PERSPECTIVES IN LOGIC

Pure Inductive Logic

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PREFACE

We have been motivated to write this monograph by a wish to provide an introduction to an emerging area of Uncertain Reasoning, Pure Inductive Logic. Starting with John Maynard Keynes's 'Treatise on Probability' in 1921 there have been many books on, or touching on, Inductive Logic as we now understand it, but to our knowledge this one is the first to treat and develop that area as a discipline within Mathematical Logic, as opposed to within Philosophy. It is timely to do so now because of the subject's recent rapid development and because, by collecting together in one volume what we perceive to be the main results to date in the area, we would hope to encourage the subject's continued growth and good health.

This is primarily a text aimed at mathematical logicians, or philosophical logicians with a good grasp of Mathematics. However the subject itself gains its direction and motivation from considerations of rational reasoning which very much lie within the province of Philosophy, and it should also be relevant to Artificial Intelligence. For this reason we would hope that even at a somewhat more superficial and circumspect reading it will have something worthwhile to say to that wider community, and certainly the link must be maintained if the subject is not to degenerate into simply doing more mathematics just for the sake of it. Having said that however we will not be lingering particularly on the more philosophical aspects and considerations nor will we speculate about possible application within AI. Rather we will mostly be proving theorems and leaving the reader to interpret them in her or his lights.

This monograph is divided into three parts. In the first we have tried to give a rather gentle introduction and this should be reasonably accessible to quite a wide audience. In the second part, which deals with 'classical' Unary Pure Inductive Logic, the mathematics required is a little more demanding, with more being left for the reader to fill in, and this trend continues in the final, 'post classical', part on Polyadic Pure Inductive Logic. In presenting the results we have tried to keep the material reasonably self contained though on a few occasions it has proved necessary to import a 'big theorem' from outside; this happens in particular in the third part where Nonstandard Analysis is used. In all such cases however we provide source references.

The chapters have been arranged, hardly surprisingly, so that they can be read straight through from start to finish. However an alternative approach would be to familiarize oneself with Part 1, though even there Chapter 5 on the Dutch Book could be omitted, and then simply dip in to selected later chapters, only referring back to earlier material as subsequently proved necessary. To aid this approach we have included at the end a rather extensive list of symbols and abbreviations used in the text and have generally tried to outline the contents of a chapter in its first paragraph or so.

Much of the material we will present has previously appeared in journals or books. What we have endeavoured is to collect and present it in a unified notation and fashion, in so doing making it, we hope, more easily accessible to a wider audience. Whilst most of the later chapters derive from research of ourselves and our collaborators, and here we should make a special mention of Jürgen Landes and Chris Nix, this monograph naturally owes a debt in its first two parts to the seminal contributions of the founding fathers of the area, in particular W.E. Johnson, Rudolf Carnap, and more recently Haim Gaifman and Jakko Hintikka. Doubtless there are many other results and attributions we should also have included and for this we can only apologize and, to quote Dr Samuel Johnson, blame 'pure ignorance'.

We would also like to thank Jon Williamson, Teddy Groves, our colleague George Wilmers and our recent research students, Martin Adamčik, Alex Hill, Hykel Hosni, Elizabeth Howarth, Malte Kließ, Jürgen Landes, Soroush Rafiee Rad and Tahel Ronel for diligently reading earlier drafts and spotting errors. Any that remain are, of course, entirely our fault.

Part 1

THE BASICS

Chapter 1

INTRODUCTION TO PURE INDUCTIVE LOGIC

Before a cricket match can begin the tradition is that the umpire tosses a coin and one of the captains calls, heads or tails, whilst the coin is in the air. If the captain gets it right s/he chooses which side opens the batting. There never seems to be an issue as to which captain actually makes this call (otherwise we would have to toss a coin and make a call to decide who makes the call, and in turn toss a coin and make a call to decide who makes that call and so on) since it seems clear that this procedure is fair. In other words both captains are giving equal probability to the coin landing heads as to it landing tails no matter which of them calls it. The obvious explanation for this is that both captains are, subconsciously perhaps, appealing to the *symmetry* of the situation.

At the same time they are, it seems, also tacitly making the assumption that all the other information they possess about the situation, for example the weather, the gender of the referee, even past successes at coin calling, is *irrelevant*, at least if it doesn't involve some specific knowledge about this particular coin or the umpires's ability to influence the outcome. Of course if we knew that on the last 8 occasions on which this particular umpire had tossed up this same coin the result had been heads we might well consider that that *was relevant.*

Forming beliefs, or subjective probabilities, in this way by considering symmetry, irrelevance, relevance, can be thought of as *logical* or *rational* inference. This is something different from statistical inference. The perceived fairness of the coin toss is clearly not based on the captains' knowledge of a long run of past tosses by the umpire which have favoured heads close to half the time. Indeed it is conceivable that this long run frequency might not give an average of close to half heads, maybe this coin is, contrary to appearances, biased. Nevertheless even if the captains knew that the coin was biased, provided that they also knew that the caller was not privy to which side of the coin was favoured, they would surely still consider the process as fair.

This illustrates another feature of probabilities that are inferred on logical grounds: they certainly need not agree with the long term frequency probability, if this even exists, and of course in many situations in which we form subjective probabilities no such probability does exist; for example when assigning odds in a horse race.

The aim of this monograph is to investigate this process of assigning logical, as opposed to statistical, probabilities by attempting to formulate the underlying notions, such as symmetry, irrelevance, relevance on which they appear to depend. Much has already been written by philosophers on these matters and doubtless much still remains to be said. Our approach here however will be that of mathematical, rather than philosophical, logicians. So instead of spending a significant time discussing these notions at length in the context of specific examples we shall largely consider ways in which they might be given a purely mathematical formulation and then devote our main effort to considering the mathematical and logical consequences which ensue.

In this way then we are proposing, or at least reviving since Rudolf Carnap had already introduced the notion in [14], an area of Mathematical Logic, *Pure Inductive Logic*, PIL for short.[1] It is not Philosophy as such but there are close connections. Firstly most of the logical, aka rational, principles we consider are motivated by philosophical considerations, frequently having an already established presence in the literature within that subject. Secondly we would hope that the mathematical results included here may feed back and contribute to the continuing debates within Philosophy itself, if only by clarifying that *if* you subscribe to A, B, C *then* you must, by dint of mathematical proof, accept D.

There is a parallel here with Set Theory. In that case we propose axioms based on our intuitions concerning the nature of sets and then investigate their consequences. These axioms have philosophical content and considering this is part of the picture but so also is drawing out their mathematical relationships and consequences. And as we go deeper into the subject we are led to propose or investigate axioms which initially might not have entered our minds, not least because we may well not have possessed the language or notions to even express them. And at the end of the day most of us would like to think that discoveries in Set Theory were telling us something about the universe of sets, or at least about possible universes of sets, and thus feeding back into the philosophical debate (and not simply generating yet more mathematics 'because it is there'!). Hopefully Pure Inductive Logic, PIL, will similarly tell us something about the universe of uncertain reasoning.

As far as the origins of PIL are concerned, whilst one may hark back to Keynes, Mill and even as far as Aristotle, in the more recent history of logical probability as we see it, W.E. Johnson's posthumous 1932 paper [58] in *Mind* was the first important contribution in the general spirit of what we

[1] For a gentle introduction to PIL see also [100].

are proposing here. It contains an initial assertion of mathematical conditions capturing intuitively attractive principles of uncertain reasoning and a derivation from these of what subsequently became known as Carnap's Continuum of Inductive Methods. Independently Carnap was to follow a similar line of enquiry in his [9], [12], which he developed further with [13], [15], [16], [17] into the subject he dubbed 'Inductive Logic'. Already in 1946 however N. Goodman's so called 'grue' paradox, see [35], [36] (to which Carnap responded with [10], [11]) threatened to capsize the whole venture by calling into question the very possibility of a purely logical basis for inductive inference[2]. Notwithstanding Carnap maintained his commitment to the idea of an Inductive Logic till his death in 1970 and to the present day his vision encourages a small but dedicated band of largely philosopher logicians to continue the original venture in a similar spirit, albeit in the ubiquitous shadow of 'grue'.

From the point of view of this text however 'grue' is no paradox at all, it is just the result of failing to make explicit all the assumptions that were being used. There is no isomorphism between premises involving grue and green (a point we will touch on again later in the footnote on page 177) because we have different background knowledge concerning grue and green etc. and it is precisely this knowledge which the paradox subsequently uses to announce a contradiction.[3] Indeed in his initial response to 'grue' Carnap had also stressed the importance of having all the assumptions up front from the start, what he called the 'Principle of Total Evidence', see [10, p138], [12, p211], known earlier as 'Bernoulli's Maxim', see [6, footnote 1, p215], [65, p76, p313].

Even so, 'grue' is relevant to this monograph in that it highlights a divergence within Carnap's Inductive Logic as to its focus or subject matter between *Pure Inductive Logic*, which is our interest in this monograph, and *Applied Inductive Logic*, which is the practical concern of many philosophers. The former was already outlined by Carnap in [14]; it aims to

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[2]For the reader unfamiliar with this 'paradox' here is a pared down mathematician's version: Let *grue* stand for 'green before the 1st of next month, blue after'. Now consider the following statements:

All the emeralds I have ever seen have been green, so I should give high probability that any emerald I see next month will be green.

All the emeralds I have ever seen have been grue, so I should give high probability that any emerald I see next month will be grue.

The conclusion that advocates of this 'paradox' would have us conclude is that Carnap's hope of determining such probabilities by purely logical or rational considerations cannot succeed. For here are 'isomorphic' premises with different (contradictory even) conclusions so the conclusion cannot simply be a logical function of the available information.

[3]For example we learnt in school that emeralds are green and never heard anything about this possibly changing at some future date. In contrast if we had been talking here about UK Road Fund Licence discs changing to a new colour next January 1st there would have been a 'paradox' for the contrary reason!

study formal systems in the mathematical sense, devoid of explicit interpretation. Assumptions must be stated within the formal language and conclusions drawn only on the basis of explicitly given rules. On the other hand Applied Inductive Logic is intended as a tool, in particular, to sanction degrees of confirmation, within particular contexts. The language therein is interpreted and so carries with it knowledge and assumptions. What Goodman's Paradox points out is that applied in this fashion the conclusions of any such logic may be language dependent (see [136] for a straightforward amplification of this point), a stumbling block which has spawned a considerable literature, for example [131], [137], and which, within PIL, we thankfully avoid. In short then we might draw a parallel here with the aims and methods of Pure Mathematics as opposed to those of Applied Mathematics.

In the latter we begin with an immensely complicated real world situation, cut it down to manageable size by choosing what we consider to be the relevant variables and the relevant constraints, so ignoring a wealth of other information which we judge irrelevant, and then, *drawing on existing mathematical theories and apparatus*, we hopefully come up with some predictive or explicative formula. Similarly with Inductive Logic the applied arm has been largely concerned with proposing formulae in such contexts - prior probability functions, to provide answers. The value of these answers and the whole enterprize has been subject to near century long debate, some philosophers feeling that the project is fundamentally flawed. On the other hand it clearly finds new challenges with the advent of designing artificial reasoning agents. Be it as it may, PIL is not out to *prescribe* priors. Rather it is an investigation into the various notions of 'rationality' in the context of forming beliefs as probabilities. It is in this foundational sense that we hope this monograph may be of interest to philosophers and to the Artificial Intelligence community. Similarly to other mathematical theories, we would hope that it would serve to aid any researcher contemplating actual problems related to rational reasoning.

A rough plan of this monograph is as follows. In the early chapters we shall introduce the basic notation and general results about probability functions for predicate languages, as well as explaining what we see as the most attractive justification (de Finetti's Dutch Book argument) for identifying degrees of belief with probability. We will then investigate principles based on considerations of symmetry, relevance, irrelevance and analogy, amongst others, for *Unary Pure Inductive Logic*, that is for predicate languages with only unary relation symbols. This was the context for Inductive Logic in which Carnap et al worked and with only a very few exceptions it remained so until the end of the 20th century. In the second half of this monograph we will enlarge the framework to *Polyadic Pure Inductive Logic* and return to reconsider symmetry, relevance and irrelevance within this wider context.

The style of this monograph is mathematical. When introducing various purportedly rational principles we will generally give some fairly brief explanation of why they might be considered 'rational', in the sense that a rational agent should, or could, adhere to them, but there will not be an extended philosophical discussion.[4] It will not be our aim to convince the reader that each of them really is 'rational'; indeed that would be difficult since in combination they are often inconsistent. We merely seek to show that they might be candidates for an expression of 'rationality', a term which we will therefore feel free to leave at an intuitive level. As these principles are introduced we will prove theorems relating them to each other and attempt to characterize the probability functions which satisfy them. Most proofs will be given in full although we sometimes import well known results from outside of Mathematical Logic itself. On a few occasions giving the proof of a theorem in detail would just be too extravagant and in that case we will refer the reader to the relevant paper and content ourselves instead by explaining the key ideas behind the proof. In any case it is our intention that this monograph will still be accessible even to someone who wishes to treat the proofs simply as the 'small print'.

[4]There are a number of books which do provide extended philosophical discussions of some of the general principles we shall investigate, for example [12], [13], [15], [16], [17], [18], [22], [27], [29], [33], [37], [38], [40], [57], [72], [127], [135], [140], [148].

Chapter 2

CONTEXT

For the mathematical setting we need to make the formalism completely clear. Whilst there are various possible choices here the language which seems best for our study, and corresponds to most of the literature, including Carnap's, is where we work with a first order language L with variables $x_1, x_2, x_3, \dots$, relation symbols $R_1, R_2, \dots, R_q$, say of finite arities $r_1, r_2, \dots, r_q$ respectively, and constants a_n for $n \in \mathbb{N}^+ = \{1, 2, 3, \dots\}$, and no function symbols nor (in general) the equality symbol.[5] The intention here is that the a_i name all the individuals in some population though there is no prior assumption that they necessarily name different individuals. We identify L with the set $\{R_1, R_2, \dots, R_q\}$.

Let SL denote the set of first order sentences of this language L and $QFSL$ the quantifier free sentences of this language. Similarly let FL, $QFFL$ denote respectively the set of formulae, quantifier free formulae, of L. We use θ, ϕ, ψ etc. for elements of FL and adopt throughout the convention that, unless otherwise stated, when *introducing*[6] a formula $\theta \in FL$ as $\theta(a_{i_1}, a_{i_2}, \dots, a_{i_m}, x_{j_1}, x_{j_2}, \dots, x_{j_m})$, or $\theta(\vec{a}, \vec{x})$, *all* the constant symbols (respectively free variables) in θ are amongst these a_i (x_j) and that they are distinct. In particular if we write a formula as $\theta(\vec{x})$ then it will be implicit that no constant symbols appear in θ. To avoid double subscripts we shall sometimes use $b_1, b_2, \dots$ etc. in place of $a_{i_1}, a_{i_2}, \dots$.

Let $\mathcal{T}L$ denote the set of structures for L with universe $\{a_1, a_2, a_3, \dots\}$, with the obvious interpretation of the a_i as a_i itself. Notice that if $\Gamma \subseteq SL$ is consistent[7] and infinitely many of the constants a_i are not mentioned in any sentence in Γ then there is $M \in \mathcal{T}L$ such that $M \models \Gamma$. This follows since the countability of the language L means that Γ must have a countable model, and hence, by re-interpreting the constant symbols not mentioned in any sentence in Γ, a model in which every element of its universe is the interpretation of at least one of the constant symbols.

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[5] We shall add equality in Chapter 37 but will omit function symbols throughout, that being a topic which, in our opinion, is still deserving of more investigation and thought.

[6] So this will not apply if we introduce a sentence as $\exists x\, \psi(x)$ and then pass to $\psi(a_n)$. In this case there may be other constants mentioned in $\psi(a_n)$.

[7] Because of the Completeness Theorem for the Predicate Calculus we shall use consistent and satisfiable interchangeably according to which seems most appropriate in the context.

To capture the underlying problem that PIL aims to address we can imagine an agent who inhabits some structure $M \in \mathcal{T}L$ but knows nothing about what is true in M. Then the problem is,

Q: *In this situation of zero knowledge, logically, or rationally, what belief should our agent give to a sentence $\theta \in SL$ being true in M?*

There are several terms in this question which need explaining. Firstly 'zero knowledge' means that the agent has no intended interpretation of the a_i nor the R_j. To mathematicians this seems a perfectly easy idea to accept; we already do it effortlessly when proving results about, say, an arbitrary group. In these cases all you can assume is the axioms and you are not permitted to bring in new facts because they happen to hold in some particular group you have in mind. Unfortunately outside of Mathematics this sometimes seems to be a particularly difficult idea to embrace and much confusion has found its way into the folklore as a result.[8]

In a way this is at the heart of the difference between the 'Pure Inductive Logic' proposed here as Mathematics and the 'Applied Inductive Logic' of Philosophy. For many philosophers would argue that in this latter the language is intended to carry with it an interpretation and that without it one is doing Pure Mathematics not Philosophy. It is the reason why Grue is a paradox in Philosophy and simply an invalid argument in Mathematics. Nevertheless, mathematicians or not, we all need to be on our guard against allowing interpretations to slip in subconsciously. Carnap himself was very well aware of this distinction, and the dangers presented by ignoring it, and spent some effort explaining it in [14]. Indeed in that paper he describes Inductive Logic as the study of the rational beliefs of just such a zero knowledge agent, a 'robot' as he terms it.

A second unexplained term is 'logical' and its synonym (as far as this text is concerned) 'rational'. In this case, as already mentioned, we shall offer no definition; they are to be taken as intuitive, something we recognize when we see it without actually being able to give it a definition. This will not be a great problem, for our purpose is to propose and mathematically investigate principles for which it is enough that we may simply *entertain* the idea that they are logical or rational. The situation parallels that of the intuitive notion of an 'effective process' in recursion theory, and similarly we may hope that our investigations will ultimately lead to a clearer understanding.

The third unexplained term above is 'belief'. For the present we shall identify belief, or more precisely degree of belief, with (subjective) probability and only later provide a justification, the Dutch Book Argument, for this identification. The main reason for proceeding in this way is that in order to give this argument in full we actually need to have already developed some of the apparatus of probability functions, a task we now move on to.

[8]See for example [106] and the issue of the representation dependence of maxent.

Chapter 3

PROBABILITY FUNCTIONS

A function $w : SL \to [0,1]$ is a probability function on SL if for all $\theta, \phi, \exists x\, \psi(x) \in SL$,

(P1) $\models \theta \Rightarrow w(\theta) = 1$.
(P2) $\theta \models \neg\phi \Rightarrow w(\theta \vee \phi) = w(\theta) + w(\phi)$.
(P3) $w(\exists x\, \psi(x)) = \lim_{n\to\infty} w(\psi(a_1) \vee \psi(a_2) \vee \cdots \vee \psi(a_n))$.

Condition (P3) is often referred to as *Gaifman's Condition*, see [30], and is a special addition to the conventional conditions (P1), (P2) appropriate to this context. It intends to capture the idea that the $a_1, a_2, a_3, \ldots$ exhaust the universe.[9]

Having now made precise what we mean by a probability function we can reformulate our question $\mathcal{Q}$ asking what probability $w(\theta)$ a rational agent with zero knowledge should assign to a sentence θ of L. It is implicit in this statement that the same probability function[10] w is being referred to throughout, so stated more precisely $\mathcal{Q}$ amounts to the driving problem of PIL:

$\mathcal{Q}$: *In the situation of zero knowledge, logically, or rationally, what probability function* $w : SL \to [0,1]$ *should an agent adopt when* $w(\theta)$ *is to represent the agent's probability that a sentence* $\theta \in SL$ *is true in the ambient structure* M?

Our long term aim in the chapters which follow will be to posit rational principles which, arguably, such a probability function should obey and then investigate the consequences of imposing these principles and the relationships between them. Before that however we need to get acquainted with probability functions and their properties. We start with some actual examples.

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[9]It can be shown, see [103], that if $z : SL \to [0,1]$ satisfies (P1–2) and L^+ is the language formed by adding additional constants $a_1', a_2', a_3', \ldots$ to L then there is a probability function w on SL^+ satisfying (P1–2) and the corresponding version of (P3) for this extended language such that $z = w \restriction SL$ ($= w$ restricted to SL). In this sense then Gaifman's Condition (P3) does not essentially alter the notion of probability that we are dealing with.

[10]Carnap calls this the agent's *initial credence function*, see for example [14, p310].

A particularly simple example of a probability function is the function $V_M : SL \to \{0, 1\}$, where $M \in \mathcal{T}L$, defined by

$$V_M(\theta) = \begin{cases} 1 & \text{if } M \vDash \theta, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In turn, since convex sums of functions satisfying (P1–3) also satisfy these conditions, convex sums of probability functions are also probability functions. So for $M_i \in \mathcal{T}L$ and $e_i \geq 0$ such that $\sum_i e_i = 1$, $\sum_i e_i V_{M_i}$ is a probability function.

More generally for $\theta \in SL$ let

$$[\theta] = \{ M \in \mathcal{T}L \mid M \vDash \theta \},$$

let $\mathcal{B}$ be the σ-algebra of subsets of $\mathcal{T}L$ generated by these subsets $[\theta]$ and let μ be a countably additive measure[11] on $\mathcal{B}$. Then it is straightforward to check that

$$w = \int_{\mathcal{T}L} V_M \, d\mu(M)$$

defines a probability function on SL.

Indeed as we shall see later (Corollary 7.2) every probability function has a representation of this form. Putting it another way, for every probability function w on SL there is a countably additive measure μ_w on the algebra $\mathcal{B}$ such that for any $\theta \in SL$,

$$w(\theta) = \int_{\mathcal{T}L} V_M(\theta) \, d\mu_w(M). \tag{2}$$

We shall shortly consider other, generally more useful ways of specifying particular families of probability functions. For the present however our next step will be to derive some of their simple algebraic properties.

PROPOSITION 3.1. *Let w be a probability function on SL. Then for $\theta, \phi \in SL$,*

(a) $w(\neg\theta) = 1 - w(\theta)$.
(b) $\vDash \neg\theta \Rightarrow w(\theta) = 0$.
(c) $\theta \vDash \phi \Rightarrow w(\theta) \leq w(\phi)$.
(d) $\theta \equiv \phi \Rightarrow w(\theta) = w(\phi)$.
(e) $w(\theta \vee \phi) = w(\theta) + w(\phi) - w(\theta \wedge \phi)$.

PROOF. (a) We have that $\vDash \theta \vee \neg\theta$ and $\theta \vDash \neg\neg\theta$ so by (P1) and (P2),

$$1 = w(\theta \vee \neg\theta) = w(\theta) + w(\neg\theta).$$

(b) If $\vDash \neg\theta$ then $w(\neg\theta) = 1$ by (P1) so from (a), $w(\theta) = 0$.

(c) If $\theta \vDash \phi$ then $\neg\phi \vDash \neg\theta$ so from (P2), (a) and the fact that w takes values in $[0, 1]$,

$$1 \geq w(\neg\phi \vee \theta) = w(\neg\phi) + w(\theta) = 1 - w(\phi) + w(\theta)$$

from which the required inequality follows.

[11] All measures will be assumed to be normalized and countably additive unless otherwise indicated.

(d) If $\theta \equiv \phi$ then $\theta \vDash \phi$ and $\phi \vDash \theta$. By (c), $w(\theta) \leq w(\phi)$ and $w(\phi) \leq w(\theta)$ so $w(\theta) = w(\phi)$.

(e) Since $\theta \vee \phi \equiv \theta \vee (\neg\theta \wedge \phi)$ and $\theta \vDash \neg(\neg\theta \wedge \phi)$ (P2) and (d) give

$$w(\theta \vee \phi) = w(\theta \vee (\neg\theta \wedge \phi)) = w(\theta) + w(\neg\theta \wedge \phi). \tag{3}$$

Also $\phi \equiv (\theta \wedge \phi) \vee (\neg\theta \wedge \phi)$ and $\theta \wedge \phi \vDash \neg(\neg\theta \wedge \phi)$ so by (P2) and (d),

$$w(\phi) = w((\theta \wedge \phi) \vee (\neg\theta \wedge \phi)) = w(\theta \wedge \phi) + w(\neg\theta \wedge \phi). \tag{4}$$

Eliminating $w(\neg\theta \wedge \phi)$ from (3), (4) now gives

$$w(\theta \vee \phi) = w(\theta) + w(\phi) - w(\theta \wedge \phi). \qquad \dashv$$

In fact (c) and (d) above turn out to be the best we can do in general:

PROPOSITION 3.2. *For $\theta, \phi \in SL$, $w(\theta) \leq w(\phi)$ for all probability functions w on SL if and only if $\theta \vDash \phi$. Similarly $w(\theta) = w(\phi)$ for all probability functions w on SL if and only if $\theta \equiv \phi$.*

PROOF. In view of Proposition 3.1(c),(d) we only need to show that if $\theta \nvDash \phi$ then there is a probability function w such that $w(\phi) < w(\theta)$. But in this case $\{\theta, \neg\phi\}$ is consistent, so has a countable model, and hence a model $M \in \mathcal{T}L$. Then $V_M(\theta) = V_M(\neg\phi) = 1$ but $V_M(\phi) = 1 - V_M(\neg\phi) = 0$, as required. $\dashv$

It is worthwhile observing here that nowhere in the proof of Proposition 3.1 did we use the property (P3), each of (a) - (e) holds even if we only assume (P1) and (P2). Indeed if we restrict θ, ϕ here to quantifier free sentences then we only need assume (P1) and (P2) in Proposition 3.1 for quantifier free sentences. A second consequence of this observation is that just having (P1) and (P2) holding for $\theta, \phi \in SL$ already assures, by (c), 'half' of (P3), namely:

$$\lim_{n\to\infty} w(\psi(a_1) \vee \psi(a_2) \vee \cdots \vee \psi(a_n)) \leq w(\exists x\, \psi(x)). \tag{5}$$

We now derive some equivalents of (P3) in the presence of (P1) and (P2) (for $\theta, \phi \in SL$).

PROPOSITION 3.3. *Let $w : SL \to [0, 1]$ satisfy* (P1), (P2). *Then condition* (P3) *is equivalent to*:

(P3') $w(\exists x\, \psi(x)) = \sum_{n=1}^{\infty} w\Big(\psi(a_n) \wedge \neg \bigvee_{i=1}^{n-1} \psi(a_i)\Big)$

for $\exists x\, \psi(x) \in SL$.

PROOF. Since

$$\psi(a_1) \vee \psi(a_2) \vee \cdots \vee \psi(a_n) \equiv \bigvee_{j=1}^{n} \Big(\psi(a_j) \wedge \neg \bigvee_{i=1}^{j-1} \psi(a_i)\Big)$$

by Proposition 3.1(e) and the above remark,

$$w(\psi(a_1) \vee \psi(a_2) \vee \cdots \vee \psi(a_n)) = w\Big(\bigvee_{j=1}^{n} \big(\psi(a_j) \wedge \neg \bigvee_{i=1}^{j-1} \psi(a_i)\big) \Big)$$
$$= \sum_{j=1}^{n} w\Big(\psi(a_j) \wedge \neg \bigvee_{i=1}^{j-1} \psi(a_i)\Big)$$

by repeated use of (P2) since the disjuncts here are all disjoint. The required equivalence of (P3), (P3') follows. ⊣

Condition (P3) is expressed in terms of the probability of existential sentences. However it could equally well have been expressed in terms of universal sentences as the next proposition shows.

PROPOSITION 3.4. *Let* $w : SL \to [0, 1]$ *satisfy* (P1), (P2). *Then condition* (P3) *is equivalent to*:

$$w(\forall x\, \psi(x)) = \lim_{n\to\infty} w\Big(\bigwedge_{i=1}^{n} \psi(a_i)\Big) \tag{6}$$

for $\forall x\, \psi(x) \in SL$.

