Category Theory

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Cambridge University
Mathematical Tripos Part III
Michaelmas 2015

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Last updated June 5, 2016.

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1 Introduction

Category theory has been around for about half a century now, invented in the 1940's by Eilenberg and MacLane. Eilenberg was an algebraic topologist and MacLane was an algebraist. They realized that they were doing the same calculations in different areas of mathematics, which led them to develop category theory. Category theory is really about building bridges between different areas of mathematics.

1.1 Definitions and examples

This is just about setting up the terminology. There will be no theorems in this chapter.

Definition 1.1. A **category** $\mathbf{C}$ consists of

(i) a collection $\operatorname{ob}\mathbf{C}$ of objects $A, B, C, \ldots$

(ii) a collection $\operatorname{mor}\mathbf{C}$ of morphisms $f, g, h \ldots$

(iii) two operations, called $\operatorname{dom}(-)$ and $\operatorname{cod}(-)$, from morphisms to objects. We write $A \xrightarrow{f} B$ or $f \colon A \to B$ for $f \in \operatorname{mor}\mathbf{C}$ and $\operatorname{dom}(f) = A$ and $\operatorname{cod}(f) = B$;

(iv) an operation $A \mapsto 1_A$ from objects to morphisms, such that $A \xrightarrow{1_A} A$;

(v) an operation $\circ \colon (f, g) \mapsto f \circ g$ from pairs of morphisms (so long as we have $\operatorname{dom} f = \operatorname{cod} g$) to morphisms, such that $\operatorname{dom}(fg) = \operatorname{dom}(g)$ and $\operatorname{cod}(fg) = \operatorname{cod}(f)$.

These data must satisfy:

(vi) for all $f \colon A \to B$, $f1_A = 1_B f = f$;

(vii) composition is associative. If fg and gh are defined, then $f(gh) = (fg)h$.

Remark 1.2.

(a) We don't require that $\operatorname{ob}\mathbf{C}$ and $\operatorname{mor}\mathbf{C}$ are sets.

(b) If they are sets, then we call $\mathbf{C}$ a **small category**.

(c) We can get away without talking about objects, since $A \mapsto 1_A$ is a bijection from $\operatorname{ob}\mathbf{C}$ to the collection of morphisms f satisfying $fg = g$ and $hf = h$ whenever teh composites are defined. Essentially, we can represent objects by their identity arrows.

Example 1.3.

(a) The category **Set** whose objects are sets and whose arrows are functions. Technically, we should specify the codomain for the functions because really the definition of a function doesn't specify a codomain. So morphisms are pairs (f, B), where B is the codomain of the function f.

(b) **Gp** is the category of groups and group homomorphisms;

(c) **Ring** is the category of rings and ring homomorphisms;

(d) R-Mod is the category of R-modules and R-module homomorphisms;

(e) **Top** is the category of topological spaces and continuous maps;

(f) **Mf** is the category of smooth manifolds and smooth maps;

(g) The homotopy category of topological spaces **Htpy** has the same objects as **Top**, but the morphisms $X \to Y$ are homotopy classes of continuous maps;

(h) for any category **C**, we can turn the arrows around to make the opposite category $\mathbf{C}^{\text{op}}$.

Example (h) leads to the **duality principle**, which is a kind of "two for the price of one" deal in category theory.

Theorem 1.4 (The Duality Principle)**.** If ϕ is a valid statement about categories, so is the statement ϕ^* obtained by reversing all the morphisms.

Example (g) above gives rise to the following definition.

Definition 1.5. In general, an equivalence relation $\sim$ on the collection of all morphisms of a category is called a **congruence** if

(i) $f \sim g \implies \operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{cod} f = \operatorname{cod} g$;

(ii) $f \sim g \implies fh \sim gh$ and $kf \sim kg$ whenever the composites are defined.

There's a category $\mathbf{C}/\sim$ with the same objects as **C** but $\sim$-equivalence classes as morphisms.

Example 1.6. Continued from Example 1.3.

(i) A category **C** with one object $*$ must have $\operatorname{dom} f = \operatorname{cod} f = *$ for all $f \in \operatorname{mor} \mathbf{C}$. So all composites are defined, and (if $\operatorname{mor} \mathbf{C}$ is a set), $\operatorname{mor} \mathbf{C}$ is just a **monoid** (which is a semigroup with identity).

(j) In particular, a group can be considered as a small category with one object, in which every morphism is an isomorphism.

(k) A **groupoid** is a category in which all morphisms are isomorphisms. For a topological space X, the **fundamental groupoid** $\pi(X)$ is the "basepointless fundamental group;" the objects are points of X and the morphisms $x \to y$ are homotopy classes paths from x to y. (Homotopy classes are required so that each path has an inverse).

(l) a category whose only morphisms are identites is called **discrete**. A category in which, for any two objects A, B there is at most one morphism $A \to B$ is called **preorder**, i.e. it's a collection of objects with a reflexive and transitive relation. In particular, a **partial order** is a preorder in which the only isomorphisms are identities.

(m) The category **Rel** has the same objects as **Set**, but the morphisms are relations instead of functions. Precisely, a morphism $A \to B$ is a triple (A, R, B) where $R \subseteq A \times B$. The composite $(B, S, C)(A, R, B)$ is $(A, R \circ S, C)$ where

$$R \circ S = \{(a, c) \mid \exists b \in B \text{ s.t. } (a, b) \in R \text{ and } (b, c) \in S\}.$$

Note that **Set** is a subcategory of **Rel** and $\mathbf{Rel} \cong \mathbf{Rel}^{\mathrm{op}}$.

Let's continue with the examples.

Example 1.7. Continued from Example 1.3.

(n) Let K be a field. The category $\mathbf{Mat}_K$ has natural numbers as objects. A morphism $n \to p$ is a $p \times n$ matrix with entries in K. Composition is just matrix multiplication. Note that, once again, $\mathbf{Mat}_K \cong \mathbf{Mat}_k^{\mathrm{op}}$, via transposition of matrices.

(o) An example from logic. Suppose you have some formal theory $\mathbb{T}$. The category $\mathbf{Det}_{\mathbb{T}}$ of derivations relative to $\mathbb{T}$ has formulae in the language of $\mathbb{T}$ as objects, and morphisms $\phi \to \psi$ are derivations

$$\begin{array}{c} \phi \\ \vdots \\ \psi \end{array}$$

and composition is just concatenation. The identity 1_ϕ is the one-line derivation ϕ.

Definition 1.8. Let **C** and **D** be categories. A **functor** $F \colon \mathbf{C} \to \mathbf{D}$ consists of

(i) an operation $A \mapsto F(A)$ from $\operatorname{ob} \mathbf{C}$ to $\operatorname{ob} \mathbf{D}$;

(ii) an operation $f \mapsto F(f)$ from $\operatorname{mor} \mathbf{C}$ to $\operatorname{mor} \mathbf{D}$,

satisfying

(i) $\operatorname{dom} F(f) = F(\operatorname{dom} f), \operatorname{cod} F(f) = F(\operatorname{cod} f)$ for all f;

(ii) $F(1_A) = 1_{F(A)}$ for all A;

(iii) and $F(fg) = F(f)F(g)$ whenever fg is defined.

Let's see some examples again.

Example 1.9. (a) The **forgetful functor** $\mathbf{Gp} \to \mathbf{Sets}$ which sends a group to its underlying set, and any group homomorphism to itself as a function. Similarly, there's one $\mathbf{Ring} \to \mathbf{Set}$, and $\mathbf{Ring} \to \mathbf{Ab}$, and $\mathbf{Top} \to \mathbf{Set}$

(b) There are lots of constructions in algebra and topology that turn out to be functors. For example, the free group construction. Let FA denote the free group on a set A. It comes equipped with an inclusion map

$\eta_A \colon A \to FA$, and any $f \colon A \to G$, where G is a group, extends uniquely to a homomorphism $FA \to G$.

$$\begin{tikzcd} FA \arrow[r, dashed] & G \\ A \arrow[u, "\eta_A"] \arrow[ur, "f"'] & \end{tikzcd}$$

F is a functor from **Set** to **Gp**, and given $g \colon A \to B$, we define Fg to be the unique homomorphism extending the composite $A \xrightarrow{g} B \xrightarrow{\eta_B} FB$.

(c) The **abelianization** of an arbitrary group G is the quotient G/G' of G by it's **derived subgroup** $G' = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle$. This gives the largest quotient of G which is abelian. If $\phi \colon G \to H$ is a homomorphism, then it maps the derived subgroup of G to the derived subgroup of H, so the abelianization is functorial $\mathbf{Gp} \to \mathbf{Ab}$.

(d) The powerset functor. For any set A, let PA denote the set of all subsets of A. P is a functor $\mathbf{Set} \to \mathbf{Set}$; given $f \colon A \to B$, we define $Pf(A') = \{f(x) \mid x \in A'\}$ for $A' \subseteq A$.

But we also make P into a functor $P^* \colon \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ (or $\mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$) by setting $P^*f(B') = f^{-1}(B')$ for $B' \subseteq B$.

This last example is what we call a contravariant functor.

Definition 1.10. A **contravariant functor** $F \colon \mathbf{C} \to \mathbf{D}$ is a functor $F \colon \mathbf{C} \to \mathbf{D}^{\mathrm{op}}$ (equivalently, $\mathbf{C}^{\mathrm{op}} \to \mathbf{D}$). The term **covariant functor** is used sometimes to make it clear that a functor is not contravariant.

Example 1.11. Continued from Example 1.9

(e) The dual space of a vector space over K defines a contravariant functor k-**Mod** $\to$ k-**Mod**. If $\alpha \colon V \to W$ is a linear map, then $\alpha^* \colon W^* \to V^*$ is the operation of composing linear maps $W \to K$ with α.

(f) Let **Cat** denote the category of small categories and functors between them. Then $\mathbf{C} \mapsto \mathbf{C}^{\mathrm{op}}$ is *covariant* functor $\mathbf{Cat} \to \mathbf{Cat}$.

(g) If M and N are monoids, regarded as one-object categories, what is a functor between them? It's just a monoid homomorphism from M to N: it preserves the identity element and composition. In particular, if M, N are groups, then the functor is a group homomorphism. Hence, we may think of **Gp** is a subcategory of **Cat**.

(h) Similarly, if P and Q are partially ordered sets, regarded as categories, a functor $P \to Q$ is just an order-preserving map.

(i) Let G be a group, regarded as a category. A functor $F \colon G \to \mathbf{Sets}$ picks out a set as the image of the one object in G, and each morphism of G is an isomorphism so gets mapped to a bijection of this set. So this is a group action $G \circlearrowright F(G)$. If we replace **Sets** by k-**Vect** for k a field, we get linear representations of G.

(j) In algebraic topology, there are many functors. For example, the fundamental group $\pi_1(X, x)$ defines a functor from $\mathbf{Top}_*$ (the category of **pointed topological spaces**, i.e., those with a distinguished basepoint) to $\mathbf{Gp}$. Similarly, homology groups are functors $H_n\colon \mathbf{Top} \to \mathbf{Ab}$ (or more commonly, $\mathbf{Htpy} \to \mathbf{Ab}$).

There's another layer too. There are morphisms between functors, called **natural transformations**.

Definition 1.12. Let $\mathbf{C}$ and $\mathbf{D}$ be categories and $F, G\colon \mathbf{C} \to \mathbf{D}$. A **natural transformation** $\alpha\colon F \to G$ is an operation $A \mapsto \alpha_A$ from $\operatorname{ob}\mathbf{C}$ to $\operatorname{mor}\mathbf{D}$, such that $\operatorname{dom}(\alpha_A) = F(A)$, $\operatorname{cod}(\alpha_A) = G(A)$ for all A, and the following diagram commutes.

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd}$$

Again, we should mention some examples of natural transformations.

Example 1.13. (a) There's a natural transformation $\alpha\colon 1_{k\text{-}\mathbf{Mod}} \to **$, where $*$ is the dual space functor. This is the statement that a vector space is canonically isomorphic to it's double dual. $\alpha_V\colon V \to V^{**}$ sends $r \in V$ to the "evaluate at r" map $V^* \to k$. If we restrict to finite-dimensional spaces, then α becomes a **natural isomorphism**, i.e. an isomorphism in the category $[k\text{-}\mathbf{fgMod}, k\text{-}\mathbf{fgMod}]$, where $[\mathbf{C}, \mathbf{D}]$ denotes the category of all functors $\mathbf{C} \to \mathbf{D}$ with natural transformations as arrows.

Remark 1.14. Note that if α is a natural transformation, and each α_A is an isomorphism, then the inverses β_A of the α_A also form a natural transformation, because

$$\beta_B \circ Gf = \beta_B \circ Gf \circ \alpha_A \circ \beta_A = \beta_B \circ \alpha_B \circ Ff \circ \beta_A = Ff \circ \beta_A.$$

Example 1.15. Continued from Example 1.13

(b) Let $F\colon \mathbf{Sets} \to \mathbf{Gp}$ be the free group functor, and let $U\colon \mathbf{Gp} \to \mathbf{Set}$ be the forgetful functor. The inclusion of generators $\eta_A\colon A \to UFA$ is the A-component of a natural transformation $1_{\mathbf{Set}} \to UF$.

(c) For any set A, the mapping $a \mapsto \{a\}$ is a function $\{-\}_A\colon A \to P(A)$. We see that $\{-\}$ is a natural transformation $1_{\mathbf{Set}} \to P$, since for any $f\colon A \to B$, we have $Pf(\{a\}) = \{f(a)\}$.

(d) Suppose given two groups G, H and two homomorphisms $f, f'\colon G \to H$. A natural transformation $f \to f'$ is an element $h \in H$ such that $hf(g) = f'(g)h$ for all $g \in G$, or equivalently, $hf(g)h^{-1} = f'(g)$. So such a transformation exists if and only if f and f' are conjugate.

(e) For any space X with a base point x, there's a natural homomorphism $h_{(X,x)}\colon \pi_1(X,x) \to H_1(X)$ called the **Hurewicz homomorphism**. This is the (X,x)-component of the natural transformation h from π_1 to the composite

$$\mathbf{Top}_* \xrightarrow{U} \mathbf{Top} \xrightarrow{H_1} \mathbf{Ab} \xrightarrow{I} \mathbf{Gp},$$

where U is the forgetful functor and I is the inclusion.

It's not often useful to say that functors are injective or surjective on objects. Generally, a functor might output some object which is isomorphic to a bunch of others, but might not actually be surjective – it could be surjective up to isomorphism. This is is similar to the idea that equality is not useful when comparing groups, but rather isomorphism.

Definition 1.16. Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. We say F is

(1) **faithful** if, given $f, g \in \operatorname{mor}\mathbf{C}$, the three equations $\operatorname{dom}(f) = \operatorname{dom}(g)$, $\operatorname{cod}(f) = \operatorname{cod}(g)$, and $Ff = Fg$ imply $f = g$;

(2) **full** if, given $g\colon FA \to FB$ in $\mathbf{D}$, there exists $f\colon A \to B$ in $\mathbf{C}$ with $Ff = g$.

We say a subcategory $\mathbf{C}'$ of $\mathbf{C}$ is **full** if the inclusion functor $\mathbf{C}' \to \mathbf{C}$ is full.

Example 1.17.

(a) **Ab** is a full subcategory of **Gp**;

(b) The category **Lat** of lattices (that is, posets with top element 1, bottom element 0, binary joint $\vee$, binary meet $\wedge$) is a non-full subcategory of **Posets**.

Likewise, equality of categories is a very rigid idea. Isomorphism of categories, as well, is a little bit too rigid. We might have several objects in a category $\mathbf{C}$ which are isomorphic in $\mathbf{C}$ and all mapped to the same object in $\mathbf{D}$ – in this case, we want to consider these categories somehow the same. If we require isomorphism of categories, we cannot insist on even the number of objects being the same. See Example 1.20 for a concrete realization of this.

Definition 1.18. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. An **equivalence of categories** between $\mathbf{C}$ and $\mathbf{D}$ is a pair of functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ together with natural isomorphisms $\alpha\colon 1_{\mathbf{C}} \to GF$, $\beta\colon FG \to 1_{\mathbf{D}}$.

The notation for this is $\mathbf{C} \simeq \mathbf{D}$.

Definition 1.19. We say that a property of categories is a **categorical property** if whenever $\mathbf{C}$ has property P and $\mathbf{C} \simeq \mathbf{D}$, then $\mathbf{D}$ has P as well.

Example 1.20.

(a) Given an object B of a category $\mathbf{C}$, we write $\mathbf{C}/B$ for the category whose objects are morphisms $A \xrightarrow{f} B$ with codomain B, and whose morphisms

$g\colon (A \xrightarrow{f} B) \longrightarrow (A' \xrightarrow{f'} B)$ are commutative triangles

$$\begin{tikzcd}
A \arrow[rr, "g"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\
& B &
\end{tikzcd}$$

The category $\mathbf{Sets}/B$ is equivalent to the category $\mathbf{Sets}^B$ of B-indexed families of sets. In one direction, we send $(A \xrightarrow{f} B)$ to $(f^{-1}(b) \mid b \in B)$, and in the other direction we send $(C_b \mid b \in B)$ to

$$\bigcup_{b \in B} C_b \times \{b\} \xrightarrow{\pi_2} B.$$

Composing these two functors doesn't get us back to where we started, but it does give us something clearly isomorphic.

(b) Let $1/\mathbf{Set}$ be the category of pointed sets (A, a), and let **Part** be the subcategory of **Rel** whose morphisms are partial functions, i.e. relations R such that $(a, b) \in R$ and $(a, b') \in R$ implies $b = b'$.

Then $1/\mathbf{Set} \simeq \mathbf{Part}$: in one direction we send (A, a) to $A \backslash \{a\}$ and $f\colon (A, a) \to (B, b)$ to

$$\{(x, y) \mid x \in A, y \in B, f(x) = y, y \neq b\}$$

In the other direction we send A to $(A \cup \{A\}, A)$ and a partial function $f \cdot A \rightharpoonup B$ (apparently that's the notation for partial functions) to the function $\overline{f}$ defined by

$$\overline{f}(a) = \begin{cases} f(a) & a \in \operatorname{dom} f \\ B & a \in A \backslash \operatorname{dom} f \\ B & a = A \end{cases}$$

(c) The category $\mathbf{fdMod}_k$ of finite dimensional vector spaces over k is equivalent to $\mathbf{fdMod}_k^{\mathrm{op}}$ by the dual functors

$$\mathbf{fdMod}_k \underset{*}{\overset{*}{\rightleftarrows}} \mathbf{fdMod}_k^{\mathrm{op}},$$

and the natural isomorphism $1_{\mathbf{fdMod}_k} \to **$. This is an equivalence but not an isomorphism of categories.

(d) The category $\mathbf{fdMod}_k$ is also equivalent to $\mathbf{Mat}_k$: in one direction send an object of n of $\mathbf{Mat}_k$ to k^n, and a morphism A to the linear map with matrix A relative to the standard basis. In the other direction, send a vector space V to $\dim V$ and choose a basis for each V to send a linear transformation $\theta\colon V \to W$ to the matrix representing θ with respect to the chosen bases.

The composite $\mathbf{Mat}_k \to \mathbf{fdMod}_k \to \mathbf{Mat}_k$ is the identity; the other composite is isomorphic to the identity via the isomorphisms sending the chosen bases to the standard basis of $k^{\dim V}$.

There's another notion slightly weaker than surjectivity of a functor. Some call it "surjective up to isomorphism."

Definition 1.21. We say a functor $F\colon \mathbf{C} \to \mathbf{D}$ is **essentially surjective** if for every object D of $\mathbf{D}$, there exists an object C of $\mathbf{C}$ such that $D \cong F(C)$.

The next lemma somehow uses a more powerful version of the axiom of choice and is beyond usual set theory.

Lemma 1.22. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is part of an equivalence between $\mathbf{C}$ and $\mathbf{D}$ if and only if F is full, faithful, and essentially surjective.

Proof ($\Longrightarrow$). Suppose given $G\colon \mathbf{D} \to \mathbf{C}$, $\alpha\colon 1_{\mathbf{C}} \to GF$ and $\beta\colon FG \to 1_{\mathbf{D}}$ as in the definition of equivalence of categories. Then $B \cong FGB$ for all B, so F is clearly essentially surjective.

Let's prove faithfulness. Now suppose given $f, g\colon A \to B \in \mathbf{C}$ such that $Ff = Fg$. Then $GFf = GFg$. Using the naturality of α, the following diagram commutes:

$$\begin{tikzcd} GFA \arrow[r, "GFf=GFg"] & GFB \\ A \arrow[u, "\alpha_A"] \arrow[r, "f"] & B \arrow[u, "\alpha_B"'] \end{tikzcd} \tag{1}$$

Now

$$\begin{aligned} f &= \alpha_B^{-1}(GFf)\alpha_A \\ &= \alpha_B^{-1}(GFg)\alpha_A \\ &= g, \end{aligned}$$

the last line by the naturality of α with g along the bottom arrow of (1) instead of f. Therefore, f is faithful.

For fullness, suppose given $g\colon FA \to FB$ in $\mathbf{D}$. Define

$$f = \alpha_B^{-1} \circ (Gg) \circ \alpha_A\colon A \to GFA \to GFB \to B.$$

Observe that $Gg = \alpha_B \circ f \circ \alpha_A^{-1}$. Similarly, the following square commutes:

$$\begin{tikzcd} GFA \arrow[r, "GFf"] & GFB \\ A \arrow[u, "\alpha_A"] \arrow[r, "f"] & B \arrow[u, "\alpha_B"'] \end{tikzcd}$$

by the naturality of α. Therefore, $GFf = \alpha_B \circ f \circ \alpha_A^{-1}$ as well. Hence,

$$GFf = \alpha_B \circ f \circ \alpha_A^{-1} = Gg.$$

Applying the argument for faithfulness of F to the functor G shows that G is faithful. Therefore, $GFf = Gg$ implies that $Ff = g$. Hence, the functor F is full.

($\Longleftarrow$). We have to define the functor $G\colon \mathbf{D} \to \mathbf{C}$.

For each object D of $\mathbf{D}$, there is some $C \in \operatorname{ob}\mathbf{C}$ such that $FC \cong D$, because F is essentially surjective. Define $GD = C$ for some choice of C, and also choose an isomorphism $\beta_D \colon FGD \to D$ for each D. This defines G on objects.

To define G on morphisms, suppose we are given $g \colon X \to Y$ in $\mathbf{D}$. Since F is full and faithful, there is a unique $f \colon GX \to GY$ such that $Ff = g$. We could define G in this way, but we won't because we want β to be a natural isomorphism. Instead, we define $Gg \colon GX \to GY$ to be the unique morphism in $\mathbf{C}$ whose image under F is

$$FGX \xrightarrow{\beta_X} X \xrightarrow{g} Y \xrightarrow{\beta_Y^{-1}} FGY.$$

This definition guarantees that β is a natural isomorphism.

To check that G is a functor, we can apply faithfulness of F to assert that $G(g) \circ G(h) = G(gh)$ whenever gh is defined, since these two morphisms of $\mathbf{C}$ have the same image under F. So G is a functor, and β is a natural isomorphism $FG \to 1_{\mathbf{D}}$ by construction.

To define $\alpha_A \colon A \to GFA$, take it to be the unique map in $\mathbf{C}$ such that $F\alpha_A = \beta_{FA}^{-1}$. This is an isomorphism since the unique morphism mapped to β_{FA} is a two-sided inverse for it. We just need to check that α is a natural transformation. Given a (not necessarily commuting) square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha_A"] & B \arrow[d, "\alpha_B"] \\ GFA \arrow[r, "GFf"] & GFB \end{tikzcd} \tag{2}$$

apply F to it to get the square

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "F\alpha_A = \beta_{FA}^{-1}"'] & FB \arrow[d, "F\alpha_B = \beta_{FB}^{-1}"] \\ FGFA \arrow[r, "FGFf"] & FGFB \end{tikzcd}$$

which commutes by the naturality of β. In particular, this gives us the equality

$$\begin{aligned} F(\alpha_B \circ f) &= F\alpha_B \circ Ff \\ &= \beta_{FB}^{-1} \circ Ff \\ &= FGFf \circ \beta_{FA}^{-1} \\ &= FGFf \circ F\alpha_A = F(GFf \circ \alpha_A) \end{aligned}$$

Faithfulness of F then implies that $\alpha_B \circ f = GFf \circ \alpha_A$, so in particular the square (2) commutes. Hence, α is a natural transformation. □

Definition 1.23.

(a) A category $\mathbf{C}$ is **skeletal** if for any isomorphism f in $\mathbf{C}$, $\operatorname{dom}(f) = \operatorname{cod}(f)$;

(b) By a **skeleton** of a category $\mathbf{C}$, we mean a full subcategory $\mathbf{C}'$ containing exactly one object from each isomorphism class of $\mathbf{C}$.

Note that by Lemma 1.22, if $\mathbf{C}'$ is a skeleton of $\mathbf{C}$ then the inclusion functor $\mathbf{C}' \to \mathbf{C}$ is part of an equivalence.

Remark 1.24. More or less any statement you make about skeletons of small categories is equivalent to the (set-theoretic) axiom of choice, including each of the following statements:

(a) Every small category has a skeleton;

(b) Every small category is equivalent to any of it's skeletons;

(c) Any two skeletons of a given small category are isomorphic.

How do we discuss morphisms within a category being surjective and injective? The correct category-theoretic generalization of these notions are epimorphism and monomorphism.

Definition 1.25. A morphism $f \colon A \to B$ is a category $\mathbf{C}$ is

(a) a **monomorphism**, or **monic**, if, given any $g, h \colon C \to A$ with $fg = fh$, we have $g = h$;

(b) an **epimorphism**, or **epic**, if given any $k, \ell \colon B \to D$ with $kf = \ell f$, we have $k = \ell$.

We write $f \colon A \rightarrowtail B$ to indicate that f is monic, and $f \colon A \twoheadrightarrow B$ to indicate that f is epic.

A category $\mathbf{C}$ is called **balanced** if every $f \in \operatorname{mor} \mathbf{C}$ which is both monic and epic is an isomorphism.

Example 1.26.

(a) In **Set**, f is monic if and only if injective, and f is epic if and only if surjective. Therefore, **Set** is a balanced category.

(b) In **Gp**, monomorphisms are injective and epimorphisms are surjective (nontrivial to show that every epimorphism is surjective). So **Gp** is balanced.

(c) In **Rings**, every monomorphism is injective, but not all epimorphisms are surjective. For example, the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both monic and epic, but clearly not an isomorphism.

(d) In **Top**, monic and epic are equivalently injective and surjective. But **Top** isn't balanced.

(e) In a poset, **every** morphism is both monic and epic, so the only balanced posets are discrete ones.

2 The Yoneda Lemma

Remark 2.1. It may seem odd that we're devoting a whole few lectures to just one lemma, but Yoneda's Lemma is really much more. It's an entire way of thinking about category theory! In fact, it's both much more and much less than a lemma.

Remark 2.2. The Yoneda Lemma should probably not be attributed to Yoneda, because he never wrote it down! It was discovered by many category theorists in the early work on the subject, but who first wrote it down we don't know. It's called the Yoneda lemma because Saunders MacLane attributed it to Yoneda in his book, but the paper he cited doesn't contain the lemma! In the next edition, MacLane instead attributed it to private correspondence, which means Yoneda told it to him once while they were waiting for a train.

Definition 2.3. We say a category $\mathbf{C}$ is **locally small** if, for any two objects A, B, the morphisms $A \to B$ in $\mathbf{C}$ form (are parameterized by) a set $\mathbf{C}(A, B)$, which is sometimes written $\mathrm{Hom}_{\mathbf{C}}(A, B)$.

