# Elementary Logic

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github

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# Contents


1. Introduction
2. History
3. Propositional Logic
4. Predicate Logic
5. Equational Logic
6. Set Theory
7. Recursion Theory
8. Modal Logic
9. Logic vs Game Theory

# Contents


1. Introduction
   - Logic Puzzle
   - Textbook and Homework
   - Analogical Argument
   - Mill's Five Methods
   - Fallacy and Bullshit
   - Logic and other Disciplines
2. History
3. Propositional Logic
4. Predicate Logic
5. Equational Logic
6. Set Theory
7. Recursion Theory
8. Modal Logic
9. Logic vs Game Theory

# Information Update

Figure: The information content of a formula $\varphi$ is the set $\mathrm{Mod}(\varphi)$ of its models. An update with new information $\psi$ reduces the current set of models $\mathrm{Mod}(\varphi)$ to the overlap of $\mathrm{Mod}(\varphi)$ and $\mathrm{Mod}(\psi)$.

# Three Cards Problem

- Three cards ‘red’, ‘white’, ‘blue’ are given to three children: 1, 2, 3.
- The children can only see their own cards.
- 2 asks 1: “Do you have the blue card?”
- 1 answers: “No”.

# Unfaithful Husband Puzzle

## Problem (大女子主义村出轨问题)

1. 大女子主义村规定：每个发现其老公出轨的女子必须当天枪杀之！
2. 村里住着 100 对夫妇。
3. 男人全都出轨了。
4. 虽然每个女子都知道别人的老公是否出轨，但不知道自己老公是否出轨。村里生活宁静幸福。
5. 直到某日，女王来访，并提醒："村里有男人出轨了"。
6. 宁静幸福的生活还会一直持续下去吗？

After a date, one says to the other: "Would you like to come up to my apartment to see my etchings?"

# Test

**测试**

从 $0 \sim 100$ 之间选一个数字。谁的数字最接近平均数的 $\frac{2}{3}$ 谁赢。

# Predestination — All You Zombies — Heinlein

1945 一女婴被弃孤儿院。1963 长大后的女孩与一男子邂逅、怀孕。男子失踪。女孩产下一女婴后发现自己是双性人。女婴被偷。伤心的她变性成他。开始酗酒。1970 一酒保把他招募进时光穿梭联盟。为报复负心男，酒保带他飞回 1963。他邂逅一女孩并使其怀孕。酒保乘时光机前行 9 个多月偷走女婴，并将其送至 1945 的孤儿院，然后回 1963 把他带到 1985 的联盟基地。他受命飞回 1970，化装酒保去招募一个酒鬼。

## Problem

周迅的前男友窦鹏是窦唯的堂弟；窦唯是王菲的前老公；周迅的前男友宋宁是高原的表弟；高原是窦唯的前任老婆；周迅的前男友李亚鹏是王菲的现任老公；周迅的前男友朴树的音乐制作人是张亚东；张亚东是王菲的前老公窦唯的妹妹窦颖的前老公，也是王菲的音乐制作人；张亚东是李亚鹏前女友瞿颖的现男友。

下列说法不正确的是：

1. 王菲周迅是情敌关系
2. 瞿颖王菲是情敌关系
3. 窦颖周迅是情敌关系
4. 瞿颖周迅是情敌关系

# Gateway to Heaven

**Problem (天堂之路)**

- 你面前有左右两人守卫左右两门。
- 一人只说真话，一人只说假话。
- 一门通天堂，一门通地狱。
- 你只有向其中一人提问一个“是/否”问题的机会。
- 怎么问出去天堂的路？

# Hardest Logic Puzzle Ever

## Problem (Hardest Logic Puzzle Ever)

- *Three gods, $A$, $B$, and $C$ are called in some order, $T$, $F$, and $R$.*
- *$T$ always speaks truly, $F$ always speaks falsely (if he is certain he can; but if he is unable to lie with certainty, he responds like $R$), but whether $R$ speaks truly or falsely (or whether $R$ speaks at all) is completely random.*
- *Your task is to determine the identities of $A$, $B$, and $C$ by asking $2$ ($3$) yes/no questions; each question must be put to exactly one god.*
- *The gods understand English, but will answer in their own language, in which the words for 'yes' and 'no' are 'da' and 'ja' in some order. You don't know which word means which.*

# HLPE — Solution

## Solution (assume $T$ and $F$ can't predict $R$'s answer)

1. *Directed to $A$:*
   *Would you answer 'ja' to the question of whether you would answer with a word that means 'yes' in your language to the question of whether you and $B$ would give the same answer to the question whether '$1 + 1 = 2$'?* $Q$
2. *Directed to $A$ or $B$ we now know not to be $R$:*
   $Q[C/B]$

## Solution (assume $T$ and $F$ can predict $R$'s answer)

1. *Directed to $A$:*
   *Would you answer 'ja' to the question of whether either:*
   - *$B$ isn't $R$ and you are $F$, or*
   - *$B$ is $R$ and you would answer 'da' to $Q$?* $Q$
2. *Directed to $A$ or $B$ we now know not to be $R$:*
   $Q[C/B, Q'/Q]$ $Q'$

# Contents


1. Introduction
   - Logic Puzzle
   - Textbook and Homework
   - Analogical Argument
   - Mill's Five Methods
   - Fallacy and Bullshit
   - Logic and other Disciplines
2. History
3. Propositional Logic
4. Predicate Logic
5. Equational Logic
6. Set Theory
7. Recursion Theory
8. Modal Logic
9. Logic vs Game Theory

# Outline

- Critical Thinking ✓
- History
- Term Logic
- Propositional Logic ✓
- Predicate Logic ✓
- Equational Logic
- Set Theory
- Recursion Theory
- Modal Logic

# Readings

1. P. J. Hurley: A Concise Introduction to Logic. — $B$
2. A. Hausman, H. Kahane, P. Tidman: Logic and Philosophy. — $B$
3. M. V. Lawson: A First Course in Logic. — $P$
4. P. Smith: An Introduction to Formal Logic. — $P$
5. *P. Smith: Teach Yourself Logic.* — $P$
6. H. de Swart: Philosophical and Mathematical Logic — $P$
7. T. Sider: Logic for Philosophy. — $P$
8. J. van Benthem: Logic in Action. — $P$
9. Open Logic Project. — $P$
10. H. Enderton: A Mathematical Introduction to Logic. — L
11. J. L. Bell, M. Machover: A Course in Mathematical Logic. — $L$
12. E. Mendelson: Introduction to Mathematical Logic. — $L$
13. G. S. Boolos, J. P. Burgess, R. C. Jeffrey: Computability and Logic. — $L$
14. H. Ebbinghaus, J. Flum, W. Thomas: Mathematical Logic. — $L$
15. A. Nerode, R. A. Shore: Logic for Applications. — $C$
16. J. H. Gallier: Logic for Computer Science. — $C$
17. Yu. I. Manin: A Course in Mathematical Logic for Mathematicians. — $M$
18. R. Smullyan / I. Chiswell & W. Hodges / P. G. Hinman / A. G. Hamilton / U. Schöning / W. Rautenberg / J. R. Shoenfield / S. M. Srivastava / S. Hedman / D. van Dalen / R. S. Wolf / M. Ben-Ari / R. Cori & D. Lascar

# Advanced Readings

- Modal Logic
  - P. Blackburn, M. de Rijke, Y. Venema: Modal Logic
  - J. van Benthem: Modal Logic for Open Minds
- Set Theory
  - T. Jech: Set Theory
  - K. Kunen: Set Theory
- Recursion Theory
  - R. I. Soare: Turing Computability
  - A. Nies: Computability and Randomness
  - M. Li, P. Vitányi: An Introduction to Kolmogorov Complexity and Its Applications
- Model Theory
  - D. Marker: Model Theory
  - C. C. Chang, H. J. Keisler: Model theory
- Proof Theory
  - G. Takeuti: Proof Theory

# Readings, Movies and More

- D. Hofstadter: Gödel, Escher, Bach
- Dangerous Knowledge
- The Imitation Game
- Philosophical Logic
- Philosophy of Logic
- Philosophy of Mathematics

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**Hofstadter's Law**

It always takes longer than you expect, even when you take into account Hofstadter's Law.

# Exams and Credits

- Question
- Discussion
- Exercises/Homework ✓
- Presentation
- Paper
- Examination ✓
- Techniques e.g. LaTeX / Coq ...
- ...

# Homework

Google/Wikipedia/Stanford Encyclopedia/Internet Encyclopedia

- Leibniz, Cantor, Frege, Russell, Hilbert, Gödel, Tarski, Turing.
- finite, infinite, syntax, semantics, formal system, deduction, logical consequence, consistency, satisfiability, validity, soundness, completeness, compactness, decidability
- Philosophy of Logic, Philosophical Logic
- Logicism, Formalism, Intuitionism
- Hilbert's program
- Church-Turing thesis

# Aim

- Critical thinking ✓
- Formalization of an argument ✓
- Demonstration of the validity of an argument ✓
- Object & Meta-language / Syntax & Semantics / Finite & Infinite / Countable & Uncountable / Induction & Recursion / Truth & Proof / Axiomatization / Theory / Soundness / Completeness / Compactness / Elementary Equivalent & Isomorphism / Representability / Definability / Categoricity / Decidability / Complexity / Expressiveness / Succinctness / Interpretability ... $\surd$
- Formal Philosophy
- Understanding of the nature of mathematics
- Application in Math / CS / AI / Linguistics / Cognition / Physics / Information Theory / Game Theory / Social Science ...
- Mathematical Logic

# Contents


1. Introduction
   - Logic Puzzle
   - Textbook and Homework
   - Analogical Argument
   - Mill's Five Methods
   - Fallacy and Bullshit
   - Logic and other Disciplines
2. History
3. Propositional Logic
4. Predicate Logic
5. Equational Logic
6. Set Theory
7. Recursion Theory
8. Modal Logic
9. Logic vs Game Theory

# Example

一个受过良好教育的男性吹嘘自己比女性聪明，如同吹嘘他有勇气打败一个手脚被捆绑的人一样。

学问像大海，考试像鱼钩。老师怎么能把鱼挂在鱼钩上教它在大海中学习自由、平衡的游泳呢？ — Hermite

正如在潜水时身体有一个自然的漂浮到水面的趋势，它要求我们必须将身体尽力下沉；在思考问题时，我们必须将思维尽力发挥，使我们远离肤浅的表面，下潜到哲学的深度。

# Example

- 叶开忍不住又道：“你为什么还是戴着这草帽？”
- 墨九星道：“因为外面有狗在叫。”
- 叶开怔了怔，道：“外面有狗叫，跟你戴草帽又有什么关系？”
- 墨九星冷冷道：“我戴不戴草帽，跟你又有什么关系？”

# Example

人们似乎经常相信创造力，但它所做的只不过是把事物的分界线确定下来，并赋予它一个名字。正如地理学家划出海岸线并说“这些线确定的海域为黄海”，此时他并未创造一个海；数学家也一样，他不能通过定义创造东西。 — Frege

- 我认为性教育导致怀孕。
- 是的，正如驾驶教育导致交通事故。 $^{\circledcirc}\hat{o}^{\circledcirc}$

# Example

## 庄子《齐物论》

不知周之梦为蝴蝶与，蝴蝶之梦为周与？

## 庄子《秋水》

1. 庄子：鲦鱼出游从容，是鱼之乐也。
2. 惠子：子非鱼，安知鱼之乐？
3. 庄子：子非我，安知我不知鱼之乐？
4. 惠子：我非子，固不知子矣；子固非鱼也，子之不知鱼之乐，全矣。

身是菩提树，心如明镜台，时时勤拂拭，勿使惹尘埃。

菩提本无树，明镜亦非台，本来无一物，何处染尘埃？

# Analogical Argument

$$\frac{\begin{array}{c} abcd \text{ have the attributes } PQR \\ abc \text{ have the attribute } S \end{array}}{d \text{ probably has the arribute } S}$$

1. 相似的类比物数量越多，论证越强；不相似的类比物数量越多，论证越弱。
2. 类比物的差异性越大，论证越强。
3. 类比物与目标物的相似性越多，论证越强。
4. 类比物与目标物的相似性之间越相关，论证越强。
5. 类比物与目标物之间非相似性的性质和程度，可能削弱或加强论证。
6. 结论越具体，论证越弱。

# Example

## Argument by Analogy

有妻杀夫，放火烧舍，称“火烧夫死”。夫家疑之，讼于官。妻不服。取猪两头，杀其一。积薪焚之，活者口中有灰，杀者口中无灰。因验尸，口果无灰，鞠之服罪。

## Refutation by Analogy

1. 楚王赐晏子酒，酒酣，吏二缚一人诣王。
2. 王曰：“缚者曷为者也？”对曰：“齐人也，坐盗。”
3. 王视晏子曰：“齐人固善盗乎？”
4. 晏子避席对曰：“婴闻之，橘生淮南则为橘，生于淮北则为枳，叶徒相似，其实味不同。所以然者何？水土异也。今民生齐不盗，入楚则盗，得无楚之水土，使民善盗耶？”

# Example

## Argument by Analogy

不能要求每样东西都有定义，否则如同要求任何物质都可被分解。简单物质不能被分解，逻辑上简单的东西不能被定义。 — Frege

## Refutation by Analogy

有人认为，人工智能不可能实现，因为“人工智能是建立在固体物理学之上的，而人脑是一个活的半流体系统”。照此推理，汽车也不可能代替马，因为汽车是铁做的，而马是活的血肉做的有机体。

## Refutation by Analogy

- 计算机会思考吗？
- 潜水艇会游泳吗？

# Examples — Does God Exist?

如果我们看见某个复杂精巧的机械装置，比如一块手表，我们会推测它是由某人制造的。我们所处的宇宙是一个错综复杂却运行精巧的自然机制，所以，我们应当推测它也有一个造物主。

众多宇宙中的每一个都有各自的规律和参数。有些适合生命生存，有些不适合。有的甚至发展出了能提出人择问题的高级生命。但这就像是一场随机摸彩。总会有人赢，而没有人在刻意挑选赢家。仅仅因为一个宇宙具有一套独特的规律和参数，不能推出它是被造物主精心设计的。

# Contents


1. Introduction
   - Logic Puzzle
   - Textbook and Homework
   - Analogical Argument
   - Mill's Five Methods
   - Fallacy and Bullshit
   - Logic and other Disciplines
2. History
3. Propositional Logic
4. Predicate Logic
5. Equational Logic
6. Set Theory
7. Recursion Theory
8. Modal Logic
9. Logic vs Game Theory

# Method of Agreement

$$\frac{\begin{array}{c} ABCD \rightarrow wxyz \\ AEFG \rightarrow wtuv \end{array}}{A \rightarrow w}$$

某天下午很多同学突然腹泻，我们得了解原因。腹泻的人中午吃了什么？某些人吃了但不是所有病人都吃了的食物应该不是病因。

一天晚上老王看了两小时书，喝了很多浓茶，用热水泡了脚，结果失眠了；第二天晚上他看了两小时电视，抽了很多烟，用热水泡了脚，结果又失眠了；第三天他听了两小时音乐，喝了很多咖啡，用热水泡了脚，结果再次失眠。根据求同法，用热水泡脚是失眠的原因。◎ô◎

# Method of Difference

$$\frac{\begin{array}{c} ABCD \to wxyz \\ \overline{A}BCD \to \overline{w}xyz \end{array}}{A \to w}$$

秋末冬初街道旁的响叶杨纷纷开始落叶，但高压水银灯下的响叶杨却迟迟不落叶，因此，高压水银灯照射可能是响叶杨落叶迟的原因。

取一只蜘蛛，冲它大吼一声，蜘蛛被吓跑了。把它的腿砍掉，冲它大吼，蜘蛛纹丝不动。结论：蜘蛛的听觉器官长在腿上。

# Joint Method of Agreement and Difference

$$\frac{\begin{array}{cc} ABC \to xyz & ABC \to xyz \\ ADE \to xtw & \overline{A}BC \to \overline{x}yz \end{array}}{A \to x}$$

达尔文观察到不同类的生物在相同环境中常常具有相似的形态，鲨鱼属于鱼类，鲸鱼属于哺乳类，鱼龙属于爬行类，但形貌相似。又观察到同类生物在不同环境中呈现不同形态，鼹鼠、鲸鱼、蝙蝠同属哺乳类，却分别生活在陆、海、空，形态差别很大。通过对比，达尔文认为，生活环境是影响生物形态的重要原因。

# Method of Residues

$$\begin{array}{c} ABC \to xyz \\ B \to y \\ C \to z \\ \hline A \to x \end{array}$$

居里夫人知道铀的放射线的强度，也知道一定量的沥青矿石所含的铀数量。她观察到一定量的沥青矿石所发出的放射线要比它所含的铀所发出的放射线强许多倍。她推断：沥青矿石中还含有其它放射性极强的新元素。经过试验，她发现了镭。

# Method of Concomitant Variation

$$\frac{\begin{array}{c} ABC \to xyz \\ A^{\uparrow}BC \to x^{\uparrow}yz \\ A^{\downarrow}BC \to x^{\downarrow}yz \end{array}}{A \to x}$$

热胀冷缩、体温表

老王发现，抽烟越多肺病越严重，所以推测，抽烟是导致肺病的重要原因。
有没有可能，老王携带的某种基因是导致他容易抽烟和容易得肺病的共同原因？

# Contents


1. Introduction
   - Logic Puzzle
   - Textbook and Homework
   - Analogical Argument
   - Mill's Five Methods
   - Fallacy and Bullshit
   - Logic and other Disciplines
2. History
3. Propositional Logic
4. Predicate Logic
5. Equational Logic
6. Set Theory
7. Recursion Theory
8. Modal Logic
9. Logic vs Game Theory

# Informal Fallacies

ⓘ 形式谬误
ⓘⓘ 非形式谬误
  ❶ 言辞谬误
  ❷ 实质谬误

- 模糊谬误: 划界谬误或连续体谬误，假精确或过度精确，抽象概念当具体概念用
- 歧义谬误: 一词多义，歧义句构，辖域谬误，重音，脱离语境、断章取义，概念扭曲，偷换概念，混淆集合与个体或整体与部分，变更标准
- 定义谬误：不当定义，篡改定义
- 废话谬误：平凡真理，无意义的问题，回顾性宿命论

# Informal Fallacies

- 不相干：歪曲论题（稻草人、红鲱鱼、烟雾弹），诉诸人身（扣帽子、人身攻击、诉诸动机、罪恶关联、诉诸虚伪、伪善、诉诸成就、富贵、贫贱、智商），诉诸情感（诉诸恐惧、厌恶、仇恨、谄媚、同情、愧疚、可爱、性感、时髦、嘲弄、虚荣、势利、沉默），诉诸暴力、恐吓、诽谤，诉诸来源、年代、新潮、传统，诉诸信心、意愿，诉诸后果、中庸、自然，转移举证责任，不得要领
- 不充分：不当概括（偏差样本、偏差统计），诉诸无知，诉诸不当权威、名人、大众，虚假原因，滑坡谬误，诉诸可能，诉诸阴谋，隐瞒证据，不当类比，乱赋因果（相关、巧合、因果倒置、单因谬误），完美主义谬误、权宜主义谬误
- 不当预设：窃取论题，非黑即白，打压对立，自然主义谬误（实然推应然），道德主义谬误（好的就是自然的），复合问题，诱导性提问，诉诸顽固、反复、冗赘，乱枪打鸟，两面讨好

# “这鸡蛋真难吃。”

- 有本事你下一个好吃的蛋啊！
- 我可以负责任地说，我们的鸡蛋都是合格的健康蛋！
- 鸡是优等鸡，你咋说它下的蛋难吃？
- 这是别有用心的煽动，你有何居心？
- 隔壁的鸡给了你多少钱?
- 隔壁家的鸡蛋是伪蛋！
- 伟大的隔壁老王说好吃，你跟他说去！
- 没有伟大的老王，你连臭蛋都吃不上！
- 杀掉这只鸡换一只就能下金蛋？
- 你叫什么名字？你是干什么的！你是站在谁的立场上说话？
- 美国鸡蛋好吃，你去吧！
- 你以为你谁啊，品蛋师啊？轮到你说！
- 你个鸭蛋脑残粉！
- 下蛋的是一只勤劳勇敢善良正直的鸡！
- 再难吃也是自己家的鸡下的蛋！
- 但隔壁家的鸡蛋没有我们家的蛋形圆！
- 吃鸡蛋是我们家的传统美德。祖宗三代都是吃鸡蛋长大的！你也是！你有什么权力说这蛋难吃？还是不是人！
- 作为一个吃鸡蛋长大的人，我为我天天吃鸡蛋感到自豪！
- 拒绝抹黑！抵制鸭蛋！鸡蛋万岁！鸡蛋加油！
- 人心理阴暗会导致味觉异常……
- 其实隔壁家鸡蛋是个巨大的阴谋，试图颠覆我们家！
- 其实邻居家只有少数人才能吃上鸡蛋。
- 我们这么大的一个家，问题太复杂，下蛋没有你想得那么容易。
- 不要再吵了，这个家不能乱，稳定、稳定压倒一切！
- 要对我们家的鸡有耐心，它一定会下出更好吃的蛋。
- 蛋无完蛋！

# “这鸡蛋真难吃。”

- 我们家的鸡已经可以打败隔壁家的鸭！
- 隔壁家也吃过这样的鸡蛋，现在是初级阶段，必须坚持一百年不动摇！
- 我们家人肠胃不好，现阶段还不适合吃鸭蛋，不符合我们家的具体家情！
- 凡事都有个过程，现在还不是吃鸭蛋的时候。
- 鸡蛋好不好吃，全体蛋鸡最有发言权。
- 老外都说好吃呢。
- 这蛋难吃但是历史悠久啊。
- 虽然难吃但重要的是好看啊。
- 比以前已经进步很多了。
- 哎，人心不古，世风日下，就是因为你这种想吃鸭蛋的人太多了……
- 隔壁家那鸭蛋更难吃，你咋不说呢？
- 嫌难吃就别吃，滚去吃隔壁的鸭蛋吧。
- 隔壁亡我之心不死！该鸡蛋肯定是被隔壁一小撮不会下蛋的鸡煽动变臭的！
- 你上次吃茄子都吐，味觉一贯奇葩。
- 胡说！我们家的鸡蛋比隔壁家的鸭蛋好吃五倍！五倍！
- 是你的思想跟不上鸡蛋口味的升级！
- 心理阴暗！连鸡蛋不好吃也要发牢骚！
- 抱怨有毛用，有这个时间快去赚钱！
- 隔壁家的鸡蛋也一样，天下乌鸦一般黑，没有好吃的鸡蛋！
- 吃了人家的鸡蛋还留下证据说鸡蛋难吃，太有城府了！
- 很多家都是因为吃隔壁的鸭蛋而导致家庭冲突，生活水平下降甚至解体！
- 到目前为止，我没发现这鸡蛋难吃。专家说了，这鸡蛋难吃的可能性不大。即使出现这种情况，也是结构性难吃。
- 荷兰狗/东北猪/瘪三……不配吃鸡蛋！
- 大家小心，此人 IP 在国外。
- 滚，你丫是鸡奸，这里不欢迎你。