PROOF. Assume (P3). Then

$$\begin{aligned} w(\forall x\, \psi(x)) &= 1 - w(\exists x\, \neg\psi(x)) && \text{by Proposition 3.1(a)(d),} \\ &= 1 - \lim_{n\to\infty} w\Big(\bigvee_{i=1}^{n} \neg\psi(a_i)\Big) && \text{by (P3),} \\ &= \lim_{n\to\infty} \Big(1 - w\Big(\bigvee_{i=1}^{n} \neg\psi(a_i)\Big)\Big) \\ &= \lim_{n\to\infty} w\Big(\bigwedge_{i=1}^{n} \psi(a_i)\Big) && \text{by Proposition 3.1(a)(d).} \end{aligned}$$

The converse follows similarly. ⊣

We conclude this chapter by developing some inequalities (and an application of them, Lemma 3.8) which will be useful later though for the present the reader could blithely skip them. The first of these, given by Adams in [1, Theorem 15] and derived earlier[12] in [2] extends Proposition 3.1(c) to multiple premises. To formulate and prove the result we will need some notation. For $\psi \in SL$ let ψ^1, ψ^0, alternatively $+\psi, -\psi$, denote $\psi, \neg\psi$ respectively. Now suppose that $\theta_1, \theta_2, \ldots, \theta_n, \phi \in SL$ and

$$\Gamma = \{\theta_1, \theta_2, \ldots, \theta_n\} \models \phi.$$

Call a subset Δ of Γ *essential* if $\Gamma - \Delta \not\models \phi$ but for any $\Omega \subset \Delta$, $\Gamma - \Omega \models \phi$ ($\subset$ always denotes *strict subset*). Define the *essentialness*, $e(\theta_i)$, of θ_i to

[12] Using methods from Linear Analysis.

be 0 if θ_i is not an element of any essential subset of Γ and otherwise m_i^{-1} where m_i is the smallest number of elements in an essential subset of Γ containing θ_i.

THEOREM 3.5. *For a probability function w and for Γ, $\theta_1, \theta_2, \dots, \theta_n, \phi$ and e as above,*

$$w(\neg\phi) \leq \sum_{i=1}^{n} e(\theta_i)w(\neg\theta_i),$$

equivalently,

$$w(\phi) \geq 1 + \sum_{i=1}^{n} e(\theta_i)w(\theta_i) - \sum_{i=1}^{n} e_i(\theta_i).$$

PROOF. Let β_j for $j = 1, 2, \dots, r$ enumerate the consistent sentences of the form

$$\theta_1^{\varepsilon_1} \wedge \theta_2^{\varepsilon_2} \wedge \cdots \wedge \theta_n^{\varepsilon_n} \wedge \phi^{\varepsilon_{n+1}},$$

where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n+1} \in \{0, 1\}$, and for $\psi \in SL$ set

$$\chi(\beta_j, \psi) = \begin{cases} 1 & \text{if } \beta_j \vDash \psi, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that for $\psi \in \{\pm\theta_1, \pm\theta_2, \dots, \pm\theta_n, \pm\phi\}$

$$\psi \equiv \bigvee_{\beta_j \vDash \psi} \beta_j$$

so

$$w(\psi) = \sum_{j=1}^{r} \chi(\beta_j, \psi)w(\beta_j). \tag{7}$$

Suppose that $\beta_j \vDash \neg\phi$. Then

$$\Delta_j = \{\theta_i \mid \beta_j \vDash \neg\theta_i\} = \{\theta_i \mid \chi(\beta_j, \neg\theta_i) = 1\}$$

is such that $\Gamma - \Delta_j \nvDash \phi$ (since $\beta_j \vDash \theta_k$ for every $\theta_k \in \Gamma - \Delta_j$ but $\beta_j \nvDash \phi$, otherwise β_j would not be consistent). Clearly then there is an essential $\Omega_j \subseteq \Delta_j$ and

$$\begin{aligned} w(\beta_j) &= \sum_{\theta_i \in \Omega_j} \frac{w(\beta_j)}{|\Omega_j|} \cdot \chi(\beta_j, \neg\theta_i) \\ &\leq \sum_{\theta_i \in \Omega_j} \frac{w(\beta_j)}{m_i} \cdot \chi(\beta_j, \neg\theta_i), \text{ since } m_i \leq |\Omega_j| \text{ because } \theta_i \in \Omega_j, \\ &\leq \sum_{i=1}^{n} e(\theta_i)w(\beta_j)\chi(\beta_j, \neg\theta_i). \end{aligned} \tag{8}$$

Summing both sides of (8) over the β_j such that $\beta_j \vDash \neg\phi$ gives

$$\begin{aligned} w(\neg\phi) &\le \sum_{\beta_j \vDash \neg\phi} \sum_{i=1}^{n} e(\theta_i) w(\beta_j) \chi(\beta_j, \neg\theta_i) \\ &\le \sum_{j=1}^{r} \sum_{i=1}^{n} e(\theta_i) w(\beta_j) \chi(\beta_j, \neg\theta_i) \\ &= \sum_{i=1}^{n} e(\theta_i) \sum_{j=1}^{r} w(\beta_j) \chi(\beta_j, \neg\theta_i) \\ &= \sum_{i=1}^{n} e(\theta_i) w(\neg\theta_i). \end{aligned}$$

⊣

The e_i in this theorem are bounded above by 1 and using this bound with $\phi = \bigwedge_{i=1}^{n} \theta_i$ gives as a corollary the following practically useful *Suppes' Lemma*, see [138], for which however we now give a much simpler direct proof.

LEMMA 3.6. *Let $\theta_1, \theta_2, \dots, \theta_n \in SL$ and w a probability function on SL. Then*

$$w\Big(\bigwedge_{i=1}^{n} \theta_i\Big) \ge \sum_{i=1}^{n} w(\theta_i) - (n-1).$$

In particular if $w(\theta_i) = 1$ for $i = 2, 3, \dots, n$, $(n \ge 1)$ then

$$w\Big(\bigwedge_{i=1}^{n} \theta_i\Big) = w(\theta_1).$$

PROOF. First notice that by Proposition 3.1(e)

$$w(\theta \vee \phi) = w(\theta) + w(\phi) - w(\theta \wedge \phi) \le w(\theta) + w(\phi)$$

so by repeated use of this,

$$w\Big(\bigvee_{i=1}^{n} \neg\theta_i\Big) \le \sum_{i=1}^{n} w(\neg\theta_i) \tag{9}$$

Hence

$$\begin{aligned} w\Big(\bigwedge_{i=1}^{n} \theta_i\Big) &= 1 - w\Big(\neg \bigwedge_{i=1}^{n} \theta_i\Big) && \text{by Proposition 3.1(a)} \\ &= 1 - w\Big(\bigvee_{i=1}^{n} \neg\theta_i\Big) && \text{by Proposition 3.1(d)} \\ &\ge 1 - \sum_{i=1}^{n} w(\neg\theta_i) && \text{by (9)} \end{aligned}$$

$$= 1 - \sum_{i=1}^{n}(1 - w(\theta_i)) \qquad \text{by Proposition 3.1(a)}$$

$$= \sum_{i=1}^{n} w(\theta_i) - (n-1). \tag{10}$$

For the last part, (10) gives

$$w\Big(\bigwedge_{i=1}^{n} \theta_i\Big) \geq w(\theta_1)\,.$$

Since

$$\bigwedge_{i=1}^{n} \theta_i \vDash \theta_1$$

by Proposition 3.1(c) we also have

$$w\Big(\bigwedge_{i=1}^{n} \theta_i\Big) \leq w(\theta_1)$$

so the result follows. ⊣

Whilst Theorem 3.5 can give a significant improvement on Suppes' Lemma 3.6, for example when the θ_i are all logically equivalent, in applications it is usually the case that each of the θ_i are essential to the proof of $\bigwedge_{i=1}^{n} \theta_i$ from $\{\theta_1, \dots, \theta_n\}$ so that the e_i are all 1 and the two estimates are equal.

LEMMA 3.7. *If w is a probability function on SL, $\theta_i, \phi_i \in SL$, $q_i \in \mathbb{R}$ (the set of real numbers) and $w(\theta_i \leftrightarrow \phi_i) \geq q_i$ for $i = 1, 2, \dots, n$ then*

$$\Big| w\Big(\bigvee_{i=1}^{n} \theta_i\Big) - w\Big(\bigvee_{i=1}^{n} \phi_i\Big) \Big|,\ \Big| w\Big(\bigwedge_{i=1}^{n} \theta_i\Big) - w\Big(\bigwedge_{i=1}^{n} \phi_i\Big) \Big| \leq \sum_{i=1}^{n} (1 - q_i).$$

PROOF. Since

$$\bigvee_{i=1}^{n} \theta_i \wedge \bigwedge_{i=1}^{n} (\theta_i \leftrightarrow \phi_i) \vDash \bigvee_{i=1}^{n} \phi_i$$

by Proposition 3.1(c) and Suppes' Lemma 3.6,

$$w\Big(\bigvee_{i=1}^{n} \phi_i\Big) \geq w\Big(\bigvee_{i=1}^{n} \theta_i \wedge \bigwedge_{i=1}^{n} (\theta_i \leftrightarrow \phi_i)\Big) \geq w\Big(\bigvee_{i=1}^{n} \theta_i\Big) + \sum_{i=1}^{n} q_i - n.$$

Consequently,

$$w\Big(\bigvee_{i=1}^{n} \theta_i\Big) - w\Big(\bigvee_{i=1}^{n} \phi_i\Big) \leq \sum_{i=1}^{n} (1 - q_i).$$

The required conclusion now follows since clearly we can interchange the θ_i and ϕ_i.

The case for conjunction is proved similarly. ⊣

We now give an application of this last inequality which will be needed later.

LEMMA 3.8. *For* $\exists x_1, \dots, x_k\, \theta(x_1, \dots, x_k, \vec{a}) \in SL$ *and* w *a probability function on* SL,

$$w(\exists x_1, \dots, x_k\, \theta(x_1, \dots, x_k, \vec{a})) = \lim_{n\to\infty} w\Big(\bigvee_{i_1, i_2, \dots, i_k \le n} \theta(a_{i_1}, a_{i_2}, \dots, a_{i_k}, \vec{a})\Big), \quad (11)$$

$$w(\forall x_1, \dots, x_k\, \theta(x_1, \dots, x_k, \vec{a})) = \lim_{n\to\infty} w\Big(\bigwedge_{i_1, i_2, \dots, i_k \le n} \theta(a_{i_1}, a_{i_2}, \dots, a_{i_k}, \vec{a})\Big). \quad (12)$$

PROOF. Notice that by Proposition 3.1(c) the terms on the right hand side of (11) are increasing in n, and at most the left hand side. It is enough to show that these terms can be make arbitrarily close to the left hand side for large n.

The proof is by induction on k. If $k = 1$ the result is immediate from (P3). Assume the result for $k - 1$. By (P3) for any $\varepsilon > 0$ there is a j_0 such that for any $j \ge j_0$,

$$w\Big(\bigvee_{i_1 \le j} \exists x_2, \dots, x_k\, \theta(a_{i_1}, x_2, \dots, x_k, \vec{a})\Big)$$

is within ε of

$$w(\exists x_1, x_2, \dots, x_k\, \theta(x_1, x_2, \dots, x_k, \vec{a})).$$

Hence

$$w(\exists x_1, \dots, x_k\, \theta(x_1, \dots, x_k, \vec{a}) \wedge \neg \bigvee_{i_1 \le j} \exists x_2, \dots, x_k\, \theta(a_{i_1}, x_2, \dots, x_k, \vec{a})) \le \varepsilon \quad (13)$$

since

$$\bigvee_{i_1 \le j} \exists x_2, \dots, x_k\, \theta(a_{i_1}, x_2, \dots, x_k, \vec{a}) \models \exists x_1, \dots, x_k\, \theta(x_1, \dots, x_k, \vec{a})\,. \quad (14)$$

Again from (14),

$$\neg\Big(\bigvee_{i_1 \le j} \exists x_2, \dots, x_k\, \theta(a_{i_1}, x_2, \dots, x_k, \vec{a}) \leftrightarrow \exists x_1, \dots, x_k\, \theta(x_1, \dots, x_k, \vec{a})\Big)$$

is logically equivalent to

$$\exists x_1, \dots, x_k\, \theta(x_1, \dots, x_k, \vec{a}) \wedge \neg \bigvee_{i_1 \le j} \exists x_2, \dots, x_k\, \theta(a_{i_1}, x_2, \dots, x_k, \vec{a})\,,$$

giving by Proposition 3.1(a)(d) and (13) that

$$w\Big(\exists x_1,\dots,x_k\,\theta(x_1,\dots,x_k,\vec{a}) \leftrightarrow \bigvee_{i_1\le j} \exists x_2,\dots,x_k\,\theta(a_{i_1},x_2,\dots,x_k,\vec{a})\Big) \geq 1-\varepsilon\,. \quad (15)$$

Similarly for each $1 \le i_1 \le j$ there is, by inductive hypothesis, a number r_{i_1} such that

$$w\Big(\exists x_2,\dots,x_k\,\theta(a_{i_1},x_2,\dots,x_k,\vec{a}) \leftrightarrow \bigvee_{i_2,\dots,i_k\le n_{i_1}} \theta(a_{i_1},a_{i_2},\dots,a_{i_k},\vec{a})\Big) \geq 1-j^{-1}\varepsilon \quad (16)$$

whenever $n_{i_1} \ge r_{i_1}$.

Taking r to be the maximum of these r_{i_1} for $1 \le i_1 \le j$ we have by Lemma 3.7 that for $n \ge r$,

$$\Big| w\Big(\bigvee_{i_1\le j}\ \bigvee_{i_2,\dots,i_k\le n} \theta(a_{i_1},\dots,a_{i_k},\vec{a})\Big) - w\Big(\bigvee_{i_1\le j} \exists x_2,\dots,x_k\,\theta(a_{i_1},x_2,\dots,x_k,\vec{a})\Big)\Big| \le \sum_{i=1}^{j}(1-(1-j^{-1}\varepsilon))\,.$$

Hence with (15),

$$\Big| w\Big(\bigvee_{i_1\le j}\ \bigvee_{i_2,\dots,i_k\le n} \theta(a_{i_1},\dots,a_{i_k},\vec{a})\Big) - w(\exists x_1,\dots,x_k\,\theta(x_1,\dots,x_k,\vec{a}))\Big| \le 2\varepsilon$$

and the result follows by our remarks which initiated this proof.

(12) now follows directly by taking negations. ⊣

Chapter 4

CONDITIONAL PROBABILITY

Given a probability function w on SL and $\phi \in SL$ with $w(\phi) > 0$ we define the *conditional probability function* $w(.\,|\,\phi) : SL \to [0, 1]$ (said 'w conditioned on ϕ') by

$$w(\theta\,|\,\phi) = \frac{w(\theta \wedge \phi)}{w(\phi)}. \tag{17}$$

Proposition 4.1. *Let w be a probability function on SL, $\phi \in SL$ and $w(\phi) > 0$. Then $w(.\,|\,\phi)$ is a probability function and $w(\theta\,|\,\phi) = 1$ whenever $\phi \vDash \theta$.*

Proof. To show (P1) suppose that $\vDash \theta$. Then $\phi \equiv \theta \wedge \phi$ so $w(\theta \wedge \phi) = w(\phi)$ by Proposition 3.1(d) and in turn $w(\theta\,|\,\phi) = 1$. Notice that all we really need for this argument is the logical equivalence $\phi \equiv \theta \wedge \phi$ so since it is enough to have $\phi \vDash \theta$ for this hold we immediately get the final part of the proposition too.

For (P2) suppose that $\theta \vDash \neg\eta$. Then $\theta \wedge \phi \vDash \neg(\eta \wedge \phi)$ so since

$$(\theta \vee \eta) \wedge \phi \equiv (\theta \wedge \phi) \vee (\eta \wedge \phi),$$

$$\begin{aligned} w((\theta \vee \eta) \wedge \phi) &= w((\theta \wedge \phi) \vee (\eta \wedge \phi)), && \text{by Proposition 3.1(d)},\\ &= w(\theta \wedge \phi) + w(\eta \wedge \phi), && \text{by (P2) for } w, \end{aligned}$$

and dividing by $w(\phi)$ gives the result.

For (P3), note that

$$\exists x\, \psi(x) \wedge \phi \equiv \exists x\, (\psi(x) \wedge \phi),$$

$$\Big(\bigvee_{i=1}^{n} \psi(a_i)\Big) \wedge \phi \equiv \bigvee_{i=1}^{n} (\psi(a_i) \wedge \phi),$$

so using Proposition 3.1(d) and (P3) for w,

$$\begin{aligned} w(\exists x\, \psi(x) \wedge \phi) &= w(\exists x\, (\psi(x) \wedge \phi))\\ &= \lim_{n\to\infty} w\Big(\bigvee_{i=1}^{n} (\psi(a_i) \wedge \phi)\Big) = \lim_{n\to\infty} w\Big(\Big(\bigvee_{i=1}^{n} \psi(a_i)\Big) \wedge \phi\Big) \end{aligned}$$

and the result follows after dividing both sides by $w(\phi)$. ⊣

So far we have quoted the properties of w coming from (P1–3) and Proposition 3.1 that we were using at each step. From now on however we shall assume that the reader has these properties at his/her fingertips.

Conditional probabilities appear frequently in this subject in large part because it is generally assumed that they model 'updating'. To explain this suppose that on the basis of some knowledge (possibly empty) about the structure s/he is inhabiting our agent has settled on w as his/her subjective assignment of probabilities. Now suppose that the agent learns that $\phi \in SL$ is also true in this world. How should the agent 'update' w? The 'received wisdom'[13] is that the agent should replace w by $w(. \mid \phi)$ (tacitly assuming that the agent had anticipated this eventuality by arranging that $w(\phi) > 0$).

Accepting this interpretation of conditional probability as probability moderated by given knowledge has a major consequence for the motivating problem in PIL, to wit:

Q: *In the situation of zero knowledge, logically, or rationally, what probability function* $w : SL \to [0,1]$ *should an agent adopt when* $w(\theta)$ *is to represent the agent's probability that a sentence* $\theta \in SL$ *is true in the ambient structure* M?

For if we consider an agent who *does* have some knowledge K of the form

$$\{\phi_1, \phi_2, \ldots, \phi_n ,\}$$

we can imagine that this agent first chose his/her subjective probability function w on the basis of *zero* knowledge and then learnt K and updated w to $w(\,.\mid \bigwedge_{i=1}^{n} \phi_i)$, assuming of course that the agent gave non-zero probability to $\bigwedge_{i=1}^{n} \phi_i$ in the first place. Accepting this scenario means that Q is really the key issue, at least for knowledge K of this form, and in part justifies our focusing on this question.[14]

With this interpretation of conditional probability in mind we mention an interesting result due to Huzurbazar, [55], subsequently dubbed the 'First Induction Theorem' by Good [34, p62] (see also [149, p288]):

PROPOSITION 4.2. *For* w *a probability function on* SL *and* $\forall x\, \theta(x) \in SL$, *if* $w(\forall x\, \theta(x)) > 0$ *then*

$$\lim_{n\to\infty} w\Big(\theta(a_{n+1}) \mid \bigwedge_{i=1}^{n} \theta(a_i)\Big) = 1. \tag{18}$$

[13]Not surprisingly this issue is a minefield within Philosophy, especially in more expressive languages.

[14]The corresponding issue has been the subject of rather detailed investigation in the case of *Propositional Uncertain Reasoning*, even for the case of knowledge K which itself expresses uncertainties. We refer the reader to [97] (especially chapter 8) and to the references therein.

Proof. Since

$$\forall x\, \theta(x) \models \bigwedge_{i=1}^{n} \theta(a_i),$$

by Proposition 3.1(c),

$$0 < w(\forall x\, \theta(x)) \leq w\Big(\bigwedge_{i=1}^{n} \theta(a_i)\Big),$$

and again by this proposition the sequence $w(\bigwedge_{i=1}^{n} \theta(a_i))$ is decreasing. It must therefore reach a strictly positive limit ($w(\forall x\, \theta(x))$ in fact) and the result follows by elementary analysis. $\dashv$

In the other direction this proposition need not hold, we can have (18) holding with $w(\bigwedge_{i=1}^{n} \theta(a_i)) > 0$ but $w(\forall x\, \theta(x)) = 0$. For example this happens for an consistent but non-tautologous $\theta \in QFSL$ for the probability functions c_λ^L $(\lambda > 0)$ to be defined in (103).

Whilst we have given above the standard interpretation of conditional probability as the result of updating on some information, an alternative which in our view fits more comfortably with question $\mathcal{Q}$ and our picture of a rational agent is to treat $w(\,.\,|\,\phi)$ as what the agent *imagines* s/he would update his/her probability function to *if* s/he learnt ϕ. Thus conditional probabilities, which will frequently figure in later rational principles we propose, are to be treated as referring to thought experiments of the agent.

One difficulty that arises when presenting principles centred around conditional probabilities is that the conditioning sentence, ϕ in the account above, may have zero probability, leading to the need to be constantly introducing provisos. A useful convention to avoid this, *which we shall adopt throughout*, is to agree that an expression such as

$$w(\theta \,|\, \phi) = c, \quad \text{ or } \quad w(\theta \,|\, \phi) = w(\eta \,|\, \zeta),$$

is shorthand for

$$w(\theta \wedge \phi) = c\, w(\phi), \quad \text{ or } \quad w(\theta \wedge \phi)\, w(\zeta) = w(\eta \wedge \zeta)\, w(\phi) \quad \text{respectively.}$$

Clearly if the denominator(s) are non-zero these amount to the same thing whilst if some of the denominators is zero the expression still has meaning (and, interestingly, usually the meaning we would wish it to have).

In the chapters which follow we shall have occasion to look at several principles asserting that under certain conditions the probability one gives to θ should increase, or at least not decrease, on receipt of information ϕ, which we will formally capture by the inequality

$$w(\theta \,|\, \phi) \geq w(\theta).$$

Notice that by (P2) this will hold if $\phi \models \theta$, since in this case

$$w(\phi \wedge \theta) = w(\phi) \geq w(\phi) \cdot w(\theta).$$

Interestingly this inequality also holds, by the same argument, if $\theta \models \phi$. In other words learning a consequence ϕ of θ does not decrease your probability that θ holds.

Chapter 5

THE DUTCH BOOK ARGUMENT

Having derived some of the basic properties of probability functions we will now take a short diversion to give what we consider to be the most compelling argument in this context, namely the Dutch Book argument originating with Ramsey [122] and de Finetti [25], in favour of an agent's 'degrees of belief' satisfying (P1–3), and hence being identified with a probability function, albeit *subjective* probability since it is ostensibly the property of the agent in question. Of course this could really be said to be an aside to the purely mathematical study of PIL and hence dispensable. The advantage of considering this argument however is that by linking belief and subjective probability it better enables us to appreciate and translate into mathematical formalism the many rational principles we shall later encounter.[15]

The idea of the Dutch Book argument is that it identifies 'belief' with willingness to bet. So suppose, as in the context of PIL explained above, we have an agent[16] inhabiting some unknown structure $M \in \mathcal{T}L$ (which one imagines will eventually be revealed to decide the wager) and that $\theta \in SL$, $0 \leq p \leq 1$ and for a stake $s > 0$ the agent is offered a choice of one of two wagers:

($\mathrm{Bet1}_p$) Win $s(1-p)$ if $M \models \theta$, lose sp if $M \nvDash \theta$.
($\mathrm{Bet2}_p$) Win sp if $M \nvDash \theta$, lose $s(1-p)$ if $M \models \theta$.

If the agent would not be happy to accept $\mathrm{Bet1}_p$ we assume that it is because the agent thinks that the bet is to his/her disadvantage and hence to the advantage of the bookmaker. But in that case $\mathrm{Bet2}_p$ allows the agent to swap roles with the bookmaker so s/he should now see that bet as being to his/her advantage, and hence acceptable. In summary then, we may suppose that for any $0 \leq p \leq 1$ at least one of $\mathrm{Bet1}_p$ and $\mathrm{Bet2}_p$

[15]There are several other rational justifications for belief being identified with probability, notably L. Savage's based on *Expected Utility*, [128], and J. Joyce's based on *Accuracy Domination*, [59]. See also R. Pettigrew and H. Leitgeb's [82], [83], [119].

[16]For this purpose we may suppose that we have selected an agent who does not suffer from any aversion to gambling, is fully committed to winning, and his/her decisions are not affected by the size of the stakes.

is acceptable to the agent. In particular we may assume that $\mathrm{Bet1}_0$ and $\mathrm{Bet2}_1$ are acceptable since in both cases the agent has nothing to lose and everything to gain.

Now suppose that $\mathrm{Bet1}_p$ was acceptable to the agent and $0 \leq q < p$. Then $\mathrm{Bet1}_q$ should also be acceptable to the agent[17] since it would result in a greater return, $s(1-q)$, than $\mathrm{Bet1}_p$ if θ turns out to be true in M and a smaller loss, sq, than $\mathrm{Bet1}_p$ if θ turns out to be false. Similarly if $\mathrm{Bet2}_p$ is acceptable to the agent and $p < q \leq 1$ then $\mathrm{Bet2}_q$ should be acceptable to the agent.

From this it follows that those $p \in [0, 1]$ for which $\mathrm{Bet1}_p$ is acceptable to the agent form an initial segment of the interval $[0, 1]$, those for which $\mathrm{Bet2}_p$ is acceptable form a final segment and every p is in one or other of these segments, possibly even both.

Define $Bel(\theta)$ to be the supremum of those $p \in [0, 1]$ for which $\mathrm{Bet1}_p$ is acceptable to the agent. We can argue that $Bel(\theta)$ is a measure of the agent's willingness to bet on θ and in turn take this to quantify the agent's belief that θ is true in M. For if the agent strongly believes that θ is true then s/he would be willing to risk a small potential gain of $s(1-p)$ against a large potential loss of sp, simply because s/he strongly expects that gain, albeit small. In other words the agent would favour $\mathrm{Bet1}_p$ even for p quite close to 1. From this viewpoint then $Bel(\theta)$ represents the top limit of the agent's belief in θ.[18]

We now impose a further rationality requirement on the agent: That s/he does not allow a (possibly infinite) set of simultaneous bets each of which is acceptable to him/her but whose combined effect is to cause the agent certain loss no matter what the ambient structure $M \in \mathcal{T}L$ turns out to be. In gambling parlance that the agent cannot be 'Dutch Booked'.

To formalize this idea first observe that if the agent accepts $\mathrm{Bet1}_p$ s/he will in the event of the ambient structure being M gain

$$s(1-p)V_M(\theta) - sp(1 - V_M(\theta)) = s(V_M(\theta) - p)$$

where V_M was defined on page 12 and loss is identified with negative gain. Clearly in $\mathrm{Bet2}_p$ the gain is minus this, i.e. $-s(V_M(\theta) - p)$.

So the agent could be certainly be Dutch Booked if there were countable sets A, B and sentences θ_i, stakes $s_i > 0$, $p_i \in [0, Bel(\theta_i))$[19] for $i \in A$, and sentences ϕ_j, stakes $t_j > 0$, $q_j \in (Bel(\phi_j), 1]$ for $j \in B$, and $K > 0$

[17] Assuming that s/he is 'rational'.

[18] A common assumption in accounts of this material is that the agent is also willing to accept $\mathrm{Bet1}_p$ on θ when $p = Bel(\theta)$. This seems to us unjustified and in any case is not necessary for the subsequent derivations in this chapter.