Definition 2.4. Let $\mathbf{C}$ be a locally small category. Given $A \in \mathrm{ob}\,\mathbf{C}$, we have a functor $\mathbf{C}(A, -)\colon \mathbf{C} \to \mathbf{Set}$ sending B to $\mathbf{C}(A, B)$, and a morphism $g\colon B \to C$ to the pullback function $g_*\colon \mathbf{C}(A, B) \to \mathbf{C}(A, C)$ given by $g_*(f) = gf$. This is functorial by associativity of composition in $\mathbf{C}$. This is the **covariant representable functor**.

Similarly we can make $A \mapsto \mathbf{C}(A, B)$ into a functor $\mathbf{C}(-, B)\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, which is called the **contravariant representable functor**.

Lemma 2.5 (Yoneda Lemma)**.** Let $\mathbf{C}$ be a locally small category, let A be an object of C, and let $F\colon \mathbf{C} \to \mathbf{Set}$ be a functor. Then

(i) there is a bijection between natural transformations $\mathbf{C}(A, -) \to F$ and elements of FA; and
(ii) the bijection in (i) is natural in both F and A.

Remark 2.6. Note that we haven't assumed that $\mathbf{C}$ is a small category! So the content of Lemma 2.5 transcends set theory.

Proof of Lemma 2.5.

(i) We have to construct a bijection between the set FA and natural transformations $\mathbf{C}(A, -) \to F$. To that end, given any natural transformation $\alpha\colon \mathbf{C}(A, -) \to F$, we define $\Phi(\alpha) = \alpha_A(1_A)$, which is an element of FA.

Conversely, given $x \in FA$, we define $\Psi(x)\colon \mathbf{C}(A, -) \to F$ by

$$\begin{aligned} \Psi(x)_B\colon \mathbf{C}(A, B) &\longrightarrow FB \\ f &\longmapsto Ff(x) \end{aligned}$$

To verify that $\Psi(x)$ is genuinely a natural transformation, let $g\colon B \to C$ be a morphism in $\mathbf{C}$, and then consider the diagram:

$$\begin{tikzcd} \mathbf{C}(A,B) \arrow[r, "\mathbf{C}(A,g)"] \arrow[d, "\Psi(x)_B"] & \mathbf{C}(A,C) \arrow[d, "\Psi(X)_C"] \\ FB \arrow[r, "Fg"] & FC \end{tikzcd} \tag{3}$$

Let $f \in \mathbf{C}(A,B)$. Then going around the diagram (3) clockwise,

$$\Psi(x)_C(\mathbf{C}(A,g)(f)) = \Psi(x)_C(g \circ f) = F(g \circ f)(x)$$

and going around (3) counterclockwise,

$$F(g) \circ (\Psi(x)_B(f)) = F(g) \circ F(f)(x) = F(g \circ f)(x).$$

This verifies that $\Psi(x)$ is a natural transformation, for each $x \in FA$.

We should check that Ψ and Φ are inverses. So:

$$\Phi\Psi(x) = \Psi(x)(1_A) = F(1_A)(x) = 1_{FA}(x) = x.$$

$$\begin{aligned} \Psi(\Phi(\alpha))_B(f) &= F(f)(\Phi(\alpha)) \\ &= F(f)(\alpha_A(1_A)) \\ &= \alpha_B(\mathbf{C}(A,f)(1_A)) \\ &= \alpha_B(f \circ 1_A) \\ &= \alpha_B(f) \end{aligned}$$

for all B and $f\colon A \to B$, so $\Psi(\Phi(\alpha)) = \alpha$. Therefore, Φ and Ψ are inverse.

(ii) If $\mathbf{C}$ is small (so that $[\mathbf{C},\mathbf{Set}]$ is locally small) then we have two functors:

$$\begin{array}{rcl} \mathbf{C} \times [\mathbf{C},\mathbf{Set}] & \longrightarrow & \mathbf{Set} \\ (A,F) & \longmapsto & FA \end{array}$$

$$\begin{array}{rcl} \mathbf{C} \times [\mathbf{C},\mathbf{Set}] & \longrightarrow & \mathbf{Set} \\ (A,F) & \longmapsto & [\mathbf{C},\mathbf{Set}](\mathbf{C}(A,-),F) \end{array}$$

where $[\mathbf{C},\mathbf{Set}](\mathbf{C}(A,-),F)$ is a confusing notation for the set of natural transformations between $\mathbf{C}(A,-)$ and F. The assertion of (ii) is that Φ is a natural isomorphism among these functors.

For naturality in A, suppose given $f\colon A' \to A$. We want to show that the following square commutes:

$$\begin{tikzcd} {[\mathbf{C},\mathbf{Set}](\mathbf{C}(A',-),F)} \arrow[r, "\Theta"] \arrow[d, "\Phi_{A'}"] & {[\mathbf{C},\mathbf{Set}](\mathbf{C}(A',-),F)} \arrow[d, "\Phi_A"] \\ FA' \arrow[r, "Ff"] & FA \end{tikzcd} \tag{4}$$

where $\Theta(\alpha) = \alpha \circ \mathbf{C}(f, -)$.

Suppose we are given $\alpha\colon \mathbf{C}(A', -) \to F$. We will chase the image of α around the diagram (4) in two different ways, and show they are equal. Going counterclockwise,

$$Ff(\Phi(\alpha)) = Ff(\alpha_{A'}(1_{A'})) = \alpha_A(f)$$

and going clockwise,

$$\Phi(\Theta(\alpha)) = \Phi(\alpha \circ \mathbf{C}(f, -)) = \alpha_A(\mathbf{C}(f, -)_A(1_{A'})) = \alpha_A(f)$$

This verifies that Φ is natural in A.

To show that Φ is natural in F, suppose given a natural transformation $\eta\colon F \to G$. We want to show that the following diagram commutes:

$$\begin{tikzcd}
{[\mathbf{C}, \mathbf{Set}](\mathbf{C}(A, -), F)} \arrow[r, "\eta\circ -"] \arrow[d, "\Phi_F"] & {[\mathbf{C}, \mathbf{Set}](\mathbf{C}(A, -), G)} \arrow[d, "\Phi_G"] \\
FA \arrow[r, "\eta_A"] & GA
\end{tikzcd} \tag{5}$$

Once again, let $\alpha\colon \mathbf{C}(A, -) \to F$ and chase the diagram (5) around counterclockwise

$$\eta_A(\Phi_F(\alpha)) = \eta_A(\alpha_A(1_A))$$

and clockwise

$$\Phi_G(\eta \circ \alpha) = (\eta \circ \alpha)_A(1_A) = \eta_A(\alpha_A(1_A))$$

to see that it commutes. Hence, Φ is natural in F as well. □

Corollary 2.7. The functor $Y\colon \mathbf{C}^{\mathrm{op}} \to [\mathbf{C}, \mathbf{Set}]$ given by $A \mapsto \mathbf{C}(A, -)$ is full and faithful. Hence, every locally small category is equivalent to a subcategory of a functor category $[\mathbf{C}, \mathbf{Set}]$.

Proof. Note that for $f\colon B \to A$, $Y(f) = \mathbf{C}(f, -)$.

By Lemma 2.5(i), natural transformations $\mathbf{C}(A, -) \to \mathbf{C}(B, -)$ correspond bijectively to elements of $\mathbf{C}(B, A)$. But it's not clear that this bijection comes from the map Y; if it does, then Y is full and faithful. So we want to show that Y is the inverse of $\Phi\colon [\mathbf{C}, \mathbf{Set}](\mathbf{C}(A, -), \mathbf{C}(B, -)) \to \mathbf{C}(B, A)$, where Φ is the natural transformation as in Lemma 2.5

To that end, let $\alpha\colon \mathbf{C}(A, -) \to \mathbf{C}(B, -)$ be a natural transformation. Then

$$Y(\Phi(\alpha)) = Y(\alpha_A(1_A)) = \mathbf{C}(\alpha_A(1_A), -)$$

is a natural transformation. Now for any $g\colon A \to C$,

$$Y(\Phi(\alpha))_C(g) = \mathbf{C}(\alpha_A(1_A), C)(g) = g \circ \alpha_A(1_A). \tag{6}$$

Because α is a natural transformation, the following square commutes.

$$\begin{tikzcd} \mathbf{C}(A,A) \arrow[r, "\alpha_A"] \arrow[d, "g\circ-"] & \mathbf{C}(B,A) \arrow[d, "g\circ-"] \\ \mathbf{C}(A,C) \arrow[r, "\alpha_C"] & \mathbf{C}(B,C) \end{tikzcd}$$

Chasing this diagram starting with 1_A in the top-left, we see that

$$g \circ \alpha_A(1_A) = \alpha_C(g \circ 1_A) = \alpha_C(g)$$

Therefore, substituting into (6), we have

$$Y(\Phi(\alpha))_C(g) = g \circ \alpha_A(1_A) = \alpha_C(g).$$

This holds for any C and any $g\colon A \to C$, so it follows that $Y(\Phi(\alpha)) = \alpha$, so Y is a left-inverse to Φ.

Since Φ is a bijection, this means that Y is a right-inverse to Φ as well, and therefore Y is also a bijection between $\mathbf{C}(B,A)$ and natural transformations $\mathbf{C}(A,-) \to \mathbf{C}(B,-)$. □

Definition 2.8. We call this functor $Y\colon \mathbf{C}^{\mathrm{op}} \to [\mathbf{C}, \mathbf{Set}]$ the **Yoneda Embedding**.

This is not unlike the Cayley representation theorem in group theory, which says that every finite group is a subgroup of a symmetric group. In fact, the Cayley representation theorem is just a special case of the Yoneda embedding!

Definition 2.9. We say a functor $\mathbf{C} \to \mathbf{Set}$ is **representable** if it's isomorphic to $\mathbf{C}(A,-)$ for some A. By a **representation** of $F\colon \mathbf{C} \to \mathbf{Set}$, we mean a pair (A,x) with $x \in FA$ such that $\Psi(x)$ is a natural isomorphism $\mathbf{C}(A,-) \to F$. We also call x a **universal element** of F.

Corollary 2.10 (Representations are unique up to unique isomorphism)**.** If (A,x) and (B,y) are both representations of F, then there's a unique isomorphism $f\colon A \to B$ such that $Ff(x) = y$.

Proof. The composite $\mathbf{C}(B,-) \xrightarrow{\Psi(y)} F \xrightarrow{\Psi(x)^{-1}} \mathbf{C}(A,-)$ is an isomorphism, so it's of the form $\mathbf{C}(f,-)$ for a unique isomorphism $f\colon A \to B$ by Corollary 2.7. So the following diagram commutes.

$$\begin{tikzcd} \mathbf{C}(B,-) \arrow[rr, "\mathbf{C}(f,-)"] \arrow[dr, "\Psi(y)"'] & & \mathbf{C}(A,-) \arrow[dl, "\Psi(x)"] \\ & F & \end{tikzcd}$$

In particular, plug in B to this diagram

$$\begin{tikzcd} \mathbf{C}(B,B) \arrow[rr, "\mathbf{C}(f,B)"] \arrow[dr, "\Psi(y)_B"'] & & \mathbf{C}(A,B) \arrow[dl, "\Psi(x)_B"] \\ & FB & \end{tikzcd}$$

and chase $1_B \in \mathbf{C}(B,B)$ around the diagram in two ways:

$$(\Psi(x)_B \circ \mathbf{C}(f,B))(1_B) = \Psi(x)_B(1_B \circ f) = \Psi(x)_B(f) = Ff(x)$$
$$(\Psi(x)_B \circ \mathbf{C}(f,B))(1_B) = \Psi(y)_B(1_B) = F(1_B)(y) = 1_{FB}(y) = y$$

Therefore, $Ff(x) = y$. □

Often we will abuse terminology and talk about *the* representation of a functor, but this is okay by Corollary 2.10, because any two representations are uniquely isomorphic. Representable functors appear everywhere, as the next example shows.

Example 2.11.

(a) The forgetful functor **Gp** → **Set** is representable by $(\mathbb{Z}, 1)$;

(b) The forgetful functor **Ring** → **Set** is representable by $(\mathbb{Z}[x], x)$;

(c) The forgetful functor **Top** → **Set** is representable by $(\{*\}, *)$.

(d) The contravariant power-set functor $P^*\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is representable by $(\{0,1\}, \{1,\})$ since there's a natural bijection between subsets $A' \subseteq A$ and functions $\chi_A\colon A \to \{0,1\}$.

(e) The dual-vector-space functor k-$\mathbf{Mod}^{\mathrm{op}} \to k$-**Mod**, when composed with the forgetful functor k-**Mod** → **Set**, is representable by $(k, 1_k)$.

(f) For a group G, the unique (up to representation) representable functor $G \to \mathbf{Set}$ is the left-regular representation of G, that is, G acts on the set G with left-multiplication. This is the Cayley representation theorem of group theory. Note that the endomorphisms of this object of $[G, \mathbf{Set}]$ are just the right multiplications $h \mapsto hk$ for some fixed $k \in G$. So they form a group isomorphic to G^{op}.

Definition 2.12. Given two objects A, B of a locally small category **C**, we can form the functor $\mathbf{C}(-,A) \times \mathbf{C}(-,B)\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. A representation of this functor is a (categorical) **product** of A and B.

What does this look like? It consists of an object $A \times B$ and two maps $\pi_1\colon A \times B \to A$, and $\pi_2\colon A \times B \to B$ such that, given any pair $(f\colon C \to A, g\colon C \to B)$, there's a unique $\langle f, g\rangle\colon C \to A \times B$ such that $\pi_1\langle f,g\rangle = f$ and $\pi_2\langle f,g\rangle = g$.

Definition 2.13. Given a parallel pair of maps $A \underset{g}{\overset{f}{\rightrightarrows}} B$, the assignment $\mathcal{E}(\mathbf{C}) = \{h\colon C \to A \mid fh = gh\}$. defines a subfunctor $\mathcal{E}$ of $\mathbf{C}(-,A)$. A representation of $\mathcal{E}$, if it exists, is called an **equalizer** of f and g: it consists of an object E and a morphism $e\colon E \to A$ such that $fe = ge$ such that every $h\colon C \to A$ with $fh = gh$ factors uniquely through e.

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
C \arrow[u, dashed] \arrow[ur, "h"'] & &
\end{tikzcd}$$

Remark 2.14. If $e\colon E \to A$ is an equalizer of the two maps $f, g\colon A \to B$, then it's necessarily monic, since any morphism $h\colon C \to A$ factors through e in at most one way. In particular, given $a, b\colon C \to E$ such that $ea = eb$, we have that ea factors through e in at most one way. One such way that it does factor is as written: $e \circ a$. Another such way is $e \circ b$, but the way it factors must be unique so it must be that $a = b$.

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \\ C \arrow[u, shift left, "a"] \arrow[u, shift right, "b"'] & & \end{tikzcd}$$

Definition 2.15. We call a monomorphism $f\colon A \rightarrowtail B$

(i) **regular** if it occurs as an equalizer;

(ii) **split** if there is $g\colon B \to A$ with $gf = 1_A$.

Lemma 2.16.

(i) A split monomorphism is regular monic.

(ii) A morphism which is epic and regular monic is an isomorphism.

Proof. (i) Let f be split monic with left-inverse g. Claim that f is the equalizer of fg and 1_B. If $gf = 1_A$ then $fgf = f1_A = 1_B f$. And if $h\colon C \to B$ satisfies $fgh = 1_B h$, then h factors through f via gh. If $h = fk$ is another such factorization, then $fgh = h = fk$ implies that $k = gh$ because f is monic, so the factorization is unique. Hence f is an equalizer.

$$\begin{tikzcd} A \arrow[r, tail, "f"] & B \arrow[r, shift left, "fg"] \arrow[r, shift right, "1_B"'] & B \\ C \arrow[u, "gh"] \arrow[ur, "h"'] & & \end{tikzcd}$$

(ii) Suppose $e\colon E \to A$ is epic and an equalizer of the maps $f, g\colon A \to B$. Then $ef = eg$, which means that $f = g$ because e is epic. We know that 1_A is another map such that $f1_A = g1_A$, so the map 1_A factors through e as $ek = 1_A$ for some map $k\colon A \to E$. Therefore, e has a right-inverse and so is split epic.

The dual of statement (i) shows that a split epimorphism is a regular epimorphism, so e is both monic and regular epic. The dual of the argument in the previous paragraph then produces a left-inverse $\ell\colon A \to E$ such that $\ell e = 1_E$, and moreover

$$\ell = \ell 1_A = \ell e k = 1_E k = k$$

so the inverses are equal. Hence, e is an isomorphism. □

Definition 2.17. Let $\mathbf{C}$ be a category and let $\mathcal{G}$ be a collection of objects in $\mathbf{C}$.

(a) We say $\mathcal{G}$ is a **separating family** if, whenever we have that $f, g\colon A \to B$ in $\mathbf{C}$ such that $fh = gh$ for all $h\colon G \to A$ with $G \in \mathcal{G}$, then $f = g$.

(b) We say $\mathcal{G}$ is a **detecting family** if given $f\colon A \to B$ such that every morphism $h\colon G \to B$ with $G \in \mathcal{G}$ factors uniquely through f, then f is an isomorphism.

(c) If $\mathcal{G} = \{G\}$ is a singleton, we call G a **separator** or **detector** for $\mathbf{C}$, depending on which case we're in.

Remark 2.18. Here is an equivalent definition of separating and detecting families for a locally small category $\mathbf{C}$. A family $\mathcal{G}$ is separating if and only if the collection of functors $\{\mathbf{C}(G, -) \mid G \in \mathcal{G}\}$ are jointly faithful and $\mathcal{G}$ is detecting if and only if $\{\mathbf{C}(G, -) \mid G \in \mathcal{G}\}$ jointly reflect isomorphisms.

Lemma 2.19.

(i) If $\mathbf{C}$ has equalizers, then every detecting family for $\mathbf{C}$ is separating.

(ii) If $\mathbf{C}$ is balanced (mono + epi $\implies$ iso), then every separating family is detecting.

Proof.

(i) Suppose that $\mathcal{G}$ is a detecting family and we have maps $f, g\colon A \to B$ such that if $fh = gh$ for all $h\colon G \to A$ for all $G \in \mathcal{G}$, then $f = g$. Let $e\colon E \to A$ be an equalizer of f and g; then every $h\colon G \to A$ with $G \in \mathcal{G}$ factors uniquely through e, so e is an isomorphism and therefore $f = g$.

$$\begin{tikzcd} & G \arrow[dl, dashed] \arrow[d, "h"] & \\ E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

(ii) Suppose $\mathcal{G}$ is a separating family and $f\colon A \to B$ satisfying the hypotheses of Definition 2.17(b). If $k, \ell\colon B \to C$ satisfy $kf = \ell f$, then $kh = \ell h$ for all $h\colon G \to B$ with $G \in \mathcal{G}$, so $k = \ell$. Hence, f is epic.

Similarly, if $p, q\colon D \to A$ satisfy $fp = fq$, then for any $n\colon G \to D$ we have $fpn = fqn$, so both pn and qn are factorizations of fpn through f, and hence equal. Because $\mathcal{G}$ is a separating family, this means in turn that $p = q$. Hence, f is monic.

By the assumption that $\mathbf{C}$ is balanced, f is therefore an isomorphism. □

Example 2.20.

(a) ob $\mathbf{C}$ is always both separating and detecting.

(b) If $\mathbf{C}$ is locally small, then $\{\mathbf{C}(A, -) \mid A \in \operatorname{ob}\mathbf{C}\}$ is both separating and detecting for $[\mathbf{C}, \mathbf{Set}]$.

(c) $\mathbb{Z}$ is a separator and a detector for **Gp**. The functor that it represents is the forgetful functor $\mathbf{Gp} \to \mathbf{Set}$, and that functor is faithful and respects isomorphisms.

(d) In all sensible algebraic categories, the free object on one generator is both separator and detector.

(e) $\{*\}$ is a separator, but not a detector, for **Top**. In fact, **Top** has no detecting family since for any cardinal κ, we can find a set X and topologies $\tau_1 \subsetneq \tau_2$ on X that agree on any subset of X with cardinality less than κ. Also $\mathbf{Top}^{\mathrm{op}}$ has a detector, namely $X = \{x, y, z\}$ with topology $\{X, \varnothing, \{x, y\}\}$.

Definition 2.21. We say an object P is **projective** in $\mathbf{C}$ if, whenever we're given

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[d, "f"] \\ A \arrow[r, two heads, "e"] & B \end{tikzcd}$$

with e epic, then there is $g\colon P \to A$ with $eg = f$. Dually, P is **injective** if it's projective on $\mathbf{C}^{\mathrm{op}}$.

If the condition holds not for all epimorphisms e but for some class $\mathcal{E}$ of epis, we say that P is **$\mathcal{E}$-projective**. (For example, if $\mathcal{E}$ is the regular epimorphisms, then P is **regular-projective**).

This definition generalizes the algebraic notion of projective objects.

Lemma 2.22. Representable functors are projective as elements of the functor category. More precisely, for any locally small $\mathbf{C}$, the functors $\mathbf{C}(A, -)$ are all $\mathcal{E}$-projective in $[\mathbf{C}, \mathbf{Set}]$, where $\mathcal{E}$ is the class of pointwise-surjective natural transformations.

We can't prove this for $\mathcal{E}$ the set of all epimorphisms of $[\mathbf{C}, \mathbf{Set}]$ yet, because we don't know what epimorphisms are in this category. But it will turn out that they're exactly the pointwise-surjective natural transformations, so what we prove below suffices.

Proof of Lemma 2.22. Use the Yoneda lemma. Given the diagram

$$\begin{tikzcd} & \mathbf{C}(A, -) \arrow[d, "\alpha"] \\ F \arrow[r, two heads, "\beta"] & G \end{tikzcd}$$

let $y \in GA$ correspond to α. Then there is $x \in FA$ with $\beta_A(x) = y$, and the corresponding $\Psi(x)\colon \mathbf{C}(A, -) \to F$ satisfies $\beta \circ \Psi(x) = \alpha$. □

Making the analogy with algebra, this says that the functors $\mathbf{C}(A, -)$ are kind of like the free objects of $[\mathbf{C}, \mathbf{Set}]$ (insofar as free objects are projective).

3 Adjunctions

This theory was first developed by D.M. Kan in the paper *Adjoint Functors* which appeared in TAMS in 1958. The real difficulty in formalizing the idea that had been around for a few years was finding the right level of abstraction: maximally useful but sufficiently general. The definition we give is the one that Kan gave in that paper.

Definition 3.1. Let $\mathbf{C}$ and $\mathbf{D}$ be categories and $F\colon \mathbf{C} \to \mathbf{D}$, $G\colon \mathbf{D} \to \mathbf{C}$ be two functors. An **adjunction** between F and G is a natural bijection between morphism $FA \to B$ in $\mathbf{D}$ and morphisms $A \to GB$ in $\mathbf{C}$, for all $A \in \operatorname{ob} \mathbf{C}$ and $B \in \operatorname{ob} \mathbf{D}$.

If $\mathbf{C}$ and $\mathbf{D}$ are locally small, this is a natural isomorphism between the functors $\mathbf{D}(F(-), -)$ and $\mathbf{C}(-, G(-))$. These are both functors $\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$.

Notice that the definition of adjunction has a definite direction: we say that F is **left-adjoint** to G or that G is **right-adjoint** to F, and write $F \dashv G$.

Example 3.2. Examples of Adjunctions

(a) The free group functor $F\colon \mathbf{Set} \to \mathbf{Gp}$ is left-adjoint to the forgetful functor $U\colon \mathbf{Gp} \to \mathbf{Set}$. For any set A and any group G, each function $A \to UG$ extends to a unique homomorphism $FA \to G$, and this correspondence is natural.

(b) The forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$ has a left adjoint D, where DA is A with the discrete topology. Any function $A \to UX$ becomes continuous as a map $DA \to X$. U also has a right adjoint I, which is $IA = A$ with the indiscrete topology $\{A, \varnothing\}$. Any map into an indiscrete space is continuous, so any map $UX \to A$ is continuous as a map $X \to IA$. So we have adjunctions $D \dashv U \dashv I$.

(c) Consider the functor $\operatorname{ob}\colon \mathbf{Cat} \to \mathbf{Set}$. This has a left adjoint given by the discrete category functor $D\colon \mathbf{Set} \to \mathbf{Cat}$, where DA is the discrete category with the objects same as elements of A; any mapping $A \to \operatorname{ob} \mathbf{C}$ defines a unique functor $DA \to \mathbf{C}$. In particular, $D \dashv \operatorname{ob}$.

    It also has a right adjoint I, which is the indiscrete functor. IA has objects those elements of A with one morphism $a \to b$ for each $(a, b) \in A \times A$. Again, a functor $\mathbf{C} \to IA$ is determined by its effect on objects. $\operatorname{ob} \dashv I$

    D has a left adjoint the connected components functor π_0, where $\pi_0 \mathbf{C} = (\operatorname{ob} C)/\sim$ is the smallest equivalence relation such that $U \sim V$ whenever there exists $f\colon U \to V$ in $\mathbf{C}$. Any functor $\mathbf{C} \to DA$ is constant on each $\sim$-equivalence class, but any function $\pi_0 \mathbf{C} \to A$ can occur.

(d) Let **Idem** be the category whose objects are pairs (A, e), for some set A and idempotent $e\colon A \to A$, $e^2 = e$. The morphisms $f\colon (A, e) \to (A', e')$ in this category are those functions satisfying $e'f = fe$.

Let $G\colon \mathbf{Idem} \to \mathbf{Set}$ be the functor sending (A, e) to $\{a \in A \mid e(a) = a\}$ and a morphism f to its restriction $f|_{G(A,e)}$. Let $F\colon \mathbf{Set} \to \mathbf{Idem}$ be the functor sending A to (A, id_A).

Now $F \dashv G$ since any morphism $FA \to (B, e)$ takes values in $G(B, e)$. And furthermore, $G \dashv F$ since any morphism $(A, e) \xrightarrow{f} FB$ is determined by its effect on $G(A, e)$, since $f(a) = f(e(a))$ for all $a \in G(A, e)$.

(e) For any $\mathbf{C}$, there's a unique functor $\mathbf{C} \to 1$ (where 1 is the category with one object and one morphism). A left adjoint for this specifies an object I of $\mathbf{C}$ such that there's a unique morphism $I \to A$ for any A, i.e. an **initial object** of $\mathbf{C}$. Dually, a right adjoint to $\mathbf{C} \to 1$ specifies a **terminal object**.

(f) Let A and B be sets and $f\colon A \to B$. Then we have order-preserving maps $Pf\colon PA \to PB$ and $P^*f\colon PB \to PA$ and $Pf \dashv P^*f$ since $Pf(A') \subseteq B'$ if and only if $\forall x \in A', f(x) \in B'$ if and only if $A \subseteq P^*f(B')$.

(g) Suppose given two sets A, B and a relation $R \subseteq A \times B$. We define order-reversing maps

$$L\colon PB \to PA \text{ by } L(B') = \{a \in A \mid \forall b \in B', (a, b) \in \mathbb{R}\}$$

$$R\colon PA \to PB \text{ by } R(B') = \{b \in B \mid \forall a \in A', (a, b) \in \mathbb{R}\}$$

For any subsets A', B' we have $A' \subseteq L(B') \Leftrightarrow A' \times B' \subseteq R \Leftrightarrow B' \subseteq R(A')$.

We say a pair (F, G) of contravariant $\mathbf{C} \leftrightarrow \mathbf{D}$ are **adjoint on the right** if $F\colon \mathbf{C} \to \mathbf{D}^{\mathrm{op}} \dashv G\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{C}$.

(h) The contravariant power-set functor P^* is self-adjoint on the right, since functions $A \to PB$ correspond to subsets $R \subseteq A \times B$, and hence to functions $B \to PA$.