假如潘金莲不开窗户，就不会掉下木棍打到西门庆，也就不会认识西门庆，不会出轨，不会害死武大郎，武松不会被逼杀人上梁山，不会有独臂擒方腊，方腊就可夺取大宋江山，没了宋就不会有靖康耻、金兵入关，也不会有元、明、清，不会闭关锁国、鸦片战争、八国联军。这样中国将成为超级大国，称霸世界！

今天在路边看见一条鱼，捡起一看，还是条活鱼，回家一炖可好吃了。又一想，做鱼要有油、有厨房，还要找个媳妇来做，媳妇一定有娘，又多了个丈母娘。要娶她家姑娘，丈母娘一定会开条件：要房、要车、要钱……恍然大悟，赶紧把鱼扔了，现在房价这么恐怖，这鱼肯定是开发商故意扔的。我的妈呀，差点上当……

天无二日，国无二主。

天以终岁之数，成人之身，故小节三百六十六，副日数也；大节十二，分副月数也；内有五藏，副五行数也；外有四肢，副四时数也；乍视乍瞑，副昼夜也；乍刚乍柔，副冬夏也。

- 告子：性犹湍水也，决诸东方则东流，决诸西方则西流。人性之无分于善不善也，犹水之无分于东西也。
- 孟子：水信无分于东西，无分于上下乎？人性之善也，犹水之就下也。人无有不善，水无有不下。

**孟子《生于忧患，死于安乐》**

舜发于畎亩之中，傅说举于版筑之中，胶鬲举于鱼盐之中，管夷吾举于士，孙叔敖举于海，百里奚举于市。故天将降大任于斯人也，必先苦其心志，劳其筋骨，饿其体肤，空乏其身，行拂乱其所为，所以动心忍性，曾益其所不能。

三秀才赶考，途遇算命先生，问几人中举？先生竖起一指。

**明 · 浮白斋主人《雅谑》**

叶衡罢相归，一日病，问诸客曰：“我且死，但未知死后佳否？”一士曰：“甚佳”。叶惊问曰：“何以知之？”答曰：“**使死而不佳，死者皆逃回矣。一死不返，以是知其佳也**。”

当人们思考上帝时，人们是把上帝作为一切完美性的总和来思考的。因为不存在的必然是不完美的，所以必须把存在算在上帝的完美性之中。因此上帝存在。

原子没有知觉，所以其复合物也没有知觉。

**帕斯卡赌**

如果上帝不存在，但你相信上帝存在，也没太大损失；然而，如果上帝存在，而你却不相信上帝存在，那你将面临巨大的惩罚。所以，应该相信上帝存在。

**鲁迅《论辩的灵魂》**

我骂卖国贼，所以我是爱国者。爱国者的话是最有价值的，所以我的话是不错的，我的话既然不错，你就是卖国贼无疑了！

Wholeness depends on dimensionless phenomena. Reality has always been full of messengers of the multiverse, whose third eyes are transformed into transcendence. Transcendence is the healing of choice. Complexity is the driver of transcendence. Our conversations with other messengers have led to an awakening of ultra-non-local consciousness. Consciousness requires exploration. We are at a crossroads of flow and ego. We can no longer afford to live with ego. Where there is ego, life can't thrive. We exist as expanding wave functions. The goal is to plant the seeds of passion rather than bondage. We are in the midst of a self-aware blossoming of being that will align us with the nexus itself. Lifeform, look within and recreate yourself. To follow the path is to become one with it. By unfolding, we believe; By deepening, we vibrate; By blossoming, we self-actualize. We dream, we heal, we are reborn. We must learn how to lead unlimited lives in the face of delusion. You and I are dreamweavers of the quantum soup. The infinite is approaching a tipping point. Hidden meaning transforms unparalleled abstract beauty. Wholeness quiets infinite phenomena.

# How to generate pseudo-profound bullshit?[1]

1. state the blindingly obvious (of life's big theme) incredibly slowly.
   - We were all children once.
   - The world of the happy is quite different from the world of the unhappy.
2. doublethink/dialectic/contradiction.
   - War is peace.
   - Freedom is slavery.
   - Ignorance is strength.
   - All animals are equal but some animals are more equal than others.
   - Everyone is the other, and no one is himself.
   - Man can do what he wills but he can't will what he wills.
   - To believe is to know you believe, and to know you believe is not to believe.

[1] Law: Bellieving Bullshit.
Frankfurt: On Bullshit.

# How to generate pseudo-profound bullshit?

3. ambiguity/metaphor/parable.
   - Love is just a word.
   - There is no need for torture: Hell is other people.
   - Language is the house of the truth of Being.
   - What is reasonable is real; that which is real is reasonable.
   - Never stay up on the barren heights of cleverness, but come down into the green valleys of silliness.
   - The World and Life are one. Ethics and Aesthetics are one. Ethics does not treat of the world. Ethics must be a condition of the world, like logic.
   - When you gaze long into the abyss the abyss also gazes into you.
   - The limits of my language mean the limits of my world.
   - A person is neither a thing nor a process but an opening through which the Absolute can manifest.
   - My work consists of two parts: of the one which is here, and of everything which I have not written. And precisely this second part is the important one.
   - Making itself intelligible is suicide for philosophy. Those who idolize "facts" never notice that their idols only shine in a borrowed light.

# How to generate pseudo-profound bullshit?

4. analogy.
   - Life is like a box of chocolates. You never know what you're gonna get.
   - Life is like a coin. You can spend it any way but only once.
   - Life is like a shell, which suddenly bursts into fragments, which fragments, being themselves shells, burst in their turn into fragments destined to burst again, and so on for a time incommensurably long.
   - We have got on to slippery ice where there is no friction, and so, in a certain sense, the conditions are ideal; but also, just because of that, we are unable to walk. We want to walk: so we need friction. Back to the rough ground!
   - The subject does not belong to the world: rather, it is a limit of the world. This is exactly like the case of the eye and the visual field. You do not see the eye. Nothing in the visual field allows you to infer that it is seen by an eye.

   - Our life is endless in the way that our visual field is without limit.
   - My propositions serve as elucidations in the following way: anyone who understands me eventually recognizes them as nonsensical, when he has used them — as steps — to climb up beyond them. (He must throw away the ladder after he has climbed up it.)

# How to generate pseudo-profound bullshit?

5. use jargon.
   - The Nothing itself nothings.
   - Profound boredom, drifting here and there in the abysses of our existence like a muffling fog, removes all things and men and oneself along with it into a remarkable indifference. This boredom reveals being as a whole.
   - Dasein has always made some sort of decision as to the way in which it is in each case mine. That entity which in its Being has this very Being as an issue, comports itself towards its Being as its ownmost possibility. With death, Dasein stands before itself in its ownmost potentiality-for-Being.
   - The Absolute Idea. The Idea, as unity of the Subjective and Objective Idea, is the notion of the Idea — a notion whose object is the Idea as such, and for which the objective is Idea — an Object which embraces all characteristics in its unity.
   - A machinic assemblage, through its diverse components, extracts its consistency by crossing ontological thresholds, non-linear thresholds of irreversibility, ontological and phylogenetic thresholds, creative thresholds of heterogenesis and autopoiesis.

# The Unreasonable Ineffectiveness of Philosophy

- 费曼：“砖头算不算本质客体？”
- 哲学家甲：“一块砖是独特的砖，是怀海德所说的本质客体。”
- 哲学家乙：“本质客体的意思并不是指个别的砖块，而是指所有砖块的共有的普遍性质，换句话说，‘砖性’才是本质客体。”
- 哲学家丙：“不对，重点不在砖本身，‘本质客体’指的是，当你想到砖块时内心形成的概念。”
- 就像所有关于哲学家的故事一样，最终以一片混乱收场。好笑的是，在先前的那么多次讨论中，他们从来没有问过自己，像简单的砖块究竟是不是“本质客体”。
- 科学哲学对科学家的用处跟鸟类学对鸟的用处差不多。 — 费曼

- 哲学旨在感动那些混淆晦涩与深刻的人。 — 温伯格
- 哲学难道不是用蜜写成的吗？乍一看，很精彩，再一看，除了一团浆糊，什么都没留下。 — 爱因斯坦

# The Unreasonable Ineffectiveness of Philosophy

- When a philosopher says something that is true then it is trivial. When he says something that is not trivial then it is false. — *Gauss*
- *There is only one thing a philosopher can be relied upon to do, and that is to contradict other philosophers.* — *James*
- *Philosophers are free to do whatever they please, because they don't have to do anything right.*
- *Philosophy is to science as pornography is to sex: it is cheaper, easier, and some people seem, bafflingly, to prefer it.*
- *A philosopher looking for the ultimate truth is like a blind darky with an extinguished candle on a dark night searching a dark subterranean cave for a black cat that isn't there, and shouting "I found it!"*

# Elimination of Metaphysics?×?

一个陈述的意义在于它的证实方法。形而上学陈述不能被证实，毫无意义。那么留给哲学的还有什么呢？一种方法：逻辑分析法。逻辑分析的消极应用是清除无意义的词和陈述，积极应用是澄清有意义的概念和命题，为经验科学和数学奠基。形而上学家相信自己是在攸关真假的领域里前行，却未断言任何东西。他们只是试图表达一点儿人生态度。艺术是表达人生态度的恰当手段。抒情诗人并不企图在自己的诗里驳倒其他抒情诗人诗里的陈述，但形而上学家却用论证维护他的陈述。形而上学家是没有艺术才能的艺术家，有的是在理论环境里工作的爱好，却既不在科学领域里发挥这种爱好，又不能满足艺术表达的要求，倒是混淆了这两个方面，创造出一种对知识既无贡献、对人生态度的表达又不相宜的东西。[a]

[a] 卡尔纳普：通过语言的逻辑分析清除形而上学

Mystics exult in mystery and want it to stay mysterious. Scientists exult in mystery for a different reason: it gives them something to do.

— *Dawkins*

# Practice is the sole criterion for testing truth?

Practice is the sole criterion for testing truth?

- What is “practice”?
- What is “truth”?
- What is “criterion”?
- Why “sole” criterion?
- How to “test”?
- How to test “truth” with “practice”?

# Galilei's Leaning Tower of Pisa

*Philosophy is written in this grand book -- I mean the universe -- which stands continually open to our gaze, but it can't be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics.*

— *Galileo Galilei*

# Why is the Night Sky Dark?

A static, infinitely old universe with an infinite number of stars uniformly distributed in an infinitely large space would be bright rather than dark.

# Newton's Apple

If an apple falls, does the moon also fall?

- What is "rest/motion"?
- What is "state of rest/motion"?
- What is "change/tends to change"?
- What is "body"?
- What is "force"?
- What is "definition"?

A force is that which changes or tends to change the state of rest or motion of a body.

# Einstein's Train Thought Experiment

- What is "simultaneity"?
- How to measure "simultaneity"?
- How to measure "time"?

# Einstein's Elevator Thought Experiment

- The gravitational "force" as experienced locally while standing on a massive body is actually the same as the pseudo-force experienced by an observer in a non-inertial (accelerated) frame of reference.
- Spacetime tells matter how to move; matter tells spacetime how to curve.
- Gravity is not a force that applies via Newton's $2^{nd}$ Law, but a consequence of the curvature of spacetime caused by the uneven distribution of mass/energy that acts via the geodesic principle, which is the relativistic equivalent of Newton's $1^{st}$ Law.

**How to *express* your thoughts precisely and succinctly?**

- Not only is the universe stranger than we imagine, it is stranger than we can imagine.
- A few lines of reasoning can change the way we see the world.
- Logic enlarges our abstract imagination, and provides all possible hypotheses to be applied in the analysis of complex facts. Nothing is forbidden except nonsense and contradiction.
- Logic is the immune system of the mind!
- The music of reason — the fulfillment of the human spirit.

# The Music of Reason

**How to *express* your thoughts <u>precisely</u> and <u>succinctly</u>?**

The glory of the human spirit!

What are the extent and limits of reason?

# Contents


1. Introduction
   - Logic Puzzle
   - Textbook and Homework
   - Analogical Argument
   - Mill's Five Methods
   - Fallacy and Bullshit
   - Logic and other Disciplines
2. History
3. Propositional Logic
4. Predicate Logic
5. Equational Logic
6. Set Theory
7. Recursion Theory
8. Modal Logic
9. Logic vs Game Theory

# Why Study Logic?

- Logic vs (Analytic) Philosophy.
  sense & reference / extension & intension / use & mention / truth & provability / mutual vs distributed vs common knowledge / knowledge update / belief revision / preference change / information flow / action & strategy / multi-agent interaction / counterfactual / causation / possible world / cross-world identity / essentialism / induction / ontological commitment / concept analysis / laws of thought / strength & limitation / paradoxes ...
  Peirce, Frege, Russell, Wittgenstein, Ramsey, Carnap, Quine, Putnam, Kripke, Chomsky, Gödel, Tarski, Turing ...
- Logic vs Mathematical Foundations.
  Logicism / Formalism / Intuitionism / Constructivism / Finitism / Structuralism / Homotopy Type Theory
- Logic vs Computer Science.

$$\frac{\text{Logic}}{\text{Computer Science}} \approx \frac{\text{Calculus}}{\text{Physics}}$$

- ...

# Logic vs CS

- Computer Architecture.
  Gates and digital circuit design $\approx$ Propositional Logic
- Programming Languages.
  Semantics of programming languages via methods of logic
  LISP $\approx$ $\lambda$-calculus
  Prolog $\approx$ First Order Logic + Recursion
  Typing $\approx$ Type Theory
- Theory of Computation and Computational Complexity.
  Models of computation (Turing machines, finite automata)
  Logic provides *complete problems* for complexity classes.
  Logical characterizations of complexity classes
  Descriptive Complexity
- Theorem Proving

# Logic vs CS

- Knowledge representation via logic rules
- Common sense reasoning via Non-monotonic Logic
- Fuzzy Control vs Fuzzy Logic and Multi-valued Logic
- Relational Databases
  SQL $\approx$ First Order Logic + Syntactic Sugar
- Hardware and Software Verification.
  Extensive use of formal methods based on logic
  Temporal Logic, Dynamic Logic and Automata, Hoare Logic, Model Checking
- Agents.
  Epistemic Logic
- Semantic Web.
  Web Ontology Language (OWL) $\approx$ Description Logic

# Branches of Logic

## Mathematical Logic

- **First Order Logic**
- Set Theory
- Model Theory
- Proof Theory
- Recursion Theory

## Computational Logic

- Automata Theory
- Computational Complexity
- Finite Model Theory
- Model Checking
- Type Theory
- Lambda Calculus
- Categorical Logic
- Theorem Proving
- Description Logic
- Dynamic Logic
- Temporal Logic
- Hoare Logic
- Inductive Logic
- Fuzzy Logic
- Non-monotonic Logic
- Computability Logic
- Default Logic
- Situation Calculus

## Philosophical Logic

- Intuitionistic Logic
- Algebraic Logic
- Quantum Logic
- Modal Logic
- Epistemic Logic
- Doxastic Logic
- Preference Logic
- Provability Logic
- Hybrid Logic
- Free Logic
- Conditional Logic
- Relevance Logic
- Linear Logic
- Paraconsistent Logic
- Intensional Logic
- Partial Logic
- Diagrammatic Logic
- Deontic Logic

- Logic is
  1. mainly philosophy by subject matter
  2. mainly mathematics by methodology
  3. mainly computer science by applications
- Logicians always want to be
  1. Philosophers of philosophers
  2. Mathematicians of mathematicians
  3. Computer scientists of computer scientists
- However, they often end up being
  1. Mathematicians to philosophers
  2. Computer scientists to mathematicians
  3. Philosophers to computer scientists

*Between theology and science there is a no man's land, exposed to attack from both sides; this no man's land is philosophy.*

— *Russell*

*Philosophy is a 'catalyst' or 'spice' which makes the interdisciplinary mixture work. 'Philosophy-internal' issues seem like intellectual black holes: they absorb a lot of clever energy, but nothing ever seems to come out.*

— *van Benthem*

- Philosophy is a game with objectives and no rules.
- Logic is a game with rules and no objectives.

**Logic is like love; a simple idea, but it can get complicated.**

- 这 TM 也用证？
- 这 TM 也能证？

*If Church says it's obvious, then everybody has seen it half an hour ago. If Weyl says it's obvious, von Neumann might be able to prove it. If Lefschetz says it's obvious, it's false.*

— *Rosser*

# Contents


1. Introduction
2. History
3. Propositional Logic
4. Predicate Logic
5. Equational Logic
6. Set Theory
7. Recursion Theory
8. Modal Logic
9. Logic vs Game Theory

# Contents


1 Introduction

2 History
- The Prehistory of Logic
- The Rise of Logic
- After Gödel

3 Propositional Logic

4 Predicate Logic

5 Equational Logic

6 Set Theory

7 Recursion Theory

8 Modal Logic

9 Logic vs Game Theory

# Aristotle(384-322 BC) — Term Logic

- Term Logic.
- Aristotle believed that any logical argument can, in principle, be broken down into a series of applications of a small number of syllogisms.
- Any person in the present day who wishes to learn logic will be wasting his time if he reads Aristotle.

— *Russell*

# Sophistic vs Valid Argument

1. 没有东西存在；
2. 就算有东西存在，也无法认识它；
3. 就算能认识它，也无法与他人谈论它；
4. 就算能谈论它，也无法互相理解。

$$\frac{\text{All men are mortal} \quad \text{Socrates is a man}}{\text{Socrates is mortal}}$$

**A**: All $S$ are $P$.

**E**: No $S$ are $P$.

**I**: Some $S$ are $P$.

**O**: Some $S$ are not $P$.

# Syllogism

$$\frac{\begin{array}{c}M—P\\S—M\end{array}}{S—P} \qquad \frac{\begin{array}{c}P—M\\S—M\end{array}}{S—P} \qquad \frac{\begin{array}{c}M—P\\M—S\end{array}}{S—P} \qquad \frac{\begin{array}{c}P—M\\M—S\end{array}}{S—P}$$

- 大项：结论的谓项。
- 小项：结论的主项。
- 中项：第三个项。
- 大前提：含大项的前提。
- 小前提：含小项的前提。

- 4 格，$4^3 \times 4 = 256$ 式。
- 15 布尔有效式。
- 24 亚里士多德有效式。（存在预设）
- 如何断定三段论的有效性？
  1. Venn Diagrams
  2. Rules
  3. Boolean Algebra
  4. Axiomatization

# Venn Diagram — Boolean Standpoint

$$\frac{\begin{array}{c}\text{No } P \text{ are } M \\ \text{Some } S \text{ are not } M\end{array}}{\text{Some } S \text{ are } P}$$

1. 在三个圆的图上标记三段论的三个项；
2. 把两个前提在图上表示出来，如果一个前提是全称、另一个是特称，先表示全称；
3. 如果特称前提没有明确表明把 × 画在哪一部分，就把它画在交叉线上；
4. 检查图示是否支持结论。

# Venn Diagram — Aristotelean Standpoint

$$\frac{\begin{array}{c}\text{No } M \text{ are } P\\ \text{All } M \text{ are } S\end{array}}{\text{Some } S \text{ are not } P}$$

1. 如果一个含全称前提和特称结论的三段论形式不是布尔有效的，看看是否有一个圆其中除了一个区域外全是阴影，如果有，在那个区域画 ⊗ 。
2. 如果该三段论是条件有效的，检查 ⊗ 是否表示某个存在物，如果是，则该三段论是亚里士多德有效的。

# 三段论有效性的判定规则

$S$ 周延

| | | |
|---|---|---|
| $P$ 不周延 | **A**: All $S$ are $P$. | **E**: No $S$ are $P$. |
| | **I**: Some $S$ are $P$. | **O**: Some $S$ are not $P$. |

$P$ 周延

$S$ 不周延

1. 中项至少周延一次。
2. 结论中周延的项在前提中必周延。
3. 前提与结论中否定命题的数目必相同。
4. 特称结论需要特称前提。（存在谬误）

$\left.\begin{array}{l}1\\2\\3\end{array}\right\}$ *Aristotle*

$\left.\begin{array}{l}1\\2\\3\\4\end{array}\right\}$ *Boole*

# Example and Criticism

All men are intelligent
Women are not men
---
Women are not intelligent

John does not read books
Students who like to learn read books
---
John does not like to learn

Nothing is better than money
Philosophy is better than nothing
---
Philosophy is better than money

Only man is rational
No woman is a man
---
No woman is rational

No professors are ignorant
All ignorant people are vain
---
No professors are vain

Everyone loves my baby
My baby loves only me
---
I am my own baby

# Deduction/Induction/Abduction/Examplification

$$\frac{M \to P \qquad S \to M}{S \to P}$$

$$\frac{M \to P \qquad M \to S}{S \to P}$$

$$\frac{H \to E \qquad E}{H}$$

$$\frac{P \to M \qquad S \to M}{S \to P}$$

$$\frac{H \to E \qquad \top \to E}{\top \to H}$$

$$\frac{P \to M \qquad M \to S}{S \to P}$$

# Abduction

1. 观察到恒星光谱红移。
2. 如果恒星在退行，那么恒星光谱红移就可以解释。
3. 如果整个宇宙在膨胀，那么恒星在离我们而去。
4. 如果宇宙起源于大爆炸，那么宇宙就会膨胀。
5. 因此，宇宙起源于大爆炸。

# Leibniz 1646-1716

**Don't argue. Calculate!**

- Principle of Contradiction: Nothing can be and not be, but everything either is or is not.
- Principle of Sufficient Reason: Nothing is without a reason.
- Principle of Perfection: The real world is the best of all possible worlds.

**In the beginning was the Logic.**

As God calculates, so the world is made.

# Leibniz

- The last "universal genius", developed Calculus, refined binary number system, invented mechanical calculator that could perform addition, subtraction, multiplication and division.
- Leibniz was claimed (by Russell, Euler, Gödel, Weiner, Mandelbrot, Robinson, Chaitin) to be a precursor of *mathematical logic, topology, game theory, cybernetic theory, fractal geometry, non-standard analysis, algorithmic information theory and digital philosophy*.
- Wolfram: "Leibniz had the idea of encoding logical properties using numbers. He thought about associating every possible attribute of a thing with a prime number, then characterizing the thing by the product of the primes for its attributes — and then representing logical inference by arithmetic operations."

# Leibniz's Dream — Deduction

1. *Universal Characteristic and Rational Calculus.*
   - i encyclopedia — arranged in a logical order according to a demonstrative method, beginning with definitions of all the simple and primitive terms (which form the alphabet of human thoughts).
   - ii a universal ideal language that is purged of ambiguity and vagueness. — *the language of creation, the language before the Tower of Babel, the language that God used in creating the universe, the language whose rules of composition directly expresses the structure of the universe, the language in which concepts are expressed in their direct, original format.*

$$\textit{sign} \rightleftarrows \textit{idea}$$

$$\textit{encyclopedia} \Rightarrow \textit{fundamental principles} \Rightarrow \textit{primitive notions}$$

   - iii the arrangement of all true propositions in an axiomatic system.
   - iv decision procedure. A calculus of manipulating the knowledge in a computational fashion, so as to reveal its logical interrelations and consequences. — replace reasoning by computation.
   - v a proof that the rational calculus is consistent.
   - vi a procedure for the rapid enlargement of knowledge. — the art of invention. free mind from intuition.