[19] If $Bel(\theta_i) = 0$ then we take this interval to be $\{0\}$ since we have already assumed that $\mathrm{Bet1}_0$ on θ_i is always acceptable to the agent. We adopt a similar convention for the other possible extreme belief values.

such that the series below converge[20],

$$\sum_{i \in A} s_i(V_M(\theta_i) - p_i) + \sum_{j \in B} (-t_j)(V_M(\phi_j) - q_j) < 0 \tag{19}$$

and

$$\Big|\sum_{i \in A} s_i(V_M(\theta_i) - p_i)\Big|, \ \Big|\sum_{j \in B} t_j(V_M(\phi_j) - q_j)\Big| < K \tag{20}$$

for all $M \in \mathcal{T}L$. For in this case the agent would accept $\mathrm{Bet1}_{p_i}$ for θ_i at stake s_i for each $i \in A$ and similarly would accept $\mathrm{Bet2}_{q_j}$ for ϕ_j at stake t_j for each $j \in B$, and condition (20) ensures that it would be feasible[21] to so do, but the result of simultaneously placing all these bets would, by (19), be negative no matter what $M \in \mathcal{T}L$ was.

We now show that imposing the condition that there exists no Dutch Book[22], that is that no such p_i, s_i, θ_i etc. exist, forces Bel to satisfy (P1–3) and hence to be a probability function according to our definition. The original proof of this is due to de Finetti, [25], for (P1–2). Williamson, see [145, Theorem 5.1], gives a Dutch Book argument for (P3) though the one we present here is somewhat different.

Theorem 5.1. *Suppose that for* $Bel : SL \to [0, 1]$ *there are no countable sets* A, B, *sentences* $\theta_i \in SL$, $p_i \in [0, Bel(\theta_i))$, *stakes* s_i *for* $i \in A$ *etc. such that* (19), (20) *hold for all* $M \in \mathcal{T}L$. *Then* Bel *satisfies* (P1–3).

Proof. For (P1) suppose that $\theta \in SL$ and $\vDash \theta$ but $Bel(\theta) < 1$. Then for $Bel(\theta) < q < 1$ the agent accepts $\mathrm{Bet2}_q$. But since $V_M(\theta) = 1$ for all $M \in \mathcal{T}L$ we have that with stake 1,

$$(-1)(V_M(\theta) - q) = q - 1 < 0$$

which gives an instance of (19), contradiction.

Now suppose that (P2) fails, say $\theta, \phi \in SL$ are such that $\theta \vDash \neg\phi$ but

$$Bel(\theta) + Bel(\phi) < Bel(\theta \vee \phi).$$

Then $\theta \vDash \neg\phi$ forces that at most one of θ, ϕ can be true in any $M \in \mathcal{T}L$ so

$$V_M(\theta \vee \phi) = V_M(\theta) + V_M(\phi).$$

Pick $p > Bel(\theta)$, $q > Bel(\phi)$, $r < Bel(\theta \vee \phi)$ such that $p + q < r$. Then with stakes 1,1,1,

$$(-1)(V_M(\theta) - p) + (-1)(V_M(\phi) - q) + (V_M(\theta \vee \phi) - r) = (p + q) - r < 0$$

[20] A simple condition sufficient for the convergence and (20) is that the sum of all the stakes involved be finite.

[21] In the sense of the agent or the bookmaker only being exposed to a finite loss even in the worst possible case.

[22] Synonymously that the system of bets is 'coherent'.

giving an instance of (19) and contradicting our assumption. A similar argument when

$$Bel(\theta) + Bel(\phi) > Bel(\theta \vee \phi)$$

shows that this cannot hold either so we must have equality here.

Finally suppose that $\exists x\, \psi(x) \in SL$. By Proposition 3.3 and the fact that we have already proved that (P1), (P2) hold for Bel, it is enough to derive a contradiction from the assumption that

$$\sum_{n=1}^{\infty} Bel\Big(\psi(a_n) \wedge \neg \bigvee_{i=1}^{n-1} \psi(a_i)\Big) \neq Bel(\exists x\, \psi(x)).$$

Notice that since the sentences on the left hand side here are disjoint both sides are bounded by 1.

We cannot have $>$ here since then that would hold for the sum of a finite number of terms on the left hand side, contradicting Proposition 3.1(c) etc.. So we may suppose that we have $<$ here. In this case we can pick

$$p_n > Bel\Big(\psi(a_n) \wedge \neg \bigvee_{i=1}^{n-1} \psi(a_i)\Big) \quad \text{for } n = 1, 2, \dots$$

and $r < Bel(\exists x\, \psi(x))$ with $\sum_{n=1}^{\infty} p_n < r$. Since for $M \in \mathcal{T}L$,

$$V_M(\exists x\, \psi(x)) = \sum_{n=1}^{\infty} V_M\Big(\psi(a_n) \wedge \neg \bigvee_{i=1}^{n-1} \psi(a_i)\Big)$$

we get, as with the argument above for (P2), that for all stakes 1,

$$(V_M(\exists x\, \psi(x)) - r) + \sum_{n=1}^{\infty}(-1)\Big(V_M\Big(\psi(a_n) \wedge \neg \bigvee_{i=1}^{n-1} \psi(a_i)\Big) - p_n\Big)$$
$$= -r + \sum_{n=1}^{\infty} p_n < 0,$$

giving an instance of (19) (and (20)) in contradiction to our assumption. $\dashv$

The Dutch Book argument can also be extended to conditional bets to justify the standard definition of the derived conditional probability given by (17). The idea is that not only is the agent offered unconditional bets as above but also bets on $\theta \in SL$ being true in M *given* that $\phi \in SL$ has turned out to be true in M. Similarly to the above unconditional case then for $\theta, \phi \in SL$, $0 \leq p \leq 1$ and for a stake $s > 0$ the agent is offered a choice of one of two wagers:

(CBet1$_p$) If $M \vDash \phi$ win $s(1-p)$ if $M \vDash \theta$, lose sp if $M \nvDash \theta$;

(CBet2$_p$) If $M \vDash \phi$ win sp if $M \nvDash \theta$, lose $s(1-p)$ if $M \vDash \theta$;

with all bets null and void if $M \nvDash \phi$. Now define $Bel(\theta \mid \phi)$ to be the supremum of those $p \in [0, 1]$ for which CBet1_p is acceptable to the agent[23]. Notice that in this case accepting CBet1_p for a stake $s > 0$ means gaining (negative gain equals loss)

$$sV_M(\phi)(V_M(\theta) - p)$$

whilst accepting CBet2_p means gaining minus this.

With this extension the agent's belief function Bel could be Dutch Booked if there were countable sets A, B, C, D, sentences θ_i, stakes $s_i > 0$, $p_i \in [0, Bel(\theta_i))$ for $i \in A$, sentences ϕ_i, stakes $t_i > 0$, $q_i \in (Bel(\phi_i), 1]$ for $i \in B$, sentences η_i, ψ_i, stakes $u_i > 0$, $r_i \in [0, Bel(\eta_i \mid \psi_i))$ for $i \in C$, sentences ζ_i, ξ_i, stakes $v_i > 0$, $m_i \in (Bel(\zeta_i \mid \xi_i), 1]$ for $i \in D$, and $K > 0$ such that the series below converge[24] and for all $M \in \mathcal{T}L$ we have

$$\sum_{i \in A} s_i(V_M(\theta_i) - p_i) + \sum_{i \in B}(-t_i)(V_M(\phi_i) - q_i) + \\ \sum_{i \in C} u_i V_M(\psi_i)(V_M(\eta_i) - r_i) + \sum_{i \in D}(-v_i) V_M(\xi_i)(V_M(\zeta_i) - m_i) < 0 \tag{21}$$

and

$$\Big|\sum_{i \in A} s_i(V_M(\theta_i) - p_i)\Big|, \Big|\sum_{i \in B} t_i(V_M(\phi_i) - q_i)\Big|, \\ \Big|\sum_{i \in C} u_i V_M(\psi_i)(V_M(\eta_i) - r_i)\Big|, \Big|\sum_{i \in D} v_i V_M(\xi_i)(V_M(\zeta_i) - m_i)\Big| < K. \tag{22}$$

THEOREM 5.2. *Suppose that* $Bel : SL \to [0, 1]$, $Bel(.\,|\,.) : SL \times SL \to [0, 1]$ *cannot be Dutch Booked in the above sense* (21), (22). *Then* $Bel(\theta \mid \phi) \cdot Bel(\phi) = Bel(\theta \wedge \phi)$.

PROOF. By Theorem 5.1 we can already assume that Bel is a probability function. Suppose first that

$$Bel(\theta \mid \phi) \cdot Bel(\phi) < Bel(\theta \wedge \phi). \tag{23}$$

If $Bel(\theta \mid \phi) < Bel(\theta \wedge \phi)$ then picking $Bel(\theta \mid \phi) < r < p < Bel(\theta \wedge \phi)$ gives

$$-V_M(\phi)(V_M(\theta) - r) + (V_M(\theta \wedge \phi) - p) = rV_M(\phi) - p \leq r - p < 0$$

for any M, since $V_M(\phi)V_M(\theta) = V_M(\theta \wedge \phi)$, contradicting the given no Dutch Book condition. Hence with (23), $Bel(\phi) < 1$. We also have $Bel(\theta \mid \phi) < 1$ since otherwise $Bel(\phi) < Bel(\theta \wedge \phi)$, contradicting Bel

[23]When $\vDash \neg\phi$, this appears to require an impossible stretch of imagination on the part of the agent. In this case we can define $Bel(\theta \mid \phi)$ arbitrarily.

[24]Again, note that this convergence and (22) are guaranteed e.g. by the sum of all the stakes involved being finite.

being a probability function (Proposition 3.1(c)). Hence we can pick $Bel(\theta \mid \phi) < r$, $Bel(\phi) < q$, $p < Bel(\theta \wedge \phi)$ with $qr < p$. But then considering the corresponding wagers with stakes $1, r, 1$ gives

$$-V_M(\phi)(V_M(\theta) - r) - r(V_M(\phi) - q) + (V_M(\theta \wedge \phi) - p)$$

and furnishes a Dutch Book since it is straightforward to check that its value is $rq - p < 0$ regardless of M.

We have shown that (23) cannot hold. So if the required equality fails it must be because

$$Bel(\theta \mid \phi) \cdot Bel(\phi) > Bel(\theta \wedge \phi). \tag{24}$$

But in this case pick $Bel(\theta \mid \phi) > r$, $Bel(\phi) > q$, $p > Bel(\theta \wedge \phi)$ with $qr > p$ and obtain a Dutch Book via

$$V_M(\phi)(V_M(\theta) - r) + r(V_M(\phi) - q) - (V_M(\theta \wedge \phi) - p). \qquad \dashv$$

Theorems 5.1 and 5.2 tell us then that the beliefs of a rational agent, rational in the sense of not allowing him/herself to be Dutch Booked, may be quantified as a probability function with conditional probability satisfying (17). That however raises the question whether there might be additional properties (P4), (P5) etc. of *Bel* which follow from there being no Dutch Book. The answer to this is 'No', as the next theorem (due to Kemeny, [63], and Lehman, [81], for (P1–2)) shows.

THEOREM 5.3. *Suppose that* $Bel : SL \to [0, 1]$ *is a probability function. Then Bel cannot be (conditionally) Dutch Booked.*

PROOF. Suppose on the contrary that there were countable sets A, B, C, D etc. such that (21), (22) held. Let μ_{Bel} be the measure[25] as in (2) such that for $\theta \in SL$,

$$Bel(\theta) = \int_{\mathcal{T}L} V_M(\theta)\, d\mu_{Bel}(M).$$

We have

$$\sum_{i \in A} s_i(Bel(\theta_i) - p_i) + \sum_{i \in B}(-t_i)(Bel(\phi_i) - q_i) +$$
$$\sum_{i \in C} u_i(Bel(\eta_i \wedge \psi_i) - r_i Bel(\psi_i)) +$$
$$\sum_{i \in D}(-v_i)(Bel(\zeta_i \wedge \xi_i) - m_i Bel(\xi_i)) =$$

[25] This identity will be derived as Corollary 7.2 and will not depend on the theorem we are currently proving!

$$\sum_{i \in A} s_i \int_{\mathcal{T}L} (V_M(\theta_i) - p_i)\, d\mu_{Bel}(M) \; +$$
$$\sum_{i \in B} (-t_i) \int_{\mathcal{T}L} (V_M(\phi_i) - q_i)\, d\mu_{Bel}(M) +$$
$$\sum_{i \in C} u_i \int_{\mathcal{T}L} (V_M(\eta_i \wedge \psi_i) - r_i V_M(\psi_i))\, d\mu_{Bel}(M) +$$
$$\sum_{i \in D} (-v_i) \int_{\mathcal{T}L} (V_M(\zeta_i \wedge \xi_i) - m_i V_M(\xi_i))\, d\mu_{Bel}(M)\,.$$

By (22) and Lebesgue's Dominated Convergence Theorem, using the facts that $V_M(\eta_i \wedge \psi_i) = V_M(\eta_i)V_M(\psi_i)$, $V_M(\xi_i \wedge \zeta_i) = V_M(\xi_i)V_M(\zeta_i)$ this further equals

$$\int_{\mathcal{T}L} \Bigg(\sum_{i \in A} s_i (V_M(\theta_i) - p_i) + \sum_{i \in B} (-t_i)(V_M(\phi_i) - q_i) +$$
$$\sum_{i \in C} u_i V_M(\psi_i)(V_M(\eta_i) - r_i) + \sum_{i \in D} (-v_i) V_M(\xi_i)(V_M(\zeta_i) - m_i) \Bigg)\, d\mu_{Bel}\,.$$

By (21) this is strictly negative. Hence also the expression we have started with is strictly negative, but that is impossible since $Bel(\theta_i) - p_i > 0$, $Bel(\phi_i) - q_i < 0$,

$$Bel(\eta_i \wedge \psi_i) - r_i Bel(\psi_i) = Bel(\psi_i)(Bel(\eta_i \mid \psi_i) - r_i) \geq 0,$$
$$Bel(\zeta_i \wedge \xi_i) - m_i Bel(\xi_i) = Bel(\xi_i)(Bel(\zeta_i \mid \xi_i) - r_i) \leq 0. \qquad \dashv$$

Chapter 6

SOME BASIC PRINCIPLES

In the foregoing discussion we have frequently mentioned rational principles, that is principles we might entertain our agent adhering to because of their purported rationality, but so far we have not actually specified any such principles. In this chapter we shall put that to rights by mentioning three principles which are so basic, and in a sense reasonable, that we shall frequently take them as assumptions in what follows.

As we shall see in the chapters which follow most of the principles which have been proposed to date are justified by considerations of symmetry, relevance and irrelevance. Of these it is the principles based on symmetry[26] (such as our original coin tossing example) which seem the most compelling, the argument being that if we can demonstrate a symmetry in the situation then it would be irrational of the agent to break that symmetry when assigning probabilities.

One obvious such symmetry relates to the constants $a_1, a_2, a_3, \ldots$. The agent has no reason to treat these any differently[27], a consideration which leads to:[28]

THE CONSTANT EXCHANGEABILITY PRINCIPLE, EX.
For $\theta(b_1, b_2, \ldots, b_m) \in SL$[29] *and (distinct) constants* $b_1', b_2', \ldots, b_m'$

$$w(\theta(b_1, b_2, \ldots, b_m)) = w(\theta(b_1', b_2', \ldots, b_m')). \tag{25}$$

Equivalently, for any permutation $\sigma \in \mathrm{S}_{\mathbb{N}^+}$ (= *the set of permutations of* $\mathbb{N}^+$),

$$w(\theta(a_1, a_2, \ldots, a_m)) = w(\theta(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(m)})). \tag{26}$$

[26] In Inductive Logic the word 'symmetry' is sometimes used where 'loose analogy' might in our opinion be a more appropriate expression, see for example page 123.

[27] The subscripts on the a's are simply to allow us to list them easily. The agent is not supposed to 'know' that a_1 comes before a_2 which comes before ... in our list.

[28] Which in our context corresponds to Johnson's *Permutation Postulate*, Carnap's *Axiom of Symmetry* (see A7 of his Axiom System for Inductive Logic, [130, page 974]), Kuiper's *Strong Principle of Order Indifference,* (see [73])

[29] Recall the convention of occasionally using $b_1, b_2, \ldots$ in place of $a_{i_1}, a_{i_2}, \ldots$ etc. in order to avoid double subscripts.

Given the reasonableness of Ex *we shall henceforth assume that it holds of all the probability functions we consider*, only re-iterating this on occasion for emphasis.

It is worth noting here that in the statement of Ex it is enough that it holds just for quantifier free formulae since we can then show it holds generally by induction on the quantifier complexity of the Prenex Normal Form equivalent of arbitrary $\phi(a_1, a_2, \dots, a_m) \in SL$.[30] The key step in this argument is when

$$\theta(a_1, a_2, \dots, a_m) = \exists x_1, x_2, \dots, x_k \, \psi(x_1, x_2, \dots, x_k, a_1, a_2, \dots, a_m).$$

In this case notice that for any $n \in \mathbb{N}$ there are $n < h < s \in \mathbb{N}$ such that

$$\{a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}\} \subseteq \{a_1, a_2, \dots, a_h\} \subseteq \{a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(s)}\}$$

so,

$$\begin{aligned} w\Big(& \bigvee_{i_1, i_2, \dots, i_k \leq n} \psi(a_{i_1}, a_{i_2}, \dots, a_{i_k}, a_1, a_2, \dots, a_m)\Big) \qquad (27) \\ & = w\Big(\bigvee_{i_1, i_2, \dots, i_k \leq n} \psi(a_{\sigma(i_1)}, a_{\sigma(i_2)}, \dots, a_{\sigma(i_k)}, a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)})\Big) \end{aligned}$$

by inductive hypothesis,

$$\begin{aligned} & \leq w\Big(\bigvee_{i_1, i_2, \dots, i_k \leq h} \psi(a_{i_1}, a_{i_2}, \dots, a_{i_k}, a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)})\Big) \\ & \leq w\Big(\bigvee_{i_1, i_2, \dots, i_k \leq s} \psi(a_{\sigma(i_1)}, a_{\sigma(i_2)}, \dots, a_{\sigma(i_k)}, a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)})\Big) \\ & = w\Big(\bigvee_{i_1, i_2, \dots, i_k \leq s} \psi(a_{i_1}, a_{i_2}, \dots, a_{i_k}, a_1, a_2, \dots, a_m)\Big), \end{aligned}$$

by inductive hypothesis, and taking limits gives by Lemma 3.8 that

$$\begin{aligned} & w(\exists x_1, \dots, x_k \, \psi(\vec{x}, a_1, a_2, \dots, a_m)) \\ & \qquad \leq w(\exists x_1, \dots, x_k \, \psi(\vec{x}, a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)})) \\ & \qquad \leq w(\exists x_1, \dots, x_k \, \psi(\vec{x}, a_1, a_2, \dots, a_m)). \end{aligned}$$

In what follows we shall observe that many of our principles have this property; that it is enough to state them simply for quantifier free sentences because they then extend to all sentences.

[30] In more detail, let $\vec{Q}_1 \vec{x_1} \vec{Q}_2 \vec{x_2} \dots \vec{Q}_n \vec{x_n} \psi(\vec{x_1}, \vec{x_2}, \dots, \vec{x_n}, \vec{a})$ be a sentence in Prenex Normal Form logically equivalent to ϕ, so $\psi \in QFSL$ and the $\vec{Q}_i$ are blocks of like quantifiers with the blocks being alternately $\exists$'s and $\forall$'s. In this case the quantifier complexity of this Prenex Normal Form sentence is n, the number of alternating blocks. The reason for applying induction over this n rather than, say, the length of ϕ, is that the disjunction in (27) has lower quantifier complexity than ϕ despite being longer.

As in the justification of the rationality of Ex we may argue that it would, in the zero knowledge situation we are limiting ourselves to, be irrational of an agent to treat any two relation symbols of the same arity any differently. This leads to the following principle which in our context corresponds to Carnap's axioms A8 and A9 from his Axiom System for Inductive Logic see [130, page 974].

The Principle of Predicate Exchangeability, Px.
If R, R' are relation symbols of L with the same arity then for $\theta \in SL$,

$$w(\theta) = w(\theta')$$

where θ' is the result of simultaneously replacing R by R' and R' by R throughout θ.

Although in the statement of this principle we only transposed two relation symbols repetition of this action (and the fact that any permutation of a finite set can be represented as compositions of transpositions) gives that if $R_1, R_2, \ldots, R_k$ are relation symbols of L of the same arity, $\sigma \in \mathsf{S}_k$ (the set of permutations of $\{1, 2, \ldots, k\}$) and $\theta \in SL$ then $w(\theta) = w(\theta')$ where θ' is the result of simultaneously replacing R_i by $R_{\sigma(i)}$ throughout θ for $i = 1, 2, \ldots, k$.

Our final symmetry motivated principle in this chapter is perhaps slightly more questionable. In this case the contention is that there is a symmetry between any relation symbol R of L and its negation $\neg R$ and so our agent has no reason to treat R any differently than $\neg R$. Since $\neg\neg R$ is logically equivalent to R this finds expression in[31]:

The Strong Negation Principle, SN.
For $\theta \in SL$,

$$w(\theta) = w(\theta')$$

where θ' is the result of replacing each occurrence of R in θ by $\neg R$.

Just as with Ex in the definitions of Px and SN we could equally well have restricted θ to being quantifier free since the given versions then follow.

Obviously requiring w to satisfy these three principles severely restricts the possible choices for w, an issue which we shall return to later. Notice that for each of Ex, Px, SN if the probability function w on SL satisfies the principle then so does its restriction $w \restriction SL'$ to a sublanguage L'.[32]

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[31] There is also a weaker version of this principle, the *Weak Negation Principle*, motivated by considering the symmetry as existing only when *all* relation symbols of L are being replaced by their negations. Precisely,

The Weak Negation Principle, WN: For $\theta \in SL$, $w(\theta) = w(\theta')$, where θ' is the result of simultaneously replacing each occurrence of a relation symbol in θ by its negation.

In this monograph we do not study WN further.

[32] By L' being a sublanguage of L we mean that the set of relation symbols in L' is a subset of the set of relation symbols in L but L' still has all the constant symbols $a_1, a_2, a_3, \ldots$.

At this point we shall prove a useful technical lemma concerning probability functions satisfying Ex and illustrate an application of it with a corollary. Both the lemma and corollary are due to Gaifman, [30].

Lemma 6.1. *Let $\exists x_1, \dots, x_k\, \psi(x_1, \dots, x_k, \vec{a}) \in SL$ and let $b_1, b_2, b_3, \dots$ be an infinite sequence of distinct constants including each of the a_i in $\vec{a}$. Then for w a probability function on SL satisfying Ex,*

$$w(\exists x_1, \dots, x_k\, \psi(x_1, \dots, x_k, \vec{a})) = \lim_{n \to \infty} w\Big(\bigvee_{i_1, i_2, \dots, i_k \leq n} \psi(b_{i_1}, b_{i_2}, \dots, b_{i_k}, \vec{a})\Big), \tag{28}$$

$$w(\forall x_1, \dots, x_k\, \psi(x_1, \dots, x_k, \vec{a})) = \lim_{n \to \infty} w\Big(\bigwedge_{i_1, i_2, \dots, i_k \leq n} \psi(b_{i_1}, b_{i_2}, \dots, b_{i_k}, \vec{a})\Big). \tag{29}$$

In other words, in Lemma 3.8 we need not take the limit over all $a_{i_1}, a_{i_2}, \dots, a_{i_k}$, just some infinite set of constants will suffice provided it contains all those appearing in $\vec{a}$.

Proof. Let $\vec{a} = \langle a_{m_1}, a_{m_2}, \dots, a_{m_r} \rangle$ and let $j \geq n \geq \max\{m_1, \dots, m_r\}$ be such that

$$\{a_{m_1}, a_{m_2}, \dots, a_{m_r}\} \subseteq \{b_1, b_2, \dots, b_n\} \subseteq \{a_1, a_2, \dots, a_j\}.$$

Then there is a permutation of $\{a_1, a_2, \dots, a_j\}$ which is fixed on $\vec{a}$ and maps $\{b_1, b_2, \dots, b_n\}$ to $\{a_1, a_2, \dots, a_n\}$. Hence by Ex,

$$w\Big(\bigvee_{i_1, i_2, \dots, i_k \leq n} \psi(b_{i_1}, b_{i_2}, \dots, b_{i_k}, \vec{a})\Big) = w\Big(\bigvee_{i_1, i_2, \dots, i_k \leq n} \psi(a_{i_1}, a_{i_2}, \dots, a_{i_k}, \vec{a})\Big)$$

and the result follows from Lemma 3.8 for $\exists \vec{x}\, \psi(\vec{x}, \vec{a})$. The case for $\forall \vec{x}\, \theta(\vec{x}, \vec{a})$ follows by taking negations. $\dashv$

Corollary 6.2. *Suppose that the probability function w on SL satisfies Ex and $w(\theta \wedge \phi) = w(\theta) \cdot w(\phi)$ whenever $\theta, \phi \in QFSL$ mention no constants in common. Then $w(\theta \wedge \phi) = w(\theta) \cdot w(\phi)$ for any $\theta, \phi \in SL$ which do not mention any constants in common.*

Proof. The proof is by induction on the quantifier complexity of $\theta \wedge \phi$ when written in Prenex Normal Form (recall the footnote on page 34). By assumption the result is true if $\theta \wedge \phi$ is quantifier free. To show the induction step it is enough to consider

$$\exists x_1, \dots, x_k\, \psi(x_1, \dots, x_k, \vec{a}) \wedge \exists x_1, \dots, x_h\, \eta(x_1, \dots, x_h, \vec{a}')$$

where $\vec{a}, \vec{a}'$ have no elements in common, the remaining cases being similar.

Let $b_1, b_2, b_3, \ldots$ be distinct constants containing those in $\vec{a}$ and $b'_1, b'_2, b'_3, \ldots$ a disjoint set of distinct constants containing those in $\vec{a}'$. By Lemma 6.1 for n large,

$$w\Big(\Big(\bigvee_{i_1,i_2,\ldots,i_k\leq n} \psi(b_{i_1}, b_{i_2}, \ldots, b_{i_k}, \vec{a})\Big) \leftrightarrow \exists x_1, \ldots, x_k\, \psi(x_1, \ldots, x_k, \vec{a})\Big)$$

and

$$w\Big(\Big(\bigvee_{i_1,i_2,\ldots,i_h\leq n} \eta(b'_{i_1}, b'_{i_2}, \ldots, b'_{i_h}, \vec{a}')\Big) \leftrightarrow \exists x_1, \ldots, x_h\, \eta(x_1, \ldots, x_h, \vec{a}')\Big)$$

are close to 1 so by Lemma 3.7,

$$w(\exists x_1, \ldots, x_k\, \psi(x_1, \ldots, x_k, \vec{a}) \wedge \exists x_1, \ldots, x_h\, \eta(x_1, \ldots, x_h, \vec{a}'))$$

is close to

$$w\Big(\bigvee_{i_1,\ldots,i_k\leq n} \psi(b_{i_1}, \ldots, b_{i_k}, \vec{a}) \wedge \bigvee_{i_1,\ldots,i_h\leq n} \eta(b'_{i_1}, \ldots, b'_{i_h}, \vec{a}')\Big),$$

which equals

$$w\Big(\bigvee_{i_1,\ldots,i_k\leq n} \psi(b_{i_1}, \ldots, b_{i_k}, \vec{a})\Big) \cdot w\Big(\bigvee_{i_1,\ldots,i_h\leq n} \eta(b'_{i_1}, \ldots, b'_{i_h}, \vec{a}')\Big)$$

by inductive hypothesis, and hence in turn is close to

$$w(\exists x_1, \ldots, x_k\, \psi(x_1, \ldots, x_k, \vec{a})) \cdot w(\exists x_1, \ldots, x_h\, \eta(x_1, \ldots, x_h, \vec{a}')).$$

The result follows. ⊣

Chapter 7

SPECIFYING PROBABILITY FUNCTIONS

On the face of it it might appear that because of the great diversity of sentences in SL probability functions would be very complicated objects and not easily described. In fact this is not the case as we shall now explain. The first step in this direction is the following theorem of Gaifman, [30]:

THEOREM 7.1. *Suppose that* $w^- : QFSL \to [0, 1]$ *satisfies* (P1) *and* (P2) *for* $\theta, \phi \in QFSL$. *Then* w^- *has a unique extension to a probability function* w *on* SL *satisfying* (P1–3) *for any* $\theta, \phi, \exists x\, \psi(x) \in SL$. *Furthermore if* w^- *satisfies Ex, Px, SN* (*respectively*) *on* $QFSL$ *then so will its extension* w *to* SL.

