then covers non-introductory topics in set theory, of the kind we take up again in Ch. 7.)

10. Judith Roitman, *Introduction to Modern Set Theory*** (Wiley, 1990: now freely downloadable, or available as an inexpensive paperback via Amazon). This relatively short, and very engagingly written, book manages to cover quite a bit of ground – we've reached the constructible universe by p. 90 of the downloadable pdf version, and there's even room for a concluding chapter on 'Semi-advanced set theory' which says something about large cardinals and infinite combinatorics. A few quibbles aside, this could make excellent revision material as Roitman is particularly good at highlighting key ideas without getting bogged down in too many details.

11. Winfried Just and Martin Weese, *Discovering Modern Set Theory I: The Basics* (American Mathematical Society, 1996). This covers overlapping ground to Enderton, patchily but perhaps more zestfully and with a little more discussion of conceptually interesting issues. It is at some places more challenging – the pace can be uneven. But this is evidently written by enthusiastic teachers, and the book is very engaging. (The story continues in a second volume.)

    I like the style a lot, and think it works very well. I don't mean the occasional (slightly laboured?) jokes: I mean the in-the-classroom feel of the way that proofs are explored and motivated, and also the way that teach-yourself exercises are integrated into the text. For instance there are exercises that encourage you to produce proofs that are in fact non-fully-justified, and then the discussion explores what goes wrong and how to plug the gaps.

I guess I should mention another book for beginners which has been warmly recommended by some, but which I do find less successful:

12. George Tourlakis, *Lectures in Logic and Set Theory, Volume 2: Set Theory* (CUP, 2003). Although this is the second of two volumes, it is a stand-alone text. Indeed Tourlakis goes as far as giving a 100 page outline of the logic covered in the first volume as the long opening chapter in this volume. Assuming you have already studied FOL, you can initially skip this chapter, consulting if/when needed. That still leaves over 400 pages on basic set theory, with long chapters on the usual axioms, on the Axiom of Choice, on the natural numbers, on order and ordinals, and on cardinality. (The final chapter on forcing should be omitted at this stage, and strikes me as less clear than what precedes it.)

    As the title suggests, Tourlakis aims to retain something of the relaxed style of the lecture room, complete with occasional asides and digressions.

And as the length suggests, the pace is quite gentle and expansive, with room to pause over questions of conceptual motivation etc. However, there is a certain quite excessive and unnecessary formalism that many will find off-putting (e.g. it takes Tourlakis a page to prove that if x is a set and $x \subseteq \{\varnothing\}$, then $x = \varnothing$ or $x = \{\varnothing\}$). And simple constructions and results take a long time to arrive. We don't meet the von Neumann ordinals for three hundred pages, and we don't get to Cantor's theorem on the uncountability of $\mathcal{P}(\omega)$ until p. 455!

So while this book might be worth dipping into for some of the motivational explanations, I don't positively recommend it.

Some of the books I have mentioned so far are pretty weighty. So here, to conclude, are just two more notably short books aimed at beginners that are also excellent and could also be very useful additional reading to add to the boxed main recommendations above:

13. A. Shen and N. K. Vereshchagin, *Basic Set Theory* (American Mathematical Society, 2002), just over 100 pages, and mostly about ordinals. But very readable, with 151 'Problems' as you go along to test your understanding. Potentially *very* helpful by way of revision/consolidation.

14. Ernest Schimmerling, *A Course on Set Theory* (CUP, 2011) is slightly mistitled: it is just 160 pages, again introductory but with some rather different emphases and occasional forays into what Roitman would call 'semi-advanced' material. Quite an engaging supplementary read at this level.

Finally, what about the introductory chapters on set theory in those Big Books on Mathematical Logic? I'm not convinced that any are now to be particularly recommended compared with the stand-alone treatments I have mentioned.

4.4 Extras: two variant logics

4.4.1 Second-order logic

At some fairly early point we must look at a familiar *extension* of first-order classical logic, namely second-order logic, where we also allow generalizations which quantify into predicate position.

Consider, for example, the intuitive principle of arithmetical induction. Take any property X; if 0 has it, and for any n it is passed down from n to $n+1$, then *all* numbers must have X. It is very natural to regiment this as follows:

$$\forall X[(X0 \wedge \forall n(Xn \to X(n+1)] \to \forall n\, Xn]$$

where the second-order quantifier $\forall X$ quantifies 'into predicate position' and supposedly runs over all properties of numbers. But this is illegitimate in standard first-order logic.

Historical aside Note that the earliest presentations of quantificational logic, in Frege and in *Principia Mathematica*, were of logics that did allow this kind of higher-order quantification. The concentration on first-order logic which has become standard was a later development, and its historical emergence is a tangled story. It is well worth one day chasing up some of this history. See, for example, José Ferreiros, 'The road to modern logic – an interpretation'. Bulletin of Symbolic Logic 7 (2001): 441–484. And for the broader setting see also Paolo Mancosu, Richard Zach and Calixto Badesa, 'The Development of Mathematical Logic from Russell to Tarski: 1900–1935' in Leila Haaparanta, ed., The History of Modern Logic (OUP, 2009, pp. 318–471).

To resume How should we handle apparent second-order quantifiers? One option is to keep your logic first-order but go set-theoretic and write the induction principle instead as

$$\forall X[(0 \in X \wedge \forall n(n \in X \to (n+1) \in X] \to \forall n\, n \in X]$$

where the variable 'X' is now a sorted *first-order* variable running over sets. But arguably this changes the subject (our ordinary principle of arithmetical induction doesn't *seem* to be about sets), and there are other issues too. So why not take things at face value and allow that the 'natural' logic of informal mathematical discourse often deploys second-order quantifiers that range over properties (expressed by predicates) as well as first-order quantifiers that range over objects (denoted by names), i.e. why not allow quantification into predicate position as well as into name position?

For a brief but very informative overview of second-order logic, see the article

1. Herbert Enderton, 'Second-order and Higher-order Logic', *The Stanford Encyclopedia of Philosophy.*

You could then try one or both of

2. Herbert Enderton *A Mathematical Introduction to Logic* (Academic Press 1972, 2002), Ch. 4.
3. Dirk van Dalen, *Logic and Structure*, Ch. 4,

Any logician ought to know at least what these recommendations together explain – i.e. the difference between full and Henkin semantics, why the compactness and Löwenheim-Skolem theorems fail for second-order logic with full semantics, and some consequences of this. But having got this far, philosophers in particular should want to dive into

4. Stewart Shapiro, *Foundations without Foundationalism: A Case for Second-Order Logic*, Oxford Logic Guides 17 (Clarendon Press, 1991), Chs. 3–5.

And indeed it would be a pity, while you have Shapiro's wonderfully illuminating book in your hands, to skip the initial philosophical/methodological discussion in the first two chapters here. This whole book is a modern classic, remarkably accessible, and important too for the contrasting side-light it throws on FOL.

4.4.2 Intuitionist logic

(a) Why should we endorse the principle that $\varphi \vee \neg\varphi$ is always true no matter the domain? Even prescinding from issues of vagueness, does the world have to cooperate to determine any given proposition to be true or false?

Could there, for example, be domains – mathematics, for example – where truth is in some good sense a matter of provability-in-principle, and falsehood a matter of refutability-in-principle? And if so, would every proposition from such a domain be either true or false, i.e. provable-in-principle or refutable-in-principle? Why so? Perhaps we shouldn't suppose that the principle that $\varphi \vee \neg\varphi$ always holds true no matter the domain. (Maybe the principle, even when it does hold for some domain, doesn't hold as a matter of *logic* but e.g. as a matter of metaphysics or of the nature of truth for that domain.)

Thoughts like this give rise to one kind of challenge to classical two-valued logic, which of course does assume excluded middle across the board. For more on this intuitionist challenge, see e.g. the brief remarks in Theodore Sider,'s *Logic for Philosophy* (OUP, 2010), §3.5.

Now, particularly in a natural deduction framework, the syntax/proof theory of intuitionist logic is straightforward: putting it crudely, we have to suitably tinker with the rules of your favourite proof system so as to block the derivation of excluded middle while otherwise keeping as much as possible. Things get interesting when we look at the semantics. The now-standard version due to Kripke is a brand of 'possible-world semantics' of a kind that is also used in modal logic. Philosophers might like, therefore, to cover intuitionism after first looking at modal logic more generally.

Mathematicians, however, should have no problem diving straight into

1. Dirk van Dalen, *Logic and Structure** (Springer, 4th edition 2004), §§5.1–5.3.
2. M. Fitting, *Intuitionistic Logic, Model Theory, and Forcing* (North Holland, 1969). The first part of the scarily-titled book is a particularly attractive stand-alone introduction to the semantics and a proof-system for intuitionist logic (the second part of the book concerns an application of this to a construction in set theory, but don't let that put you off the brilliantly clear Chs 1–6!).

If, however, you want to approach intuitionistic logic *after* looking at some modal logic, then you could instead start with

3. Graham Priest, *An Introduction to Non-Classical Logic** (CUP, much expanded 2nd edition 2008), Chs. 6, 20. These chapters of course flow on naturally from Priest's treatment in that book of modal logics, first propositional and then predicate.

And then, if you want to pursue things further (though this is ratcheting up the difficulty level), you could try the following wide-ranging essay, which is replete with further references:

4. Dirk van Dalen, 'Intuitionistic Logic', in the *Handbook of Philosophical Logic*, Vol. 5, ed. by D. Gabbay and F. Guenthner, (Kluwer 2nd edition 2002).

(b) One theme not highlighted in these initial readings is that intuitionistic logic seemingly has a certain naturalness compared with classical logic, from a more proof-theoretic point of view. Suppose we think of the natural deduction introduction rule for a logical operator as fixing the meaning of the operator (rather than a prior semantics fixing what is the appropriate rule). Then the corresponding elimination rule surely ought to be in harmony with the introduction rule, in the sense of just 'undoing' its effect, i.e. giving us back from a wff φ with O as its main operator no more than what an application of the O-introduction rule to justify φ would have to be based on. For this idea of harmony see e.g. Neil Tennant's *Natural Logic*, §4.12. From this perspective the characteristic classical excluded middle rule is seemingly not 'harmonious'. There's a significant literature on this idea: for some discussion, and pointers to other discussions, you could start with Peter Milne, 'Classical harmony: rules of inference and the meaning of the logical constants', *Synthese* vol. 100 (1994), pp. 49–94.

For an introduction to intuitionistic logic in a related spirit, see

5. Stephen Pollard, *A Mathematical Prelude to the Philosophy of Mathematics* (Springer, 2014), Ch. 7, 'Intuitionist Logic'.

(c) Note, by the way, that what we've been talking about is intuitionist *logic* not intuitionist *mathematics*. For something on the relation between these, see e.g.

6. Rosalie Iemhoff, 'Intuitionistic Logic', *The Stanford Encyclopedia of Philosophy*,

7. Joan Moschovakis, 'Intuitionism in the Philosophy of Mathematics', *The Stanford Encyclopedia of Philosophy*.

And then the stand-out recommendation is

8. Michael Dummett, *Elements of Intuitionism*, Oxford Logic Guides 39 (OUP 2nd edn. 2000). A classic – but (it has to be said) quite tough. The final chapter, 'Concluding philosophical remarks', is very well worth looking at, even if you bale out from reading all the formal work that precedes it.

But perhaps this really needs to be set in the context of a wider engagement with varieties of constructive mathematics.

Chapter 5

Modal and other logics

Here's the menu for this chapter, which is probably of particular interest to philosophers rather than mathematicians:

5.1 We start with modal logic – like second-order logic, an *extension* of classical logic – for two reasons. First, the basics of modal logic don't involve anything mathematically more sophisticated than the elementary first-order logic covered in Chiswell and Hodges (indeed to make a start on modal logic you don't even need as much as that). Second, and much more importantly, philosophers working in many areas surely *ought* to know a little modal logic.

5.2 Classical logic demands that all terms denote one and one thing – i.e. it doesn't countenance empty terms which denote nothing, or plural terms which may denote more than one thing. In this section, we look at logics which remain classical in spirit (retaining the usual sort of definition of logical consequence) but which do allow empty and/or plural terms.

5.3 Among variant logics which are non-classical in spirit, we have already mentioned intutionist logic. Here we consider some other deviations from the classical paradigm, starting with those which require that conclusions be related to their premisses by some connection of *relevance* (so the classical idea that a contradiction entails anything is dropped).

5.1 Getting started with modal logic

Basic modal logic is the logic of the one-place propositional operators '$\Box$' and '$\Diamond$' (read these as 'it is necessarily true that' and 'it is possibly true that'); it adopts new principles like $\Box\varphi \to \varphi$ and $\varphi \to \Diamond\varphi$, and investigates more disputable principles like $\Diamond\varphi \to \Box\Diamond\varphi$.

There are some nice introductory texts written for philosophers, though I think the place to start is clear:

1. Rod Girle, *Modal Logics and Philosophy* (Acumen 2000; 2nd edn. 2009). Girle's logic courses in Auckland, his enthusiasm and abilities as a teacher, are justly famous. Part I of this book provides a particularly lucid introduction, which in 136 pages explains the basics, covering both trees and natural deduction for some propositional modal logics, and extending to the beginnings of quantified modal logic. Philosophers may well want to go on to read Part II of the book, on applications of modal logic.

Also introductory, though perhaps a little brisker than Girle at the outset, is

2. Graham Priest, *An Introduction to Non-Classical Logic** (CUP, much expanded 2nd edition 2008): read Chs 2–4 for propositional modal logics, Chs 14–18 for quantified logics. This book – which is a terrific achievement and enviably clear and well-organized – systematically explores logics of a wide variety of kinds, using trees throughout in a way that can be very illuminating indeed. Although it starts from scratch, however, it would be better to come to the book with a prior familiarity with logic via trees, as in Chs 16–18, 25 and 29 of my *IFL*. We will be mentioning Priest's book again in later sections for its excellent coverage of other non-classical themes.

If you do start with Priest's book, then at some point you will want to supplement it by looking at a treatment of natural deduction proof systems for modal logics. One option is to dip into Tony Roy's comprehensive 'Natural Derivations for Priest, *An Introduction to Non-Classical Logic*' which presents natural deduction systems corresponding to the propositional logics presented in tree form in the first edition of Priest (so the first half of the new edition). Another possible way in to ND modal systems would be via the opening chapters of

3. James Garson, *Modal Logic for Philosophers** (CUP, 2006; 2nd end. 2014). This again is certainly intended as a gentle introductory book: it deals with both ND and semantic tableaux (trees), and covers quantified modal logic. It is reasonably accessible, but not – I think – as attractive as Girle.

We now go a step up in sophistication (and the more mathematical might want to start here):

4. Melvin Fitting and Richard L. Mendelsohn, *First-Order Modal Logic* (Kluwer 1998). This book starts again from scratch, but then does go rather more snappily, with greater mathematical elegance (though it should certainly be accessible to anyone who is modestly on top of non-modal first-order logic). It still also includes a good amount of philosophically interesting material. Recommended.

A few years ago, I would have said that getting as far as Fitting and Mendelsohn will give most philosophers a good enough grounding in basic modal logic. But e.g. Timothy Williamson's book *Modal Logic as Metaphysics* (OUP, 2013) calls on rather more, including second-order modal logics. If you need to sharpen your knowledge of the technical background here, I guess there is nothing for it but to tackle

5. Nino B. Cocchiarella and Max A. Freund, *Modal Logic: An Introduction to its Syntax and Semantics* (OUP, 2008). The blurb announces that "a variety of modal logics at the sentential, first-order, and second-order levels are developed with clarity, precision and philosophical insight". However, when I looked at this book with an eye to using it for a graduate seminar a couple of years back, I confess I didn't find it very appealing: so I do suspect that many philosophical readers will indeed find the treatments in this book rather relentless. However, the promised wide coverage could make the book of particular interest to determined philosophers concerned with the kind of issues that Williamson discusses.

Finally, I should certainly draw your attention to the classic book by Boolos mentioned at the end of §6.4, where modal logic gets put to use in exploring results about provability in arithmetic, Gödel's Second Incompleteness Theorem, and more.

Postscript for philosophers Old hands learnt their modal logic from G. E. Hughes and M. J. Cresswell *An Introduction to Modal Logic* (Methuen, 1968). This was at the time of original publication a unique book, enormously helpfully bringing together a wealth of early work on modal logic in an approachable way. Nearly thirty years later, the authors wrote a heavily revised and updated version, *A New Introduction to Modal Logic* (Routledge, 1996). This newer version like the original one concentrates on *axiomatic* versions of modal logic, which doesn't make it always the most attractive introduction from a modern point of view. But it is still an admirable book at an introductory level (and going beyond), and a book that enthusiasts can still learn from.

I didn't recommend the first part of Theodore Sider's *Logic for Philosophy** (OUP, 2010). However, the second part of the book which is entirely devoted to modal logic (including quantified modal logic) and related topics like Kripke semantics for intuitionistic logic is significantly better. Compared with the early chapters with their inconsistent levels of coverage and sophistication, the discussion here develops more systematically and at a reasonably steady level of exposition. There is, however, a lot of (acknowledged) straight borrowing from Hughes and Cresswell, and – like those earlier authors – Sider also gives axiomatic systems. But if you just want a brisk and pretty clear explanation of Kripke semantics, and want to learn e.g. how to search systematically for countermodels, Sider's treatment in his Ch. 6 could well work as a basis. And then the later treatments of quantified modal logic in Chs 9 and 10 (and some of the conceptual issues they raise) are also brief, lucid and approachable.

Postscript for the more mathematical Here are a couple of good introductory modal logic books with a mathematical flavour:

6. Sally Popkorn, *First Steps in Modal Logic* (CUP, 1994). The author is, at least in this possible world, identical with the mathematician Harold Simmons. This book, which entirely on propositional modal logics, is written for computer scientists. The Introduction rather boldly says 'There are few books on this subject and even fewer books worth looking at. None of these give an acceptable mathematically correct account of the subject. This book is a first attempt to fill that gap.' This considerably oversells the case: but the result is illuminating and readable.

7. Also just on propositional logic, I'd recommend Patrick Blackburn, Maarten de Ricke and Yde Venema's *Modal Logic* (CUP, 2001). This is one of the Cambridge Tracts in Theoretical Computer Science: but again don't let that provenance put you off – it is (relatively) accessibly and agreeably written, with a lot of signposting to the reader of possible routes through the book, and interesting historical notes. I think it works pretty well, and will certainly give you an idea about how non-philosophers approach modal logic.

Going in a different direction, if you are interested in the relation between modal logic and intuitionistic logic (see §4.4.2), then you might want to look at

Alexander Chagrov and Michael Zakharyaschev *Modal Logic* (OUP, 1997). This is a volume in the Oxford Logic Guides series and again concentrates on propositional modal logics. Written for the more mathematically minded reader, it tackles things in an unusual order, starting with an extended discussion of intuitionistic logic, and is pretty demanding. But enthusiasts should take a look.

Finally, if you want to explore even more, there's the giant *Handbook of Modal Logic*, edited by van Bentham et al. (Elsevier, 2005). You can get an idea of what's in the volume by looking at this page of links to the opening pages of the various contributions.