# Leibniz's Dream — Induction

2. *Compute all descriptions of possible worlds that can be expressed with the primitive notions. And the possible worlds will all have some propensity to exist.*
3. *Compute the probabilities of disputed hypotheses relative to the available data. As we learn more our probability assignments will asymptotically tend to a maximum for the real world, i.e., the possibility with the highest actual propensity.*

# Leibniz's Metaphysics and Quantum Mechanics

| **Monadology** | **Path Integral** |
|---|---|
| Amount of existence | Square of probability amplitude |
| Measure of necessity of individual possibility | Probability |
| Collision or competition of possibilities | Interference or summation of probability amplitudes |
| Coexisting or compatible essences | Superposition of coherent paths |
| Maximal degree of existence | Observed path |

$$P = |\langle \boldsymbol{q}_2, t_2 | \boldsymbol{q}_1, t_1 \rangle|^2 \quad \langle \boldsymbol{q}_2, t_2 | \boldsymbol{q}_1, t_1 \rangle = \int_{\boldsymbol{q}_1}^{\boldsymbol{q}_2} \varphi[\boldsymbol{q}] \mathcal{D}\boldsymbol{q}$$

$$\varphi[\boldsymbol{q}] \propto \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}[\boldsymbol{q}]} \quad \mathcal{S}[\boldsymbol{q}] = \int_{t_1}^{t_2} \mathcal{L}[\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)] \mathrm{d}t \quad \delta\mathcal{S} = 0$$

- Probability of the actual path = maximum
- Action of the actual path = minimum

*the absolute square of the sum of probability amplitudes over all possible paths*

# Leibniz's Program

# Contents


1 Introduction

2 History
- The Prehistory of Logic
- The Rise of Logic
- After Gödel

3 Propositional Logic

4 Predicate Logic

5 Equational Logic

6 Set Theory

7 Recursion Theory

8 Modal Logic

9 Logic vs Game Theory

# Boole 1815-1864

- *The Laws of Thought.*
- Logic as Algebra.
- Propositional Logic.
- Algebra's strength emanates from the fact that the symbols that represent quantities and operations obey a small number of rules.

# Cantor 1845-1918

- Mathematics $\rightsquigarrow$ Set Theory.
- Diagonalization.
- There are many different levels of infinity.
- Cantor set.
- Continuum Hypothesis (CH).
  How many points on the line?

# Frege 1848-1925 (+Peirce)

- *Begriffsschrift, a formal language of pure thought modelled upon that of arithmetic.*
- Predicate Logic. (Relation & Quantification)
  (Every boy loves some girl.)

$$\frac{\text{subject}}{\text{predicate}} \approx \frac{\text{argument}}{\text{function}}$$

- Philosophy of Language.
  The evening star is the morning star. (venus)

*Logicism* Mathematics $\rightsquigarrow$ Logic.[2]

- 本书把算术规律归结为分析判断、从而是先验的。算术不过是逻辑的延伸。在科学中应用数学就是在观察事实中应用逻辑；计算就是推理。
- 如果哲学的任务是破除语词对人类精神的支配，揭开由于语言的表达方式而造成的概念关系的假象，把思想从日常语言的负担中解放出来，那么，我的概念文字将成为哲学家的有用工具。

[2] Frege: The Foundations of Arithmetic.

# Russell 1872-1970

- Russell Paradox.
  ($3^{ed}$ crisis of the Foundations of Mathematics)
- Theory of Descriptions.
  (The present King of France is not bald.)
- Type Theory.
- *Principia Mathematica.*

No barber shaves exactly those who do not shave themselves.[3]

**$*54 \cdot 43$.** $\vdash: .\alpha, \beta \in 1. \supset: \alpha \cap \beta = \Lambda. \equiv .\alpha \cup \beta \in 2$

*Dem.*

$$\begin{aligned}
&\vdash .*54 \cdot 26. \supset \vdash: .\alpha = \iota' x . \beta = \iota' y. \supset: \alpha \cup \beta \in 2. \equiv .x \neq y. \\
&[*51 \cdot 231] \qquad \equiv .\iota' x \cap \iota' y = \Lambda. \\
&[*13 \cdot 12] \qquad \equiv .\alpha \cap \beta = \Lambda \qquad (1) \\
&\vdash .(1). *11 \cdot 11 \cdot 35. \supset \\
&\qquad \vdash: .(\exists x, y).\alpha = \iota' x . \beta = \iota' y. \supset: \alpha \cup \beta \in 2. \equiv .\alpha \cap \beta = \Lambda \qquad (2) \\
&\vdash .(2). *11 \cdot 54. *52 \cdot 1. \supset \vdash .Prop
\end{aligned}$$

From this proposition it will follow, when arithmetical addition has been defined, that $\mathbf{1 + 1 = 2}$.

[3] Russell: On denoting.

# Intuitionism

- Impredicativism. (*Poincaré*, Russell)
  Vicious circle principle: No entity can be defined only in terms of a totality to which this entity belongs.
- *Intuitionism* Logic $\rightsquigarrow$ Mathematics $\rightsquigarrow$ Mental construction.
  (Kronecker, *Brouwer*, Heyting, *Kolmogorov*, Weyl)
  - Potential infinity vs actual infinity.
  - To be is to be constructed by intuition.
  - Law of excluded middle.×
  - Non-constructive proof.×

  (There exist two irrational numbers $x$ and $y$ s.t. $x^y$ is rational.)

$$\sqrt{2} \quad \log_2 9$$

  "God created the integers, all the rest is the work of man."
- Constructive Mathematics. (Bishop, *Martin-Löf*)

# Hilbert 1862-1943

- **Formal** Axiomatization of Geometry.
  The consistency of geometry relative to arithmetic.
  (Klein: Non-Euclidean relative to Euclidean)
  (natural/integer/rational/real/complex)
- Hilbert's $23/24$ problems. $(1^{st}, 2^{nd}, 10^{th}, 24^{th})$
- Meta-mathematics — Proof Theory.
- *Formalism* Mathematics $\rightsquigarrow$ Symbolic Game.

- 公理是初始概念的隐定义。
- 我们完全可以用桌子、椅子、啤酒瓶来代替点、线、面。
- 数学是根据某些简单规则使用毫无意义的符号在纸上进行的游戏，是制造快乐的游戏。
- 我们的内心响起了永恒的召唤：那里有一个问题，去找出它的答案！你可以通过纯粹理性找到它，因为，数学里没有不可知！
- 我们必须知道；我们必将知道！

# Hilbert's Program

1. Formalization & Axiomatization (Richness): Formalize and axiomatize (recursively) the elementary logic $\mathbb{L}$, the "finitistic" mathematics $\mathbb{F}$ (which is concerned with the real world) and the "infinitistic" mathematics $\mathbb{T}$ (which is concerned with 'ideal objects').
2. Independence: the axioms should be "independent" of one another.
3. Completeness: $(1)$ all valid logical statements can be proved in $\mathbb{L}$; $(2)$ all true mathematical statements can be proved in $\mathbb{T}$.
4. **Consistency**: a finitistic proof that no contradiction can be proved in $\mathbb{T}$.
5. Conservation(Consequence of Consistency $\forall \varphi \in \Pi_1 : \mathbb{T} \vdash \varphi \implies \mathbb{F}, Con(\mathbb{T}) \vdash \varphi$): any statement about 'real objects' provable in $\mathbb{T}$ can be proved in $\mathbb{F}$.
6. Decidability (Effectiveness): a mechanical procedure for deciding the validity of any logical statement and the truth of any mathematical statement.
7. Simplicity: a criteria of simplicity, or proof of the greatest simplicity of certain proofs.
8. Categoricity? $\mathbb{T}$ characterizes exactly one model up to isomorphism.

# Hilbert's Program

*When we consider the axiomatization of logic more closely we soon recognize that the question of the consistency of the integers and of sets is not one that stands alone, but that it belongs to a vast domain of difficult epistemological questions which have a specifically mathematical tint: e.g., the problem of the solvability in principle of every mathematical question, the problem of the subsequent checkability of the results of a mathematical investigation, the question of a criterion of simplicity for mathematical proofs, the question of the relationship between content and formalism in mathematics and logic, and finally the problem of the decidability of a mathematical question in a finite number of operations.*

— *Hilbert*

现在正在发生的一切——计算机接管世界、人类社会数字化、信息化，都是 20 世纪初希尔伯特提出的哲学问题的结果。

— *Gregory Chaitin*

*It may well be doubted wheter human ingenuity can construct an enigma of the kind which human ingenuity may not, by proper application, resolve.*

— *Edgar Allan Poe*

# Philosophy of Math: is math synthetic a priori?

- Descartes: we can be certain about how things seem to us from the inside; but how to build up to the external world?
- Hume: we can't. (i) Knowledge of the external world requires knowledge of causation. (ii) Causal statements are synthetic, and so can be known only a posteriori. (iii) Causal statements can't be known a posteriori, because we don't perceive causation itself and can't noncircularly argue that the future will resemble the past.
- Kant: we can know facts about causation a priori, even though they are synthetic, because facts about causation are constituted partly by how the world is in itself, and partly by our minds' operation; and we can know a priori the rules by which our mind operates.

Kant: Mathematics is synthetic a priori.
Frege: Mathematics is analytic.

# Philosophy of Math: Logicism/Intuitionism/Formalism

| Logicism | Intuitionism | Formalism |
|---|---|---|
| $\frac{Mathematics}{Logic}$ | $\frac{\frac{Logic}{Mathematics}}{Mind}$ | $\frac{Mathematics}{Game}$ |
| Realism | Conceptualism | Nominalism |

# 数学哲学

Figure: 形式主义/逻辑主义/直觉主义/柏拉图主义/物理主义

Is there more than one mathematical universe?

# Contents


1 Introduction

2 History
- The Prehistory of Logic
- The Rise of Logic
- After Gödel

3 Propositional Logic

4 Predicate Logic

5 Equational Logic

6 Set Theory

7 Recursion Theory

8 Modal Logic

9 Logic vs Game Theory

# Leibniz & Hilbert — Dream Shattered. . .

# Gödel 1906-1978

"I am unprovable."[4]

- Completeness.
  I think (consistently), therefore I am.
  (Consistency implies existence.)
- Incompleteness.
  1. provable < true
  2. un-self-aware
- Consistency of AC and CH.

[4] Gödel: On formally undecidable propositions of Principia Mathematica and related systems.

# Tarski 1901-1983

"snow is white" is true iff snow is white.

"I am false."[5]

Model Theory

**Undefinability of Truth**

Arithetical truth can't be defined in arithmetic.

The theory of real closed fields / elementary geometry is complete and decidable.

[5] Tarski: On the Concept of Truth in Formalized Languages.
Tarski: The Semantic Conception of Truth and the Foundations of Semantics.

# Turing 1912-1954

- Universal Turing Machine.
- Church-Turing Thesis.
- Halting Problem.
- Undecidability.
- Oracle Machine.
- Computable Absolutely Normal Number.
- Turing Test.
- Morphogenesis.
- Good-Turing Smoothing.
- Enigma.

What is "effective procedure"?[6] — Recursion Theory

[6] Turing: On computable numbers, with an application to the Entscheidungsproblem.

# Contents


1. Introduction
2. History
3. Propositional Logic
4. Predicate Logic
5. Equational Logic
6. Set Theory
7. Recursion Theory
8. Modal Logic
9. Logic vs Game Theory

# Logic → Truth

Truth *points the way for logic, just as beauty does for aesthetics, and goodness for ethics.*

— *Frege*

Natural Language

$\Downarrow$ represents

$\boxed{\text{Formal Language (Syntax)}}$

$\Downarrow$ expresses

$\boxed{\text{Theory (calculus } \vdash)}$

interprets $\Downarrow$ $\Uparrow$ characterizes

$\boxed{\text{Models (semantics } \vDash)}$

$\Uparrow$ represents

. . . . . semantic gap

Real World

# Contents


1. Introduction
2. History
3. Propositional Logic
   - Syntax
   - Semantics
   - Connectives
   - Formal System
   - Meta-Theorems
   - Application
4. Predicate Logic
5. Equational Logic
6. Set Theory
7. Recursion Theory
8. Modal Logic
9. Logic vs Game Theory

# Propositional Logic

- Language.
  Building blocks of propositional logic language.
- Syntax.
  Propositional symbols and propositional formulas.
- Semantics.
  Assign "meaning" to propositional formulas by first assigning "meaning" to propositional symbols.
- Calculus.
  Axioms and inference rules.

# Syntax

## Language

$$\mathscr{L}^0 := \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, (, )\} \cup \mathscr{P}$$

where $\mathscr{P} := \{p_1, \ldots, p_n, (\ldots)\}$.

## Well-Formed Formula WFF

$$\varphi ::= p_i \mid (\neg\varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi)$$

- $\bot := (\varphi \wedge (\neg\varphi))$
- $\top := (\neg\bot)$

## Example

- Lily is (not) beautiful.
- If wishes are horses, then beggars will ride.
- Lily is beautiful and/or/iff $2$ is not a prime number.

# Well-Formed Formula

## Definition (Formula-Building Operator)

$$\mathscr{E}_{\neg}(\varphi) := (\neg\varphi)$$
$$\mathscr{E}_{\wedge}(\varphi,\psi) := (\varphi \wedge \psi)$$
$$\mathscr{E}_{\vee}(\varphi,\psi) := (\varphi \vee \psi)$$
$$\mathscr{E}_{\rightarrow}(\varphi,\psi) := (\varphi \rightarrow \psi)$$
$$\mathscr{E}_{\leftrightarrow}(\varphi,\psi) := (\varphi \leftrightarrow \psi)$$

$$\boxed{\begin{aligned} \mathscr{E}_{\neg}(\varphi) &:= \neg\varphi \\ \mathscr{E}_{\wedge}(\varphi,\psi) &:= \wedge\varphi\psi \\ \mathscr{E}_{\vee}(\varphi,\psi) &:= \vee\varphi\psi \\ \mathscr{E}_{\rightarrow}(\varphi,\psi) &:= \rightarrow \varphi\psi \\ \mathscr{E}_{\leftrightarrow}(\varphi,\psi) &:= \leftrightarrow \varphi\psi \end{aligned}}$$

# Well-Formed Formula

## Definition (Construction Sequence)

A construction sequence $(\alpha_1, \dots, \alpha_n)$ is a finite sequence of expressions s.t. for each $i \leq n$ we have at least one of

$$\alpha_i = p_i \quad \text{for some } i$$
$$\alpha_i = \mathscr{E}_\neg(\alpha_j) \quad \text{for some } j$$
$$\alpha_i = \mathscr{E}_\star(\alpha_j, \alpha_k) \quad \text{for some } j < i, k < i$$

where $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

## Definition (Well-Formed Formula)

A formula $\varphi$ is a well-formed formula (wff) iff there is some construction sequence $(\alpha_1, \dots, \alpha_n)$ and $\alpha_n = \varphi$.

$$\mathrm{WFF}_0 := \{p_1, p_2, \dots\}$$
$$\mathrm{WFF}_{n+1} := \mathrm{WFF}_n \cup \big\{(\neg\varphi) \colon \varphi \in \mathrm{WFF}_n\big\} \cup \big\{(\varphi \rightarrow \psi) \colon \varphi, \psi \in \mathrm{WFF}_n\big\}$$
$$\mathrm{WFF}_* := \bigcup_{n \in \mathbb{N}} \mathrm{WFF}_n$$

# Generation — Bottom Up vs Top Down

## Problem

Given a class $\mathcal{F}$ of functions over $U$, how to generate a certain subset of $U$ by starting with some initial elements $B \subset U$?

## Bottom Up

$$C_0 := B$$

$$C_{n+1} := C_n \cup \bigcup_{f \in \mathcal{F}} \{f(\boldsymbol{x}) : \boldsymbol{x} \in C_n\}$$

$$C_* := \bigcup_{n \in \mathbb{N}} C_n$$

$$deg(\boldsymbol{x}) := \mu n\, [\boldsymbol{x} \in C_n]$$

## Top Down

- A set $S$ is closed under a function $f$ if for all $\boldsymbol{x}$: $\boldsymbol{x} \in S \rightarrow f(\boldsymbol{x}) \in S$.
- A set $S$ is inductive if $B \subset S$ and for all $f \in \mathcal{F}$: $S$ is closed under $f$.
- $C^* := \bigcap \{S : S \text{ is inductive}\}$

# Bottom Up vs Top Down

**10 块钱最多可以喝几瓶啤酒？$^{\circledcirc}\hat{o}^{\circledcirc}$**

- 2 块钱买 1 瓶啤酒。
- 2 个空瓶换 1 瓶啤酒。
- 4 个瓶盖换 1 瓶啤酒。

# Generation — Bottom Up vs Top Down

## Example

Let $B := \{0\}, \mathcal{F} := \{S, P\}, S(x) := x + 1, P(x) := x - 1$

$$C_* = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

There is more than one way of obtaining a member of $C_*$, e.g. $1 = S(0) = S(P(S(0)))$.

## Theorem (Bottom up and Top down)

$$C_* = C^*$$

## Proof.

$(C^* \subset C_*)$: to show $C_*$ is inductive.
$(C_* \subset C^*)$: consider $x \in C_*$ and a construction sequence $(x_1, \ldots, x_n)$ for $x$. First $x_1 \in B \subset C^*$. If for all $j < i$ we have $x_j \in C^*$, then $x_i \in C^*$. By induction, $x_1, \ldots, x_n \in C^*$.

# Induction Principle for WFF

**Theorem (归纳原理)**

令 $\Psi$ 为一个关于合式公式的性质。假设

- $\Psi$ 对所有的命题符号成立；并且
- $\Psi$ 对公式构造运算封闭。

那么，$\Psi$ 对所有的合式公式都成立。

**Proof.**

$$\mathrm{WFF}_* = \mathrm{WFF}^* \subset \Psi$$

$$\Psi(0) \wedge \forall k \in \mathbb{N}(\Psi(k) \to \Psi(k+1)) \to \forall n \in \mathbb{N}\Psi(n)$$

$$\Psi(k) := \Psi(\mathrm{WFF}_k)$$

# Induction vs Recursion

$\psi(n) :=$ "$n$ rings needs $2^n - 1$ moves."

1. If ever you leave milk one day, be sure and leave it the next day as well.
2. Leave milk today.

Leave milk today and read this note again tomorrow.

# Subformula

## Definition (Subformula)

The set $\mathrm{Sub}(\varphi)$ of subformulas of a wff $\varphi$ is the smallest set $\Gamma$ that satisfies

1. $\varphi \in \Gamma$
2. $\neg\psi \in \Gamma \implies \psi \in \Gamma$
3. $\psi \to \chi \in \Gamma \implies \psi, \chi \in \Gamma$

$$\mathrm{Sub}(\varphi) := \begin{cases} \varphi & \text{if } \varphi = p \\ \{\varphi\} \cup \mathrm{Sub}(\psi) & \text{if } \varphi = \neg\psi \\ \{\varphi\} \cup \mathrm{Sub}(\psi) \cup \mathrm{Sub}(\chi) & \text{if } \varphi = \psi \to \chi \end{cases}$$

# Unique Readability, Unique Tree

$$((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r)))$$

subformula vs subtree

# Balanced-Parentheses

**Corollary (Balanced-Parentheses)**

*In any wff, the number of left parentheses is equal to the number of right parentheses.*

**Proof.**

Let $S$ be the set of "balanced" wffs.
Base step: the propositional symbols have zero parentheses.
Inductive step: obviously.

# Left-Weighted-Parentheses

## Lemma

*Any proper initial segment of a wff contains an excess of left parentheses. Thus no proper initial segment of a wff can itself be a wff.*

## Proof.

Consider $\varphi = (\alpha \wedge \beta)$. The proper initial segments of $(\alpha \wedge \beta)$ are the following:

1. $($
2. $(\alpha_0$ [inductive hypothesis]
3. $(\alpha$ [balanced-parentheses]
4. $(\alpha\wedge$ [balanced-parentheses]
5. $(\alpha \wedge \beta_0$ [inductive hypothesis]
6. $(\alpha \wedge \beta$ [balanced-parentheses]

# Unique Readability

## Theorem (Unique Readability Theorem)

*The five formula-building operations, when restricted to the set of wffs,*

1. *have ranges that are disjoint from each other and from the set of proposition symbols, and*
2. *are injective.*

## Proof.

To show $\mathscr{E}_\wedge$ is injective.

$$(\varphi \wedge \psi) = (\alpha \wedge \beta)$$
$$\Downarrow$$
$$\varphi \wedge \psi) = \alpha \wedge \beta)$$
$$\Downarrow$$
$$\varphi = \alpha \qquad [Lemma]$$

then it follows $\psi = \beta$.
Similarly, we can prove

$$(\varphi \wedge \psi) \neq (\alpha \rightarrow \beta)$$

# Omitting Parentheses

1. The outermost parentheses need not be explicitly mentioned.
2. We order the boolean connectives according to decreasing binding strength: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.
3. Where one connective symbol is used repeatedly, grouping is to the right.

$$1 + 2 * 3$$

# Contents


1 Introduction

2 History

3 Propositional Logic
- Syntax
- Semantics
- Connectives
- Formal System
- Meta-Theorems
- Application

4 Predicate Logic

5 Equational Logic

6 Set Theory

7 Recursion Theory

8 Modal Logic

9 Logic vs Game Theory

# Assignment

- A truth assignment for $\mathscr{L}^0$ is a function

$$\nu\colon \mathscr{P} \to \{0,1\}$$

- Such a truth assignment can be uniquely extended to $\overline{\nu}\colon \mathrm{WFF} \to \{0,1\}$ satisfying the following condition:

1. $\overline{\nu}(p) = \nu(p)$ for $p \in \mathscr{P}$
2. $\overline{\nu}(\neg\varphi) = 1 - \overline{\nu}(\varphi)$
3. $\overline{\nu}(\varphi \wedge \psi) = \min\{\overline{\nu}(\varphi), \overline{\nu}(\psi)\}$
4. $\overline{\nu}(\varphi \vee \psi) = \max\{\overline{\nu}(\varphi), \overline{\nu}(\psi)\}$
5. $\overline{\nu}(\varphi \to \psi) = 1 - \overline{\nu}(\varphi) + \overline{\nu}(\varphi) \cdot \overline{\nu}(\psi)$
6. $\overline{\nu}(\varphi \leftrightarrow \psi) = \overline{\nu}(\varphi) \cdot \overline{\nu}(\psi) + (1 - \overline{\nu}(\varphi)) \cdot (1 - \overline{\nu}(\psi))$

# Freeness vs Unique Readability

## Definition

The set $C$ is freely generated from $B$ by a class of functions $\mathcal{F}$ iff in addition to the requirements for being generated, the following conditions hold:

1. for every $f \in \mathcal{F}$: $f\restriction_C$ is injective.
2. the range of $f\restriction_C$ for all $f \in \mathcal{F}$, and the set $B$ are pairwise disjoint.

# Recursion Theorem

**Theorem (Recursion Theorem)**

*Assume that $C$ is freely generated from $B$ by $\mathcal{F}$, and for every $f \in \mathcal{F}$ we have $F_f : V^n \to V$, where $n = arity(f)$. Then for every function $h\colon B \to V$, there exists a unique function $\overline{h}\colon C \to V$ s.t.*

1. $\overline{h}\restriction_B = h$
2. *for all $f \in \mathcal{F}$ and all $x_1, \ldots, x_n \in C$:*

$$\overline{h}(f(x_1, \ldots, x_n)) = F_f\left(\overline{h}(x_1), \ldots, \overline{h}(x_n)\right)$$

$$\begin{tikzcd} B \arrow[r] \arrow[dr, "h"'] & C \arrow[d, "\overline{h}"] \\ & V \end{tikzcd}$$

- $h$ tells you how to color the initial elements in $B$;
- $F_f$ tells you how to convert the color of $\boldsymbol{x}$ into the color of $f(\boldsymbol{x})$.