PROOF. Let w^- be as in the statement of the theorem. For $\theta \in QFSL$ the subsets

$$[\theta] = \{\, M \in \mathcal{T}L \mid M \vDash \theta \,\}$$

of $\mathcal{T}L$ form an algebra, $\mathcal{A}$ say, of sets and μ_{w^-} defined by

$$\mu_{w^-}([\theta]) = w^-(\theta) \quad \text{for } \theta \in QFSL$$

is easily seen to be a finitely additive measure on this algebra.

Indeed μ_{w^-} is (trivially) a pre-measure. For suppose $\theta, \phi_i \in QFSL$ for $i \in \mathbb{N}$ with the $[\phi_i]$ disjoint and

$$\bigcup_{i \in \mathbb{N}} [\phi_i] = [\theta]. \tag{30}$$

Then it must be the case that for some finite n

$$\bigcup_{i \leq n} [\phi_i] = [\theta],$$

otherwise

$$\{\, \neg\phi_i \mid i \in \mathbb{N} \,\} \cup \{\theta\}$$

would be finitely satisfiable and hence, by the Compactness Theorem for the Predicate Calculus, would be satisfiable in some structure for L. Although this particular structure need not be in $\mathcal{T}L$ its substructure with universe the $\{a_1, a_2, a_3, \ldots\}$ will be, and will satisfy the same quantifier free sentences, thus contradicting (30). So from the disjointness of the $[\phi_i]$ we must have that $[\phi_i] = \emptyset$ for $i > n$ (so $\mu_{w^-}([\phi_i]) = 0$), giving

$$\mu_{w^-}([\theta]) = \sum_{i \leq n} \mu_{w^-}([\phi_i]) = \sum_{i \in \mathbb{N}} \mu_{w^-}([\phi_i]),$$

and confirming the requirement to be a pre-measure.

Hence by Carathéodory's Extension Theorem (see for example [4]) there is a unique extension μ_w of μ_{w^-} defined on the σ-algebra $\mathcal{B}$ generated by $\mathcal{A}$. Notice that for $\exists x\, \psi(x) \in SL$ (where there may be some constants appearing in $\psi(x)$)

$$\begin{aligned}
[\exists x\, \psi(x)] &= \{\, M \in \mathcal{T}L \mid M \vDash \exists x\, \psi(x) \,\} \\
&= \{\, M \in \mathcal{T}L \mid M \vDash \psi(a_i),\ \text{some } i \in \mathbb{N}^+ \,\} \\
&= \bigcup_{i \in \mathbb{N}^+} \{\, M \in \mathcal{T}L \mid M \vDash \psi(a_i) \,\} \\
&= \bigcup_{i \in \mathbb{N}^+} [\psi(a_i)] \qquad (31)
\end{aligned}$$

so since $\mathcal{B}$ is closed under complements and countable unions $\mathcal{B}$ contains all the sets $[\theta]$ for $\theta \in SL$.

Now define a function w on SL by setting

$$w(\theta) = \mu_w([\theta]).$$

Notice that w extends w^- as μ_w extends μ_{w^-}. Since μ_w is a measure w satisfies (P1–2) and also (P3) from (31) and the fact that μ_w is countably additive.

This probability function must be the unique extension of w^- to SL satisfying (P1–3). For suppose that there was another such probability function, u say. By Proposition 3.1(d) it is enough to show that u and w agree on sentences θ in Prenex Normal Form. But using Lemma 3.8 this is straightforward to prove by induction on the quantifier complexity of θ (recall the footnote on page 34).

The last part for Ex is clear by the earlier remarks on page 34 but alternatively it can be shown as follows: Suppose on the contrary that $\sigma \in S_{\mathbb{N}^+}$ and $\theta(a_1, a_2, \dots, a_m) \in SL$ was such that

$$w(\theta(a_1, a_2, \dots, a_m)) \neq w(\theta(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)})) \qquad (32)$$

though w did satisfy Ex on $QFSL$. Define $v : SL \to [0, 1]$ by

$$v(\phi(a_{i_1}, a_{i_2}, \dots, a_{i_k})) = w(\phi(a_{\sigma^{-1}(i_1)}, a_{\sigma^{-1}(i_2)}, \dots, a_{\sigma^{-1}(i_k)})).$$

Clearly v is also a probability function on SL and agrees with w on $QFSL$ since w satisfies Ex on here. Hence by the previously proved uniqueness we must have $v = w$ on SL. But that contradicts (32). The cases of Px and SN are similar. ⊣

We are now in a position to justify the assertion (2) which we made earlier.

Corollary 7.2. *Suppose that w is a probability function on SL. Then for some countably additive measure μ_w on the algebra $\mathcal{B}$ of subsets of $\mathcal{T}L$,*

$$w = \int_{\mathcal{T}L} V_M \, d\mu_w(M).$$

Proof. If we start with $u^- = w \restriction QFSL$ then the proof of Theorem 7.1 shows that there is a countably additive measure μ_u on $\mathcal{B}$ such that for $\theta \in SL$,

$$u(\theta) = \mu_u([\theta]) = \int_{\mathcal{T}L} V_M(\theta) \, d\mu_u(M).$$

But by uniqueness u must equal w and we can take $\mu_w = \mu_u$. $\dashv$

It is easy to see, by considering $w(\theta \wedge \theta)$ when $0 < w(\theta) < 1$, that the V_M are precisely the probability functions on SL which satisfy $w(\theta \wedge \phi) = w(\theta) \cdot w(\phi)$ for all $\theta, \phi \in SL$. So this corollary is saying that *all* probability functions on SL are *mixtures* of such highly independent probability functions. We shall see this feature repeated in a number of future representation theorems for more limited classes of probability functions: The probability functions in these classes are all mixtures of certain highly 'independent' members of that class, for example Theorems 9.1, 25.1, 32.1.

An immediate consequence of Corollary 7.2 is that in our question $\mathcal{Q}$ the agent's problem of how to rationally assign probabilities $w(\theta)$ to sentences θ is equivalent to the problem of determining how to rationally assign measure $\mu_w(A)$ to Borel subsets A of $\mathcal{T}L$. In other words how to choose μ_w. The considerations however which we might choose to invoke in attempting to answer this latter question would seem to be *statistical* rather than *logical* so that in practice this aspect is generally neglected (but see page 122 for an example where it does seem to have been influential).

From this it follows that to specify a probability function on SL it is enough to say how it acts on the quantifier free sentences. We now explain how even that task can be simplified.

As usual let L be our default language with relation symbols $R_1, R_2, \dots, R_q$ of arities $r_1, r_2, \dots, r_q$ respectively. For distinct constants $b_1, b_2, \dots, b_m$ coming from $a_1, a_2, \dots$, a *State Description* for $b_1, b_2, \dots, b_m$ is a sentence of L of the form

$$\bigwedge_{i=1}^{q} \bigwedge_{c_1, c_2, \dots, c_{r_i}} \pm R_i(c_1, c_2, \dots, c_{r_i})$$

where the $c_1, c_2, \dots, c_{r_i}$ range over all (not necessarily distinct) choices from $b_1, b_2, \dots, b_m$ and $\pm R_i$ stands for either R_i or $\neg R_i$.

A state description for $b_1, b_2, \ldots, b_m$ then tells us which of the $R_i(c_1, c_2, \ldots, c_{i_r})$ hold and which do not hold for R_i a relation symbol from our language and any arguments from $b_1, b_2, \ldots, b_m$. Hence any two distinct[33] (inequivalent) state descriptions for $\vec{b}$ are exclusive in the sense that their conjunction is inconsistent. We allow here the possibility that $m = 0$ in which case the sole state description for these constants is a tautology, for which we use the symbol $\top$.

As an example here, if L has just the binary relation symbol R and the unary relation (or predicate) symbol P then

$$P(a_1) \wedge \neg P(a_2) \wedge \neg R(a_1, a_1) \wedge R(a_1, a_2) \wedge R(a_2, a_1) \wedge R(a_2, a_2)$$

is a state description for a_1, a_2.

Henceforth upper case Θ, Φ, Ψ etc. will *always* be used for state descriptions. In particular in an expression such as $\sum_{\Theta(\vec{a})} \ldots$ it will be taken that the sum is over all state descriptions for $\vec{a}$.

By the Disjunctive Normal Form Theorem (see for example [23, p49]) any $\theta(b_1, b_2, \ldots, b_m) \in QFSL$ is logically equivalent to a disjunction of state descriptions for $b_1, b_2, \ldots, b_m$, say

$$\theta(\vec{b}) \equiv \bigvee_{i \in S} \Theta_i(\vec{b}).$$

Hence, since state descriptions are exclusive, using (P2) repeatedly,

$$w(\theta(\vec{b})) = \sum_{i \in S} w(\Theta_i(\vec{b})). \tag{33}$$

From this it follows that the probability function w is determined on $QFSL$, and hence on all of SL by Theorem 7.1, by its values on state descriptions. Indeed since in (33) not all the $b_1, b_2, \ldots, b_m$ need to actually appear in $\theta(\vec{b})$ to determine w it is enough to know w just on the state descriptions for $a_1, a_2, \ldots, a_m$, $m \in \mathbb{N}$. Again from (33) if w satisfies Ex on state descriptions then it will on all $\theta \in QFSL$, and hence all $\theta \in SL$ by the remarks on page 34, and similarly for the principles Px and SN.

In the other direction, if the function w is defined on the state descriptions $\Theta(a_1, a_2, \ldots, a_m)$, $m \in \mathbb{N}$, to satisfy:

(i) $w(\Theta(a_1, a_2, \ldots, a_m)) \geq 0$,

(ii) $w(\top) = 1$,

(iii) $w(\Theta(a_1, a_2, \ldots, a_m)) =$

$$\sum_{\Phi(a_1, \ldots, a_{m+1}) \models \Theta(a_1, \ldots, a_m)} w(\Phi(a_1, a_2, \ldots, a_{m+1})), \tag{34}$$

[33] Following standard practice we shall, unless otherwise indicated, identify two state descriptions if they are the same *up to the ordering of their conjuncts*. Since throughout we will only really be concerned with sentences up to logical equivalence this abuse should not cause any distress.

(and notice that these are all properties that do hold for a probability function on SL) then w extends to a probability function on $QFSL$, and hence on SL, by setting (unambiguously by (iii))

$$w(\theta(b_1, b_2, \dots, b_m)) = \sum_{\Theta(a_1,\dots,a_k)\models\theta(b_1,\dots,b_m)} w(\Theta(a_1, a_2, \dots, a_k)) \tag{35}$$

where k is sufficiently large that all of the b_i are amongst $a_1, a_2, \dots, a_k$.

In the coming chapters we will rather often use this scheme to define probability functions, and the following lemma will prove useful.

Lemma 7.3. *Let w be a probability distribution and assume that whenever $m \in \mathbb{N}$, $\Phi(a_1, \dots, a_m)$ is a state description and $\tau \in \mathsf{S}_m$ then*

$$w(\Phi(a_1, \dots, a_m)) = w(\Phi(a_{\tau(1)}, \dots, a_{\tau(m)}))\,. \tag{36}$$

Then w satisfies Ex.

Proof. If $\Theta(a_1, \dots, a_n)$ is a state description and $\sigma \in \mathsf{S}_{\mathbb{N}^+}$ then there is a permutation $\tau \in \mathsf{S}_m$, where

$$m = \max\{\sigma(1), \dots, \sigma(n)\}(\geq n),$$

such that $\tau(i) = \sigma(i)$ for $i = 1, 2, \dots, n$. So

$$\begin{aligned}
w(\Theta(a_1, \dots, a_n)) &= \sum_{\Phi(a_1,\dots,a_m)\models\Theta(a_1,\dots,a_n)} w(\Phi(a_1, \dots, a_m)) \\
&= \sum_{\Phi(a_1,\dots,a_m)\models\Theta(a_1,\dots,a_n)} w(\Phi(a_{\tau(1)}, \dots, a_{\tau(m)})), \quad \text{by (36)}, \\
&= \sum_{\Phi(a_{\tau(1)},\dots,a_{\tau(m)})\models\Theta(a_{\tau(1)},\dots,a_{\tau(n)})} w(\Phi(a_{\tau(1)}, \dots, a_{\tau(m)})) \\
&= \sum_{\Psi(a_1,\dots,a_m)\models\Theta(a_{\tau(1)},\dots,a_{\tau(n)})} w(\Psi(a_1, \dots, a_m)) \\
&= w(\Theta(a_{\tau(1)}, \dots, a_{\tau(n)})) \\
&= w(\Theta(a_{\sigma(1)}, \dots, a_{\sigma(n)}))\,.
\end{aligned}$$

It follows that w satisfies (26) on state descriptions and hence by virtue of (33) on QFSL and consequently by Theorem 7.1 on all sentences. ⊣

For this monograph we shall be almost exclusively concerned with probability functions w on SL. However as the proof of Theorem 7.1 shows w has a unique extension w^+ to the $L_{\omega_1,\omega}$ sentences Λ of L given by $w^+(\Lambda) = \mu_w([\Lambda])$, where

$$[\Lambda] = \{M \in \mathcal{T}L \mid M \models_{\omega_1,\omega} \Lambda\} \in \mathcal{B}$$

so we could instead have made this infinitary logic our setting, as indeed is the case for the formative Scott-Krauss paper [132]. Our reason for not wishing to do this is that in the context of question $\mathcal{Q}$ it seems to overburden our agent, at least at this early stage in the development of his/her rationality.

We end this chapter by briefly sketching[34] a property of the μ_w which is actually a special case of Gaifman-Snir's [32, Theorem 2.1]. Let w be a probability function on SL. From the standard proof of Carathéodory's Extension Theorem (see [4]), as used in the proof of Theorem 7.1, for every $\varepsilon > 0$ and $\theta \in SL$ there is an open covering $\bigcup_{i\in\mathbb{N}}[\phi_i]$ of $[\theta]$ with the $\phi_i \in QFSL$ such that

$$\mu_w([\theta]) \leq \sum_{i\in\mathbb{N}} \mu_w([\phi_i]) < \mu_w([\theta]) + \varepsilon/2.$$

Hence, by taking $\psi = \bigvee_{i\leq n} \phi_i$ for some suitably large n we have that

$$\mu_w([\theta \leftrightarrow \psi]) = w(\theta \leftrightarrow \psi) > 1 - \varepsilon. \tag{37}$$

In other words *every* sentence of L can, in the sense of w, be approximated arbitrarily closely by a *quantifier free* sentence of L.

Building on this one can show (see [32] for more details) the following, where the ϕ_i, $i \in \mathbb{N}$, enumerate $QFSL$ and for $M \in \mathcal{T}L$,

$$\phi_i^{(M)} = \begin{cases} \phi_i & \text{if } M \vDash \phi_i, \\ \neg\phi_i & \text{if } M \vDash \neg\phi_i. \end{cases}$$

THEOREM 7.4. *For w a probability function on SL and $\theta \in SL$, for almost all $M \in \mathcal{T}L$ (with respect to the measure μ_w),*

$$\lim_{n\to\infty} w\Big(\theta \mid \bigwedge_{i\leq n} \phi_i^{(M)}\Big) = \begin{cases} 1 & \text{if } M \vDash \theta, \\ 0 & \text{if } M \vDash \neg\theta, \end{cases}$$

where the limit is taken as undefined if the conditioning sentences have probability zero eventually.

In other words, for almost all $M \in \mathcal{T}L$ (with respect to the measure μ_w) it is the case that

$$\{\, \theta \in SL \mid \lim_{n\to\infty} w\Big(\theta \mid \bigwedge_{i\leq n} \phi_i^{(M)}\Big) = 1 \,\}$$

is the complete theory of M, and hence in our context determines M uniquely.

PROOF. First notice that for almost all M we must have $w\big(\bigwedge_{i\leq n} \phi_i^{(M)}\big) > 0$ so for almost all M, $w\big(\theta \mid \bigwedge_{i\leq n} \phi_i^{(M)}\big)$ is defined.

Thus if $w(\theta) = 0$ then for almost all M we will have $w\big(\theta \mid \bigwedge_{i\leq n} \phi_i^{(M)}\big) = 0$ and the result follows directly in this case. So henceforth assume that $w(\theta) > 0$.

[34]This result will not be needed again in the following chapters, we include it simply out of interest.

By (37), given $\varepsilon > 0$ pick $\phi_k \in QFSL$ such that $w(\theta \leftrightarrow \phi_k) > 1 - \varepsilon^2$. Let Δ be the set of sentences ξ of the form[35]

$$\bigwedge_{i \leq n} \phi_i^{\varepsilon_i},$$

for some $n > k$, with $\varepsilon_k = 1$ and such that

$$w(\xi)(1 - 2\varepsilon) \geq w(\theta \wedge \xi). \tag{38}$$

but

$$w\Big(\bigwedge_{i \leq s} \phi_i^{\varepsilon_i}\Big)(1 - 2\varepsilon) < w\Big(\theta \wedge \bigwedge_{i \leq s} \phi_i^{\varepsilon_i}\Big)$$

for all $k < s < n$. Then these ξ are disjoint and we must have

$$\sum_{\xi \in \Delta} w(\xi) < \varepsilon, \tag{39}$$

otherwise

$$w\big(\neg(\theta \leftrightarrow \phi_k)\big) \geq \sum_{\xi \in \Delta} \big(w(\xi) - w(\theta \wedge \xi)\big) \geq 2\varepsilon^2$$

from (38). From this it follows that for each $n > k$,

$$\mu_w\Big(\{M \vDash \phi_k \mid \forall k < s \leq n,\ w\big(\bigwedge_{i \leq s} \phi_i^{(M)}\big)(1 - 2\varepsilon) \leq w\big(\theta \wedge \bigwedge_{i \leq s} \phi_i^{(M)}\big)\}\Big)$$

is at least $w(\phi_k) - \varepsilon$. Hence replacing ϕ_k by θ and taking the intersection over all n,

$$\mu_w\Big(\{M \vDash \theta \mid \forall k < n,\ w\big(\bigwedge_{i \leq n} \phi_i^{(M)}\big) > 0 \text{ and } (1 - 2\varepsilon) \leq w\big(\theta \mid \bigwedge_{i \leq n} \phi_i^{(M)}\big)\}\Big)$$

is at least $w(\theta) - 2\varepsilon$. Starting with other $\varepsilon > 0$ we obtain a similar lower bound for the measure of the set of $M \vDash \theta$ for which for all n eventually

$$w\Big(\bigwedge_{i \leq n} \phi_i^{(M)}\Big) > 0 \text{ and } (1 - 2\varepsilon) \leq w\Big(\theta \mid \bigwedge_{i \leq n} \phi_i^{(M)}\Big)$$

and intersecting these over a suitable decreasing sequence of ε gives that

$$\mu_w\Big(\{M \vDash \theta \mid \lim_{n \to \infty} w\big(\theta \mid \bigwedge_{i \leq n} \phi_i^{(M)}\big) = 1\}\Big) = w(\theta).$$

It now follows that for only a measure zero set of $M \in \mathcal{T}L$ do we have $M \vDash \theta$ but not $\lim_{n \to \infty} w\big(\theta \mid \bigwedge_{i \leq n} \phi_i^{(M)}\big) = 1$ for *some* θ, from which the required result follows. $\dashv$

[35] Recall the notation introduced on page 14.

Part 2

UNARY PURE INDUCTIVE LOGIC

Chapter 8

INTRODUCTION TO UNARY PURE INDUCTIVE LOGIC

In the initial investigations of Johnson and Carnap the language was taken to be *unary*. That is, in the notation we are adopting here the relation symbols $R_1, R_2, \dots, R_q$ of the language L all had arity 1, and so would more commonly be referred to as 'predicate' symbols. Carnap, [12, p123–4], and later Kemeny [64, p733–4], did comment that generalizing to polyadic relations was a future goal but with a few isolated exceptions there was almost no movement in that direction for the next 60–70 years.

There seem to be at least three reasons for this. Firstly the subject, and particularly the notation, becomes much more involved when we introduce non-unary relation symbols, a point which Kemeny acknowledged in [64] (appearing in 1963 though actually written in 1954). Secondly one's intuitions about what comprises rationality appear less well developed with regard to binary, ternary, etc. relations, possibly because in the real world they are not so often encountered, so the need to capture those fainter whispers which do exist seems less immediately pressing. Thirdly, for those working in the area there were more than enough issues to be resolved within the purely unary Pure Inductive Logic.

For all of these reasons it seems quite natural to concentrate first on the unary case. In addition many of the notions that this will lead us to study will reappear later in the polyadic case where this prior familiarity, in a relatively simple context, will provide an easy introduction and stepping stone to what will follow.

Our method in this, and the next chapter, will be to introduce various principles that one *might* argue our agent should observe on the grounds of their perceived rationality, though we will not set very high demands on what this amounts to. It will be enough that one is willing to entertain the idea that these principles are somehow rational. As already indicated these principles will largely arise from considerations of symmetry, relevance and irrelevance. At the same time as introducing these principles we shall derive relationships between them, and ideally characterize via a representation theorem the probability functions satisfying them, at least in the presence of some standing, basic principles, such as Constant Exchangeability, Ex.

Before setting out on this path we will briefly review the notation of the previous chapter as it applies in our new context of L having just unary relation symbols $R_1, R_2, \ldots, R_q$. In this case a state description $\Theta(b_1, b_2, \ldots, b_m)$ has the form

$$\bigwedge_{i=1}^{m} \bigwedge_{j=1}^{q} \pm R_j(b_i),$$

equivalently[36], the form

$$\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i) \tag{40}$$

where $\alpha_1(x), \alpha_2(x), \ldots, \alpha_{2^q}(x)$ are the 2^q *atoms* of L, that is formulae of the form

$$\pm R_1(x) \wedge \pm R_2(x) \wedge \cdots \wedge \pm R_q(x).$$

We will adopt the convention that the atoms are listed in the 'lexicographic' order with negated occurrences of predicate symbols coming later, that is e.g. for $L = \{R_1, R_2\}$ we take $\alpha_1(x) = R_1(x) \wedge R_2(x)$, $\alpha_2(x) = R_1(x) \wedge \neg R_2(x)$, $\alpha_3(x) = \neg R_1(x) \wedge R_2(x)$, $\alpha_4(x) = \neg R_1(x) \wedge \neg R_2(x)$.

Notice that these atoms are pairwise disjoint or exclusive, i.e. $\vDash \neg(\alpha_i(x) \wedge \alpha_k(x))$ for $i \neq k$ and also exhaustive in the sense that

$$\vDash \forall x \bigvee_{j=1}^{2^q} \alpha_j(x).$$

In the light of Lemma 7.3 the simplification afforded by state descriptions being now just conjunctions of atoms allows the Constant Exchangeability Principle, Ex, to be expressed in a particularly elegant form:

THE CONSTANT EXCHANGEABILITY PRINCIPLE (UNARY VERSION).

$w\Big(\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i)\Big)$ *depends only the signature* $\langle m_1, m_2, \ldots, m_{2^q}\rangle$ *of this state description, where* $m_j = |\{i \mid h_i = j\}|$. (41)

To see the equivalence note that if w satisfies (41) then the condition from Lemma 7.3 holds since for a permutation $\tau \in S_m$, the state descriptions $\bigwedge_{i=1}^{m} \alpha_{h_i}(a_i)$ and $\bigwedge_{i=1}^{m} \alpha_{h_i}(a_{\tau(i)})$ have the same signature. Hence w satisfies Ex. Conversely, if w satisfies Ex and $a_{k_1}, \ldots, a_{k_m}$, $a_{j_1}, \ldots, a_{j_m}$ are distinct constants and $\bigwedge_{i=1}^{m} \alpha_{h_i}(a_{k_i})$, $\bigwedge_{i=1}^{m} \alpha_{g_i}(a_{j_i})$ are state descriptions with the same signature then there is a bijection $\sigma : \{k_1, \ldots, k_m\} \to \{j_1, \ldots, j_m\}$ (extendable to $\sigma \in S_{\mathbb{N}^+}$) such that $\bigwedge_{i=1}^{m} \alpha_{g_i}(a_{j_i})$ is

[36] Up to logical equivalence, which is all that will matter.

$\bigwedge_{i=1}^{m} \alpha_{h_i}(a_{\sigma(k_i)})$ (up to a permutation of the conjuncts), and hence (41) is satisfied.

An important point to note in this unary case is that once we know which atom a constant b satisfies then we know everything there is to know about b. This is different from the situation where we have, say, a binary relation symbol R. For in that case knowing the state description $\Theta(b_1, b_2, \dots, b_m)$ satisfied by $b_1, b_2, \dots, b_m$ tells us nothing about whether or not $R(b_1, b_{m+1})$.

This simple situation also makes it comparatively easy to specify various probability functions. To give an example for a probability function which we shall employ frequently in what follows let

$$\mathbb{D}_{2^q} = \{ \langle x_1, x_2, \dots, x_{2^q} \rangle \in \mathbb{R}^{2^q} \mid x_1, \dots, x_{2^q} \geq 0, \sum_{i=1}^{2^q} x_i = 1 \}$$

and let

$$\vec{c} = \langle c_1, c_2, \dots, c_{2^q} \rangle \in \mathbb{D}_{2^q}.$$

Now define $w_{\vec{c}}$ on state descriptions (40) as above by

$$w_{\vec{c}}\Big(\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i) \Big) = \prod_{i=1}^{m} w_{\vec{c}}(\alpha_{h_i}(b_i)) = \prod_{i=1}^{m} c_{h_i} = \prod_{j=1}^{2^q} c_j^{m_j} \tag{42}$$

where $m_j = |\{ i \mid h_i = j \}|$ for $j = 1, 2, \dots, 2^q$ and the atoms are enumerated in the lexicographic ordering as explained on page 50. It seems clear at this point that $w_{\vec{c}}$ extends uniquely to a probability function on $QFSL$ satisfying (P1–2) and then, again uniquely by Theorem 7.1, to a probability function on SL.

Nevertheless, we should at least be aware of an imagined problem here: We would certainly be alright if we had defined $w_{\vec{c}}$ successively on the state descriptions $\Theta_1(a_1), \Theta_2(a_1, a_2), \Theta_3(a_1, a_2, a_3), \dots$ as in scheme (34). However by defining $w_{\vec{c}}$ on state descriptions for general strings of (distinct) constants $b_1, b_2, \dots, b_m$ are we not risking over-specifying $w_{\vec{c}}$? Fortunately we do not. To see this suppose that we did define $w_{\vec{c}}$ according to the scheme (34) and let n be sufficiently large that $b_1, \dots, b_m$ are all amongst $a_1, \dots, a_n$. Then

$$w_{\vec{c}}\Big(\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i) \Big) = \sum_{\Phi(a_1, \dots, a_n) \vDash \bigwedge_{i=1}^{m} \alpha_{h_i}(b_i)} w_{\vec{c}}(\Phi(a_1, \dots, a_n)),$$

where the Φ are state descriptions as usual. But since in these Φ there is a free choice of which atoms are satisfied by the a_j which do not appear amongst $b_1, \dots, b_m$ (and just one choice for those that do) this sum is just what we get if we multiply out $A_1 A_2 \dots A_n$ where

$$A_j = \begin{cases} c_{h_i} & \text{if } a_j = b_i, \\ \sum_{k=1}^{2^q} c_k & \text{if } a_j \notin \{b_1, \dots, b_m\}, \end{cases}$$

and since $\sum_{k=1}^{2^q} c_k$ is 1 this gives $\prod_{i=1}^{m} c_{h_i}$ and there is no over specification.