5.2 Free logic, plural logic

We next look at what happens if you stay first-order (keep your variables running over objects), stay classical in spirit (keep the same basic notion of logical consequence) but allow terms that fail to denote (free logic) or allow terms that refer to more than one thing (plural logic).

5.2.1 Free Logic

Classical logic assumes that any term denotes an object in the domain of quantification, and in particular assumes that all functions are total, i.e. defined for every argument – so an expression like '$f(c)$' always denotes. But mathematics cheerfully countenances partial functions, which may lack a value for some arguments. Should our logic accommodate this, by allowing terms to be free of existential commitment? In which case, what would such a 'free' logic look like?

For some background and motivation, see the gently paced and accessible

1. David Bostock, *Intermediate Logic* (OUP 1997), Ch. 8.

Then for more detail, we have a helpful overview article in the ever-useful Stanford Encyclopedia:

2. John Nolt, 'Free Logic', *The Stanford Encyclopedia of Philosophy.*

For formal treatments in, respectively, natural deduction and tableau settings, see:

3. Neil Tennant, *Natural Logic*** (Edinburgh UP 1978, 1990), §7.10.
4. Graham Priest, *An Introduction to Non-Classical Logic** (CUP, 2nd edition 2008), Ch. 13.

If you want to explore further (going rather beyond the basics), you could make a start on

5. Ermanno Bencivenga, 'Free Logics', in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic, vol. III: Alternatives to Classical Logic* (Reidel, 1986). Reprinted in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, 2nd edition, vol. 5 (Kluwer, 2002).

Postscript Rolf Schock's *Logics without Existence Assumptions* (Almqvist & Wiskell, Stockholm 1968) is still well worth looking at on free logic after all this time. And for a collection of articles of interest to philosophers, around and about the topic of free logic, see Karel Lambert, *Free Logic: Selected Essays* (CUP 2003).

5.2.2 Plural logic

In ordinary mathematical English we cheerfully use plural denoting terms such as '2, 4, 6, and 8', 'the natural numbers', 'the real numbers between 0 and 1', 'the complex solutions of $z^2 + z + 1 = 0$', 'the points where line L intersects curve C', 'the sets that are not members of themselves', and the like. Such locutions are entirely familiar, and we use them all the time without any sense of strain or logical impropriety. We also often generalize by using plural quantifiers like 'any natural numbers' or 'some reals' together with linked plural pronouns such as 'they' and 'them'. For example, here is a version of the Least Number Principle: given any natural numbers, one of them must be the least. By contrast, there are some reals – e.g. those strictly between 0 and 1 – such that no one of *them* is the least.

Plural terms and plural quantifications appear all over the place in everyday mathematical argument. It is surely a project of interest to logicians to regiment and evaluate the informal modes of argument involving such constructions. This is the business of plural logic, a topic of much recent discussion.

For an introduction, see

6. Øystein Linnebo, 'Plural Quantification', *The Stanford Encyclopedia of*

Philosophy.

And do read at least these two of the key papers listed in Linnebo's expansive bibliography:

7. Alex Oliver and Timothy Smiley, 'Strategies for a Logic of Plurals', *Philosophical Quarterly* (2001) pp. 289–306.
8. George Boolos, 'To Be Is To Be a Value of a Variable (or to Be Some Values of Some Variables)', *Journal of Philosophy* (1984) pp. 430–50. Reprinted in Boolos, *Logic, Logic, and Logic* (Harvard University Press, 1998).

(Oliver and Smiley give reasons why there is indeed a real subject here: you can't readily eliminate all plural talk in favour e.g. of singular talk about sets. Boolos's classic will tell you something about the possible relation between plural logic and second-order logic.) Then, for much more about plurals, you can follow up more of Linnebo's bibliography, or could look at

9. Thomas McKay, *Plural Predication* (OUP 2006),

which is clear and approachable. However:

Real enthusiasts for plural logic will want to dive into the long-awaited (though occasionally rather idiosyncratic)

10. Alex Oliver and Timothy Smiley, *Plural Logic* (OUP 2013: revised and expanded second edition, 2016).

5.3 Relevance logics (and wilder logics too)

(a) Classically, if $\varphi \vdash \psi$, then $\varphi, \chi \vdash \psi$ (read '$\vdash$' as 'entails': irrelevant premisses can be added without making a valid entailment invalid). And if $\varphi, \chi \vdash \psi$ then $\varphi \vdash \chi \rightarrow \psi$ (that's the Conditional Proof rule in action, a rule that seems to capture something essential to our understanding of the conditional). Presumably we have $P \vdash P$. So we have $P, Q \vdash P$. Whence $P \vdash Q \rightarrow P$. It seems then that classical logic's carefree attitude to questions of relevance in deduction and its dubious version of the conditional are tied closely together.

Classically, we also have $\varphi, \neg\varphi \vdash \psi$. But doesn't the inference from P and $\neg P$ to Q commit another fallacy of relevance? And again, if we allow it and also allow conditional proof, we will have $P \vdash \neg P \rightarrow Q$, another seemingly unhappy result about the conditional.

Can we do better? What does a more relevance-aware logic look like?

For useful introductory reading, see

1. Edwin Mares, 'Relevance Logic', *The Stanford Encycl. of Philosophy.*
2. Graham Priest, Koji Tanaka, Zach Weber, 'Paraconsistent logic', *The Stanford Encyclopedia of Philosophy.*

These two articles have many pointers for further reading across a range of topics, so I can be brief here. But I will mention two wide-ranging introductory texts:

3. Graham Priest, *An Introduction to Non-Classical Logic** (CUP, much expanded 2nd edition 2008). Look now at Chs. 7–10 for a treatment of propositional logics of various deviant kinds. Priest starts with relevance logic and goes on to also treat logics where there are truth-value gaps, and – more wildly – logics where a proposition can be both true and false (there's a truth-value glut), Then, if this excites you, carry on to look at Chs. 21–24 where the corresponding quantificational logics are presented. This book really is a wonderful resource.
4. J. C. Beall and Bas van Fraassen's *Possibilities and Paradox* (OUP 2003), also covers a range of logics. In particular, Part III of the book covers relevance logic and also non-standard logics involving truth-value gaps and truth-value gluts. (It is worth looking too at the earlier parts of the book on logical frameworks generally and on modal logic.)

The obvious next place to go is then the very lucid

5. Edwin Mares, *Relevant Logic: A Philosophical Interpretation* (CUP 2004). As the title suggests, this book has very extensive conceptual discussion alongside the more formal parts elaborating what might be called the mainstream tradition in relevance logics.

(b) Note, however, that although they get discussed in close proximity in the books by Priest and by Beall and van Fraasen, there's no tight connection between (i) the reasonable desire to have a more relevance-aware logic (e.g. without the principle that a contradiction implies everything) and (ii) the highly revisionary proposal that there can be propositions which are both true and false at the same time.

At the risk of corrupting the youth, if you are interested in exploring the latter immodest proposal further, then I can point you to

6. Graham Priest, 'Dialetheism', *The Stanford Encyclopedia of Philosophy.*

(c) There is however a minority tradition on relevance that I myself find extremely appealing, developed by Neil Tennant in scattered papers and episodes in books. You will need to know a little proof theory to appreciate it – though you can get a flavour of the approach from the early programmatic paper

6. Neil Tennant, 'Entailment and proofs', *Proceedings of the Aristotelian Society* LXXIX (1979) 167–189.

You can then follow this up by looking at papers on Tennant's website with titles involving key words and phrases like 'entailment' 'relevant logic' and 'core logic': Tennant has a new book to be published this year aiming to bring things neatly together:

7. Neil Tennant, *Core Logic* (OUP, 2017).

Chapter 6

More advanced reading on some core topics

In this chapter, there are some suggestions for more advanced reading on a selection of topics in and around the core mathematical logic curriculum we looked at in Chs. 3 and 4 (other than set theory, which we return to in the next chapter). Three points before we begin:

- Before tackling this often significantly more difficult material, it could be very well worth first taking the time to look at one or two of the wider-ranging Big Books on mathematical logic which will help consolidate your grip on the basics at the level of Chapter 4 and/or push things on just a bit. See the slowly growing set of Book Notes for some guidance on what's available.

- I did try to be fairly systematic in Chapter 4, aiming to cover the different core areas at comparable levels of detail, depth and difficulty. The coverage of various topics from here on is more varied: the recommendations can be many or few (or non-existent!) depending on my own personal interests and knowledge.

- I do, however, still aim to cluster suggestions within sections or subsections in rough order of difficulty. And I now use boxes to set off a number of acknowledged classics that perhaps any logician ought to read one day, whatever their speciality.

And a warning to those philosophers still reading: some of the material I point to is inevitably mathematically quite demanding!

6.1 Proof theory

Proof theory has been (and continues to be) something of a poor relation in the standard Mathematical Logic curriculum: the usual survey textbooks don't discuss it. Yet this is a fascinating area, of interest to philosophers, mathematicians, and computer scientists who all *ought* to be concerned with the notion of proof! So let's start to fill this gap next.

(a) I mentioned in §3.3 the introductory book by Jan von Plato, *Elements of Logical Reasoning** (CUP, 2014), which approaches elementary logic with more of an eye on proof theory than is at all usual: you might want to take a look at that book if you didn't before. Philosophers may also find the first two chapters of a – regrettably, very partial – draft book by Greg Restall, *Proof Theory and Philosophy*, helpful.

However, you should start serious work by reading this extremely useful encyclopaedia entry:

1. Jan von Plato, 'The development of proof theory', *The Stanford Encyclopedia of Philosophy.*

This will give you orientation and introduce you to some main ideas: there is also an excellent bibliography which you can use to guide further exploration.

That biblio perhaps makes the rest of this section a bit redundant; but for what they are worth, here are my less informed suggestions. Everyone will agree that you should certainly read the little hundred-page classic

2. Dag Prawitz, *Natural Deduction: A Proof-Theoretic Study** (Originally published in 1965: reprinted Dover Publications 2006).

And if you want to follow up in more depth Prawitz's investigations of the proof theory of various systems of logic, the next place to look is surely

3. Sara Negri and Jan von Plato, *Structural Proof Theory* (CUP 2001). This is a modern text which is neither too terse, nor too laboured, and is generally very clear. When we read it in a graduate-level reading group, we did find we needed to pause sometimes to stand back and think a little about the motivations for various technical developments. So perhaps a few more 'classroom asides' in the text would have made a rather good text even better. But this is still *extremely* helpful.

Then in a more mathematical style, there is the editor's own first contribution to

4. Samuel R. Buss, ed., *Handbook of Proof Theory* (North-Holland, 1998). Later chapters of this very substantial handbook do get pretty hard-core; but the 78 pp. opening chapter by Buss himself, a 'Introduction to Proof Theory'**, is readable, and freely downloadable. (Student health warming: there are, I am told, some confusing misprints in the cut-elimination proof.)

(b) And now the path through proof theory forks. In one direction, the path cleaves to what we might call classical themes (I don't mean themes simply concerning classical logic, as intuitionistic logic was also treated as central from the start: I mean themes explicit in the early classic papers in proof theory, in particular in Gentzen's work). It is along this path that we find e.g. Gentzen's famous proof of the consistency of first-order Peano Arithmetic using proof-theoretic ideas. One obvious text on these themes remains

5. Gaisi Takeuti, *Proof Theory** (North-Holland 1975, 2nd edn. 1987: reprinted Dover Publications 2013). This is a true classic – if only because for a while it was about the only available book on most of its topics. Later chapters won't really be accessible to beginners. But you could/should try reading Ch. 1 on logic, §§1–7 (and perhaps the beginnings of §8, pp. 40–45, which is easier than it looks if you compare how you prove the completeness of a tree system of logic). Then on Gentzen's proof, read Ch. 2, §§9–11 and §12 up to at least p. 114. This isn't exactly plain sailing – but if you skip and skim over some of the more tedious proof-details you can pick up a very good basic sense of what happens in the consistency proof.

Gentzen's proof of the consistency of depends on transfinite induction along ordinals up to ε_0; and the fact that it requires just so much transfinite induction to prove the consistency of first-order PA is an important characterization of the strength of the theory. The project of 'ordinal analysis' in proof theory aims to provide comparable characterizations of other theories in terms of the amount of transfinite induction that is needed to prove *their* consistency. Things do get quite hairy quite quickly, however.

6. For a glimpse ahead, you could look at (initial segments of) these useful notes for mini-courses by Michael Rathjen, on 'The Realm of Ordinal Analysis' and 'Proof Theory: From Arithmetic to Set Theory'.

Turning back from these complications, however, let's now glance down the other path from the fork, where we investigate not the proof theory of theories constructed in familiar logics but rather investigate non-standard logics themselves. Reflection on the structural rules of classical and intuitionistic proof systems naturally raises the question of what happens when we tinker with these rules. We noted before the inference which takes us from the trivial $P \vdash P$ by 'weakening' to $P, Q \vdash P$ and on, via 'conditional proof', to $P \vdash Q \to P$. If we want a conditional that conforms better to intuitive constraints of relevance, then we need to block that proof: is 'weakening' the culprit? The investigation of what happens if we tinker with standard structural rules such as weakening belongs to substructural logic, outlined in

7. Greg Restall, 'Substructural Logics', *The Stanford Encyclopedia of Philosophy.*

(which again has an admirable bibliography). And the place to continue exploring these themes at length is the same author's splendid

8. Greg Restall, *An Introduction to Substructural Logics* (Routledge, 2000), which will also teach you a lot more about proof theory generally in a very accessible way. Do read at least the first seven chapters.

(You could note again here the work on Neil Tennant mentioned at the very end of §5.3.)

(c) For the more mathematically minded, here are a few more books of considerable interest. I'll start with a couple that in fact aim to be accessible to beginners. They wouldn't be my recommendations of texts to start from, but they could be very useful if you already know a bit of proof theory.

9. Jean-Yves Girard, *Proof Theory and Logical Complexity. Vol. I* (Bibliopolis, 1987) is intended as an introduction [Vol. II was never published]. With judicious skipping, which I'll signpost, this is readable and insightful, though some proofs are a bit arm-waving.

   So: skip the 'Foreword', but do pause to glance over 'Background and Notations' as Girard's symbolic choices need a little explanation. Then the long Ch. 1 is by way of an introduction, proving Gödel's two incompleteness theorem and explaining 'The Fall of Hilbert's Program': if you've read some of the recommendations in §4.2 above, you can probably skim this pretty quickly, just noting Girard's highlighting of the notion of 1-consistency.

   Ch. 2 is on the sequent calculus, proving Gentzen's *Hauptsatz*, i.e. the crucial cut-elimination theorem, and then deriving some first consequences (you can probably initially omit the forty pages of annexes to this chapter). Then also omit Ch. 3 whose content isn't relied on later. But Ch. 4 on

'Applications of the *Hauptsatz*' is crucial (again, however, at a first pass you can skip almost 60 pages of annexes to the chapter). Take the story up again with the first two sections of Ch. 6, and then tackle the opening sections of Ch. 7. A bumpy ride but very illuminating.

10. A. S. Troelstra and H. Schwichtenberg's *Basic Proof Theory* (CUP 2nd ed. 2000) is a volume in the series Cambridge Tracts in Computer Science. Now, one theme that runs through the book indeed concerns the computer-science idea of formulas-as-types and invokes the lambda calculus: however, it is in fact possible to skip over those episodes in you aren't familiar with the idea. The book, as the title indicates, is intended as a first foray into proof theory, and it *is* reasonably approachable. However it is perhaps a little cluttered for my tastes because it spends quite a bit of time looking at slightly different ways of doing natural deduction and slightly different ways of doing the sequent calculus, and the differences may matter more for computer scientists with implementation concerns than for others. You could, however, with a bit of skipping, very usefully read just Chs. 1–3, the first halves of Chs. 4 and 6, and then Ch. 10 on arithmetic again.

And now for three more advanced offerings:

11. I have already mentioned the compendium edited by Samuel R. Buss, *Handbook of Proof Theory* (North-Holland, 1998), and the fact that you can download its substantial first chapter. You can also freely access Ch. 2 on 'First-Order Proof-Theory of Arithmetic'. Later chapters of the Handbook are of varying degrees of difficulty, and cover a range of topics (though there isn't much on ordinal analysis).

12. Wolfram Pohlers, *Proof Theory: The First Step into Impredicativity* (Springer 2009). This book has introductory ambitions, to say something about so-called ordinal analysis in proof theory as initiated by Gentzen. But in fact I would judge that it requires quite an amount of mathematical sophistication from its reader. From the blurb: "As a 'warm up' Gentzen's classical analysis of pure number theory is presented in a more modern terminology, followed by an explanation and proof of the famous result of Feferman and Schütte on the limits of predicativity." The first half of the book is probably manageable if (but only if) you already have done some of the other reading. But then the going indeed gets pretty tough.

13. H. Schwichtenberg and S. Wainer, *Proofs and Computations* (Association of Symbolic Logic/CUP 2012) "studies fundamental interactions between

proof-theory and computability". The first four chapters, at any rate, will be of wide interest, giving another take on some basic material and should be manageable given enough background. Sadly, I found the book to be not particularly well written and it sometimes makes heavier weather of its material than seems really necessary. Worth the effort though.

There is a recent more introductory text by Katalin Bimbó, *Proof Theory: Sequent Calculi and Related Formalisms* (CRC Press, 2014); but having looked at it, I'm not minded to recommend this.

6.2 Beyond the model-theoretic basics

(a) If you want to explore model theory beyond the entry-level material in §4.1, why not start with a quick warm-up, with some reminders of headlines and some useful pointers to the road ahead:

1. Wilfrid Hodges, 'First-order model theory', *The Stanford Encyclopedia of Philosophy*.

Now, we noted before that e.g. the wide-ranging texts by Hedman and Hinman eventually cover a substantial amount of model theory. But you will do even better with two classic stand-alone treatments of the area which really choose themselves. Both in order of first publication and of eventual difficulty we have:

2. C. Chang and H. J. Keisler, *Model Theory** (originally North Holland 1973: the third edition has been inexpensively republished by Dover Books in 2012). This is the Old Testament, the first systematic text on model theory. Over 550 pages long, it proceeds at an engagingly leisurely pace. It is particularly lucid and is extremely nicely constructed with different chapters on different methods of model-building. A really fine achievement that still remains a good route in to the serious study of model theory.
3. Wilfrid Hodges, *A Shorter Model Theory* (CUP, 1997). The New Testament is Hodges's encyclopedic original *Model Theory* (CUP 1993). This shorter version is half the size but still really full of good things. It does get tougher as the book progresses, but the earlier chapters of this modern classic, written with this author's characteristic lucidity, should certainly be readily manageable.

My suggestion would be to read the first three long chapters of Chang and Keisler, and then pause to make a start on

4. J. L. Bell and A. B. Slomson, *Models and Ultraproducts** (North-Holland 1969; Dover reprint 2006). Very elegantly put together: as the title suggests, the book focuses particularly on the ultra-product construction. At this point read the first five chapters for a particularly clear introduction.

You could then return to Ch. 4 of C&K to look at (some of) their treatment of the ultra-product construction, before perhaps putting the rest of their book on hold and turning to Hodges.