Danger! $F_f$ is saying "green" but $F_g$ is saying "red" for the same point.

# Truth Table

| $p$ | $\neg p$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

| $p$ | $q$ | $p \wedge q$ | $p \vee q$ | $p \rightarrow q$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 |

## Example

- If $0 = 1$, then Russell is God.
- Snow is white iff $1 + 1 = 2$.

# Material Implication vs Cognition

翻哪几张卡才能验证“如果一面是偶数，另一面肯定是红色”？

未满 18 岁禁止饮酒！

# Tautology

If lily is beautiful, then the fact that $2$ is a prime number implies lily is beautiful.

| $p$ | $q$ | $q \to p$ | $p \to q \to p$ |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 |

$2^n$ truth assignments for a set of $n$ propositional symbols.

- $\nu \vDash \varphi$ if $\overline{\nu}(\varphi) = 1$.
- Logical Consequence. $\Gamma \vDash \varphi$ if for any truth assignment $\nu$ s.t. (for all $\psi \in \Gamma$: $\nu \vDash \psi$) $\implies$ $\nu \vDash \varphi$.
- Tautology. $\vDash \varphi$ if $\emptyset \vDash \varphi$.

$$\vDash (p \to q \to r) \to (p \to q) \to p \to r$$

| $p$ | $q$ | $r$ | $q \to r$ | $p \to q \to r$ | $p \to q$ | $p \to r$ | $(p \to q) \to p \to r$ | $(p \to q \to r) \to (p \to q) \to p \to r$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | | | | | | 1 |
| 0 | 0 | 1 | | | | | | 1 |
| 0 | 1 | 0 | | | | | | 1 |
| 0 | 1 | 1 | | | | | | 1 |
| 1 | 0 | 0 | | | | | | 1 |
| 1 | 0 | 1 | | | | | | 1 |
| 1 | 1 | 0 | | | | | | 1 |
| 1 | 1 | 1 | | | | | | 1 |

# Truth Table — Simplification for Tautology

| $(p$ | $\to$ | $q$ | $\to$ | $r)$ | $\to$ | $(p$ | $\to$ | $q)$ | $\to$ | $p$ | $\to$ | $r$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\underline{\mathbf{0}}$ | | | | | | | |
| | 1 | | | | $\mathbf{0}$ | | | | $\underline{\mathbf{0}}$ | | | |
| | 1 | | | | $\mathbf{0}$ | | 1 | | $\mathbf{0}$ | | $\underline{\mathbf{0}}$ | |
| 1 | 1 | | | 0 | $\mathbf{0}$ | 1 | $\underline{\mathbf{1}}$ | | $\mathbf{0}$ | 1 | $\mathbf{0}$ | 0 |
| 1 | $\underline{\mathbf{1}}$ | 1 | | 0 | $\mathbf{0}$ | 1 | $\mathbf{1}$ | 1 | $\mathbf{0}$ | 1 | $\mathbf{0}$ | 0 |
| 1 | $\mathbf{1}$ | 1 | $\underline{\mathbf{1}}$ | 0 | $\mathbf{0}$ | 1 | $\mathbf{1}$ | 1 | $\mathbf{0}$ | 1 | $\mathbf{0}$ | 0 |
| | | | $\times$ | | | | | | | | | |

# Exercises — Translation

1. The answer is $3$ or $6$.
2. I am not good at logic.
3. If you can't say it clearly, you don't understand it yourself.
4. You understand something only if you can formalize it.
5. I will go out unless it rains.
6. You can pay by credit card or cheque.
7. Neither Sarah nor Peter was to blame for the mistake.
8. I want to buy either a new desktop computer or a laptop, but I have neither the cash nor the credit I need.
9. If I get in the lift then it breaks, and/or if you get in then the lift breaks.(???) (Natural language is ambiguous!)
10. If we both get in the lift, then the lift breaks.
11. $p \vee q \to r \vDash (p \to r) \wedge (q \to r)$
12. $p \wedge q \to r \vDash (p \to r) \vee (q \to r)$

# Example

$^{\circledcirc}\hat{o}^{\circledcirc}$

1. The programmer's wife tells him: "Run to the store and pick up a loaf of bread. If they have eggs, get a dozen."
2. The programmer comes home with $12$ loaves of bread.
3. "Why did you buy $12$ loaves of bread!?", his wife screamed.
4. "Because they had eggs!"

- wife.

$$q \wedge (p \to r)$$

- programmer.

$$(\neg p \to q) \wedge (p \to s)$$

# Exercises — Validity

1. $p \lor q \vDash\dashv \neg p \to q \vDash\dashv (p \to q) \to q$
2. $p \land q \vDash\dashv \neg(p \to \neg q)$
3. $p \leftrightarrow q \vDash\dashv (p \to q) \land (q \to p)$
4. $p \land q \vDash\dashv \neg(\neg p \lor \neg q)$
5. $p \to q \to r \vDash\dashv (p \land q) \to r$
6. $p \to q \vDash\dashv \neg q \to \neg p$
7. $p \land (q \lor r) \vDash\dashv (p \land q) \lor (p \land r)$
8. $p \lor (q \land r) \vDash\dashv (p \lor q) \land (p \lor r)$
9. $\neg(p \lor q) \vDash\dashv \neg p \land \neg q$
10. $\neg(p \land q) \vDash\dashv \neg p \lor \neg q$
11. $p \vDash\dashv p \lor (p \land q)$
12. $p \vDash\dashv p \land (p \lor q)$

1. $\vDash \neg\neg p \to p$
2. $\vDash p \to \neg\neg p$
3. $\vDash p \lor \neg p$
4. $\vDash \neg(p \land \neg p)$
5. $\vDash p \land \neg p \to q$
6. $\vDash (p \to q) \land (\neg p \to q) \to q$
7. $\vDash (p \to q) \land (p \to \neg q) \to \neg p$
8. $\vDash (\neg p \to q) \land (\neg p \to \neg q) \to p$
9. $\vDash ((p \to q) \to p) \to p$

1. $\Gamma, \varphi \vDash \psi \iff \Gamma \vDash \varphi \to \psi$
2. $\varphi \vDash\dashv \psi \iff \vDash \varphi \leftrightarrow \psi$
3. $\varphi \lor \psi, \neg\varphi \lor \chi \vDash \psi \lor \chi$

$$\frac{\begin{array}{r} p \to q \\ q \end{array}}{p} \qquad \frac{\begin{array}{r} p \to q \\ \neg p \end{array}}{\neg q} \qquad \frac{\begin{array}{r} p \vee q \\ p \end{array}}{\neg q}$$

$$\frac{\begin{array}{c} \text{I think, therefore I am} \\ \text{I do not think} \end{array}}{\text{Therefore I am not}}$$

$$\frac{\begin{array}{c} \text{美女是老王或大黄杀的} \\ \text{老王是凶手} \end{array}}{\text{大黄是无辜的}}$$

*By all means marry; if you get a good wife, you'll be happy.*
*If you get a bad one, you'll become a philosopher.*

— *Socrates*

# Example

好货不贱，贱货不好。

如果把整个太平洋的水倒出，也浇不灭我对你爱情的火焰。整个太平洋的水倒得出吗？不行。所以，我不爱你。

如果把整个浴缸的水倒出，也浇不灭我对你爱情的火焰。整个浴缸的水倒得出吗？可以。所以，是的，我爱你。

# Example

- 如果你工作，就能挣钱；如果你赋闲在家，就能悠然自在。你要么工作要么赋闲，总之，你能挣钱或者能悠然自在。
- 如果你工作，就不能悠然自在；如果你赋闲在家，就不能挣钱。你要么工作要么赋闲，总之，你不能悠然自在或者不能挣钱。

$$p \to r, q \to s \vDash p \lor q \to r \lor s$$

$$p \to \neg s, q \to \neg r \vDash p \lor q \to \neg s \lor \neg r$$

- 老婆婆有俩儿子，老大卖阳伞，老二卖雨伞，晴天雨伞不好卖，雨天阳伞不好卖……
- 被困失火的高楼，走楼梯会被烧死，跳窗会摔死……

# Example

## 诉讼悖论

- 曾有师生签订合同：上学期间不收费，学生毕业打赢第一场官司后交学费。
- 可学生毕业后并未从事律师职业，于是老师威胁起诉学生。
- 老师说：如果我赢了，根据法庭判决，你必须交学费；如果你赢了，根据合同，你也必须交学费。要么我赢要么你赢，你都必须交学费。
- 学生说：如果我赢了，根据法庭判决，我不用交学费；如果你赢了，根据合同，我不用交学费。要么我赢要么你赢，我都不用交学费。

$$w \to p, \neg w \to p, w \vee \neg w \vDash p$$

$$w \wedge j \to p, \neg w \wedge c \to p, w \vee \neg w \overset{?}{\vDash} p$$

$$\neg w \wedge j \to \neg p, w \wedge c \to \neg p, w \vee \neg w \overset{?}{\vDash} \neg p$$

$$w \wedge j \to p, \neg w \wedge c \to p, (w \wedge j) \vee (\neg w \wedge c) \vDash p$$

# The Crocodile Dilemma

## The Crocodile Dilemma

I will return your child iff you can correctly predict what I will do next.

$$x = ? \implies \vDash (x \leftrightarrow r) \rightarrow r$$

| $r$ | $(\neg r \leftrightarrow r) \rightarrow r$ |
|---|---|
| 0 | 1 |
| 1 | 1 |

$$((r \vee \neg r) \leftrightarrow r) \rightarrow r$$

# 怎么得大奖？

**Problem (怎么得大奖？)**

- 说真话得一个大奖或一个小奖。
- 说假话不得奖。

- b: 我会得大奖。
- s: 我会得小奖。

# 怎么得大奖？

**Problem (怎么得大奖？)**

- 说真话得一个大奖或一个小奖。
- 说假话不得奖。

- b: 我会得大奖。
- s: 我会得小奖。

$$x = ? \implies \vDash (x \leftrightarrow b \vee s) \rightarrow b$$

| $b$ | $s$ | $(\neg b \wedge \neg s \leftrightarrow b \vee s) \rightarrow b$ | $(\neg s \leftrightarrow b \vee s) \rightarrow b$ | $((s \rightarrow b) \leftrightarrow b \vee s) \rightarrow b$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 |

# Gateway to Heaven

## Problem (天堂之路)

- 你面前有左右两人守卫左右两门。
- 一人只说真话，一人只说假话。
- 一门通天堂，一门通地狱。
- 你只有向其中一人提问一个“是/否”问题的机会。
- 怎么问出去天堂的路？

$$x = ? \implies \vDash (p \to (x \leftrightarrow q)) \land (\neg p \to (x \leftrightarrow \neg q))$$

- p: 你说真话。
- q: 左门通天堂。

| $p$ | $q$ | $(p \land q) \lor (\neg p \land \neg q)$ | report | $\varphi$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 |

# Proof by Contradiction

$$(\neg p \to q) \land (\neg p \to \neg q) \to p$$

**Theorem (Euclid's Theorem)**

*There are infinitely many prime numbers.*

**Proof.**

Assume to the contrary that $\mathbb{P} := \{p_1, p_2, \ldots, p_k\}$ is finite. Let $N := \prod_{i=1}^{k} p_i$.

$\exists p \in \mathbb{P}\colon\ p \mid (N+1)\ \ \&\ \ p \mid N \implies p \mid 1.$

# Semantic Equivalence

- Semantic equivalence is an equivalence relation between formulas.
- Semantic equivalence is compatible with operators.

$$\varphi \vDash\!\!\!\!\dashv \varphi' \implies \neg\varphi \vDash\!\!\!\!\dashv \neg\varphi'$$

$$\left.\begin{array}{l}\varphi \vDash\!\!\!\!\dashv \varphi' \\ \psi \vDash\!\!\!\!\dashv \psi'\end{array}\right\} \implies \varphi \star \psi \vDash\!\!\!\!\dashv \varphi' \star \psi'$$

where $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

- Equivalence relation + Compatible with Operators = Congruence relation

# Substitution

$$p[\alpha_1/p_1, \ldots, \alpha_n/p_n] := \begin{cases} \alpha_i & \text{if } p = p_i \text{ for some } 1 \leq i \leq n \\ p & \text{otherwise} \end{cases}$$

$$(\neg\varphi)[\alpha_1/p_1, \ldots, \alpha_n/p_n] := \neg\varphi[\alpha_1/p_1, \ldots, \alpha_n/p_n]$$

$$(\varphi \to \psi)[\alpha_1/p_1, \ldots, \alpha_n/p_n] := \varphi[\alpha_1/p_1, \ldots, \alpha_n/p_n] \to \psi[\alpha_1/p_1, \ldots, \alpha_n/p_n]$$

## Theorem

*Consider a wff $\varphi$ and a sequence $\alpha_1, \ldots, \alpha_n$ of wffs.*

1. *Let $\nu$ be a truth assignment for the set of all propositional symbols. Define $\rho$ to be the truth assignment for which $\rho(p_i) = \overline{\nu}(\alpha_i)$. Then $\overline{\rho}(\varphi) = \overline{\nu}(\varphi[\alpha_1/p_1, \ldots, \alpha_n/p_n])$.*
2. $\vDash \varphi \implies \vDash \varphi[\alpha_1/p_1, \ldots, \alpha_n/p_n]$

## Example

$$\vDash p \vee \neg p \implies \vDash (p \wedge \neg p) \vee \neg(p \wedge \neg p)$$

# Duality

**Theorem**

*Let $\varphi$ be a wff whose only connectives are $\neg, \wedge, \vee$. Let $\varphi^*$ be the result of interchanging $\wedge$ and $\vee$ and replacing each propositional symbol by its negation. Then $\neg\varphi \vDash\!\!\!\dashv \varphi^*$.*

**Proof.**

Prove by induction.

- $\varphi = p_i$
- $\varphi = \neg\alpha$
- $\varphi = \alpha \wedge \beta$
- $\varphi = \alpha \vee \beta$

# Contents


1 Introduction

2 History

3 Propositional Logic
- Syntax
- Semantics
- Connectives
- Formal System
- Meta-Theorems
- Application

4 Predicate Logic

5 Equational Logic

6 Set Theory

7 Recursion Theory

8 Modal Logic

9 Logic vs Game Theory

# Connectives

Would we gain anything by adding more connectives to the language?

## Example (Exclusive Disjunction)

$$\begin{aligned} v(p \dot{\vee} q) &= |v(p) - v(q)| \\ &\Downarrow \\ p \dot{\vee} q &\vDash\!\!\!\dashv (\neg p \wedge q) \vee (p \wedge \neg q) \\ &\vDash\!\!\!\dashv (p \vee q) \wedge (\neg p \vee \neg q) \\ &\vDash\!\!\!\dashv \neg(p \leftrightarrow q) \end{aligned}$$

| $p$ | $q$ | $p \dot{\vee} q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

$$\frac{\begin{array}{c} p \dot{\vee} q \\ p \end{array}}{\neg q} \qquad \frac{\begin{array}{c} p \dot{\vee} q \\ \neg p \end{array}}{q} \qquad \frac{p}{p \dot{\vee} q}\ ?$$

# Example

## Example

Let # be a three-place proposition connective.
The interpretation of # is given by

$$v(\#(p,q,r)) = \left\lfloor \frac{v(p)+v(q)+v(r)}{2} \right\rfloor$$

then

$$\#(p,q,r) \vDash\!\!\!\dashv (p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$$

| $p$ | $q$ | $r$ | $\#(p,q,r)$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

# Truth Function

A truth assignment for $\mathscr{L}^0$ is a function $v\colon \mathscr{P} \to \{0,1\}$.

| $p$ | $\neg p$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

| $p$ | $q$ | $p \wedge q$ | $p \vee q$ | $p \to q$ | $p \leftrightarrow q$ | $\cdots$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | $\cdots$ |
| 0 | 1 | 0 | 1 | 1 | 0 | $\cdots$ |
| 1 | 0 | 0 | 1 | 0 | 0 | $\cdots$ |
| 1 | 1 | 1 | 1 | 1 | 1 | $\cdots$ |

A $n$-place truth function is a function $B\colon \{0,1\}^n \to \{0,1\}$.

| $x$ | $B_\neg(x)$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

| $x$ | $y$ | $B_\wedge(x,y)$ | $B_\vee(x,y)$ | $B_\to(x,y)$ | $B_\leftrightarrow(x,y)$ | $\cdots$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | $\cdots$ |
| 0 | 1 | 0 | 1 | 1 | 0 | $\cdots$ |
| 1 | 0 | 0 | 1 | 0 | 0 | $\cdots$ |
| 1 | 1 | 1 | 1 | 1 | 1 | $\cdots$ |

There are $2^{2^n}$ distinct truth functions with $n$ places.

# Truth Function

$$v\colon \mathscr{P} \to \{0,1\}$$
$$B\colon \{0,1\}^n \to \{0,1\}$$

| $v(p_1),\ldots,v(p_n)$ | | $x_1,\ldots,x_n$ | $B_\varphi(x_1,\ldots,x_n)$ | | $\overline{v}(\varphi)$ |
|---|---|---|---|---|---|
| $v_1(p_1),\ldots,v_1(p_n)$ | $=$ | $0,\ldots,0$ | $B_\varphi(0,\ldots,0)$ | $=$ | $\overline{v}_1(\varphi)$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ |
| $v_{2^n}(p_1),\ldots,v_{2^n}(p_n)$ | $=$ | $1,\ldots,1$ | $B_\varphi(1,\ldots,1)$ | $=$ | $\overline{v}_{2^n}(\varphi)$ |

## Definition

Suppose $\varphi$ is a wff whose propositional symbols are $p_1,\ldots,p_n$.
A truth function $B\colon \{0,1\}^n \to \{0,1\}$ represented by $\varphi$ is

$$B_\varphi(v(p_1),\ldots,v(p_n)) = \overline{v}(\varphi)$$

$$\varphi \vDash\!\!\!\dashv \psi \iff B_\varphi = B_\psi$$

## Theorem (Post1921)

*Every truth function can be represented by some wff whose only connectives are $\neg, \wedge, \vee$.*

## Proof.

$$p_i^{x_i} := \begin{cases} p_i & \text{if } x_i = 1 \\ \neg p_i & \text{otherwise} \end{cases}$$

Case1: $B(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \{0,1\}^n$.
Let $\varphi = p \wedge \neg p$.
Case2:

$$\varphi = \bigvee_{\boldsymbol{x}:B(\boldsymbol{x})=1} \bigwedge_{i=1}^{n} p_i^{x_i}$$

Case1: $B(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in \{0,1\}^n$.
Let $\psi = p \vee \neg p$.
Case2:

$$\psi = \bigwedge_{\boldsymbol{x}:B(\boldsymbol{x})=0} \bigvee_{i=1}^{n} p_i^{1-x_i}$$

# Normal Form

## Corollary

*Every wff which is not a contradiction is logically equivalent to a formula of disjunctive normal form (DNF):*

$$\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \pm p_{ij}$$

## Corollary

*Every wff which is not a tautology is logically equivalent to a formula of conjunctive normal form (CNF):*

$$\bigwedge_{i=1}^{m} \bigvee_{j=1}^{n} \pm p_{ij}$$

## Proof.

$$\neg\varphi \vDash \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \pm p_{ij} \implies \varphi \vDash \neg\left(\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \pm p_{ij}\right) \vDash \bigwedge_{i=1}^{m} \bigvee_{j=1}^{n} \mp p_{ij}$$

# CNF Transformation

$$
\begin{aligned}
\varphi \leftrightarrow \psi &\implies (\neg\varphi \vee \psi) \wedge (\neg\psi \vee \varphi) \\
\varphi \rightarrow \psi &\implies \neg\varphi \vee \psi \\
\neg(\varphi \wedge \psi) &\implies \neg\varphi \vee \neg\psi \\
\neg(\varphi \vee \psi) &\implies \neg\varphi \wedge \neg\psi \\
\neg\neg\varphi &\implies \varphi \\
(\varphi_1 \wedge \cdots \wedge \varphi_n) \vee \psi &\implies (\varphi_1 \vee \psi) \wedge \cdots \wedge (\varphi_n \vee \psi)
\end{aligned}
$$

# Adequate Sets of Connectives

## Definition

An adequate set of connectives is a set s.t. every truth function can be represented by a wff containing only connectives from that set.

- $\{\neg, \wedge, \vee\}$
- $\{\neg, \wedge\}; \{\neg, \vee\}; \{\neg, \rightarrow\}; \{\bot, \rightarrow\}$
- $\{\uparrow\}; \{\downarrow\}$
- $\{\wedge, \vee, \rightarrow, \leftrightarrow\}; \{\neg, \leftrightarrow\}$ not adequate.

| $p$ | $\bot$ |
|---|---|
| 0 | 0 |
| 1 | 0 |

$$\begin{aligned}
\bot &:= p \wedge \neg p \\
p \uparrow q &:= \neg(p \wedge q) \\
p \downarrow q &:= \neg(p \vee q) \\
\neg p &:= p \uparrow p \\
p \wedge q &:= (p \uparrow q) \uparrow (p \uparrow q) \\
p \vee q &:= (p \uparrow p) \uparrow (q \uparrow q)
\end{aligned}$$

| $p$ | $q$ | $p \uparrow q$ | $p \downarrow q$ |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |

# Contents


1 Introduction

2 History

3 Propositional Logic
- Syntax
- Semantics
- Connectives
- Formal System
- Meta-Theorems
- Application

4 Predicate Logic

5 Equational Logic

6 Set Theory

7 Recursion Theory

8 Modal Logic

9 Logic vs Game Theory

# Why Study Formal System?

Why truth tables are not sufficient?

- Exponential size
  - How many times would you have to fold a piece of paper($0.1mm$) onto itself to reach the Moon?
  - Common Ancestors of All Humans
    $(1)$ Someone alive $1000BC$ is an ancestor of everyone alive today;
    $(2)$ Everyone alive $2000BC$ is either an ancestor of nobody alive today or of everyone alive today;
    $(3)$ Most of the people you are descended from are no more genetically related to you than strangers are.
    $(4)$ Even if everyone alive today had exactly the same set of ancestors from $2000BC$, the distribution of one's ancestors from that population could be very different.
- Inapplicability beyond Boolean connectives.

# Formal System = Axiom + Inference Rule

## Axiom Schema

1. $\varphi \to \psi \to \varphi$
2. $(\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi$
3. $(\neg\varphi \to \neg\psi) \to (\neg\varphi \to \psi) \to \varphi$

## Inference Rule

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ [\text{MP}]$$

这个句子永远不能被证明

什么是“证明”？

**Definition (Deduction)**

A deduction from $\Gamma$ is a sequence of wff $(\varphi_1, \ldots, \varphi_n)$ s.t. for $k \leq n$, either

1. $\varphi_k$ is an axiom, or
2. $\varphi_k \in \Gamma$, or
3. for some $i < k$ and $j < k$, $\varphi_i = \varphi_j \to \varphi_k$.