We have spent some time here explaining this vacuous, as it turns out, concern. The reason for doing so is because we will meet similar situations in future where, without further explanation, the reader will be expected to see that they are similarly vacuous.

Notice that $w_{\vec{c}}$ satisfies Ex but not Px nor SN except for rather special choices of $\vec{c}$.

The $w_{\vec{c}}$ are characterized by satisfying Ex and the following principle from [44]:[37]

THE CONSTANT IRRELEVANCE PRINCIPLE, IP.
If $\theta, \phi \in QFSL$ have no constant symbols in common then

$$w(\theta \wedge \phi) = w(\theta) \cdot w(\phi)$$

PROPOSITION 8.1. *Let w be a probability function on SL satisfying Ex. Then w satisfies IP just if $w = w_{\vec{c}}$ for some $\vec{c} \in \mathbb{D}_{2^q}$.*

PROOF. First notice that for $\vec{c} \in \mathbb{D}_{2^q}$ and state descriptions $\bigwedge_{i=1}^{n} \alpha_{h_i}(a_{j_i})$, $\bigwedge_{i=1}^{m} \alpha_{g_i}(a_{k_i})$ with no constant symbols in common,

$$\begin{aligned} w_{\vec{c}}\Big(\bigwedge_{i=1}^{n} \alpha_{h_i}(a_{j_i}) \wedge \bigwedge_{i=1}^{m} \alpha_{g_i}(a_{k_i})\Big) &= \prod_{i=1}^{n} c_{h_i} \cdot \prod_{i=1}^{m} c_{g_i} \\ &= w_{\vec{c}}\Big(\bigwedge_{i=1}^{n} \alpha_{h_i}(a_{j_i})\Big) \cdot w_{\vec{c}}\Big(\bigwedge_{i=1}^{m} \alpha_{g_i}(a_{k_i})\Big). \end{aligned}$$

Hence if $\theta, \phi \in QFSL$ have no constant symbols in common and $\theta \equiv \bigvee_{j=1}^{r} \Theta_j$, $\phi \equiv \bigvee_{t=1}^{s} \Phi_t$ with the Θ_j, Φ_t state descriptions (for the a_i in θ, ϕ respectively) then

$$\begin{aligned} w_{\vec{c}}(\theta \wedge \phi) = w_{\vec{c}}\Big(\bigvee_{j=1}^{r} \Theta_j \wedge \bigvee_{t=1}^{s} \Phi_t\Big) &= w_{\vec{c}}\Big(\bigvee_{j=1}^{r} \bigvee_{t=1}^{s} \Theta_j \wedge \Phi_t\Big) \\ = \sum_{j=1}^{r} \sum_{t=1}^{s} w_{\vec{c}}(\Theta_j \wedge \Phi_t) &= \sum_{j=1}^{r} \sum_{t=1}^{s} w_{\vec{c}}(\Theta_j) \cdot w_{\vec{c}}(\Phi_t) \\ = \sum_{j=1}^{r} w_{\vec{c}}(\Theta_j) \cdot \sum_{t=1}^{s} w_{\vec{c}}(\Phi_t) &= w_{\vec{c}}(\theta) \cdot w_{\vec{c}}(\phi). \end{aligned}$$

Conversely if w satisfies Ex and IP then by repeated application

$$w\Big(\bigwedge_{i=1}^{n} \alpha_{h_i}(a_{j_i})\Big) = \prod_{i=1}^{n} w(\alpha_{h_i}(a_{j_i})) = \prod_{i=1}^{n} w(\alpha_{h_i}(a_1)) = \prod_{i=1}^{n} c_{h_i}$$

where $c_i = w(\alpha_i(a_1))$ for $i = 1, 2, \ldots, 2^q$. Since w is determined by its values on state descriptions this forces $w = w_{\vec{c}}$, as required. $\dashv$

[37] Such probability functions are also referred to as 'product probability functions', for example in [69].

Recall that by Corollary 6.2 if the probability function w on SL satisfies $w(\theta \wedge \phi) = w(\theta) \cdot w(\phi)$ for all $\theta, \phi \in QFSL$ mentioning no constants in common then this also holds even for $\theta, \phi \in SL$. Consequently by the above proposition for any $\theta, \phi \in SL$ mentioning no constants in common, $w_{\vec{c}}(\theta \wedge \phi) = w_{\vec{c}}(\theta) \cdot w_{\vec{c}}(\phi)$. In particular then for $\theta \in SL$ not mentioning any constants at all

$$w_{\vec{c}}(\theta) = w_{\vec{c}}(\theta \wedge \theta) = w_{\vec{c}}(\theta)^2$$

so $w_{\vec{c}}(\theta)$ is either 0 or 1.

We end this chapter by introducing a convenient abbreviation which on some future occasions will spare us having to write out excessively long sentences. Namely, we may express a state description with signature $\langle m_1, m_2, \dots, m_{2^q} \rangle$ in the form

$$\alpha_1^{m_1} \alpha_2^{m_2} \dots \alpha_{2^q}^{m_{2^q}} \qquad \text{or} \qquad \bigwedge_{i=1}^{2^q} \alpha_i^{m_i}$$

where the unmentioned instantiating constants are implicitly assumed to avoid any unintended duplication with other constants appearing in the context. Notice that by our standing assumption of Ex as far as probability values are concerned it does not otherwise matter which (distinct) constants these are.

Chapter 9

DE FINETTI'S REPRESENTATION THEOREM

As observed in the previous chapter for w a probability function satisfying Ex (as usual) the value of w on a state description,

$$w\Big(\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i)\Big)$$

does not depend on the order of the $\alpha_{h_i}(b_i)$ (by Proposition 3.1(d)) nor on particular instantiating constants $b_1, b_2, \dots, b_m$. All it depends on is the *signature* of this state description, in other words the vector $\langle m_1, m_2, \dots, m_{2^q}\rangle$ where $m_j = |\{\, i \mid h_i = j\,\}|$ is the number of times that $\alpha_j(x)$ appears amongst the $\alpha_{h_1}(x), \alpha_{h_2}(x), \dots, \alpha_{h_m}(x)$.

This dependence on the signature is seen clearly in the following theorem, a version of de Finetti's Representation Theorem for Exchangeable Probability Functions, see [26], appropriate to our present context.[38]

De Finetti's Representation Theorem 9.1. *Let L be a unary language with*[39] *q relation symbols and let w be a probability function on SL satisfying Ex. Then there is a measure*[40] *μ on the Borel subsets of $\mathbb{D}_{2^q}$ such that*

$$\begin{aligned} w\Big(\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i)\Big) &= \int_{\mathbb{D}_{2^q}} \prod_{j=1}^{2^q} x_j^{m_j}\, d\mu(\vec{x}) \\ &= \int_{\mathbb{D}_{2^q}} w_{\vec{x}}\Big(\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i)\Big)\, d\mu(\vec{x}), \qquad (43) \end{aligned}$$

where $m_j = |\{i \mid h_i = j\}|$ for $j = 1, 2, \dots, 2^q$.

Conversely, given a measure μ on the Borel subsets of $\mathbb{D}_{2^q}$ the function w defined by (43) *extends* (*uniquely*) *to a probability function on SL satisfying Ex.*

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[38] The forthcoming proof of Theorem 25.1 will, as a special case, provide an alternative proof of de Finetti's Theorem to the one given here.

[39] The default assumption throughout is that we are working in a language L with q relation symbols.

[40] Recall that all measures considered in this monograph will be assumed to be normalized and countably additive unless otherwise stated.

PROOF. To simplify the notation assume that $q = 1$, the full case being an immediate generalization. So there are just two atoms, $\alpha_1(x) = R_1(x)$ and $\alpha_2(x) = \neg R_1(x)$. Let w be a probability function satisfying Ex and define

$$w(n,k) = w\Big(\bigwedge_{i=1}^{n+k} \alpha_{h_i}(b_i)\Big) \tag{44}$$

where $n = |\{i \mid h_i = 1\}|$ and $k = |\{i \mid h_i = 2\}|$. Notice that since w satisfies Ex the order of the h_i and the choice of distinct constants $b_1, b_2, \ldots, b_{n+k}$ here is immaterial. Hence for fixed n, k there are $\binom{n+k}{n}$ distinct possibilities for the ordering of the $h_1, h_2, \ldots, h_{n+k}$. Let $r > n+k$. Then from (33)

$$1 = w(\top) = \sum_{r_1+r_2=r} \binom{r}{r_1} w(r_1, r_2) \tag{45}$$

and

$$w(n,k) = \sum_{\substack{r_1+r_2=r \\ n \le r_1,\, k \le r_2}} \binom{r-n-k}{r_1-n} w(r_1, r_2). \tag{46}$$

From (45) let μ_r be the discrete measure on $\mathbb{D}_2$ which puts measure

$$\binom{r}{r_1} w(r_1, r_2)$$

on the point $\langle r_1/r, r_2/r\rangle \in \mathbb{D}_2$. Note that from (46) we obtain that $w(n,k)$ equals

$$\sum_{\substack{r_1+r_2=r \\ n \le r_1,\, k \le r_2}} \binom{r-n-k}{r_1-n} \binom{r}{r_1}^{-1} \binom{r}{r_1} w(r_1, r_2). \tag{47}$$

We shall show that

$$\left| \binom{r-n-k}{r_1-n} \binom{r}{r_1}^{-1} - \left(\frac{r_1}{r}\right)^n \left(\frac{r_2}{r}\right)^k \right| \tag{48}$$

tends to 0 as $r \to \infty$ uniformly in r_1, r_2.

Notice that the left hand term in (48) can be written as

$$\left(\frac{r_1}{r}\right)^n \left(\frac{r_2}{r}\right)^k \frac{(1-r_1^{-1})\cdots(1-(n-1)r_1^{-1})(1-r_2^{-1})\cdots(1-(k-1)r_2^{-1})}{(1-r^{-1})\cdots(1-(n+k-1)r^{-1})}. \tag{49}$$

We now consider cases. If $n = k = 0$ then (48) is zero. If $k > 0$ and $r_2 \le \sqrt{r}$ then both terms in (48) are less than $r^{-k/2} \le r^{-1/2}$, and similarly if $n > 0$ and $r_1 \leq \sqrt{r}$. If $k > 0$ and $r_2 > \sqrt{r}$ and either $n = 0$ or $r_1 > \sqrt{r}$ then using (49) and the fact that $r_1/r, r_2/r \leq 1$ we see that (48) is at most

$$1 - \frac{(1-\sqrt{r}^{\,-1})\cdots(1-(n-1)\sqrt{r}^{\,-1})(1-\sqrt{r}^{\,-1})\cdots(1-(k-1)\sqrt{r}^{\,-1})}{(1-r^{-1})\cdots(1-(n+k-1)r^{-1})}.$$

Similarly if $n > 0$ and $r_1 > \sqrt{r}$ and either $k = 0$ or $r_2 > \sqrt{r}$, and together we have covered all cases.

Hence from (45) and (47) $w(n, k)$ equals the limit as $r \to \infty$ of

$$\sum_{\substack{r_1+r_2=r \\ n\leq r_1,\, k\leq r_2}} \left(\frac{r_1}{r}\right)^n \left(\frac{r_2}{r}\right)^k \mu_r(\{\langle r_1/r, r_2/r\rangle\}). \tag{50}$$

In turn this equals the limit of the same expressions but summed simply over $0 \leq r_1, r_2,\ r_1 + r_2 = r$ since from (45) (or trivially if $n = 0$),

$$\sum_{\substack{r_1+r_2=r \\ r_1<n,\, k\leq r_2}} \left(\frac{r_1}{r}\right)^n \left(\frac{r_2}{r}\right)^k \mu_r(\{\langle r_1/r, r_2/r\rangle\}), \quad \text{etc.}$$

tends to zero as $r \to \infty$.

In other words,

$$w(n,k) = \lim_{r\to\infty} \int_{\mathbb{D}_2} x_1^n x_2^k \, d\mu_r(\langle x_1, x_2\rangle). \tag{51}$$

By Prohorov's Theorem, see for example [7, Theorem 5.1], since $\mathbb{D}_2$ is compact the μ_r have a subsequence μ_{i_r} weakly convergent to a countably additive measure μ, meaning that for any continuous function $f(x_1, x_2)$

$$\lim_{r\to\infty} \int_{\mathbb{D}_2} f(x_1,x_2)\, d\mu_{i_r}(\langle x_1,x_2\rangle) = \int_{\mathbb{D}_2} f(x_1,x_2)\, d\mu(\langle x_1,x_2\rangle).$$

Using this the required result follows from (51).

Finally the converse result, that functions w defined by (43) extend to probability functions on SL satisfying Ex is entirely straightforward. ⊣

From (43) it follows that the integrals

$$\int_{\mathbb{D}_2} f(x_1,x_2)\, d\mu(\langle x_1,x_2\rangle)$$

are uniquely determined by w for *any* polynomial $f(x_1, x_2)$, and hence (see for example [7]) that μ must be the unique measure satisfying (43). We shall call this measure the *de Finetti prior of* w.

De Finetti's Theorem generalizes directly to SL and indeed in what follows we shall use that name in this extended sense. Precisely:

COROLLARY 9.2. *Let w be a probability function on SL satisfying Ex. Then there is a measure μ on $\mathbb{D}_{2^q}$ (the de Finetti prior of w in fact) such that* *for $\theta \in SL$,*

$$w(\theta) = \int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\theta)\, d\mu(\vec{x}). \tag{52}$$

Conversely given a measure μ on $\mathbb{D}_{2^q}$, w defined by (52) is a probability function on SL satisfying Ex.

In other words every probability function w on SL is a convex mixture

$$w = \int_{\mathbb{D}_{2^q}} w_{\vec{x}}\, d\mu(\vec{x}), \tag{53}$$

of the $w_{\vec{c}}$ for $\vec{c} \in \mathbb{D}_{2^q}$.

Proof. Theorem 9.1 gives this for θ a state description, hence immediately for $\theta \in QFSL$, and then in turn for any $\theta \in SL$ by induction on quantifier complexity, Lemma 3.8 and Lebesgue's Dominated Convergence Theorem. The converse follows by checking (P1–3) noting that the functions $\vec{x} \mapsto w_{\vec{x}}(\theta)$ are measurable. ⊣

In what follows we shall see a number of analogous representation theorems and will customarily make this step from state descriptions to arbitrary sentences, eventually without explicit mention of the (common) underlying justification.

As we shall subsequently see de Finetti's Representation Theorem, and theorems like it, can be particularly powerful since on occasions it is enough to establish that a property holds for a probability function w by showing that it holds for the (relatively simple) probability functions $w_{\vec{c}}$ and then observing that the property is preserved by convex combinations, and so holds too for w. Indeed this is a general feature of such representation theorems which we shall exploit in the chapters to come.

As a simple example of such a result notice that for real numbers b, c,

$$b^2x_1^2 + c^2x_2^2 \geq 2bcx_1x_2$$

since $(bx_1 - cx_2)^2 \geq 0$. Hence

$$b^2w_{\vec{x}}(\alpha_1(a_1)\wedge\alpha_1(a_2))+c^2w_{\vec{x}}(\alpha_2(a_1)\wedge\alpha_2(a_2)) \geq 2bcw_{\vec{x}}(\alpha_1(a_1)\wedge\alpha_2(a_2))$$

and from (53), for w satisfying Ex,

$$b^2w(\alpha_1(a_1) \wedge \alpha_1(a_2)) + c^2w(\alpha_2(a_1) \wedge \alpha_2(a_2)) \geq 2bcw(\alpha_1(a_1) \wedge \alpha_2(a_2)). \tag{54}$$

Thus at least one of

$$b^2w(\alpha_1(a_1) \wedge \alpha_1(a_2)) \geq bcw(\alpha_1(a_1) \wedge \alpha_2(a_2)),$$
$$c^2w(\alpha_2(a_1) \wedge \alpha_2(a_2)) \geq bcw(\alpha_1(a_1) \wedge \alpha_2(a_2)),$$

must hold. Putting $b = c = 1$ and using Ex to permute constants this gives that at least one of

$$w(\alpha_1(a_2) \mid \alpha_1(a_1)) \geq w(\alpha_2(a_2) \mid \alpha_1(a_1)),$$
$$w(\alpha_2(a_2) \mid \alpha_2(a_1)) \geq w(\alpha_1(a_2) \mid \alpha_2(a_1)),$$

must hold. Similarly putting $b = w(\alpha_2(a_1))$, $c = w(\alpha_1(a_1))$ it gives that at least one of

$$w(\alpha_1(a_2) \mid \alpha_1(a_1)) \geq w(\alpha_1(a_2) \mid \alpha_2(a_1)),$$
$$w(\alpha_2(a_2) \mid \alpha_2(a_1)) \geq w(\alpha_2(a_2) \mid \alpha_1(a_1)),$$

must hold. In other words we cannot have $\alpha_2(a_1)$ being stronger evidence for $\alpha_1(a_2)$ than $\alpha_1(a_1)$ is *and* at the same time have $\alpha_1(a_1)$ being stronger evidence for $\alpha_2(a_2)$ than $\alpha_2(a_1)$ is (though it is easy to see that we can have both separately).

At this point it will be useful to mention a fact about de Finetti priors which will be used on occasion in the forthcoming chapters. Suppose that $L' = \{R_1, R_2, \ldots, R_q, R_{q+1}\}$ is the result of adding one new (unary) relation symbol to L, so we may take the atoms of L' to be, in order,

$$\alpha_1(x) \wedge R_{q+1}(x), \alpha_1(x) \wedge \neg R_{q+1}(x), \alpha_2(x) \wedge R_{q+1}(x), \alpha_2(x) \wedge \neg R_{q+1}(x), \\ \ldots, \alpha_{2^q}(x) \wedge R_{q+1}(x), \alpha_{2^q}(x) \wedge \neg R_{q+1}(x).$$

Let w' be a probability function on SL' satisfying Ex and with de Finetti prior μ' on

$$\mathbb{D}_{2^{q+1}} = \{ \langle x_1, x_2, \ldots, x_{2^{q+1}} \rangle \in \mathbb{R}^{2^{q+1}} \mid x_1, \ldots, x_{2^{q+1}} \geq 0, \sum_{i=1}^{2^{q+1}} x_i = 1 \}$$

and let $w = w' \restriction SL$, the restriction of w' to SL. Then clearly w also satisfies Ex and with the notation of the above theorem,

$$w\Big(\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i)\Big) = \int_{\mathbb{D}_{2^{q+1}}} \prod_{j=1}^{2^q} (y_{2j-1} + y_{2j})^{m_j}\, d\mu'(\vec{y})$$
$$= \int_{\mathbb{D}_{2^q}} \prod_{j=1}^{2^q} x_j^{m_j}\, d\mu(\vec{x}) \tag{55}$$

where μ is the measure defined on the Borel subsets A of $\mathbb{D}_{2^q}$ by

$$\mu(A) =$$
$$\mu'\{\langle y_1, y_2, \ldots, y_{2^{q+1}} \rangle \in \mathbb{R}^{2^{q+1}} \mid \langle y_1+y_2, y_3+y_4, \ldots, y_{2^{q+1}-1}+y_{2^{q+1}} \rangle \in A\}$$

(see for example [126]). By uniqueness μ must be the de Finetti prior of w.

Recall that we are assuming of all the probability functions we consider that they satisfy Ex, only mentioning it explicitly for emphasis or as a reminder. Thus we will always have de Finetti's Representation Theorem at our disposal when working, as in this part, with unary probability functions.

This theorem has to date proved to be by far the most important tool we have for studying PIL. Moreover it often provides insights which we cannot (currently) imagine gaining by any other route. In the following chapters we shall give further examples of this but for the moment we shall look at two notions, *regularity* and *universal certainty*.

Chapter 10

REGULARITY AND UNIVERSAL CERTAINTY

A rational principle assumed by Carnap (see for example [12, §55A]) was:

THE PRINCIPLE OF REGULARITY, REG.
For satisfiable $\theta \in QFSL$, $w(\theta) > 0$.

Notice that since any satisfiable $\theta \in QFSL$ is logically equivalent to a *non-empty* disjunction of state descriptions (each of which is obviously satisfiable) we could equally require θ to be a state description in the statement of the Regularity Principle.

The obvious justification for this principle is that if it is possible for θ to hold then the agent should give θ at least some non-zero probability. After all the agent seemingly has no grounds on which to dismiss θ as impossible.[41,42] Apart from this assuming Reg has the practical advantage that all probabilities conditioned on satisfiable quantifier free sentences, such as state descriptions, are well defined without the need to invoke 'the convention'.

Clearly Reg is not dependent on the language being purely unary and indeed we shall later blithely carry it over to polyadic languages.

The plan now is to give a characterization, in terms of their de Finetti priors, of those probability functions for a unary language which satisfy Reg. At the same time this will illuminate why Reg is restricted to quantifier free sentences. First however we need some notation and an observation. We begin with the notation.

Let w be a probability function on SL and μ its de Finetti prior. For $\emptyset \neq T \subseteq \{1, 2, \dots, 2^q\}$ let

$$N_T = \{\, \vec{c} \in \mathbb{D}_{2^q} \mid c_i > 0 \iff i \in T \,\}.$$

[41] One might feel that the agent would similarly not have any grounds for dismissing as impossible a satisfiable non-quantifier free sentence either, but as we shall shortly see on further reflection that is not quite so evident.

[42] In [133] Shimony gives a 'strict' Dutch Book argument for *Bel* to satisfy Reg (strengthened to an equivalence in [63], [81]). However this also requires the additional assumption mentioned in a footnote on page 26 that even for $p = Bel(\theta)$ the agent is willing to accept $\mathrm{Bet}_1(p)$ on θ.

If $\mu(N_T) > 0$ let μ_T be the measure on $\mathbb{D}_{2^q}$ such that for a Borel set $A \subseteq \mathbb{D}_{2^q}$,

$$\mu_T(A) = \frac{\mu(A \cap N_T)}{\mu(N_T)},$$

whilst if $\mu(N_T) = 0$ let μ_T be any measure on $\mathbb{D}_{2^q}$ (this is just for notational convenience, it will not actually figure in what follows). Let w_T be the probability function whose de Finetti prior is μ_T. Then

$$\mu = \sum_T \mu(N_T)\, \mu_T \tag{56}$$

(where the summation is over $\emptyset \neq T \subseteq \{1, 2, \dots, 2^q\}$) and

$$\begin{aligned} w &= \int_{\mathbb{D}_{2^q}} w_{\vec{x}}\, d\mu(\vec{x}) \\ &= \int_{\mathbb{D}_{2^q}} w_{\vec{x}}\, d\Big(\sum_T \mu(N_T)\mu_T(\vec{x})\Big) \\ &= \sum_T \mu(N_T) \int_{\mathbb{D}_{2^q}} w_{\vec{x}}\, d\mu_T(\vec{x}) \\ &= \sum_T \mu(N_T)\, w_T. \end{aligned} \tag{57}$$

Turning to the 'observation' let $\psi \in SL$, say the constants mentioned in ψ are amongst $b_1, b_2, \dots, b_m$. By the Prenex Normal Form Theorem ψ is logically equivalent to some quantifier free formula $\phi(x_1, x_2, \dots, x_n, \vec{b})$ preceded by a block $\overrightarrow{Qx}$ of quantifiers. Since L is unary, by the Disjunctive Normal Form Theorem,

$$\overrightarrow{Qx}\, \phi \equiv \overrightarrow{Qx} \bigvee_{k=1}^{r} \Big(\bigwedge_{j=1}^{n} \alpha_{g_{kj}}(x_j) \wedge \bigwedge_{i=1}^{m} \alpha_{h_{ki}}(b_i) \Big) \tag{58}$$

for some g_{kj}, h_{ki}. By repeated use of standard logical equivalences, for example

$$\forall x_1 \exists x_2\, (\alpha_1(x_1) \wedge \alpha_2(x_2)) \equiv \forall x_1\, \alpha_1(x_1) \wedge \exists x_2\, \alpha_2(x_2),$$

$$\forall x_1\, (\alpha_1(x_1) \vee \alpha_2(x_1)) \equiv \bigwedge_{i=3}^{2^q} \neg \exists x_1\, \alpha_i(x_1),$$

on the right hand side formula in (58) and changing variables if necessary we can replace this by a boolean combination of the $\pm\exists x\, \alpha_j(x)$, for *all* $j = 1, 2, \dots, 2^q$ and the $\alpha_h(b_i)$ which appear in ψ to produce an equivalent of ψ of the form

$$\bigvee_{k=1}^{l} \Big(\bigwedge_{j=1}^{2^q} \pm\exists x\, \alpha_j(x) \wedge \bigwedge_{i=1}^{m} \alpha_{f_{ki}}(b_i) \Big) \tag{59}$$

where all these disjuncts are disjoint and *satisfiable*. (Note that if ψ is not satisfiable then this will be the empty disjunction, which as usual we identify with a contradiction.)[43]

In this case the value of w on ψ will be the sum of its values on these disjuncts, each of which will be the weighted sum of the w_T on this disjunct for $\mu(N_T) \neq 0$. Consider such a disjunct, say

$$w_T\Big(\bigwedge_{j\in S} \exists x\, \alpha_j(x) \wedge \bigwedge_{j\notin S} \neg\exists x\, \alpha_j(x) \wedge \bigwedge_{i=1}^{m} \alpha_{f_{ki}}(b_i)\Big), \tag{60}$$

where we have collected together the negative $\neg\exists x\, \alpha_j(x)$, and similarly the positive $\exists x\, \alpha_j(x)$, conjuncts.

PROPOSITION 10.1. *Under the assumption that $\mu(N_T) \neq 0$ the value of* (60) *is non-zero if and only if $S = T$.*

PROOF. First suppose that $S = T$. Then because this term is satisfiable all of the f_{ki} must be in S, equivalently T, so

$$\begin{aligned} w_T\Big(\bigwedge_{i=1}^{m} \alpha_{f_{ki}}(b_i)\Big) &= \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{m} x_{f_{ki}}\, d\mu_T(\vec{x}) \\ &= \int_{N_T} \prod_{i=1}^{m} x_{f_{ki}}\, d\mu_T(\vec{x}) \end{aligned}$$

is greater than zero since $\prod_{i=1}^{m} x_{f_{ki}}$ is greater than zero on N_T.

Now let $j \notin S = T$. Then by Proposition 3.1(d),

$$\begin{aligned} w_T(\neg\exists x\, \alpha_j(x)) &= w_T(\forall x\, \neg\alpha_j(x)) \\ &= \lim_{r\to\infty} w_T\Big(\bigwedge_{i=1}^{r} \neg\alpha_j(a_i)\Big) \\ &= \lim_{r\to\infty} w_T\Big(\bigwedge_{i=1}^{r} \bigvee_{t\neq j} \alpha_t(a_i)\Big) \\ &= \lim_{r\to\infty} \int_{N_T} \Big(\sum_{t\neq j} x_t\Big)^r d\mu_T(\vec{x}) \\ &= \lim_{r\to\infty} \int_{N_T} 1^r\, d\mu_T(\vec{x}) \end{aligned} \tag{61}$$

which of course equals 1.