(b) A level up again, here are two more books. The first has been around long enough to have become regarded as a modern standard text. The second is more recent but also comes well recommended. Their coverage is significantly different – so those wanting to get seriously into model theory will take a look at both:

5. David Marker, *Model Theory: An Introduction* (Springer 2002). Despite its title, this book would surely be hard going if you haven't already tackled some model theory (at least read Manzano first). But despite being sometimes a rather bumpy ride, this highly regarded text will teach you a great deal. Later chapters, however, probably go far over the horizon for all except those most enthusiastic readers of this Guide who are beginning to think about specializing in model theory – it isn't published in the series 'Graduate Texts in Mathematics' for nothing!

6. Katrin Tent and Martin Ziegler, *A Course in Model Theory* (CUP, 2012). From the blurb: "This concise introduction to model theory begins with standard notions and takes the reader through to more advanced topics such as stability, simplicity and Hrushovski constructions. The authors introduce the classic results, as well as more recent developments in this vibrant area of mathematical logic. Concrete mathematical examples are included throughout to make the concepts easier to follow." Again, although it starts from the beginning, it could be a bit of challenge to readers without any prior exposure to the elements of model theory – though I, for one, find it more approachable than Marker's book.

(c) So much for my principal suggestions. Now for an assortment of additional/alternative texts. I should, begin, perhaps by mentioning an old fifty-page survey essay (though note that model theory has advanced a lot in the intervening years):

7. H. J. Keisler, 'Fundamentals of Model Theory', in J. Barwise, editor, *Handbook of Mathematical Logic*, pp. 47–103 (North-Holland, 1977). This is surely going to be too terse for most readers to read with full understanding at the outset, which is why I didn't highlight it before. However, you could helpfully read it at an early stage to get a half-understanding of where you want to be going, and then re-read it at a later stage to check your understanding of some of the fundamentals.

And now here are two more books which aim to give general introductions:

8. Philipp Rothmaler's *Introduction to Model Theory* (Taylor and Francis 2000) is, overall, comparable in level of difficulty with, say, the first half of Hodges. As the blurb puts it: "This text introduces the model theory of first-order logic, avoiding syntactical issues not too relevant to model theory. In this spirit, the compactness theorem is proved via the algebraically useful ultraproduct technique (rather than via the completeness theorem of first-order logic). This leads fairly quickly to algebraic applications, ..." Now, the opening chapters are indeed very clear: but oddly the introduction of the crucial ultraproduct construction in Ch. 4 is done very briskly (compared, say, with Bell and Slomson). And thereafter it seems to me that there is some unevenness in the accessibility of the book. But others have recommended this text, so I mentioned it as a possibility worth checking out.

9. Bruno Poizat's *A Course in Model Theory* (English edition, Springer 2000) starts from scratch and the early chapters give an interesting and helpful account of the model-theoretic basics, and the later chapters form a rather comprehensive introduction to stability theory. This often-recommended book is written in a rather distinctive style, with rather more expansive class-room commentary than usual: so an unusually engaging read at this sort of level.

Another book which is often mentioned in the same breath as Poizat, Marker, and now Tent and Ziegler as a modern introduction to model theory is *A Guide to Classical and Modern Model Theory*, by Annalisa Marcja and Carlo Toffalori (Kluwer, 2003) which also covers a lot: but I prefer the previously mentioned books.

The next two suggestions are of books which are helpful on particular aspects of model theory:

10. Kees Doets's short *Basic Model Theory** (CSLI 1996) highlights so-called Ehrenfeucht games. This is enjoyable and very instructive.

11. Chs. 2 and 3 of Alexander Prestel and Charles N. Delzell's *Mathematical Logic and Model Theory: A Brief Introduction* (Springer 1986, 2011) are brisk but clear, and can be recommended if you wanting a speedy review of model theoretic basics. The key feature of the book, however, is the sophisticated final chapter on applications to algebra, which might appeal to mathematicians with special interests in that area. For a very little more on this book, see my Book Note.

Indeed, as we explore model theory, we quickly get entangled with algebraic questions. And as well as going (so to speak) in the direction from logic to algebra, we can make connections the other way about, starting from algebra. For something on this approach, see the following short, relatively accessible, and illuminating book:

12. Donald W. Barnes and John M. Mack, *An Algebraic Introduction to Mathematical Logic* (Springer, 1975).

(d) Let me also briefly allude to the sub-area of Finite Model Theory which arises particularly from consideration of problems in the theory of computation (where, of course, we are interested in *finite* structures – e.g. finite databases and finite computations over them). What happens, then, to model theory if we restrict our attention to finite models? Trakhtenbrot's theorem, for example, tells that the class of sentences true in any finite model is not recursively enumerable. So there is no deductive theory for capturing such finitely valid sentences (that's a surprise, given that there's a complete deductive system for the valid sentences!). It turns out, then, that the study of finite models is surprisingly rich and interesting (at least for enthusiasts!). So why not dip into one or other of

13. Leonard Libkin, *Elements of Finite Model Theory* (Springer 2004).

14. Heinz-Dieter Ebbinghaus and Jörg Flum, *Finite Model Theory* (Springer 2nd edn. 1999).

Either is a very good standard text to explore the area with, though I prefer Libkin's.

(e) Two afterthoughts. First, it is illuminating to read something about the history of model theory: there's a good, and characteristically lucid, unpublished piece by a now-familiar author here:

15. W. Hodges, ‘Model Theory’.

Second, one thing you will have noticed if you tackle a few texts beyond the level of Manzano’s is that the absolutely key compactness theorem (for example) can be proved in a variety of ways – indirectly via the completeness proof, via a more direct Henkin construction, via ultraproducts, etc. How do these proofs inter-relate? Do they generalize in different ways? Do they differ in explanatory power? For a quite excellent essay on this – on the borders of mathematics and philosophy (and illustrating that there is indeed very interesting work to be done in that border territory), see

16. Alexander Paseau, ‘Proofs of the Compactness Theorem’, *History and Philosophy of Logic* 31 (2001): 73–98.

6.3 Computability

In §4.2 we took a first look at the related topics of computability, Gödelian incompleteness, and theories of arithmetic. In this and the next two main sections, we return to these topics, taking them separately (though this division is necessarily somewhat artificial).

6.3.1 Computable functions

(a) Among the Big Books on mathematical logic, the one with the most useful treatment of computability is probably

1. Peter G. Hinman, *Fundamentals of Mathematical Logic* (A. K. Peters, 2005). Chs. 4 and 5 on recursive functions, incompleteness etc. strike me as the best written, most accessible (and hence most successful) chapters in this very substantial book. The chapters could well be read after my *IGT* as somewhat terse revision for mathematicians, and then as sharpening the story in various ways. Ch. 8 then takes up the story of recursion theory (the author’s home territory).

However, good those these chapters are, I’d still recommend starting your more advanced work on computability with

2. Nigel Cutland, *Computability: An Introduction to Recursive Function Theory* (CUP 1980). This is a rightly much-reprinted classic and is beautifully lucid and well-organized. This *does* have the look-and-feel of a

traditional maths text book of its time (so with fewer of the classroom asides we find in some discursive books). However, if you got through most of e.g. Boolos, Burgess and Jeffrey without too much difficulty, you ought certainly to be able to tackle this as the next step. Very warmly recommended.

And of more recent books covering computability this level (i.e. a step up from the books mentioned in §4.2, I also particularly like

3. S. Barry Cooper, *Computability Theory* (Chapman & Hall/CRC 2004: a second edition is announced for 2107). This is a very nicely done modern textbook. Read at least Part I of the book (about the same level of sophistication as Cutland, but with some extra topics), and then you can press on as far as your curiosity takes you, and get to excitements like the Friedberg-Muchnik theorem.

(b) The inherited literature on computability is huge. But, being *very* selective, let me mention three classics from different generations, two to dip into, one you really ought to study:

4. Rósza Péter, *Recursive Functions* (originally published 1950: English translation Academic Press 1967). This is by one of those logicians who was 'there at the beginning'. It has that old-school slow-and-steady un-flashy lucidity that makes it still a considerable pleasure to read. It remains very worth looking at.

5. Hartley Rogers, Jr., *Theory of Recursive Functions and Effective Computability* (McGraw-Hill 1967) is a heavy-weight state-of-the-art-then classic, written at the end of the glory days of the initial development of the logical theory of computation. It quite speedily gets advanced. But the opening chapters are still excellent reading and are action-packed. At least take it out of the library, read a few chapters, and admire!

6. Piergiorgio Odifreddi, *Classical Recursion Theory*, Vol. 1 (North Holland, 1989) is well-written and discursive, with numerous interesting asides. It's over 650 pages long, so it goes further and deeper than other books on the main list above (and then there is Vol. 2). But it certainly starts off quite gently paced and very accessible and can be warmly recommended for consolidating and extending your knowledge.

(c) A number of books we've already mentioned say something about the fascinating historical development of the idea of computability: as we noted before, Richard Epstein offers a very helpful 28 page timeline on 'Computability and Undecidability' at the end of the 2nd edn. of Epstein/Carnielli (see §4.2). Cooper's short first chapter on 'Hilbert and the Origins of Computability Theory' also gives some of the headlines. Odifreddi too has many historical details. But here are two more good essays on the history:

7. Robert I. Soare, 'The History and Concept of Computability', in E. Griffor, ed., *Handbook of Computability Theory* (Elsevier 1999).

8. Robin Gandy, 'The Confluence of Ideas in 1936' in R. Herken, ed., *The Universal Turing Machine: A Half-century Survey* (OUP 1988). Seeks to explain why so many of the absolutely key notions all got formed in the mid-thirties.

6.3.2 Computational complexity

Computer scientists are – surprise, surprise! – interested in the theory of feasible computation, and any logician should be interested in finding out at least a little about the topic of computational complexity.

1. Shawn Hedman *A First Course in Logic* (OUP 2004): Ch. 7 on 'Computability and complexity' has a nice review of basic computability theory before some lucid sections introducing notions of computational complexity.

2. Michael Sipser, *Introduction to the Theory of Computation* (Thomson, 2nd edn. 2006) is a standard and very well regarded text on computation aimed at computer scientists. It aims to be very accessible and to take its time giving clear explanations of key concepts and proof ideas. I think this is very successful as a general introduction and I could well have mentioned the book before. But I'm highlighting the book in this subsection because its last third is on computational complexity.

3. Ofed Goldreich, *P, NP, and NP-Completeness* (CUP, 2010). Short, clear, and introductory stand-alone treatment of computational complexity.

4. Ashley Montanaro, *Computational Complexity*. Excellent 2012 lecture notes, lucid and detailed and over 100 pages (also include a useful quick guide to further reading).

5. You could also look at the opening chapters of the pretty encyclopaedic Sanjeev Arora and Boaz Barak *Computational Complexity: A Modern Approach*** (CUP, 2009). The authors say 'Requiring essentially no background apart from mathematical maturity, the book can be used as a reference for self-study for anyone interested in complexity, including physicists, mathematicians, and other scientists, as well as a textbook for a variety of courses and seminars.' And it at least starts very readably. A late draft of the book can be freely downloaded.

6.4 Incompleteness and related matters

(a) If you have looked at my book and/or Boolos and Jeffrey you should now be in a position to appreciate the terse elegance of

1. Raymond Smullyan, *Gödel's Incompleteness Theorems*, Oxford Logic Guides 19 (Clarendon Press, 1992). This is delightully short – under 140 pages – proving some beautiful, slightly abstract, versions of the incompleteness theorems. This is a modern classic which anyone with a taste for mathematical elegance will find rewarding.

2. Equally short and equally elegant is Melvin Fitting's, *Incompleteness in the Land of Sets** (College Publications, 2007). This approaches things from a slightly different angle, relying on the fact that there is a simple correspondence between natural numbers and 'hereditarily finite sets' (i.e. sets which have a finite number of members which in turn have a finite number of members which in turn ... where all downward membership chains bottom out with the empty set).

In terms of difficulty, these two lovely brief books could easily have appeared among our introductory readings in Chapter 4. I have put them here because (as I see it) the simpler, more abstract, stories they tell can probably only be fully appreciated if you've first met the basics of computability theory and the incompleteness theorems in a more conventional treatment.

You ought also at some stage read an even briefer, and still officially introductory, treatment of the incompleteness theorems,

3. Craig Smoryński, 'The incompleteness theorems' in J. Barwise, editor, *Handbook of Mathematical Logic*, pp. 821–865 (North-Holland, 1977), which covers a lot very compactly.

After these, where should you go if you want to know more about matters more or less directly to do with the incompleteness theorems?

4. Raymond Smullyan's *Diagonalization and Self-Reference*, Oxford Logic Guides 27 (Clarendon Press 1994) is an investigation-in-depth around and about the idea of diagonalization that figures so prominently in proofs of limitative results like the unsolvability of the halting problem, the arithmetical undefinability of arithmetical truth, and the incompleteness of arithmetic. Read at least Part I.

5. Torkel Franzén, *Inexaustibility: A Non-exhaustive Treatment* (Association for Symbolic Logic/A. K. Peters, 2004). The first two-thirds of the book gives another take on logic, arithmetic, computability and incompleteness. The last third notes that Gödel's incompleteness results have a positive consequence: 'any system of axioms for mathematics that we recognize as correct can be properly extended by adding as a new axiom a formal statement expressing that the original system is consistent. This suggests that our mathematical knowledge is inexhaustible, an essentially philosophical topic to which this book is devoted.' Not always easy (you will need to know something about ordinals before you read this), but very illuminating.

6. Per Lindström, *Aspects of Incompleteness* (Association for Symbolic Logic/A. K. Peters, 2nd edn., 2003). This is probably for enthusiasts. A terse book, not always reader-friendly in its choices of notation and the brevity of its argument, but the more mathematical reader will find that it again repays the effort.

(b) Going in a rather different direction, you will recall from my *IGT2* or other reading on the second incompleteness theorem that we introduced the so-called derivability conditions on $\Box\varphi$ where this is an abbreviation for (or at any rate, is closely tied to) $\mathsf{Prov}(\ulcorner\varphi\urcorner)$, which expresses the claim that the wff φ, whose Gödel number is $\ulcorner\varphi\urcorner$, is provable in some given theory. The '$\Box$' here functions rather like a modal operator: so what is its modal logic? This is investigated in:

7. George Boolos, *The Logic of Provability* (CUP, 1993). From the blurb: "What [the author] does is to show how the concepts, techniques, and methods of modal logic shed brilliant light on the most important logical discovery of the twentieth century: the incompleteness theorems of Kurt Gödel and the 'self-referential' sentences constructed in their proof. The book explores the effects of reinterpreting the notions of necessity and

possibility to mean provability and consistency." This is a wonderful modern classic.

6.5 Theories of arithmetic

The readings in §4.2 will have introduced you to the canonical first-order theory of arithmetic, first-order Peano Arithmetic, as well as to some subsystems of PA (in particular, Robinson Arithmetic) and second-order extensions. And you will already know that first-order PA has non-standard models (in fact, it even has uncountably many non-isomorphic models which can be built out of natural numbers!).

So what to read next? You should get to more about models of PA. For a taster, you could look at these nice lecture notes:

1. Jaap van Oosten, 'Introduction to Peano Arithmetic: Gödel Incompleteness and Nonstandard Models'. (1999)

But for a fuller story, you need

2. Richard Kaye's *Models of Peano Arithmetic* (Oxford Logic Guides, OUP, 1991) which tells us a great deal about non-standard models of PA. This will reveal more about what PA can and can't prove, and will introduce you to some non-Gödelian examples of incompleteness. This does get pretty challenging in places, and it is probably best if you've already done a little model theory. Still, this is a terrific book, and deservedly a modern classic.

(There's also another volume in the Oxford Logic Guides series which can be thought of as a sequel to Kaye's for real enthusiasts with more background in model theory, namely Roman Kossak and James Schmerl, *The Structure of Models of Peano Arithmetic*, OUP, 2006. But this is much tougher.)

Next, going in a rather different direction, and explaining a lot about arithmetics weaker than full PA, here's another modern classic:

3. Petr Hájek and Pavel Pudlák, *Metamathematics of First-Order Arithmetic*** (Springer 1993). Now freely available from projecteuclid.org. This is pretty encyclopaedic, but the long first three chapters, say, actually do remain surprisingly accessible for such a work. This is, eventually, a must-read if you have a serious interest in theories of arithmetic and incompleteness.

And what about going beyond first-order PA? We know that full second-order PA (where the second-order quantifiers are constrained to run over all possible sets of numbers) is unaxiomatizable, because the underlying second-order logic is unaxiomatiable. But there are axiomatizable subsystems of second order arithmetic. These are wonderfully investigated in another encyclopaedic modern classic:

4. Stephen Simpson, *Subsystems of Second-Order Logic* (Springer 1999; 2nd edn CUP 2009). The focus of this book is the project of 'reverse mathematics' (as it has become known): that is to say, the project of identifying the weakest theories of numbers-and-sets-of-numbers that are required for proving various characteristic theorems of classical mathematics.

   We know that we can reconstruct classical analysis in pure set theory, and rather more neatly in set theory with natural numbers as unanalysed 'urelemente'. But just *how much* set theory is needed to do the job, once we have the natural numbers? The answer is: stunningly little. The project of exploring what's needed is introduced very clearly and accessibly in the first chapter, which is a must-read for anyone interested in the foundations of mathematics. This introduction is freely available at the book's website.

Chapter 7

Serious set theory

In §4.3, we gave suggestions for readings on the elements of set theory. These will have introduced you to the standard set theory ZFC, and the iterative hierarchy it seeks to describe. They also explained e.g. how we can construct the real number system in set theoretic terms (so giving you a sense of what might be involved in saying that set theory can be used as a 'foundation' for another mathematical theory). You will have in addition learnt something about the role of the axiom of choice, and about the arithmetic of infinite cardinal and ordinal numbers.

If you looked at the books by Fraenkel/Bar-Hillel/Levy or by Potter, however, you will also have noted that while standard ZFC is the market leader, it is certainly not the only set theory on the market.

So where do we go next? We'll divide the discussion into three.

- We start by focusing again on our canonical theory, ZFC. The exploration eventually becomes seriously hard mathematics – and, to be honest, it becomes of pretty specialist interest (very well beyond 'what every logician ought to know'). But it isn't clear where to stop, in a Guide like this, even if I have no doubt overdone it!

- So next we backtrack from those excursions towards the frontiers to consider old questions about the Axiom of Choice (as this is of particular conceptual interest).

- Then we will say something about non-standard set theories, rivals to ZFC (again, the long-recognised possibility of different accounts, with different degrees of departure from the canonical theory, is of considerable conceptual interest and you don't need a huge mathematical background to understand some of the options).

7.1 ZFC, with all the bells and whistles

7.1.1 A first-rate overview

One option is immediately to go for broke and dive in to the modern bible, which is highly impressive not just for its size:

1. Thomas Jech, *Set Theory*, The Third Millennium Edition, Revised and Expanded (Springer, 2003). The book is in three parts: the first, Jech says, every student should know; the second part every budding set-theorist should master; and the third consists of various results reflecting 'the state of the art of set theory at the turn of the new millennium'. Start at page 1 and keep going to page 705 (or until you feel glutted with set theory, whichever comes first).