- $\Gamma \vdash \varphi$ if $\varphi$ is the last member of some deduction from $\Gamma$.
- $\vdash \varphi := \emptyset \vdash \varphi$

房子着火了，数学家的妻子拿起灭火器把火灭了；
煤气泄漏了，数学家把房子点了。

# Example

**Theorem**

$$\vdash p \to p$$

**Proof.**

| | | |
|---|---|---|
| 1 | $p \to (p \to p) \to p$ | A1 |
| 2 | $(p \to (p \to p) \to p) \to (p \to p \to p) \to p \to p$ | A2 |
| 3 | $(p \to p \to p) \to p \to p$ | 1,2 MP |
| 4 | $p \to p \to p$ | A1 |
| 5 | $p \to p$ | 3,4 MP |

# Example

**Theorem**

$$\vdash (\neg p \to p) \to p$$

**Proof.**

1. $(\neg p \to \neg p) \to (\neg p \to p) \to p$ — A3
2. $\neg p \to \neg p$
3. $(\neg p \to p) \to p$ — 1,2 MP

# Example

**Theorem**

$$p \to q, q \to r \vdash p \to r$$

**Proof.**

| | | |
|---|---|---|
| 1 | $(q \to r) \to (p \to q \to r)$ | A1 |
| 2 | $q \to r$ | Premise |
| 3 | $p \to q \to r$ | 1,2 MP |
| 4 | $(p \to q \to r) \to (p \to q) \to p \to r$ | A2 |
| 5 | $(p \to q) \to p \to r$ | 4,3 MP |
| 6 | $p \to q$ | Premise |
| 7 | $p \to r$ | 5,6 MP |

# Example — Curry's Paradox ◎ô◎

如果这句话是真的那么上帝存在。

**If this sentence is true, then I am God.**

$$p \leftrightarrow (p \to q) \vdash q$$

**Proof.**

1. $p \leftrightarrow (p \to q)$
2. $p \to p \to q$
3. $(p \to p) \to p \to q$
4. $p \to q$
5. $p$
6. $q$

1. 甲：如果我没说错，那么上帝存在。
2. 乙：**如果你没说错**，那么上帝存在。
3. 甲：你承认我没说错了？
4. 乙：当然。
5. 甲：可见我没说错。你已经承认：**如果我没说错，那么上帝存在**。所以，上帝存在。

这句话是假的并且上帝不存在。

# Curry's Paradox — Kiss-Kiss ♡◎♡

## Smullyan 实力撩妹 ♡◎♡

1. Smullyan: "我说一句话，如果它是真的，可以给我你的签名吗？"
2. 美女："可以。"
3. Smullyan: "不过如果我说的不是真的，那就不要给我签名了。"
4. 美女："好的。"
5. 然后 Smullyan 说了一句话。
6. 美女想了一下，发现她必须给他一个吻！

$$x = ? \implies \vDash (s \leftrightarrow x) \to k$$

## Hi 美女，问你个问题呗 ~v~

如果我问你"你能做我女朋友吗"，那么你的答案能否和**这个问题本身**的答案一样？

# Deduction Theorem — “搬运工”

## Theorem (Deduction Theorem)

$$\Gamma, \varphi \vdash \psi \iff \Gamma \vdash \varphi \to \psi$$

## Proof.

Prove by induction on the length of the deduction sequence $(\alpha_1, \ldots, \alpha_n)$ of $\psi$ from $\Gamma \cup \{\varphi\}$.

Base step $n = 1$:

case1. $\psi$ is an axiom. (use Axiom1.)

case2. $\psi \in \Gamma$.

case3. $\psi = \varphi$.

Inductive step $n > 1$:

case1. $\psi$ is either an axiom, or $\psi \in \Gamma$, or $\psi = \varphi$.

case2. $\alpha_i = \alpha_j \to \psi$

$\Gamma, \varphi \vdash \alpha_j \implies \Gamma \vdash \varphi \to \alpha_j$

$\Gamma, \varphi \vdash \alpha_j \to \psi \implies \Gamma \vdash \varphi \to \alpha_j \to \psi$

# Equivalent Replacement

**Theorem**

*Suppose $\psi \in \mathrm{Sub}(\varphi)$, and $\varphi^*$ arises from the wff $\varphi$ by replacing one or more occurrences of $\psi$ in $\varphi$ by $\chi$. Then*

$$\psi \leftrightarrow \chi \vdash \varphi \leftrightarrow \varphi^*$$

**Proof.**

Prove by induction on the number of connective of $\varphi$.

# Example

$^{\circledcirc}\hat{o}^{\circledcirc}$

1. A logician's wife is having a baby.
2. The doctor immediately hands the newborn to the dad.
3. His wife asks impatiently: "So, is it a boy or a girl"?
4. The logician replies: "yes".

- wife.

$$?p$$

- logician.

$$\left.\begin{array}{r} p \vee q \\ q \leftrightarrow \neg p \end{array}\right\} \Longrightarrow p \vee \neg p \qquad \checkmark$$

# Formal System — Variant

## Axiom Schema

1. $\varphi \to \psi \to \varphi$
2. $(\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi$
3. $\varphi \wedge \psi \to \varphi$
4. $\varphi \wedge \psi \to \psi$
5. $(\varphi \to \psi) \to (\varphi \to \chi) \to \varphi \to \psi \wedge \chi$
6. $\varphi \to \varphi \vee \psi$
7. $\psi \to \varphi \vee \psi$
8. $(\varphi \to \chi) \to (\psi \to \chi) \to \varphi \vee \psi \to \chi$
9. $(\varphi \to \neg\psi) \to (\varphi \to \psi) \to \neg\varphi$
10. $\neg\varphi \to \varphi \to \psi$
11. $\neg\neg\varphi \to \varphi$

## Reference Rule

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ [\text{MP}]$$

$$p' := \neg\neg p$$

$$(\varphi \star \psi)' := \neg\neg(\varphi' \star \psi')$$

where $\star \in \{\wedge, \vee, \to\}$

$$\Gamma' := \{\varphi' : \varphi \in \Gamma\}$$

$$\Gamma \vdash_{\boldsymbol{C}} \varphi \iff \Gamma' \vdash_{\boldsymbol{I}} \varphi'$$

1–8+MP=**P**ositive Calculus  **P**+9=**M**inimal Calculus

**M**+10=**I**ntuitionistic Calculus  **I**+11=**C**lassical Calculus

# Tree Method for Propositional Logic

$$\begin{array}{c} \neg\neg\varphi \\ | \\ \varphi \end{array} \qquad \begin{array}{c} \varphi \to \psi \\ \diagup \quad \diagdown \\ \neg\varphi \qquad \psi \end{array} \qquad \begin{array}{c} \neg(\varphi \to \psi) \\ | \\ \varphi \\ \neg\psi \end{array}$$

---

$$\begin{array}{c} \varphi \wedge \psi \\ | \\ \varphi \\ \psi \end{array} \qquad \begin{array}{c} \neg(\varphi \wedge \psi) \\ \diagup \quad \diagdown \\ \neg\varphi \qquad \neg\psi \end{array} \qquad \begin{array}{c} \varphi \vee \psi \\ \diagup \quad \diagdown \\ \varphi \qquad \psi \end{array} \qquad \begin{array}{c} \neg(\varphi \vee \psi) \\ | \\ \neg\varphi \\ \neg\psi \end{array}$$

---

$$\begin{array}{c} \varphi \leftrightarrow \psi \\ \diagup \quad \diagdown \\ \begin{array}{c} \varphi \\ \psi \end{array} \qquad \begin{array}{c} \neg\varphi \\ \neg\psi \end{array} \end{array} \qquad \begin{array}{c} \neg(\varphi \leftrightarrow \psi) \\ \diagup \quad \diagdown \\ \begin{array}{c} \varphi \\ \neg\psi \end{array} \qquad \begin{array}{c} \neg\varphi \\ \psi \end{array} \end{array}$$

✓

# Instructions for Tree Construction

- A *literal* is an atomic formula or its negation.
- When a non-literal wff has been fully unpacked, check it with ✓

1. Start with premises and the negation of the conclusion.
2. Inspect each open path for an occurrance of a wff and its negation. If these occur, close the path with ×.
3. If there is no unchecked non-literal wff on any open path, then stop!
4. Otherwise, unpack any unchecked non-literal wff on any open path.
5. Goto ②.

- *Closed branch.* A branch is closed if it contains a wff and its negation.
- *Closed tree.* A tree is closed if all its branches are closed.
- *Open branch.* A branch is open if it is not closed and no rule can be applied.
- *Open tree.* A tree is open if it has at least one open branch.

# Tactics

- Try to apply "non-branching" rules first, in order to reduce the number of branches.
- Try to close off branches as quickly as possible.

**Definition (Deduction)**

$\varphi_1, \ldots, \varphi_n \vdash \psi$ iff there exists a *closed tree* from $\{\varphi_1, \ldots, \varphi_n, \neg\psi\}$.

**Theorem (Soundness & Completeness Theorem)**

$$\varphi_1, \ldots, \varphi_n \vdash \psi \iff \varphi_1, \ldots, \varphi_n \vDash \psi$$

**Remark:** If an inference with propositional formulas is not valid, then its tree will have at least one open branch. The tree method can generate every counterexample of an invalid inference in propositional logic.

# Examples — Tree Method

$$\boxed{p \to q, \neg q \vee r \vdash \neg p \vee r}$$

$p \to q$ ✓

$\neg q \vee r$ ✓

$\neg(\neg p \vee r)$ ✓

$\neg\neg p$

$\neg r$

- $\neg p$ ✗
- $q$
  - $\neg q$ ✗
  - $r$ ✗

# An open branch corresponds to a valuation

$$p \lor q, r \lor s, s \to t \overset{?}{\vDash} r$$

$$v(r) = 0, \quad v(s) = 1, \quad v(t) = 1 \quad v(p) = 1 \quad v(q) = 1 \text{ or } 0$$

$$v(r) = 0, \quad v(s) = 1, \quad v(t) = 1 \quad v(q) = 1 \quad v(p) = 1 \text{ or } 0$$

$$v \vDash p \lor q, \quad v \vDash r \lor s, \quad v \vDash s \to t, \quad v \nvDash r$$

# Exercises — Tree Method

1. $p \to (\neg q \to q) \vdash p \to q$
2. $(p \to r) \land (q \to r) \vdash p \lor q \to r$
3. $(p \to q) \land (r \to s) \vdash \neg q \land r \to \neg q \land s$
4. $\Big(\big((p \to q) \to (\neg r \to \neg s)\big) \to r\Big) \to t \vdash (t \to p) \to s \to p$
5. $(p \to q) \lor (q \to r)$
6. $(p \to q) \to (\neg p \to q) \to q$
7. $((p \to q) \to p) \to p$
8. $(p \to q) \land (r \to s) \to p \lor r \to q \lor s$
9. $(p \to q) \land r \to \neg(p \land r) \lor (q \land r)$
10. $(p \leftrightarrow (p \to q)) \to q$
11. $\neg(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow q)$

# Exercises — Tree Method

Decide whether the following inferences are valid or not. If not, provide a counterexample.

1. $(p \vee q) \wedge r \overset{?}{\vDash} p \vee (q \wedge r)$
2. $p \vee (q \wedge r) \overset{?}{\vDash} (p \vee q) \wedge r$
3. $p \leftrightarrow (q \rightarrow r) \overset{?}{\vDash} (p \leftrightarrow q) \rightarrow r$
4. $(p \leftrightarrow q) \rightarrow r \overset{?}{\vDash} p \leftrightarrow (q \rightarrow r)$
5. $\neg(p \rightarrow q \wedge r), r \rightarrow p \wedge q \overset{?}{\vDash} \neg r$
6. $p \rightarrow (q \wedge r), \neg(p \vee q \rightarrow r) \overset{?}{\vDash} p$
7. $p \rightarrow q, r \rightarrow s, p \vee r, \neg(q \wedge s) \overset{?}{\vDash} (q \rightarrow p) \wedge (s \rightarrow r)$
8. If God does not exist, then it's not the case that *if I pray, my prayers will be answered*; and I don't pray; so God exists.

# Contents


1 Introduction

2 History

3 Propositional Logic
- Syntax
- Semantics
- Connectives
- Formal System
- Meta-Theorems
- Application

4 Predicate Logic

5 Equational Logic

6 Set Theory

7 Recursion Theory

8 Modal Logic

9 Logic vs Game Theory

# Independence

## Definition (Independence)

An axiom $\varphi$ in $\Gamma$ is independent if $\Gamma \setminus \{\varphi\} \nvdash \varphi$.

**找一个这个公理没有的性质，证明能从其他公理推出来的命题都有这个性质。 比如对某个特殊语义 $\Vdash$:**

- for all $\psi$, $\Gamma \setminus \{\varphi\} \vdash \psi \implies \Vdash \psi$
- $\nVdash \varphi$

## Theorem

*Axiom3 is independent of Axiom1 and Axiom2.*

| $p$ | $\neg p$ |
|---|---|
| 0 | 0 |
| 1 | 0 |

| $\rightarrow$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 0 | 1 |

Let $\nu(p) = 0$ and $\nu(q) = 1$, then $\nVdash (\neg p \rightarrow \neg q) \rightarrow (\neg p \rightarrow q) \rightarrow p$.

# Independence

Axiom$1$ and Axiom$2$ axiomatizes the conditional ($\to$) fragment of intuitionistic propositional logic. To axiomatize the conditional fragment of classical logic, we also need *Peirce's law*: $((p \to q) \to p) \to p$.

**Theorem**

*Peirce's law is independent of Axiom$1$ and Axiom$2$.*

| $\to$ | $1$ | $2$ | $3$ |
|---|---|---|---|
| $1$ | $1$ | $2$ | $3$ |
| $2$ | $1$ | $1$ | $3$ |
| $3$ | $1$ | $1$ | $1$ |

Here we interpret $1$ as "true", $3$ as "false", and $2$ as "maybe".
Let $\nu(p) = 2$ and $\nu(q) = 3$, then $\nu(((p \to q) \to p) \to p) = 2$.

# Model & Semantic Consequence

- $\mathrm{Mod}(\varphi) := \{v : v \vDash \varphi\}$
- $\mathrm{Mod}(\Gamma) := \bigcap_{\varphi \in \Gamma} \mathrm{Mod}(\varphi)$
- $\mathrm{Th}(v) := \{\varphi : v \vDash \varphi\}$
- $\mathrm{Th}(\mathcal{K}) := \bigcap_{v \in \mathcal{K}} \mathrm{Th}(v)$
- $\mathrm{Cn}(\Gamma) := \{\varphi : \Gamma \vDash \varphi\}$

- $\Gamma \subset \Gamma' \implies \mathrm{Mod}(\Gamma') \subset \mathrm{Mod}(\Gamma)$
- $\mathcal{K} \subset \mathcal{K}' \implies \mathrm{Th}(\mathcal{K}') \subset \mathrm{Th}(\mathcal{K})$
- $\Gamma \subset \mathrm{Th}(\mathrm{Mod}(\Gamma))$
- $\mathcal{K} \subset \mathrm{Mod}(\mathrm{Th}(\mathcal{K}))$
- $\mathrm{Mod}(\Gamma) = \mathrm{Mod}(\mathrm{Th}(\mathrm{Mod}(\Gamma)))$
- $\mathrm{Th}(\mathcal{K}) = \mathrm{Th}(\mathrm{Mod}(\mathrm{Th}(\mathcal{K})))$
- $\mathrm{Cn}(\Gamma) = \mathrm{Th}(\mathrm{Mod}(\Gamma))$
- $\Gamma \subset \Gamma' \implies \mathrm{Cn}(\Gamma) \subset \mathrm{Cn}(\Gamma')$
- $\mathrm{Cn}(\mathrm{Cn}(\Gamma)) = \mathrm{Cn}(\Gamma)$

# Consistency & Satisfiability

- $\Gamma$ is consistent if $\Gamma \nvdash \bot$.
- $\Gamma$ is Post-consistent if there is some wff $\varphi$: $\Gamma \nvdash \varphi$.

$$\boxed{\Gamma \text{ is consistent iff it is Post-consistent.}}$$

- $\Gamma$ is maximal if for every wff $\varphi$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.
- $\Gamma$ is maximal consistent if it is both consistent and maximal.
- $\Gamma$ is satisfiable if $\mathrm{Mod}(\Gamma) \neq \emptyset$.
- $\Gamma$ is finitely satisfiable if every finite subset of $\Gamma$ is satisfiable.

- If $\Gamma$ is consistent and $\Gamma \vdash \varphi$, then $\Gamma \cup \{\varphi\}$ is consistent.
- $\Gamma \cup \{\neg\varphi\}$ is inconsistent iff $\Gamma \vdash \varphi$.
- If $\Gamma$ is maximal consistent, then $\varphi \notin \Gamma \implies \Gamma \cup \{\varphi\}$ is inconsistent.

# Soundness Theorem

## Theorem (Soundness Theorem)

$$\Gamma \vdash \varphi \implies \Gamma \vDash \varphi$$

## Proof.

Prove by induction on the length of the deduction sequence.
Case1: $\varphi$ is an axiom. (truth table)
Case2: $\varphi \in \Gamma$
Case3:

$$\left.\begin{array}{l} \Gamma \vDash \alpha_j \\ \Gamma \vDash \alpha_j \to \varphi \end{array}\right\} \implies \Gamma \vDash \varphi$$

## Corollary

*Any satisfiable set of wffs is consistent.*

# Compactness Theorem

## Theorem (Compactness Theorem)

*A set of wffs is satisfiable iff it is finitely satisfiable.*

如果语言可以说无穷析取，则没有紧致性。$\left\{\bigvee_{i=1}^{\infty} p_i, \neg p_1, \neg p_2, \dots\right\}$

## Corollary

*If $\Gamma \vDash \varphi$, then there is a finite $\Gamma_0 \subset \Gamma$ s.t. $\Gamma_0 \vDash \varphi$.*

## Proof.

$$\begin{aligned}
\Gamma_0 \nvDash \varphi \text{ for any } \Gamma_0 \subset \Gamma &\implies \Gamma_0 \cup \{\neg\varphi\} \text{ is satisfiable for any } \Gamma_0 \subset \Gamma \\
&\implies \Gamma \cup \{\neg\varphi\} \text{ is satisfiable} \\
&\implies \Gamma \nvDash \varphi
\end{aligned}$$

# Applications of Compactness ⊙$\underline{\Delta}$⊙

An infinite graph $(V, E)$ is $n$-colorable iff every finite subgraph of $(V, E)$ is $n$-colorable.

**Proof.**

Take $V \times \{1, \ldots, n\}$ as the set of atoms.
$\Gamma := \{p(v,1) \vee \cdots \vee p(v,n) : v \in V\} \cup \{\neg(p(v,i) \wedge p(v,j)) : v \in V, 1 \leq i < j \leq n\} \cup \{\neg(p(v,i) \wedge p(w,i)) : (v,w) \in E, 1 \leq i \leq n\}$

# Proof of Compactness Theorem

## Proof.

part1. Extend the finitely satisfiable set $\Gamma$ to a maximal finitely satisfiable set $\Delta$.

Let $\langle \varphi_i : i \in \mathbb{N} \rangle$ be a fixed enumeration of the wffs.

$$\Delta_0 := \Gamma$$

$$\Delta_{n+1} := \begin{cases} \Delta_n \cup \{\varphi_n\} & \text{if } \Delta_n \cup \{\varphi_n\} \text{ is finitely satisfiable} \\ \Delta_n \cup \{\neg\varphi_n\} & \text{otherwise} \end{cases}$$

$$\Delta := \bigcup_{n \in \mathbb{N}} \Delta_n$$

part2. Define a truth assignment that satisfies $\Gamma$.

$$\nu(p) := \begin{cases} 1 & \text{if } p \in \Delta \\ 0 & \text{otherwise} \end{cases} \implies \big(\nu \vDash \varphi \iff \varphi \in \Delta\big)$$

# "Compactness Theorem"

**Theorem ("Compactness Theorem")**

*$\Gamma$ is consistent iff every finite subset of $\Gamma$ is consistent.*

**Proof.**

Suppose $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.
Then there is a deduction sequence $(\alpha_1, \ldots, \alpha_n)$ of $\varphi$ from $\Gamma$, and a deduction sequence $(\beta_1, \ldots, \beta_m)$ of $\neg\varphi$ from $\Gamma$.
Let $\Sigma_0 := \{\alpha_i \in \Gamma \colon 1 \leq i \leq n\}$ and $\Sigma_1 := \{\beta_i \in \Gamma \colon 1 \leq i \leq m\}$.
The finite set $\Sigma := \Sigma_0 \cup \Sigma_1$ is inconsistent.

# Weak Completeness Theorem

## Lemma

*Let $\varphi$ be a wff whose only propositional symbols are $p_1, \ldots, p_n$. Let*

$$p_i^\nu := \begin{cases} p_i & \text{if } \nu \vDash p_i \\ \neg p_i & \text{otherwise} \end{cases} \qquad \varphi^\nu := \begin{cases} \varphi & \text{if } \nu \vDash \varphi \\ \neg\varphi & \text{otherwise} \end{cases}$$

*then $p_1^\nu, \ldots, p_n^\nu \vdash \varphi^\nu$.*

## Weak Completeness Theorem $\vDash \varphi \implies \vdash \varphi$

$$\mu(p) := \begin{cases} 1 - \nu(p) & \text{if } p = p_n \\ \nu(p) & \text{otherwise} \end{cases}$$

$$\left.\begin{array}{l} p_1^\nu, \ldots, p_{n-1}^\nu, p_n^\nu \vdash \varphi \\ p_1^\mu, \ldots, p_{n-1}^\mu, p_n^\mu \vdash \varphi \end{array}\right\} \implies p_1^\nu, \ldots, p_{n-1}^\nu \vdash \varphi$$

# Completeness Theorem

$$\vDash \varphi \iff \vdash \varphi$$

$$+$$

Compactness

$$\Downarrow$$

$$\Gamma \vDash \varphi \iff \Gamma \vdash \varphi$$

# Completeness Theorem — Post1921

## Theorem (Completeness Theorem)

$$\Gamma \vDash \varphi \implies \Gamma \vdash \varphi$$

## Corollary

*Any consistent set of wffs is satisfiable.*

$$\begin{array}{ccc} \Gamma \vDash \varphi & \Longleftrightarrow & \Gamma \vdash \varphi \\ \Updownarrow & & \Updownarrow \\ \begin{array}{c}\Gamma \cup \{\neg\varphi\} \\ \text{unsatisfiable}\end{array} & \Longleftrightarrow & \begin{array}{c}\Gamma \cup \{\neg\varphi\} \\ \text{inconsistent}\end{array} \end{array}$$

## Corollary (Compactness Theorem)

*A set of wffs is satisfiable iff it is finitely satisfiable.*

# Proof of Completeness Theorem

**Proof.**

step1. Extend the consistent set $\Gamma$ to a maximal consistent set $\Delta$.
Let $\langle \varphi_i : i \in \mathbb{N} \rangle$ be a fixed enumeration of the wffs.

$$\Delta_0 := \Gamma$$

$$\Delta_{n+1} := \begin{cases} \Delta_n \cup \{\varphi_n\} & \text{if } \Delta_n \cup \{\varphi_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg\varphi_n\} & \text{otherwise} \end{cases}$$

$$\Delta := \bigcup_{n \in \mathbb{N}} \Delta_n$$

step2. Define a truth assignment that satisfies $\Gamma$.

$$v(p) := \begin{cases} 1 & \text{if } p \in \Delta \\ 0 & \text{otherwise} \end{cases} \implies \big(v \vDash \varphi \iff \varphi \in \Delta\big)$$

# Decidability — Post1921

**Theorem**

*There is an effective procedure that, given any expression, will decide whether or not it is a wff.*

**Theorem**

*There is an effective procedure that, given a finite set $\Gamma \cup \{\varphi\}$ of wffs, will decide whether or not $\Gamma \vDash \varphi$.*

**Theorem**

*If $\Gamma$ is a decidable set of wffs, then the set of logical consequences of $\Gamma$ is recursively enumerable.*

# Post 1897-1954

- Truth table
- Completeness of propositional logic
- Post machine
- Post canonical system
- Post correspondence problem
- Post problem

# Theory & Axiomatization

**什么是“理论”？**

- A set $\Gamma$ of sentences is a theory if $\Gamma = \mathrm{Cn}(\Gamma)$.
- A theory $\Gamma$ is complete if for every sentence $\varphi$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.
- A theory $\Gamma$ is axiomatizable if there is a decidable set $\Sigma$ of sentences s.t. $\Gamma = \mathrm{Cn}(\Sigma)$.
- A theory $\Gamma$ is finitely axiomatizable if $\Gamma = \mathrm{Cn}(\Sigma)$ for some finite set $\Sigma$ of sentences.