Given a functor $G\colon \mathbf{D} \to \mathbf{C}$, we want to know if it possibly has a left adjoint. To that end, we define a category $(A \downarrow G)$ called the comma category (because it was originally written (A, G) with a comma instead of an arrow)

Definition 3.3. The **comma/arrow category** $(A \downarrow G)$ has objects all pairs (B, f) with $B \in \mathrm{ob}\,\mathbf{D}$ and $f\colon A \to GB$. The morphisms $(B, f) \to (B', f')$ are morphisms $g\colon B \to B'$ in $\mathbf{D}$ which make the diagram below commute.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ GB \arrow[rr, "Gg"] & & GB' \end{tikzcd}$$

Theorem 3.4. Specifying a left-adjoint for $\mathbf{G}$ is equivalent to specifying an initial object of $(A \downarrow G)$ for each $A \in \mathrm{ob}\,\mathbf{C}$.

Proof. ($\Longrightarrow$) Suppose $F \dashv G$. For any A, $FA \xrightarrow{1_{FA}} FA$ corresponds to a morphism $\eta_A \colon A \to GFA$. I claim that (FA, η_A) is initial in $(A \downarrow G)$. Given an object (B, f) in $(A \downarrow G)$, the assertion that

$$\begin{tikzcd} & A \arrow[dl, "\eta_A"'] \arrow[dr, "f"] & \\ GFA \arrow[rr, "Gg"] & & GB \end{tikzcd}$$

commutes is equivalent to saying that

$$\begin{tikzcd} & FA \arrow[dl, "1_{FA}"'] \arrow[dr, "\overline{f}"] & \\ FA \arrow[rr, "g"] & & B \end{tikzcd}$$

commutes, where $\overline{f}$ corresponds to f under the adjunction. So the unique morphism $(FA, \eta_A) \to (B, f)$ is $\overline{f}$. $\square$

Last time we were in the middle of proving Theorem 3.4. We proved one direction last time, so let's start by finishing the proof.

Proof of Theorem 3.4, continued. ($\Longleftarrow$). Suppose given an initial object (B_A, η_A) in $(A \downarrow G)$, for each A. Define $FA = B_A$ for all objects A of $\mathbf{C}$. To define F on morphisms, let $f \colon A \to A'$ in $\mathbf{C}$. We define $Ff \colon FA \to FA'$ to be the unique morphism $FA \to FA'$ making the following commute.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"] & GFA \arrow[d, "GFf"] \\ A' \arrow[r, "\eta_{A'}"] & GFA' \end{tikzcd}$$

The uniqueness of this morphism comes from the fact that (FA, η_A) is initial. This uniqueness ensures that F is functorial: if $f' \colon A' \to A''$, then $(Ff)(Ff')$ and $F(ff')$ both fit into the same naturality square for η, and so by the uniqueness of morphisms that fit into this square, they're equal.

Furthermore, by construction, this makes $\eta \colon 1 \to GF$ a natural transformation.

Given a morphism $y \colon A \to GB$, we know that there is a unique map $x \colon FA \to B$ such that the following commutes:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "y"'] & GFA \arrow[d, "Gx"] \\ & GB \end{tikzcd}$$

In particular, this gives a bijection between morphisms $x \colon FA \to B$ and morphisms $y \colon A \to GB$ sending $x \colon FA \to B$ to $\Phi(x) = (Gh)\eta_A$. To show this is natural in B, suppose given $g\colon B \to B'$. We want to show the following commutes.

$$\begin{tikzcd} \mathbf{D}(FA,B) \arrow[r, "\Phi"] \arrow[d, "g\circ-"] & \mathbf{C}(A,GB) \arrow[d, "Gg\circ-"] \\ \mathbf{D}(FA,B) \arrow[r, "\Phi"] & \mathbf{C}(A,GB) \end{tikzcd}$$

But this follows because for any $h\colon FA \to B$, we have

$$\Phi(gh) = G(gh)\eta_A = (Gg)(Gh)\eta_A = (Gg)\Phi(h).$$

Naturality in A follows from the fact that η is natural. More precisely, given $f\colon A' \to A$, we want to show that the following commutes

$$\begin{tikzcd} \mathbf{D}(FA,B) \arrow[r, "\Phi"] \arrow[d, "-\circ Ff"] & \mathbf{C}(A,GB) \arrow[d, "-\circ f"] \\ \mathbf{D}(FA',B) \arrow[r, "\Phi"] & \mathbf{C}(A',GB) \end{tikzcd}$$

But this is true because for any $h\colon FA \to B$, we have

$$\Phi(h\circ Ff) = G(h\circ Ff)\circ\eta_A = G(h)\circ GF(f)\circ\eta_A = G(h)\circ\eta_A\circ f = \Phi(h)\circ f.$$

□

This way of thinking about adjoints turns out to be quite useful.

Corollary 3.5. If F and F' are both left adjoints for G, then F is naturally isomorphic to F'.

Proof. Let η_A be the map $A \to GFA$ that corresponds to $\mathrm{id}_{FA}\colon FA \to FA$, and likewise let η'_A be the map $A \to GF'A$ that corresponds to $\mathrm{id}_{F'A}\colon F'A \to F'A$. Then (FA,η_A) and $(F'A,\eta'_A)$ are both initial in $(A \downarrow G)$, which means that they must be isomorphic as objects in this category. hence, there is a unique isomorphism $\alpha_A\colon (FA,\eta_A) \to (F'A,\eta'_A)$ that is natural in A: given $f\colon A \to A'$, then there two ways around the naturality square

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"] & F'A \arrow[d, "F'f"] \\ FA' \arrow[r, "\alpha_{A'}"] & F'A' \end{tikzcd}$$

are both morphisms $(FA,\eta_A) \to (F'A', G(\alpha_{A'})\circ\eta_{A'}\circ f)$ in $(A \downarrow G)$, and this morphism is unique because (FA,η_A) is initial. □

Lemma 3.6. Given functors

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "F"] & \mathbf{D} \arrow[l, shift left, "G"] \arrow[r, shift left, "H"] & \mathbf{E} \arrow[l, shift left, "K"] \end{tikzcd}$$

with $F \dashv G$ and $H \dashv K$, we have $HF \dashv GK$.

Proof. Given $A \in \operatorname{ob}\mathbf{C}$, $C \in \operatorname{ob}\mathbf{E}$, we have bijections $\mathbf{E}(HFA, C) \longleftrightarrow \mathbf{D}(FA, KC)$ and $\mathbf{D}(FA, KC) \longleftrightarrow \mathbf{C}(A, GKC)$. Compose these bijections to get the result. □

Corollary 3.7. Suppose given a commutative square of categories and functors

$$\begin{tikzcd} \mathbf{C} \arrow[r, "F"] \arrow[d, "H"] & \mathbf{D} \arrow[d, "G"] \\ \mathbf{E} \arrow[r, "K"] & \mathbf{F} \end{tikzcd}$$

in which all the functors have left adjoints. Then the diagram of left-adjoints commutes up to natural isomorphism.

Proof. The two ways round it are both left-adjoint to $GF = KH$ by Lemma 3.6, so they're isomorphic by Corollary 3.5. □

Example 3.8. A functor with a right adjoint preserves initial objects, if they exist. If $F\colon \mathbf{C} \to \mathbf{D} \dashv G\colon \mathbf{D} \to \mathbf{C}$, then the diagram below commutes.

$$\begin{tikzcd} \mathbf{D} \arrow[rr, "G"] \arrow[dr] & & \mathbf{C} \arrow[dl] \\ & 1 & \end{tikzcd}$$

But a left adjoint for $\mathbf{C} \to 1$ picks out an initial object of $\mathbf{C}$ by Example 3.2(e). So F maps it to an initial object of $\mathbf{D}$.

Theorem 3.9. Suppose given $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$. Specifying an adjunction $F \dashv G$ is equivalent to specifying two natural transformations $\eta\colon 1_{\mathbf{C}} \to GF$ and $\varepsilon\colon FG \to 1_{\mathbf{D}}$ such that

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\varepsilon_F"] \\ & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd} \tag{7}$$

both commute. η and ε are called the **unit** and **counit** of the adjunction, and the two diagrams in Equation 7 are called the **triangular identities**.

Proof. ($\Longrightarrow$). Given $F \dashv G$, with a natural bijection $\Theta_{A,B}\colon \mathbf{D}(FA, B) \to \mathbf{C}(A, GB)$, define

$$\eta_A = \Theta_{A,FA}(1_{FA}) \qquad \text{and} \qquad \varepsilon_B = \Theta^{-1}_{GB,B}(1_{GB})$$

We want to show that both η and ε are natural in A and B, respectively. Note that the definition of ε is dual to η, so it suffices to check that η is natural and the naturality of ε will follow dually.

To check the naturality of η, suppose given $f\colon A \to A'$. We want to show that

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"] & GFA \arrow[d, "GFf"] \\ A' \arrow[r, "\eta_{A'}"] & GFA' \end{tikzcd}$$

commutes. We can use the naturality of Θ to show this. In particular, consider the following diagram

$$\begin{tikzcd}
\mathbf{D}(FA, FA) \arrow[r, "\Theta_{A,FA}"] \arrow[d, "Ff\circ-"] & \mathbf{C}(A, GFA) \arrow[d, "GFf\circ-"] \\
\mathbf{D}(FA, FA') \arrow[r, "\Theta_{A,FA'}"] & \mathbf{C}(A, GFA')
\end{tikzcd}$$

which commutes by the naturality of Θ. Chase 1_{FA} around the diagram starting in the upper left to see that $\Theta_{A,FA'}(Ff) = GF(f)$.

$$\begin{tikzcd}
1_{FA} \arrow[r, mapsto] \arrow[d, mapsto] & \Theta_{A,FA}(1_{FA}) = \eta_A \arrow[d, mapsto] \\
Ff \circ 1_{FA} \arrow[r, mapsto] & \Theta_{A,FA'}(Ff) = GF(f) \circ \eta_A
\end{tikzcd} \tag{8}$$

Now what is $\Theta_{A,FA'}(Ff)$? To answer this question, consider the diagram

$$\begin{tikzcd}
\mathbf{D}(FA, FA') \arrow[r, "\Theta_{A,FA'}"] & \mathbf{C}(A, GFA') \\
\mathbf{D}(FA', FA') \arrow[r, "\Theta_{A',FA'}"] \arrow[u, "-\circ Ff"] & \mathbf{C}(A', GFA') \arrow[u, "-\circ f"']
\end{tikzcd}$$

which again commutes by naturality of Θ. Now chase $1_{FA'}$ around this diagram, starting in the lower left.

$$\begin{tikzcd}
1_{FA'} \circ Ff \arrow[r, mapsto] & \Theta_{A,FA'}(Ff) = \eta_{A'} \circ f \\
1_{FA'} \arrow[r, mapsto] \arrow[u, mapsto] & \eta_{A'} = \Theta_{A',FA'}(1_{FA'}) \arrow[u, mapsto]
\end{tikzcd} \tag{9}$$

Therefore, combining (8) and (9), we get that η_A is natural, because

$$\eta_{A'} \circ f = \Theta_{A,FA'}(Ff) = GFf \circ \eta_A.$$

Finally, it remains to check the triangular identities. We can only check one of them; the other follows dually. Let $g\colon B \to B'$ in $\mathbf{D}$ and let $f\colon FA \to B$. By commutativity the following diagram (which follows again from naturality of Θ)

$$\begin{tikzcd}
\mathbf{D}(FA, B) \arrow[r, "\Theta_{A,B}"] \arrow[d, "g\circ-"] & \mathbf{C}(A, GB) \arrow[d, "Gg\circ-"'] \\
\mathbf{D}(FA, B') \arrow[r, "\Theta_{A,B'}"] & \mathbf{C}(A, GB')
\end{tikzcd}$$

we can conclude that $\Theta(g \circ f) = Gg \circ \Theta(f)$. In particular, for $B = FGA$, $f = 1_{FA}$, $g = \varepsilon_A\colon FGA \to A$, we conclude one of the triangular identities.

$$1_{GB} = \Theta(\Theta^{-1}(1_{GB})) = \Theta(\varepsilon_B) = \Theta(\varepsilon_B \circ 1_{FGB}) = G\varepsilon_B \circ \Theta_{1_{FGB}} = G\varepsilon_B \circ \eta_{GB}$$

The other one follows dually.

($\Longleftarrow$). Conversely, suppose given η and ε satisfying the triangular identities. We need to establish a natural bijection between maps $FA \to B$ and $A \to GB$ for $A \in \mathbf{C}$, $B \in \mathbf{D}$. Given $h\colon FA \to B$, define $\phi(h) = (Gh)\eta_A\colon A \to GFA \to GB$, and given $k\colon A \to GB$, define $\psi(k) = \varepsilon_B(Fk)\colon FA \to FGB \to B$. These are natural in both A and B (check this!) and they're inverse to each other by the triangular identities:

$$\psi\phi(h) = \varepsilon_B(F\phi(h)) = \varepsilon_B(FGh)F\eta_A = h\varepsilon_{FA}BF\eta_A = h,$$

and similarly for $\phi \circ \psi(k)$. □

Is every equivalence of categories $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$, with maps $\alpha\colon 1_{\mathbf{C}} \xrightarrow{\sim} FG$ and $\beta\colon FG \xrightarrow{\sim} 1_{\mathbf{D}}$ an adjunction? The answer to this question is yes-and-no, assuming certain conditions.

Lemma 3.10 (*Every Equivalence is an adjoint equivalence*)**.** Suppose given F, G, α, β as above. Then there exist isomorphisms $\alpha'\colon 1_{\mathbf{C}} \xrightarrow{\sim} GF$, $\beta'\colon FG \xrightarrow{\sim} 1_{\mathbf{D}}$ satisfying the triangular identities. In particular, $F \dashv G$ and $G \dashv F$.

Proof. Define $\alpha' = \alpha$ and let β' be the composite

$$\beta'\colon FG \xrightarrow{FG\beta}{}^{-1} FGFG \xrightarrow{(F\alpha_G)^{-1}} FG \xrightarrow{\beta} 1_{\mathbf{D}}.$$

Note that

$$\begin{array}{ccc} FGFG & \xrightarrow{FG\beta} & FG \\ \big\downarrow{\scriptstyle \beta_{FG}} & & \big\downarrow{\scriptstyle \beta} \\ FG & \xrightarrow{\beta} & 1_{\mathbf{D}} \end{array}$$

commutes by the naturality of β, and β is monic, so $FG\beta = \beta_{FG}$. Now

$$\beta'_F F\alpha = F \xrightarrow{F\alpha} FGF \xrightarrow{(FG\beta_F)^{-1} = \beta_{FGF}^{-1}} FGFGF \xrightarrow{F\alpha_{GF}} FGF \xrightarrow{\beta_F} F$$

We can rewrite this by

$$F \xrightarrow{\beta_F^{-1}} FGF \xrightarrow{FGF\alpha} FGFGF \xrightarrow{(F\alpha_{GF})^{-1} = (FGF\alpha)^{-1}} FGF \xrightarrow{\beta_F} F,$$

and so everything cancels! Therefore, $\beta'_F F\alpha = 1_F$.

To see the other way, $G\beta' \circ \alpha'_G$ is the composite

$$\begin{aligned} G\beta'\alpha'_G &= G \xrightarrow{\alpha_G} GFG \xrightarrow{(GFG\beta)^{-1}} GFGFG \xrightarrow{(GF\alpha_G)^{-1}} GFG \xrightarrow{G\beta} G \\ &= G \xrightarrow{(G\beta)^{-1}} GFG \xrightarrow{\alpha_{GFG}} GFGFG \xrightarrow{(\alpha_{GFG})^{-1}} GFG \xrightarrow{G\beta} G \\ &= 1_G \end{aligned}$$

□

Lemma 3.11. Let $G\colon \mathbf{D} \to \mathbf{C}$ be a functor having a left adjoint F, with counit $\varepsilon\colon FG \to 1_{\mathbf{D}}$. Then

(i) G is faithful $\iff$ ε is (pointwise) epic. (We say "(pointwise) epic" because it will turn out that arrows in a functor category are epic iff they are pointwise epic. But we don't know that yet).

(ii) G is full and faithful $\iff$ ε is an isomorphism.

Proof. (i) Let $f\colon B \to B'$ be a morphism in $\mathbf{D}$. The composite $f\varepsilon_B\colon FGB \to B \to B'$ corresponds under the adjunction to $Gf\colon GB \to GB'$.

So ε_B is an epimorphism for all B $\iff$ for all B, B', composition with ε_B is an injection $\mathbf{D}(B, B') \to \mathbf{D}(FGB, B')$ $\iff$ for all B, B', application of G is an injection $\mathbf{D}(B, B') \to \mathbf{C}(GB, GB')$ $\iff$ G is faithful.

(ii) Similarly, G is full and faithful $\iff$ for all B, composition with ε_B is a bijection $\mathbf{D}(B, B') \to \mathbf{D}(FGB, B')$. This clearly holds if ε_B is an isomorphism for all B.

Conversely, if G is full and faithful, then $1_{FGB} = f\varepsilon_B$ for some $f\colon B \to FGB$. By (i), we know that ε_B is epic and this shows that ε_B is split monic. Hence, ε_B is an isomorphism.

□

Definition 3.12. (a) An adjunction $(F \dashv G)$ is called a **reflection** if G is full and faithful.

(b) We say that a full subcategory $\mathbf{C}' \subseteq \mathbf{C}$ is **reflective** if the inclusion $\mathbf{C}' \to \mathbf{C}$ has a left adjoint.

This comes with a caveat, that this terminology isn't fully standard. Some people don't require that a reflective subcategory is full, but I think it makes more sense to talk about reflective subcategories when they correspond to the reflections.

Example 3.13.

(a) $\mathbf{Ab}$ is reflective in $\mathbf{Gp}$: the left adjoint to the inclusion sends G to it's abelianization G/G'.

(b) An abelian group is **torsion** if all of its elements have finite order. In any abelian group A, the elements of finite order form a subgroup called the torsion subgroup A_T, and any homomorphism $B \to A$ where B is torsion takes values in A_T. So $A \mapsto A_T$ defines a coreflection from $\mathbf{Ab}$ to the full subcategory $\mathbf{Ab}_T$ of torsion groups.

Similarly, $A \mapsto A/A_T$ defines reflection to the subcategory of $\mathbf{Ab}$ that consists of torsion-free groups.

(c) There many examples of this in topology, and the most important of these is the Stone-Čech compactification. Let **KHaus** $\subseteq$ **Top** be the full subcategory of compact Hausdorff spaces. The inclusion **KHaus** $\to$ **Top** has a left adjoint β, called the **Stone-Čech compactification**. Interestingly, Stone and Čech gave different constructions of the compactification that now has their name, and essentially the only way to show that these two constructions are equal is to show that they are both left-adjoint to the forgetful functor.

4 Limits and Colimits

To talk about limits, we need to formally define what a diagram is. We've been drawing diagrams but we don't quite yet know what they are.

Definition 4.1. Let **J** be a category (usually small, and often finite). A **diagram of shape** J **in C** is a functor $D\colon \mathbf{J} \to \mathbf{C}$.

Example 4.2.

- If $\mathbf{J} = \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$, then a diagram of shape **J** is a commutative square in **C**.

- If $\mathbf{J} = \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$, then a diagram of shape **J** is a square in **C** that need not commute.

Definition 4.3. Given $D\colon J \to \mathbf{C}$, a **cone** over D consists of an object $C \in \operatorname{ob} \mathbf{C}$ (the **apex** of the cone) together with morphisms $\lambda_j\colon C \to D(j)$ (the **legs** of the cone) for each $j \in \operatorname{ob} \mathbf{J}$ such that

$$\begin{tikzcd} & C \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ D(i) \arrow[rr, "D(\alpha)"] & & D(j) \end{tikzcd}$$

commutes for all $\alpha\colon j \to j'$ in J.

If we write ΔC for the **constant diagram** of shape **J** sending all $j \in \operatorname{ob} J$ to **C** and all $\alpha\colon j \to j'$ to 1_C, then a cone over **D** with apex **C** is a natural transformation $\Delta C \to D$. Then Δ is a functor $\mathbf{C} \to [J, \mathbf{C}]$.

Definition 4.4.

(a) The **category of cones over** D is the arrow/comma category $(\Delta \downarrow D)$, defined dually to Theorem 3.4.

(b) The **category of cones under** D is the arrow/comma category $(D \downarrow \Delta)$, as defined in Theorem 3.4.

Definition 4.5. By a **limit** for $D\colon \mathbf{J} \to \mathbf{C}$, we mean a terminal object of $(\Delta \downarrow D)$. A **colimit** for D is an initial object of $(D \downarrow \Delta)$.

We say that **C has limits** (resp. **colimits**) of shape J if $\Delta\colon \mathbf{C} \to [\mathbf{J}, \mathbf{C}]$ has a right (resp. left) adjoint.

Example 4.6. (a) Suppose $J = \varnothing$. Then there's a unique $D\colon \varnothing \to \mathbf{C}$, and $(\Delta \downarrow D) \cong \mathbf{C}$. So a limit (resp. colimit) for D is a terminal (resp. initial) object of $\mathbf{C}$.

In **Set**, $1 = \{*\}$ is terminal and $\varnothing$ is initial. Similarly in **Top**. In **Gp**, the trivial group $\{1\}$ is both initial and terminal, and in **Ring** $\mathbb{Z}$ is initial.

(b) Let $\mathbf{J}$ be the discrete category with two objects. A diagram of shape $\mathbf{J}$ is a pair of objects (A_1, A_2), a limit for this is a **product**

$$\begin{tikzcd} & A_1 \times A_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ A_1 & & A_2 \end{tikzcd}$$

and a colimit for $\mathbf{J}$ is a **coproduct**

$$\begin{tikzcd} A_1 \arrow[dr, "i_1"'] & & A_2 \arrow[dl, "i_2"] \\ & A_1 + A_2 & \end{tikzcd}$$

In **Set**, **Gp**, **Ring**, **Top**, ..., the product are cartesian products (with suitable structure).

In **Set** and **Top**, corproducts are disjoint unions $A_1 \sqcup A_2$.

In **Gp**, coproducts are free products $G_1 * G_2$.

In **Ab**, finite coproducts coincide with finite products.

Last time we were talking about limits and colimits, and giving some examples. We saw the product of two objects, but we can also define the product of many objects.

Definition 4.7.

(i) More generally, let $\mathbf{J}$ be any (small) discrete category. A diagram of shape J is a J-indexed family of objects $(A_j \mid j \in J)$. A limit for it is a **product** $\prod_{j \in \mathbf{J}} A_j$ equipped with projections $\pi_i\colon \prod_{j \in \mathbf{J}} A_j \to A_i$. Dually, a **coproduct** $\sum_{j \in \mathbf{J}} A_j$ with $\nu_i\colon A_i \to \sum_{j \in \mathbf{J}} A_j$.

(ii) Let $\mathbf{J}$ be the category $\bullet \rightrightarrows \bullet$. A diagram of shape J is a parallel pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$; a cone over it looks like a diagram

$$\begin{tikzcd} x & C \arrow[dl, "\lambda_1"'] \arrow[dr, "\lambda_2"] & \\ A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B \end{tikzcd}$$

satisfying $f\lambda_1 = \lambda_2 = g\lambda_1$; equivalently, it's $C \xrightarrow{\lambda} A$ satisfying $f\lambda = g\lambda$. So a limit over J is an **equalizer** of f and g; dually a colimit over J is a **coequalizer**.

(iii) Let J be the category $\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$. A diagram of shape **J** looks like

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"] & D \end{tikzcd}$$

satisfying $fh = \ell = gk$. A cone over it consists of C and the arrows as in the diagram below

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "k"] \arrow[dr, "\ell"] & A \\ B & D \end{tikzcd}$$

Equivalently, this is a way of completing this diagram to a commutative square

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "k"] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & D \end{tikzcd}$$

A limit for the diagram is called a **pullback** of the pair f, g.

(iv) Colimits of shape $\mathbf{J}^{\mathrm{op}}$ are called **pushouts**; they can similarly be constructed from coproducts and coequalizers.

Example 4.8.

(i) Products / coproducts in **Set** are cartesian products / disjoint unions. Likewise in **Top**. In algebraic categories like **Gp**, **Ab**, **Ring**, R-**Mod**, etc. products are cartesian products but coproducts vary.

(ii) In **Set**, the equalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is the arrow $A' \xrightarrow{i} A$ where A' is the set $A' = \{a \in A \mid f(a) = g(a)\}$ and i is the inclusion. The coequalizer of this pair is $B \xrightarrow{q} B/\sim$, where $\sim$ is the smallest equivalence relation on B under which $f(a) \sim g(a)$, and q is the quotient map.

(iii) In **Set** (more generally, any category with binary products and equalizers) we may construct it by first forming the product $A \times B$ and then the equalizer $P \to A \times B$ of $f\pi_A \colon A \times B \to D$ and $g\pi_B \colon A \times B \to D$.

(iv) Let $\mathbf{J} = \mathbb{N}$ with it's usual total ordering. A diagram of shape J is just a sequence

$$A_0 \to A_1 \to A_2 \to A_3 \to \cdots$$

the **direct limit** of this diagram is confusingly a colimit which is an object A_∞ and maps $A_n \to A_\infty$ for all n that commute with all of the maps $A_i \to A_{i+1}$, and A_∞ is initial among such objects.

$$\begin{tikzcd} A_0 \arrow[r] \arrow[drrr] & A_1 \arrow[r] \arrow[drr] & A_2 \arrow[r] \arrow[dr] & A_3 \arrow[r] \arrow[d] & \cdots \arrow[dl] \\ & & & A_\infty & \end{tikzcd}$$

This is where the name limit comes from, because it looks like the limit of this sequence.

Dually, a limit of shape $\mathbb{N}^{\mathrm{op}}$ is called an **inverse limit**.

Theorem 4.9 (Freyd)**.**

(i) Suppose **C** has equalizers and all small products. Then **C** has all small limits.

(ii) Suppose **C** has equalizers and all finite products. Then **C** has all finite limits.

(iii) Suppose **C** has pullbacks and a terminal object. Then **C** has all small limits.

Proof. The proofs of (i) and (ii) are identical; just replace all occurances of "small" with "finite." So we'll do them simultaneously.

Let **J** be a small (resp. finite) category and let $D\colon \mathbf{J} \to \mathbf{C}$ be a diagram. Form the products

$$P = \prod_{j \in \mathrm{ob}\, J} D(j)$$

and

$$Q = \prod_{\alpha \in \mathrm{mor}\, J} D(\mathrm{cod}\,\alpha).$$

Let $f, g\colon P \rightrightarrows Q$ be the morphisms defined by

$$\begin{aligned} \pi_\alpha f &= \pi_{\mathrm{cod}\,\alpha}\colon P \to D(\mathrm{cod}\,\alpha) \\ \pi_\alpha g &= D(\alpha)\pi_{\mathrm{dom}\,\alpha}\colon P \to D(\mathrm{dom}\,\alpha) \to D(\mathrm{cod}\,\alpha) \end{aligned}$$

and finally, let $e\colon L \to P$ be the equalizer of f and g, and set $\lambda_j = \pi_j e\colon L \to D(j)$. We claim that the λ_j form a limit cone over D.

To see that the λ_j form a cone over D, note that

$$D(\alpha)\lambda_{\mathrm{dom}\,\alpha} = D(\alpha)\pi_{\mathrm{dom}\,\alpha}e = \pi_\alpha g e = \pi_\alpha f e = \pi_{\mathrm{cod}\,\alpha}e = \lambda_{\mathrm{cod}\,\alpha}$$

for all α.