   This is indeed a masterly achievement by a great expositor. And if you've happily read e.g. the introductory books by Enderton and then Moschovakis mentioned earlier in the Guide, then you should be able to cope pretty well with Part I of the book while it pushes on the story a little with some material on small large cardinals and other topics. Part II of the book starts by telling you about independence proofs. The Axiom of Choice is consistent with ZF and the Continuum Hypothesis is consistent with ZFC, as proved by Gödel using the idea of 'constructible' sets. And the Axiom of Choice is independent of ZF, and the Continuum Hypothesis is independent with ZFC, as proved by Cohen using the much more tricky idea of 'forcing'. The rest of Part II tells you more about large cardinals, and about descriptive set theory. Part III is indeed for enthusiasts.

Now, Jech's book is wonderful, but let's face it, the sheer size makes it a trifle daunting. It goes quite a bit further than many will need, and to get there it does in places speed along a bit faster than some will feel comfortable with. So what other options are there for if you want to take things more slowly?

7.1.2 More slowly, towards forcing

Why not start with some preliminary historical orientation. If you looked at the old book by Fraenkel/Bar-Hillel/Levy which was also recommended earlier in the Guide, then you will already know something of the early days. Alternatively, there is a nice short introductory overview

2. José Ferreirós, 'The early development of set theory', *The Stanford Encycl. of Philosophy.* (Ferreirós has also written a terrific book *Labyrinth of Thought: A History of Set Theory and its Role in Modern Mathematics* (Birkhäuser 1999), which at some stage in the future you might well want to read.)

You could also browse through the substantial article

3. Akhiro Kanamori, 'The Mathematical Development of Set Theory from Cantor to Cohen', *The Bulletin of Symbolic Logic* (1996) pp. 1-71, a revised version of which is downloadable here. (You will very probably need to skip chunks of this at a first pass: but even a partial grasp will help give you a good sense of the lie of the land for when you work on the technicalities.)

Then to start filling in details, a much admired older book (still a fine first treatment of its topic) is

4. Frank R. Drake, *Set Theory: An Introduction to Large Cardinals* (North-Holland, 1974), which – at a gentler pace? – overlaps with Part I of Jech's bible, but also will tell you about Gödel's Constructible Universe and some more about large cardinals.

But the crucial next step – that perhaps marks the point where set theory gets challenging – is to get your head around Cohen's idea of forcing used in independence proofs. However, there is not getting away from it, this is tough. In the admirable

5. Timothy Y. Chow, 'A beginner's guide to forcing',

(and don't worry if initially even this beginner's guide looks puzzling), Chow writes

> All mathematicians are familiar with the concept of *an open research problem.* I propose the less familiar concept of *an open exposition problem.* Solving an open exposition problem means explaining a mathematical subject in a way that renders it totally perspicuous. Every step should be motivated and clear; ideally, students should feel that they could have arrived at the results themselves. The proofs should be 'natural' . . . [i.e., lack] any ad hoc constructions or brilliancies. I believe that it is an open exposition problem to explain forcing.

In short: if you find that expositions of forcing tend to be hard going, then join the club.

Here though is a very widely used and much reprinted textbook, which nicely complements Drake's book and which has (inter alia) a pretty good first presentation of forcing:

6. Kenneth Kunen, *Set Theory: An Introduction to Independence Proofs* (North-Holland, 1980). If you have read (some of) the introductory set theory books mentioned in the Guide, you should actually find much of this text now pretty accessible, and can probably speed through some of the earlier chapters, slowing down later, until you get to the penultimate chapter on forcing which you'll need to take slowly and carefully. This is a rightly admired classic text.

Kunen has since published another, totally rewritten, version of this book as *Set Theory** (College Publications, 2011). This later book is quite significantly longer, covering an amount of more difficult material that has come to prominence since 1980. Not just because of the additional material, my current sense is that the earlier book may remain the slightly more approachable read.

7.1.3 Pausing for problems

At this point mathematicians could very usefully look at many of the problem sets in the excellent

7. Péver Komjáte and Vilmos Totik, *Problems and Theorems in Classical Set Theory* (Springer, 2006). From the blurb: "Most of classical set theory is covered, classical in the sense that independence methods are not used, but classical also in the sense that most results come from the period between 1920–1970. Many problems are also related to other fields of mathematics .... Rather than using drill exercises, most problems are challenging and require work, wit, and inspiration." Look at the problems that pique your interest: the authors give answers, often very detailed.

7.1.4 Pausing for more descriptive set-theory

Early on, it was discovered that the Axiom of Choice implied the existence of 'pathological' subsets of the reals, sets lacking desirable properties like being measurable. In reaction, their developed the study of nice, well-behaved, 'definable' sets – the topic of descriptive set theory. This has already been touched on in e.g. Kunen's book. For more see e.g.

8. Yiannis Moschovakis, *Descriptive Set Theory* (North Holland, 1980: second edition AMS, 2009, freely available from the author's website).

9. Alexander Kechris, *Classical Descriptive Set Theory* (Springer-Verlag, 1994).

7.1.5 Forcing further explored

To return, though, to the central theme of independence proofs and other results that can be obtained be forcing: Kunen's classic text takes a 'straight down the middle' approach, starting with what is basically Cohen's original treatment of forcing, though he does relate this to some variant approaches. Here are two of them:

10. Raymond Smullyan and Melvin Fitting, *Set Theory and the Continuum Problem* (OUP 1996, Dover Publications 2010). This medium-sized book is divided into three parts. Part I is a nice introduction to axiomatic set theory. The shorter Part II concerns matters round and about Gödel's consistency proofs via the idea of constructible sets. Part III gives a different take on forcing (a variant of the approach taken in Fitting's earlier *Intuitionistic Logic, Model Theory, and Forcing*, North Holland, 1969). This is beautifully done, as you might expect from two writers with a quite enviable knack for wonderfully clear explanations and an eye for elegance.

11. Keith Devlin, *The Joy of Sets* (Springer 1979, 2nd edn. 1993) Ch. 6 introduces the idea of Boolean-Valued Models and their use in independence proofs. The basic idea is fairly easily grasped, but details perhaps get hairy. For more on this theme, see John L. Bell's classic *Set Theory: Boolean-Valued Models and Independence Proofs* (Oxford Logic Guides, OUP, 3rd edn. 2005). The relation between this approach and other approaches to forcing is discussed e.g. in Chow's paper and the last chapter of Smullyan and Fitting.

Here are three further, more recent, books which highlight forcing ideas, one very short, the others much more wide-ranging:

10. Nik Weaver, *Forcing for Mathematicians* (World Scientific, 2014) is less than 150 pages (and the first applications of the forcing idea appear after

just 40 pages: you don't have to read the whole book to get the basics). From the blurb: "Ever since Paul Cohen's spectacular use of the forcing concept to prove the independence of the continuum hypothesis from the standard axioms of set theory, forcing has been seen by the general mathematical community as a subject of great intrinsic interest but one that is technically so forbidding that it is only accessible to specialists ... This is the first book aimed at explaining forcing to general mathematicians. It simultaneously makes the subject broadly accessible by explaining it in a clear, simple manner, and surveys advanced applications of set theory to mainstream topics." And this does strike me as a clear and very helpful attempt to solve Chow's basic exposition problem.

11. Lorenz J. Halbeisen, *Combinatorial Set Theory, With a Gentle Introduction to Forcing* (Springer 2011, with a late draft freely downloadable from the author's website). This is particularly attractively written for a set theory book. From the blurb "This book provides a self-contained introduction to modern set theory and also opens up some more advanced areas of current research in this field. The first part offers an overview of classical set theory wherein the focus lies on the axiom of choice and Ramsey theory. In the second part, the sophisticated technique of forcing, originally developed by Paul Cohen, is explained in great detail. With this technique, one can show that certain statements, like the continuum hypothesis, are neither provable nor disprovable from the axioms of set theory. In the last part, some topics of classical set theory are revisited and further developed in the light of forcing." True, this book gets quite hairy towards the end: but the first two parts of the book ('Topics in Combinatorial Set Theory' and 'From Martin's Axiom to Cohen Forcing') should be within reach to any mathematician. I can well imagine this book being spoken of as a modern classic of exposition to put alongside Kunen's and Jech's books: certainly, it has been strongly recommended by others.

12. Ralf Schindler, *Set Theory: Exploring Independence and Truth* (Springer, 2014). The book's theme is "the interplay of large cardinals, inner models, forcing, and descriptive set theory". It doesn't presume you already know any set theory, though it does proceed at a cracking pace in a brisk style. But, if you already have some knowledge of set theory, this seems a clear and interesting exploration of some themes highly relevant to current research.

7.1.6 The higher infinite

And then what next? You want more?? Back to finish Jech's doorstep of a book, perhaps. And then – oh heavens! – there is another blockbuster awaiting you:

13. Akihiro Kanamori, *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings* (Springer, 1997, 2nd edn. 2003).

7.2 The Axiom of Choice

But now let's leave the Higher Infinite and get back down to earth, or at least to less exotic mathematics! In fact, to return to the beginning, we might wonder: is ZFC the 'right' set theory?

Start by thinking about the Axiom of Choice in particular. It is comforting to know from Gödel that AC is consistent with ZF (so adding it doesn't lead to contradiction). But we also know from Cohen's forcing argument that AC is independent with ZF (so accepting ZF doesn't commit you to accepting AC too). So why buy AC? Is it an optional extra?

Some of the readings already mentioned will have touched on the question of AC's status and role. But for an overview/revision of some basics, see

1. John L. Bell, 'The Axiom of Choice', *The Stanford Encyclopedia of Philosophy.*

And for a short book also explaining some of the consequences of AC (and some of the results that you need AC to prove), see

2. Horst Herrlich, *Axiom of Choice* (Springer 2006), which has chapters rather tantalisingly entitled 'Disasters without Choice', 'Disasters with Choice' and 'Disasters either way'.

That already probably tells you more than enough about the impact of AC: but there's also a famous book by H. Rubin and J.E. Rubin, *Equivalents of the Axiom of Choice* (North-Holland 1963; 2nd edn. 1985) which gives over two hundred equivalents of AC! Then next there is the nice short classic

3. Thomas Jech, *The Axiom of Choice** (North-Holland 1973, Dover Publications 2008). This proves the Gödel and Cohen consistency and independence results about AC (without bringing into play everything needed to prove the parallel results about the Continuum Hypothesis). In particular,

there is a nice presentation of the so-called Fraenkel-Mostowski method of using 'permutation models'. Then later parts of the book tell us something about what mathematics without choice, and about alternative axioms that are inconsistent with choice.

And for a more recent short book, taking you into new territories (e.g. making links with category theory), enthusiasts might enjoy

4. John L. Bell, *The Axiom of Choice** (College Publications, 2009).

7.3 Other set theories?

From earlier reading you should have picked up the idea that, although ZFC is the canonical modern set theory, there are other theories on the market. I mention just a selection here (I'm not suggesting you follow up all these – the point is to stress that set theory is not quite the monolithic edifice that some presentations might suggest).

For a brisk overview, putting many of the various set theories we'll consider below into some sort of order (and mentioning yet further alternatives) see

1. M. Randall Holmes, 'Alternative axiomatic set theories', *The Stanford Encyclopedia of Philosophy.*

At this stage, you'll probably find this *too* brisk and allusive, but it is useful to give you a preliminary sense of the range of possibilities here.

NBG You will have come across mention of this already (e.g. even in the early pages of Enderton's set theory book). And in fact – in many of the respects that matter – it isn't really an 'alternative' set theory. So let's get it out of the way first. We know that the universe of sets in ZFC is not itself a set. But we might think that this universe is a *sort* of big collection. Should we explicitly recognize, then, two sorts of collection, sets and (as they are called in the trade) proper classes which are too big to be sets? NBG (named for von Neumann, Bernays, Gödel: some say VBG) is one such theory of collections. So NBG in *some* sense recognizes proper classes, objects having members but that cannot be members of other entities. NBG's principle of class comprehension is predicative; i.e. quantified variables in the defining formula can't range over proper classes but range only over sets, and we get a conservative extension of ZFC (nothing in the language of sets can be proved in NBG which can't already be proved in ZFC). See

1. Michael Potter, *Set Theory and Its Philosophy* (OUP 2004) Appendix C, for more on NBG and on other theories with classes as well as sets.

2. The Wikipedia article 'Von Neumann–Bernays–Gödel set theory is useful.

3. Elliott Mendelson, *Introduction to Mathematical Logic* (CRC, 4th edition 1997), Ch.4. is a classic and influential textbook presentation of set-theory via NBG.

SP This again is by way of reminder. Recall, earlier in the Guide, we very warmly recommended Michael Potter's book which we just mentioned again. This presents a version of an axiomatization of set theory due to Dana Scott (hence 'Scott-Potter set theory'). This axiomatization is consciously guided by the conception of the set theoretic universe as built up in levels (the conception that, supposedly, also warrants the axioms of ZF). What Potter's book aims to reveal is that we can get a rich hierarchy of sets, more than enough for mathematical purposes, without committing ourselves to *all* of ZFC (whose extreme richness comes from the full Axiom of Replacement). If you haven't read Potter's book before, now is the time to look at it.

ZFA (i.e. ZF − AF + AFA) Here again is the now-familiar hierarchical conception of the set universe: We start with some non-sets (maybe zero of them in the case of pure set theory). We collect them into sets (as many different ways as we can). Now we collect what we've already formed into sets (as many as we can). Keep on going, as far as we can. On this 'bottom-up' picture, the Axiom of Foundation is compelling (any downward chain linked by set-membership will bottom out, and won't go round in a circle). But now here's another alternative conception of the set universe. Think of a set as a gadget that points you at some some things, its members. And those members, if sets, point to *their* members. And so on and so forth. On this 'top-down' picture, the Axiom of Foundation is not so compelling. As we follow the pointers, can't we for example come back to where we started? It is well known that in much of the usual development of ZFC the Axiom of Foundation AF does little work. So what about considering a theory of sets which drops AF and instead has an Anti-Foundation Axiom (AFA), which allows self-membered sets? To explore this idea,

1. Start with Lawrence S. Moss, 'Non-wellfounded set theory', *The Stanford Encycl. of Philosophy.*

2. Keith Devlin, *The Joy of Sets* (Springer, 2nd edn. 1993), Ch. 7. The last

chapter of Devlin's book, added in the second edition of his book, starts with a very lucid introduction, and develops some of the theory.

3. Peter Aczel's, *Non-well-founded sets*, (CSLI Lecture Notes 1988) is a very readable short classic book.

4. Luca Incurvati, 'The graph conception of set' *Journal of Philosophical Logic* (2014) pp. 181-208, very illuminatingly explores the motivation for such set theories.

NF Now for a much more radical departure from ZF. Standard set theory lacks a universal set because, together with other standard assumptions, the idea that there is a set of all sets leads to contradiction. But by tinkering with those other assumptions, there are coherent theories with universal sets. For very readable presentations concentrating on Quine's NF ('New Foundations'), and explaining motivations as well as technical details, see

1. T. F. Forster, *Set Theory with a Universal Set* Oxford Logic Guides 31 (Clarendon Press, 2nd edn. 1995). A classic: very worth reading even if you are a committed ZF-iste.

2. M. Randall Holmes, *Elementary Set Theory with a Universal Set*** (Cahiers du Centre de Logique No. 10, Louvain, 1998). Now freely available here.

ETCS Famously, Zermelo constructed his theory of sets by gathering together some principles of set-theoretic reasoning that seemed actually to be used by working mathematicians (engaged in e.g. the rigorization of analysis or the development of point set topology), hoping to get a theory strong enough for mathematical use while weak enough to avoid paradox. But does he overshoot? We've already noted that SP is a weaker theory which may suffice. For a more radical approach, see

1. Tom Leinster, 'Rethinking set theory', gives an advertising pitch for the merits of Lawvere's Elementary Theory of the Category of Sets, and ...

2. F. William Lawvere and Robert Rosebrugh, *Sets for Mathematicians* (CUP 2003) gives a very accessible presentation which in principle doesn't require that you have already done any category theory.

But to *fully* appreciate what's going on, you will have to go on to start engaging with some more category theory.

IZF, CZF ZF/ZFC has a classical logic: what if we change the logic to intuitionistic logic? what if we have more general constructivist scruples? The place to start exploring is

1. Laura Crosilla, 'Set Theory: Constructive and Intuitionistic ZF', *The Stanford Encyclopedia of Philosophy*.

Then for one interesting possibility, look at the version of constructive ZF in

2. Peter Aczel and Michael Rathjen, *Constructive Set Theory* (Draft, 2010).

IST Leibniz and Newton invented infinitesimal calculus in the 1660s: a century and a half later we learnt how to rigorize the calculus without invoking infinitely small quantities. Still, the idea of infinitesimals retains a certain intuitive appeal, and in the 1960s, Abraham Robinson created a theory of hyperreal numbers: this yields a rigorous formal treatment of infinitesimal calculus (you will have seen this mentioned in e.g. Enderton's *Mathematical Introduction to Logic*, §2.8, or van Dalen's *Logic and Structure*, p. 123). Later, a simpler and arguably more natural approach, based on so-called Internal Set Theory, was invented by Edward Nelson. As put it, 'IST is an extension of Zermelo-Fraenkel set theory in that alongside the basic binary membership relation, it introduces a new unary predicate 'standard' which can be applied to elements of the mathematical universe together with some axioms for reasoning with this new predicate.' Starting in this way we can recover features of Robinson's theory in a simpler framework.

1. Edward Nelson, 'Internal set theory: a new approach to nonstandard analysis' *Bulletin of The American Mathematical Society* 83 (1977), pp. 1165–1198.

2. Nader Vakin, *Real Analysis through Modern Infinitesimals* (CUP, 2011). A monograph developing Nelson's ideas whose early chapters are quite approachable and may well appeal to some.

Yet more? Well yes, we *can* keep on going. Take a look, for example, at SEAR. But we must call a halt! Though you could round things out by taking a look at a piece that could be thought of as an expanded version of Randall Holmes's *Stanford Encyclopedia* piece that we mentioned at the the beginning of this section:

2. M. Randall Holmes, Thomas Forster and Thierry Libert. 'Alternative Set Theories'. In Dov Gabbay, Akihiro Kanamori, and John Woods, eds. *Handbook of the History of Logic, vol. 6, Sets and Extensions in the Twentieth Century*, pp. 559-632. (Elsevier/North-Holland 2012).

Chapter 8

What else?

Mathematical logicians and philosophers interested in the philosophy of maths will want to know about yet more areas that fall outside the traditional math logic curriculum. For example:

- With roots going right back to *Principia Mathematica*, there's the topic of *modern type theories*.
- Relatedly, we should explore *the lambda calculus*.
- Again relatedly, of central computer-science interest, there's the business of *logic programming* and *formal proof verification*.
- Even in elementary model theory we relax the notion of a language to allow e.g. for uncountably many names: what if we further relax and allow for e.g. sentences which are infinite conjunctions? Pursuing such questions leads us to consider *infinitary logics*.
- Going in the opposite direction, as it were, an intuitionist worries about applying classical reasoning to infinite domains, opening up the whole topic of constructive reasoning and *constructive mathematics*.