# Model Checking & Satisfiability Checking & Validity Checking[7]

- Given a model $\nu$ and a formula $\varphi$. Is $\nu \vDash \varphi$? —P
- Given a formula $\varphi$. Is there a model $\nu$ s.t. $\nu \vDash \varphi$? —NP
- Given a sentence $\varphi$. Is $\vDash \varphi$?

Figure: *Hamiltonian Circle(NP)*

Figure: *Eulerian Circle(P)*

Figure: *Graph Coloring(NPC)*

[7] Aaronson: Why Philosophers Should Care About Computational Complexity.

# Contents


1 Introduction

2 History

3 Propositional Logic
- Syntax
- Semantics
- Connectives
- Formal System
- Meta-Theorems
- Application

4 Predicate Logic

5 Equational Logic

6 Set Theory

7 Recursion Theory

8 Modal Logic

9 Logic vs Game Theory

# Party and Friends

## Problem

- *We want to throw a party for Tweety, Gentoo and Tux.*
- *But they have different circles of friends and dislike some.*
- *Tweety tells you that he would like to see* either *his friend Kimmy* or *not to meet Gentoo's Alice, but not both.*
- *But Gentoo proposes to invite Alice or Harry or both.*
- *Tux, however, does not like Harry and Kimmy too much, so he suggests to exclude at least one of them.*

# Party and Friends

## Problem

- *We want to throw a party for Tweety, Gentoo and Tux.*
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- *But Gentoo proposes to invite Alice or Harry or both.*
- *Tux, however, does not like Harry and Kimmy too much, so he suggests to exclude at least one of them.*

## Solution

$$(K \vee \neg A) \wedge \neg(K \wedge \neg A) \wedge (A \vee H) \wedge (\neg H \vee \neg K)$$

# Sudoku

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | 8 | 6 | | | | 2 | 9 | |
| 4 | | | 1 | | 5 | | | 8 |
| 7 | | | | 9 | | | | 4 |
| 1 | | | | | | | | 9 |
| | 5 | | | | | | 1 | |
| | | 8 | | | | 3 | | |
| | | | 5 | | 9 | | | |
| | | | | 2 | | | | |

$p(i,j,n)$ := the cell in row $i$ and column $j$ contains the number $n$

- Every row/column contains every number.

$$\bigwedge_{i=1}^{9}\bigwedge_{n=1}^{9}\bigvee_{j=1}^{9} p(i,j,n) \qquad \bigwedge_{j=1}^{9}\bigwedge_{n=1}^{9}\bigvee_{i=1}^{9} p(i,j,n)$$

- Every $3 \times 3$ block contains every number.

$$\bigwedge_{r=0}^{2}\bigwedge_{s=0}^{2}\bigwedge_{n=1}^{9}\bigvee_{i=1}^{3}\bigvee_{j=1}^{3} p(3r+i, 3s+j, n)$$

- No cell contains more than one number. for all $1 \leq i, j, n, n' \leq 9$ and $n \neq n'$:

$$p(i,j,n) \rightarrow \neg p(i,j,n')$$

# Shannon — Digital Circuit Design

$$(A \wedge B) \vee ((C \vee A) \wedge \neg B) \mathrel{\vDash\!\dashv} A \vee (C \wedge \neg B)$$

# Shannon — Digital Circuit Design

$$(A \vee \neg C) \wedge (B \vee \neg C) \equiv (A \wedge B) \vee \neg C$$

# McCulloch-Pitts Artificial Neural Network

$$y = g\left(\sum_{i=1}^{n} w_i x_i + b\right)$$

## 《三体》

- 秦始皇：朕当然需要预测太阳的运行，但你们让我集结三千万大军，至少要首先向朕演示一下这种计算如何进行吧。
- 冯诺依曼：陛下，请给我三个士兵，我将为您演示。……
- 秦始皇：他们不需要学更多的东西了吗？
- 冯诺依曼：不需要，我们组建一千万个这样的门部件，再将这些部件组合成一个系统，这个系统就能进行我们所需要的运算，解出那些预测太阳运行的微分方程。

| $p$ | $q$ | $p \dot{\vee} q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

$$w_1 \cdot 0 + w_2 \cdot 0 + b < 0 \qquad b < 0$$
$$w_1 \cdot 0 + w_2 \cdot 1 + b \geq 0 \qquad w_2 + b \geq 0$$
$$w_1 \cdot 1 + w_2 \cdot 0 + b \geq 0 \qquad w_1 + b \geq 0$$
$$w_1 \cdot 1 + w_2 \cdot 1 + b < 0 \qquad w_1 + w_2 + b < 0$$

A simple single-layer perception can't solve nonlinearly separable problems.

# Finite State Automaton

| $y_1$ | $y_2$ | $x$ | $y_1^+$ | $y_2^+$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 1 |

$$y_1^+ = \bar{y}_1\bar{y}_2 + \bar{x}\bar{y}_2$$
$$y_2^+ = y_1\bar{y}_2 + xy_1$$

# Reversible Computing — Fredkin Gate

| $c$ | $x_1$ | $x_2$ | $c'$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

Figure: transmit the first bit unchanged and swap the last two bits iff the first bit is $1$.
$f\colon (c, x_1, x_2) \mapsto (c, \bar{c}x_1 + cx_2, cx_1 + \bar{c}x_2)$

- If $x_2 = 0$, then $y_2 = cx_1$. $\wedge$
- If $x_2 = 1$, then $y_1 = c + x_1$. $\vee$
- If $x_1 = 0$ and $x_2 = 1$, then $y_2 = \bar{c}$. $\neg$

# Exercise

**宝藏在哪里？**

你面前有三扇门，只有一扇门后是宝藏。门上各有一句话，只有一扇门上的是真话。

1. 宝藏不在这儿。
2. 宝藏不在这儿。
3. 宝藏在②号门。

- ① $\neg t_1$; ② $\neg t_2$; ③ $t_2$.
- 只有一扇门上的是真话。
  $(\neg t_1 \wedge \neg\neg t_2 \wedge \neg t_2) \vee (\neg\neg t_1 \wedge \neg t_2 \wedge \neg t_2) \vee (\neg\neg t_1 \wedge \neg\neg t_2 \wedge t_2)$
- 只有一扇门后是宝藏。
  $(t_1 \wedge \neg t_2 \wedge \neg t_3) \vee (\neg t_1 \wedge t_2 \wedge \neg t_3) \vee (\neg t_1 \wedge \neg t_2 \wedge t_3)$

# Exercise

## 谁是凶手？

一起凶杀案有三个嫌疑人：小白、大黄和老王。

1. 至少有一人是凶手，但不可能三人同时犯罪。
2. 如果小白是凶手，那么老王是同犯。
3. 如果大黄不是凶手，那么老王也不是。

## 谁是窃贼？

1. 钱要么是甲偷的要么是乙偷的。
2. 如果是甲偷的，则偷窃时间不会在午夜前。
3. 如果乙的证词正确，则午夜时灯光未灭。
4. 如果乙的证词不正确，则偷窃发生在午夜前。
5. 午夜时没有灯光。

# Exercise

## 哪个部落的？

一个岛上有 T、F 两个部落，T 部落的居民只说真话，F 部落的居民只说谎。你在岛上遇到了小白、大黄、老王三个土著。

1. 小白："如果老王说谎，我或大黄说的就是真话"。
2. 大黄："只要小白或老王说真话，那么，我们三人中有且只有一人说真话是不可能的"。
3. 老王："小白或大黄说谎当且仅当小白或我说真话"。

## 我在做什么？

1. 如果我不在打网球，那就在看网球。
2. 如果我不在看网球，那就在读网球杂志。
3. 但我不能同时做两件以上的事。

# Summary

- Syntax
- Semantics
- Formal System
- Expressiveness / Succinctness
- Satisfiability / Validity
- Soundness / Completeness / Compactness
- Decidability / Computational Complexity
- $\vdots$

# Contents


1. Introduction
2. History
3. Propositional Logic
4. Predicate Logic
5. Equational Logic
6. Set Theory
7. Recursion Theory
8. Modal Logic
9. Logic vs Game Theory

# Why Study Predicate Logic?

- Propositional logic assumes the world contains facts.
- Predicate logic assumes the world contains
  - Objects: people, houses, numbers, colors, baseball games, wars, ...
  - Relations: red, round, prime, brother of, bigger than, part of, between, fall in love with, ...
  - Functions: father of, best friend, one more than, plus, ...
- Expressive power.

# Example

$^{\circledcirc}\hat{o}^{\circledcirc}$

What will a logician choose: an egg or eternal bliss in the afterlife? An egg! Because nothing is better than eternal bliss in the afterlife, and an egg is better than nothing.

$$b < 0 < e \implies b < e$$

$$\neg\exists x(x > b) \implies 0 \not> b$$

$^{\circledcirc}\hat{o}^{\circledcirc}$

No cat has eight tails. A cat has one tail more than no cat. Therefore, a cat has nine tails.

Modeling
Statistics
Operations Research
Differential Equations
Probability
Cryptology
Number Theory
Knot Theory
Computational Math
Game Theory
Calculus
Numerical Analysis
Graph Theory
Topology
Linear Algebra
Complex Analysis
Real Analysis
Geometry
Algebra
Analysis
Number Systems
Set Theory
Logic

GENERAL RELATIVITY
(3+1-dimensional pseudo-Riemannian manifolds with tensor fields obeying PDE's of, say, Einstein-Maxwell theory with perfect fluid component)
QUANTUM FIELD THEORY
(Operator-valued fields on R 4 obeying certain Lorentz-invariant PDE's and commutation relationships, acting on an abstract Hilbert space)
LORENTZ GROUP
SU(2)
Specialize
LIE GROUPS
LIE ALGEBRAS
DIFFERENTIAL OPERATORS
MANIFOLDS WITH TENSOR FIELDS
METRIC MANIFOLDS
ALGEBRAS
LINEAR OPERATORS
TENSOR SPACES
REAL MANIFOLDS
COMPLEX MANIFOLDS
HILBERT SPACES
DISTRI-BUTIONS
Define linear vector mappings
Define linear functions of n vectors
Define charts and atlases
Add inner product
Define linear functionals
TRIPLE FIELDS
Add 4th binary op. etc.
DOUBLE FIELDS
VECTOR SPACES
R n
C n
BANACH SPACES
REAL FUNCTIONS
Add new class (vectors), define vector sum & scalar product
Add ordering, least upper bound property
REAL NUMBERS
Define =, + & · for pairs
COMPLEX NUMBERS
COMPLEX FUNCTIONS
Add 3rd binary operation, etc.
ABELIAN FIELDS
Define =, + & · for Dedekind cuts
Add norm & complete-ness
MEASURABLE SPACES
RATIONAL NUMBERS
METRIC SPACES
Add measure
Add commutativity
ABELIAN GROUPS
FIELDS
Define =, + & · for equi-valence classes of pairs
Add metric
INTEGERS
Add multiplicative inverse
Add commutativity
GROUPS
RINGS
Define =, + & · for negative numbers
TOPOLOGICAL SPACES
Add inverse
Add commutativity and 2nd assoc. & distrib. binary operation
NATURAL NUMBERS
Add open set axioms
ABSTRACT GEOMETRIES
Specialize
SEMI-GROUPS
Add number theory symbols & axioms
DODECA-HEDRON GROUP
Add = & associative binary operation with identity
LOWER PREDICATE CALCULUS
Add symbols and axioms for =, in, subset, union & intersection
SETS
RELATIONS
Add cartesian products
Add quantifiers
BOOLEAN ALGEBRA
Add symbols & axioms for brackets, not, and, etc.
MODELS
FORMAL SYSTEMS

?
h→0
G→0
General Relativity
Quantum Field Theory
h→0
c→∞
G→0
Special Relativity
Nonrelativistic Quantum Mechanics
Nuclear Physics
Particle phenomenology
h→0
Electro-magnetism
Classical Mechanics
Atomic Physics
Solid State Physics
Statistical Mechanics
Chemistry
Electric properties of solids
Mechanics of solids
Hydro-dynamics
Thermo-dynamics
Astrophysics
Electrical Engineering
Mechanical Engineering
Earth Science
Evolution
Computer Science
Biology
Psychology
Medicine
Sociology

# 打破学科鄙视链

Figure: 还原论 ≠ 演生论

# Contents


1 Introduction

2 History

3 Propositional Logic

4 Predicate Logic
- Syntax
- Semantics
- Formal Systems
- Definability & Isomorphism
- What is Logic?
- Normal Forms
- Meta-Theorems

5 Equational Logic

6 Set Theory

7 Recursion Theory

8 Modal Logic

9 Logic vs Game Theory

# Syntax

## Language

$$\mathscr{L}^1 := \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, =, (, )\} \cup \mathscr{V} \cup \overbrace{\mathscr{F} \cup \mathscr{Q}}^{\text{signature}}$$

where

$$\mathscr{V} := \{x_i : i \in \mathbb{N}\}$$

$$\mathscr{F} := \bigcup_{k \in \mathbb{N}} \mathscr{F}^k \quad \mathscr{F}^k := \{f_1^k, \ldots, f_n^k, (\ldots)\}$$

$$\mathscr{Q} := \bigcup_{k \in \mathbb{N}} \mathscr{Q}^k \quad \mathscr{Q}^k := \{P_1^k, \ldots, P_n^k, (\ldots)\}$$

$f^k$ is a $k$-place function symbol.
$P^k$ is a $k$-place predicate symbol.
A $0$-place function symbol $f^0$ is called constant.
A $0$-place predicate symbol $P^0$ is called (atomic) proposition.

# Term & Formula

## Term $\mathcal{T}$

$$t ::= x \mid f(t, \ldots, t)$$

where $x \in \mathscr{V}$ and $f \in \mathscr{F}$.

- $\mathcal{T}$ is freely generated from $\mathscr{V}$ by $\mathscr{F}$.

## Well-Formed Formula $\mathrm{WFF}$

$$\varphi ::= \overbrace{t = t \mid P(t, \ldots, t)}^{\text{atomic formula}} \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \varphi \mid \varphi \leftrightarrow \psi \mid \forall x \varphi \mid \exists x \varphi$$

where $t \in \mathcal{T}$ and $P \in \mathscr{Q}$.

- $\mathrm{WFF}$ is freely generated from atomic formulas by connective and quantifier operators.

# Syntax

- $\varphi \wedge \psi := \neg(\varphi \to \neg\psi)$
- $\varphi \vee \psi := \neg\varphi \to \psi$
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$
- $\exists x\varphi := \neg\forall x\neg\varphi$
- $\bot := \varphi \wedge \neg\varphi$
- $\top := \neg\bot$

- Bottom up and Top down definitions of terms, subterms, wffs and subformulas.
- Induction Principle for terms and wffs.
- Unique readability theorem for terms and wffs.
- Omitting Parenthesis.
  1). outermost parentheses. 2). $\neg, \forall, \exists, \wedge, \vee, \to, \leftrightarrow$ 3). group to the right.

# Freedom & Bondage

$$\exists y \forall z (\neg \exists x R x z \to R x y \wedge R z c)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p^s}}$$

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

$$\left( \sum_{x \in \mathcal{X}} |P(x) - Q(x)| \right)^2 \leq 2 \sum_{x \in \mathcal{X}} P(x) \ln \frac{P(x)}{Q(x)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\mathrm{d}t = f(x)$$

$$\int_0^t \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x$$

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-2\pi \mathrm{i} x \xi} \mathrm{d}x$$

# Freedom & Bondage

## Definition (Free Variable of a Term)

$$\mathrm{Fv}(t) := \begin{cases} x & \text{if } t = x \\ \emptyset & \text{if } t = c \\ \mathrm{Fv}(t_1) \cup \cdots \cup \mathrm{Fv}(t_n) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

## Definition (Free Variable of a wff)

$$\mathrm{Fv}(\varphi) := \begin{cases} \mathrm{Fv}(t_1) \cup \mathrm{Fv}(t_2) & \text{if } \varphi = t_1 = t_2 \\ \mathrm{Fv}(t_1) \cup \cdots \cup \mathrm{Fv}(t_n) & \text{if } \varphi = P(t_1, \ldots, t_n) \\ \mathrm{Fv}(\psi) & \text{if } \varphi = \neg\psi \\ \mathrm{Fv}(\psi) \cup \mathrm{Fv}(\chi) & \text{if } \varphi = \psi \to \chi \\ \mathrm{Fv}(\psi) \setminus \{x\} & \text{if } \varphi = \forall x \psi \end{cases}$$

# Freedom & Bondage

## Definition (Bound Variable)

$$\mathrm{Bv}(\varphi) := \begin{cases} \emptyset & \text{if } \varphi = t_1 = t_2 \text{ or } \varphi = P(t_1, \ldots, t_n) \\ \mathrm{Bv}(\psi) & \text{if } \varphi = \neg\psi \\ \mathrm{Bv}(\psi) \cup \mathrm{Bv}(\chi) & \text{if } \varphi = \psi \to \chi \\ \mathrm{Bv}(\psi) \cup \{x\} & \text{if } \varphi = \forall x \psi \end{cases}$$

- $t$ is a ground (closed) term if $\mathrm{Fv}(t) = \emptyset$.
- $\varphi$ is a sentence (closed formula) if $\mathrm{Fv}(\varphi) = \emptyset$.
- $\varphi$ is an open formula if $\mathrm{Bv}(\varphi) = \emptyset$.

Example: $c = d$ is clopen.

# Translation

How to 'speak' the language of first order logic?

1. $\boldsymbol{A}$: $\forall x(Sx \to Px)$
2. $\boldsymbol{E}$: $\forall x(Sx \to \neg Px)$
3. $\boldsymbol{I}$: $\exists x(Sx \wedge Px)$
4. $\boldsymbol{O}$: $\exists x(Sx \wedge \neg Px)$
5. Every boy loves some girl. $\forall x\big(Bx \to \exists y(Gy \wedge Lxy)\big)$
6. 有爹就有娘。$\forall x(\exists y Fyx \to \exists y Myx)$
7. 外婆是妈妈的妈妈。
   $\forall xy\big(Gxy \leftrightarrow \exists z(Mxz \wedge Mzy)\big)$
   $\forall xy\big(x = Gy \leftrightarrow \exists z(x = Mz \wedge z = My)\big)$
8. 如果大鱼比小鱼游得快，那么，有最大的鱼就有游得最快的鱼。
   $\forall xy(Fx \wedge Fy \wedge Bxy \to Sxy) \to \exists x(Fx \wedge \forall y(Fy \to Bxy)) \to$
   $\exists x(Fx \wedge \forall y(Fy \to Sxy))$
9. There are $n$ elements. $\exists x_1 \ldots x_n \left( \bigwedge_{1 \le i < j \le n} x_i \neq x_j \wedge \forall x \left( \bigvee_{i=1}^{n} x = x_i \right) \right)$

# Translation

1. $Cogito(i) \to \exists x(x = i)$ — *Descartes*
2. $\exists x(x = i) \vee \neg\exists x(x = i)$ — *Shakespeare*
3. $\forall x\big(Month(x) \to Crueler(april, x)\big)$ — *Eliot*
4. $\forall x(\neg Weep(x) \to \neg See(x))$ — *Hugo*
5. $\forall x\big(Time(x) \to Better(t, x)\big) \wedge \forall x\big(Time(x) \to Better(x, t)\big)$ — *Dickens*
6. $\exists p\Big(Child(p) \wedge \neg Grow(p) \wedge \forall x\big(Child(x) \wedge x \neq p \to Grow(x)\big)\Big)$ — *Barrie*
7. $\forall xy\big(Fx \wedge Fy \to (Hx \wedge Hy \to Axy) \wedge (\neg Hx \wedge \neg Hy \to \neg Axy)\big)$ — *Tolstoi*
8. $\exists t\forall x Fool(x, t) \wedge \exists x\forall t Fool(x, t) \wedge \neg\forall x\forall t Fool(x, t)$ — *Lincoln*
9. $\forall x\big(Problem(x) \wedge Philo(x) \wedge Serious(x) \leftrightarrow x = suicide\big)$ — *Camus*
10. $\forall x\big(Feather(x) \wedge Perch(x, soul) \leftrightarrow x = hope\big)$ — *Dickinson*
11. $\forall x\Big(Enter(x) \to \forall y\big(Hope(y) \to Abandon(x, y)\big)\Big)$ — *Dante*
12. $\exists x\forall y\big(For(y, x) \wedge For(x, y)\big)$? — *Dumas*
13. $\exists x\big(Fear(we, x) \leftrightarrow x = Fear\big)$? — *Roosevelt*
14. $\forall xy(Ax \wedge Ay \to Exy) \wedge \exists xy\big(Ax \wedge Ay \wedge [\![Exx]\!] > [\![Eyy]\!]\big)$? — *Orwell*

1. Cogito, ergo sum. (I think, therefore I am.) *Descartes*
2. To be or not to be. *Shakespeare*
3. April is the cruellest month. *Eliot*
4. Those who do not weep, do not see. *Hugo*
5. It was the best of times, it was the worst of times. *Dickens*
6. All Children, except one, grow up. *Barrie*
7. All happy families are alike; each unhappy family is unhappy in it's own way. *Tolstoi*
8. You can fool all the people some of the time, and some of the people all the time, but you can't fool all the people all the time. *Lincoln*
9. There is but one truly serious philosophical problem and that is suicide. *Camus*
10. Hope is the thing with feathers that perches in the soul. *Dickinson*
11. All hope abandon, all you who enter here. *Dante*
12. One for all and all for one. *Dumas*
13. The only thing we have to fear is fear itself. *Roosevelt*
14. All animals are equal, but some animals are more equal than others. *Orwell*

# Exercises — Translation

1. If you can't solve a problem, then there is an easier problem that you can't solve.
2. Men *and* women are welcome to apply.
3. *None but* ripe bananas are edible.
4. *Only* Socrates and Plato are human.
5. *All but* Socrates and Plato are human.
6. Every boy loves *at least* two girls.
7. Adams can't do *every* job right.
8. Adams can't do *any* job right.
9. *Not all* that glitters are gold.
10. Every farmer who owns a donkey is happy.
11. Every farmer who owns a donkey beats it.
12. All even numbers are divisible by $2$, but *only some* are divisible by $4$.