---

[43] A possibly slightly surprising consequence of this representation is that up to logical equivalence there are only finitely many different sentences $\psi(a_{i_1}, \dots, a_{i_n})$ for a fixed choice of $i_1, \dots, i_n$ and hence the set of values

$$\{w(\psi(a_{i_1}, \dots, a_{i_n})) \mid \psi(a_{i_1}, \dots, a_{i_n}) \in SL\}$$

is finite. Indeed under our standing assumption of Ex it is the same set for all choices of (distinct) $a_{i_1}, \dots, a_{i_n}$. This does not hold in general for polyadic languages, see [53].

Now suppose that $j \in S = T$. Then as above we obtain that

$$w_T(\neg\exists x\, \alpha_j(x)) = \lim_{r\to\infty} \int_{N_T} \Big(\sum_{t\neq j} x_t\Big)^r d\mu_T(\vec{x}).$$

However now $\sum_{t\neq j} x_t < 1$ on N_T so the countable additivity of μ_T ensures that this limit is

$$\int_{N_T} \lim_{r\to\infty} \Big(\sum_{t\neq j} x_t\Big)^r d\mu_T(\vec{x}) = \int_{N_T} 0\; d\mu_T(\vec{x}) = 0. \tag{62}$$

Putting these three parts together and using Lemma 3.6 we obtain that

$$w_T\Big(\bigwedge_{j\in T} \exists x\, \alpha_j(x) \wedge \bigwedge_{j\notin T} \neg\exists x\, \alpha_j(x) \wedge \bigwedge_{i=1}^{m} \alpha_{f_{ki}}(b_i)\Big) = w_T\Big(\bigwedge_{i=1}^{m} \alpha_{f_{ki}}(b_i)\Big) > 0,$$

as required.

Now suppose that $S \neq T$. If $j \in S - T$ then by the argument in (61) $w_T(\neg\exists x\, \alpha_j(x)) = 1$ so $w_T(\exists x\, \alpha_j(x)) = 0$ and since this is a conjunct in (60) that too must get zero probability according to w_T.

Finally suppose that $j \in T - S$. Then by the argument in (62) $w_T(\neg\exists x\, \alpha_j(x)) = 0$ so again (60) must get probability 0. ⊣

From this proposition we can draw several conclusions.

COROLLARY 10.2. *For* $\emptyset \neq T \subseteq \{1, 2, \ldots, 2^q\}$ *let*

$$\eta_T = \bigwedge_{j\in T} \exists x\, \alpha_j(x) \wedge \bigwedge_{j\notin T} \neg\exists x\, \alpha_j(x). \tag{63}$$

Then $w(\eta_T) = \mu(N_T)$ *and when* $\mu(N_T) > 0$, $w_T(\theta) = w(\theta \mid \eta_T)$ *and*

$$w = \sum_T w(\eta_T) w_T. \tag{64}$$

PROOF. From the proof of Proposition 10.1 it follows that when $w(N_T) > 0$, $w_T(\eta_S) = 1$ if $S = T$ and 0 otherwise. From (57) we now obtain (64) and $w_T(\theta) = w(\theta \mid \eta_T)$ then follows directly. ⊣

COROLLARY 10.3. *The probability function* w *satisfies Reg if and only if*

$$\mu(\{\, \vec{c} \in \mathbb{D}_{2^q} \mid c_1, c_2, \ldots, c_{2^q} > 0 \,\}) > 0 \tag{65}$$

where μ *is the de Finetti prior of* w.

PROOF. If (65) holds then for $T = \{1, 2, \ldots, 2^q\}$, $\mu(N_T) > 0$ and

$$w_T\Big(\bigwedge_{i=1}^{m} \alpha_{f_{ki}}(b_i)\Big) \geq w_T\Big(\bigwedge_{j=1}^{2^q} \exists x\, \alpha_j(x) \wedge \bigwedge_{i=1}^{m} \alpha_{f_{ki}}(b_i)\Big) > 0,$$

by the above proposition. So from (57)

$$w\Big(\bigwedge_{i=1}^{m} \alpha_{f_{ki}}(b_i)\Big) > 0$$

and w satisfies Reg since it gives each state description non-zero probability.

In the other direction suppose that (65) fails. Then since

$$\bigwedge_{i=1}^{2^q} \alpha_i(a_i) \equiv \bigwedge_{j=1}^{2^q} \exists x\, \alpha_i(x) \wedge \bigwedge_{i=1}^{2^q} \alpha_i(a_i),$$

by Proposition 10.1 the only w_T which could possibly give a non-zero contribution to

$$w\Big(\bigwedge_{i=1}^{2^q} \alpha_i(a_i)\Big)$$

according to (57) is when $T = \{1, 2, \ldots, 2^q\}$. But by assumption, for this T, $\mu(N_T) = 0$ so

$$w\Big(\bigwedge_{i=1}^{2^q} \alpha_i(a_i)\Big) = 0$$

and w does not satisfy Reg. ⊣

By a directly similar argument we also obtain that:

COROLLARY 10.4.

$$w\Big(\forall x\, \Big(\bigvee_{i=1}^{r} \alpha_{h_i}(x)\Big)\Big) > 0$$

if and only if

$$\mu(\{\,\vec{c} \in \mathbb{D}_{2^q} \mid c_j = 0 \textit{ for } j \notin \{h_1, h_2, \ldots, h_r\}\}) > 0,$$

where μ is the de Finetti prior of w.

Together with the representation (59) we now have

THEOREM 10.5. *For w a probability function with de Finetti prior μ, $w(\psi) > 0$ for all consistent $\psi \in SL$ just if $\mu(N_T) > 0$ for all $\emptyset \neq T \subseteq \{\,1, 2, \ldots, 2^q\,\}$.*

In other words, for w to satisfy regularity for *all* consistent sentences, which we shall refer to as *Super Regularity, SReg,*[44] (sometimes called *Universal Certainty* in the literature) its de Finetti prior μ must give every subset N_T of $\mathbb{D}_{2^q}$ for $\emptyset \neq T \subseteq \{1, 2, \ldots, 2^q\}$ non-zero measure, a situation which possibly seems perverse given that except for $T = \{1, 2, \ldots, 2^q\}$ each of these sets is of lower dimension than that of the set $\mathbb{D}_{2^q}$.

Perhaps not surprisingly then few of the popular proposals for w within Inductive Logic satisfy SReg. This is viewed as a serious problem by those who feel that regularity should hold beyond just the quantifier free level, that if ψ is consistent it should be blessed with non-zero probability, even if it is not a tautology. In particular it is argued that a rational choice

[44] As with Regularity we shall blithely carry over this notion to polyadic languages when the time comes.

of w should give some non-zero probability to all the a_i satisfying the same atom or proper subset of the atoms. A number of suggestions have been made for w with this property by specifically arranging that its de Finetti prior μ satisfies the requirement of the above theorem, see for example Hintikka [48], [47], [50], Hintikka & Niiniluoto [51], Niiniluoto [92] (and for surveys [22, p87], [72] and [93]), though from the standpoint of PIL these would still seem to be somewhat ad hoc (on this point see the remarks on page 121 after Theorem 17.7). Certainly deriving such a w from other, more obviously logical, sources would seem to us preferable.

As already remarked most of the probability functions we naturally meet in PIL actually have de Finetti priors which give all the sets N_T for $T \neq \{1, 2, \dots, 2^q\}$ measure zero. For such probability functions sentences of L act as if they are quantifier free, as captured in the following proposition.

PROPOSITION 10.6. *Suppose that w has de Finetti prior μ and $\mu(N_T) = 0$ for all $T \subset \{1, \dots, 2^q\}$. Then for $\psi \in SL$ there is a $\phi \in QFSL$ such that for all $\theta \in SL$, $w(\psi \wedge \theta) = w(\phi \wedge \theta)$.*

PROOF. Given the representation of ψ in the form (59) let ϕ be the disjunction of those

$$\bigwedge_{i=1}^{m} \alpha_{f_{ki}}(b_i)$$

where all the earlier conjuncts $\pm\exists x\, \alpha_j(x)$ were actually $\exists x\, \alpha_j(x)$. Using Lemma 3.7 it is straightforward to see that this ϕ has the required properties. ⊣

We conclude this chapter with a lemma which will be required in Chapter 17.

Let T be such that $w(\eta_T) > 0$ and let Υ_T^m be the disjunction of all state descriptions

$$\bigwedge_{i=1}^{m} \alpha_{h_i}(a_i)$$

for which $\{h_1, h_2, \dots, h_m\} = T$.

LEMMA 10.7.

(i) $\lim_{m\to\infty} w(\Upsilon_T^m \leftrightarrow \eta_T) = 1$.

(ii) *For $\theta \in SL$,*

$$\lim_{m\to\infty} w(\Upsilon_T^m \wedge \theta) = w(\eta_T \wedge \theta) = w(\eta_T)w_T(\theta).$$

PROOF. (i) By considering structures for the language L (not necessarily in $\mathcal{T}L$) we see that

$$\bigwedge_{j=1}^{2^q} \left(\exists x\, \alpha_j(x) \leftrightarrow \bigvee_{i=1}^{m} \alpha_j(a_i)\right)$$

logically implies $\Upsilon_T^m \leftrightarrow \eta_T$ and hence $\neg(\Upsilon_T^m \leftrightarrow \eta_T)$ logically implies

$$\bigvee_{j=1}^{2^q} \neg\Big(\exists x\, \alpha_j(x) \leftrightarrow \bigvee_{i=1}^{m} \alpha_j(a_i)\Big).$$

Consequently

$$\begin{aligned} w(\neg(\Upsilon_T^m \leftrightarrow \eta_T)) &\leq w\Bigg(\bigvee_{j=1}^{2^q} \neg\Big(\exists x\, \alpha_j(x) \leftrightarrow \bigvee_{i=1}^{m} \alpha_j(a_i)\Big)\Bigg) \\ &\leq \sum_{j=1}^{2^q} w\Bigg(\neg\Big(\exists x\, \alpha_j(x) \leftrightarrow \bigvee_{i=1}^{m} \alpha_j(a_i)\Big)\Bigg). \end{aligned}$$

But since for any j the corresponding summand in this expression equals

$$w\Big(\exists x\, \alpha_j(x) \wedge \neg \bigvee_{i=1}^{m} \alpha_j(a_i)\Big) = w\Big(\exists x\, \alpha_j(x)\Big) - w\Big(\bigvee_{i=1}^{m} \alpha_j(a_i)\Big),$$

which by (P3) tends to zero as $m \to \infty$, it must be that

$$\lim_{m\to\infty} w(\Upsilon_T^m \leftrightarrow \eta_T) = 1\,.$$

(ii) By Lemma 3.7,

$$|w(\Upsilon_T^m \wedge \theta) - w(\eta_T \wedge \theta)| \leq 1 - w(\Upsilon_T^m \leftrightarrow \eta_T)\,,$$

so the result follows from part (i). $\dashv$

Chapter 11

RELEVANCE

In our initial example in the introduction, of forming beliefs about the outcome of a coin toss, we noted three important considerations: symmetry, irrelevance and relevance. We have already considered to some extent the first two of these (though we shall have more to say later) and we now come to consider the third, relevance. In that example we observed that knowing that the previous eight coin tosses by this umpire had all been heads would surely have seemed relevant as far as the imminent toss was concerned.

Such an example seems to be a rather typical source of 'relevance': Namely that the more frequently we (or our rational agent) have seen some event in the past the more frequently we expect to see it in the future. For example if I am looking out of my window and of the 6 birds I see fly past all but the third are headed left to right I might reasonably expect that more than likely the next bird observed will also be heading that way, though precisely quantifying what I mean by 'more than likely' might still be far from obvious.

Such examples have been very much philosophers' main focus of study in Carnap's Inductive Logic, embodying as they do the very essence of 'induction', with less attention being paid to our problem of the assignment of beliefs, or probabilities, in general. In these examples Carnap et al assumed the position that such past evidence should be incorporated via conditioning (as explained already on page 22). Hence formalizing the above bird example for illustration, in its simplest form 'relevance' tells us that we should have

$$w(R_1(a_7) \mid R_1(a_1) \wedge R_1(a_2) \wedge \neg R_1(a_3) \wedge R_1(a_4) \wedge R_1(a_5) \wedge R_1(a_6)) > 1/2. \tag{66}$$

A problem here of course is saying by how much the left hand side should exceed one half.

As something of an aside at this point it might have appeared, from the time that question Q was first posed, that it had an obvious answer: Namely, if nothing is known then there is no reason to suppose that any $R_j(a_i)$ is any more probable than $\neg R_j(a_i)$ (so both get probability 1/2), nor any reason to suppose that these $R_j(a_i)$ are in any way (stochastically) dependent. It is easy to see that there is just one probability function[45] whose assignment of values satisfies this property. Unfortunately it gives, for example, equality in (66) rather the desired inequality. So despite its naive appeal this (apparent) flaw means that the subject of Carnap's Inductive Logic has not immediately accepted the supremacy of this probability function (and packed up and gone home as a result!) but instead has blossomed into more esoteric considerations.

Of course the examples above in support of 'relevance' are based on particular interpretations, which as we have stressed would not be available to the zero knowledge agent. On the other hand the agent could certainly introspect along the lines that 'if I learnt $R_1(a_1) \wedge R_1(a_2) \wedge \neg R_1(a_3) \wedge R_1(a_4) \wedge R_1(a_5) \wedge R_1(a_6)$ then would it not be reasonable for me to entertain (66)?'.[46]

It turns out that there are several ways of formally capturing this idea that 'the more times we have seen something in the past the more times we should expect to see it in the future' as a principle within PIL. We shall introduce some of these in later chapters (in particular the *Unary Principle of Induction* on page 161) but for the present we consider a proposal by Carnap (see [16, chapter 13], [130, page 975]) which, with some simplifications, can be formulated as:

The Principle of Instantial Relevance, PIR.
For $\theta(a_1, a_2, \dots, a_n) \in SL$ and atom $\alpha(x)$ of L,

$$w(\alpha(a_{n+2}) \mid \alpha(a_{n+1}) \wedge \theta(a_1, a_2, \dots, a_n)) \geq w(\alpha(a_{n+2}) \mid \theta(a_1, a_2, \dots, a_n)). \tag{67}$$

This principle then captures the informal idea that in the presence of 'evidence' $\theta(a_1, a_2, \dots, a_n)$ the observation that a_{n+1} satisfies $\alpha(x)$ should enhance one's belief that a_{n+2} will also satisfy $\alpha(x)$.[47]

It is one of the pleasing successes of this subject that in fact PIR is a consequence simply of Ex, a result first proved by Gaifman [31] (later simplified by Humburg [54]).

Theorem 11.1. *Ex implies PIR*

Proof. We will write $\vec{a}$ for $a_1, a_2, \dots, a_n$. Let the probability function w on SL satisfy Ex. Employing the notation of (67), let $\alpha(x) = \alpha_1(x)$ and denote $A = w(\theta(\vec{a}))$. Then for μ the de Finetti prior for w (using

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[45] Which we shall soon meet as c_∞^L.

[46] To counter this however the agent could argue that according to the naively favoured c_∞^L such biased conditioning evidence is rather unlikely in the first place so such a thought experiment should not cause one to particularly alter one's views!

[47] Of course since we are assuming Ex the particular constants a_i used here are irrelevant.

the fact that by Proposition 8.1 the $w_{\vec{x}}$ satisfy IP)

$$
\begin{aligned}
A = w(\theta(\vec{a})) &= \int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\theta(\vec{a}))\, d\mu(\vec{x})\,, \\
w(\alpha_1(a_{n+1}) \wedge \theta(\vec{a})) &= \int_{\mathbb{D}_{2^q}} x_1 w_{\vec{x}}(\theta(\vec{a}))\, d\mu(\vec{x}), \\
w(\alpha_1(a_{n+2}) \wedge \alpha_1(a_{n+1}) \wedge \theta(\vec{a})) &= \int_{\mathbb{D}_{2^q}} x_1^2 w_{\vec{x}}(\theta(\vec{a}))\, d\mu(\vec{x})
\end{aligned}
$$

and (67) amounts to

$$
\Big(\int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\theta(\vec{a}))\, d\mu(\vec{x})\Big) \cdot \Big(\int_{\mathbb{D}_{2^q}} x_1^2 w_{\vec{x}}(\theta(\vec{a}))\, d\mu(\vec{x})\Big) \geq \\
\Big(\int_{\mathbb{D}_{2^q}} x_1 w_{\vec{x}}(\theta(\vec{a}))\, d\mu(\vec{x})\Big)^2. \tag{68}
$$

If $A = 0$ then this clearly holds (because the other two integrals are less or equal to A and greater equal zero) so assume that $A \neq 0$. In that case (68) is equivalent to

$$
\int_{\mathbb{D}_{2^q}} \Big(x_1 A - \int_{\mathbb{D}_{2^q}} x_1 w_{\vec{x}}(\theta(\vec{a}))\, d\mu(\vec{x})\Big)^2 w_{\vec{x}}(\theta(\vec{a}))\, d\mu(\vec{x}) \,\geq\, 0 \tag{69}
$$

as can be seen by multiplying out the square and dividing by A. But obviously, being an integral of a non-negative function, (69) holds, as required. ⊣

Clearly we can obtain more from this proof. For we can only have equality in (69) if up to a measure zero set x_1 is constant on

$$
\{\vec{x} \in \mathbb{D}_{2^q} \mid w_{\vec{x}}(\theta(\vec{a})) > 0\}.
$$

It is also interesting to note that we do not need to take an atom $\alpha(x)$ in the statement of PIR, a minor revision of the above proof using the observation following the proof of Proposition 8.1 gives that

THE EXTENDED PRINCIPLE OF INSTANTIAL RELEVANCE, EPIR.
For $\theta(a_1, a_2, \ldots, a_n) \in SL, \phi(x_1) \in FL$,

$$
w(\phi(a_{n+2}) \mid \phi(a_{n+1}) \wedge \theta(a_1, a_2, \ldots, a_n)) \geq w(\phi(a_{n+2}) \mid \theta(a_1, a_2, \ldots, a_n)). \tag{70}
$$

Theorem 11.1 is of interest for the following reason. A straight-off argument one might think of for accepting support by induction (i.e. that the more we've observed something in the past the more we should expect to observe it in the future) is that it often seems to work in the real world. However it can be argued that this reasoning is circular, that one is basing one's support on inductions often working in the future because ... they often worked in the past![48] What Theorem 11.1 shows is that the past would be a reasonable guide to the future if one assumed exchangeability, that globally the world was more or less uniform. Of course one could in turn respond that this amounts to saying that the future will be more or less similar to the past (cf for example Zabell [148]), and put like that this 'justification for induction' again seems to be no more than blatantly assuming what one wants to prove. Nevertheless when induction is formalized by PIR and the uniformity of the world by Ex we see that this implication becomes Theorem 11.1 and there is something to prove. Indeed, in view of its reliance on de Finetti's Theorem (or the like) the proof is surprisingly complicated given the professed transparency of its 'blatantly assuming what one wants to prove'.[49]

[48] For an introduction to the 'Problem(s) of Induction', first highlighted in 1793 in David Hume's 'A Treatise on Human Nature', see [80], [142].

[49] The result does not go the other way however, see [117, footnote 6] for an example of a probability function satisfying EPIR but not Ex.

Chapter 12

ASYMPTOTIC CONDITIONAL PROBABILITIES

The purpose of this chapter is to use de Finetti's Representation Theorem to prove a technical result (see also, for example, [42], [44], [129, Theorem 7.78]) about the behaviour of certain conditional probabilities

$$w\Big(\bigwedge_{i=1}^{r}\alpha_{k_i}(a_{m+i}) \mid \bigwedge_{i=1}^{m}\alpha_{h_i}(a_i)\Big)$$

as $m \to \infty$ which will provide a powerful tool in later applications.[50]

The intuition behind this result is as follows. Suppose w was our chosen well behaved rational probability function and we then make (or imagine making) a long sequence of observations

$$\alpha_{h_1}(a_1), \alpha_{h_2}(a_2), \alpha_{h_3}(a_3), \dots, \alpha_{h_m}(a_m).$$

In that case we might feel that our initial subjective input into w should be largely subordinate to this *statistical evidence* and in consequence that the probability that a_{m+1} will satisfy $\alpha_i(x)$ should be close to m_i/m where m_i is the number of times that α_i has occurred amongst the $\alpha_{h_1}, \dots, \alpha_{h_m}$. More generally for *fixed* r the probability of

$$\bigwedge_{i=1}^{r}\alpha_{k_i}(a_{m+i})$$

should be close to

$$\prod_{i=1}^{r}\Big(\frac{m_{k_i}}{m}\Big).$$

To within some minor conditions on w this intuition is correct as the forthcoming Corollary 12.2 will show. To set the scene for this result let w satisfy (as usual) Ex and have de Finetti prior μ. Notice then that with the above notation

$$w\Big(\bigwedge_{i=1}^{r}\alpha_{k_i}(a_{m+i}) \mid \bigwedge_{i=1}^{m}\alpha_{h_i}(a_i)\Big) = \frac{\int_{\mathbb{D}_{2^q}}\prod_{i=1}^{2^q} x_i^{m_i+r_i}\,d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}}\prod_{i=1}^{2^q} x_i^{m_i}\,d\mu(\vec{x})}$$

[50]Since this result, and its development in the next chapter, are indeed rather technical the trusting reader could simply note the main theorems and jump to the gentler Chapter 14.

where r_i is the number of times that α_i appears amongst the $\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_r}$. We shall assume that all the right hand side denominators here are non-zero, as will indeed invariably be the case in the results which follow.

Let $\vec{b} = \langle b_1, b_2, \ldots, b_{2^q} \rangle \in \mathbb{D}_{2^q}$ be in the *support* of μ, meaning that for all $\varepsilon > 0$, $\mu(N_\varepsilon(\vec{b})) > 0$, where

$$N_\varepsilon(\vec{b}) = \{ \vec{x} \in \mathbb{D}_{2^q} \mid |\vec{x} - \vec{b}| < \varepsilon \}.$$

As usual let $[mb_i]$ be the integer part of mb_i, that is the largest integer less or equal mb_i. The following lemmas and corollaries in this chapter are taken from [42].

LEMMA 12.1. *With the above notation and assumptions,*

$$\lim_{m\to\infty} \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[mb_i]+r_i} \, d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[mb_i]} \, d\mu(\vec{x})} = \prod_{i=1}^{2^q} b_i^{r_i}.$$

PROOF. We begin by showing that for any natural numbers $p_1, p_2, \ldots, p_{2^q}$, possibly equal to 0, and for any $\nu > 0$

$$\left| \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{mb_i+n_i} \, d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{mb_i} \, d\mu(\vec{x})} - \prod_{i=1}^{2^q} b_i^{n_i} \right| < \nu. \tag{71}$$

for all m eventually, where $n_i = [(m+s)b_i] - (m+s)b_i + sb_i + p_i$ for some fixed $s \geq 2/b_i$ whenever $b_i \neq 0$ and $n_i = p_i$ whenever $b_i = 0$, for $i = 1, 2, \ldots, 2^q$. Let $h \geq 1$ be an upper bound on the n_i (for all m) and let

$$A_i = \begin{cases} [b_i, h] & \text{if } b_i \neq 0, \\ \{b_i, p_i\} & \text{if } b_i = 0. \end{cases}$$

Notice that the A_i do not depend on m and $n_i \in A_i$ for all m, $i = 1, 2, \ldots, 2^q$.

Since the function

$$f : \langle x_1, x_2, \ldots, x_{2^{q+1}} \rangle \mapsto \prod_{i=1}^{2^q} x_i^{x_{2^q+i}}$$

is uniformly continuous on $\mathbb{D}_{2^q} \times \prod_{i=1}^{2^q} A_i$ we can pick $0 < \varepsilon < \nu$ such that for $\vec{z}, \vec{t} \in \mathbb{D}_{2^q} \times \prod_{i=1}^{2^q} A_i$,

$$\left| \prod_{i=1}^{2^q} z_i^{z_{2^q+i}} - \prod_{i=1}^{2^q} t_i^{t_{2^q+i}} \right| < \nu/2 \text{ whenever } |\vec{z} - \vec{t}| < \varepsilon. \tag{72}$$

Also the function $\vec{x} \mapsto \prod_{i=1}^{2^q} x_i^{b_i}$ is concave on $\mathbb{D}_{2^q}$, taking its maximum value at $\vec{x} = \vec{b}$, so there is a $\delta > 0$ such that

$$\prod_{i=1}^{2^q} b_i^{b_i} > \prod_{i=1}^{2^q} y_i^{b_i} + 2\delta \text{ whenever } |\vec{y} - \vec{b}| \geq \varepsilon, \vec{y} \in \mathbb{D}_{2^q}.$$

Again by the uniform continuity of the function f we can choose $0 < \tau < \varepsilon$ such that for $\vec{z}, \vec{t} \in \mathbb{D}_{2^q} \times \prod_{i=1}^{2^q} A_i$,

$$\left| \prod_{i=1}^{2^q} z_i^{z_{2^q+i}} - \prod_{i=1}^{2^q} t_i^{t_{2^q+i}} \right| < \delta \text{ whenever } |\vec{z} - \vec{t}| < \tau. \tag{73}$$

Hence for any $\vec{x}, \vec{y} \in \mathbb{D}_{2^q}$ with $|\vec{x} - \vec{b}| < \tau$, $|\vec{y} - \vec{b}| \geq \varepsilon$,

$$\left| \prod_{i=1}^{2^q} x_i^{b_i} - \prod_{i=1}^{2^q} y_i^{b_i} \right| \geq \left| \left| \prod_{i=1}^{2^q} b_i^{b_i} - \prod_{i=1}^{2^q} y_i^{b_i} \right| - \left| \prod_{i=1}^{2^q} b_i^{b_i} - \prod_{i=1}^{2^q} x_i^{b_i} \right| \right| > |2\delta - \delta| = \delta.$$

For any such $\vec{x}, \vec{y}$ then,

$$\prod_{i=1}^{2^q} y_i^{b_i} < \prod_{i=1}^{2^q} x_i^{b_i} - \delta \leq \prod_{i=1}^{2^q} x_i^{b_i} (1 - \delta)$$

so

$$\prod_{i=1}^{2^q} y_i^{mb_i} < \prod_{i=1}^{2^q} x_i^{mb_i} (1 - \delta)^m.$$

Let I_m denote the integral

$$\frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{mb_i + n_i} \, d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{mb_i} \, d\mu(\vec{x})}.$$

Then

$$I_m = \frac{\int_{N_\varepsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i + n_i} \, d\mu(\vec{x}) + \int_{\neg N_\varepsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i + n_i} \, d\mu(\vec{x})}{\int_{N_\varepsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i} \, d\mu(\vec{x}) + \int_{\neg N_\varepsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i} \, d\mu(\vec{x})}, \tag{74}$$

where $\neg N_\varepsilon(\vec{b})$ is short for $\mathbb{D}_{2^q} - N_\varepsilon(\vec{b})$.