Any of these areas may or may not get a section in a later version of this Guide. But there is one more topic which *does* get its own supplementary webpage, including a reading list for philosophers, and links to a lot of freely available material, including my own notes:

- If set theory traditionally counts as part of mathematical logic, because of its generality, breadth and foundational interest, then there is an argument for including some *category theory* too.

Meanwhile, those who feel that – even with the Big Books mentioned in the Book Notes – they haven't got enough to read or think that the mainstream diet so far is a bit restricted can always consult the long, wide-ranging, articles in the seventeen(!) volume *Handbook of Philosophical Logic*. This collection, edited by the indefatigable Dov Gabbay together with F. Guenthner, is somewhat mistitled: a lot of the articles are straight mathematics which might be of interest to technically minded philosophers but could be of at least equal interest to mathematicians too. Or you can try some of the articles in the fascinating ten-volumes-and-counting of the *Handbook of the History of Logic* (edited by Dov Gabbay together with John Woods, with contents listed here), and/or dip into the five volumes of the *Handbook of Logic in Computer Science* (edited by S. Abramsky, Dov Gabbay – again! – and T. S. E. Maibaum).

But enough already!

Index of authors

Teach Yourself Logic: Appendix

Some Big Books on Mathematical Logic

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December 14, 2015

Pass it on, ... That's the game I want you to learn. Pass it on.

Alan Bennett, *The History Boys*

The latest versions of the main Guide and this Appendix can always be downloaded from logicmatters.net/students/tyl/

URLs in blue are live links to web-pages or PDF documents. Internal cross-references to sections etc. are also live. But to avoid the text being covered in ugly red rashes these internal links are *not* coloured – if you are reading onscreen, your cursor should change as you hover over such live links: *try here*.

The Guide's layout is optimized for reading on-screen: try, for example, reading in iBooks on an iPad, or in double-page mode in Adobe Reader on a laptop.

Contents

Introduction

The traditional menu for a first serious Mathematical Logic course is basic first order logic with some model theory, the basic theory of computability and related matters (like Gödel's incompleteness theorem), and introductory set theory. The Teach Yourself Logic Study Guide makes some topic-by-topic recommendations for reading yourself into some back knowledge. The plan in this Appendix is to look at some of the Big (or sometimes not so big) Books that aim to cover an amount of this core menu, to give an indication of what they cover and – more importantly – how they cover it, while commenting on style, accessibility, etc.

These wider-ranging books don't always provide the best introductions now available to this or that particular area: but they can still be very useful aids to widening and deepening your understanding and can reveal how topics from different areas fit together. I don't promise to discuss [eventually] every worthwhile Big Book, or to give a similar level of coverage to those I do consider. But I'm working on the principle that even a patchy guide, very much work in progress, might still be useful.

Sections start with a general indication of the coverage of the book, and some general remarks. Then there is usually, in small print, a more detailed description of contents, and more specific comments. Finally, there's a summary verdict. The books are listed in chronological order of first publication.

1 Kleene, 1952

First published sixty years ago, Stephen Cole Kleene's *Introduction to Metamathematics** (North-Holland, 1962; reprinted Ishi Press 2009: pp. 550) for a while held the field as a survey treatment of first-order logic (without going much past the completeness theorem), and a more in-depth treatment of the theory of computable functions, and Gödel's incompleteness theorems.

In a 1991 note about writing the book, Kleene notes that up to 1985, about 17,500 copies of the English version of his text were sold, as were thousands of various translations (including a sold-out first print run of 8000 of the Russian translation). So this is a book with a quite pivotal influence on the education of later logicians, and on their understanding of the fundamentals of recursive function theory and the incompleteness theorems in particular.

But it isn't just nostalgia that makes old hands continue to recommend it. Kleene's book remains particularly lucid and accessible: it is often discursive, pausing to explain the motivation behind formal ideas. It is still a pleasure to read (or at least, it ought to be a pleasure for anyone interested in logic enough to be reading this Appendix to the Guide!). And, modulo relatively superficial presentational matters, you'll probably be struck by a sense of familiarity when reading through, as aspects of his discussions evidently shape many later textbooks (not least my own Gödel book). The *Introduction to Metamathematics* remains a really impressive achievement: and not one to be admired only from afar, either.

Some details Chs. 1–3 are introductory. There's a little about enumerability and countability (Cantor's Theorem); then a chapter on natural numbers, induction, and the axiomatic method; then a little tour of the paradoxes, and possible responses.

Chs. 4–7 are a gentle introduction to the propositional and predicate calculus and a formal system which is in fact first-order Peano Arithmetic (you need to be aware that the identity rules are treated as part of the arithmetic, not the logic). Although Kleene's official system is Hilbert-style, he shows that 'natural deduction' introduction and elimination rules can be thought of as derived rules in his system, so it all quickly becomes quite user-friendly. (He doesn't at this point prove the completeness theorem for his predicate logic: as I said, things go quite gently at the outset!)

Ch. 8 starts work on 'Formal number theory', showing that his formal arithmetic has nice properties, and then defines what it is for a formal predicate to capture ('numeralwise represent') a numerical relation. Kleene then proves Gödel's incompleteness theorem, assuming a Lemma – eventually to be proved in his Chapter 10 – about the capturability of the relation 'm numbers a proof [in Kleene's system] of the sentence with number n'.

Ch. 9 gives an extended treatment of primitive recursive functions, and then Ch. 10 deals with the arithmetization of syntax, yielding the Lemma needed for the incompleteness theorem.

Chs. 11-13 then give a nice treatment of general (total) recursive functions, partial recursive functions, and Turing computability. This is all very attractively done.

The last two chapters, forming the last quarter of the book, go under the heading 'Additional Topics'. In Ch. 14, after proving the completeness theorem for the predicate calculus without and then with identity, Kleene discusses the decision problem. And the final Ch. 15 discusses Gentzen systems, the normal form theorem, intuitionistic systems and Gentzen's consistency proof for arithmetic.

Summary verdict Can still be warmly recommended as an enjoyable and illuminating presentation of this fundamental material, written by someone who was himself so closely engaged in the early developments back in the glory days. It should be entirely accessible if you have managed e.g. Chiswell and Hodges and then my Gödel book, and will enrich and broaden your understanding.

2 Mendelson, 1964, 2009

Elliot Mendelson's *Introduction to Mathematical Logic* (Van Nostrand, 1964: pp. 300) was first published in the distinguished and influential company of The University Series in Undergraduate Mathematics. It has been much used in graduate courses for philosophers since: a 4th edition was published by Chapman Hall in 1997 (pp. 440), with a slightly expanded 5th edition being published in 2009. I will here compare the first and fourth editions, as these are the ones I know.

Even in the later editions this isn't, in fact, a very *big* Big Book (many of the added pages of the later editions are due to there now being answers to exercises): the length is kept under control in part by not covering a great deal, and in part by a certain brisk terseness. As the Series title suggests, the intended level of the book is upper undergraduate mathematics, and the book does broadly keep to that aim. Mendelson is indeed pretty clear; however, his style is of the times, and will strike many modern readers as dry and rather old-fashioned. (Some of the choices of typography are not wonderfully pretty either, and this can make some pages look as if they will be harder going than they really turn out to be.)

Some details After a brief introduction, Ch. 1 is on the propositional calculus. It covers semantics first (truth-tables, tautologies, adequate sets of connectives), then an axiomatic proof system. The treatments don't change much between editions, and will probably only be of interest if you've never encountered a Hilbert-style axiomatic system before. The fine print of how Mendelson regards his symbolic apparatus is interesting: if you read him carefully, you'll see that the expressions in his formal systems are not sentences, not expressions of the kind that – on interpretation – can be true or false – but are *schemata*, what he calls statement forms. But this relatively idiosyncratic line about how the formalism is to be read, which for a while (due to Quine's influence) was oddly popular among philosophers, doesn't much affect the development.

Ch. 2 is on quantification theory, again in an axiomatic style. The fourth edition adds to the end of the chapter more sections on model theory: there is a longish section on ultra-powers and non-standard analysis, then there's (too brief) a nod to semantic trees, and finally a new discussion of quantification allowing empty domains. The extra sections in the fourth edition are a definite bonus: without them, there is nothing special to recommend this chapter, if you have worked through the suggestions in §2.2 of the Guide, and in particular, perhaps, the chapters in van Dalen's book.

Ch. 3 is titled 'Formal number theory'. It presents a formal version of first-order Peano Arithmetic, and shows you can prove some expected arithmetic theorems within it. Then Mendelson defines the primitive recursive and the (total) recursive functions, shows that these are representable (capturable) in PA.

It then considers the arithmetization of syntax, and proves Gödel's first incompleteness theorem and Rosser's improvement. The chapter then proves Church's Theorem about the decidability of arithmetic. One difference between editions is that the later proof of Gödel's theorem goes via the Diagonalization Lemma; another is that there is added a brief treatment of Löb's Theorem. At the time of publication of the original addition, this Chapter was a quite exceptionally useful guide thorough the material. But now – at least if you've read my Gödel book or the equivalent – then there is nothing to divert you here, except that Mendelson does go through *every* single stage of laboriously showing that the relation *m-numbers-a-PA-proof-of-the-sentence-numbered-n* is primitive recursive.

Ch. 4 is on set theory, and – unusually for a textbook – the system presented is NBG (von Neumann/Bernays/Gödel) rather than ZF(C). In the first edition, this chapter is under fifty pages, and evidently the coverage can't be very extensive and it also probably goes too rapidly for many readers. The revised edition doesn't change the basic treatment (much) but adds sections comparing NBG to a number of other set theories. So while this chapter certainly can't replace the introductions to set theory recommended in §**??**, it could be worth skimming briskly through the chapter in later editions to learn about NBG and other deviations from ZF.

The original Ch. 5 on effective computability starts with a discussion of Markov algorithms (again, unusual for a textbook), then treats Turing algorithms, then Herbrand-Gödel computability and proves the equivalence of the three approaches. There are discussions of recursive enumerability and of the Kleene-Mostowski hierarchy. And the chapter concludes with a short discussion of undecidable problems. In the later edition, the material is significantly rearranged, with Turing taking pride of place and other treatments of computability relegated to near the end of the chapter; also more is added on decision problems. Since the introductory texts mentioned in the Guide don't talk about Markov or Herbrand-Gödel computability, you might want to dip into the chapter briefly to round out your education!

I should mention the appendices. The first edition has a very interesting though brisk appendix giving a version of Schütte's variation on a Gentzen-style consistency proof for PA. Rather sadly, perhaps, this is missing from later editions. The fourth edition has instead an appendix on second-order logic.

Summary verdict Moderately accessible and very important in its time, but there is now not so much reason to plough through this book end-to-end. It doesn't have the charm and readability of Kleene 1952, and there are now better separate introductions to each of the main topics . You could *skim* the early chapters if you've never seen axiomatic systems of logic being used in earnest: it's good for the soul. The appendix that appears only in the first edition is interesting for enthusiasts. Look at the section on non-standard analysis in the revised editions. If set theory is your thing, you should dip in to get the headline news about NBG. And some might want to expand their knowledge of definitions of computation by looking at Ch. 5.

3 Shoenfield, 1967

Joseph R. Shoenfield's *Mathematical Logic* (Addison-Wesley, 1967: pp. 334) is officially intended as 'a text for a first-year [maths] graduate course'. It has, over the years, been much recommended and much used (a lot of older logicians first learnt their serious logic from it).

This book, however, is hard going – a significant step or two up in level from Mendelson – though the added difficulty in mode of presentation seems to me not always to be necessary. I recall it as being daunting when I first encountered it as a student. Looking back at the book after a very long time, and with the benefit of greater knowledge, I have to say I am not any more enticed: it is still a tough read.

So this book can, I think, only be recommended to hard-core mathmos who already know a fair amount and can cherry-pick their way through the book. It does have heaps of hard exercises, and some interesting technical results are in fact buried there. But whatever the virtues of the book, they don't include approachability or elegance or particular student-friendliness.

Some details Chs. 1–4 cover first order logic, including the completeness theorem. It has to be said that the logical system chosen is rebarbative. The primitives are $\neg$, $\vee$, $\exists$, and $=$. Leaving aside the identity axioms, the axioms are the instances of excluded middle, instances of $\varphi(\tau) \to \exists\xi\varphi(\xi)$, and then there are five rules of inference. So this neither has the cleanness of a Hilbert system nor the naturalness of a natural deduction system. Nothing is said to motivate this seemingly horrible choice as against others.

Ch. 5 is a brisk introduction to some model theory getting as far as the Ryll-Nardjewski theorem. I believe that the algebraic criteria for a first-order theory to admit elimination of quantifiers given here are original to Shoenfield. But this is surely all done very rapidly (unless you are using it as a terse revision course from quite an advanced base, going beyond what you will have picked up from the reading suggested in our §**??** above).

Chs. 6–8 cover the theory of recursive functions and formal arithmetic. The take-it-or-leave-it style of presentation continues. Shoenfield defines the recursive functions as those got from an initial class by composition and regular minimization: again, no real motivation for the choice of definition is given (and e.g. the definition of the primitive recursive functions is relegated to the exercises). Unusually for a treatment at this sort of level, the discussion of recursion theory in Ch. 8 goes far enough to cover a Gödelian 'Dialectica'-style proof of the consistency of arithmetic, though the presentation once more wins no prizes for accessibility.

Ch. 9 on set theory is perhaps the book's real original *raison d'être*; in fact, it is a quarter of the whole text. The discussion starts by briskly motivating the ZF axioms by appeal to the conception of the set universe as built in stages (an approach that has become very common but at the time of publication was I think much less usually articulated); but this isn't the place to look for an in depth development of that idea. For a start, there is Shoenfield's own article 'The axioms of set theory', *Handbook of mathematical logic*, ed. J. Barwise, (North-Holland, 1977) pp. 321–344.

We get a brusque development of the elements of set theory inside ZF (and then ZFC), and something about the constructible universe. Then there is the first extended textbook presentation of Cohen's 1963 independence results via forcing, published just four years previous to the publication of this book: set theory enthusiasts might want to look at this to help round out their understanding of the forcing idea. The discussion also touches on large cardinals.

This last chapter was in some respects a highly admirable achievement in its time: but it is equally surely not *now* the best place to start with set theory in general or forcing in particular, given the availability of later presentations.

Summary verdict This is pretty tough going. Now surely only for *very* selective dipping into by already-well-informed enthusiasts.

4 Kleene, 1967

In the preface to his *Mathematical Logic** (John Wiley 1967, Dover reprint 2002: pp. 398), Stephen Cole Kleene writes

> After the appearance in 1952 of my *Introduction to Metamathematics*, written for students at the first-year graduate level, I had no expectation of writing another text. But various occasions arose which required me to think about how to present parts of the same material more briefly, to a more general audience, or to students at an earlier educational level. These newer expositions were received well enough that I was persuaded to prepare the present book for undergraduate students in the Junior year.

You'd expect, therefore, that this later book would be more accessible, a friendlier read, than Kleene's remarkable *IM*. But in fact, this doesn't actually strike me as the case. I'd still recommend reading the older book, augmented by one chapter of this later 'Little Kleene'. To explain:

Some details The book divides into two parts. The first part, 'Elementary Mathematical Logic' has three chapters. Ch. 1 is on the propositional calculus (including a Kalmár-style completeness proof). This presents a Hilbert-style proof system with an overlay of derived rules which look rather natural-deduction-like (but aren't the real deal) There is a lot of fussing over details in rather heavy-handed ways. I couldn't recommend anyone nowadays *starting* here, while if you've already read a decent treatment of the propositional calculus (and e.g. looked at Mendelson to see how things work in a Hilbert-style framework) you won't get much more out of this.

Much the same goes for the next two chapters. Ch. 2 gives an axiomatic version of the predicate calculus without identity, and Ch. 3 adds identity. (Note, a completeness proof doesn't come to the final chapter of the book). Again, these chapters are not done with a sufficiently light touch to make them a particularly attractive read now.

The second part of the book is titled 'Mathematical Logic and the Foundations of Mathematics'. Ch.4 is basically an abridged version of the opening three chapters of *IM*, covering the paradoxes, the idea of an axiomatic system, introducing formal number theory. You might like to read in particular §§36–37 on Hilbert vs. Brouwer and 'metamathematics'.

Ch. 5 is a sixty page chapter on 'Computability and Decidability'. Kleene is now on his home ground, and he presents the material (some original to him) in an attractive and illuminating way, criss-crossing over some of the same paths trodden in later chapters of *IM.* In particular, he uses arguments for incompleteness and undecidability turning on use of the Kleene T-predicate (compare §33.7 of *IGT1* or §43.8 of *IGT2*). This chapter is certainly worth exploring.

Finally, the long Ch. 6 proves the completeness theorem for predicate logic by Beth/Hintikka rather than by Henkin (as we would now think of it, he in effect shows the completeness of a tree system for logic in the natural way). But nicer versions of this approach are available. The last few sections cover some supplementary material (on Gentzen systems, Herbrand's Theorem, etc.) but again I think all of it is available more accessibly elsewhere

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Summary verdict Do read Chapter 5 on computability, incompleteness, decidability and closely related topics. This is nicely done, complements Kleene's earlier treatment of the same material, and takes an approach which is interestingly different from what you will mostly see elsewhere.

5 Robbin, 1969

Joel W. Robbin's *Mathematical Logic: A First Course** (W. A Benjamin, 1969, Dover reprint 2006: pp. 212) is not exactly a 'Big Book'. The main text is just 170 pages long. But it does range over both formal logic (first-order and second-order), and formal arithmetic, primitive recursive functions, and Gödelian incompleteness. Robbin, as you might guess, has to be quite brisk (in part he achieves brevity by leaving a lot of significant results to be proved as more or less challenging exercises). However, the book remains approachable and has some nice and unusual features for which it can be recommended.

Some details Ch. 1 is on the propositional calculus. Robbin presents an axiomatic system whose primitives are $\to$ and $\perp$ – or rather, in his notation, $\supset$ and $\mathfrak{f}$. The system, including the 'dotty' syntax which gives us wffs like $p_1 \supset p_2 \mathbin{\centerdot\!\supset\!\centerdot} p_1 \supset \mathfrak{f}$, is a version of Alzono Church's system in his *Introduc-*

tion to Mathmatical Logic, Vol. 1 (1944/1956), except that where Church lays down three specific wffs as axioms and has a substitution rule for deriving variant wffs, Robbin lays down three axiom schemas. [Perhaps I should say something about Church's classic book in this Appendix: but that's for another day.]

As in later chapters, Robbin buries some interesting results in the extensive exercises. Here's one, pointed out to me by David Auerbach. Robbin defines negation in the obvious way from his two logical primitives, so that $\sim \varphi =_{\text{def}} (\varphi \rightarrow \mathfrak{f})$. And then his three axiom schemas can all be stated in terms of $\supset$ and $\sim$, and his one rule is modus ponens. This system is complete. However, if we take the alternative language with $\supset$ and $\sim$ *primitive*, then the same deductive system (with the same axioms and rules) is *not* complete. That's a nice little surprise, and it is worth trying to work out just why it is true.