Now given any cone $\left(C \xrightarrow{\mu_j} D(j) \mid j \in \mathrm{ob}\, J\right)$, we get a unique $\mu\colon C \to P$ satisfying $\pi_j \mu = \mu_j$ for all j. And since the μ_j form a cone, we have $\pi_\alpha f \mu = \pi_\alpha g \mu$

for all α, so $f\mu = g\mu$. And μ factors uniquely as $e\nu$. Then ν is the unique morphism of cones

$$\nu\colon \left(C \xrightarrow{\mu_j} D(j) \,\middle|\, j \in \operatorname{ob} j\right) \to \left(L \xrightarrow{\lambda_j} D(j) \,\middle|\, j \in \operatorname{ob} J\right)$$

Finally, to prove (iii), we want to apply (ii). This means we have to construct finite products and equalizers from the terminal object and pullbacks.

And given a terminal object 1 and pullbacks, we can form $A \times B$ as the pullback of $A \to 1 \leftarrow B$. Note that we can construct $\prod_{i=1}^n A_i$ for any $n \geqslant 3$ as the iterated product of the A_i. Hence we have all finite products.

To form the equalizer of $f, g\colon A \rightrightarrows B$, consider the diagram

$$\begin{tikzcd} & A \arrow[d, "(1_A, f)"] \\ A \arrow[r, "(1_A, g)"] & A \times B \end{tikzcd}$$

A cone over it consists of $A \xleftarrow{k} C \xrightarrow{h} A$ satisfying $1_A h = 1_A k$ and $fh = gk$, or equivalently, a map $h\colon C \to A$ satisfying $fh = gh$. So the pullback for this is an equalizer for (f, g). Hence, we have all equalizers. □

Definition 4.10. We say a category $\mathbf{C}$ is **complete** (or dually, **cocomplete**) if it has all small limits (dually, small colimits).

Example 4.11. $\mathbf{Set}, \mathbf{Gp}, \mathbf{Ab}, \mathbf{Top}$ are all both complete and cocomplete.

Definition 4.12. Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor, $\mathbf{J}$ a diagram shape.

(a) We say that F **preserves limits** of shape $\mathbf{J}$ if, given $D\colon \mathbf{J} \to \mathbf{C}$ and a limit cone

$$\left(L \xrightarrow{\lambda_j} D(j) \,\middle|\, j \in \operatorname{ob} J\right)$$

in $\mathbf{C}$, there is also a limit cone

$$\left(FL \xrightarrow{F\lambda_j} FD(j) \,\middle|\, j \in \operatorname{ob} \mathbf{J}\right)$$

in $\mathbf{D}$.

(b) We say that F **reflects limits** of shape $\mathbf{J}$, if given $D\colon \mathbf{J} \to \mathbf{C}$ and a cone $\left(L \xrightarrow{\lambda_j} D(j)\right)$ such that $\left(FL \xrightarrow{F\lambda_j} FD(j) \,\middle|\, j \in \operatorname{ob} \mathbf{J}\right)$ is a limit cone in $\mathbf{D}$, the original cone is a limit in $\mathbf{C}$.

(c) We say that F **creates limits** of shape $\mathbf{J}$ if, given $D\colon \mathbf{J} \to \mathbf{D}$ and a limit cone

$$\left(M \xrightarrow{\mu_j} FD(j) \,\middle|\, j \in \mathbf{J}\right)$$

for FD, there is a cone $\left(L \xrightarrow{\lambda_j} D(j) \,\middle|\, j \in \mathbf{J}\right)$ whose image under F is isomorphic to $\left(M \xrightarrow{\mu_j} FD(j) \,\middle|\, j \in \mathbf{J}\right)$, and any such cone is a limit for D.

Remark 4.13. If $F\colon \mathbf{C} \to \mathbf{D}$ creates limits of shape J, then it preserves and reflects them provided limits of shape J exist in $\mathbf{D}$.

Moreover, in any of the three statements of Theorem 4.9, there words "$\mathbf{C}$ has" can be replaced by either "$\mathbf{C}$ has and $F\colon \mathbf{C} \to \mathbf{D}$ preserves" or by "$\mathbf{D}$ has and $F\colon \mathbf{C} \to \mathbf{D}$ creates."

Example 4.14.

(a) The forgetful functor $U\colon \mathbf{Gp} \to \mathbf{Set}$ creates all small limits (in the strict sense) but it doesn't preserve colimits (e.g. coproduct of a pair of groups is not the disjoint union as sets).

(b) The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ preserves all small limits and colimits. We need only look at (co)equalizers and (co)products to see this by Theorem 4.9. This doesn't reflect limits, since the (co)limit in $\mathbf{Set}$ can be equipped with topologies other that that which makes it a (co)limit in $\mathbf{Top}$.

(c) The inclusion functor $\mathbf{Ab} \to \mathbf{Gp}$ reflects binary coproducts, but it doesn't preserve them. The coproduct of two groups A and B is the free product of groups, which is never abelian unless A or B is the trivial group 0, and $0 \to B \xleftarrow{1} B$ is a coproduct cone in both categories.

Let's say we want to construct limits in a functor category $[\mathbf{C}, \mathbf{D}]$. It's enough to have limits of this shape in $\mathbf{D}$.

Lemma 4.15. The forgetful functor $[\mathbf{C}, \mathbf{D}] \to \mathbf{D}^{\operatorname{ob}\mathbf{C}}$ creates all limits and colimits. In particular, $\mathbf{D}$ has (co)limits of shape J then so does $[\mathbf{C}, \mathbf{D}]$.

Proof. Given a diagram $D\colon \mathbf{J} \to [\mathbf{C}, \mathbf{D}]$, we can consider it as a curryed functor $\mathbf{C} \times \mathbf{J} \to \mathbf{D}$. Suppose we're given, for each $A \in \operatorname{ob}\mathbf{C}$, a limit cone $\big(LA \xrightarrow{\lambda_{A,j}} (A,j) \mid j \in \operatorname{ob} J\big)$ over the diagram $D(A,-)\colon \mathbf{J} \to \mathbf{D}$.

For each $f\colon A \to B$ in $\mathbf{C}$, the composites $LA \xrightarrow{\lambda_{A,j}} D(A,j) \xrightarrow{D(f,j)} D(B,j)$ for $j \in \operatorname{ob} J$ form a cone over $D(B,-)$, since for any $\alpha\colon j \to j'$, the square

$$\begin{tikzcd} D(A,j) \arrow[r, "D(f{,}1_j)"] \arrow[d, "D(1_A{,}\alpha)"'] & D(B,j) \arrow[d, "D(1_B{,}\alpha)"] \\ D(A,j') \arrow[r, "D(f{,}1_{j'})"'] & D(B,j') \end{tikzcd}$$

commutes as the image of the commutative square

$$\begin{tikzcd} (A,j) \arrow[r, "(f{,}1_j)"'] \arrow[d, "(1_A{,}\alpha)"] & (B,j) \arrow[d, "(1_B{,}\alpha)"] \\ (A,j') \arrow[r, "(f{,}1_{j'})"'] & (B,j') \end{tikzcd}$$

under the functor $\mathbf{D}(-,-)$, viewed as a curryed functor.

So there's a unique $Lf\colon LA \to LB$ making

$$\begin{tikzcd} LA \arrow[r, "\lambda_{A,j}"] \arrow[d, "Lf"'] & D(A,j) \arrow[d, "D(f,1_j)"] \\ LB \arrow[r, "\lambda_{B,j}"] & D(B,j) \end{tikzcd}$$

Uniqueness ensures that $f \mapsto Lf$ is functorial, and the $\lambda_{-,j}$ form natural transformations $L \to D(-,j)$. Hence, $\big(L \xrightarrow{\lambda_{-,j}} D(-,j) \mid j \in \operatorname{ob}\mathbf{J}\big)$ forms a cone over D. We want to show that L is actually the limit cone.

To that end, suppose given any cone $\big(C \xrightarrow{\mu_{-,j}} D(-,j) \mid j \in \operatorname{ob} J\big)$, each $\big(CA \xrightarrow{\mu_{A,j}} D(A,j) \mid j \in \operatorname{ob} J\big)$ factors uniquely as

$$\left(CA \xrightarrow{\nu_A} LA \xrightarrow{\lambda_{A,j}} D(A,j) \,\middle|\, j \in \operatorname{ob} j\right)$$

and any square

$$\begin{tikzcd} CA \arrow[r, "\nu_A"] \arrow[d, "Cf"'] & LA \arrow[d, "Lf"] \\ CB \arrow[r, "\nu_B"] & LB \end{tikzcd}$$

since the two ways around the diagram are factorizations of the same cone over $\mathbf{D}(B,-)$.

So the $\lambda_{-,j}$ are a limit cone in $[\mathbf{C},\mathbf{D}]$. □

Now we can finally fulfil the promise I gave earlier to prove that the monos and epis in a functor category are precisely the pointwise monos and epis. This follows from the following remark.

Remark 4.16. In any category, a morphism $f\colon A \to B$ is monic if and only if the diagram

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

is a pullback. Hence, provided $\mathbf{D}$ has pullbacks (resp. pushouts), a morphism $\alpha\colon F \to G$ in $[\mathbf{C},\mathbf{D}]$ is monic (resp. epic) if and only if each $\alpha_A\colon FA \to GA$ is monic (resp. epic). This is because functors preserve this diagram in a category.

4.1 The Adjoint Functor Theorems

Now that we've seen how functors interact with limits, we can see how adjunctions interact with limits.

Theorem 4.17 (RAPL: Right Adjoints Preserve Limits)**.** If $G\colon \mathbf{C} \to \mathbf{D}$ has a left adjoint $F\colon \mathbf{C} \to \mathbf{D}$, then G preserves limits.

We'll give two proofs: a high-power conceptual one and an elementary one.

High Power proof of Theorem 4.17. Assume limits of shape **J** exist in both **C** and **D**. Then composition with F and G induce functors $F_*(D) = FD$ and $G_*(D) = GD$,

$$\begin{tikzcd} {[\mathbf{J},\mathbf{C}]} \arrow[r, shift left, "F_*"] & {[\mathbf{J},\mathbf{D}]} \arrow[l, shift left, "G_*"] \end{tikzcd} ,$$

and it's easy to see that $F_* \dashv G_*$.

The diagram

$$\begin{tikzcd} \mathbf{C} \arrow[r, "F"] \arrow[d, "\Delta"] & \mathbf{D} \arrow[d, "\Delta"] \\ {[\mathbf{J},\mathbf{C}]} \arrow[r, "F_*"] & {[\mathbf{J},\mathbf{D}]} \end{tikzcd}$$

commutes, so

$$\begin{tikzcd} {[\mathbf{J},\mathbf{D}]} \arrow[r, "G_*"] \arrow[d, "\lim_{\mathbf{J}}"] & {[\mathbf{J},\mathbf{C}]} \arrow[d, "\lim_{\mathbf{J}}"] \\ \mathbf{D} \arrow[r, "G"] & \mathbf{C} \end{tikzcd}$$

commutes up to isomorphism by Corollary 3.7. □

Elementary proof of Theorem 4.17. Suppose given $D\colon \mathbf{J} \to \mathbf{D}$ and a limit cone $\left(L \xrightarrow{\lambda_j} D(j) \mid j \in \operatorname{ob}\mathbf{J}\right)$. Given a cone $\left(C \xrightarrow{\mu_j} GD(j) \mid j \in \operatorname{ob}\mathbf{J}\right)$ over GD in **C**, the naturality of the bijection $\mu \mapsto \overline{\mu}$ ensures that there is a cone over D, $\left(FC \xrightarrow{\overline{\mu_j}} D(j) \mid j \in \operatorname{ob}\mathbf{J}\right)$.

So there's a unique $FC \xrightarrow{\overline{\nu}} L$ such that $\lambda_j\overline{\nu} = \overline{\mu}_j$ for all j, and then $C \xrightarrow{\nu} GL$ is the unique morphism such that $(F\lambda_j)\nu = \mu_j$ for all j. □

The 'Primeval Adjoint Functor Theorem' asserts that the converse of Theorem 4.17 is morally true modulo some stupid set theory issues. If G preserves all limits, then it *should* have a left adjoint.

Lemma 4.18. Suppose **D** has, and $G\colon \mathbf{D} \to \mathbf{C}$ preserves limits of shape **J**. Then for any object A of **C**, the arrow/comma category $(A \downarrow G)$ has limits of shape **J** and the forgetful functor $U\colon (A \downarrow G) \to \mathbf{D}$ creates them.

Proof. Suppose given $D\colon J \to (A \downarrow G)$ and a limit cone $\left(L \xrightarrow{\lambda_j} UD(j) \mid j \in \operatorname{ob} J\right)$ over UD. Then $\left(GL \xrightarrow{G\lambda_j} GUD(j) \mid j \in \operatorname{ob} J\right)$ is a limit for GUD. But if we write $D(j) = (UD(j), f_j)$, the f_j form a cone over GUD with apex A, and therefore induce a unique map $f\colon A \to GL$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f_j"'] & GL \arrow[d, "G\lambda_j"] \\ & GUD(j) \end{tikzcd}$$

commutes for all j.

Hence, $\big((L, f) \xrightarrow{\lambda_j} (UD(j), f_j) \mid j \in \operatorname{ob} J\big)$ form a cone over D in $(A \downarrow G)$. The proof that this is a limit cone in $(A \downarrow G)$ is just like Lemma 4.15. □

Lemma 4.19. Let $\mathbf{A}$ be a category. Specifying an initial object of $\mathbf{A}$ is equivalent to specifying a limit for the diagram $1_{\mathbf{A}} \colon \mathbf{A} \to \mathbf{A}$.

Proof. ($\Rightarrow$). Let I be initial. The unique maps $\iota_A \colon I \to \mathbf{A}$ form a cone over $1_{\mathbf{A}}$, since for any $f \colon A \to B$ we have that $f\iota_A = \iota_B$ because the map $I \to B$ is unique. Given any other cone $(\mu_A \colon C \to A \mid A \in \operatorname{ob}\mathbf{A})$, the morphism $\mu_I \colon C \to I$ satisfies $\iota_A \mu_I = \mu_A$ for all A, so μ_I is a factorization of the cone $(\mu_A \mid A \in \operatorname{ob}\mathbf{A})$ through $(\iota_A \mid A \in \operatorname{ob}\mathbf{A})$. And it's the only one, because if ν is any such factorization, then $\nu = \iota_i \nu = \mu_I$, so $(\iota_A \mid A \in \operatorname{ob}\mathbf{A})$ is a limit cone.

($\Leftarrow$). Suppose given a limit cone $(\iota_A \colon I \to A \mid A \in \operatorname{ob}\mathbf{A})$ for $1_{\mathbf{A}}$. We know that I is **weakly initial** in that there is a map $I \to A$ for every object of $\mathbf{A}$, but not that these maps are unique. If $f \colon I \to A$ is any morphism, then the diagram below commutes:

$$\begin{tikzcd} & I \arrow[dl, "\iota_I"'] \arrow[dr, "\iota_A"] & \\ I \arrow[rr, "f"] & & A \end{tikzcd}$$

In particular, $\iota_A \iota_I = \iota_A$ for all A, so ι_I is a factorization of the limit cone through itself. Hence, $\iota_I = 1_I$, and hence $f = \iota_A$ for all $f \colon I \to A$. So I is initial. □

Now combining Lemma 4.18 and Lemma 4.19 with Theorem 3.4, we've proved the Primeval Adjoint Functor Theorem. However, there's a catch. If $\mathbf{D}$ has limits over all diagrams "as big as itself," then it must be a preorder! (Example sheet 2, question 6). There are also examples of applications of the Primeval Adjoint Functor Theorem to ordered sets on the example sheet.

We desire a better version of the Adjoint Functor Theorem that are applicable to all categories. To get other versions of this, we will cut down to small limits and locally small categories.

Theorem 4.20 (General Adjoint Functor Theorem)**.** Suppose $\mathbf{D}$ is complete and locally small. Then $G \colon \mathbf{D} \to \mathbf{C}$ has a left adjoint $\iff$ G preserves all small limits and satisfies the **solution set condition**: for any $A \in \operatorname{ob}\mathbf{C}$, there is a set of morphisms $\{f_i \colon A \to GB_i \mid i \in I\}$ such that every map $f \colon A \to GB$ factors as

$$A \xrightarrow{f_i} GB_i \xrightarrow{Gg_i} GB$$

for some $g_i \colon B_i \to B$.

Proof. ($\Rightarrow$). If G has a left-adjoint, then by RAPL (Theorem 4.17) it preserves limits, and $\{A \xrightarrow{\eta_A} GFA\}$ is a singleton solution set for $A \in \operatorname{ob}\mathbf{C}$, by Theorem 3.4.

($\Leftarrow$). By Lemma 4.18, the fact that G preserves all small limits indicates that the arrow categories $(A \downarrow G)$ are complete, and they inherit the local smallness from $\mathbf{D}$. So we need to show that if $\mathbf{A}$ is complete and locally small, and has a weakly initial set of objects $\{B_j \mid j \in J\}$, then it has an initial object. Then we have an initial object in each $(A \downarrow G)$, which is equivalent to specifying a left adjoint to G by Theorem 3.4.

To this end, first form $P = \prod_{j \in J} B_j$; then there are morphisms $\pi_j \colon P \to B_j$ for each j, so P is weakly initial. Now form the limit $i \colon I \to P$ of the diagram

$$\begin{tikzcd} P \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & P \end{tikzcd}$$

whose edge are all the endomorphisms of P. Clearly, I is weakly initial: given $f, g \colon I \to A$, we can form their equalizer $e \colon E \to I$ and find a morphism $h \colon P \to E$. Now ieh and 1_P are both endomorphisms of P, so $iehi = i$. But i is monic (just like an ordinary equalizer) so $ehi = 1_I$. Hence e is split epic, and $f = g$. Therefore, I is initial.

$$\begin{tikzcd} & P \arrow[dl, "h"'] & \\ E \arrow[r, "e"'] & I \arrow[u, tail, "i"'] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \end{tikzcd}$$

□

Example 4.21.

(a) Consider the forgetful functor $U \colon \mathbf{Gp} \to \mathbf{Set}$. $\mathbf{Gp}$ is complete and locally small, and U preserves all small limits. To obtain a solution set for U at A, observe that any $f \colon A \to UG$ factors as $A \to UG' \to UG$, where G' is the subgroup of G generated by $\{f(a) \mid a \in A\}$, and the cardinality of G' is bounded by the max of $\aleph_0$ and the cardinality of A.

Fix a set B of this cardinality, and consider all subsets $B' \subseteq B$ and all possible group structures on B', and all possible functions $A \to B'$. This is the solution set for U.

Thereby, we say that U has a left adjoint, which is the free group functor. But to get the cardinality bound on G', we need to say something about words in G'! And this is entirely useless to say anything about free groups.

(b) Here's an example where the solution set condition fails. Consider the forgetful functor $U \colon \mathbf{CLat} \to \mathbf{Set}$, where $\mathbf{CLat}$ is the category of **complete lattices** with all joins and meets, and the morphisms are functions that preserve all joins and meets. Just like $\mathbf{Gp}$, $\mathbf{CLat}$ is locally small and complete, and U preserves (indeed creates) all small limits.

But A.W. Hales (1965) shows that the solution set can fail when A is just a three-element set! He proved that for any cardinal κ, there is a complete lattice L_κ with $\mathrm{card}(L_\kappa) \geqslant \kappa$ which is generated by a 3-element subset. Given any solution set, there is no cardinal bound on the set of lattices that you need, and so the solution set condition for U fails at $A = \{a, b, c\}$.

Therefore, there is no left adjoint.

Definition 4.22. By a **subobject** of an object A in a category $\mathbf{C}$, we mean a monomorphism with codomain A, $A' \overset{m}{\rightarrowtail} A$. The subobjects of A form a preorder $\mathrm{Sub}_{\mathbf{C}}(A)$ with $(A' \overset{m}{\rightarrowtail} A) \leqslant (A'' \overset{m'}{\rightarrowtail} A)$ if m factors through m'.

Definition 4.23. We say that $\mathbf{C}$ is **well-powered** if, for every A, $\mathrm{Sub}_{\mathbf{C}}(A)$ is equivalent to a partially ordered *set*.

Remark 4.24. An equivalent definition of well-poweredness is as follows: $\mathbf{C}$ is **well-powered** if there exists a set $\{A_i \overset{m_i}{\rightarrowtail} A \mid i \in I\}$ of subobjects of A such that for each subobjects $B \overset{f}{\rightarrowtail} A$ of A there is $i \in I$ such that f factors through m_i.

$$\begin{tikzcd} B \arrow[r, tail, "f"] \arrow[d] & A \\ A_i \arrow[ur, tail, "m_i"'] & \end{tikzcd}$$

Morally, this is taking one object for each isomorphism class of subobjects.

Example 4.25. **Set** is well-powered, since $\mathrm{Sub}_{\mathbf{C}}(A) \simeq PA$, the power set of A. **Set** is also well-copowered, since $\mathrm{Sub}_{\mathbf{Set}^{\mathrm{op}}}(A)$ is equivalent to the poset of equivalence relations on A.

Lemma 4.26. Suppose given a pullback square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "k"] & B \arrow[d, tail, "f"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

with f monic. Then k is also monic.

Proof. Suppose given $\ell, m \colon E \to A$ with $k\ell = km$, then $gk\ell = gkm$, but $gk = fh$, so $fh\ell = fhm$, Since f is monic, then $h\ell = hm$. Therefore, ℓ and m are factorizations of the cone $(h\ell, k\ell)$ through the limit. Hence, $\ell = m$. Therefore, k is monic. $\square$

We're now ready to prove the special adjoint functor theorem.

Theorem 4.27 (Special Adjoint Functor Theorem (SAFT))**.** Let $\mathbf{C}$ be locally small and let $\mathbf{D}$ be locally small, complete, and well-powered, with a coseparating set of objects. Then $G \colon \mathbf{D} \to \mathbf{C}$ has a left adjoint if and only if G preserves all small limits.

Proof. $(\Longrightarrow)$. is just the same as in the proof of Theorem 4.17.

$(\Longleftarrow)$. The converse is the interesting part. Let $A \in \mathrm{ob}\,\mathbf{C}$. As in Theorem 4.20, $(A \downarrow G)$ inherits completeness and local smallness from $\mathbf{D}$. It also inherits well-poweredness: the subobjects of (B, f) are just the subobjects $B' \rightarrowtail B$ in $\mathbf{D}$ for which f factors through $GB' \rightarrowtail GB$; see Remark 4.24. (Note that the forgetful functor $(A \downarrow G) \to \mathbf{D}$ preserves and reflects monos by Remark 4.16.)

It also inherits a coseparating set: if $\mathcal{S}$ is a coseparating set for $\mathbf{D}$, then $\{(S, f) \mid S \in \mathcal{S}, f \in \mathbf{C}(A, GS)\}$ is a coseparating set for $(A \downarrow G)$. This is because, given a parallel pair $(B, h) \overset{k}{\underset{\ell}{\rightrightarrows}} (B', h')$ in $(A \downarrow G)$ with $k \neq \ell$, we can always find $m\colon B' \to S$ with $mk \neq m\ell$. Then m is also a morphism $(B', h') \to (S, (Gm)h')$ in $(A \downarrow G)$.

$$\begin{tikzcd} & GB' \arrow[dr, "Gm"] & \\ A \arrow[ur, "h'"] \arrow[rr, dashed] & & GS \end{tikzcd}$$

So we've reduced the theorem to proving that if some category $\mathbf{A}$ (in our case, the category $(A \downarrow G)$) is locally small, complete, and well-powered, and has a coseparating set, then $\mathbf{A}$ has an initial object.

To that end, let $\{S_j \mid j \in J\}$ be a coseparating set, and form $P = \prod_{j \in J} S_j$. Form now the limit of the diagram

$$\begin{tikzcd} P'_{k_1} \arrow[dr, tail] & \\ P'_{k_2} \arrow[r, tail] & P \\ \vdots & \\ P'_{k_i} \arrow[uur, tail] & \end{tikzcd}$$

whose edges are a representative set of subobjects of P. By an easy extension of Lemma 4.26, the legs $I \to P'_k$ of the limit cone are monic, so the composite $I \to P'_k \rightarrowtail P$ is also monic, and also a least element of $\mathrm{Sub}_{\mathbf{A}}(P)$, that is, a least subobject of P.

We claim that I is the initial object we seek. Let's check uniqueness and existence of these arrows from $I \to A$ in $\mathbf{A}$.

For uniqueness, suppose we had $I \overset{f}{\underset{g}{\rightrightarrows}} A$ in $\mathbf{A}$; then the equalizer $E \rightarrowtail I$ of f and g is a subobject of P contained in $I \rightarrowtail P$, so $E \to I$ is an isomorphism and thus $f = g$.

We now need the existence of $I \to A$ for each $A \in \operatorname{ob} \mathbf{A}$. This is the hard part. Fix $A \in \operatorname{ob} \mathbf{A}$. Form the product

$$Q = \prod_{j \in J f\colon A \to S_j} S_j$$

and let $h\colon A \to Q$ be defined by $\pi_{j,f} h = f$. Since the S_j form a coseparating set, h is monic. Let $g\colon P \to Q$ be defined by $\pi_{j,f} g = \pi_j$, and form the pullback

$$\begin{tikzcd} B \arrow[r, "\ell"] \arrow[d, "m"'] & A \arrow[d, tail, "h"] \\ P \arrow[r, "g"] & Q \end{tikzcd}$$

Now m is monic by Lemma 4.26, so there's a factorization

$$\begin{tikzcd} & B \arrow[dr, tail] & \\ I \arrow[ur, "n"] \arrow[rr, tail] & & P \end{tikzcd}$$

and $\ell n\colon I \to A$ is the desired map. So I has a map to each other object in **A**, and is thus initial. □

Example 4.28 (Stone-Čech Compactification)**.** Consider the inclusion functor from compact Hausdorff spaces to all topological spaces, $\mathbf{KHaus} \xrightarrow{I} \mathbf{Top}$. Both categories are locally small, and **KHaus** has and I preserves all small products by Tychonoff's Theorem. Similarly, **KHaus** has and I preserves all equalizers, since the equalizer in **Top** of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ with Y Hausdorff is a closed subspace of X.

Next, **KHaus** is well-powered, since the subobjects of X correspond (up to isomorphism) to closed subsets of X. Finally, $[0,1]$ is a coseparator for **KHaus**, by Urysohn's Lemma: given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ with $f \neq g$, we can choose $x \in X$ with $f(x) \neq g(x)$, and then find $h\colon Y \to [0,1]$ with $hf(x) = 0$ and $hg(x) = 1$, so $hf \neq hg$.

So I satisfies the conditions of SAFT (Theorem 4.27), so I has left-adjoint β.

Remark 4.29.

(a) Čech's original construction of β is virtually identical to this. Given X, he forms
$$P = \prod_{f\colon X \to [0,1]} [0,1]$$
and the map $h\colon X \to P$ defined by $\pi_f h = f$. Then defines βX to be the image of h, which has the appropriate universal property. But this is the smallest subobject of the product of members of a coseparating set for $(X \downarrow I)$. It seems like the proof of Theorem 4.27 is modelled on this construction.

(b) We could have constructed β using Theorem 4.20 to obtain a solution set for I at X, because it's enough to consider maps $f\colon X \to Y$ with Y compact Hausdorff where $\operatorname{im} f$ is dense in Y. If X has cardinality κ, then we can show that Y has cardinality bounded by 2^{2^κ}.

5 Monads

This is the last chapter of material that forms the core of the course. Suppose we have an adjunction $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$. How much of this structure can we recover from $\mathbf{C}, F, G$ without ever mentioning $\mathbf{D}$? We know $T = GF\colon \mathbf{C} \longrightarrow \mathbf{C}$, and we also have the unit of the adjunction, which is a natural transformation $\eta\colon 1_{\mathbf{C}} \longrightarrow GF = T$. We don't know the counit ε because it lives in $\mathbf{D}$, but we do have the natural transformation $\mu = G\varepsilon_F\colon TT = GFGF \longrightarrow GF = T$.