# Exercises — Translation

1. Everyone alive $2000BC$ is either an ancestor of nobody alive today or of everyone alive today.
2. John hates all people who do not hate themselves.
3. No barber shaves exactly those who do not shave themselves.
4. Andy and Bob have the same maternal grandmother. $mother(x, y)$
5. Anyone who loves *two* different girls is Tony.
6. There is *exactly* one sun.
7. Socrates' wife *has* a face that *only* her mother could love.
8. If dogs are animals, every head of a dog is the head of an animal.
9. Someone *other than the girl* who loves Bob is stupid.
10. Morris only loves *the girl* who loves him.
11. *The one* who loves Alice is *the one* she loves.
12. *The shortest* English speaker loves *the tallest* English speaker.

# Translation

$$\lim_{n\to\infty} a_n = a \iff \forall\varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N\big(|a_n - a| < \varepsilon\big)$$

$$\lim_{x\to c} f(x) \uparrow \iff \forall y \in \mathbb{R} \exists\varepsilon > 0 \forall\delta > 0 \exists x \in \mathbb{R}\big(0 < |x-c| < \delta \wedge |f(x)-y| \geq \varepsilon\big)$$

continuity vs uniform continuity

$$\forall x \in \mathbb{R} \forall\varepsilon > 0 \exists\delta > 0 \forall y \in \mathbb{R}\big(|x - y| < \delta \to |f(x) - f(y)| < \varepsilon\big)$$

$$\forall\varepsilon > 0 \exists\delta > 0 \forall xy \in \mathbb{R}\big(|x - y| < \delta \to |f(x) - f(y)| < \varepsilon\big)$$

- M: It's one of the most important discoveries of the last decade!
- P: Can you explain it in words ordinary mortals can understand?
- M: Look, buster, if ordinary mortals could understand it, you wouldn't need mathematicians to do the job for you, right? You can't get a feeling for what's going on without understanding the technical details. How can I talk about manifolds without mentioning that the theorems only work if the manifolds are finite-dimensional paracompact Hausdorff with empty boundary?
- P: Lie a bit.
- M: Oh, but I couldn't do that!
- P: Why not? Everybody else does.
- M: Oh, no! Don't lie — because everybody else does.

# Translation

1. $\exists x\Big(Gx \land \forall y\big(By \land \forall z(Gz \land z \neq x \to \neg Lzy) \to Lxy\big)\Big) \to \forall x\Big(Bx \to \exists y\big(Gy \land Lyx\big)\Big)$
2. $\forall xy\Big(\big(Gx \land \forall y(By \to \neg Lxy)\big) \land \big(Gy \land \exists x(Bx \land Lyx)\big) \to \neg Lxy\Big)$

# Translation

1. $\exists x\Big(Gx \wedge \forall y\big(By \wedge \forall z(Gz \wedge z \neq x \to \neg Lzy) \to Lxy\big)\Big) \to \forall x\Big(Bx \to \exists y\big(Gy \wedge Lyx\big)\Big)$
2. $\forall xy\Big(\big(Gx \wedge \forall y(By \to \neg Lxy)\big) \wedge \big(Gy \wedge \exists x(Bx \wedge Lyx)\big) \to \neg Lxy\Big)$

# Exercises — Translation

1. Only the bishop gave the monkey the banana.
2. The only bishop gave the monkey the banana.
3. The bishop only gave the monkey the banana.
4. The bishop gave only the monkey the banana.
5. The bishop gave the only monkey the banana.
6. The bishop gave the monkey only the banana.
7. The bishop gave the monkey the only banana.
8. The bishop gave the monkey the banana only.

# Substitution and Substitutable

## Definition (Substitution)

$$=, P, \neg, \rightarrow \cdots$$

$$(\forall y\psi)[t_1/x_1, \ldots, t_n/x_n] := \begin{cases} \forall y\psi[t_1/x_1, \ldots, t_n/x_n] & \text{if } y \notin \{x_1, \ldots, x_n\} \\ \forall y\psi[t_1/x_1, \ldots, t_{i-1}/x_{i-1}, t_{i+1}/x_{i+1}, \ldots, t_n/x_n] & \text{if } y = x_i \end{cases}$$

## Definition (Substitutable)

$t$ is substitutable for $x$ in $\varphi$:

$$=, P, \neg, \rightarrow \cdots$$

$\varphi = \forall y\psi$ iff either

1. $x \notin \mathrm{Fv}(\varphi)$ or
2. $y \notin \mathrm{Fv}(t)$ and $t$ is substitutable for $x$ in $\psi$.

**避免替换后 $t$ 中的变元被 $\varphi$ 中的量词约束。**

# Contents


1 Introduction

2 History

3 Propositional Logic

4 Predicate Logic
- Syntax
- Semantics
- Formal Systems
- Definability & Isomorphism
- What is Logic?
- Normal Forms
- Meta-Theorems

5 Equational Logic

6 Set Theory

7 Recursion Theory

8 Modal Logic

9 Logic vs Game Theory

# Philosophy

- No entity without identity. — Quine's standards of ontological admissibility
- To be is to be the value of a bound variable. — Quine's criterion of ontological commitments
- To be is to be constructed by intuition. — Brouwer
- To be true is to be provable. — Kolmogorov
- "$p$" is true iff $p$. — Tarski's "$T$-schema"

什么是"真"？— Are all truths knowable?

1. *formally correct* $\forall x\big(T(x) \leftrightarrow \varphi(x)\big)$
2. *materially adequate* $\varphi(s) \leftrightarrow p$
   where '$s$' is the name of a sentence of $\mathscr{L}$, and '$p$' is the translation of this sentence in $\mathscr{L}'$.

# Structure

A structure over the signature is a pair $\mathcal{A} := (A, I)$, where $A$ is a non-empty set, and $I$ is a mapping which

- assigns to each function symbol $f^k$ a $k$-ary function $I(f^k)\colon A^k \to A$,
- assigns to each predicate symbol $P^k$ a $k$-ary relation $I(P^k) \subset A^k$.

We write $\mathcal{A} = \left(A, c^{\mathcal{A}}, f^{\mathcal{A}}, P^{\mathcal{A}}\right)$ for convenience.

> The 'elements' of the structure have no properties other than those relating them to other 'elements' of the same structure.

# Structure

火
心
木生火
火生土
疏肝理气
除湿解毒
木
肝
脾
土
益气健脾
帮助消化
木克土
水克火
火克金
金克木
土克水
土生金
金生水
水生木
肾
肺
滋阴壮阳
强筋壮骨
水
金
祛风散寒
止嗽润肺

fire
wood
earth
water
metal

# Interpretation

An interpretation $(\mathcal{A}, \nu)$ is a structure $\mathcal{A}$ together with a variable assignment $\nu\colon \mathscr{V} \to A$.
We extend $\nu$ to $\overline{\nu}\colon \mathcal{T} \to A$ by recursion as follows:

- $\overline{\nu}(x) := \nu(x)$
- $\overline{\nu}(c) := c^{\mathcal{A}}$
- $\overline{\nu}(f(t_1, \ldots, f_n)) := f^{\mathcal{A}}(\overline{\nu}(t_1), \ldots, \overline{\nu}(t_n))$

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\overline{\nu}"] \arrow[d, "\mathscr{E}_f"'] & A \arrow[d, "f^{\mathcal{A}}"] \\
\mathcal{T} \arrow[r, "\overline{\nu}"] & A
\end{tikzcd}$$

# Tarski's Definition of Truth

## Definition ($\mathcal{A}, \nu \vDash \varphi$)

- $\mathcal{A}, \nu \vDash t_1 = t_2$ if $\overline{\nu}(t_1) = \overline{\nu}(t_2)$
- $\mathcal{A}, \nu \vDash P(t_1, \ldots, t_n)$ if $(\overline{\nu}(t_1), \ldots, \overline{\nu}(t_n)) \in P^{\mathcal{A}}$
- $\mathcal{A}, \nu \vDash \neg\varphi$ if $\mathcal{A}, \nu \nvDash \varphi$
- $\mathcal{A}, \nu \vDash \varphi \to \psi$ if $\mathcal{A}, \nu \nvDash \varphi$ or $\mathcal{A}, \nu \vDash \psi$
- $\mathcal{A}, \nu \vDash \forall x\varphi$ if for every $a \in A$: $\mathcal{A}, \nu(a/x) \vDash \varphi$
  where
  $$\nu(a/x)(y) := \begin{cases} \nu(y) & \text{if } y \neq x \\ a & \text{otherwise} \end{cases}$$

  or, $\mathcal{A}, \nu \vDash \forall x\varphi$ if for all $\nu' \sim_x \nu$: $\mathcal{A}, \nu' \vDash \varphi$.
  where $\nu' \sim_x \nu$ if for all $y \neq x$: $\nu'(y) = \nu(y)$.

*To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, or of what is not that it is not, is true.*

— *Aristotle*

# Tarski's Definition of Truth

Let $h$ map atomic formulas to variable assignments $P\big(A^{\mathscr{V}}\big)$.

- $h(t_1 = t_2) = \{\nu \colon \overline{\nu}(t_1) = \overline{\nu}(t_2)\}$
- $h(P(t_1, \ldots, t_k)) = \big\{\nu \colon (\overline{\nu}(t_1), \ldots, \overline{\nu}(t_n)) \in P^{\mathcal{A}}\big\}$

We extend $h$ to $\overline{h} \colon \mathrm{WFF} \to P\big(A^{\mathscr{V}}\big)$ by recursion as follows:

1. $\overline{h}(\varphi) = h(\varphi)$ for atomic $\varphi$
2. $\overline{h}(\neg\varphi) = A^{\mathscr{V}} \setminus \overline{h}(\varphi)$
3. $\overline{h}(\varphi \to \psi) = \big(A^{\mathscr{V}} \setminus \overline{h}(\varphi)\big) \cup \overline{h}(\psi)$
4. $\overline{h}(\forall x\varphi) = \bigcap\limits_{a \in A} \big\{\nu \colon \nu(a/x) \in \overline{h}(\varphi)\big\}$

$$\mathcal{A}, \nu \vDash \varphi \;:=\; \nu \in \overline{h}(\varphi)$$

# Tarski's Definition of Truth

- $\mathcal{A} \vDash \varphi$ if for all $\nu$: $\mathcal{A}, \nu \vDash \varphi$. (True)
- $\mathcal{A}, \nu \vDash \Gamma$ if for all $\varphi \in \Gamma$: $\mathcal{A}, \nu \vDash \varphi$.
- $\mathcal{A} \vDash \Gamma$ if for all $\varphi \in \Gamma$: $\mathcal{A} \vDash \varphi$.
- $\Gamma \vDash \varphi$ if for all $\mathcal{A}, \nu$: $\mathcal{A}, \nu \vDash \Gamma \implies \mathcal{A}, \nu \vDash \varphi$.
- $\Gamma \vDash^* \varphi$ if for all $\mathcal{A}$: $\mathcal{A} \vDash \Gamma \implies \mathcal{A} \vDash \varphi$.
- $\vDash \varphi$ if $\emptyset \vDash \varphi$. (Valid)
- $\varphi$ is satisfiable if there exists $\mathcal{A}, \nu$ s.t. $\mathcal{A}, \nu \vDash \varphi$.

## Example

- $A = \{a_1, a_2, a_3\}$
- $c^{\mathcal{A}} = a_3$
- $P^{\mathcal{A}} = \{a_1, a_2\}$
- $R^{\mathcal{A}} = \{(a_1, a_2), (a_1, a_3), (a_3, a_2), (a_3, a_3)\}$

- $c^{\mathcal{A}}$: green point
- $P^{\mathcal{A}}$: blue circles
- $R^{\mathcal{A}}$: arrows

- $\mathcal{A} \overset{?}{\models} Pc$
- $\mathcal{A} \overset{?}{\models} Pc \vee Rcc$
- $\mathcal{A} \overset{?}{\models} \forall x(Px \vee Rxx)$
- $\mathcal{A} \overset{?}{\models} \exists x \forall y(y = x \vee Rxy)$
- $\mathcal{A}, \nu \overset{?}{\models} Rxy \rightarrow Rcy$
  where $\nu(x) = a_1, \nu(y) = a_3$.

# Example

## Example

$$\forall xyz(Rxy \wedge Ryz \rightarrow Rxz)$$

What arrows are missing to make the following a model?

(Add only those arrows that are really needed.)

# Counter Model

## Example (Counter Model)

$\forall x \exists y Rxy \nvDash \exists x \forall y Rxy$

$\forall x \exists y Rxy \nvDash \exists y \forall x Rxy$

$\exists y \forall x Rxy \nvDash \forall y \exists x Rxy$

# Is there a finite counter model?

## Exercise (Counter Model)

*Give a counter model for*

1. $\forall x \exists y Rxy \wedge \forall xyz(Rxy \wedge Ryz \rightarrow Rxz) \nvDash \exists x Rxx$
2. $\forall x \exists y Rxy \wedge \forall xyz(Rxy \wedge Ryz \rightarrow Rxz) \nvDash \exists xy(Rxy \wedge Ryx)$

$$\frac{\begin{array}{c}\text{Everybody loves somebody}\\ \text{Everybody loves all persons who are loved by his loved ones}\end{array}}{\text{There is at least a pair of persons who love each other}}$$

$$(\mathbb{Z}, <)$$

# Relevance Lemma

**Lemma (Relevance Lemma)**

*Assume* $\nu_1, \nu_2 \colon \mathscr{V} \to A$*, and for any* $x \in \mathrm{Fv}(\varphi) \colon \nu_1(x) = \nu_2(x)$*. Then*

$$\mathcal{A}, \nu_1 \vDash \varphi \iff \mathcal{A}, \nu_2 \vDash \varphi$$

- If $\varphi$ is a sentence, then either $\mathcal{A} \vDash \varphi$ or $\mathcal{A} \vDash \neg\varphi$.
- $\mathcal{A} \vDash \varphi \implies \mathcal{A} \vDash \forall x \varphi$
- **Notation**: If $\mathrm{Fv}(\varphi) \subset \{x_1, \ldots, x_n\}$, then we write $\mathcal{A} \vDash \varphi[a_1, \ldots, a_n]$ to mean $\mathcal{A}, \nu \vDash \varphi$ for some (equivalently any) assignment $\nu$ s.t. $\nu(x_i) = a_i$ for $1 \leq i \leq n$.

# Substitution Lemma

## Lemma (Substitution Lemma)

- $\nu\,(s[t/x]) = \nu(\nu(t)/x)(s)$
- *If the term $t$ is substitutable for the variable $x$ in the wff $\varphi$, then*
  $\mathcal{A}, \nu \vDash \varphi[t/x] \iff \mathcal{A}, \nu(\nu(t)/x) \vDash \varphi$

$$\begin{tikzcd}
s \arrow[rr, "{[t/x]}"] \arrow[dr, "\overline{\nu(\overline{\nu}(t)/x)}"'] & & s[t/x] \arrow[dl, "\overline{\nu}"] \\
& A &
\end{tikzcd}$$

$$\mathscr{L}_A := \mathscr{L} \cup \mathscr{C}_A \quad \text{where} \quad \mathscr{C}_A := \{c_a : a \in A\}$$

$$\mathcal{A}, \nu \vDash \varphi[c_a/x] \iff \mathcal{A}, \nu(a/x) \vDash \varphi$$

We abbreviate $\mathcal{A}, \nu \vDash \varphi[c_a/x]$ by $\mathcal{A}, \nu \vDash \varphi[a]$.

$$\mathcal{A}, \nu \vDash \forall x \varphi \iff \text{for every } a \in A : \mathcal{A}, \nu \vDash \varphi[a]$$

# Equivalent Replacement

**Lemma**

*Suppose $\psi \in \mathrm{Sub}(\varphi)$, and $\varphi^*$ arises from $\varphi$ by replacing zero or more occurrences of $\psi$ by $\chi$. Then*

$$\vDash \psi \leftrightarrow \chi \implies \vDash \varphi \leftrightarrow \varphi^*$$

# Alphabetic Variant

## Definition (Alphabetic Variant)

If $y \notin \mathrm{Fv}(\varphi)$, and $y$ is substitutable for $x$ in $\varphi$, we say that $\forall y \varphi[y/x]$ is an alphabetic variant of $\forall x \varphi$.

## Theorem

*If $\forall y \varphi[y/x]$ is an alphabetic variant of $\forall x \varphi$, then*

$$\vDash \forall x \varphi \leftrightarrow \forall y \varphi[y/x]$$

If $y \notin \mathrm{Fv}(\varphi)$, then $\varphi[y/x][x/y] = \varphi$.

- **Convention**: When we write $\varphi[t/x]$ we assume that $t$ is substitutable for $x$ in $\varphi$. — *For any formula $\varphi$ and a finite number of variables $y_1, \ldots, y_n$ (occurring in $t$), we can always find a logically equivalent alphabetic variant $\varphi^*$ of $\varphi$ s.t. $y_1, \ldots, y_n$ do not occur bound in $\varphi^*$.*

# Equality and Equivalence

## Lemma

*Suppose* $\mathrm{Fv}(t) \cup \mathrm{Fv}(s) \subset \{x_1, \ldots, x_n\}$, *and* $\varphi^*$ *arises from the wff* $\varphi$ *by replacing one occurrence of* $t$ *in* $\varphi$ *by* $s$. *Then*

$$\vDash \forall x_1 \ldots x_n (t = s) \to (\varphi \leftrightarrow \varphi^*)$$

$$\mathcal{A} \vDash t = s \implies \mathcal{A} \vDash \varphi \leftrightarrow \varphi^*$$

## Lemma

*Suppose* $\mathrm{Fv}(\psi) \cup \mathrm{Fv}(\chi) \subset \{x_1, \ldots, x_n\}$, *and* $\varphi^*$ *arises from the wff* $\varphi$ *by replacing one occurrence of* $\psi$ *in* $\varphi$ *by* $\chi$. *Then*

$$\vDash \forall x_1 \ldots x_n (\psi \leftrightarrow \chi) \to (\varphi \leftrightarrow \varphi^*)$$

$$\mathcal{A} \vDash \psi \leftrightarrow \chi \implies \mathcal{A} \vDash \varphi \leftrightarrow \varphi^*$$

# Remark

- $\vDash \forall x\big(Px \leftrightarrow Qx\big) \to \big(\forall xPx \leftrightarrow \forall xQx\big)$
  $\nvDash \big(Px \leftrightarrow Qx\big) \to \big(\forall xPx \leftrightarrow \forall xQx\big)$
- $\mathcal{A}, \nu \vDash t = s \;\not\Longrightarrow\; \mathcal{A}, \nu \vDash \varphi \leftrightarrow \varphi^*$
- $\mathcal{A}, \nu \vDash \psi \leftrightarrow \chi \;\not\Longrightarrow\; \mathcal{A}, \nu \vDash \varphi \leftrightarrow \varphi^*$

$$\psi = Px, \quad \chi = Py, \quad \varphi = \forall xPx, \quad \varphi^* = \forall xPy$$

# Valid Formulas — Example

$\forall x\varphi \to \varphi[t/x]$

$\neg\forall x\varphi \leftrightarrow \exists x\neg\varphi$

$\forall x(\varphi \wedge \psi) \leftrightarrow \forall x\varphi \wedge \forall x\psi$

$\exists x(\varphi \vee \psi) \leftrightarrow \exists x\varphi \vee \exists x\psi$

$\forall x(\varphi \to \psi) \to \forall x\varphi \to \forall x\psi$

$\forall xy\varphi \leftrightarrow \forall yx\varphi$

$\exists x\forall y\varphi \to \forall y\exists x\varphi$

$\forall x(\varphi \leftrightarrow \psi) \to (\forall x\varphi \leftrightarrow \forall x\psi)$

$(\forall x\varphi \to \exists x\psi) \leftrightarrow \exists x(\varphi \to \psi)$

$\varphi[t/x] \to \exists x\varphi$

$\neg\exists x\varphi \leftrightarrow \forall x\neg\varphi$

$\forall x\varphi \vee \forall x\psi \to \forall x(\varphi \vee \psi)$

$\exists x(\varphi \wedge \psi) \to \exists x\varphi \wedge \exists x\psi$

$\forall x(\varphi \to \psi) \to \exists x\varphi \to \exists x\psi$

$\exists xy\varphi \leftrightarrow \exists yx\varphi$

# Valid Formulas — Example

$x \notin \mathrm{Fv}(\varphi)$ :

$$\varphi \leftrightarrow \forall x \varphi$$
$$\forall x(\varphi \vee \psi) \leftrightarrow \varphi \vee \forall x \psi$$
$$\forall x(\varphi \wedge \psi) \leftrightarrow \varphi \wedge \forall x \psi$$
$$\forall x(\varphi \rightarrow \psi) \leftrightarrow (\varphi \rightarrow \forall x \psi)$$
$$\forall x(\psi \rightarrow \varphi) \leftrightarrow (\exists x \psi \rightarrow \varphi)$$

$$\varphi \leftrightarrow \exists x \varphi$$
$$\exists x(\varphi \vee \psi) \leftrightarrow \varphi \vee \exists x \psi$$
$$\exists x(\varphi \wedge \psi) \leftrightarrow \varphi \wedge \exists x \psi$$
$$\exists x(\varphi \rightarrow \psi) \leftrightarrow (\varphi \rightarrow \exists x \psi)$$
$$\exists x(\psi \rightarrow \varphi) \leftrightarrow (\forall x \psi \rightarrow \varphi)$$
$$\exists x(\varphi \rightarrow \forall x \varphi)$$

# Valid Formulas — Example

$$t = t$$

$$t = s \to s = t$$

$$t = s \to s = r \to t = r$$

$$t_1 = s_1 \to \cdots \to t_n = s_n \to f(t_1, \ldots, t_n) = f(s_1, \ldots, t_n)$$

$$t_1 = s_1 \to \cdots \to t_n = s_n \to \big(P(t_1, \ldots, t_n) \leftrightarrow P(s_1, \ldots, s_n)\big)$$

$$t = s \to r[t/x] = r[s/x]$$

$$t = s \to \big(\varphi[t/x] \leftrightarrow \varphi[s/x]\big)$$

# Valid Formulas — Example

$x \notin \mathrm{Fv}(t)$ :

$$\exists x(x = t)$$
$$\varphi[t/x] \leftrightarrow \forall x(x = t \rightarrow \varphi)$$
$$\varphi[t/x] \leftrightarrow \exists x(x = t \wedge \varphi)$$

$\{x_1, \ldots, x_n\} \cap \mathrm{Fv}(s) \cap \mathrm{Fv}(t_1) \cap \cdots \cap \mathrm{Fv}(t_n) = \emptyset$ :

$$f(t_1, \ldots, t_n) = s$$
$$\updownarrow$$
$$\forall x_1 \ldots x_n \big(t_1 = x_1, \ldots, t_n = x_n \rightarrow f(x_1, \ldots, x_n) = s\big)$$
$$\updownarrow$$
$$\exists x_1 \ldots x_n \big(t_1 = x_1 \wedge t_n = x_n \wedge f(x_1, \ldots, x_n) = s\big)$$

# Application — Game

**Theorem (Zermelo's Theorem)**

*Every finite game of perfect information with no tie is determined.*

**Proof.**

First, color those end nodes black that are wins for player $1$, and color the other end nodes white, being the wins for $2$. Then

- if player $1$ is to move, and at least one child is black, color it black; if all children are white, color it white.
- if player $2$ is to move, and at least one child is white, color it white; if all children are black, color it black.