Ch. 2 briefly covers first-order logic, including the completeness theorem. Then Ch. 3 introduces what Robbin calls 'First-order (Primitive) Recursive Arithmetic' (RA). Robbin defines the primitive recursive functions, and then defines a language which has a function expression for each p.r. function f (the idea is to have a complex function expression built up to reflect a full definition of f by primitive recursion and/or composition ultimately in terms of the initial functions). RA has axioms for the logic plus axioms governing the expressions for the initial functions, and then there are axioms for dealing with complex functional expressions in terms of their constituents. RA also has all instances of the induction schema for open wffs of the language (so – for cognoscenti – this is a stronger theory than what is usually called Primitive Recursive Arithmetic these days, which normally has induction only for quantifier-free wffs).

Ch. 4 explores the arithmetization of syntax of RA. Since RA has every p.r. function built in, we don't then have to go through the palaver of showing that we are dealing with a theory which can represent all p.r. functions (in the way we have to if we take standard PA as our base theory of interest). So in Ch. 5 Robbin can prove Gödel's incompleteness theorem for RA in a more pain-free way.

Ch. 6 then turns to second-order logic, introduces a version of second-order PA_2 with just the successor relation as primitive non-logical vocabulary. Robbin shows that all the p.r. functions can be explicitly defined in PA_2, so the incompleteness theorem carries over.

Summary verdict Robbin's book offers a different route through a rather different selection of material than is usual, accessibly written and still worth reading (you will be able to go though quite a bit of it pretty rapidly if you are up to speed with the relevant basics. Look especially at Robbin's Ch. 3 for the unusually detailed story about how to build a language with a function expression for every p.r. function, and the last chapter for how in effect to do the same in PA_2.

6 Enderton, 1972, 2002

The first edition of Herbert B. Enderton's *A Mathematical Introduction to Logic* (Academic Press, 1972: pp. 295) rapidly established itself as much-used textbook among the mathematicians it was aimed towards. But it has also been used too in math. logic courses offered to philosopers. A second edition was published in 2002, and a glance at the section headings indicates much the same overall structure: but there are many local changes and improvements, and I'll comment here on this later version of the book (which by now should be equally widely available in libraries). The author died in 2010, but his webpages live on, including one with his own comments on his second edition: http://www.math.ucla.edu/~hbe/.

Enderton's text deals with first order-logic and a smidgin of model theory, followed by a look at formal arithmetic, recursive functions and incompleteness. A final chapter covers second-order logic and some other matters.

A Mathematical Introduction to Logic eventually became part of a logical trilogy, with the publication of the wonderfully lucid *Elements of Set Theory* (1977) and *Computability Theory* (2010). The later two volumes strike me as masterpieces of exposition, providing splendid 'entry level' treatments of their material. The first volume, by contrast, is *not* the most approachable first pass through its material. It is good (often *very* good), but I'd say at a notch up in difficulty from what you might be looking for in an *introduction* to the serious study of first-order logic and/or incompleteness.

Some details After a brisk Ch. 0 ('Some useful facts about sets', for future reference), Enderton starts with a 55 page Ch. 1, 'Sentential Logic'. Some might think this chapter to be slightly odd. For the usual motivation for separating off propositional logic and giving it an extended treatment at the beginning of a book at this level is that this enables us to introduce and contrast the key ideas of semantic entailment and of provability in a formal deductive system, and then explain strategies for soundness and completeness proofs, all in a helpfully simple and uncluttered initial framework. But (except for some indications in final exercises) there is no formal proof system mentioned in Enderton's chapter.

So what does happen in this chapter? Well, we do get a proof of the expressive completeness of $\{\wedge, \vee, \neg\}$, etc. We also get an exploration (which can be postponed) of the idea of proofs by induction and the Recursion Theorem, and based on these we get proper proofs of unique readability and the uniqueness of the extension of a valuation of atoms to a valuation of a set of sentences containing them (perhaps not the most inviting things for a beginner to be pausing long over). We get a direct proof of compactness. And we get a first look at the ideas of effectiveness and computability.

The core Ch. 2, 'First-Order Logic', is over a hundred pages long, and covers a good deal. It starts with an account of first-order languages, and then there is a lengthy treatment of the idea of truth in a structure. This is pretty clearly done and mathematicians should be able to cope quite well (but does Enderton forget his officially intended audience on p. 83 where he throws in an unexplained commutative diagram?). Still, readers might sometimes appreciate rather more explanation (for example, surely it would be worth saying a bit more than that 'In order to define 'σ is true in $\mathfrak{A}$' for sentences σ and structures $\mathfrak{A}$, we will find it desirable [sic] first to define a more general concept involving wffs', i.e. satisfaction by sequences).

Enderton then at last introduces a deductive proof system (110 pages into the book). He chooses a Hilbert-style presentation, and if you are not already used to such a system, you won't get much of a feel for how they work, as there are very few examples before the discussion turns to metatheory (even Mendelson's presentation of a similar Hilbert system is here more helpful). Then, as you'd expect, we get the soundness and completeness theorems. The proof of the latter by Henkin's method *is* nicely chunked up into clearly marked stages, and again a serious mathematics student should cope well: but this is still not, I think, a 'best buy' among initial presentations.

The chapter ends with a little model theory – compactness, the LS theorems, interpretations between theorems – all rather briskly done, and there is an application to the construction of infinitesimals in non-standard analysis which is surely going to be *too* compressed for a first encounter with the ideas.

Ch. 3, 'Undecidability', is also a hundred pages long and again covers a great deal. After a preview introducing three somewhat different routes to (versions of) Gödel's incompleteness theorem, we initially meet:

1. A theory of natural numbers with just the successor function built in (which is shown to be complete and decidable, and a decision procedure by elimination of quantifiers is given).
2. A theory with successor and the order relation (also shown to admit elimination of quantifiers and to be complete).
3. Presburger arithmetic (shown to be decidable by a quantifier elimination procedure, and shown not to define multiplication)
4. Robinson Arithmetic with exponentiation.

The discussion then turns to the notions of definability and representability. We are taken through a long catalogue of functions and relations representable in Robinson-Arithmetic-with-exponentiation, including functions for encoding and decoding sequences. Next up, we get the arithmetization of syntax done at length, leading as you'd expect to the incompleteness and undecidability results.

But we aren't done with this chapter yet. We get (sub)sections on recursive enumerability, the arithmetic hierarchy, partial recursive functions, register machines, the second incompleteness theorem for Peano Arithmetic, applications to set theory, and finally we learn how to use the β-function trick so we can get take our results to apply to any nicely axoimatized theory containing plain Robinson Arithmetic.

As is revealed by that quick description there really is a *lot* in Ch. 3. To be sure, the material here is not mathematically difficult in itself (indeed it is one of the delights of this area that the initial Big Results come so quickly). However, I do doubt that such an action-packed presentation is the best way to first meet this material. It would, however, make for splendid revision-consolidation-extension reading after tackling e.g. my Gödel book.

The final Ch. 4 is much shorter, on 'Second-Order Logic'. This goes *very* briskly at the outset. It again wouldn't be my recommended introduction for this material, though it could make useful supplementary reading for those wanting to get clear about the relation between second-order logic, Henkin semantics, and many-sorted first-order logic.

Summary verdict To repeat, *A Mathematical Introduction to Logic* is good in many ways, but is – in my view – often a step or two more difficult in mode of presentation than will suit many readers wanting an introduction to the material it covers. However, if you have already read an entry-level presentation of first order logic (e.g. Chiswell/Hodges) then you could read Chs 1 and 2 as revision/consolidation. And if you have already read an entry-level presentation on incompleteness (e.g. my book) then it could be well worth reading Ch. 3 as bringing the material together in a somewhat different way.

7 Ebbinghaus, Flum and Thomas, 1978, 1994

We now turn to consider H.-D. Ebbinghaus, J. Flum and W. Thomas, *Mathematical Logic* (Springer, 2nd end. 1994: pp. 289). This is the English translation of a book first published in German in 1978, and appears in a series 'Undergraduate Texts in Mathematics', which indicates the *intended* level: parts of the book, however, do seem more than a little ambitious for most undergraduates.

EFT's book is often warmly praised and is (I believe) quite widely used. But revisiting it, I can't find myself wanting to recommend it as a good place to start, certainly not for philosophers but I'm not sure I would recommend it for mathematicians either. The presentation of the core material on the syntax and semantics of first-order logic in the first half of the book is done more accessibly and more elegantly elsewhere. In the second half of the book, the chapters do range widely across interesting material. But again most of the discussions will go too quickly if you haven't encountered the topics before, and – if you want revision/amplification of what you already know – you will mostly do better elsewhere.

Some details The book is divided into two parts. EFT start Part A with a gentle opening chapter talking about a couple of informal mathematical theories (group theory, the theory of equivalence relations), giving a couple of simple informal proofs in those theories. They then stand back to think about what goes on in the proofs, and introduce the project of formalization. So far, so good.

Ch. 2 describes the syntax of first-order languages, and Ch. 3 does the semantics. The presentation goes at a fairly gentle pace, with some useful asides (e.g. on handling the many-sorted languages of informal mathematics using a many-sorted calculus vs. use restricted quantifiers in a single-sorted calculus). EFT though do make quite heavy work of some points of detail.

Ch. 4 is called 'A Sequent Calculus'. The version chosen strikes me as really not very nice. For a start (albeit a minor point that only affects readability), instead of writing a sequent as '$\Gamma \vdash \varphi$', or '$\Gamma \Rightarrow \varphi$', or even '$\Gamma\colon \varphi$', EFT just write an unpunctuated '$\Gamma\ \varphi$'. Much more seriously, they adopt a system of rules which many would say runs together a classical logical rule for the connectives with a structural rule in an unprincipled way that hides e.g. the relation between classical and intuitionistic logic. [To be more specific, they introduce a classical 'Proof by Cases' rule that takes us from the sequents (in their notation) $\Gamma\ \psi\ \varphi$ and $\Gamma\ \neg\psi\ \varphi$ to $\Gamma\ \varphi$, and then then show that this almost immediate yields Cut as a derived rule. This seems to me to muddy waters that usual presentations of the sequent calculus strive to keep clear.]

Ch. 5 gives a Henkin completeness proof for first-order logic. For my money, there's too much symbol-bashing and not enough motivating chat here. (I don't think it is good exegetical policy to complicate matters as EFT do by going straight for a proof for the predicate calculus with identity: but they are not alone in this.)

Ch. 6 is briskly about The Löwenheim-Skolem Theorem, compactness, and elementarily equivalent structures (but probably fine if you've met this stuff before).

Ch. 7, 'The Scope of First-Order Logic' is really rather odd. It briskly argues that first-order logic is the logic for mathematics (readers of Shapiro's book on second-order logic won't be so quickly convinced!). The reason given is that we can reconstruct (nearly?) all mathematics in first-order ZF set theory – which the authors then proceed to give the axioms for. These few pages surely wouldn't help if you have never seen the axioms before and don't already know about the project of doing-maths-inside-set-theory.

Finally in Part A, there's rather ill-written chapter on normal forms, on extending theories by definitions, and (badly explained) on what the authors call 'syntactic interpretations'.

Part B of the book discusses a number of rather scattered topics. It kicks off with a nice little chapter on extensions of first-order logic, more specifically on second-order logic, on $\mathcal{L}_{\omega_1\omega}$ [which allows infinitely long conjunctions and disjunctions], and $\mathcal{L}_Q$ [logic with quantifier Qx, 'there are uncountably many x such that …'].

Then Ch. 10 is on 'Limitations of the Formal Method', and in under forty pages aims to talk about register machines, the halting problem for such machines, the undecidability of first-order logic, Trahtenbrot's theorem and the incompleteness of second-order logic, Gödel's incompleteness theorems, and more. This would just be far too rushed if you'd not seen this material before, and if you have then there are plenty of better sources for revising/consolidating/extending your knowledge.

Ch. 11 is by some way the longest in the book, on 'Free Models and Logic Programming'. This is material we haven't covered in this Guide. But again it doesn't strike me as a particular attractive introduction.

Ch. 12 is back to core model theory, Fraïssé's Theorem and Ehrenfeucht Games: but (I'm sorry – this is getting repetitious!) you'll again find better treatments elsewhere, this time in books dedicated to model theory.

Finally, there is an interesting (though quite tough) concluding chapter on Lindström's Theorems which show that there is a sense in which standard first-order logic occupies a unique place among logical theories.

Summary verdict The core material in Part A of the book is covered better (more accessibly, more elegantly) elsewhere.

Of the supplementary chapters in Part B, the two chapters that stand out as worth looking at are perhaps Ch. 9 on extensions of first-order logic, and Ch. 13 (though not easy) on Lindström's Theorems.

8 Van Dalen, 1980, 2012

Dirk van Dalen's popular *Logic and Structure* (Springer: 263 pp. in the most recent edition) was first published in 1980, and has now gone through a number of editions. It is widely used and has a lot to recommend it. A very substantial chapter on incompleteness was added in the fourth edition in 2004. A fifth edition published in 2012 adds a further new section on ultraproducts. Comments here apply to these last two editions.

Some details Ch. 1 on 'Propositional Logic' gives a presentation of the usual truth-functional semantics, and then a natural deduction system (initially with primitive connectives $\rightarrow$ and $\perp$). This is overall pretty clearly done – though really rather oddly, although van Dalen uses in his illustrative examples of deductions the usual practice of labelling a discharged premiss with numbers and using a matching label to mark the inference move at which that premiss is discharged, he doesn't pause to explain the practice in the way you would expect. Van Dalen then gives a standard Henkin proof of completeness for this cut-down system, before re-introducing the other connectives into his natural deduction system in the last section of the chapter. Compared with Chiswell and Hodges, this has a somewhat less friendly, more conventional, mathematical look-and-feel: but this is still an accessible treatment, and will certainly be very readily manageable if you've read C&H first. (It should be noted that van Dalen can be surprisingly slapdash. For example, a tautology is defined is defined to be an (object-language) proposition which is always true. But then the meta-linguistic schema $\varphi \wedge \psi \leftrightarrow \psi \wedge \varphi$ is said to be a tautology. He means the instances are tautologies; compare Mendelson who really *does* think tautologies are schemata. Again, van Dalen presents his natural deduction system, says he is going to give some 'concrete cases', but then presents not arguments in the object language, but schematic templates for arguments, written out using 'φ's and 'ψ's again.)

Ch. 2 describes the syntax of a first-order language, gives a semantic story, and then presents a natural deduction system, first adding quantifier rules, then adding identity rules. Overall, this is pretty clearly done (though van Dalen reuses variables as parameters, which isn't the nicest way of setting things up). The approach to the semantics is to consider an extension of a first order language L with domain A to an augmented language L^A which has a constant $\overline{a}$ for every element $a \in A$; and then we can say $\forall x \varphi(x)$ is true if all $\varphi(\overline{a})$ are true. Fine: though it would have been good if van Dalen had paused to say a little more about the pros and cons of doing things this way rather than the more common Tarskian way that students will encounter. (Let's complain some more about van Dalen's slapdash ways. For example, he talks about '*the* language of a similarity type' in §2.3, but gives examples of different languages of the same similarity type in §2.7. He fusses unclearly about different uses of the identity sign in §2.3, before going on to make use of the symbolism ':=' in a way that isn't explained, and is different from the use made of it in the previous chapter. This sort of thing could upset the more pernickety reader.)

Ch. 3, 'Completeness and Applications', gives a pretty clear presentation of a Henkin-style completeness proof, and then the compactness and L-S theorems. The substantial third section on model theory goes rather more speedily, and you'll need some mathematical background to follow some of the illustrative examples. The final section newly added in the fifth edition on the ultraproduct construction speeds up again and is probably too quick to be useful to many.

Ch. 4 is quite short, on second-order logic. If you have already seen a presentation of the basic ideas, this quick presentation of the formalities could be helpful.

Ch. 5 is on intuitionism – this is a particular interest of van Dalen's, and his account of the BHK interpretation as motivating intuitionistic deduction rules, his initial exploration of the resulting logic, and his discussion of the Kripke semantics are quite nicely done (though again, this chapter will probably work better if you have already seen the main ideas in a more informal presentation before).

Ch. 6 is on proof theory, and in particular on the idea that natural deduction proofs (both classical and intutitionistic) can be normalized. Most readers will find a more expansive and leisurely treatment much to be preferred.

The final 50 page Ch. 7 *is* more leisurely. It starts by introducing the ideas of primitive recursive and partial recursive functions, and the idea of recursively enumerable sets, leading up to a proof that there exist effectively inseparable r.e. sets. We then turn to formal arithmetic, and prove that recursive functions are representable (because his version of PA does have the exponential built in, van Dalen doesn't need to tangle with the β-function trick.). Next we get the arithmetization of syntax and proofs that the numerical counterparts of some key syntactic properties and relations are primitive recursive. Then, as you would expect, we get the diagonalization lemma, and that is used to prove Gödel's first incompleteness theorem. We then get another proof relying on the earlier result that there are effectively inseparable r.e. sets, and going via the undecidability of arithmetic. The chapter finishes by announcing that there is a finitely axiomatized arithmetic strong enough to represent all recursive properties/relations, so the undecidability of arithmetic implies the undecidability of first-order logic. There's nothing, however, about the second theorem: so most students who get this far will want that bit more more. However, this chapter is all done pretty clearly, could probably be managed by good students as a first introduction to its topics, and would be very good revision/consolidatory reading for those who've already encountered this material.

Summary verdict Revisiting this book, I find it a rather patchily uneven read. Although intended for beginners in mathematical logic, the level of difficulty of the discussions rather varies, and the amount of more relaxed motivational commentary also varies. As noted there are occasional lapses where van Dalen's exposition isn't as tight as it could be. So this is probably best treated as a book to be read *after* you've had a first exposure to the material in the various chapters: but then it should indeed prove pretty helpful for consolidating/expanding your initial understanding and then pressing on a few steps.

9 Prestel and Delzell, 1986, 2011

Alexander Prestell and Charles N. Delzell's *Mathematical Logic and Model Theory: A Brief Introduction* (Springer, 1986; English version 2011: pp. 193) is advertised as offering a 'streamlined yet easy-to-read introduction to mathematical logic and basic model theory'. Easy-to-read, perhaps, for those with a fair amount of mathematical background in algebra, for – as the Preface makes clear – the aim of the book is make available to interested mathematicians 'the best known model theoretic results in algebra'. The last part of the book develops a complete proof of Ax and Kochen's work on Artin's conjecture about Diophantine properties of p-adic number fields.

So this book is not really aimed at a likely reader of this Guide. Still, Ch. 1 is a crisp and clean 60 page introduction to first-order logic, that could be used as brisk and helpful revision material. And Ch. 2, 'Model constructions' gives a nice if pacey introduction to some basic model theoretic notions: again – at least for readers of this Guide – it could serve well to consolidate and somewhat extend ideas if you have already encountered at least some of this material before.

The remaining two chapters, 'Properties of Model Classes' and 'Model Theory of Several Algebraic Theories' are tougher going, and belong with the more advanced reading like Marker's book. But, for those who want to work through this material, it does strike me as well presented.