Of course, these data satisfy certain identities inherited from the triangular identities from η and ε. In particular, the following diagrams commute; the one of the left from the triangular identities and the one on the right by naturality of ε.

$$\begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr, "1_T"'] & TT \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "1_T"] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu_T"] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"] & T \end{tikzcd} \tag{10}$$

We could also consider the dual notion, where we know $\mathbf{D}, F, G$ but not $\mathbf{C}$. This motivates the following definition.

Definition 5.1. By a **monad** in $\mathbf{C}$, we mean a triple $\mathbb{T} = (T, \eta, \mu)$ where $T\colon \mathbf{C} \to \mathbf{C}$ is a functor and $\eta\colon 1_{\mathbf{C}} \to T$, $\mu\colon TT \to T$ are natural transformations satisfying the three commutative diagrams in (10).

Dually, a **comonad** $\mathbb{R} = (R, \varepsilon, \delta)$ has $\varepsilon\colon R \to 1_{\mathbf{C}}$ and $\delta\colon R \to RR$ satisfying the diagrams dual to (10).

Remark 5.2. The name "monad" is a long time coming. Originally, people referred to these things as just "the standard construction," which is really an admission of defeat in naming conventions. Later, they were called "triples," but that requires that comonads had the awkward name "co-triples." MacLane popularized the term monad, but there are still some people who insist on calling them triples. Mostly these people live in Montreal.

Recall the definition of Monads from (10). Let's number the three equations for ease of reference as

(11) $\mu \circ \eta_T = 1_T$;

(12) $\mu \circ T_\eta = 1_T$;

(13) $\mu \circ \mu_T = \mu \circ T\mu$.

$$\begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr, "1_T"'] & TT \arrow[d, "\mu"] \\ & T \end{tikzcd} \tag{11}$$

$$\begin{tikzcd} TT \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "1_T"] \\ T & \end{tikzcd} \tag{12}$$

$$\begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu_T"] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"] & T \end{tikzcd} \tag{13}$$

Before we continue, let's talk a little more about examples.

Example 5.3. (a) The monad $\mathbb{T}$ on $\mathbf{C}$ induced by an adjunction $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} D$, $F \dashv G$, has $T = GF$, η is the unit of the adjunction, and $\mu = G\varepsilon_F$.

(b) Given a monad M, we can make $A \mapsto M \times A$ into a functor $M \times - \colon \mathbf{Set} \to \mathbf{Set}$. This functor has monad structure, with $\eta_A \colon A \to M \times A$ defined by $\eta_A(a) = (1, a)$, and $\mu_A \colon M \times M \times A \to M \times A$ defined by $\mu_A(m, m', a) = (mm', a)$.

(c) In any category $\mathbf{C}$ with binary products, given $A \in \operatorname{ob} \mathbf{C}$, we can make $A \times -$ into a functor $\mathbf{C} \to \mathbf{C}$, and it has a comonad structure given by $\varepsilon_B = \pi_2 \colon A \times B \to B$ and $\delta_B = (\pi_1, \pi_1, \pi_2) \colon A \times B \to A \times A \times B$.

Given Example 5.3(a), we might ask "Does every monad arise from an adjunction?" The answer is yes. Eilenberg-Moore (1965) observed that Example 5.3(b) arises from the adjunction $\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} [M, \mathbf{Set}]$ where $FA = M \times A$ with M action on the left-factor.

Definition 5.4. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathbf{C}$. A **$\mathbb{T}$-algebra** is a pair (A, α) where $A \in \operatorname{ob} \mathbf{C}$ and $\alpha \colon TA \to A$ satisfies

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \tag{14}$$

$$\begin{tikzcd} TTA \arrow[r, "T\alpha"] \arrow[d, "\mu_A"] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"] & A \end{tikzcd} \tag{15}$$

A **homomorphism of $\mathbb{T}$-algebras** $f \colon (A, \alpha) \to (B, \beta)$ is a morphism $f \colon A \to B$ making the following diagram commute.

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"] & B \end{tikzcd} \tag{16}$$

The category of $\mathbb{T}$ algebras and $\mathbb{T}$-algebra homomorphisms is denoted $\mathbf{C}^{\mathbb{T}}$. This is called the **Eilenberg-Moore category**.

Lemma 5.5. The forgetful functor $\mathbf{C}^{\mathbb{T}} \to \mathbf{C}$ has left adjoint $F^{\mathbb{T}} \colon \mathbf{C} \to \mathbf{C}^{\mathbb{T}}$ and the adjunction induces the monad $\mathbb{T}$.

Proof. We define $F^{\mathbb{T}}A = (TA, \mu_A)$ and $F^{\mathbb{T}}(A \xrightarrow{f} B) = Tf$, which is an algebra by Equation 11 and Equation 13.

Clearly $G^{\mathbb{T}}F^{\mathbb{T}} = T$, and we have a natural transformation $\eta \colon 1_{\mathbf{C}} \to G^{\mathbb{T}}F^{\mathbb{T}}$.

We define $\varepsilon \colon F^{\mathbb{T}}G^{\mathbb{T}} \to 1_{\mathbf{C}^{\mathbb{T}}}$ by $\varepsilon_{(A,\alpha)} = \alpha \colon TA \to A$. This is a homomorphism by Equation 15, and natural by Equation 16.

We just need to check the triangular identities now. The composite $(G^{\mathbb{T}}\varepsilon)(\eta_{G^{\mathbb{T}}})$ is the identity by Equation 14. And $\eta_{F^{\mathbb{T}}A}(F^{\mathbb{T}}\eta_A) = \mu_A T\eta_A = 1_{TA}$ by Equation 12.

Finally, the multiplication $G^{\mathbb{T}}\varepsilon_{F^{\mathbb{T}}}$ of the induced monad is μ. $\square$

So this was the Eilenberg-Moore approach to Monads. Kleisli instead took a minimalist approach. If $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$ induces $\mathbb{T}$, we may replace $\mathbf{D}$ by its full subcategory $\mathbf{D}'$ on objects of the form FA. So we may as well assume that F is surjective on objects (up to equivalence, it may as well be bijective). Also, morphisms $FA \to FB$ in $\mathbf{D}$ correspond bijectively to morphisms $A \to TB$ in $\mathbf{C}$. Kleisli's idea was to take this as the definition.

Definition 5.6. Let $\mathbb{T}$ be a monad on $\mathbf{C}$. The **Kleisli Category** $\mathbf{C}_{\mathbb{T}}$ of $\mathbb{T}$ has $\operatorname{ob}\mathbf{C}_{\mathbb{T}} = \operatorname{ob}\mathbf{C}$. Morphisms $A \xrightarrow{f} B$ in $\mathbf{C}_{\mathbb{T}}$ are morphisms $A \xrightarrow{f} TB$ in $\mathbf{C}$. The identity $A \to A$ in $\mathbf{C}_{\mathbb{T}}$ is $A \xrightarrow{\eta_A} TA$. The composite $A \xrightarrow{f} B \xrightarrow{g} C$ is $A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC$.

Remark 5.7. To avoid confusion, since $\operatorname{ob}\mathbf{C} = \operatorname{ob}\mathbf{C}_{\mathbb{T}}$, morphisms in $\mathbf{C}_{\mathbb{T}}$ are written in blue, while morphisms in $\mathbf{C}$ retain this lovely black color.

We should really verify that $\mathbf{C}_{\mathbb{T}}$ is a category, so consider the diagrams

$$
\begin{tikzcd}
A \arrow[r, "f"] & TB \arrow[r, "T\eta_B"] \arrow[dr, "1_{TB}"'] & TTB \arrow[d, "\mu_B"] \\
 & & TB
\end{tikzcd}
\qquad
\begin{tikzcd}
TA \arrow[r, "Tf"] & TTB \arrow[r, "\mu_B"] & TB \\
A \arrow[u, "\eta_A"] \arrow[r, "f"] & TB \arrow[u, "\eta_{TB}"] \arrow[ur, "1_{TB}"'] &
\end{tikzcd}
$$

Given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, consider the diagram

$$
\begin{tikzcd}
A \arrow[r, "f"] & TB \arrow[r, "Tg"] & TTC \arrow[r, "\mu_C"] \arrow[d, "TTh"] & TC \arrow[d, "Th"] \\
 & & TTTD \arrow[r, "\mu_{TD}"] \arrow[d, "T\mu_D"] & TTD \arrow[d, "\mu_D"] \\
 & & TTD \arrow[r, "\mu_D"] & TD
\end{tikzcd}
$$

The upper way round is $h(gf)$, and the lower way route is $(hg)f$.

Lemma 5.8. There exists an adjunction $\mathbf{C} \underset{G_{\mathbb{T}}}{\overset{F_{\mathbb{T}}}{\rightleftarrows}} \mathbf{C}_{\mathbb{T}}$ with $F_{\mathbb{T}} \dashv G_{\mathbb{T}}$, inducing $\mathbb{T}$.

Proof. We define $F_{\mathbb{T}}A = A$ and $F_{\mathbb{T}}(A \xrightarrow{f} B) = A \xrightarrow{f} B \xrightarrow{\eta_B} TB$. Clearly, $F_{\mathbb{T}}$ preserves identities, and the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "\eta_B"] \arrow[d, "g"] & TB \arrow[d, "Tg"] & \\
& C \arrow[r, "\eta_C"] & TC \arrow[r, "T\eta_C"] \arrow[dr, "1_{TC}"'] & TTC \arrow[d, "\mu_C"] \\
& & & TC
\end{tikzcd}$$

shows that $F_{\mathbb{T}}(gf) = F_{\mathbb{T}}g \circ F_{\mathbb{T}}f$. (The composition is in $\mathbf{C}_{\mathbb{T}}$, which is why it's blue.)

Define $G_{\mathbb{T}}A = TA$ and $G_{\mathbb{T}}(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$. Again, this is functorial: $G_{\mathbb{T}}(1_A) = 1_{TA}$ by Equation 12, and given $A \xrightarrow{f} B \xrightarrow{g} C$, consider the diagram

$$\begin{tikzcd}
TA \arrow[r, "Tf"] & TTB \arrow[r, "\mu_B"] \arrow[d, "TTg"] & TB \arrow[d, "Tg"] \\
& TTTC \arrow[r, "\mu_{TC}"] \arrow[d, "T\mu_C"] & TTC \arrow[d, "\mu_C"] \\
& TTC \arrow[r, "\mu_C"] & TC
\end{tikzcd}$$

which shows that $G_{\mathbb{T}}g \circ G_{\mathbb{T}}f = G_{\mathbb{T}}(gf)$.

Once again, we have $G_{\mathbb{T}}F_{\mathbb{T}} = T$, and so η of $\mathbb{T} = (T, \eta, \mu)$ is a natural transformation $1_{\mathbf{C}} \to G_{\mathbb{T}}F_{\mathbb{T}}$. This will be the unit of this adjunction. The counit map $F_{\mathbb{T}}G_{\mathbb{T}}A \to A$ is $1_{TA}\colon TA \to TA$. The fact that the counit map is natural is left as an exercise.

Let's check the triangular identities to show that this is actually an adjunction. $(G_{\mathbb{T}}\varepsilon_A)(\eta_{G_{\mathbb{T}}A})$ is the composite

$$TA \xrightarrow{\eta_{TA}} TTA \xrightarrow{\mu_A} TA$$

is equal to $1_{F_{\mathbb{T}}A}$. On the other hand, $(\varepsilon_{F_{\mathbb{T}}A})(F_{\mathbb{T}}\eta_A)$ is the composite

$$A \xrightarrow{\eta_A} TA \xrightarrow{\eta_{TA}} TTA \xrightarrow{F_{\mathbb{T}A}} TTA \xrightarrow{\mu_A} = \eta_A = 1_{F_{\mathbb{T}}A}.$$

Finally, $G_{\mathbb{T}}\varepsilon_{F_{\mathbb{T}}A} = G_{\mathbb{T}}(1_{TA}) = \mu_A$, so the induced monad is $\mathbb{T}$. □

Last time we defined the Eilenberg-Moore and Kleisli categories for an adjunction. We'll see shortly that they're the two extreme categories of this sort. First, we'll need a few definitions.

Definition 5.9. Given a monad $\mathbb{T}$ on $\mathbf{C}$, let $\mathbf{Adj}(\mathbb{T})$ be the category whose objects are adjunctions $\left(\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}\right)$ inducing $\mathbb{T}$, and whose morphisms

$$\left(\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}\right) \longrightarrow \left(\mathbf{C} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathbf{D}'\right)$$

are functors $H\colon \mathbf{D} \to \mathbf{D}'$ satisfying $HF = F'$ and $G'H = G$.

Theorem 5.10. The Kleisli adjunction $F_{\mathbb{T}} \dashv G_{\mathbb{T}}$ is initial in $\mathbf{Adj}(\mathbb{T})$ and the Eilenberg-Moore adjunction $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$ is terminal.

Proof. We'll do the case of Eilenberg-Moore first, because as always the Eilenberg-Moore category is easier to work with.

Given $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$ with $F \dashv G$, we define the **(Eilenberg-Moore) comparison functor** $K\colon \mathbf{D} \to \mathbf{C}^{\mathbb{T}}$ by $KB = (GB, G\varepsilon_B)$ and $K(B \xrightarrow{g} B') = Gg$ (which is a homomorphism of $\mathbb{T}$-algebras by naturality of ε), where ε is the counit of the adjunction $F \dashv G$.

Note that $G\varepsilon_B \circ \eta_{GB} = 1_{GB}$ is one of the triangular identities for $F \dashv G$, and

$$G\varepsilon_B GFG\varepsilon_B = G\varepsilon_B \mu_{GB} = G\varepsilon_B G\varepsilon_{FGB}$$

by naturality of ε.

Clearly $G^{\mathbb{T}}K = G$ and

$$KFA = (GFA, G\varepsilon_{FA}) = (TA, \mu_A) = F^{\mathbb{T}}A.$$

So $G^{\mathbb{T}}K$ agrees with $F^{\mathbb{T}}$on objects, and

$$KF(A \xrightarrow{f} A') = GFf = TF = F^{\mathbb{T}}f,$$

so it also agrees on arrows. Therefore, K is a morphism from $(F \dashv G)$ to $(F^{\mathbb{T}} \dashv G^{T})$ in $\mathbf{Adj}(\mathbb{T})$.

Now suppose $H\colon \mathbf{D} \to \mathbf{C}^{\mathbb{T}}$ is another such morphism. We want to show that $H = K$, to demonstrate that $(F^{\mathbb{T}} \dashv G^{\mathbb{T}})$ is terminal. Then $HB = (GB, \beta_B)$ for some structure map $\beta_B\colon GFGB \to GB$ and $Hg = Gg$. Moreover, we have that $\beta_{FA} = \mu_A = G\varepsilon_{FA}$ for all $A \in \operatorname{ob}\mathbf{C}$.

Now consider the square

$$\begin{tikzcd}
GFGFGB \arrow[r, "GFG\varepsilon_B"] \arrow[d, "\mu_{GB}"] & GFGB \arrow[d, "\beta_B"] \\
GFGB \arrow[r, "G\varepsilon_B"] & GB
\end{tikzcd}$$

this commutes becasue $G\varepsilon_B$ is a $\mathbb{T}$-algebra homomorphism $H(FGB) \to HB$. But this diagram would also commute with β_B replaced with $G\varepsilon_B$ on the right, and the top edge is split epic by a triangular identity.

So $\beta_B = G\varepsilon_B$, so therefore $H = K$. Thus, $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$ is terminal in $\mathbf{Adj}(\mathbb{T})$.

We should now show that the Kleisli category $\mathbf{C}_{\mathbb{T}}$ is initial. To that end, we define $L\colon \mathbf{C}_{\mathbb{T}} \to \mathbf{D}$ by

$$LA = FA$$
$$L(A \xrightarrow{f} B) = FA \xrightarrow{Ff} FGFB \xrightarrow{\varepsilon_{FB}} FB$$

Note that Lf is the morphism corresponding to f under the adjunction $(F \dashv G)$, and so is always full and faithful. Moreover, $L(\eta_A) = \varepsilon_{FA}$, and $F\eta_A = 1_{FA}$.

Now given $A \xrightarrow{f} B \xrightarrow{g} C$, consider

$$\begin{tikzcd} FA \arrow[r, "Ff"] & FGFB \arrow[r, "FGFg"] \arrow[d, "\varepsilon_{FB}"] & FGFGFC \arrow[r, "F\mu_C"] \arrow[d, "\varepsilon_{FGFC}"] & FGFC \arrow[d, "\varepsilon_{FC}"'] \\ & FB \arrow[r, "Fg"] & FGFC \arrow[r, "\varepsilon_{FC}"] & FC \end{tikzcd}$$

This proves that $L(fg) = (Lf)(Lg)$.

We also see $LF_{\mathbb{T}}A = FA$ and

$$LF_{\mathbb{T}}(A \xrightarrow{f} B) = \begin{tikzcd} FA \arrow[r, "Ff"] & FB \arrow[r, "F\eta_B"] \arrow[dr, "1_{FB}"] & FGFB \arrow[d, "\varepsilon_{FB}"] \\ & & FB \end{tikzcd}$$

so $LF_{\mathbb{T}} = F$.

Additionally, $GLA = GFA = TA = G_{\mathbb{T}}A$ and

$$GL(A \xrightarrow{f} B) = GFA \xrightarrow{GFf} GFGFB \xrightarrow{G\varepsilon_{FB}} GFB = \mu_B Tf = G_{\mathbb{T}}f$$

so L is a morphism of $\mathbf{Adj}(\mathbb{T})$.

Finally, suppose that $H\colon \mathbf{C}_{\mathbb{T}} \to \mathbf{D}$ is any morphism from $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$ to $(F \dashv G)$. We want to check that $H = L$, that is, that $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$ is initial.

Given this H, we know that $HA = FA$ for all A, and $H(TA \xrightarrow{1_{TA}} A) = FGFA \xrightarrow{\varepsilon_{FA}} FA$. But for any $A \xrightarrow{f} B$, we have that $f = A \xrightarrow{F_{\mathbb{T}}f} TB \xrightarrow{1_{TB}} B$, so $H(f) = \varepsilon_{FA}(Ff) = Lf$. This shows that $H = L$, so the Kleisli category is initial in $\mathbf{Adj}(\mathbb{T})$. □

This is really the last time that we'll see the Kleisi category in action, mostly because it's a huge pain to work with. Even if $\mathbf{C}$ has nice properties, it's not always the case that $\mathbf{C}_{\mathbb{T}}$ has these properties. However, $\mathbf{C}_{\mathbb{T}}$ inherits coproducts from $\mathbf{C}$, since $F_{\mathbb{T}}$ is bijective on objects and preserves colimits. In general, though, it has few other limits or colimits. For example, the Kleisi category on $\mathbf{Gp}_{\mathbb{T}}$, where $\mathbb{T}$ comes from the free $\dashv$ forgetful adjunction, is the category of all free groups and homomorphisms between them. But the product of two free groups need not be free, so it doesn't have even binary products!

The Eilenberg-Moore category $\mathbf{C}^{\mathbb{T}}$ is much nicer to work with.

Theorem 5.11. Let $\mathbf{C}^{\mathbb{T}}$ be the Eilenberg-Moore category for a monad $\mathbb{T}$ in $\mathbf{C}$.

(i) The forgetful functor $\mathbf{C}^{\mathbb{T}} \xrightarrow{G^{\mathbb{T}}} \mathbf{C}$ creates all limits which exist in $\mathbf{C}$.

(ii) If $T\colon \mathbf{C} \to \mathbf{C}$ preserves all colimits of shape J, then $G^{\mathbb{T}}\colon \mathbf{C}^{\mathbb{T}} \to \mathbf{C}$ creates them.

[Note: from now on we'll write $G = G^{\mathbb{T}}$]

Proof.

(i) Suppose given $D\colon \mathbf{J} \to \mathbf{C}^{\mathbb{T}}$ and a limit cone $\big(\lambda_j\colon L \to GD(j) \mid j \in \operatorname{ob}\mathbf{J}\big)$ for GD in $\mathbf{C}$. Write $D(j) = (GD(j), \delta_j)$, where δ_j is the $\mathbb{T}$-algebra structure map of $GD(j)$. Now the composites

$$TL \xrightarrow{T\lambda_j} TGD(j) \xrightarrow{\delta_j} GD(j)$$

form a cone over GD since the edges of GD are $\mathbb{T}$-algebra homomorphisms. Therefore, these induce a unique $\lambda\colon TL \to L$ such that $\lambda_j \nu = \delta_j(T\lambda_j)$ for all j. Moreover, λ satisfies the equations for a $\mathbb{T}$-algebra structure, since for example the associativity condition asserts the equality of any two morphisms $TTL \rightrightarrows L$ which are both factorizations of the limit cone over GD.

So L has a unique $\mathbb{T}$-algebra structure ν making all the λ_j into homomorphisms of $\mathbb{T}$-algebras.

Finally, we should check that given any cone $(\beta_j\colon (A, \alpha) \to D(j) \mid j \in \operatorname{ob}\mathbf{J})$ over D in $\mathbf{C}^{\mathbb{T}}$, this factors through the limit cone corresponding to L. But we know that there is a unique $\beta\colon A \to L$ in $\mathbf{C}$ such that $\lambda_j\beta = \beta_j$ for all j. Once again,

$$\begin{tikzcd} TA \arrow[r, "T\beta"] \arrow[d, "\alpha"] & TL \arrow[d, "\lambda"] \\ A \arrow[r, "\beta"] & L \end{tikzcd}$$

So $(\lambda_j\colon (L, \lambda) \to D(j) \mid j \in \operatorname{ob}\mathbf{J})$ is a limit cone in $\mathbf{C}^{\mathbb{T}}$.

(ii) Simialrly, if $D\colon \mathbf{J} \to \mathbf{C}^{\mathbb{T}}$and $(\lambda_j\colon GD(j) \to L \mid j \in \operatorname{ob}\mathbf{J})$ is a colimit for GD in $\mathbf{C}$, then $(T\lambda_j\colon TGD(j) \to TL \mid j \in \operatorname{ob}\mathbf{J})$ is also a colimit because T preserves colimits, so the composites

$$TGD(j) \xrightarrow{\delta_j} GD(j) \xrightarrow{\lambda_j} L$$

induce a unique $\lambda\colon TL \to L$. It's a good exercise to check that λ is an algebra structure using the fact that TTL is also a colimit, and the λ_j form a colimit cone in $\mathbf{C}^{\mathbb{T}}$.

□

5.1 Monadicity Theorems

We saw that the Eilenberg-Moore category and the adjunction coming from a monad was particularly nice to work with. We may want to know when arbitrary adjunctions look like this particularly nice example.

Definition 5.12. Let $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$ be an adjunction inducing a monad $\mathbb{T}$ on $\mathbf{C}$. We say that $(F \dashv G)$ is **monadic** if the comparison functor $K\colon \mathbf{D} \to \mathbf{C}^{\mathbb{T}}$ is part of an equivalence of categories. We say $\mathbf{D} \xrightarrow{G} \mathbf{C}$ is **monadic** if it has a left adjoint and the adjunction is monadic.

Today, our goal is to prove the monadicity theorem which characterizes these adjunctions. The "Primeval Monadicity Theorem" asserts that the Eilenberg-Moore adjunction is characterized up to equivalence by the fact that

$$FGFGB \underset{FG\varepsilon_B}{\overset{\varepsilon_{FGB}}{\rightrightarrows}} FGB \xrightarrow{\varepsilon_B} B$$

is a coequalizer for every object B of $\mathbf{D}$.

Before we state these theorems, let's give the key lemma. This is at the heart of every monadicity theorem, but it's not often stated on it's own. However, it it quite useful by its own merit.

Lemma 5.13. Let $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$ be an adjunction $F \dashv G$ inducing a monad $\mathbf{T}$, and suppose that for every $\mathbb{T}$-algebra (A, α), the pair

$$FGFA \underset{\varepsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$$

has a coequalizer. Then $K\colon \mathbf{D} \to \mathbf{C}^{\mathbb{T}}$ has a left adjoint.

Proof. We will find a functor $L\colon \mathbf{C}^{\mathbb{T}} \to \mathbf{D}$ such that $L \dashv K$. Define $L(A, \alpha)$ to be the coequalizer of

$$FGFA \underset{\varepsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA \longrightarrow L(A, \alpha).$$

Given a homomorphism $f\colon (A, \alpha) \to (A', \alpha')$, we have

$$\begin{tikzcd}
FGFA \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\varepsilon_{FA}"'] \arrow[d, "FGFf"'] & FA \arrow[r] \arrow[d, "Ff"] & L(A,\alpha) \arrow[d, "Lf"] \\
FGFA' \arrow[r, shift left, "F\alpha'"] \arrow[r, shift right, "\varepsilon_{FA'}"'] & FA' \arrow[r] & L(A',\alpha')
\end{tikzcd}$$

commutes. Uniqueness implies L is functorial.

Now, given $B \in \operatorname{ob} \mathbf{D}$, morphisms $L(A, \alpha) \to B$ correspond bijectively to morphisms $f\colon FA \to B$ satisfying $f(F\alpha) = f\varepsilon_{FA}$. But morphisms $f\colon FA \to B$

correspond bijectively to morphisms $\overline{f}\colon A \to GB$ satisfying $\overline{f}\alpha = Gf$. But we can also write f in terms of $\overline{f}$ as $f = \varepsilon_B \circ F\overline{f}$, so we see that

$$\overline{f}\alpha = Gf = G(\varepsilon_B \circ F\overline{f}) = G\varepsilon_B \circ GF(\overline{f}).$$

This means that $\overline{f}$ is a $\mathbb{T}$-algebra homomorphism $\overline{f}\colon (A, \alpha) \to KB = (GB, G\varepsilon_B)$ in $\mathbf{C}^{\mathbb{T}}$.

Check for yourself that this bijection is natural in (A, α) and in B as well, so $(L \dashv K)$. □

Definition 5.14.

(a) We say a parallel pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is **reflexive** if there is some $r\colon B \to A$ such that $fr = gr = 1_B$.

(b) By a **split coequalizer diagram** in $\mathbf{C}$, we mean a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend left=50, "t"] \arrow[r, "h"] & C \arrow[l, bend left=50, "s"] \end{tikzcd}$$

satisfying $hf = hg$, $hs = 1_C$, $gt = 1_B$ and $ft = sh$.

Note that the pair $(F\alpha, \varepsilon_{FA})$ in the statement of Lemma 5.13 is reflexive, with common splitting $F\eta_A$. Note also that reflexive coequalizers (i.e. coequalizers of reflexive pairs) are colimits of shape $\mathbf{J}$, where

$$\mathbf{J} = \begin{tikzcd} \bullet \arrow[loop above, "d"] \arrow[loop below, "e"] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \bullet \arrow[l, "r"] \end{tikzcd}$$

satisfying $fr = gr = 1$, $rf = d$, $rg = e$.

Noe that the equations of a split coequalizer imply that $A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{h} C$ is a coequalizer: given $k\colon B \to D$ with $kf = kg$, then $k = kgt = kft = ksh$, so k factors through h and the factorization is unique since h is split epic.

Note also that *any* functor preserves split coequalizers – this is the property of an **absolute colimit**.

Definition 5.15. Given $G\colon \mathbf{D} \to \mathbf{C}$, we say that a pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathbf{D}$ is G**-split** if there is a split coequalizer diagram

$$\begin{tikzcd} GA \arrow[r, shift left, "Gf"] \arrow[r, shift right, "Gg"'] & GB \arrow[l, bend left=50, "t"] \arrow[r, "h"] & C \arrow[l, bend left=50, "s"] \end{tikzcd}$$

Note that the pair in the statement of Lemma 5.13 is G-split, since

$$\begin{tikzcd} GFGFA \arrow[r, shift left, "GF\alpha"] \arrow[r, shift right, "G\varepsilon_{FA}"'] & GFA \arrow[r, "\alpha"] \arrow[l, bend left, "\eta_{GFA}"] & A \arrow[l, bend left, "\eta_A"] \end{tikzcd}$$

is a split coequalizer diagram.