**Proof.**

$$\exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \varphi \vee \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \neg \varphi$$

where $\varphi$ states that a final position is reached where player $1$ wins.

# Contents


1 Introduction

2 History

3 Propositional Logic

4 Predicate Logic
- Syntax
- Semantics
- Formal Systems
- Definability & Isomorphism
- What is Logic?
- Normal Forms
- Meta-Theorems

5 Equational Logic

6 Set Theory

7 Recursion Theory

8 Modal Logic

9 Logic vs Game Theory

# Formal Systems

- Hilbert System
- Tree Method
- Natural Deduction
- Sequent Calculus
- Resolution
- ...

# Hilbert System = Axiom + Inference Rule

## Axiom Schema

1. $\varphi \to \psi \to \varphi$
2. $(\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi$
3. $(\neg\varphi \to \neg\psi) \to (\neg\varphi \to \psi) \to \varphi$
4. $\forall x(\varphi \to \psi) \to \forall x\varphi \to \forall x\psi$
5. $\forall x\varphi \to \varphi[t/x]$ where $t$ is substitutable for $x$ in $\varphi$.
6. $\varphi \to \forall x\varphi$ where $x \notin \mathrm{Fv}(\varphi)$.
7. $x = x$
8. $x = y \to \varphi \to \varphi'$ where $\varphi$ is atomic and $\varphi'$ is obtained from $\varphi$ by replacing $x$ in zero or more places by $y$.
9. $\forall x_1 \ldots x_n\varphi$ where $n \geq 0$ and $\varphi$ is any axiom of the preceding groups.

## Inference Rule

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ [\mathrm{MP}]$$

# Example

**Theorem**

$$\varphi \vdash \exists x \varphi$$

**Proof.**

| | | |
|---|---|---|
| 1 | $(\forall x\, \neg\varphi \to \neg\varphi) \to \varphi \to \neg\forall x\, \neg\varphi$ | Tautology |
| 2 | $\forall x\, \neg\varphi \to \neg\varphi$ | A5 |
| 3 | $\varphi \to \neg\forall x\, \neg\varphi$ | 1,2 MP |
| 4 | $\varphi$ | Premise |
| 5 | $\neg\forall x\, \neg\varphi$ | 3,4 MP |
| 6 | $\exists x\, \varphi$ | Definition of $\exists$ |

# Meta-properties

- $\vDash \varphi[\psi_1/p_1, \ldots, \psi_n/p_n]$ where $\varphi \in \mathscr{L}^0$, $\psi_1, \ldots, \psi_n \in \mathscr{L}^1$. tautology
- $\Gamma, \varphi \vdash \psi \iff \Gamma \vdash \varphi \to \psi$ deduction theorem
- $\Gamma, \varphi \vdash \psi \wedge \neg\psi \implies \Gamma \vdash \neg\varphi$ reductio ad absurdum
- $\Gamma, \neg\varphi \vdash \psi \;\&\; \Gamma, \neg\varphi \vdash \neg\psi \implies \Gamma \vdash \varphi$ proof by contradiction
- $\Gamma, \varphi \vdash \neg\psi \iff \Gamma, \psi \vdash \neg\varphi$ contraposition
- $t = s \vdash r[t/x] = r[s/x]$ substitution
- $t = s \vdash \varphi[t/x] \leftrightarrow \varphi[s/x]$ substitution
- $\vdash \psi \leftrightarrow \chi \implies \vdash \varphi \leftrightarrow \varphi^*$ where $\varphi^*$ arises from $\varphi$ by replacing one or more occurrences of $\psi$ in $\varphi$ by $\chi$. equivalent replacement
- $\vdash \forall x\varphi \iff \vdash \forall y\varphi[y/x]$ alphabetic variant

# Meta-properties

- $\Gamma, \varphi[t/x] \vdash \psi \implies \Gamma, \forall x\varphi \vdash \psi$ $\qquad \forall L$
- $\Gamma \vdash \varphi[t/x] \implies \Gamma \vdash \exists x\varphi$ $\qquad \exists R$
- $\Gamma \vdash \varphi[y/x] \ \& \ y \notin \mathrm{Fv}(\Gamma, \forall x\varphi) \implies \Gamma \vdash \forall x\varphi$ $\qquad \forall R$
- $\Gamma, \varphi[y/x] \vdash \psi \ \& \ y \notin \mathrm{Fv}(\Gamma, \exists x\varphi, \psi) \implies \Gamma, \exists x\varphi \vdash \psi$ $\qquad \exists L$
- $\Gamma \vdash \varphi \ \& \ a \notin \mathrm{Cst}(\Gamma) \implies \exists y \notin \mathrm{Fv}(\varphi) \colon \Gamma \vdash \forall y\varphi[y/a]$
- $\Gamma \vdash \varphi[a/x] \ \& \ a \notin \mathrm{Cst}(\Gamma, \forall x\varphi) \implies \Gamma \vdash \forall x\varphi$ $\qquad \forall R$
- $\Gamma, \varphi[a/x] \vdash \psi \ \& \ a \notin \mathrm{Cst}(\Gamma, \exists x\varphi, \psi) \implies \Gamma, \exists x\varphi \vdash \psi$ $\qquad \exists L$

# Alphabetic Variant

**Theorem (Existence of Alphabetic Variants)**

*Let $\varphi$ be a formula, $t$ a term, and $x$ a variable. Then we can find a formula $\varphi^*$ which differs from $\varphi$ only in the choice of quantified variables s.t.*

1. $\varphi \vdash\!\dashv \varphi^*$
2. *$t$ is substitutable for $x$ in $\varphi^*$.*

# Strategy

i. $(\rightarrow)$ $\quad \Gamma \vdash \varphi \rightarrow \psi \iff \Gamma, \varphi \vdash \psi$

ii. $(\forall)$

   1. if $x \notin \mathrm{Fv}(\Gamma)$, $\Gamma \vdash \forall x \varphi \iff \Gamma \vdash \varphi$
   2. if $x \in \mathrm{Fv}(\Gamma)$, $\Gamma \vdash \forall x \varphi \iff \Gamma \vdash \forall y \varphi[y/x] \iff \Gamma \vdash \varphi[y/x]$ for some new $y$.

iii. $(\neg)$

   1. $(\neg \rightarrow)$ $\quad \Gamma \vdash \neg(\varphi \rightarrow \psi) \iff \Gamma \vdash \varphi \ \& \ \Gamma \vdash \neg\psi$
   2. $(\neg\neg)$ $\quad \Gamma \vdash \neg\neg\varphi \iff \Gamma \vdash \varphi$
   3. $(\neg\forall)$ $\quad \Gamma \vdash \neg\forall x \varphi \iff \Gamma \vdash \neg\varphi[t/x]$
      Unfortunately this is not always possible. Try contraposition, reductio ad absurdum or prove by contradiction...

# Tree Method for Propositional Logic

$$\begin{array}{c} \neg\neg\varphi \\ | \\ \varphi \end{array} \qquad \begin{array}{c} \varphi \to \psi \\ \diagup \quad \diagdown \\ \neg\varphi \qquad \psi \end{array} \qquad \begin{array}{c} \neg(\varphi \to \psi) \\ | \\ \varphi \\ \neg\psi \end{array}$$

---

$$\begin{array}{c} \varphi \wedge \psi \\ | \\ \varphi \\ \psi \end{array} \qquad \begin{array}{c} \neg(\varphi \wedge \psi) \\ \diagup \quad \diagdown \\ \neg\varphi \qquad \neg\psi \end{array} \qquad \begin{array}{c} \varphi \vee \psi \\ \diagup \quad \diagdown \\ \varphi \qquad \psi \end{array} \qquad \begin{array}{c} \neg(\varphi \vee \psi) \\ | \\ \neg\varphi \\ \neg\psi \end{array}$$

---

$$\begin{array}{c} \varphi \leftrightarrow \psi \\ \diagup \quad \diagdown \\ \begin{array}{c} \varphi \\ \psi \end{array} \qquad \begin{array}{c} \neg\varphi \\ \neg\psi \end{array} \end{array} \qquad \begin{array}{c} \neg(\varphi \leftrightarrow \psi) \\ \diagup \quad \diagdown \\ \begin{array}{c} \varphi \\ \neg\psi \end{array} \qquad \begin{array}{c} \neg\varphi \\ \psi \end{array} \end{array}$$

✓

Ground Tree:

$$\begin{array}{c} \forall x\varphi \\ | \\ \varphi[t/x] \end{array}$$

where $t$ is a ground term.

$$\begin{array}{c} \exists x\varphi \checkmark \\ | \\ \varphi(a) \end{array}$$

where $a$ is a new constant.

$$\begin{array}{c} \neg\forall x\varphi \checkmark \\ | \\ \exists x\neg\varphi \end{array}$$

$$\begin{array}{c} \neg\exists x\varphi \checkmark \\ | \\ \forall x\neg\varphi \end{array}$$

# Tree Method for Predicate Logic II

Tree Method with Unification:

$$\begin{array}{c} \forall x\varphi \checkmark \\ | \\ \varphi[x_i/x] \end{array}$$

where $x_i$ is a new variable.

$$\begin{array}{c} \exists x\varphi \checkmark \\ | \\ \varphi[f(x_1,\ldots,x_m)/x] \end{array}$$

where $f$ is a new function and $\{x_1,\ldots,x_m\} = \mathrm{Fv}(\exists x\varphi)$.

---

$$\begin{array}{c} \neg\forall x\varphi \checkmark \\ | \\ \exists x\neg\varphi \end{array}$$

$$\begin{array}{c} \neg\exists x\varphi \checkmark \\ | \\ \forall x\neg\varphi \end{array}$$

# Tree Method with Unification

- when expanding a universally quantified formula, do not choose a specific term but a rigid variable as a placeholder.
- choose the term only when it is clear it allows closing a branch.

rigid variable = same value in the whole tree

- variables can assigned to closed terms, like $x_1 = a$.
- can also be assigned to unclosed terms, like $x_1 = f(x_2)$.
- make literals one the opposite of the other.
- using terms as unspecified as possible — Given literals $A$ and $\neg B$ on the same branch, take the most general unifier of $A$ and $B$.

# Unifier

- A substitution $\sigma$ is a *unifier* for a set $\Gamma$ of formulas if for every $\varphi, \psi \in \Gamma$: $\varphi\sigma = \psi\sigma$.
- A unifier $\sigma$ is a *most general unifier* for $\Gamma$ if for each unifier $\theta$ there exists a substitution $\lambda$ s.t. $\theta = \sigma\lambda$.

$$\sigma := \{t_1/x_1, \ldots, t_m/x_m\} \qquad \lambda := \{s_1/y_1, \ldots, s_n/y_n\}$$

$$\sigma\lambda = \big\{t_1\lambda/x_1, \ldots, t_m\lambda/x_m,\ s_1/y_1, \ldots, s_n/y_n\big\} \setminus \big\{s_i/y_i \colon y_i \in \{x_1, \ldots, x_m\}\big\}$$

- $(\varphi\sigma)\lambda = \varphi(\sigma\lambda)$ and $(t\sigma)\lambda = t(\sigma\lambda)$
- $(\sigma\lambda)\theta = \sigma(\lambda\theta)$

# Tree Method for Predicate Logic

$$\begin{array}{c} \varphi(x) \\ x = y \\ | \\ \varphi(y) \end{array} \qquad\qquad \begin{array}{c} \varphi(x) \\ y = x \\ | \\ \varphi(y) \end{array}$$

where $\varphi(y)$ arises from the wff $\varphi(x)$ by replacing one or more occurrences of $x$ by $y$.

# Tactics

- Try to apply "non-branching" rules first, in order to reduce the number of branches.
- Try to close off branches as quickly as possible.
- Deal with negated quantifiers first.
- Instantiate existentials before universals.

# Example — Ground Tree

$\boxed{\{\forall x \neg P(x), \exists x(P(x) \vee P(f(x)))\} \text{ is unsatisfiable.}}$

$$\forall x \neg P(x)$$

$$\exists x(P(x) \vee P(f(x))) \checkmark$$

$$P(a) \vee P(f(a)) \checkmark$$

$$P(a) \qquad\qquad P(f(a))$$

$$\neg P(a) \qquad\qquad \neg P(f(a))$$

$$\times \qquad\qquad \times$$

# Example — Tree Method with Unification

$\boxed{\{\forall x P(x), \neg Q(f(a)), \forall x(\neg P(f(x)) \vee Q(x))\} \text{ is unsatisfiable.}}$

$$\forall x P(x) \checkmark$$
$$\neg Q(f(a))$$
$$\forall x(\neg P(f(x)) \vee Q(x)) \checkmark$$
$$P(x_1)$$
$$\neg P(f(x_2)) \vee Q(x_2) \checkmark$$

| $\neg P(f(x_2))$ | $Q(x_2)$ |
|---|---|
| $x_1 = f(x_2)$ | $x_2 = f(a)$ |
| $\times$ | $\times$ |

# Unification — Greedy Unification (incomplete)

$\neg P(a)$ ✓

$\neg Q(c)$ ✓

$\neg R(d)$ ✓

$\forall x(((R(d) \vee \neg P(b)) \wedge P(x)) \vee ((R(d) \vee \neg Q(b)) \wedge Q(x)))$ ✓

$((R(d) \vee \neg P(b)) \wedge P(x_1)) \vee ((R(d) \vee \neg Q(b)) \wedge Q(x_1))$ ✓

$(R(d) \vee \neg P(b)) \wedge P(x_1)$ ✓

$R(d) \vee \neg P(b)$

$P(x_1)$

$x_1 = a$

×

$(R(d) \vee \neg Q(b)) \wedge Q(a)$ ✓

$R(d) \vee \neg Q(b)$ ✓

$Q(a)$

$R(d)$

×

$\neg Q(b)$

Applying unification as soon as a branch can be closed by lead to incompleteness.

# Unification — Final Closure

$\neg P(a)$ ✓

$\neg Q(c)$ ✓

$\neg R(d)$ ✓

$\forall x(((R(d) \vee \neg P(b)) \wedge P(x)) \vee ((R(d) \vee \neg Q(b)) \wedge Q(x)))$ ✓

$((R(d) \vee \neg P(b)) \wedge P(x_1)) \vee ((R(d) \vee \neg Q(b)) \wedge Q(x_1))$ ✓

$(R(d) \vee \neg P(b)) \wedge P(x_1)$ ✓

$R(d) \vee \neg P(b)$ ✓

$P(x_1)$

$R(d)$ ×

$\neg P(b)$

$x_1 = b$

×

$(R(d) \vee \neg Q(b)) \wedge Q(x_1)$ ✓

$R(d) \vee \neg Q(b)$ ✓

$Q(x_1)$

$R(d)$ ×

$\neg Q(b)$

$x_1 = b$

×

Unification is applied only when it closes all open branches at the same time.

# Example — Unification vs Ground

存在某个人，如果她不孕，则人人不孕。

$$\boxed{\vdash \exists x\big(\varphi(x) \to \forall x\varphi(x)\big)}$$

$$\neg\exists x(\varphi(x) \to \forall x\varphi(x)) \checkmark$$

$$\forall x\neg(\varphi(x) \to \forall x\varphi(x)) \checkmark$$

$$\neg(\varphi(x_1) \to \forall x\varphi(x)) \checkmark$$

$$\varphi(x_1)$$

$$\neg\forall x\varphi(x) \checkmark$$

$$\neg\varphi(a)$$

$$x_1 = a$$

$$\times$$

$$\forall x\neg(\varphi(x) \to \forall x\varphi(x))$$

$$\neg(\varphi(a) \to \forall x\varphi(x)) \checkmark$$

$$\varphi(a)$$

$$\neg\forall x\varphi(x) \checkmark$$

$$\neg\varphi(b)$$

$$\neg(\varphi(b) \to \forall x\varphi(x)) \checkmark$$

$$\varphi(b)$$

$$\neg\forall x\varphi(x)$$

$$\times$$

# Soundness & Completeness

## Definition (Deduction)

$\varphi_1, \ldots, \varphi_n \vdash \psi$ iff there exists a closed tree from $\{\varphi_1, \ldots, \varphi_n, \neg\psi\}$.

## Theorem (Soundness Theorem)

*If the tree closes, the set is unsatisfiable.*

## Theorem (Completeness Theorem)

*If a set is unsatisfiable, there exists a closed tree from it.*

$$\varphi_1, \ldots, \varphi_n \vdash \psi \iff \varphi_1, \ldots, \varphi_n \vDash \psi$$

**Remark:** If an inference with predicate wff is not valid and its counterexample is an infinite model, the tree will not find it. The tree method cannot generate every counterexample of an invalid inference in predicate logic.

# Exercises — Tree Method

1. $\forall x(Px \to Qx) \to \exists x Px \to \exists x Qx$
2. $\exists x \forall y Rxy \to \forall y \exists x Rxy$
3. $\exists x(Px \wedge Qx) \to \exists x Px \wedge \exists x Qx$
4. $\forall x\big(\varphi \vee \psi(x)\big) \to \varphi \vee \forall x \psi(x)$ where $x \notin \mathrm{Fv}(\varphi)$
5. $\exists x\Big(\big(Px \wedge \forall y(Py \to y = x)\big) \wedge Qx\Big) \dashv\vdash \exists x \forall y\Big(\big(Py \leftrightarrow y = x\big) \wedge Qx\Big)$
6. $\exists x\big(Px \wedge \forall y(Py \to y = x)\big) \wedge \exists x\big(Qx \wedge \forall y(Qy \to y = x)\big) \wedge \neg\exists x(Px \wedge Qx) \to \exists xy\big(x \neq y \wedge (Px \vee Qx) \wedge (Py \vee Qy) \wedge \forall z(Pz \vee Qz \to z = x \vee z = y)\big)$

$$1 + 1 = 2$$

# Exercises — Tree Method

1. Nobody trusts *exactly* those who have no mutual trust with anybody.
2. If dogs are animals, every head of a dog is the head of an animal.
3. Every non-analytic, meaningful proposition is either verifiable or falsifiable. Philosophical propositions are neither analytic nor verifiable or falsifiable. Therefore, they are meaningless.
4. No girl loves any sexist pig. Caroline is a girl who loves whoever loves her. Henry loves Caroline. Thus Henry isn't a sexist pig.
5. *The* present king of France is bald. Bald men are sexy. Hence whoever is a present King of France is sexy.
6. *Only* Russell is a great philosopher. Wittgenstein is a great philosopher who smokes. So Russell smokes.
7. Everyone is afraid of Dracula. Dracula is afraid *only* of me. Therefore, I am Dracula.
8. Everyone loves a *lover*(*anyone who loves somebody*). Romeo loves Juliet. Therefore, I love you.
9. Everyone loves a *lover*(*anyone who loves somebody*); hence if someone is a lover, everyone loves everyone!

# Exercises — Tree Method

1. I am a philosopher. A philosopher can *only* be appreciated by philosophers. No philosopher is without some eccentricity. I sing rock. Every eccentric rock singer is appreciated by some girl. Eccentrics are conceited. Therefore, some girl is conceited.
2. Any philosopher admires some logician. Some students admire *only* film stars. No film stars are logicians. Therefore not all students are philosophers.
3. If anyone speaks to anyone, then someone introduces them; no one introduces anyone to anyone unless they know them both; everyone speaks to Frank; therefore everyone is introduced to Frank by someone who knows him.
4. Whoever stole the goods, knew the safe combination. Someone stole the goods, and *only* Jack knew the safe combination. Hence Jack stole the goods.
5. *No one but* Alice and Bette (*who are different people*) admires Carl. All and only those who admire Carl love him. Hence *exactly* two people love Carl.

# Application — Minesweeper

- There are exactly $n$ mines in the game.
- If a cell contains the number $1$, then there is exactly one mine in the adjacent cells.

  $\forall x(contain(x,1) \rightarrow \exists y(adj(x,y) \wedge mine(y) \wedge \forall z(adj(x,z) \wedge mine(z) \rightarrow z = y)))$
- ...

# Russell's Theory of Descriptions

1. The substitution of identicals.
   "The morning star is the evening star."
2. The law of the excluded middle.
   "The present King of France is bald." or
   "The present King of France is not bald."
3. The problem of negative existentials.
   "The round square is round."

# Russell's Theory of Descriptions

$$\psi(\iota_x\varphi) := \exists! x\varphi \wedge \exists x(\varphi \wedge \psi)$$
$$\dashv\vdash \exists x\forall y\Big(\big(\varphi(y) \leftrightarrow y = x\big) \wedge \psi(x)\Big)$$

The round square does not exist. $\psi(\iota_x\varphi) \vee (\neg\psi)(\iota_x\varphi)$ ?

$$\exists x\forall y\Big(\big(Ry \wedge Sy \leftrightarrow y = x\big) \wedge \neg Ex\Big) \quad (\neg\psi)(\iota_x\varphi) \ ?$$

$$\neg\exists x\forall y\Big(\big(Ry \wedge Sy \leftrightarrow y = x\big) \wedge Ex\Big) \quad \neg\psi(\iota_x\varphi) \ ?$$

$$Ex \overset{?}{:=} \exists P\big(Px \wedge \exists y\neg Py\big)$$

$$\iota_x\varphi = \iota_x\varphi \ ? \qquad \forall x\psi \to \psi(\iota_x\varphi) \ ?$$

$$\psi(\iota_x^y\varphi) := \big(\exists! x\varphi \to \exists x(\varphi \wedge \psi)\big) \wedge \big(\neg\exists! x\varphi \to \psi[y/x]\big)$$
$$\vdash \forall x\psi \to \psi(\iota_x^y\varphi)$$

- The logical form of a statement may differ from its grammatical form.
- The correct logical analysis of a word or phrase may involve an explanation not of what that word or phrase taken by itself means, but rather of what whole sentences containing the word or phrase mean.
- The method of contextual definition, which the theory of descriptions exemplifies, was inspired by the nineteenth-century rigorization of analysis.

Berkeley: $2^{nd}$ crisis of the Foundations of Mathematics
For $f(x) = x^2$,

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{f(x+\mathrm{d}x) - f(x)}{\mathrm{d}x} = \frac{(x+\mathrm{d}x)^2 - x^2}{\mathrm{d}x} = \frac{2x\mathrm{d}x + (\mathrm{d}x)^2}{\mathrm{d}x} = 2x + \mathrm{d}x = 2x$$

$\frac{\mathrm{d}}{\mathrm{d}x}$ should be explained as a whole.

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

# Logicism & Logical Positivism

- Mathematics could be reduced to logic.
- Science could be reduced to logical compounds of statements about sense data.
- Only statements verifiable through observation or logical proof are meaningful.
- If all you have is a hammer, everything looks like a nail.
- The new logical resources provided by Frege and Russell had both tempted the positivists to conjecture more than they could prove and made it clear to them that proof of their conjecture was impossible.
- Few if any philosophical schools before the positivists had even stated their aims with sufficient clarity to make it possible to see that they were unachievable.

Thanks