10 Johnstone, 1987

Peter T. Johnstone's *Notes on Logic and Set Theory* (CUP, 1987: pp. 111) is very short in page length, but very big in ambition. There is an introductory chapter on universal algebra, followed by chapters on propositional and first-order logic. Then there is a chapter on recursive functions (showing that a function is register computable if and only if computable, and that such functions are representable in PA). That is followed by four chapters on set theory (introducing the axioms of ZF, ordinals, AC, and cardinal arithmetic). And there is a final chapter 'Consistency and independence' on Gödelian incompleteness and independence results in set theory.

This is a quite remarkably action-packed menu for such a short book. True, the story is filled out a bit by the substantive exercises, but still surely this isn't the book to use for a first encounter with these ideas (even though it started life as notes for undergraduate lectures for the maths tripos). However, I would warmly recommend the book for revision/consolidation: its very brevity means that the Big Ideas get highlighted in a particularly uncluttered way, and particularly snappy proofs are given.

11 Hodel, 1995

Richard E. Hodel's *An Introduction to Mathematical Logic** (PWS Publishing, 1995, reprinted Dover Publications, 2013: pp. 491) was originally launched into the world by a relatively obscure publisher, but has now been taken up and cheaply republished by Dover. I hadn't heard of the book before a few people recommended it, commenting on the lack of any mention in earlier versions of this Guide.

The book covers first-order logic and some recursion theory, with a less usual – though not unique – feature being a full textbook treatment of Hilbert's Tenth Problem (the one about whether there is an algorithm which tells us when a polynomial equation $p(x) = 0$ has a solution in the integers).

So how does Hodel fare against his competitors? Overall, the book *is* pretty clearly written, though it does have a somewhat old-fashioned feel to it (you wouldn't guess, for example, that it was written some twenty years after George Boolos and Richard Jeffrey's *Computability and Logic*). And Hodel does give impressively generous sets of exercises throughout the book – including some getting the student to prove significant results by guided stages. However, I can't really recommend his treatment of logic in the first half of the book.

Some details Ch. 1 'Background' is an unusually wide-ranging introduction to ideas the budding logician should get her head round early. We get a first informal pass at the notions of a formal system and of an axiomatized system in particular, the idea of proof by induction, a few notions about sets, functions and relations, the idea of countability, the ideas of an algorithmically computable function and of effective decidability. We even get a first pass at the idea of a recursive function (and a look at Church's Thesis about how the informal idea of being computable relates to the idea of recursiveness). This is very lucidly done, and can be recommended.

Ch. 2 is on 'The language and semantics of propositional logic' and is again pretty clearly done.

Ch. 3 turns to formal deductive systems for propositional logic. Unfortunately, Hodel chooses to work primarily with Shoenfield's system (the primitive connectives are $\neg$ and $\vee$, every instance of $\neg A \vee A$ is an axiom, and there are four rules of inference). I *really* can't see the attraction of this system among all the competitors, or why it should be thought as especially appropriate as a starting point for beginners. It neither has the naturalness of a natural deduction system, nor the austere Bauhaus lines of one of the more usual Frege-Hilbert axiomatic systems. The chapter does also consider other systems of propositional logic in the concluding two sections, but goes too quickly to be very helpful. So this key chapter is not a success, it seems to me.

Ch. 4 on 'First-order languages', including Tarskian semantics, is again pretty clear (and could be helpful to a beginner who is first encountering the ideas and is looking for reading to augment another textbook). But as we'd expect given what's gone before, when we turn to Ch. 5 on 'First-order logic' we can a continuance of the discussion of a Shoenfield-style system (except that Hodel takes $\forall$ rather than $\exists$ as primitive). Which is, by my lights, much to be regretted. The ensuing discussion of completeness, while perhaps a little laboured, is carefully structured with a good amount of signposting. But there are better presentations of first-order logic overall.

Ch. 6 is called 'Mathematics and logic' and touches on an assortment of topics about first-order theories and their limitations (and a probably rather-too-hasty look at set theory as an example).

The next two chapters form quite a nice unit. Ch. 7 discusses 'Incompleteness, undecidability and indefinability'. Recursive functions are defined Shoenfield-style as those arising from a certain class of initial functions by composition and regular minimization, which eases the proof that all recursive functions are representable (though doesn't do much to make recursiveness seem a natural idea to beginners). Then, by making the informal assumption that certain intuitively decidable relations are recursive, Hodel proves Gödel's incompleteness theorem, Church's Theorem and Tarski's Theorem. The next chapter fills in enough detail about recursive functions and relations to show how to lift that informal assumption. This seems all pretty clearly done, even if not a first-choice for real beginners.

Then Ch. 9 extends the treatment of computability by showing that the functions computable by an unlimited register machine are just the recursive ones: but, again this sort of thing is done at least as well in other books.

Finally, Ch. 10 deals with Hilbert's tenth problem (so the first five sections of Ch. 8 and then this chapter form a nice, stand-alone treatment of the negative solution of Hilbert's problem).

Summary verdict Beginners could all usefully read Hodel's opening chapter, which is better than usual in setting the scene, and supplying the student with a useful toolkit of preliminary ideas. The presentation of first-order logic in Chs. 2-6 is based around an unattractive formal system, and while the discussion of the usual meta-theoretic results is pretty clear, it doesn't stand out from the good alternatives: so overall, I wouldn't recommend this as your first encounter with serious logic.

Students, however, might find Chs. 7 and 8 provide a nice complement to other discussions of Gödelian incompleteness and Church and Tarski's Theorems. While more advanced students could revise their grip on basic definitions and results by (re)reading §§8.1–8.5 and then enjoy tackling Ch. 10 on Hilbert's Tenth Problem.

12 Goldstern and Judah, 1995

Half of Martin Goldstern and Haim Judah's *The Incompleteness Phenomenon: A New Course in Mathematical Logic* (A.K. Peters, 1995: pp. 247) is a treatment of first-order logic. The rest of the book is two long chapters (as it happens, of just the same length), one on model theory, one mostly on incompleteness and with a little on recursive functions. So the emphasis on incompleteness in the title is somewhat misleading: the book is at least equally an introduction to some model theory. I have had this book recommended to me more than once, but I seem to be immune to its supposed charms (I too often don't particularly like the way that it handles the technicalities): your mileage may vary.

Some details Ch. 1 starts by talking about inductive proofs in general, then gives a semantic account of sentential and then first-order logic, then offers a Hilbert-style axiomatic proof system.

Very early on, the authors introduce the notion of $\mathcal{M}$-terms and $\mathcal{M}$-formulae. An $\mathcal{M}$-term (where $\mathcal{M}$ is model for a given first-order language $\mathcal{L}$) is built up from $\mathcal{L}$-constants, $\mathcal{L}$-variables *and/or elements of the domain of* $\mathcal{M}$, using $\mathcal{L}$-function-expressions; an $\mathcal{M}$-formula is built up from $\mathcal{M}$-terms in the predictable way. Any half-awake student is initially going to balk at this. Re-reading the set-theoretic definitions of expressions as tuples, she will then realize that the apparently unholy mix of bits of language and bits of some mathematical domain in an $\mathcal{M}$-term is not actually incoherent. But she will right wonder what on earth is going on and *why*: our authors don't pause to explain why we might want to do things like this at the very outset. (A good student who knows other presentations of the basics of first-order semantics should be able to work out after the event what is going on in the apparent trickery of Goldstern and Judah's sort of story: but I really can't recommend *starting* like this, without a good and expansive explanation of the point of the procedure.)

Ch. 2 gives a Henkin completeness proof for the first-order deductive system given in Ch. 1. This has nothing special to recommend it, as far as I can see: there are many more helpful expositions available. The final section of the chapter is on non-standard models of arithmetic: Boolos and Jeffrey (Ch. 17 in their third edition) do this more approachably.

Ch.3 is on model theory. There are three main sections, 'Elementary substructures and chains', 'ultra products and compactness', and 'Types and countable models'. So this chapter – less than sixty pages – aims quite high to be talking e.g. about ultraproducts and about omitting types. You could indeed usefully read it after working through e.g. Manzano's book: but I certainly don't think this chapter makes for an accessible and illuminating first introduction to serious model theory.

Ch. 4 is on incompleteness, and the approach here is significantly more gentle than the previous chapter. Goldstern and Judah make things rather easier for themselves by adopting a version of Peano Arithmetic which has exponentiation built in (so they don't need to tangle with Gödel's β function). And they only prove a semantic version of Gödel's first incompleteness theorem, assuming the soundness of PA. The proof here goes as by showing directly that – via Gödel coding – various syntactic properties and relations concerning PA are expressible in the language of arithmetic with exponentiation (in other words, they don't argue that those properties and relations are primitive recursive and then show that PA can express all such properties/relations).

How well, how accessibly, is this done? The authors hack through eleven pages (pp. 207–217) of the arithmetization of syntax, but the motivational commentary is brisk and yet the proofs aren't completely done (the authors still leave to the reader the task of e.g. coming up with a predicate satisfied by Gödel numbers for induction axioms). So this strikes the present reader as really being neither one thing nor another – neither a treatment with all the details nailed down, nor a helpfully discursive treatment with a lot of explanatory arm-waving. And in the end, the diagonalization trick seems to be just pulled like a rabbit out of the hat. After proving incompleteness, they prove Tarksi's theorem and the unaxiomatizability of the set of arithmetic truths.

To repeat, the authors assume PA's soundness. They don't say anything about why we might want to prove the syntactic version of the first theorem, and don't even mention the second theorem which we prove by formalizing the syntactic version. So this could well leave students a bit mystified when they come across other treatments.

The book ends by noting that the relevant predicates in the arithmetization of syntax are Σ_1, and then *defines* a set as being recursively enenumerable if it is expressible by a Σ_1 predicates (so now talk of recursiveness etc. does get into the picture). But really, if you want to go down this route, this is surely all much better handled in Leary and Kristiansen's book.

Summary verdict The first two chapters of this book can't really be recommended either for making a serious start on first-order logic or for revision. The third chapter could be used for a brisk revision of some model theory if you have already done some reading in this area. The final chapter about incompleteness (which the title of the book might lead you to think will be a high point) isn't the most helpful introduction in this style – go for Leary and Kristiansen (2015) instead – and on the other hand doesn't go far enough for revision/consolidatory purposes.

13 Forster, 2003

Thomas Forster's *Logic, Induction and Sets* (CUP, 2003: pp. x + 234) is rather quirky, and some readers will enjoy it for just that reason. It is based on a wide-ranging lecture course given to mathematicians who – such being the oddities of the Cambridge tripos syllabus – at the beginning of the course already knew a good deal of maths but very little logic. The book is very bumpily uneven in level, and often goes skips forward very fast, so I certainly wouldn't recommend it as an 'entry level' text on mathematical logic for someone wanting a conventionally systematic approach. But it is often intriguing.

Some details Ch. 1 is called 'Definitions and notations' but is rather more than that, and includes some non-trivial exercises: but if you are dipping into later parts of the book, you can probably just consult this opening chapter on a need-to-know basis.

Ch. 2 discusses 'Recursive datatypes', defined by specifying a starter-pack of 'founders' and some constructors, and then saying the datatype is what you can get from the founders by applying and replying the constructors (and nothing else). The chapter considers a range of examples, induction over recursive datatypes, well-foundedness, well-ordering and related matters (with some interesting remarks about Horn clauses too).

Ch. 3 is on partially ordered sets, and we get a lightning tour through some topics of logical relevance (such as the ideas of a filter and an ultrafilter).

Chs. 4 and 5 deal slightly idiosyncratically with propositional and predicate logic, and could provide useful revision material (there's a slip about theories on p. 70, giving two non-equivalent definitions).

Ch. 6 is on 'Computable functions' and is another lightning tour, touching on quite a lot in just over twenty pages (getting as far as Rice's theorem). Again, could well be useful to read as revision, especially if you want to highlight again the Big Ideas and their interrelations.

Ch. 7 is on 'Ordinals'. Note that Forster gives us the elements of the theory of transfinite ordinal numbers *before* turning to set theory in the next chapter.

It's a modern doctrine that ordinals just *are* sets, and that the basic theory of ordinals is part of set theory; and in organizing his book as he does, Forster comes nearer than most to getting the correct conceptual order into clear focus (though even he wobbles sometimes, e.g. at p. 182). However, the chapter could have been done more clearly.

Ch. 8 is called 'Set Theory' and is perhaps the quirkiest of them all – though not because Forster is here banging the drum for non-standard set theories (surprisingly given his interests, he doesn't). But the chapter is oddly structured, so for example we get a quick discussion of models of set theory and the absoluteness of Δ_0 properties *before* we actually encounter the ZFC axioms. The chapter is probably only for those, then, who already know the basics.

Ch. 9 comprises answers to some of the earlier exercises – exercises are indeed scattered through the book, and some of them are rather interesting.

Summary verdict Different from the usual run of textbooks, not a good choice for beginners. However, if you already have encountered some of the material in one way or the other, Forster's book could very well be worth looking through for revision and/or to get some new perspectives.

14 Hedman, 2004

Shawn Hedman's *A First Course in Logic* (OUP, 2004: pp. xx + 214) is subtitled 'An Introduction to Model Theory, Proof Theory, Computability and Complexity'. So there's no lack of ambition in the coverage! And I do like the general tone and approach at the outset. So I wish I could be more enthusiastic about the book in general. But, as we will see, it is decidedly patchy both in terms of the level of the treatment of various topics, and in terms of the quality of the exposition.

Some details After twenty pages of mostly rather nicely done 'Preliminaries' – including an admirably clear couple of pages the **P = NP** problem, Ch. 1 is on 'Propositional Logic'. On the negative side, we could certainly quibble that Hedman is a bit murky about object-language vs meta-language niceties. The treatment of induction half way through the chapter isn't as clear as it could be. Much more importantly, the chapter offers a particularly ugly formal deductive system. It is in fact a (single conclusion) sequent calculus, but with proofs constrained to be a simple linear column of wffs. So – heavens above! – we are basically back to Lemmon's *Beginning Logic* (1965). Except that the rules are not as nice as Lemmon's (thus Hedman's $\land$-elimination rule only allows us to extract a left conjunct; so we need an additional $\land$-symmetry rule to get from $P \land Q$ to Q). I can't begin to think what recommended this system to the author out of all the possibilities on the market. On the positive side, there's quite a nice treatment of a resolution calculus for wffs in CNF form, and a proof that this is sound and complete. This gives Hedman a completeness proof for derivations in his original calculus with a finite number of premisses, and he gives a compactness proof to beef this up to a proof of strong completeness.

Ch. 2, 'Structures and first-order logic' should really be called 'Structures and first-order languages', and deals with relations between structures (like embedding) and relations between structures and languages (like being a model for a sentence). I'm not sure I quite like its way of conceiving of a structure as always some $\mathcal{V}$-structure, i.e. as having an associated first-order vocabulary $\mathcal{V}$ which it is the interpretation of – so structures for Hedman are what some would call labelled structures. But otherwise, this chapter is clearly done.

Ch. 3 is about deductive proof systems for first-order logic. The first deductive system offered is an extension of the bastardized sequent calculus for propositional logic, and hence is equally horrible. Somehow I sense that Hedman just isn't much interested in standard proof-systems for logic. His heart is in the rest of the chapter, which moves towards topics of interest to computer scientists, about Skolem normal form, the Herbrand method, unification and resolution, so-called 'SLD-resolution', and Prolog – interesting topics, but not on *my* menu of basics to be introduced at this very early stage in a first serious logic course. The discussions seem quite well done, and will be accessible to an enthusiast with an introductory background (e.g. from Chiswell and Hodges) and who has read the section of resolution in the first chapter.

Ch. 4 is on 'Properties of first-order logic'. The first section is a nice presentation of a Henkin completeness proof (for countable languages). There is then a long aside on notions of infinite cardinals and ordinals (Hedman has a policy of introducing background topics, like the idea of an inductive proof, and now these set theoretic notions, only when needed: but it can break the flow). §4.3 can use the assumed new knowledge about non-countable infinities to beef up the completeness proof, give upwards and downwards LS theorems, etc., again done pretty well. §§4.4–4.6 does some model theory under the rubrics 'Amalgamation of structures'. 'Preservation of formulas' and 'Amalgamation of vocabularies': this already gets pretty abstract and uninviting, with not enough motivating examples. §4.7 is better on 'The expressive power of first-order logic'.

The next two chapters, 'First order theories' and 'Models of countable theories', give a surprisingly (I'd say, unrealistically) high level treatment of some model theory, going well beyond e.g. Manzano's book, eventually talking about saturated models, and even ending with 'A touch of stability'. This hardly chimes with the book's prospectus as being a *first* course in logic. The chapters, however, could be useful for someone who wants to push onwards, after a first encounter with some model theory.

Ch. 6 comes sharply back to earth: an excellent chapter on 'Computability and complexity' back at a sensibly introductory level. It begins with a well done review of the standard material on primitive recursive functions, recursive functions, computing machines, semi-decidable decision problems, undecidable decision problems. Which is followed by a particularly clear introduction to ideas about computational complexity, leading up to the notion of **NP**-completeness. An excellent chapter.

Sadly, the following Ch. 8 on the incompleteness theorems again isn't very satisfactory as a first pass through this material. In fact, I doubt whether a beginning student would take away from this chapter a really clear sense of what the key big ideas are, or of how to distinguish the general results from the hack-work needed to show that they apply to this or that particular theory. And things probably aren't helped by proving the first theorem initially by Boolos's method rather than Gödel's. Still, just because it gives an account of Boolos's proof, this chapter *can* be recommended as supplementary reading for those who have already seen some standard treatments of incompleteness.

The last two chapters ratchet up the difficulty again. Ch. 9 goes 'Beyond first-order logic' by speeding through second-order logic, infinitary logics (particularly $\mathscr{L}_{\omega_1\omega}$), fixed-point logics, and Lindström's theorem, all in twenty pages. This will probably go too fast for those who haven't encountered these ideas before. It should be noted that the particularly brisk account of second-order logic gives a non-standard syntax and says nothing about Henkin vs full semantics. The treatment of fixed-point logics (logics that are 'closed under inductive definitions') is short on motivation and examples. But enthusiasts might appreciate the treatment of Lindström's theorem.

Finally, Ch. 10 is on finite model theory and descriptive complexity. Beginners doing a first course in logic will again find this quite tough going.

Summary verdict A very uneven book in level, with sections that work well at an introductory level and other sections which will only be happily managed by considerably more advanced students. An uneven book in coverage too. By my lights, this couldn't be used end-to-end as a course text: but in the body of the Guide, I've recommended parts of the book on particular topics.

15 Hinman, 2005

Now for a really *big* Big Book – Peter G. Hinman's *Fundamentals of Mathematical Logic* (A. K. Peters, 2005: pp. 878).

The author says the book was written over a period of twenty years, as he tried out various approaches 'to enable students with varying levels of interest and ability to come to a deep understanding of this beautiful subject'. But I suspect that you will need to be mathematically quite strong to really cope with this book: whatever Hinman's intentions for a wider readership, this is not for the fainthearted.