Now we're ready to give the Monadicity Theorems and prove them.

Theorem 5.16 (Precise Monadicity Theorem / Beck's Theorem)**.** A functor $G\colon \mathbf{D} \to \mathbf{C}$ is monadic if and only if

(i) G has a left adjoint;

(ii) G creates coequalizers of G-split pairs.

Proof. ($\Longrightarrow$). Assume that G is monadic. Then by definition, G has a left adjoint F, so it remains to show that G creates coequalizers of G-split pairs. Let $\mathbb{T}$ be the monad induced by the adjunction $F \dashv G$. To do this, it suffices to show that $G^{\mathbb{T}}$ creates coequalizers of $G^{\mathbb{T}}$-split pairs, since $\mathbf{C}^{\mathbb{T}} \simeq \mathbf{D}$. So assume that $(A, \alpha) \rightrightarrows (B, \beta)$ is a G-split pair, with a split coequalizer

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] \arrow[l, bend left, "t"] & C \arrow[l, bend left, "s"] \end{tikzcd}$$

in $\mathbf{C}$. Then this split coequalizer is certainly preserved by T, since split coequalizers are preserved by any functor. Then by Theorem 5.11, $G^{\mathbb{T}}$ creates the coequalizer of this $G^{\mathbb{T}}$-split pair.

($\Longleftarrow$). Assume that G has a left adjoint F and G creates coequalizers of G-split pairs. Let $\mathbb{T}$ be the monad induced by $F \dashv G$. We want to show that $\mathbf{D} \simeq \mathbf{C}^{\mathbb{T}}$, that is, we have to construct a weak inverse L to the comparison functor K.

To define the functor L, first note that for any $\mathbb{T}$-algebra (A, α), the parallel pair

$$\begin{tikzcd} FGFA \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\varepsilon_{FA}"'] & FA \end{tikzcd}$$

is G-split, because

$$\begin{tikzcd} GFGFA \arrow[r, shift left, "GF\alpha"] \arrow[r, shift right, "G\varepsilon_{FA}"'] & GFA \arrow[r, "\alpha"] \arrow[l, bend left, "\eta_{GFA}"] & A \arrow[l, bend left, "\eta_A"] \end{tikzcd} \tag{17}$$

is a split coequalizer. Hence, Lemma 5.13 applies and there is a functor $L\colon \mathbf{C}^{\mathbb{T}} \to \mathbf{D}$ left adjoint to K.

To show that L and K form an equivalence of categories, we need to show that $KL \xrightarrow{\sim} 1_{\mathbf{C}^{\mathbb{T}}}$ and $LK \xrightarrow{\sim} 1_{\mathbf{D}}$.

To see that $KL \xrightarrow{\sim} 1_{\mathbf{C}^{\mathbb{T}}}$, let (A, α) be any $\mathbb{T}$-algebra. Then $KL(A, \alpha) = (GL(A, \alpha), G\varepsilon_{L(A,\alpha)})$, where $L(A, \alpha)$ is the coequalizer

$$FGFA \underset{\varepsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA \xrightarrow{\theta} L(A, \alpha). \tag{18}$$

Note that

$$\begin{tikzcd} GFGFA \arrow[r, shift left, "G\alpha"] \arrow[r, shift right, "G\varepsilon_{FA}"'] & GFA \arrow[l, bend left, "\eta_{GFA}"] \arrow[r, "G\theta"] & GL(A, \alpha) \end{tikzcd}$$

is a coequalizer because G creates (and therefore preserves) limits. But (17) is also a coequalizer, and therefore $G\theta = \alpha$ and $A \cong GL(A, \alpha)$. It remains to show that $\alpha = G\varepsilon_{L(A,\alpha)}$. To show this, it suffices to show that $\theta = \varepsilon_{L(A,\alpha)}$, since $G\theta = \alpha$. But we have that

$$\begin{aligned} \theta \circ F\alpha &= \theta \circ \varepsilon_{FA} && \text{by (18)} \\ &= \varepsilon_{L(A,\alpha)} \circ FG\theta && \text{naturality of } \varepsilon \\ &= \varepsilon_{L(A,\alpha)} \circ F(\alpha) && \text{because } \alpha = G\theta \end{aligned}$$

Then composing both sides on the right with $F\eta_A$, and using the fact that $\alpha \circ \eta_A = 1_A$, we obtain that $\theta = e_{L(A,\alpha)}$. So $KL \xrightarrow{\sim} 1_{\mathbf{C}^{\mathbb{T}}}$.

To see that $LK \xrightarrow{\sim} 1_{\mathbf{D}}$, let B be an object of $\mathbf{D}$. Then $LK(B) = L(GB, G\varepsilon_B)$. We know that

$$FGFGB \underset{\varepsilon_{FGB}}{\overset{FG\varepsilon_B}{\rightrightarrows}} FGB \longrightarrow L(GB, G\varepsilon_B)$$

is a coequalizer diagram. But there is another coequalizer, namely

$$FGFGB \underset{\varepsilon_{FGB}}{\overset{FG\varepsilon_B}{\rightrightarrows}} FGB \xrightarrow{\varepsilon_B} B.$$

This is a coequalizer because $FG\varepsilon_B, \varepsilon_{FGB}$ is a G-split pair by (17). Since both are coequalizers, then $B \cong L(GB, G\varepsilon_B)$. So $LK \xrightarrow{\sim} 1_{\mathbf{D}}$. □

Theorem 5.17 (Crude Monadicity Theorem)**.** A functor $G\colon \mathbf{D} \to \mathbf{C}$ is monadic if

(i) G has a left adjoint;

(ii) $\mathbf{D}$ has, and G preserves, reflexive coequalizers;

(iii) G reflects isomorphisms.

Proof. Assume (i), (ii), and (iii). This proof is very very similar to the ($\Longleftarrow$) direction for Theorem 5.16. To show that G is monadic, we have to show that G has a left adjoint F and moreover that $\mathbf{C}^{\mathbb{T}} \simeq \mathbf{D}$, where $\mathbb{T}$ is the monad induced by the adjunction $F \dashv G$.

First, we want to find a functor $L\colon \mathbf{C}^{\mathbb{T}} \to \mathbf{D}$ that is a weak inverse to the Eilenberg-Moore comparison functor K. By Lemma 5.13, it is enough to show that each parallel pair

$$FGFA \underset{\varepsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA \tag{19}$$

has a coequalizer. This is true because the above pair is reflexive with splitting $F\eta_A$, and so there is a coequalizer by assumption (ii).

Hence, there is $L\colon \mathbf{C}^{\mathbb{T}} \to \mathbf{D}$ such that $L(A,\alpha)$ is the coequalizer of (19). Now we want to show that L is a weak inverse to K, meaning that $KL \cong 1_{\mathbf{C}^{\mathbb{T}}}$ and $LK \cong 1_{\mathbf{D}}$.

Let's first show that $KL \cong 1_{\mathbf{C}^{\mathbb{T}}}$. For any $\mathbb{T}$-algebra (A,α), we have that $KL(A,\alpha) = (GL(A,\alpha), G\varepsilon_{L(A,\alpha)})$. We want to show that $GL(A,\alpha) \cong A$ and moreover that $G\varepsilon_{L(A,\alpha)} = \alpha$. We know that both

$$GFGFA \underset{G\varepsilon_{FA}}{\overset{GF\alpha}{\rightrightarrows}} GFA \xrightarrow{\alpha} A$$

and

$$GFGFA \underset{G\varepsilon_{FA}}{\overset{GF\alpha}{\rightrightarrows}} GFA \xrightarrow{G\theta} GL(A,\alpha)$$

are coequalizer diagrams, the latter since G preserves the coequalizer $L(A,\alpha)$ of the reflexive pair (19). Hence, $G\theta = \alpha$ and $GL(A,\alpha) \cong A$. So it remains to show that $G\varepsilon_{L(A,\alpha)} = \alpha$. It suffices to show that $\theta = \varepsilon_{L(A,\alpha)}$. To that end,

$$\begin{aligned}
\theta \circ F\alpha &= \theta \circ \varepsilon_{FA} && \theta \text{ is coequalizer map of (19)}\\
&= \varepsilon_{L(A,\alpha)} \circ FG\theta && \text{naturality of } \varepsilon\\
&= \varepsilon_{L(A,\alpha)} \circ F(\alpha) && \text{because } \alpha = G\theta
\end{aligned}$$

Then compose both sides on the right by $F\eta_A$ to get $\theta = \varepsilon_{L(A,\alpha)}$.

Now let's show that $LK \cong 1_{\mathbf{D}}$. For any object B of $\mathbf{D}$, we have $LKB = L(GB, G\varepsilon_B)$. This is the coequalizer of the parallel pair

$$FGFGB \underset{\varepsilon_{FGB}}{\overset{GF\varepsilon_B}{\rightrightarrows}} FGB \longrightarrow L(GB, G\varepsilon_B).$$

We want to show that this is isomorphic to B. But the following is a (split) coequalizer diagram

$$\begin{tikzcd}
GFGFGB \arrow[r, shift left, "GFG\varepsilon_B"] \arrow[r, shift right, "G\varepsilon_{FGB}"'] & GFGB \arrow[l, bend left=60, "\eta_{GFGB}"] \arrow[r, "G\varepsilon_B"] & GB \arrow[l, bend left=60, "\eta_{GB}"]
\end{tikzcd}$$

and therefore (because G preserves reflexive coequzlizers) we have that $GL(GB, G\varepsilon_B) \cong GB$. And G reflects isomorphisms, so $L(GB, G\varepsilon_B) \cong B$. $\square$

Last time we proved the monadicity theorems in two versions, so let's looks at some examples so that we understand what monadic functors look like.

Example 5.18.

(a) The forgetful functors **Gp** $\to$ **Set**, **Ring** $\to$ **Set**, R-**Mod** $\to$ **Set**, **Lat** $\to$ **Set** are all monadic. We can prove this using Lemma 5.19.

Note that if we allow infinitary algebraic structures (that is, given by maps $A^I \to A$ for some infinite set I) the left adjoint needn't exist (c.f. Example 4.21(b)). Nevertheless, we can prove some things about monadicity using the precise version of the theorem. If it does, then the forgetful functor needn't preserve reflexive coequalizers, but it can be shown to be monadic using the precise version of the theorem.

Lemma 5.19. If $\begin{tikzcd} A_1 \arrow[r, shift left=2, "f_1"] \arrow[r, shift right=2, "g_1"'] & B_1 \arrow[l, "r_1" description] \arrow[r, "h_1"] & C_1 \end{tikzcd}$ and $\begin{tikzcd} A_2 \arrow[r, shift left=2, "f_2"] \arrow[r, shift right=2, "g_2"'] & B_2 \arrow[l, "r_2" description] \arrow[r, "h_2"] & C_2 \end{tikzcd}$ are reflexive coequalizer diagrams in **Set**, so is

$$\begin{tikzcd} A_1 \times A_2 \arrow[r, bend left, "f_1 \times f_2"] \arrow[r, bend right, "g_1 \times g_2"'] & B_1 \times B_2 \arrow[l, "r_1 \times r_2"'] \arrow[r, "h_1 \times h_2"] & C_1 \times C_2 \end{tikzcd} .$$

Proof. Let $S_i = \{(f_i(a), g_i(a)) \mid a \in A_i\} \subseteq B_i \times B_i$, and let R_i be the equivalence relation generated by S_i. This equivalence relations R_i is reflexive because the coequalizers are reflexive. We need to show that $R_1 \times R_2$ is the equivalence relation generated by $S_1 \times S_2$. Note that $(b_i, b_i') \in R_i$ if and only if there is a chain $(b_i = x_1, x_2, \ldots, x_n = b_i')$ with each $(x_j, x_{j+1}) \in S_i \cup S_i^{\mathrm{op}}$. Given two such chains $(b_1 = x_1, \ldots, x_n = b_i')$ and $(b_2 = y_1, \ldots, y_m = b_2')$, we can link (b_1, b_2) to (b_1', b_2') by the chain

$$(b_1 = x_1, b_2), (x_2, b_2), \ldots, (x_n = b_1', b_2 = y_1), (b_1', y_2), \ldots, (b_1', y_m = b_2')$$

where each adjacent pair is in $(S_1 \times S_2) \cup (S_1^{\mathrm{op}} \times S_2^{\mathrm{op}})$. $\square$

Example 5.20 (Continued from Example 5.18)**.**

(b) So given a reflexive coequalizer $\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C \end{tikzcd}$, where A, B have some finitary algebraic structure and f, g are homomorphisms, then for any n-ary operation α, we have

$$\begin{tikzcd} A^n \arrow[r, shift left, "f^n"] \arrow[r, shift right, "g^n"'] \arrow[d, "\alpha_A"] & B^n \arrow[r, "h^n"] \arrow[d, "\alpha_B"] & C^n \arrow[d, dashed, "\alpha_C"] \\ A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C \end{tikzcd}$$

So C acquires a unique algebraic structure (e.g. group, ring, module, etc.) making h into a homomorphism, and a coequalizer in the category of these algebraic structures (e.g. **Gp**, **Ring**, R-**Mod**, etc.).

(c) Any reflection is monadic. This can be proved directly as in question 2 in example sheet 3. But it can also be proved using Theorem 5.16. Suppose $\mathbf{D} \hookrightarrow \mathbf{C}$ is a reflective (full) subcategory and let $A \underset{g}{\overset{f}{\rightrightarrows}} B$ be a parallel pair in $\mathbf{D}$ with a split coequalizer

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend left=60, "t"] \arrow[r, "h"] & C \arrow[l, bend left=60, "s"] \end{tikzcd}$$

Then t is in $\mathbf{D}$, and hence so is the idempotent $ft = sh \colon B \to B$.

Now s is an equalizer of sh and 1_B, so it (and hence also C and h) lives in $\mathbf{D}$ (at least up to isomorphism).

(d) Consider the composite adjunction $\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Ab} \underset{I}{\overset{L}{\rightleftarrows}} \mathbf{tfAb}$, where **tfAB** is the category of torsion-free abelian groups. This is *not* monadic, since $UILF \cong UF$ because the free abelian group is already torsion frree, and the monad structure induced by $LF \dashv UI$ is the same as that induced by $F \dashv U$.

(e) Consider the forgetful functor $U \colon \mathbf{Top} \to \mathbf{Set}$. This has a left adjoint given by the discrete space functor D that endows a set with the discrete topology. (U also preserves all coequalizers, since it also has a right adjoint). But $UD = 1_{\mathbf{Set}}$, and the corresponding category of algebras is **Set**.

(f) Consider the forgetful functor $\mathbf{KHaus} \xrightarrow{U} \mathbf{Set}$. This has a left-adjoint βD, where β is the Stone-Čech compactification functor and D is as in the previous example. This is monadic; first proved by E. Manes directly but we'll prove it using Theorem 5.16.

Suppose given $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in **KHaus** and a split coequalizer $\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbf{Y} \arrow[l, bend right=60, "t"'] \arrow[r, "h"] & Z \arrow[l, bend right=60, "s"'] \end{tikzcd}$ in **Set**. The quotient topology on Z (which makes h a coequalizer in **Top** is compact, and it's the only topology on Z that could possibly be Hausdorff and make h continuous.

So we need to show that the quotient topology on Z is Hausdorff. Using the result that if Y is compact Hausdorff, and R is an equivalence relation on Y, then Y/R is Hausdorff $\iff$ R is closed in $Y \times Y$, we reduce the problem to showing that the equivalence relation generated by (f, g) is closed. Suppose $(y, y') \in R$, then $h(y) = h(y')$ so

$$ft(Y) = sh(y) = sh(y') = ft(y').$$

So $(y, y') \in R \implies \exists\ (x, x') \in X$ with $f(x) = f(x')$ and $g(x) = y$, and $g(x') = y'$. But because this is a split coequalizer, then the reverse is also true.

Now $S = \{(x, x') \in X \times X \mid f(x) = f(x')\}$ is closed in $X \times X$ and hence compact, and R is the image of S under $g \times g$, and hence closed in $Y \times Y$.

Thus we have shown that the adjunction $\beta D \dashv U$ satisfies the conditions of Theorem 5.16, so it is monadic.

(g) The contravariant powerset functor $P^* \colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is monadic. This can be proved using Theorem 5.17 (due to R. Paré). It has a left adjoint $P^* \colon \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ (see Example 3.2(h)). It reflects isos, since if $f \colon A \to B$ is such that $f^{-1} \colon PB \to PA$ is bijective, then $f^{-1}(\operatorname{im} f) = f^{-1}(B) = A$ implies f is surjective, and there is B' with $f^{-1}(B') = \{a\}$ for all $a \in A$, which implies f is injective.

To see that this preserves split coequalizers in sets, we need another lemma.

Lemma 5.21. Suppose

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

is a pullback in **Set**. Then

$$\begin{tikzcd} PB \arrow[r, "Pg"] \arrow[d, "P^*k"] & PC \arrow[d, "P^*f"] \\ PD \arrow[r, "Ph"] & PA \end{tikzcd}$$

commutes.

Exercise 5.22. Prove this lemma.

Example 5.23 (Continued from Example 5.18)**.**

(h) So, given a coreflexive equalizer diagram $\begin{tikzcd} C \arrow[r, "h"] & B \arrow[r, shift left=2, bend left, "f"] \arrow[r, shift right=2, bend right, "g"'] & A \arrow[l, "r"'] \end{tikzcd}$ in **Set**,

we note that

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "h"] & B \arrow[d, "f"] \\ B \arrow[r, "g"] & A \end{tikzcd}$$

is a pullback since $fk = g\ell$ implies $k = rfk = rg\ell = \ell$. Therefore,

$$\begin{tikzcd} PA \arrow[r, shift left, "P^*g"] \arrow[r, shift right, "P^*f"'] & PB \arrow[r, "P^*h"] \arrow[l, bend left=60, "Pg"] & PC \arrow[l, bend left=60, "Ph"] \end{tikzcd}$$

satisfies $P^*h \circ Ph = 1_{PC}$ and $P^*g \circ Pg = 1_{PB}$, since g and h are monic, and $Ph \circ P^*h = P^*f \circ Pg$ from the lemma. In particular, the left-to-right morphisms are a coequalizer.

Definition 5.24. Suppose given an adjunction $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$ where $\mathbf{D}$ has reflexive coequalizers. We can form the **monadic tower**

$$\begin{tikzcd}
& {} \arrow[d, dotted, shift left] \\
& (\mathbf{C}^{\mathbb{T}})^{\mathbb{S}} \arrow[u, dotted, shift left] \arrow[d, shift left] \arrow[dl, shift left, "L'"] \\
\mathbf{D} \arrow[ur, shift left, "K'"] \arrow[r, shift left, "K"] \arrow[dr, shift left, "G"] & \mathbf{C}^{\mathbb{T}} \arrow[u, shift left] \arrow[l, shift left, "L"] \arrow[d, shift left] \\
& \mathbf{C} \arrow[u, shift left] \arrow[ul, shift left, "F"]
\end{tikzcd}$$

where $\mathbb{T}$ is the monad induced by $F \dashv G$, K is the comparison functor (see Theorem 5.10) and $L \dashv K$ (see Lemma 5.13). $\mathbb{S}$ is the monad induced by $L \dashv K$.

We say that $F \dashv G$ has **monadic length** n if we arrive at an equivalence of categories after n steps.

Example 5.25. The adjunction Example 5.18(c) has monadic length 2, but the adjunction Example 5.18(d) has monadic length ∞.

6 Filtered Colimits

Definition 6.1. We say a category $\mathbf{J}$ is **filtered** if every finite diagram in $\mathbf{J}$ has a cocone (sometimes called a **cone under the diagram**). A filtered poset is commonly called **directed**.

Lemma 6.2. $\mathbf{J}$ is filtered if and only if

(i) $\mathbf{J}$ is nonempty;

(ii) Given $j, j' \in \operatorname{ob} \mathbf{J}$, there is a diagram $j \to j'' \leftarrow j'$;

(iii) Given $j \underset{\beta}{\overset{\alpha}{\rightrightarrows}} j'$ in $\mathbf{J}$, there exists $j' \xrightarrow{\gamma} j''$ with $\gamma\alpha = \gamma\beta$.

Proof. ($\Longrightarrow$). Each of the three conditions is a special case of having a cocone under finite diagrams in $\mathbf{J}$.

($\Longleftarrow$). Given $D\colon I \to J$ with I finite (assume $I \neq \varnothing$), by repeated use of (ii) we can find $j \in \operatorname{ob} \mathbf{J}$ with morphisms $D(i) \xrightarrow{\alpha_i} j$ for all $i \in \operatorname{ob} I$. The triangles

$$\begin{tikzcd} D(i) \arrow[r, "\alpha_i"] \arrow[d, "D(\gamma)"'] & j \\ D(i') \arrow[ur, "\alpha_i'"'] & \end{tikzcd}$$

don't necessarily commute, but by repeated use of (iii), we can find $j \xrightarrow{\beta} j'$ such that the arrows $D(i) \xrightarrow{\beta\alpha_i} j'$ form a cone under D. $\square$

Lemma 6.3. Suppose $\mathbf{C}$ has finite colimits and (small) filtered colimits. Then $\mathbf{C}$ is cocomplete.

Proof. Since $\mathbf{C}$ has coequalizers, we need only construct coproducts $\sum_{i\in I} A_i$. Let $P_f I$ be the poset of finite subsets of I, ordered by set-theoretic inclusion. This is clearly directed. For $I' \in P_f I$, let $A_{I'}$ be the coproduct $\sum_{i \in I'} A_i$; when $I' \subseteq I''$, we have that $A_{I'} \xrightarrow{f} A_{I''}$ defined by $f\nu_i = \nu_i$ for all $i \in I'$, and a colimit for this diagram of shape $P_f I$ given by the maps $f\colon A_{I'} \to A_{I''}$ has the universal property of $\sum_{i \in I} A_i$. $\square$

Suppose now given a diagram $D\colon I \times J \to \mathbf{C}$, where $\mathbf{C}$ has limits of shape I and colimits of shape J. For each $j \in \operatorname{ob}\mathbf{J}$, there is a diagram $D(-,j)\colon I \to \mathbf{C}$, and so we can form $\lim_I D(-,j)$. Now we can use the fact that D is natural in both i and j to form maps $\lim_I D(-,j) \to \lim_I D(-,j')$:

$$\begin{tikzcd} \lim_I D(-,j) \arrow[r] \arrow[d, dashed] & D(i,j) \arrow[r] \arrow[d] & D(i',j) \arrow[d] \\ \lim_I D(-,j') \arrow[r] & D(i,j') \arrow[r] & D(i',j') \end{tikzcd}$$

The limits themselves therefore form a diagram of shape J, so we can form $\operatorname{colim}_J \lim_I D$,

$$\begin{tikzcd} \lim_I D(-,j) \arrow[rr] \arrow[d] & & D(i,j) \arrow[r] \arrow[d] & D(i',j) \arrow[d] \\ \lim_I D(-,j') \arrow[rr] \arrow[d] & & D(i,j') \arrow[r] \arrow[d] & D(i',j') \arrow[d] \\ \operatorname{colim}_J \lim_I D \arrow[r] & \lim_I \operatorname{colim}_J D \arrow[r] & \operatorname{colim}_J D(i,-) \arrow[r] & \operatorname{colim}_J D(i',-) \end{tikzcd}$$

similarly, we can form $\lim_I \operatorname{colim}_J D$.

For all i the object $\operatorname{colim}_J D(i,-)$ is the apex of a cocone under the diagram consisting of the $\lim_I D(-,j)$. Therefore there is a morphism $\operatorname{colim}_J \lim_I D \to$

$\operatorname{colim}_J D(i,-)$ for each i, making $\operatorname{colim}_J \lim_I D$ into a cone over the diagram of the $\operatorname{colim}_J D(i,-)$. Hence, we get a canonical morphism

$$\operatorname{colim}_J \lim_I D \to \lim_I \operatorname{colim}_J D. \tag{20}$$

We say that limits of shape I **commute with** colimits of shape J in $\mathbf{C}$ if this morphism is always an isomorphism. Equivalently, if

$$\operatorname{colim}_J \colon [J, \mathbf{C}] \to \mathbf{C}$$

preserves limits of shape I, or dually

$$\lim_I \colon [I, \mathbf{C}] \to \mathbf{C}$$

preserves limits of shape J.

Note that in Example 5.18(a) we showed that reflexive coequalizers commute with finite product in **Set**.

Theorem 6.4. Let J be a small category. Then colim_J commutes with all finite limits in **Set** if and only if J is filtered.

Proof. ($\Longleftarrow$). Let's get the hard part of the theorem out of the way first. Given a diagram $D\colon J \to \mathbf{Set}$, its colimit is the quotient of $\coprod_{j \in \operatorname{ob} J} D(j)$ by the smallest equivalence relation identifying $x \in D(j)$ with $D(\alpha)(x) \in D(j')$ for any $\alpha\colon j \to j'$ in J. If J is filtered, then

(a) $x \in D(j)$ is identified with $x' \in D(j')$ if and only if there are maps α, β, $j \xrightarrow{\alpha} j'' \xleftarrow{\beta} j'$ such that $D(\alpha)(x) = D(\beta)(x')$.

(b) Moreover if $j = j'$, then we can take $\alpha = \beta$.

Now we will prove that the comparison map in (20) is surjective. Suppose given $x \in \lim_I \operatorname{colim}_J D$. Its images $x_i \in \operatorname{colim}_J(D(i,-))$ must come from elements $x_{ij} \in D(i,j)$ for some j, and using Lemma 6.2(ii), we may assume j is independent of i. Now for $\alpha\colon i \to i'$ in I, $D(\alpha, j)(x_{ij})$ and $x_{i'j}$ needn't be equal, but they have the same image in $\operatorname{colim}_J D(i',-)$, so they have the same image under $D(i', \beta)$ for some $\beta\colon j \to j'$ in J by condition (b), above. Doing this for each morphism of I, we arrive at $j \to j''$ such that the images $x_{i,j''}$ of x_{ij} under $D(i \to j'')$ and $D(j \to j'')$ define an element $x_{j''} \in \lim_I D(-, j'')$. The image of $x_{j''}$ in $\operatorname{colim}_J \lim_I D$ maps to $x \in \lim_I \operatorname{colim}_J D$, so the canonical map is surjective.

Now we need to show that the map in (20) is injective. Let $y, z \in \operatorname{colim}_J \lim_I D$ have the same image in $\lim_I \operatorname{colim}_J D$. We therefore have elements $y_j, z_j \in \lim_I D(-, j)$ for some j, mapping to y, z. Their images $y_{i,j}$, $z_{i,j}$ in $D(i,j)$ needn't be equal, but they have the same image in $\operatorname{colim}_J D(i,-)$. So there are maps $j \to j'$ mapping them to the same element of $D(i, j')$. Doing this for each i, we obtain $j \to j''$ such that $y_{ij''} = z_{ij''}$ for all i. Hence, $y_{j''} = z_{j''}$ in $\lim_I D(-, j'')$. So $y = z$ in $\operatorname{colim}_J \lim_I D$.

($\Longrightarrow$). Given $D\colon I \to J$, consider the functor $E\colon I^{\mathrm{op}} \times J \to \mathbf{Set}$ defined by $E(i,j) = J(D(i), j)$. We have $\operatorname{colim}_J E(i,-) = 1$ for all i, so $\lim_I \operatorname{colim}_J E = 1$.

Commutativity of limits with colimits says that $\operatorname{colim}_J \lim_I E = 1$, so $\lim_I E(-, j)$ nonempty for some j. But an element of this limit is a cone under D with apex j. Therefore, J is filtered. □

Remark 6.5. Last time we proved the theorem about conditions under which colimits and limits commute in **Set**. Here are a few remarks about that theorem.