We have that,

$$\begin{aligned} \int_{\neg N_\varepsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i + n_i} \, d\mu(\vec{x}) &\leq \int_{\neg N_\varepsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i} \, d\mu(\vec{x}) \\ &\leq \int_{\neg N_\varepsilon(\vec{b})} \inf_{N_\tau(\vec{b})} \left\{ \prod_{i=1}^{2^q} x_i^{mb_i} \right\} d\mu(\vec{x}) (1 - \delta)^m \\ &= \inf_{N_\tau(\vec{b})} \left\{ \prod_{i=1}^{2^q} x_i^{mb_i} \right\} \mu(\neg N_\varepsilon(\vec{b}))(1 - \delta)^m. \end{aligned}$$

Also,

$$\int_{N_\tau(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i}\, d\mu(\vec{x}) \geq \int_{N_\tau(\vec{b})} \inf_{N_\tau(\vec{b})} \left\{ \prod_{i=1}^{2^q} x_i^{mb_i} \right\} d\mu(\vec{x})$$
$$= \inf_{N_\tau(\vec{b})} \left\{ \prod_{i=1}^{2^q} x_i^{mb_i} \right\} \mu(N_\tau(\vec{b})),$$

so

$$\int_{\neg N_\varepsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i+n_i}\, d\mu(\vec{x}) \leq \int_{N_\tau(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i}\, d\mu(\vec{x}) \frac{\mu(\neg N_\varepsilon(\vec{b}))}{\mu(N_\tau(\vec{b}))}(1-\delta)^m$$
$$\leq \int_{N_\varepsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i}\, d\mu(\vec{x}) \frac{\mu(\neg N_\varepsilon(\vec{b}))}{\mu(N_\tau(\vec{b}))}(1-\delta)^m. \tag{75}$$

Let $\underline{d_m}$ and $\overline{d_m}$ respectively be the minimum and maximum values of $\prod_{x=1}^{2^q} x_i^{n_i}$ for $\vec{x}$ from the closure of the set $N_\varepsilon(\vec{b})$. Then

$$\underline{d_m} \leq \frac{\int_{N_\varepsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i+n_i}\, d\mu(\vec{x})}{\int_{N_\varepsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i}\, d\mu(\vec{x})} \leq \overline{d_m}$$

so there is some constant, d_m say, such that

$$d_m = \frac{\int_{N_\varepsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i+n_i}\, d\mu(\vec{x})}{\int_{N_\varepsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i}\, d\mu(\vec{x})} = \prod_{i=1}^{2^q} e_i^{n_i}$$

for some $\vec{e} \in N_\varepsilon(\vec{b})$. By (72), $|\, d_m - \prod_{i=1}^{2^q} b_i^{n_i}| < \nu/2$.

Using this, (74) and (75) we see that for sufficiently large m,

$$I_m - \prod_{i=1}^{2^q} b_i^{n_i} \leq d_m + \frac{\mu(\neg N_\varepsilon(\vec{b}))}{\mu(N_\tau(\vec{b}))}(1-\delta)^m - \prod_{i=1}^{2^q} b_i^{n_i} < \nu.$$

Also for large m,

$$I_m - \prod_{i=1}^{2^q} b_i^{n_i} \geq \frac{d_m}{1 + \frac{\mu(\neg N_\varepsilon(\vec{b}))}{\mu(N_\tau(\vec{b}))}(1-\delta)^m} - \prod_{i=1}^{2^q} b_i^{n_i}$$
$$> \frac{\prod_{i=1}^{2^q} b_i^{n_i} - \nu/2}{1+\nu/2} - \prod_{i=1}^{2^q} b_i^{n_i}$$
$$\geq -\nu.$$

This completes the proof of (71). By taking the limit of the ratio of expressions as in (71) when the $p_i = r_i$ and when the $p_i = 0$ we now obtain as required that

$$\lim_{m\to\infty} \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[(m+s)b_i]+r_i}\, d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[(m+s)b_i]}\, d\mu(\vec{x})} = \prod_{i=1}^{2^q} b_i^{r_i}.$$ ⊣

COROLLARY 12.2. *Let w be a probability function on SL with de Finetti prior μ and let $\vec{b}$ be a support point of μ. Then there exist state descriptions $\Theta_m(a_1, \dots, a_{s_m})$ such that for any $r_1, \dots, r_{2^q} \in \mathbb{N}$,*[51]

$$\lim_{m\to\infty} w\Big(\bigwedge_{i=1}^{2^q} \alpha_i^{r_i} \mid \Theta_m(a_1, \dots, a_{s_m})\Big) = \prod_{i=1}^{2^q} b_i^{r_i}.$$

PROOF. Just take

$$\Theta_m(a_1, \dots, a_{s_n}) = \bigwedge_{i=1}^{2^q} \alpha_i^{[mb_i]},$$

so $s_m = \sum_i [mb_i]$. Then by de Finetti's Theorem 9.1

$$w\Big(\bigwedge_{i=1}^{2^q} \alpha_i^{r_i} \mid \Theta_m(a_1, \dots, a_{s_m})\Big) = \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[mb_i]+r_i}\, d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[mb_i]}\, d\mu(\vec{x})}$$

and the result follows from Lemma 12.1. ⊣

This corollary is a useful tool because it shows that for $\vec{b}$ in the support of the de Finetti prior of w we can, after suitable conditioning, make w act like $w_{\vec{b}}$. Put another way it enables us to use conditioning on state descriptions to pick out and investigate individual points in the support of a de Finetti prior. Indeed if we relax the requirement that the conditioning sentence be a state description we can in a sense pick out two points in the support of μ at the same time as the corollary to the next lemma shows. First however it will be useful to introduce a little notation. We say that $\vec{b} \in \mathbb{D}_{2^q}$ is an *extreme point* (of $\mathbb{D}_{2^q}$) if each coordinate of $\vec{b}$ except one is 0, the remaining coordinate must of course be 1. In what follows we will frequently use the fact that for an extreme point $\vec{b}$, $\prod_i b_i^{b_i} = 1$ whereas for a non-extreme point of $\mathbb{D}_{2^q}$, $\prod_i b_i^{b_i} < 1$.

LEMMA 12.3. *Let the probability function w on SL have de Finetti prior μ and suppose that $\vec{b}, \vec{c}$ are non-extreme points in the support of μ. Then there are increasing sequences k_n, j_n and $0 < \lambda < \infty$ such that*

$$\lim_{n\to\infty} \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[k_n b_i]}\, d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[j_n c_i]}\, d\mu(\vec{x})} = \lambda.$$

[51] Recall the notation introduced on page 53.

PROOF. Pick small $\varepsilon > 0$. From the proof of the previous lemma there is a $v > 0$ such that sufficiently large m

$$(1+v)\int_{N_\varepsilon(\vec{b})}\prod_{i=1}^{2^q} x_i^{[mb_i]}\,d\mu(\vec{x}) \geq \int_{\mathbb{D}_{2^q}}\prod_{i=1}^{2^q} x_i^{[mb_i]}\,d\mu(\vec{x}) \geq \int_{N_\varepsilon(\vec{b})}\prod_{i=1}^{2^q} x_i^{[mb_i]}\,d\mu(\vec{x}). \quad (76)$$

The sequence (in m)

$$\int_{N_\varepsilon(\vec{b})}\prod_{i=1}^{2^q} x_i^{[mb_i]}\,d\mu(\vec{x})$$

is decreasing to 0 (strictly for infinitely many m) since $\prod_{i=1}^{2^q} b_i^{b_i} < 1$. Indeed

$$\int_{N_\varepsilon(\vec{b})}\prod_{i=1}^{2^q} x_i^{[mb_i]}\,d\mu(\vec{x}) \geq \int_{N_\varepsilon(\vec{b})}\prod_{i=1}^{2^q} x_i^{[(m+1)b_i]}\,d\mu(\vec{x}) \geq \gamma\int_{N_\varepsilon(\vec{b})}\prod_{i=1}^{2^q} x_i^{[mb_i]}\,d\mu(\vec{x}) \quad (77)$$

where $\gamma > 0$ is at most the minimum of the function $\prod_{b_i \neq 0} x_i$ on the closure of $N_\varepsilon(\vec{b})$. Similarly for $\vec{c}$, so we may assume that this same γ works there too.

Using (77) we can now produce strictly increasing sequences $j_n, k_n \in \mathbb{N}$ such that

$$\int_{N_\varepsilon(\vec{b})}\prod_{i=1}^{2^q} x_i^{[j_n b_i]}\,d\mu(\vec{x}) \geq \int_{N_\varepsilon(\vec{c})}\prod_{i=1}^{2^q} x_i^{[k_n c_i]}\,d\mu(\vec{x}) \geq \gamma\int_{N_\varepsilon(\vec{b})}\prod_{i=1}^{2^q} x_i^{[j_n b_i]}\,d\mu(\vec{x}).$$

From this inequality we obtain that for all $n > 0$,

$$1 \leq \frac{\int_{N_\varepsilon(\vec{b})}\prod_{i=1}^{2^q} x_i^{[j_n b_i]}\,d\mu(\vec{x})}{\int_{N_\varepsilon(\vec{c})}\prod_{i=1}^{2^q} x_i^{[k_n c_i]}\,d\mu} \leq \gamma^{-1}$$

and with (76)

$$(1+v)^{-1} \leq \frac{\int_{\mathbb{D}_{2^q}}\prod_{i=1}^{2^q} x_i^{[j_n b_i]}\,d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}}\prod_{i=1}^{2^q} x_i^{[k_n c_i]}\,d\mu(\vec{x})} \leq \gamma^{-1}(1+v). \quad (78)$$

The sequence in (78) has a convergent subsequence, to λ say, and the lemma follows. ⊣

COROLLARY 12.4. *Let w be a probability function with de Finetti prior μ and let $\vec{b}, \vec{c}$ be distinct non-extreme support points of μ. Then there exists $\lambda > 0$ and state descriptions ϕ_n, ψ_n such that for any $r_1, \ldots, r_{2^q} \in \mathbb{N}$,*

$$\lim_{n\to\infty} w\Big(\bigwedge_{i=1}^{2^q} \alpha_i^{r_i} \mid \phi_n(a_1, \ldots, a_{s_n}) \vee \psi_n(a_1, \ldots, a_{t_n})\Big) = \lambda(1+\lambda)^{-1} \prod_{i=1}^{2^q} b_i^{r_i} + (1+\lambda)^{-1} \prod_{i=1}^{2^q} c_i^{r_i}.$$

PROOF. Let j_n, k_n, λ be as in Lemma 12.3 and $\phi_n(a_1, \ldots, a_{s_n})$ be the conjunction of $[j_n b_i]$ copies of $\alpha_i(x)$ for $i = 1, \ldots, 2^q$ instantiated by $a_1, \ldots, a_{s_n}$, so $s_n = \sum_{i=1}^{2^q}[j_n b_i]$. Similarly let $\psi_n(a_1, \ldots, a_{t_n})$ be the conjunction of $[k_n c_i]$ copies of $\alpha_i(x)$ for $i = 1, \ldots, 2^q$ instantiated by $a_1, \ldots, a_{t_n}$, so $t_n = \sum_{i=1}^{2^q}[k_n c_i]$. Let δ_n be such that

$$\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[j_n b_i]} \, d\mu(\vec{x}) = (1+\delta_n)\lambda \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[k_n c_i]} \, d\mu(\vec{x}),$$

so $\delta_n \to 0$ as $n \to \infty$.

Then

$$\begin{aligned}
w\Big(\bigwedge_{i=1}^{2^q} \alpha_i^{r_i} \mid \phi_n(a_1, \ldots, a_{s_n}) \vee \psi_n(a_1, \ldots, a_{t_n})\Big)
&= \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[j_n b_i]+r_i} \, d\mu(\vec{x}) + \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[k_n c_i]+r_i} \, d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[j_n b_i]} \, d\mu(\vec{x}) + \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[k_n c_i]} \, d\mu(\vec{x})} \\
&= \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[j_n b_i]+r_i} \, d\mu(\vec{x})}{(1+\lambda^{-1}(1+\delta_n)^{-1}) \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[j_n b_i]} \, d\mu(\vec{x})} \\
&\quad + \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[k_n c_i]+r_i} \, d\mu(\vec{x})}{(1+\lambda(1+\delta_n)) \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[k_n c_i]} \, d\mu(\vec{x})}.
\end{aligned}$$

Taking the limit as $n \to \infty$ now gives, by Lemma 12.1,

$$\lambda(1+\lambda)^{-1} \prod_{i=1}^{2^q} b_i^{r_i} + (1+\lambda)^{-1} \prod_{i=1}^{2^q} c_i^{r_i},$$

as required. $\dashv$

In the next chapter we will derive a significant improvement of this corollary where, rather than two points, we allow finitely many such points and furthermore we prescribe the convex combination.

Chapter 13

THE CONDITIONALIZATION THEOREM

If we think of conditioning as the procedure which our agent would adopt when receiving, or at least imagining receiving, information about the ambient structure s/he is inhabiting then a natural question the agent might consider is where such a process could lead them to. After all, being aware of where one might end up if one adopted a particular zero knowledge probability function might well make one reconsider that initial choice. In this chapter we shall derive a result which sheds some light on the possible destinations conditioning can access, at least in the limit.

On the way to proving this result we shall derive two other results, on conditioning and on approximation, which seem to be of some independent interest, as well as promising to have further applications. The first of these, the so called 'Pick 'n' Mix' Theorem, which we now turn to, is an improvement on Corollary 12.4 and in proving it we will assume a familiarity with the notation of Chapter 12.

THEOREM 13.1. *Let w be a probability function with de Finetti prior μ and let $\vec{b_1}, \vec{b_2}, \dots, \vec{b_k}$ be distinct non-extreme support points of μ. Let $\gamma_1, \gamma_2, \dots, \gamma_k > 0$, $\sum_{j=1}^{k} \gamma_j = 1$. Then there exist sentences $\theta_n(a_1, \dots, a_{m_n}) \in QFSL$ such that for any $r_1, \dots, r_{2^q} \in \mathbb{N}$,*

$$\lim_{n\to\infty} w\Big(\bigwedge_{i=1}^{2^q} \alpha_i^{r_i} \mid \theta_n(a_1, \dots, a_{m_n})\Big) = \sum_{j=1}^{k} \gamma_j \prod_{i=1}^{2^q} b_{j,i}^{r_i}.$$

PROOF. Since it is quite easy to see that Lemma 12.3 and Corollary 12.4 can both be generalized from 2 to k points we shall, to avoid a nightmare of notation, give the present proof for the case of $k = 2$ in such a way that it should be clear that it generalizes. In particular we shall take the two support points to be $\vec{b}, \vec{c}$ and adopt the notation from the proof of Corollary 12.4.

Let $\theta_1(a_1, \dots, a_{e_1}), \theta_2(a_1, \dots, a_{e_2}) \in QFSL$. Then for ξ the sentence

$$((\phi_n(a_1, \dots, a_{s_n}) \wedge \theta_1(a_{s_n+1}, \dots, a_{s_n+e_1})) \vee$$
$$(\psi_n(a_1, \dots, a_{t_n}) \wedge \theta_2(a_{t_n+1}, \dots, a_{t_n+e_2})),$$

$$w\Big(\bigwedge_{i=1}^{2^q}\alpha_i^{r_i} \mid \xi\Big) =$$
$$\frac{\int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\theta_1)\prod_{i=1}^{2^q} x_i^{[j_n b_i]+r_i}\,d\mu(\vec{x}) + \int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\theta_2)\prod_{i=1}^{2^q} x_i^{[k_n c_i]+r_i}\,d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\theta_1)\prod_{i=1}^{2^q} x_i^{[j_n b_i]}\,d\mu(\vec{x}) + \int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\theta_2)\prod_{i=1}^{2^q} x_i^{[k_n c_i]}\,d\mu(\vec{x})}$$

and by dividing all these terms by

$$\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[j_n b_i]}\,d\mu(\vec{x}) + \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[k_n c_i]}\,d\mu(\vec{x})$$

we see that as $n \to \infty$ this gives in the limit

$$\frac{w_{\vec{b}}(\theta_1)\prod_{i=1}^{2^q} b_i^{r_i} + \lambda w_{\vec{c}}(\theta_2)\prod_{i=1}^{2^q} c_i^{r_i}}{w_{\vec{b}}(\theta_1) + \lambda w_{\vec{c}}(\theta_2)} \tag{79}$$

Since the $b_i, c_i < 1$ it is easy to see that we can choose θ_1, θ_2 so that

$$\frac{w_{\vec{b}}(\theta_1)}{w_{\vec{b}}(\theta_1) + \lambda w_{\vec{c}}(\theta_2)}, \quad \frac{\lambda w_{\vec{c}}(\theta_2)}{w_{\vec{b}}(\theta_1) + \lambda w_{\vec{c}}(\theta_2)} \tag{80}$$

are close to γ_1, γ_2 respectively. The required result follows. ⊣

Notice that another, informal, way of expressing this theorem is to say that by conditioning we can make w look like $\sum_{j=1}^{k}\gamma_j w_{\vec{b}_j}$ on $QFSL$, provided the $\vec{b}_j$ are non-extreme points in the support of μ. In fact we shall now show that we can do rather better than that. We first need an approximation theorem which, apart from its applicability, is of some interest in its own right.

THEOREM 13.2. *Let w be a probability function on SL with de Finetti prior μ and let $\varepsilon > 0$. Then there are points $\vec{d}_j$, $j = 1, 2, \ldots, k$, in the support of μ and $\gamma_j \geq 0$ with $\sum_j \gamma_j = 1$ such that for $v = \sum_j \gamma_j w_{\vec{d}_j}$,*

$$|w(\theta(a_1, \ldots, a_n)) - v(\theta(a_1, \ldots, a_n))| < n2^{nq}\varepsilon \textit{ for all } \theta(a_1, \ldots, a_n) \in SL.$$

PROOF. Recalling the representation

$$w = \sum_{T \in U} w(\eta_T) w_T$$

given in (64), where

$$\eta_T = \bigwedge_{j \in T} \exists x\, \alpha_j(x) \wedge \bigwedge_{j \notin T} \neg\exists x\, \alpha_j(x), \qquad U = \{T \mid w(\eta_T) > 0\},$$

it is enough to prove the result for w_T under the assumption that $w(\eta_T) > 0$. Obviously we have this immediately if $|T| = 1$ since in that case w_T is already of the form $w_{\vec{d}}$, so assume that $|T| > 1$. As in Chapter 10 set

$$N_T = \{\, \vec{c} \in \mathbb{D}_{2^q} \mid c_i > 0 \iff i \in T \,\},$$

and set μ_T to be the de Finetti prior of w_T for $T \in U$, so $\mu(N_T) = w(\eta_T) > 0$ and $\mu_T(N_T) = w_T(\eta_T) = 1$.

For $T \in U$ let $S_T \subseteq N_T$ be the set of support points of μ_T, let $\varepsilon > 0$ and pick finitely many points $\vec{d_j} \in S_T$ and pairwise disjoint Borel subsets B_j of $N_\varepsilon(\vec{d_j}) \cap N_T$ such that these B_j cover S_T. Set

$$v_T = \mu(N_T)^{-1} \sum_j \mu(B_j) w_{\vec{d_j}}.$$

It is enough to show that for $\theta(a_1, \ldots, a_n) \in SL$

$$|w_T(\theta(a_1, \ldots, a_n)) - v_T(\theta(a_1, \ldots, a_n))| < n2^{nq}\varepsilon. \tag{81}$$

Notice that $w_T(\eta_T) = v_T(\eta_T) = 1$, $w_T(\theta) = w_T(\theta \wedge \eta_T)$, $v_T(\theta) = v_T(\theta \wedge \eta_T)$, for any $\theta \in SL$ and as a result it suffices, given the representation (59), simply to show that for a state description $\bigwedge_{i=1}^n \alpha_{h_i}(a_i)$ with each $h_i \in T$,

$$\Big| w_T\Big(\bigwedge_{i=1}^n \alpha_{h_i}(a_i)\Big) - v_T\Big(\bigwedge_{i=1}^n \alpha_{h_i}(a_i)\Big)\Big| < n\varepsilon.$$

For B a Borel subset of $\mathbb{D}_{2^q}$, $\mu_T(B) = \mu(B \cap N_T)\mu(N_T)^{-1}$. In particular $\mu_T(B_j) = \mu(B_j)\mu(N_T)^{-1}$. Hence

$$v_T = \sum_j \mu_T(B_j) w_{\vec{d_j}}$$

and

$$\begin{aligned}
&\Big| w_T\Big(\bigwedge_{i=1}^n \alpha_{h_i}(a_i)\Big) - v_T\Big(\bigwedge_{i=1}^n \alpha_{h_i}(a_i)\Big)\Big| \\
&\qquad = \Big| \int_{\mathbb{D}_T} \prod_{i=1}^n x_{h_i}\, d\mu_T - \sum_j \mu_T(B_j) \prod_{i=1}^n d_{j,h_i} \Big| \\
&\qquad = \Big| \sum_j \int_{B_j} \Big(\prod_{i=1}^n x_{h_i} - \prod_{i=1}^n d_{j,h_i} \Big)\, d\mu_T \Big| \\
&\qquad \le \sum_j \int_{B_j} \Big| \prod_{i=1}^n x_{h_i} - \prod_{i=1}^n d_{j,h_i} \Big|\, d\mu_T \\
&\qquad \le \sum_j \int_{B_j} n\varepsilon\, d\mu_T = n\varepsilon
\end{aligned}$$

since for $\vec{x} \in B_j$,

$$\Big|\prod_{i=1}^{n} x_{h_i} - \prod_{i=1}^{n} d_{j,h_i}\Big| = \Big|\sum_{i=1}^{n}(x_{h_i} - d_{j,h_i})\prod_{k\leq i} x_{h_k}\prod_{k>i} d_{j,h_k}\Big|$$
$$\leq \sum_{i=1}^{n} |x_{h_i} - d_{j,h_i}| < n\varepsilon$$

The approximation (81) now follows. ⊣

The following corollary to Theorems 13.1 and 13.2, the 'Conditionalization Theorem', gives conditions under which updating by standard conditioning can take us from a probability function as close as we want to any other probability function *on quantifier free sentences involving only a preassigned number of constants.*

COROLLARY 13.3. *Let w_1, w_2 be probability functions on SL with de Finetti priors μ_1, μ_2 respectively such that the support set of μ_1 is $\mathbb{D}_{2^q}$ and $\mu_1(\{\vec{b}\}) = 0$ for any extreme point $\vec{b}$ of $\mathbb{D}_{2^q}$. Then given n and $\nu > 0$ there is a $\theta(a_1, \ldots, a_m) \in QFSL$ such that for each $\phi(a_1, \ldots, a_n) \in QFSL$,*

$$|w_1(\phi(a_{m+1}, a_{m+2}, \ldots, a_{m+n}) \mid \theta(a_1, \ldots, a_m)) - w_2(\phi(a_1, \ldots, a_n))| < \nu. \tag{82}$$

PROOF. Let $\vec{d_j}, \gamma_j$ be as in Theorem 13.2 for $w = w_2$ and $\varepsilon = 3^{-1}\nu n^{-1}2^{-nq}$. Now pick non-extreme points $\vec{b_j} \in \mathbb{D}_{2^q}$ so close to $\vec{d_j}$ for $j = 1, 2, \ldots, k$ that for any $\phi(a_1, \ldots, a_n) \in QFSL$,

$$\Big|\sum_{j=1}^{k}\gamma_j w_{\vec{b_j}}(\phi(a_1, \ldots, a_n)) - \sum_{j=1}^{k}\gamma_j w_{\vec{d_j}}(\phi(a_1, \ldots, a_n))\Big| < \nu/3. \tag{83}$$

By Theorem 13.1 applied to w_1, there is a sentence $\theta(a_1, \ldots, a_m)$ such that for all state descriptions $\Phi(a_1, \ldots, a_n)$,

$$\Big|w_1(\Phi(a_{m+1}, \ldots, a_{m+n}) \mid \theta(a_1, \ldots, a_m)) - \sum_{j=1}^{k}\gamma_j w_{\vec{b_j}}(\Phi(a_1, \ldots, a_n))\Big| < 3^{-1}\nu 2^{-nq},$$

from which (82) with bound ν and any $\phi(a_1, \ldots, a_n) \in QFSL$ follows. ⊣

Corollary 13.3 fails if $\mu_1(\{\vec{b}\}) > 0$ for some extreme point $\vec{b}$. For example for the probability functions c_λ^L, shortly to be defined in (103), if

$$w_1 = 4^{-1}(3c_0^L + c_\infty^L), \quad w_2 = 4^{-1}(c_0^L + 3c_\infty^L)$$

then for any $\theta(a_1, \ldots, a_{m_n})$, $w_1(\alpha_1^2 \mid \theta)$ is at most 2^{-2q} if $c_0^L(\alpha_1^2 \wedge \theta) = 0$ and at least $(2^{-2q} + 3 \cdot 2^{-q})(1 + 3 \cdot 2^{-q})^{-1}$ otherwise. On the other hand

$$w_2(\alpha_1^2) = 4^{-1}(3 \cdot 2^{-2q} + 2^{-q})$$

falls strictly between these two values.

The corollary also cannot be extended to $\phi(a_1, \ldots, a_n) \in SL$, as can be seen by taking $w_1 = c_2^L$, $w_2 = c_0^L$ and $\phi = \exists x\, R(x)$, where L is the language with a single unary relation symbol R.

Chapter 14

ATOM EXCHANGEABILITY

In a similar fashion to the justification for Ex, Px etc., i.e. that it would be irrational (in a situation of zero knowledge) to attribute properties to any one particular constant that one was not equally willing to ascribe to any other constant etc., we might argue that as far as our zero knowledge agent is concerned the atoms simply determine a partition of the a_i and that there is no reason to treat any one atom differently from any other.[52] This leads to:

The Atom Exchangeability Principle, Ax.
For any permutation τ of $\{1, 2, \ldots, 2^q\}$ and constants $b_1, b_2, \ldots, b_m$,

$$w\Big(\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i)\Big) = w\Big(\bigwedge_{i=1}^{m} \alpha_{\tau(h_i)}(b_i)\Big). \tag{84}$$

Equivalently, in the presence of Ex[53], Ax asserts that the left hand side of (84) depends only on the *spectrum* of the state description $\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i)$, that is on the *multiset* $\{m_1, m_2, \ldots, m_{2^q}\}$, where, as in Ex, $m_j = |\{i \mid h_i = j\}|$. Note that the spectrum is a multiset, not a vector so it does not carry the information about which atom corresponds to which entry. We usually leave out the zero entries so spectra are multisets $\{n_1, n_2, \ldots, n_k\}$ of strictly positive natural numbers with $\sum_{i=1}^{k} n_i = m$.

Just as with the principles Ex, Px and SN, it is straightforward to check that if the probability function w on SL satisfies Ax then so will its restriction $w \restriction SL'$ to any sublanguage L' of L.

It is worth observing that the principle SN alone forces $w(\alpha_i(a_k)) = w(\alpha_j(a_k))$ for any i, j, since we can simply change the signs of the

[52] In spirit this is close to Carnap's *attribute symmetry*, see [17, p77], when there is only a single family, the role occupied in our account by *atoms* being taken there by disjoint properties within that family. For example 'colours' within the family 'colour' where the idea that in the zero knowledge situation the names of colours should be entirely interchangeable seems to be one which most of us would probably find easy to accommodate. Strictly speaking however Carnap's attribute symmetry when there are multiple families is a generalization of Strong Negation, SN.

[53] We remark that the statement '*the left hand side of* (84) *depends only on the spectrum of the state description* $\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i)$' implies Ex, but Ax as formulated above does not.

individual $\pm R_n$ in $\alpha_i(x)$ to convert it to $\alpha_j(x)$, and hence all must be 2^{-q} since

$$1 = w(\top) = w\Big(\bigwedge_{i=1}^{2^q} \alpha_i(a_k)\Big) = \sum_{i=1}^{2^q} w(\alpha_i(a_k)).$$

However that trick only works when we consider single atoms. Once we take a conjunction of even two atoms we may not be able to convert $\alpha_i(x_1) \wedge \alpha_j(x_2)$ into $\alpha_{i'}(x_1) \wedge \alpha_{j'}(x_2)$ for $i \neq j$, $i' \neq j'$, even with the additional help of Px. For example[54] when $q = 2$ and using the lexicographic enumeration of atoms (see page 50) it is easy to check that it cannot be done for

$$(R_1(x_1) \wedge R_2(x_1)) \wedge (\neg R_1(x_2) \wedge \neg R_2(x_2))),$$
$$(R_1(x_1) \wedge R_2(x_1)) \wedge (R_1(x_2) \wedge \neg R_2(x_2))).$$

Indeed if we take $0 < c < \frac{1}{2}$ and define w to be

$$4^{-1}(w_{\langle c^2, c(1-c), c(1-c), (1-c)^2\rangle} + w_{\langle c(1-c), c^2, (1-c)^2, c(1-c)\rangle} + \\ w_{\langle c(1-c), (1-c)^2, c^2, c(1-c)\rangle} + w_{\langle (1-c)^2, c(1-c), c(1-c), c^2\rangle}) \quad (85)$$

then w satisfies SN and Px but not Ax since

$$w((R_1(a_1) \wedge R_2(a_1)) \wedge (\neg R_1(a_2) \wedge \neg R_2(a_2))) = c^2(1-c)^2 \neq \\ 2^{-1}(c^3(1-c)+c(1-c)^3) = w(R_1(a_1) \wedge R_2(a_1)) \wedge (R_1(a_2) \wedge \neg R_2(a_2))).$$

With the exception of the chapter on 'Analogy' most of the probability functions we will encounter in this part will satisfy Ax.