The book's daunting size is due to its very wide coverage rather than a slow pace – so after a long introduction to first-order logic (or more accurately, to its model theory) and a discussion of the theory of recursive functions and incompleteness and related results, there follows a *very* substantial survey of set theory, and then lengthy essays on more advanced model theory and on recursion theory. As too often, proof theory is the poor relation here – indeed Hinman is very little interested in deductive systems for logic, which don't make an appearance until over two hundred pages into the book.

Let me mention at the outset what strikes me as a pretty unfortunate global notational convention, which might puzzle casual browsers or readers who want to start some way through the book, Given the two-way borrowing of notation between informal mathematics and the formal languages in which logicians regiment that mathematics, it is good to have some way of visually distinguishing the formal from the informal (so we don't just rely on context). One common method is font selection. Thus, even in an informal context, we may snappily say that addition commutes by writing e.g. $\forall x \forall y\, x + y = y + x$; the counterpart wff for expressing this in a fully formalized language may then be, e.g., $\forall\mathsf{x}\forall\mathsf{y}\,\mathsf{x}+\mathsf{y}=\mathsf{y}+\mathsf{x}$. But instead of using sans serif or boldface for formal wffs, or another font selection, Hinman prefers using an '(informal) mathematical sign with a dot over it to represent a formal symbol in a formal language which denotes the informal object', so he'd write $\forall x\forall y\, x \dot{+} y \doteq y \dot{+} x$ for the formal wff. As you can imagine, this convention eventually leads to really nasty rashes of dots – for example, to take a relatively tame example from p. 459, we get

$$\dot{\bigcup} x \doteq \dot{\{}z\colon \exists v[v \mathrel{\dot{\in}} x \wedge z \mathrel{\dot{\in}} v]\}$$

(note how even opening braces in formal set-former notation get dotted). This dottiness quite surely isn't a happy choice!

Some details Hinman himself in his Preface gives some useful pointers to routes through the book, depending on your interests.

The Introduction gives a useful and approachable overview of some key notions tied up with the mathematical logician's project of formalization (and talks about a version of Hilbert's program as setting the scene for some early investigations).

Ch. 1 is on 'Propositional Logic and other fundamentals'. §§1.1, 1.3 and 1.4 are devoted to the language of propositional logic, and give the usual semantics, define the notion tautological entailment and explore its properties, giving a proof of the compactness theorem. But note, there is no discussion at all here – or in the other sections of this chapter – of a proof-system for propositional logic.

§1.2 is a rather general treatment of proofs by induction and the definition of functions by recursion (signposted as skippable at this early stage – and indeed the generality doesn't make for a particularly easy read for a section so early in the book). §§1.6 and 1.7 also cover more advanced material, mainly introducing ideas for later use: the first briskly deals e.g. with ultrafilters and ultraproducts (we get another take on compactness), and the second relates compactness to topological ideas and also introduces the idea of a Boolean algebra.

Ch. 2, 'First-order logic', presents the syntax and semantics of first-order languages, and then talks about first-order structures (isomorphisms, embeddings, extensions, etc), and proves the downward L-S theorem. We then get a general discussion of theories (thought of as sets of sentences closed under *semantic* consequence), and an extended treatment of some examples (the theory of equality, the theory of dense linear orders, and various strengths of arithmetic). There's some quite sophisticated stuff here, including discussion of quantifier elimination. But there is still no discussion yet of a proof-system for first-order logic, so the chapter could as well, if not better, have been called 'Elements of model theory'.

Ch. 3, 'Completeness and compactness', starts with a compactness proof for countable languages. Then we at last have a *very* brisk presentation of an old-school axiomatic system for first-order logic (I told you that Hinman is not interested in proof-systems!), and a proof of completeness using the Henkin construction that has already been used in the compactness proof.

We next get – inter alia – an algebraic proof of compactness for first-order consequence via ultraproducts, and a return to Boolean algebras and e.g. the Rasiowa-Sikorski theorem (§3.3); an extension of the compactness and completeness results to uncountable languages (§3.4); and some heavy-duty applications of compactness (§3.5).

Finally in this action-packed chapter, we have some rather unfriendly treatments of higher-order logic (§3.6) and infinitary logic (§3.7).

Let's pause for breath. We are now a bit over 300 pages into the book. Things have already got pretty tough. The book is not quite a relentless march along a chain of definitions/theorems/corollaries; there are just enough pauses for illustrations and helpful remarks en route to make it a bearable. But Hinman does have a taste for going straight for abstractly general formulations (and his notational choices can sometimes be unhappy too). So as indicated in my preamble, the book will probably only appeal to mathematicians already used to this sort of fairly hardcore approach. In sum, therefore, I'd only recommend the first part of the book to the mathematically minded who already know their first-order logic and a bit of model theory; but such readers might then find it quite helpful as a beginning/mid-level model theory resource.

On we go. Next we have two chapters (almost 150 pages between them) on recursive functions, Gödelian incompleteness, and related matters. Perhaps it is because these topics are conceptually easier, more 'concrete', than what's gone before, or perhaps it is because the topics are closer to Hinman's heart, but these chapters seem to me to work better as an introduction to their topics. In particular, while not my favourite treatment, Ch. 4 is clear, very sensibly structured, and should be accessible to anyone with some background in logic and who isn't put off by a certain amount of mathematical abstraction. The chapter opens with informal proofs of the undecidability of consistent extensions of Q, the first incompleteness theorem and Tarski's theorem on the undefinability of truth (as well as taking a first look at the second incompleteness theorem). These informal proofs depend on the hypothesis that effectively calculable functions are expressible or the hypothesis that such functions are representable (we don't yet have a formal story about these functions). Unsurprisingly, given I do something in the same ball-park in my Gödel book, I too think this is a good way to start and to motivate the ensuing development. There follows, as you'd expect, the necessary account of the effectively calculable in terms of recursiveness, and then we get proofs that recursive functions can be expressed/represented in arithmetic, leading on to formal versions of the theorems about undecidable and incompleteness. This presentation takes a different-enough path through the usual ideas to be worth reading even if you've already encountered the material a couple of times before.

Ch. 5 is called 'Topics in definability' and, unlike the previous rather tightly organised chapter, is something of a grab-bag of topics. §5.1 says something about the arithmetical hierarchy; §5.2 discusses inter alia the indexing of recursive functions and the halting problem; §5.3 explains how the second incompleteness theorem is proved, and – while not attempting a full proof – there is rather more detail than usual about how you can show that the HBL derivability conditions are satisfied in PA. Then §5.4 gives more evidence for Church's Thesis by considering a couple of other characterisations of computability (by equation manipulation and by abstract machines) and explains why they again pick out the recursive functions. §5.5 discusses 'Applications to other languages and theories' (e.g. the application of incompleteness to a theory like ZF which is not initially about arithmetic). These various sections are all relatively clearly done.

Pausing for breath again, we might now try to tackle Ch. 6 on set theory (whose 200 pages amount by themselves to an almost-stand-alone book). The menu covers the basics of ZF, the way we can construct mathematics inside set theory, ordinals and cardinals, then models and independence proofs, the constructible universe, models and forcing, large cardinals and determinacy. But even from the outset, this does seem quite relentlessly hard going, too short on motivation and illustrations of concepts and constructions. Dense, to say the least. The author says of the chapter that his particular mode of presentation means that 'for each of the instances where one wants to verify that something is a class model – the intuitive universe of sets V, the constructible universe L and a forcing extension $\mathsf{m}[G]$ – ... the proofs ... exhibit more of the underlying unity.' So enthusiasts who know their set theory might want to do a fast read of the chapter to see if they can glean new insights. But I can't recommend this as a way into set theory when compared with the standard set theory texts mentioned in §**??** and §**??** of this Guide.

Ch. 7 returns to more advanced model theory for another 80 pages, getting as far Morley's theorem. Again, if you want a more accessible initial treatment, you'll go for Hodges's *Shorter Model Theory*. And then why not tackle Marker's book if you are a graduate mathematician?

Finally, there's another equally long chapter on recursion theory. The opening sections on degrees and Turing reducibility are pretty approachable. The rest of the chapter gets more challenging but (at least compared with the material on model theory and set theory) should still be tolerably accessible to those willing to put in the work.

Summary verdict It is *very* ambitious to write a book with this range and depth of coverage (as it were, an expanded version of Shoenfield, forty years on – but now when there is already a wealth of textbooks on the various areas covered, at various levels of sophistication). After such a considerable labour from a good logician, it seems very churlish to say it, but the treatments of, respectively, (i) first-order logic, (ii) model theory, (iii) computability theory and incompleteness, and (iv) set theory aren't as good as the best of the familiar stand-alone textbooks on the four areas. And I can't see that these shortcomings are balanced by any conspicuous advantage in having the accounts in a single text, rather than a handful of different ones. Still, the text should be in any university library, as enthusiasts might well find parts of it quite useful supplementary/reference material. Chapters 4, 5 and 8 on computability and recursion work the best.

16 Chiswell and Hodges, 2007

Ian Chiswell and Wilfrid Hodges's *Mathematical Logic* (OUP, 2007: pp. 249) is very largely focused on first-order logic, only touching on Church's undecidability theorem and Gödel's first incompleteness theorem in a Postlude after the main chapters. So it is perhaps stretching a point to include it in a list of texts which cover more than one core area of the mathematical logic curriculum. Still, I wanted to comment on this book at some length, without breaking the flow of the Guide (where it is warmly recommended in headline terms), and this is the obvious place to do so.

Let me highlight three key features of the book, the first one not particularly unusual (though it still marks out this text from quite a few of the older, and not so old, competitors), the second very unusual but extremely welcome, the third a beautifully neat touch:

1. Chiswell and Hodges (henceforth C&H) present natural deduction proof systems and spend quite a bit of time showing how such formal systems reflect the natural informal reasoning of mathematicians in particular.

2. Instead of dividing the treatment of logic into two stages, propositional logic and quantificational logic, C&H take things in *three* stages. First, propositional logic. Then we get the quantifier-free part of first-order logic, dealing with properties and relations, functions, and identity. So at this second stage we get the idea of an interpretation, of truth-in-a-structure, and we get added natural deduction rules for identity and the handling of the substitution of terms. At both these first two stages we get a Hintikka-style completeness proof for the given natural deduction rules. Only at the third stage do quantifiers get added to the logic and satisfaction-by-a-sequence to the semantic apparatus. Dividing the treatment of first order logic into stages like this means that a lot of key notions get first introduced in the less cluttered contexts of propositional and/or quantifier-free logic, and the novelties at the third stage are easier to keep under control. This does make for a great gain in accessibility.

3. The really cute touch is to introduce the idea of polynomials and diophantine equations early – in fact, while discussing quantifier-free arithmetic – and to state (without proof!) Matiyasevich's Theorem. Then, in the Postlude, this can be appealed to for quick proofs of Church's Theorem and Gödel's Theorem.

This is all done with elegance and a light touch – not to mention photos of major logicians and some nice asides – making an admirably attractive introduction to the material.

Some details C&H start with almost 100 pages on the propositional calculus. Rather too much of a good thing? Perhaps, if you have already done a logic course at the level of my intro book or Paul Teller's. Still, you can easily skim and skip. After Ch. 2 which talks about informal natural deductions in mathematical reasoning, Ch. 3 covers propositional logic, giving a natural deduction system (with some mathematical bells and whistles along the way, being careful about trees, proving unique parsing, etc.). The presentation of the formal natural deduction system is not exactly my favourite in its way representing discharge of assumptions (I fear that some readers might be puzzled about vacuous discharge and balk at Ex. 2.4.4 at the top of p. 19): but apart from this little glitch, this is done well. The ensuing completeness proof is done by Hintikka's method rather than Henkin's.

After a short interlude, Ch. 5 treats quantifier-free logic. The treatment of the semantics without quantifiers in the mix to cause trouble is very nice and natural; likewise at the syntactic level, treatment of substitution goes nicely in this simple context. Again we get a soundness and Hintikka-style completeness proof for an appropriate natural deduction system.

Then, after another interlude, Ch 7 covers full first-order logic with identity. Adding natural deduction rules (on the syntactic side) and a treatment of satisfaction-by-finite-n-tuples (on the semantic side) all now comes very smoothly after the preparatory work in Ch. 5. The Hintikka-style completeness proof for the new logic builds very nicely on the two earlier such proofs: this is about as accessible as it gets in the literature, I think. The chapter ends with a look at the Löwenheim-Skolem theorems and 'Things that first-order logic cannot do'.

Finally, as explained earlier, material about diophantine equations introduced naturally by way of examples in earlier chapters is used in a final Postlude to give us undecidability and incompleteness results very quickly (albeit assuming Matiyasevich's Theorem).

Summary verdict C&H have written a very admirably readable and nicely structured introductory treatment of first-order logic that can be warmly recommended. The presentation of the syntax of their type of (Gentzen-Prawitz) natural deduction system is perhaps done a trifle better elsewhere (Tennant's freely available *Natural Logic* mentioned in §**??** gives a full dress version). But the core key sections on soundness and completeness proofs and associated metalogical results are second to none for their clarity and accessibility.

17 Leary and Kristiansen, 2015

Christopher C. Leary and Lars Kristiansen's *A Friendly Introduction to Mathematical Logic* (Milne Library 2015: pp. 364) is the second, significantly expanded, edition of a fine book originally just authored by Leary (Prentice Hall, 2000: pp. 218). The book is now available at a very attractive price; the main differences between the editions are a long new chapter on computability theory, and some 75 pages of solutions to exercises.

So how friendly is *A Friendly Introduction*? – meaning, of course, 'friendly' by the standard of logic books! I do like the tone a great deal (without being the least patronizing, it is indeed relaxed and inviting), and the level of exposition seems to me to be very well-judged for an introductory course. The book is officially aimed mostly at mathematics undergraduates without assuming any particular background knowledge. But as the Preface notes, it should also be accessible to logic-minded philosophers who are happy to work at following rather abstract arguments (and, I would add, who are also happy to skip over just a few inessential elementary mathematical illustrations).

What does the book cover? Basic first-order logic (up to the L-S theorems), the incompleteness theorems, and some computability theory. But by being so tightly focused, this book rarely seems to rush at what it *does* cover: the pace is pretty even. The authors do opt for a Hilbertian axiomatic system of logic, with fairly brisk explanations. (If you'd never seen before a serious formal system for first-order logic this could initially make for a somewhat dense read: if on the other hand you have been introduced to logic by trees or seen a natural deduction presentation, you would perhaps welcome a paragraph or two explaining the advantages for present purposes of the choice of an axiomatic approach here.) But the clarity is indeed exemplary.

Some details Ch. 1, 'Structures and languages', starts by talking of first-order languages (The authors make the good choice of not starting over again with propositional logic, but assume that most readers will know their truth-tables so just give quick revision). The chapter then moves on to explaining the idea of first order structures, and truth-in-a-structure. There is a good amount of motivational chat as we go through, and the exercises – as elsewhere in the book – seem particularly well-designed to aid understanding. (The solutions to exercises added to the new edition makes the book even more suitable for self-study.)

Ch. 2, 'Deductions', introduces an essentially Hilbertian logical system and proves its soundness: it also considers systems with additional non-logical axioms. The logical primitives are '$\vee$', '$\neg$', '$\forall$' and '$=$'. Logical axioms are just the identity axioms, an axiom-version of $\forall$-elimination (and its dual, $\exists$-introduction): the inference rules are $\forall$-introduction (and its dual) and a rule which allows us to infer φ from a finite set of premisses Γ if it is an instance of a tautological entailment. I don't think this is the friendliest ever logical system (and no doubt for reasons of brevity, the authors don't pause to consider alternative options); but it certainly is not horrible either. If you take it slowly, the exposition here should be quite manageable even for the not-very-mathematical.

Ch. 3, 'Completeness and compactness', gives a nice version of a Henkin-style completeness theorem for the described deductive system, then proves compactness and the upward and downward Lwenheim-Skolem theorems (the latter in the version 'if L is a countable language and $\mathfrak{B}$ is an L-structure, then $\mathfrak{B}$ has a countable elementary substructure' [the proof might be found *just* a bit tricky though]). So there is a little model theory here as well as the completeness proof: and you could well read this chapter without reading the previous ones if you are already reasonably up to speed on structures, languages, and deductive systems. And so, in a hundred pages, we wrap up what is indeed a pretty friendly introduction to FOL.

Ch. 4, 'Incompleteness, from two points of view' is a helpful bridge chapter, outlining the route ahead, and then defining Σ, Π and Δ wffs (no subscripts in their usage, and exponentials are atomic – maybe a footnote would have been wise, to help students when they encounter other uses). Then in Ch. 5, 'Syntactic Incompleteness – Groundwork', the authors (re)introduce the theory they call N, a version of Robinson Arithmetic with exponentiation built in. They then show that (given a scheme of Gdel coding) that the usual numerical properties and relations involved in the arithmetization of syntax – such as, ultimately, $Prf(m, n)$, i.e. m codes for an N-proof of the formula numbered n – can be represented in N. They do this by the direct method. That is to say, instead of [like my *IGT*] showing that those properties/relations are (primitive) recursive, and that N can represent all (primitive) recursive relations, they directly write down Δ wffs which represent them. This is inevitably gets more than a bit messy: but they have a very good stab at motivating every step working up to showing that N can express $Prf(m, n)$ by a Δ wff. If you want a full-dress demonstration of this result, then this is one of the most user-friendly available.

Ch. 6, 'The Incompleteness Theorems', is then pretty short: but all the groundwork has been done to enable the authors now to give a brisk but very clear presentation, at least after they have proved the Diagonalization Lemma. I did complain that, in the first edition, the proof of the Lemma was slightly too rabbit-out-of-a-hat for my liking. This edition I think notably softens the blow (one of many such small but significant improvements, as well as the major additions). And with the Lemma in place, the rest of the chapter goes very nicely and accessibly. We get the first incompleteness theorem in its semantic version, the undecidability of arithmetic, Tarski's theorem, the syntactic version of incompleteness and then Rosser's improvement. Then there is nice section giving Boolos's proof of incompleteness echoing the Berry paradox. Finally, the second theorem is proved by assuming (though not proving) the derivability conditions.

The newly added Ch. 7, 'Computability theory' starts with a very brief section on historical origins, mentioning Turing machines etc.: but we then settle to exploring the μ-recursive functions. We get some way, including the S-m-n theorem and a full-dress proof of Kleene's Normal Form Theorem (with due apologies for the necessary hacking though details) and meet the standard definition of the set $K = \{x \mid x \in W_x\}$ where W_x is the domain of the computable function with index x. The uncomputability of K is then used, in the usual sort of way, to prove the undecidability of the *Entscheidungsproblem*, to re-prove the incompleteness theorem, and in tackling Hilbert's 10th problem. This is all nicely done in the same spirit and with the same level of accessibility as the previous chapters.

Summary verdict If you have already briefly met a formally presented deductive system for first-order logic, and some account of its semantics, then you'll find the opening two chapters of this book very manageable (if you haven't they'll be a bit more work). The treatment of completeness etc. in Ch. 3 would make for a nice stand-alone treatment even if you don't read the first two chapters. Or you could just start the book by reading 2.8 (where N is first mentioned), and then read the excellent ensuing chapters on incompleteness and computability with a lot of profit. *A Friendly Introduction* is indeed in many ways a very unusually likeable introduction to the material it covers, and has a great deal to recommend it.