(a) Limits always commute with limits: given $D: I \times J \to \mathbf{C}$ where $\mathbf{C}$ has limits of shapes I and J, both $\lim_I \lim_J D$ and $\lim_J \lim_I D$ have the universal property of $\lim_{I\times J} D$.

(b) Filtered colimits don't commute with finite limits in $\mathbf{Set}^{\mathrm{op}}$: consider the following diagram of shape $\mathbb{N}^{\mathrm{op}} \times 2$:

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{N} \arrow[r, "s"] \arrow[d] & \mathbb{N} \arrow[r, "s"] \arrow[d] & \mathbb{N} \arrow[d] \\ \cdots \arrow[r] & 1 \arrow[r] & 1 \arrow[r] & 1 \end{tikzcd}$$

where $s(n) = n + 1$. Applying $\lim_{\mathbb{N}^{\mathrm{op}}}$, we get a map $\varnothing \to 1$, so $\lim_{\mathbb{N}^{\mathrm{op}}}$ doesn't preserve epimorphisms, and hence it doesn't preserve pushouts.

(c) There is an infinitary version of Theorem 6.4, which we'll state but not prove. Given an infinite regular cardinal κ (cannot be written as the sum of fewer than κ cardinals), we say a category I is **κ-small** if the cardinality of mor I is less than κ. We say that J is **κ-filtered** if every κ-small diagram in J has a cone under it. Then the methods of Theorem 6.4 can be used to show that J is κ filtered if and only if colimits of shape J commute with all κ-small limits in **Set**.

We can, however, extend Theorem 6.4 to finitary algebraic catgories (e.g. groups, sets, rings, modules, lattices, etc.) more-or-less without difficulty.

Definition 6.6. $\mathbf{A}$ is a a **finitary algebraic category** if the ojects of $\mathbf{A}$ are sets equipped with certain finitary operations on their elements that satisfy equations like associative laws or commutative laws, and the morphisms are homomorphisms commuting with these operations.

Corollary 6.7. Let $\mathbf{A}$ be a finitary algebraic category. Then

(i) The forgetful functor $\mathbf{A} \to \mathbf{Set}$ creates filtered colimits.

(ii) Filtered colimits commute with finite limits in $\mathbf{A}$.

Proof.

(i) Since filtered colimits commute with finite products in **Set**, (in particular for the functor $A \mapsto A^n$ from $\mathbf{Set} \to \mathbf{Set}$), this follows as in Example 5.18(a), where we showed that these functors create reflexive coequalizers.

(ii) The forgetful functor $\mathbf{A} \to \mathbf{Set}$ preserves filtered colimits and finite limits, and reflects isomorphisms. So this follows from Theorem 6.4.[1]

[1] also "ridiculously easy"

□

Definition 6.8.

(a) We say a functor $F\colon \mathbf{C} \to \mathbf{D}$ is **finitary** if it preserves filtered colimits.

(b) If $\mathbf{C}$ is locally small and has filtered colimits, an object A of $\mathbf{C}$ is called **finitely presentable** if $\mathbf{C}(A, -)\colon \mathbf{C} \to \mathbf{Sets}$ is finitary.

Example 6.9. These examples justify the choice of the names "finitary" and "finitely presentable."

(a) A finitary functor $F\colon \mathbf{Set} \to \mathbf{D}$ is determined by its restriction to the full subcategory $\mathbf{Set}_{\text{fin}}$ of finite sets, since we can write the set A as a directed colimit (union) of its finite subsets, where the colimit is over the diagram given by its finite subsets and inclusions between them.

More generally, if $I\colon \mathbf{Set}_{\text{fin}} \to \mathbf{Set}$ is the inclusion functor, then for any A the arrow category $(I \downarrow A)$ has finite colimits since $\mathbf{Set}_f$ has them and I preserves them, and A is the colimit of $(I \downarrow A) \xrightarrow{U} \mathbf{Set}_f \xrightarrow{I} \mathbf{Set}$.

So given any $F\colon \mathbf{Set}_{\text{fin}} \to \mathbf{D}$ where $\mathbf{D}$ has filtered colimits, we can extend it to a functor $\widetilde{F}\colon \mathbf{Set} \to \mathbf{D}$ by setting

$$\widetilde{F}(A) = \operatorname{colim}\left((I \downarrow A) \xrightarrow{U} \mathbf{Set}_{\text{fin}} \xrightarrow{F} \mathbf{D}\right),$$

that is, the colimit over the diagram FU. Of course, we should check that this is actually a functor but that's easy. Note that this does extend F, since if A is finite then $(I \downarrow A)$ has a terminal object $(A, 1_A)$.

In fact, $\widetilde{F}$ is the left **Kan extension** of F along the inclusion I (as in question 9 on example sheet 2), and $F \mapsto \widetilde{F}$ is itself a functor $[\mathbf{Set}_{\text{fin}}, \mathbf{D}] \to [\mathbf{Set}, \mathbf{D}]$, left adjoint to the restriction $G \mapsto G|_{\mathbf{Set}_{\text{fin}}}$.

In fact, one can show that the image of the functor $F \mapsto \widetilde{F}$ consists exactly of the finitary functors $\mathbf{Set} \to \mathbf{D}$.

(b) Let $\mathbf{A}$ be a finitary algebraic category. We say an object A of $\mathbf{A}$ is **finitely presented** (*not* the same as finitely-present*able*) if if it's a quotient of a finitely generated free algebra $F\mathbf{n}$ (where $\mathbf{n} = \{1, 2, \dots, n\}$) by a finite number of relations $s = t$ where s and t are elements of $F\mathbf{n}$. For example, group presentations.

Claim 6.10. A is finitely-presented if and only if it's finitely presentable:

Proof. ($\Longrightarrow$). Let $A = \langle G; R \rangle$ be finitely presented, and suppose given $f\colon A \to \operatorname{colim}_J D$, where J is filtered and $D\colon J \to \mathbf{A}$. For each of the generators $g_1, \dots, g_n$, $f(g_i)$ is the image of $D(j) \to \operatorname{colim}_J D$ for some j. Since there are only finitely many of these, we can choose some j such that all $f(g_i)$ are in the image of $D(j) \to \operatorname{colim}_J D$. For each relation $s_i = t_i$, the elements which are the images of s_i and t_i in $D(j)$ become equal in the colimit $\operatorname{colim}_J D$, and so are equal in $D(j')$ for some map $j \to j'$. Doing this for each relation in R, we arrive at j'' such that f factors through $D(j'') \to \operatorname{colim}_J D$. So the canonical map $\operatorname{colim}_J \mathbf{A}(A, D(-)) \to \mathbf{A}(A, \operatorname{colim}_J D)$ is surjective.

Now let's prove injectivity. Given two homomorphisms $f \colon A \to D(j)$ and $g \colon A \to D(j')$ which become equal in the colimit, we can reduce to a pair $A \underset{g'}{\overset{f'}{\rightrightarrows}} D(j'')$, and by working with the generators in turn, we get $j'' \to j'''$ such that the two arrows $A \underset{g'}{\overset{f'}{\rightrightarrows}} D(j'') \to D(j''')$ are equal. Hence, f, g represent the same element of the colimit, and hence the canonical map is injective.

($\Longleftarrow$). Suppose A is finitely presentable. WE can find a presentation $\langle G; R \rangle$ for A, and consider the set of pairs (G', R') where $G' \subseteq G$ is finite, and $R' \subseteq R$ is finite and all relations in R' involve only elements of G'. Ordering these by inclusion in each factor, we get a directed poset P and a functor $P \to \mathbf{A}$ sending (G', R') to $\langle G'; R' \rangle$, whose colimit is $\langle G, R \rangle \cong A$. So $1_A \colon A \to A$ factors through one of these finite presentations $\langle G', R' \rangle$, and A is retract of this finite presentation. But any retract of a finitely presented algebra is finitely presented, having a finite presentation

$$\langle G'; R' \cup \{g = e(g) \mid g \in G'\} \rangle,$$

where e is the idempotent endomorphism of $\langle G'; R' \rangle$ obtained from factoring 1_A through $\langle G', R' \rangle$. So this means that A is finitely presented. □

Last time we introduced finitary functors and finitary presentable objects. The last thing we want to talk about in this chapter is finitary monads on the category of sets, for which the categories of algebras are finitary algebraic categories.

Lemma 6.11. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on **Set**. Then the elements of $T\mathbf{n}$ correspond bijectively to natural transformations $G^{\mathbf{n}} \to G$, where $G \colon \mathbf{Set}^{\mathbb{T}} \to \mathbf{Set}$ is the forgetful functor.

Proof. We can think of $\omega \in T\mathbf{n}$ as a map $\mathbf{1} \to T\mathbf{n}$, so it corresponds to a map $F\mathbf{1} \to F\mathbf{n}$ in $\mathbf{Set}^{\mathbb{T}}$, where $F\mathbf{1}$ represents $G \colon \mathbf{Set}^{\mathbb{T}} \to \mathbf{Set}$, and $F\mathbf{n}$ represents $G^{\mathbf{n}}$. So the result follows from Yoneda.

Explicitly, ω corresponds to the natural transformation defined by

$$\omega_{(A,\alpha)}(a_1, \ldots, a_n) = \alpha(T\mathbf{a}(\omega)) \tag{21}$$

where $\mathbf{a} \colon \mathbf{n} \to A$ is the map $i \mapsto a_i$. □

Definition 6.12. A monad $\mathbb{T} = (T, \eta, \mu)$ is **finitary** if T is finitary, that is, preserves filtered colimits.

Theorem 6.13. Finitary algebraic categories are, up to equivalence, exactly the categories of finitary monads on **Set**.

We won't go through the full proof of this theorem because it's very very boring, so instead we'll just sketch.

Proof Sketch. If $\mathbf{A}$ is a finitary algebraic category, we saw in Example 5.18(a) that $G\colon \mathbf{A} \to \mathbf{Set}$ is monadic. Also, we saw in Corollary 6.7(i) that G is finitary; hence so is $T = GF$ since F preserves all colimits.

Conversely, let $\mathbf{T} = (T, \eta, \mu)$ be a finitary monad. We know that for any A, we have $TA = \bigcup\{TA' \mid A' \in P_f A\}$, where $P_f A$ is the poset of finite subsets of A. Essentially, knowing what T does on finite sets is enough to know it everywhere.

We define a presentation for $\mathbb{T}$-algebras by operations and equations by taking an n-ary operation ω for each $\omega \in T\mathbf{n}$, where $\mathbf{n}$ is the finite set corresponding to some integer $n \geqslant 0$. Satisfying the two equations

$$\omega_A(a_1, \ldots, a_n) = a_i \qquad \text{if } \omega = \eta_{\mathbf{n}}(i) \in T\mathbf{n}$$

$$\omega_A = A^{\mathbf{n}} \xrightarrow{(\psi(1)_A, \ldots, \psi(m)_A)} A^{\mathbf{m}} \xrightarrow{\chi_A} A \quad \text{if } \omega = \mu_{\mathbf{n}}(x) \text{ where } x \in TT\mathbf{n} \text{ satisfies } x = T\psi(\chi), \text{ where } \chi \in T\mathbf{m} \text{ and } \psi\colon \mathbf{m} \to T\mathbf{n}$$

We can show (although it's very tedious to do so and we will avoid it)

(a) if the ω_A are obtained as in Lemma 6.11 from a $\mathbb{T}$-algebra structure $\alpha\colon TA \to A$, then they satisfy these equations;

(b) if A is equipped with operations satisfying these equations, then using (21) to define α yields a $\mathbb{T}$-algebra structure;

(c) a function $: A \to B$ between underlying sets of algebras commutes with the $\mathbb{T}$-algebra structures if and only if it commutes with all the ω's.

In particular, we know that $\mathbb{T}$-algebras satisfying these equations are the same as algebras in (objects of) $\mathbf{A}$. $\square$

7 Abelian and Additive Categories

Abelian and Additive categories are those categories whose hom-sets are not only sets, but also abelian groups. This is a special case of enriched categories.

Definition 7.1. Let $\mathbf{E}$ be a category equipped with a "forgetful" functor $U\colon \mathbf{E} \to \mathbf{Set}$. We say a locally small category $\mathbf{C}$ is **enriched** over $\mathbf{E}$ if the functor

$$\begin{array}{rcl} \mathbf{C}(-,-)\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} & \longrightarrow & \mathbf{Set} \\ (A, B) & \longmapsto & \mathbf{C}(A, B) \end{array}$$

factors through U.

We're interested in three particular cases: we say $\mathbf{C}$ is

(a) a **pointed category** if it's enriched over the category $\mathbf{Set}_*$ of pointed sets;

(b) **semi-additive** if it's enriched over the category **CMon** of commutative monoids;

(c) **additive** if it's enriched over the category **Ab** of abelian groups.

Thus, in a pointed category $\mathbf{C}$, we have a distinguished element $0 \in \mathbf{C}(A, B)$ satisfying $f \circ 0 = 0 = 0 \circ g$ whenever the composites are defined. In a semi-additive category, we also have a binary operation $+$ on $\mathbf{C}(A, B)$ which is associative and commutative, has 0 as a unit element, and satisfies $f(g + h) = fg + fh$ and $g(h + k) = gh + gk$ whenever the composites fg, fh, fh, gk are defined. In an additive category, this binary operation has inverses $-f$ such that $f + -f = 0$.

Remark 7.2 (Warning!)**.** This is not totally standard terminology. Some authors use "semi-additive" for what we've called additive, and "additive" for a category enriched over abelian groups with all finite products.

There may *a priori* be many ways to factor the functors $\mathbf{C}(-, -)$ through U, so in principle many different enriched structures on a category $\mathbf{C}$. But actually, they will all coincide, and we'll prove that. The first step is this lemma.

Lemma 7.3.

(i) In a pointed category, the following are equivalent:

  (a) A is initial;

  (b) A is terminal;

  (c) $1_A = 0\colon A \to A$.

(ii) Given three objects A, B, C in a semi-additive category, the following are equivalent:

  (a) there are $\pi_1\colon C \to A$, $\pi_2\colon C \to B$ making C a product of A and B;

  (b) there are $\nu_1\colon A \to C$, $\nu_2\colon A \to C$ making C into a coproduct of A and B;

  (c) there are $\pi_1, \pi_2, \nu_1, \nu_2$ satisfying $\pi_1\nu_1 = 1_A$, $\pi_2\nu_2 = 1_B$, $\pi_1\nu_2 = 0_{BA}$, $\pi_2\nu_1 = 0_{AB}$, and $\nu_1\pi_1 + \nu_2\pi_2 = 1_{\mathbf{C}}$.

Proof.

(i) It's enough to prove $(a) \iff (c)$, since $(b) \iff (c)$ is dual.

To that end, if (a) holds, there's only one morphism $A \to A$ which must be both 1_A and 0_{AA}. Conversely, if (c) holds, then any $f\colon A \to B$ satisfies $f = f1_A = f0_{AB} = 0_{AB}$. So there is a unique map $A \to B$, namely 0_{AB}.

(ii) It's enough to prove $(a) \iff (c)$, since $(b) \iff (c)$ is dual.

To that end, if (a) holds, then given π_1 and π_2, we define ν_1 and ν_2 by the first four equations in (c). Now

$$\pi_1(\nu_1\pi_1 + \nu_2\pi_2) = \pi_1\nu_1\pi_1 + \pi_1\nu_2\pi_2 = 1_A\pi_1 + 0_{BA}\pi_2 = \pi_1 + 0 = \pi_1 1_C$$

and similarly $\pi_2(\nu_1\pi_1 + \nu_2\pi_2) = \pi_2 1_C$, so $\nu_1\pi_1 + \nu_2\pi_2 = 1_C$ by uniqueness of factorizations through the product.

Conversely, assume (c). Then given $D \xrightarrow{h} A$ and $D \xrightarrow{k} B$, consider the map $\nu_1 h + \nu_2 k \colon D \to C$. We have that

$$\pi_1(\nu_1 h + \nu_2 k) = 1_A h + 0k = h$$

and also $\pi_2(\nu_1 h + \nu_2 k) = k$. But if $\ell \colon D \to C$ satisfies $\pi_1\ell = h$, $\pi_2\ell = k$, then

$$\ell = 1_C\ell = (\nu_1\pi_1\ell + \nu_2\pi_2\ell) = \nu_1 h + \nu_2 k.$$

So the factorization is unique. $\square$

Definition 7.4. An object which is both initial and terminal is called a **zero object** and denoted 0. An object which is simultaneously a product and coproduct of A and B is called a **biproduct** and denoted $A \oplus B$.

The previous lemma Lemma 7.3 has a partial converse in the following. We want to say that if we products and coproducts coincide, then our category is semi-additive.

Lemma 7.5.

(i) A category with a zero object is pointed.

(ii) In a pointed category $\mathbf{C}$ with finite products and coproducts, suppose that the canonical map $c \colon A + B \to A \times B$ defined by

$$\pi_i c \nu_j = \delta_{ij} = \begin{cases} 1_{A_i} & \text{if } i = j \\ 0_{A_i A_j} & \text{otherwise} \end{cases}$$

where $A_1 = A$, $A_2 = B$, is an isomorphism. Then $\mathbf{C}$ has a unique semi-additive structure.

Proof.

(i) We define the zero map $A \to B$ to be the unique composite from $A \to 0 \to B$. Note that any pointed structure on a category is unique, because if there are two zero maps 0_a and 0_b, then $0_b = 0_a 0_b = 0_a$.

(ii) By convention, a morphism $f \colon \sum_{j=1}^m A_j \to \prod_{i=1}^n B_i$ is represented by a matrix (f_{ij}) where $f_{ij} = \pi_i f \nu_j$. For example, c is represented by the matrix

$$\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}.$$

So given $A \overset{f}{\underset{g}{\rightrightarrows}} B$, we define $f +_\ell g$ to be the compoiste

$$A \xrightarrow{\binom{1_A}{1_A}} A \times A \xrightarrow{c^{-1}} A + A \xrightarrow{(f,g)} B$$

and $f +_r g$ to be the composite

$$A \xrightarrow{\binom{f}{g}} B \times B \xrightarrow{c^{-1}} B + B \xrightarrow{(1_B,1_B)} B.$$

We will show that these two notions of addition are the same.

It's immediate that $h(f +_\ell g) = hf +_\ell hg$ and $(f +_r g)k = fk +_r gk$ when the composites are defined.

To show that $f +_\ell 0 = f$, consider the diagram

$$\begin{tikzcd}
 & A \times A \arrow[r, "c^{-1}"] \arrow[d, "\pi_1"] & A + A \arrow[dl, "{(1,0)}"'] \arrow[dr, "{(f,0)}"] & \\
A \arrow[ur, "\binom{1}{1}"] \arrow[r, "1_A"'] & A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

Similarly, $0 +_\ell f = f$ and dually $f +_r 0 = f = 0 +_r f$.

Given $f, g, h, k \colon A \to B$, I claim that $(f +_\ell g) +_r (h +_\ell k) = (f +_r h) +_\ell (g +_r k)$, since both are the composite

$$A \xrightarrow{\binom{1}{1}} A \times A \xrightarrow{c^{-1}} A + A \xrightarrow{\begin{pmatrix} f & g \\ h & k \end{pmatrix}} B \times B \xrightarrow{c^{-1}} B + B \xrightarrow{(1,1)} B.$$

Now apply the **Eckmann-Hilton argument**:

- putting $g = h = 0$, we get $f +_r k = f +_\ell k$;
- putting $f = k = 0$, we get $g + h = h + g$;
- putting $h = 0$, we get $(f + g) + k = f + (g + k)$.

For the uniqueness, suppose $+_a$ is a semi-additive structure. The argument of Lemma 7.3(ii) shows that c^{-1} must be $\nu_1\pi_1 +_a \nu_2\pi_2 \colon A \times B \to A + B$, so the definitions of $f +_r g$ and $f +_\ell g$ both reduce to $f +_a g$. Hence, this structure is unique. □

Corollary 7.6. Let **C** and **D** be categories with finite biproducts (and therefore semi-additive). Then a functor $F \colon \mathbf{C} \to \mathbf{D}$ is semi-additive (satisfies $F0 = 0$ and $F(f + g) = Ff + Fg$) if and only if it preserves finite biproducts.

Proof. ($\Longrightarrow$) is immediate from Lemma 7.3, and ($\Longleftarrow$) comes from Lemma 7.5. □

Remark 7.7. Note that a composite

$$A \oplus B \xrightarrow{\begin{pmatrix} f & g \\ h & k \end{pmatrix}} C \oplus D \xrightarrow{\begin{pmatrix} t & u \\ r & w \end{pmatrix}} E \oplus F$$

is given by the matrix product

$$\begin{pmatrix} t & u \\ r & w \end{pmatrix} \begin{pmatrix} f & g \\ h & k \end{pmatrix} = \begin{pmatrix} tf + uh & tg + uk \\ vf + wh & vg + wk \end{pmatrix}$$

Definition 7.8. Let $\mathbf{C}$ be a pointed category, $A \xrightarrow{f} B$ a morphism of $\mathbf{C}$. By a **kernel** of f, we mean an equalizer of $A \overset{f}{\underset{0}{\rightrightarrows}} B$. Equivalently, it's a map $k \colon K \to A$ such that $fk = 0k = 0$, and is universal among such maps.

We say a mono in $\mathbf{C}$ is **normal** if it occurs as a kernel. Note that in an additive category, every regular mono is normal, since an equalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ has the same universal property as the kernel of $f - g$.

We say that $A \xrightarrow{f} B$ is **pseudo-monic** if its kernel is a zero map, i.e. $fg = 0$ implies $g = 0$. Again, in an additive category, pseudo-monic $\iff$ monic since $fg = fh \iff f(g-h) = 0$.

Example 7.9.

(a) Consider $\mathbf{Gp}$: in this category, all injective homomorphisms are regular monic, but not all subgroup inclusions are normal. For example, $\mathbb{Z}/2\mathbb{Z} \hookrightarrow S_3$ is a non-normal monomorphism.

But any surjective homomorphism $G \xrightarrow{f} H$ is normal epic, since it's the cokernel of $\ker f \rightarrowtail G$.

(b) In $\mathbf{Set}_*$, any injection $(A', *) \hookrightarrow (A, *)$ is a normal mono: it's the kernel of the map $(A, *) \to ((A \backslash A') \cup \{*\}, *)$ given by sending everything in A' to $*$ and everything outside of A' to itself.

But not all (regular) epimorphisms are normal, since a normal epimorphism is bijective on elements not sent to the basepoint.

Also, not all pseudo-monos are monic: f is pseudo-monic if and only if $f^{-1}(*) = \{*\}$.

Lemma 7.10. If $\mathbf{C}$ is a pointed category with kernels and cokernels, then $A \xrightarrow{f} B$ is normal monic if and only if it's the kernel of its own cokernel.

Proof. ($\Longleftarrow$). This is trivial, because if f is the kernel of its own cokernel, then f is the kernel of something and hence normal monic.

($\Longrightarrow$). Suppose f is the kernel of $B \xrightarrow{g} C$. From the diagram

$$\begin{tikzcd}
K \arrow[dr, tail, "k"] & & C \\
& B \arrow[ur, "g"] \arrow[dr, "q"] & \\
A \arrow[ur, tail, "f"] & & Q
\end{tikzcd}$$

where $q = \operatorname{coker} f$ and $k = \ker q$. Then there is a map $h \colon Q \to C$ since $gf = 0$

and $\ell\colon A \to K$ since $qf = 0$.

$$\begin{tikzcd}
K \arrow[dr, tail, "k"] & & C \\
& B \arrow[ur, "g"] \arrow[dr, "q"'] & \\
A \arrow[uu, "\ell"] \arrow[ur, tail, "f"] & & Q \arrow[uu, "h"']
\end{tikzcd}$$

We get a map $m\colon K \to A$ since $gk = hqk = 0$, so k factors through $f = \ker g$.

$$\begin{tikzcd}
K \arrow[dr, tail, "k"] \arrow[dd, shift left, "m"] & & C \\
& B \arrow[ur, "g"] \arrow[dr, "q"'] & \\
A \arrow[uu, shift left, "\ell"] \arrow[ur, tail, "f"] & & Q \arrow[uu, "h"']
\end{tikzcd}$$

Now m, ℓ are inverse isos, since f and k are monic. Hence, $A \cong K$ and f is the kernel of its own cokernel. $\square$

Remark 7.11 (First Isomorphism Theorem)**.** Note that under the hypotheses of Lemma 7.10, there's a bijection between isomorphism classes of normal subobjects and normal quotients of any object.

Definition 7.12. By an **image** of $f\colon A \to B$ in any category $\mathbf{C}$ we mean a factorization $A \xrightarrow{g} I \overset{m}{\rightarrowtail} B$ of f, where m is the least among subobjects of B through which f factors.

Remark 7.13 ("A fact so obvious that I won't bother to number it.")**.** If $\mathbf{C}$ is pointed with kernels and cokernels, and all monos in $\mathbf{C}$ are normal, then every f has an image, namely the kernel of the cokernel of f. We have a factorization

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr] & & B \\
& I \arrow[ur, "\ker\operatorname{coker} f"'] &
\end{tikzcd}$$

Proof. Let $A \longrightarrow B' \overset{m}{\rightarrowtail} B$ be another factorization of f. Since m is monic, then m is the kernel of some $g\colon B \to C$ because all monos are normal. We have that $gm = 0$, and therefore g factors through the cokernel q of f, say via $h\colon Q \to C$ satisfying $hq = g$. Then $gk = hqk = 0$, hence k factors through $\ker g = m$.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr] \arrow[ddr, bend right] & & B \arrow[r, "g"] \arrow[dr, "q"'] & C \\
& I \arrow[ur, tail, "k"] \arrow[d, dashed] & & Q \arrow[u, "h"'] \\
& B' \arrow[uur, bend right, "m"'] & &
\end{tikzcd}$$

□

Given a morphism $A \xrightarrow{f} B$ in a pointed category $\mathbf{C}$ with kernels and cokernels, we can form the diagram

$$\begin{tikzcd}
A \arrow[rrr, "f"] \arrow[dr, two heads, "\operatorname{coker}\ker f"'] \arrow[drr, "\ell"] & & & B \\
& Q \arrow[r, "i"'] \arrow[urr] & K \arrow[ur, tail, "\ker\operatorname{coker} f"'] &
\end{tikzcd}$$

where I is the image of f, and Q is the coimage of f. If $\mathbf{C}$ is nice enough, the coimage and image coincide.

Lemma 7.14.

(a) if all monos in $\mathbf{C}$ are normal, then i is pseudo-epic;

(b) if all epis in $\mathbf{C}$ are normal, then i is pseudo-monic;

Proof. We will prove only (a), because (b) is dual. The map ℓ exists because $(\operatorname{coker} f)f = 0$ implies that f factors through $\ker\operatorname{coker} f$. Then

$$(\ker\operatorname{coker} f) \circ \ell \circ (\ker f) = f \circ (\ker f) = 0,$$

which implies that $\ell \circ (\ker f) = 0$ since $(\ker\operatorname{coker} f)$ is monic. This means that ℓ factors through $\operatorname{coker}\ker f$, and thus constructs the map i.

Now take the image factorization of i, $i = (\ker\operatorname{coker} i) \circ s$. Since $\ker\operatorname{coker} i$ is monic, then so too is $\ker\operatorname{coker} i \circ \ker\operatorname{coker} f$ as the composition of monos. Hence, $\ker\operatorname{coker} i \circ \ker\operatorname{coker} f$ is the kernel of some $g\colon B \to C$. Then we know that

$$\begin{aligned}
gf &= g \circ (\ker\operatorname{coker} f) \circ \ell \\
&= g \circ (\ker\operatorname{coker} f) \circ i \circ \operatorname{coker}\ker f \\
&= g \circ (\ker\operatorname{coker} f) \circ (\ker\operatorname{coker} i) \circ s \circ (\operatorname{coker}\ker f) \\
&= 0
\end{aligned}$$

because $\ker\operatorname{coker} f \circ \ker\operatorname{coker} i$ is the kernel of g. Hence, g factors through $\operatorname{coker} f$, via some t, $g = t \circ \operatorname{coker} f$. Thus,

$$g \circ (\ker\operatorname{coker} f) = t \circ (\operatorname{coker} f) \circ (\ker\operatorname{coker} f) = 0.$$

Therefore, $(\ker\operatorname{coker} f)$ factors through $\ker g = (\ker\operatorname{coker} f) \circ (\ker\operatorname{coker} i)$. Thus, we get some v such that

$$(\ker\operatorname{coker} f) = (\ker\operatorname{coker} f) \circ (\ker\operatorname{coker} i) \circ v.$$

But $(\ker\operatorname{coker} f)$ is monic, so this means that $(\ker\operatorname{coker} i) \circ v = 1_K$. Therefore, $\ker\operatorname{coker} i$ is split epic as well as monic, and therefore an isomorphism. So the kernel of the cokernel of i is an isomorphism, and therefore $\operatorname{coker} i = 0$. Hence, i is pseudo-epic. □

Note that either hypothesis in Lemma 7.14 implies that **C** is balanced, since epi & normal mono implies iso, and dually mono & normal epi implies iso.

Hence if both hypotheses (a) and (b) hold, and **C** is additive, then i an isomorphism. That is, the image and coimage factorizations of f exist and coincide.

Definition 7.15. We say a category **A** is **abelian** if

(a) **A** is additive;

(b) **A** has all finite limits and colimits (equivalently, it suffices to have finite biproducts, kernels, and cokernels);

(c) Every mono and every epi in **A** is normal (= regular, but we use different terminology because it's slightly different).

Example 7.16. (a) **Ab** is abelian.

(b) For any ring R, $\mathbf{Mod}_R$ is abelian (R need not be commutative).

(c) For any small category **C** and abelian category **A**, then $[\mathbf{C}, \mathbf{A}]$ is abelian with everything defined pointwise.

(d) If **C** is a small additive category and **A** is abelian, then the full subcategory $\mathbf{Add}(\mathbf{C}, \mathbf{A}) \subseteq [\mathbf{C}, \mathbf{A}]$ of additive functors is abelian.

(e) An additive category on one object is a ring R, and therefore example (d) contains example (b), in the case that **C** is an abelian category with one object. Notice that $\mathbf{Mod}_R \cong \mathbf{Add}(R, \mathbf{Ab})$.

Recall from Remark 4.16 that pullbacks of monos are monic, and dually pushouts of epimorphisms are epic. In an abelian category, we also have that pullbacks of epis are epic. To prove this, we need the lemma

Lemma 7.17. Suppose given a (not necessarily commutative) square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$

in an additive category with finite biproducts. Then

(i) the square commutes if and only if the composite $A \xrightarrow{\binom{f}{g}} B \oplus C \xrightarrow{(h,-k)} D$ is zero (We call this the **flattening** of this square).

(ii) The square is a pullback if and only if $\binom{f}{g} = \ker(h, -k)$.

(iii) The square is a pushout if and only if $(h, -k) = \operatorname{coker} \binom{f}{g}$. (Notice that this isn't quite dual to (ii) because of the minus sign!)

Proof.

(i) The composite $(h,-k)\binom{f}{g} = hf - kg$, so this is zero if and only if $hf = kg$.

(ii) This holds since $\ker(h,-k)$ has teh universal property of a pullback of (h,k).

(iii) Similar to (ii). □

Corollary 7.18. Let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{k} & D \end{array}$$

be a pullback square in an abelian category, with h epic. Then

(i) The square is also a pushout.

(ii) g is epic.

Proof.

(i) By Lemma 7.17(i), we have $\binom{f}{g} = \ker(h,-k)$. But h is an epimorphism, so $(h,-k)$ is epic as well. Since we're in an abelian category, $(h,-k)$ is normal epic, so $(h,-k) = \operatorname{coker}\binom{f}{g}$ by Lemma 7.10. Then by Lemma 7.17(iii), it's a pushout.

(ii) To show that g is epic, it's enough to show that g is pseudo-epic, since we're in an abelian category (which is in particular additive). So suppose given $C \xrightarrow{x} E$ with $xg = 0$. Then x together with $B \xrightarrow{0} E$ forms a cone under (f,g). Therefore, x factors through the pushout, say by $D \xrightarrow{y} E$. Now $yh = 0$ and h is epic, so $y = 0$. Hence, $x = yk = 0$. So g is epic. □

Corollary 7.19. In an abelian category, image factorizations are stable under pullback.

7.1 Homology

Definition 7.20. Given a sequence of objects and morphisms

$$\cdots \longrightarrow A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \longrightarrow \cdots$$

in an abelian category, we say the sequence is **exact** at A_n if $\ker f_n = \operatorname{im} f_{n-1}$, or equivalently, $\operatorname{coker} f_{n-1} = \operatorname{coimage} f_n$.

We say that the sequence is **exact** if it is exact at every vertex (except possibly the end vertices, if it does end).

We say that a functor $F\colon \mathbf{A} \to \mathbf{B}$ is **exact** if it preserves exact sequences.

Remark 7.21. (a) Note that $0 \to A \xrightarrow{f} B$ is exact if and only if f is monic.

(b) Dually, $A \xrightarrow{f} B \to 0$ is exact if and only if f is epic.

(c) $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $f = \ker g$.

(d) $A \xrightarrow{f} B \xrightarrow{1_B} B$ is exact if and only if $f = 0$. So an exact functor F preserves zero morphisms, and hence the zero object. Hence it also preserves kernels and cokernels; F also preserves finite biproducts since

$$0 \longrightarrow A \xrightarrow{\binom{1}{0}} A \oplus B \xrightarrow{(0,1)} B \longrightarrow 0$$

is exact, and $A \oplus B$ is characterized by the existence of such a sequence with $\binom{1}{0}$ split monic and $(0, 1)$ split epic.

(e) If F is additive and preserves kernels and cokernels, then it preserves images and coimages, so preserves all exact sequences.

Definition 7.22. We say F is **left-exact** if it's additive and preserves kernels (equivalently, preserves finite limits). Note that F is left-exact if and only if F preserves exact sequences of the form $0 \to A \to B \to C$.

F is **right-exact** if it's additive and preserves cokernels (equivalently, preserves finite colimits), or equivalently preserves exact sequences of the form $A \to B \to C \to 0$.

Lemma 7.23 (The Five Lemma)**.** Suppose given a commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "a_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "a_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "a_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "a_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "b_1"] & B_2 \arrow[r, "b_2"] & B_3 \arrow[r, "b_3"] & B_4 \arrow[r, "b_4"] & B_5
\end{tikzcd}$$

whose rows are exact. Then

(i) f_1 epic and f_2, f_4 monic $\implies$ f_3 monic.

(ii) f_2 and f_4 epic, f_5 monic $\implies$ f_3 epic.

Proof. This is not too different from chasing elements around a diagram. Notice that (i) and (ii) are dual, so we need only prove (i).

So suppose given $C \xrightarrow{x} A_3$ with $f_3 x = 0$. Then

$$f_4 a_3 x = b_3 f_3 x = 0.$$

Therefore $a_3 x = 0$ since f_4 is monic. So x factors through $\ker a_3 = \operatorname{im} a_2$. So if we form the pullback

$$\begin{tikzcd}
D \arrow[r, "z"] \arrow[d, "y"] & C \arrow[d, "x"] \\
A_2 \arrow[r, "a_2"] & A_3
\end{tikzcd}$$

with z epic because it's a pullback of coimage a_2. Now

$$b_2 f_2 y = f_3 a_2 y = f_3 x z = 0$$

since $f_3 x = 0$, so $f_2 y$ factors through $\ker b_2 = \operatorname{im} b_1$. Since f_1 is epic, we also know that $\operatorname{im} b_1 = \operatorname{im}(b_1 f_1)$. Form another pullback

$$\begin{tikzcd}
E \arrow[r, "v"] \arrow[d, "u"] & D \arrow[d, "f_2 y"] \\
A_1 \arrow[r, "b_1 f_1"] & B_2
\end{tikzcd}$$

and note that v is epic. Now

$$f_2 a_1 u = b_1 f_1 u = f_2 y v,$$

but f_2 is monic so $a_1 u = yv$. So

$$xzv = a_2 y v = a_2 a_1 u = 0$$

because $a_2 a_1 = 0$. However, z, v are both epic, so from this we conclude that $x = 0$. This establishes that f_3 is monic, since $f_3 x = 0 \implies x = 0$. □

Lemma 7.24 (Snake Lemma)**.** Suppose given the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & A_1 \arrow[r, "f_1"] \arrow[d, "u_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "u_2"] & A_3 \arrow[d, "u_3"'] \arrow[dddll, bend left, "s"] & \\
 & B_1 \arrow[r, "g_1"] \arrow[d, "v_1"] & B_2 \arrow[r, "g_2"] \arrow[d, "v_2"'] & B_3 \arrow[r] \arrow[d, "v_3"] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d, "w_1"] & C_2 \arrow[r] \arrow[d, "w_1"] & C_3 \arrow[d, "w_1"] & \\
 & D_1 \arrow[r, "k_1"] \arrow[d] & D_2 \arrow[r, "k_2"] \arrow[d] & D_3 \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in an abelian category $\mathbf{A}$ with exact rows and columns (the diagram in black). Then there are exact morphisms f_1, f_2, s, k_1, k_2 (in blue) forming an exact sequence. In addition, if $0 \to B_1 \to B_2$ (resp. $C_2 \to C_3 \to 0$) is exact, then so is $0 \to A_1 \to A_2$ (resp. $D_2 \to D_3 \to 0$).

Proof. (NON-EXAMINABLE). See handout. For the last assertion, observe that g_1 monic $\implies g_1 u_1 = u_2 f_1$ monic $\implies f_1$ monic and dually. □

Definition 7.25. By a **(chain) complex** in an abelian category $\mathbf{A}$, we mean a sequence

$$\cdots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \cdots$$

satisfying $f_n f_{n+1} = 0$ for all n.

Complexes in $\mathbf{A}$ form a category, which is a full subcategory of the functor category $[\mathbb{Z}^{\mathrm{op}}, \mathbf{A}]$. (In fact, it's $\mathbf{Add}(\mathbf{Z}, \mathbf{A})$, where $\mathbf{Z}$ is a small additive category whose objects are integers). So the category of complexes in $\mathbf{A}$ is abelian, with all structure defined pointwise.

Exercise 7.26. Figure out what the category $\mathbf{Z}$ is.

If we have a complex, we might wonder when it's exact. The homology of a complex measures it's failure to be exact.

Definition 7.27. Given a complex $A_\bullet = (\cdots \to A_{n+1} \to A_n \to A_{n-1} \to \cdots)$, we define it's n**-th homology object** $H_n(A_\bullet)$ as follows:

$$\begin{tikzcd}
A_{n+1} \arrow[rrrr, "f_{n+1}"] \arrow[rrdd, two heads] & & & & A_n \arrow[rrrr, "f_n"] \arrow[rd, two heads] & & & & A_{n-1} \\
& & & Z_n \arrow[ru, tail] \arrow[rd, two heads] & & Q_n \arrow[rrd, two heads] & & & \\
& & I_n \arrow[ru, tail] & & H_n \arrow[ru, tail] & & & I_{n-1} \arrow[ruu, tail] &
\end{tikzcd}$$

form $Z_n \rightarrowtail A_n = \ker f_n$ and $(I_n \rightarrowtail A_n) = \operatorname{im} f_{n+1}$. Then

$$(Z_n \twoheadrightarrow H_n) = \operatorname{coker}(I_n \rightarrowtail Z_n)$$

and H_n is the n-th homology object.

Remark 7.28. If we do the dual thing, it's symmetric and gives the same definition. We could define H_n symmetrically as the image of $Z_n \to A_n \to Q_n$, then

$$Z_n \twoheadrightarrow H_n = \operatorname{coker} \ker(Z_n \to Q_n) = \operatorname{coker}(I_n \to Z_n),$$

and

$$H_n \rightarrowtail Q_n = \ker \operatorname{coker}(Z_n \to Q_n) = \ker(Q_n \to I_{n-1}).$$

Remark 7.29. Clearly, a morphism of complexes $A_\bullet \to B_\bullet$ induces morphisms $Z_n(A_\bullet) \to Z_n(B_\bullet)$, $Q_n(A_\bullet) \to Q_n(B_\bullet)$, $I_n(A_\bullet) \to I_n(B_\bullet)$ and $H_n(A_\bullet) \to H_n(B_\bullet)$ for all n.

So we can regard H_n as a functor $\mathbf{Add}(\mathbf{Z}, \mathbf{A}) \to \mathbf{A}$.

Theorem 7.30 (Meyer-Vietoris)**.** Suppose given an exact sequence

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

of chain complexes. Then there exists an exact sequence

$$\cdots \to H_n(A_\bullet) \to H_n(B_\bullet) \to H_n(C_\bullet) \to H_{n-1}(A_\bullet) \to H_{n-1}(B_\bullet) \to H_{n-1}(C_\bullet) \to \cdots$$

Proof. This follows from the Snake Lemma (Lemma 7.24). We have a diagram in black with exact rows and columns, so we get a blu exact sequence.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z_n(A_\bullet) \arrow[r] \arrow[d] & Z_n(B_\bullet) \arrow[r] \arrow[d] & Z_n(C_\bullet) \arrow[d] \arrow[dddll, rounded corners] & \\
& A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[d] & \\
& Q_n(A_\bullet) \arrow[r] \arrow[d] & Q_n(B_\bullet) \arrow[r] \arrow[d] & Q_n(C_\bullet) \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Now that we have this exact sequence, consider the new diagram below in black with exact rows an columns. By the Snake Lemma (Lemma 7.24), we again get an exact sequence in blue as indicated.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& H_n(A_\bullet) \arrow[r] \arrow[d] & H_n(B_\bullet) \arrow[r] \arrow[d] & H_n(C_\bullet) \arrow[d] \arrow[dddll, rounded corners] & \\
& Q_n(A_\bullet) \arrow[r] \arrow[d] & Q_n(B_\bullet) \arrow[r] \arrow[d] & Q_n(C_\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_{n-1}(A_\bullet) \arrow[r] \arrow[d] & Z_{n-1}(B_\bullet) \arrow[r] \arrow[d] & Z_{n-1}(C_\bullet) \arrow[d] & \\
& H_{n-1}(A_\bullet) \arrow[r] \arrow[d] & H_{n-1}(B_\bullet) \arrow[r] \arrow[d] & H_{n-1}(C_\bullet) \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Moreover, the maps $H_n(A_\bullet) \to H_n(B_\bullet) \to H_n(C_\bullet)$ are exactly $H_n(A_\bullet \to B_\bullet)$ and $H_n(B_\bullet \to C_\bullet)$, so these sequences fit together. $\square$

Definition 7.31. Let $A_\bullet$ and $B_\bullet$ be complexes and $A_\bullet \overset{f_\bullet}{\underset{g_\bullet}{\rightrightarrows}} B_\bullet$ be two morphisms of complexes. By a **homotopy** from $f_\bullet$ to $g_\bullet$, we mean a sequence of morphisms $h_n \colon A_n \to B_{n+1}$

$$\begin{tikzcd}
& A_n \arrow[r, "a_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & A_{n-1} \arrow[dl, "h_{n-1}"] \\
B_{n+1} \arrow[r, "b_{n+1}"'] & B_n &
\end{tikzcd}$$

satisfying $f_n - g_n = b_{n+1}h_n + h_{n-1}a_n \colon A_n \to B_n$ for all n.

Remark 7.32. Homotopy is an equivalence relation $f \simeq f$ by the zero homotopy; if $h \colon f \simeq g$ then $-h \colon g \simeq f$; if $h \colon f \simeq g$ and $k \colon g \simeq \ell$, then $h + k \colon f \simeq \ell$.

And moreover, it's compatible with composition, so it's a congruence on $\mathbf{Add}(\mathbf{Z}, \mathbf{Ab})$.

Lemma 7.33. Homotopic maps of chain complexes induce the same morphisms on homology.

Proof. Suppose $h \colon f \simeq g$. The composite

$$Z_n(A_\bullet) \to A_n \xrightarrow{f-g} B_n$$

is equal to

$$Z_n(A_\bullet) \to A_n \xrightarrow{h_n} B_{n+1} \xrightarrow{b_n} B_n,$$

since $Z_n(A_\bullet) \to A_n \to A_{n-1} = 0$.

And moreover, the composite $Z_n(A_\bullet) \to A_n \to B_{n+1} \to H_n(B_\bullet)$ is zero, since $(B_{n+1} \to H_n(B_\bullet)) = 0$. So $H_n(f_\bullet) = H_n(g_\bullet)$.

$$\begin{tikzcd}
Z_n(A_\bullet) \arrow[r] & A_n \arrow[r, "h_n"] & B_{n+1} \arrow[rrr, "b_n"] \arrow[dr] & & & B_n \\
& & & I_n(B_\bullet) \arrow[r] & Z_n(B_\bullet) \arrow[ur] \arrow[dr, two heads] & \\
& & & & & H_n(B_\bullet)
\end{tikzcd}$$

$\square$

Definition 7.34 (Recall from Definition 2.21)**.** An object $A \in \operatorname{ob} \mathbf{C}$ is called **projective** if the functor $\mathbf{C}(A, -) \colon \mathbf{C} \to \mathbf{Set}$ preserves epis.

Remark 7.35. A functor of the form $\mathbf{C}(A, -)$ preserves any limits which exist in $\mathbf{C}$ because limits are computed on the domain. If $\mathbf{C}$ is abelian, then $\mathbf{C}(A, -)$ is a left-exact functor $\mathbf{C} \to \mathbf{Ab}$. So A projective $\iff$ $\mathbf{C}(A, -)$ is exact.

Definition 7.36. We say that $\mathbf{C}$ has **enough projectives** if, for every $A \in \operatorname{ob} \mathbf{C}$, there exists $P \twoheadrightarrow A$ with P projective.

Example 7.37. The category $\mathbf{Mod}_R$ has enough projectives, since free modules are projective and every module is a quotient of a free module. The functor $(-)^X$ is represented by the free-module FX on the set X, and this functor preserves epis.

$\mathbf{Mod}_R$ also has enough injectives, but identifying injective modules is harder.

Definition 7.38. By a **projective resolution** of an object A in an abelian category $\mathbf{A}$, we mean an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

with all P_n projective.

Remark 7.39. Equivalently, we can think of a projective resolution as defining a complex with $P_{-n} = 0$ for all n; it will no longer be exact at zero, but the homology of this complex at P_0 will be A. More precisely, we can also define a projective resolution as a complex $P_\bullet$ satisfying

(a) P_n is projective for all n;

(b) $P_n = 0$ for all $n < 0$;

(c) $H_0(P_\bullet)A$, $H_n(P_\bullet) = 0$ for $n \neq 0$.

Remark 7.40. If $\mathbf{A}$ has enough projectives, then any object has a projective resolution: given A, chose $P_0 \twoheadrightarrow A$ with P_0 projective and kernel $K_0 \rightarrowtail P_0$, say, then chose $P_1 \twoheadrightarrow K_0$ with P_1 projective, and so forth.

Lemma 7.41. Suppose given projective resolutions $P_\bullet$, $Q_\bullet$ of objects A, B. Then any map $a\colon A \to B$ induces a map of complexes $f_\bullet\colon P_\bullet \to Q_\bullet$ with $H_0(f_\bullet) = a$. Moreover, any such map is unique up to homotopy.

Proof. We have the following diagram *a priori*

$$\begin{tikzcd}
\cdots \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] & P_0 \arrow[r, two heads, "d"] & A \arrow[d, "a"] \\
\cdots \arrow[r, "q_2"] & Q_1 \arrow[r, "q_1"] & Q_0 \arrow[r, two heads, "e"] & B
\end{tikzcd}$$

Because P_0 is projective, there is f_0 with $ef_0 = ad$:

$$\begin{tikzcd}
\cdots \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] & P_0 \arrow[r, two heads, "d"] \arrow[d, "f_0"] & A \arrow[d, "a"] \\
\cdots \arrow[r, "q_2"] & Q_1 \arrow[r, "q_1"] & Q_0 \arrow[r, two heads, "e"] & B
\end{tikzcd}$$

Now $ef_0p_1 = adp_1 = 0$, so f_0 factors through $\ker e = \operatorname{im} q_1$, and there must be a map $f_1\colon P_1 \to Q_1$ with $q_1 f_1 = f_0 p_1$.

$$\begin{tikzcd}
\cdots \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] \arrow[d, "f_1"] & P_0 \arrow[r, two heads, "d"] \arrow[d, "f_0"] & A \arrow[d, "a"] \\
\cdots \arrow[r, "q_2"] & Q_1 \arrow[r, "q_1"] & Q_0 \arrow[r, two heads, "e"] & B
\end{tikzcd}$$

and so on. This settles existence.

To see uniqueness up to homotopy, suppose we had another such $g_\bullet \colon P_\bullet \to Q_\bullet$.

$$\begin{tikzcd} P_0 \arrow[r, two heads, "d"] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] & A \arrow[d, "a"] \\ Q_0 \arrow[r, two heads, "e"] & B \end{tikzcd}$$

Then $e(f_0 - g_0) = ad - ad = 0$, so $f_0 - g_0$ factors through $\ker e = \operatorname{im} q_1$. So there is some $h_0 \colon P_0 \to Q_1$ with $q_1 h_0 = f_0 - g_0$ by the projectivity of P_0

$$\begin{tikzcd} & P_0 \arrow[r, two heads, "d"] \arrow[dl, "h_0"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] & A \arrow[d, "a"] \\ Q_1 \arrow[r, "q_1"'] & Q_0 \arrow[r, two heads, "e"'] & B \end{tikzcd}$$

Now

$$\begin{aligned} q_1(f_1 - g_1 - h_0 p_1) &= q_1 f_1 - q_1 g_1 - q_1 h_0 p_1 \\ &= f_0 p_1 - g_0 p_1 - q_1 h_0 p \\ &= (f_0 - g_0 - q_1 h_0) p_1 \quad = 0, \end{aligned}$$

so there is $h_1 \colon P_1 \to Q_2$ with $q_2 h_1 = f_1 - g_1 - h_0 p_1$ by the projectivity of P_1.

$$\begin{tikzcd} & P_1 \arrow[r, "p_1"] \arrow[dl, "h_1"'] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] & P_0 \arrow[r, two heads, "d"] \arrow[dl, "h_0"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] & A \arrow[d, "a"] \\ Q_2 \arrow[r, "q_2"'] & Q_1 \arrow[r, "q_1"'] & Q_0 \arrow[r, two heads, "e"'] & B \end{tikzcd}$$

Continuing in this manner we obtain the desired homotopy maps. □

Remark 7.42. Lemma 7.41 says that any two projective resolutions of a given object A are homotopy equivalent. So we can regard a choice of projective resolutions as a functor $\mathbf{A} \to \mathbf{Add}(\mathbf{Z}, \mathbf{A})/\simeq$.

Definition 7.43. Let $F \colon \mathbf{A} \to \mathbf{B}$ be an additive functor between abelian categories, and suppose that $\mathbf{A}$ has enough projectives. The **left derived functors** $L^n F$ for $(n \geqslant 0)$ are defined by $L^n F(A) = H_n(F(P_\bullet))$, where $P_\bullet$ is any projective resolution of A.

Note that this is well-defined, by Lemma 7.41 and Lemma 7.33; and $L^n F$ is a functor $\mathbf{A} \to \mathbf{B}$.

Remark 7.44. Note also that $FP_1 \to FP_0 \to L^0 FA \to 0$ is exact, so there's a canonical natural transformation $L^0 F \to F$, which is an isomorphism if F is right exact.

Lemma 7.45. Suppose given an exact sequence $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ and projective resolutions $P_\bullet$, $R_\bullet$ of A and C. Then there is a projective resolution $Q_\bullet$ of B such that $Q_n = P_n \oplus R_n$ and

$$P_n \xrightarrow{\binom{1}{0}} P_n \oplus R_n \xrightarrow{(0,1)} R_n$$

are maps of complexes.

Remark 7.46 (Warning!). In general, $Q_\bullet \not\cong P_\bullet \oplus R_\bullet$, and $q_n \colon Q_n \to Q_{n-1}$ has the form $\begin{bmatrix} p_n & s_n \\ 0 & r_n \end{bmatrix}$ with $s_n \neq 0$ in general.

Proof of Lemma 7.45. Again, we construct the first few maps and induct upwards.

$$\begin{tikzcd}
P_0 \arrow[r, two heads, "d"] \arrow[d, "\binom{1}{0}"'] & A \arrow[d, "a"] & \\
P_0 \oplus R_0 \arrow[r, "{(ad,e')}"] \arrow[d, "{(0,1)}"'] & B \arrow[r, "x"] \arrow[d, two heads, "b"] & D \\
R_0 \arrow[ur, "e'"] \arrow[r, "e"] & C \arrow[ur, "x'"] &
\end{tikzcd}$$

The map e' exists by the projectivity of $R_)$. The two right-hand squares commute, and (ad, e') is epic: suppose that $x(ad, e') = 0$. Then $xad = 0$ so $xa = 0$. So x factors as $x'b$. Now $0 = xe' = x'be' = x'e$. So $x' = 0$, which means $x = 0$.

Now form the kernels K_0, L_0, M_0; $0 \to K_0 \to L_0 \to M_0 \to 0$ is exact by Lemma 7.24. Now proceed as before to get an epi $P_1 \oplus R_1 \twoheadrightarrow L_0$, and so on.

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
P_1 \arrow[r, two heads] \arrow[d] & K_0 \arrow[r] \arrow[d] & P_0 \arrow[r, two heads, "d"] \arrow[d, "\binom{1}{0}"'] & A \arrow[d, "a"] & \\
P_1 \oplus R_1 \arrow[r] \arrow[d] & L_0 \arrow[r] \arrow[d] & P_0 \oplus R_0 \arrow[r, "{(ad,e')}"] \arrow[d, "{(0,1)}"'] & B \arrow[r, "x"] \arrow[d, two heads, "b"] & D \\
R_1 \arrow[ur] \arrow[r, two heads] & M_0 \arrow[r] \arrow[d] & R_0 \arrow[ur, "e'"] \arrow[r, "e"] & C \arrow[ur, "x'"] & \\
 & 0 & & &
\end{tikzcd}$$

$\square$

Theorem 7.47. Let $F \colon \mathbf{A} \to \mathbf{B}$ be an additive functor between abelian categories, where $\mathbf{A}$ has enough projectives. Let $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ be an exact sequence in $\mathbf{A}$. Then there exists an exact sequence

$$\cdots \longrightarrow L^2FC \to L^1FA \xrightarrow{L^1Fa} L^1FB \xrightarrow{L^1Fb} L^1FC \to L^0FA \xrightarrow{L^0Fa} L^0FB \xrightarrow{L^0Fb} L^0FC \to 0$$

Proof. Choose projective resolutions $P_\bullet$, $R_\bullet$ of A and C, respectively, and define $Q_\bullet$ as in Lemma 7.45. F preserves the exactness of the sequences

$$0 \longrightarrow P_n \xrightarrow{\binom{1}{0}} Q_n \xrightarrow{(0,1)} R_n \longrightarrow 0$$

So we can apply the Meyer Vietoris Theorem (Theorem 7.30), which constructs this sequence for us. □

Remark 7.48. In particular, if F is right exact, then then the seqeunce of Theorem 7.47 extends the sequence $FA \to FB \to FC \to 0$. If F is exact, then $L^n F = 0$ for all $n > 0$.