Our immediate task now will be to give a simple variant of de Finetti's Representation Theorem which will enable us to characterize the probability functions satisfying Ax. First a little notation. For $\langle c_1, c_2, \ldots, c_{2^q}\rangle \in \mathbb{D}_{2^q}$ define the probability function $v_{\vec{c}}$ by

$$v_{\vec{c}} = |\mathsf{S}_{2^q}|^{-1} \sum_{\sigma \in \mathsf{S}_{2^q}} w_{\langle c_{\sigma(1)}, c_{\sigma(2)}, \ldots, c_{\sigma(2^q)}\rangle}. \quad (86)$$

In other words $v_{\vec{c}}$ is the uniform mixture of the $w_{\langle c_{\sigma(1)}, c_{\sigma(2)}, \ldots, c_{\sigma(2^q)}\rangle}$ as the σ varies over the set S_{2^q} of permutations of $\{1, 2, \ldots, 2^q\}$.

This averaging of the $w_{\langle c_{\sigma(1)}, c_{\sigma(2)}, \ldots, c_{\sigma(2^q)}\rangle}$ is a simple trick to ensure that $v_{\vec{c}}$ satisfies Ax since,

$$v_{\vec{c}}\Big(\bigwedge_{i=1}^{m} \alpha_{\tau(h_i)}(b_i)\Big) = |\mathsf{S}_{2^q}|^{-1} \sum_{\sigma \in \mathsf{S}_{2^q}} w_{\langle c_{\sigma(1)}, c_{\sigma(2)}, \ldots, c_{\sigma(2^q)}\rangle}\Big(\bigwedge_{i=1}^{m} \alpha_{\tau(h_i)}(b_i)\Big)$$
$$= |\mathsf{S}_{2^q}|^{-1} \sum_{\sigma \in \mathsf{S}_{2^q}} \prod_{i=1}^{m} c_{\sigma(\tau(h_i))}$$

[54]This example clearly generalizes to any value of q.

$$= |\mathsf{S}_{2^q}|^{-1} \sum_{\sigma\tau \in \mathsf{S}_{2^q}} \prod_{i=1}^{m} c_{\sigma(\tau(h_i))},$$

since $\sigma \mapsto \sigma\tau$ just permutes S_{2^q},

$$= |\mathsf{S}_{2^q}|^{-1} \sum_{\sigma \in \mathsf{S}_{2^q}} \prod_{i=1}^{m} c_{\sigma(h_i)},$$

$$= v_{\vec{c}}\Big(\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i) \Big).$$

Furthermore any probability function satisfying Ax is a mixture of these $v_{\vec{c}}$ as we can now show.

THEOREM 14.1. *Let L be a unary language with q relation symbols and let w be a probability function on SL satisfying Ax (and Ex). Then there is a measure μ on the Borel subsets of $\mathbb{D}_{2^q}$ such that*

$$w = \int_{\mathbb{D}_{2^q}} v_{\vec{x}}\, d\mu(\vec{x}). \tag{87}$$

Conversely, given a measure μ on the Borel subsets of $\mathbb{D}_{2^q}$ the probability function w on SL defined by (87) *satisfies Ax* (*and Ex*).

PROOF. First suppose that w satisfies Ax. From de Finetti's Representation Theorem we can see that there is a measure μ such that for a state description $\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i)$ and $\sigma \in \mathsf{S}_{2^q}$,

$$\begin{aligned} w\Big(\bigwedge_{i=1}^{m} \alpha_{\sigma(h_i)}(b_i) \Big) &= \int_{\mathbb{D}_{2^q}} w_{\langle x_1, x_2, \ldots, x_{2^q} \rangle} \Big(\bigwedge_{i=1}^{m} \alpha_{\sigma(h_i)}(b_i) \Big)\, d\mu(\vec{x}) \\ &= \int_{\mathbb{D}_{2^q}} w_{\langle x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(2^q)} \rangle} \Big(\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i) \Big)\, d\mu(\vec{x}) \end{aligned} \tag{88}$$

But since w satisfies Ax the left hand side of (88) is the same for any $\sigma \in \mathsf{S}_{2^q}$ so averaging both sides of (88) over all $\sigma \in \mathsf{S}_{2^q}$ gives (87) when we restrict w and $v_{\vec{x}}$ to state descriptions, and the general version follows as in the Corollary 9.2 to de Finetti's Theorem. (Note that the identity (87) which we have actually proved only for arguments which are state descriptions automatically holds for all sentences from SL using the observation on page 58.)

The converse result is straightforward. ⊣

In this Representation Theorem we have chosen to capture the symmetry of Ax at the level of the $v_{\vec{c}}$. This seems a natural thing to do since it means we are representing every probability function satisfying Ax as a mixture of very simple probability functions satisfying Ax. However we could alternatively (or additionally) have captured the symmetry within the measure μ. Namely we arrive as above at

$$
\begin{aligned}
w\Big(\bigwedge_{i=1}^{m}\alpha_{h_i}(b_i)\Big) &= w\Big(\bigwedge_{i=1}^{m}\alpha_{\sigma(h_i)}(b_i)\Big)\\
&= \int_{\mathbb{D}_{2^q}} w_{\langle x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(2^q)}\rangle}\Big(\bigwedge_{i=1}^{m}\alpha_{h_i}(b_i)\Big)\,d\mu(\langle x_1,x_2,\dots,x_{2^q}\rangle)\\
&= \int_{\mathbb{D}_{2^q}} w_{\langle x_1,x_2,\dots,x_{2^q}\rangle}\Big(\bigwedge_{i=1}^{m}\alpha_{h_i}(b_i)\Big)\,d\mu(\langle x_{\sigma^{-1}(1)},x_{\sigma^{-1}(2)},\dots,x_{\sigma^{-1}(2^q)}\rangle)\\
&= \int_{\mathbb{D}_{2^q}} w_{\langle x_1,x_2,\dots,x_{2^q}\rangle}\Big(\bigwedge_{i=1}^{m}\alpha_{h_i}(b_i)\Big)\,d(\sigma\mu)(\langle x_1,x_2,\dots,x_{2^q}\rangle) \qquad (89)
\end{aligned}
$$

where for a Borel subset A of $\mathbb{D}_{2^q}$,

$$\sigma\mu(A) = \mu\{\langle x_{\sigma^{-1}(1)},x_{\sigma^{-1}(2)},\dots,x_{\sigma^{-1}(2^q)}\rangle \mid \langle x_1,x_2,\dots,x_{2^q}\rangle \in A\}.$$

Averaging both sides of (89) over the $\sigma \in \mathsf{S}_{2^q}$ gives by the same reasoning as applied before

$$w = \int_{\mathbb{D}_{2^q}} w_{\vec{x}}\,d\nu(\vec{x})$$

where now ν is a measure on $\mathbb{D}_{2^q}$ invariant under permutations of the 2^q coordinates.[55] From now on *we will always assume without further mention that the de Finetti prior of a probability function satisfying Ax which we use is invariant under permutations of the coordinates*.

Apart from the above representation theorem for probability functions satisfying Ax we shall later meet, in Theorem 34.1, another representation result for such functions of a rather different form.

As a final passing comment in this chapter, related to the supposed 'isomorphism' in Goodman's 'grue' paradox (see footnote on page 5), *if* $R_1(a_i)$ were to be interpreted as 'emerald a_i is green' (as opposed to blue) and $R_2(a_i)$ as 'emerald a_i is seen before the end of the month (as opposed to after that time) then the two probabilities in this paradox would have the forms

$$w\Big(R_1(a_{n+1}) \wedge \neg R_2(a_{n+1}) \;\Big|\; \bigwedge_{i=1}^{n} R_1(a_i) \wedge R_2(a_i)\Big),$$

$$w\Big(\neg R_1(a_{n+1}) \wedge \neg R_2(a_{n+1}) \;\Big|\; \bigwedge_{i=1}^{n} R_1(a_i) \wedge R_2(a_i)\Big),$$

and under the assumption of Ax (just transpose the atoms $R_1(x) \wedge \neg R_2(x)$ and $\neg R_1(x) \wedge \neg R_2(x)$) these would be equal.[56] On the other hand an

[55] Actually we could have proved that μ must have had that property from the start but it is enough for our purpose to notice as above that we can always choose it with that property.

[56] But compare this situation with the Analogy Principle on page 166 where we drop Ax.

obvious conclusion to be drawn from the evidence $\bigwedge_{i=1}^{n} R_1(a_i) \wedge R_2(a_i)$ is $R_2(a_{n+1})$, in other words that seeing any sort of emerald, green or not, after this month's end looks rather unlikely!!

Chapter 15

REICHENBACH'S AXIOM

The following principle has been attributed to Hans Reichenbach after a suggestion by Hilary Putnam, see [17, p120]:

REICHENBACH'S AXIOM, RA.

Let $\alpha_{h_i}(x)$ for $i = 1, 2, 3, \ldots$ be an infinite sequence of atoms of L. Then for $\alpha_j(x)$ an atom of L,

$$\lim_{n \to \infty} \left(w\left(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right) - \frac{u_j(n)}{n} \right) = 0 \tag{90}$$

where $u_j(n) = |\{i \mid 1 \le i \le n \text{ and } h_i = j\}|$.

Informally then this principle asserts that as information of the atoms satisfied by the $a_1, a_2, \ldots, a_n, \ldots$ grows so w should treat this information like a *statistical sample* giving a value to the probability that the next, $n+1$'st, case revealed will be $\alpha_j(a_{n+1})$ which gets arbitrarily close to the frequency of past instances of $\alpha_j(a_i)$.

If our rational agent is something of a statistician this would surely seem like common sense advice. However this principle says nothing about the ultimate convergence of the $u_j(n)/n$, an assumption which seems to be somehow implicit in any statistical viewpoint our agent might take.

The following theorem which was stated by Gaifman in [16] though its proof, which it was said would be given in [17], never appeared (due to the inordinate time lag between these two volumes and Gaifman developing new interests in the meantime).

THEOREM 15.1. *Let w satisfy Reg. Then w satisfies RA if and only if every point in $\mathbb{D}_{2^q}$ is a support point of the de Finetti prior μ of w.*

PROOF. First assume that every point in $\mathbb{D}_{2^q}$ is a support point of μ. By de Finetti's Theorem it is enough to show that if n is large and $m_1, m_2, \ldots, m_{2^q} \in \mathbb{N}$ with sum n then

$$\frac{\int_{\mathbb{D}_{2^q}} (x_j - m_j/n) \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu} \tag{91}$$

is close to zero. We first need to introduce some notation and derive a number of estimates.

For small $\delta > 0$ set

$$E_\delta = \{\vec{x} \in \mathbb{D}_{2^q} \mid x_i \geq \delta, i = 1, 2, \dots, 2^q\},$$
$$E_\delta(\vec{c}) = \{\vec{x} \in N_{\delta/2}(\vec{c}) \mid \exists\, \vec{y} \in E_\delta\, \exists \lambda \in [0, 1], \vec{x} = \lambda\vec{c} + (1 - \lambda)\vec{y}\}.$$

Notice that for every point $\vec{d} \in E_\delta(\vec{c})$ and $i = 1, 2, \dots, 2^q$, if $c_i < \delta$ then $c_i \leq d_i$. Also there is a fixed $\xi > 0$ such that for each $\vec{c} \in \mathbb{D}_{2^q}$ there is a $\vec{d} \in E_\delta(\vec{c})$ such that $E_\delta(\vec{c})$ contains the neighbourhood $N_\xi(\vec{d})$.[57] Hence there is some $\zeta > 0$ such that

$$\forall \vec{c} \in \mathbb{D}_{2^q}\ \ \mu(E_\delta(\vec{c})) \geq \zeta, \tag{92}$$

since if not we could find a sequence of points $\vec{c}^{\,k} \in \mathbb{D}_{2^q}$ with limit point $\vec{c}$ such that $\mu(N_\xi(\vec{c}^{\,k})) \to 0$ whilst, by the assumption on the support points of μ, $\mu(N_{\xi/2}(\vec{c})) > 0$ with $N_{\xi/2}(\vec{c})) \subseteq N_\xi(\vec{c}^{\,k}))$ for k large enough.

For $\vec{d} \in E_\delta(\vec{c})$ we have that

$$\begin{aligned}&\sum_{i=1}^{2^q}(c_i \log(c_i) - c_i \log(d_i)) = \\ &-\sum_{c_i \geq \delta} c_i \log(1 + (d_i - c_i)c_i^{-1}) + \sum_{c_i < \delta} c_i \log(c_i) - c_i \log(d_i) \leq 2^{q+1}\sqrt{\delta}\end{aligned} \tag{93}$$

since if $c_i < \delta$ then $c_i \leq d_i$ and $c_i \log(c_i) - c_i \log(d_i) \leq 0$ whilst for $\delta \leq c_i$, in view of $|d_i - c_i| < \delta/2$,

$$-c_i \log(1 + (d_i - c_i)c_i^{-1}) \leq c_i \log(2),$$

which is less or equal to $\sqrt{\delta}\log(2) \leq 2\sqrt{\delta}$ in the case of $c_i < \sqrt{\delta}$, and when $c_i \geq \sqrt{\delta}$

$$-c_i \log(1 + (d_i - c_i)c_i^{-1}) \leq -c_i \log(1 - \sqrt{\delta}/2) \leq \frac{c_i\sqrt{\delta}/2}{1 - \sqrt{\delta}/2} \leq 2\sqrt{\delta}.$$

From (93) we now have that for $\vec{d} \in E_\delta(\vec{c})$,

$$\prod_{i=1}^{2^q} d_i^{c_i} \geq e^{-2^{q+1}\sqrt{\delta}} \prod_{i=1}^{2^q} c_i^{c_i}. \tag{94}$$

We now claim that for small $\varepsilon > 0$ there exists $\tau > 0$ such that whenever $\vec{c}, \vec{d} \in \mathbb{D}_{2^q}$ and $|\vec{d} - \vec{c}| \geq \varepsilon$ then

$$\sum_{i=1}^{2^q}(c_i \log(c_i) - c_i \log(d_i)) \geq \tau.$$

[57] It can be checked that a suitable choice, for δ small, is $\xi = 2^{-q-3}\delta$ when $\vec{d}$ is given by $d_i = c_i - (2^q - 1)2^{-q-2}\delta$ for some i for which $c_i = \max\{c_j \mid j = 1, 2, \dots, 2^q\}$ and $d_j = c_j + 2^{-q-2}\delta$ for the remaining $2^q - 1$ coordinates.

For if not, then since $\sum_i c_i \log(x_i)$ takes its strict maximum on $\mathbb{D}_{2^q}$ at $\vec{x} = \vec{c}$, there would be $\vec{c}, \vec{d}, \vec{c}^k, \vec{d}^k \in \mathbb{D}_{2^q}$ such that $|\vec{d}^k - \vec{c}^k| \geq \varepsilon$ for each k, $\vec{c}^k \to \vec{c}$, $\vec{d}^k \to \vec{d}$ but

$$\sum_{i=1}^{2^q} (c_i^k \log(c_i^k) - c_i^k \log(d_i^k)) \searrow 0.$$

In this case $|\vec{d} - \vec{c}| \geq \varepsilon$ but

$$\sum_{i=1}^{2^q} c_i \log(c_i) = \sum_{i=1}^{2^q} c_i \log(d_i),$$

contradiction. It follows that the required τ exists and we can conclude that

$$\prod_{i=1}^{2^q} d_i^{c_i} \leq e^{-\tau} \prod_{i=1}^{2^q} c_i^{c_i} \tag{95}$$

whenever $\vec{c}, \vec{d} \in \mathbb{D}_{2^q}$, $|\vec{d} - \vec{c}| \geq \varepsilon$.

We now return to the proof that (91) is close to zero. Given small $\varepsilon > 0$ let $\tau > 0$ be as in (95). Now pick small $\delta > 0$ such that

$$2^{q+1}\sqrt{\delta} < \tau, \varepsilon. \tag{96}$$

Then putting $c_j = m_j/n$ for $j = 1, 2, \ldots, 2^q$,

$$\begin{aligned}
&\frac{\int_{\mathbb{D}_{2^q}} (x_j - c_j) \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu} = \\
&\qquad = \frac{\int_{N_\varepsilon(\vec{c})} (x_j - c_j) \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu}{\int_{N_\varepsilon(\vec{c})} \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu + \int_{\neg N_\varepsilon(\vec{c})} \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu} && (97)\\
&\qquad\quad + \frac{\int_{\neg N_\varepsilon(\vec{c})} (x_j - c_j) \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu}{\int_{N_\varepsilon(\vec{c})} \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu + \int_{\neg N_\varepsilon(\vec{c})} \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu} && (98)
\end{aligned}$$

Concerning (97) we have that

$$\begin{aligned}
\frac{\int_{\neg N_\varepsilon(\vec{c})} \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu}{\int_{N_\varepsilon(\vec{c})} \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu} &\leq \frac{\int_{\neg N_\varepsilon(\vec{c})} \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu}{\int_{E_\delta(\vec{c})} \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu} \\
&\leq \frac{\int_{\neg N_\varepsilon(\vec{c})} e^{-n\tau} \prod_{i=1}^{2^q} c_i^{nc_i} \, d\mu}{\int_{E_\delta(\vec{c})} e^{-n2^{q+1}\sqrt{\delta}} \prod_{i=1}^{2^q} c_i^{nc_i} \, d\mu} \qquad \text{by (94), (95),} \\
&\leq \frac{e^{-n\tau}}{\zeta e^{-n2^{q+1}\sqrt{\delta}}}. && (99)
\end{aligned}$$

which by (96) is small for large n. Hence (97) is close to

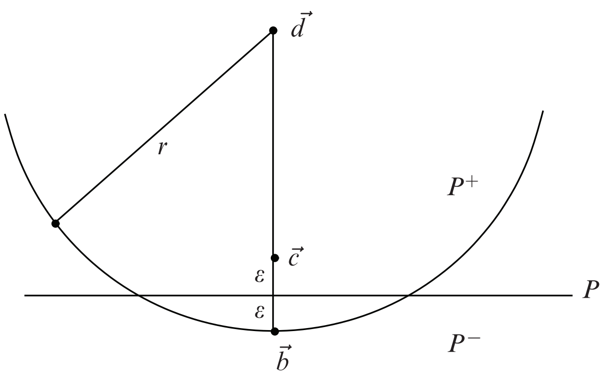
$$\frac{\int_{N_\varepsilon(\vec{c})}(x_j - c_j)\prod_{i=1}^{2^q} x_i^{m_i}\, d\mu}{\int_{N_\varepsilon(\vec{c})}\prod_{i=1}^{2^q} x_i^{m_i}\, d\mu}$$

which is between $-\varepsilon$ and ε since $|x_j - c_j| < \varepsilon$ over $N_\varepsilon(\vec{c})$.

Clearly the inequalities already given in (99) also show that (98) is small for large n and the required result follows.

Turning to the other direction of the theorem suppose that w satisfies Reg and not every point of $\mathbb{D}_{2^q}$ is a support point of the de Finetti prior μ of w. We shall sketch a proof that in this case RA fails in general, even when the sequence $u_j(n)/n$ converges.

Since the set of non-support points of μ form an open set and w satisfies Reg we can find points $\vec{b}, \vec{d} \in \mathbb{D}_{2^q}$ with no zero coordinates with $\vec{b}$ a support point of μ and $\vec{d}$ a non-support point. By considering points on the line joining $\vec{b}, \vec{d}$ we may assume that $\vec{b}$ is close to $\vec{d}$ and, by considering a nearest support point to $\vec{d}$ and then moving a distance in its direction if necessary, that no support point is as close (or closer) to $\vec{d}$ than $\vec{b}$. Let $r = |\vec{b} - \vec{d}| < s/2$ where $s = \min\{b_i, d_i \mid i = 1, 2, \ldots, 2^q\}$ and let ε be small. In the diagram below let $\vec{c}$ be on the line joining $\vec{b}, \vec{d}$ distance 2ε from $\vec{b}$, let the plane P be normal to this line distance ε from $\vec{b}$ and let P^+ be the region on the same side of P as $\vec{c}$, P^- its complement. Note that $2r < s \leq c_i$ for each i.



Then

$$\sum_{i=1}^{2^q}(c_i \log(c_i) - c_i \log(b_i))$$
$$= \sum_{i=1}^{2^q} -c_i \log\left(1 + \frac{(b_i - c_i)}{c_i}\right)$$

$$= \sum_{i=1}^{2^q} -(b_i - c_i) + \frac{(b_i - c_i)^2}{2c_i} + O(\varepsilon^3)$$

$$= \sum_{i=1}^{2^q} \frac{(b_i - c_i)^2}{2c_i} + O(\varepsilon^3) \leq 3s^{-1}\varepsilon^2 \tag{100}$$

since $\sum_{i=1}^{2^q} b_i = \sum_{i=1}^{2^q} c_i = 1$.

On the other hand let $\vec{x} \in P^+$ with $|\vec{x} - \vec{d}| \geq r$ and suppose for the moment that $|x_i - c_i| < c_i$ for each i. The distance from $\vec{x}$ to $\vec{c}$ must be at least $\sqrt{2r\varepsilon}$ so

$$\sum_{i=1}^{2^q} (c_i \log(c_i) - c_i \log(x_i))$$

$$= \sum_{i=1}^{2^q} -c_i \log\left(1 + \frac{(x_i - c_i)}{c_i}\right)$$

$$= \sum_{i=1}^{2^q} -(x_i - c_i) + \frac{(x_i - c_i)^2}{2c_i} - \frac{(x_i - c_i)^3}{3c_i^2} + \cdots$$

$$= \sum_{i=1}^{2^q} \frac{(x_i - c_i)^2}{2c_i} - \frac{(x_i - c_i)^3}{3c_i^2} + \cdots$$

$$\geq \sum_{i=1}^{2^q} \frac{(x_i - c_i)^2}{8c_i} \geq 2^{-2} r s^{-1} \varepsilon. \tag{101}$$

Furthermore the inequality (101) also holds for any $\vec{x} \in P^+$ with $|\vec{x} - \vec{d}| \geq r$ since the function $\sum_{i=1}^{2^q} (c_i \log(c_i) - c_i \log(x_i))$ is increasing along straight lines emanating from $\vec{c}$. As in the first half of this proof, using (100), (101) it now follows that if

$$\left\langle \frac{u_1(n)}{n}, \frac{u_2(n)}{n}, \ldots, \frac{u_{2^q}(n)}{n} \right\rangle \to \vec{c}$$

as $n \to \infty$ then

$$\frac{\int_{\mathbb{D}_{2^q}} x_j \prod_{i=1}^{2^q} x_i^{u_i(n)}\, d\mu}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{u_i(n)}\, d\mu} - \frac{\int_{P^-} x_j \prod_{i=1}^{2^q} x_i^{u_i(n)}\, d\mu}{\int_{P^-} \prod_{i=1}^{2^q} x_i^{u_i(n)}\, d\mu} \tag{102}$$

tends to 0 as $n \to \infty$. But even assuming the limit e_j of the left hand side of (102) exists for $j = 1, 2, \ldots, 2^q$ then, because of the equality with the right hand side of (102), would have to have $\vec{e} \in P^-$ so $\vec{e} \neq \vec{c}$. Either way RA fails, as required. ⊣

In fact the forward direction of the above proof has shown an ostensibly stronger result, that under the given assumptions RA holds *uniformly*. Precisely:

COROLLARY 15.2. *Let w satisfy Reg and suppose that every point in $\mathbb{D}_{2^q}$ is a support point of the de Finetti Prior μ of w. Then for $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that for any sequence $\alpha_{h_i}(x)$ of atoms of L and $n \geq k$*

$$\left| w\left(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right) - \frac{u_j(n)}{n} \right| < \varepsilon,$$

where $u_j(n) = |\{i \mid 1 \leq i \leq n \text{ and } h_i = j\}|$.

We have assumed in Theorem 15.1 that w satisfies Reg. Clearly this is necessary for RA to make sense *in general* since without Reg we can have the conditional in (90) undefined and (according to our convention) the assertion in RA would be rather meaningless. Notwithstanding, in [16] Gaifman states a generalization of Theorem 15.1 appropriate to this case.

If the probability function w satisfies RA and Reg then it satisfies an analogous version of RA for consistent non-tautological $\theta(a_1) \in QFSL$. Namely

$$\lim_{n\to\infty} \left(w\left(\theta(a_{n+1}) \mid \bigwedge_{i=1}^{n} \theta^{\varepsilon_i}(a_i)\right) - \frac{u(n)}{n} \right) = 0$$

where $u(n) = \sum_{i=1}^{n} \varepsilon_i$. To see this notice that the map

$$\vec{x} \in \mathbb{D}_{2^q} \mapsto w_{\vec{x}}(\theta(a_1))$$

is continuous and onto [0, 1] so for μ the de Finetti prior of w the measure ν on $\mathbb{D}_2$ defined by

$$\nu(A) = \mu\{\vec{x} \mid \langle w_{\vec{x}}(\theta(a_1)), 1 - w_{\vec{x}}(\theta(a_1))\rangle \in A\}$$

has every point in $\mathbb{D}_2$ as a support point and by the IP property of the $w_{\vec{x}}$ we have

$$\begin{aligned}\int_{\mathbb{D}_2} w_{\langle x_1, x_2\rangle}\left(\bigwedge_{i=1}^{n} R_1^{\varepsilon_i}(a_i)\right) d\nu(\langle x_1, x_2\rangle) &= \int_{\mathbb{D}_2} x_1^{n_1} x_2^{n_2}\, d\nu(\langle x_1, x_2\rangle)\\ &= \int_{\mathbb{D}_{2^q}} w_{\vec{x}}\left(\bigwedge_{i=1}^{n} \theta^{\varepsilon_i}(a_i)\right) d\mu(\vec{x})\end{aligned}$$

where $n_1 = \sum_{i=1}^{n} \varepsilon_i$, $n_2 = n - n_1$. The required conclusion now follows by applying Theorem 15.1 to this ν.

Notice in particular[58] then that in this case (and trivially if θ is a tautology)

$$\lim_{n\to\infty} w\left(\theta(a_{n+1}) \mid \bigwedge_{i=1}^{n} \theta(a_i)\right) = 1.$$

[58]Compare this with Proposition 4.2.

Chapter 16

CARNAP'S CONTINUUM OF INDUCTIVE METHODS

In this chapter we introduce a family of probability functions for unary languages which to date have played a central role in the development of Inductive Logic.

For $0 \leq \lambda \leq \infty$ the probability function c_λ^L for the unary language L ($= \{R_1, R_2, \dots, R_q\}$ as usual) is defined by

$$c_\lambda^L\Big(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^n \alpha_{h_i}(a_i)\Big) = \frac{m_j + \lambda 2^{-q}}{n + \lambda} \tag{103}$$

where $m_j = |\{i \mid h_i = j\}|$, the number of times the atom $\alpha_j(x)$ occurs amongst the $\alpha_{h_i}(x)$ and we identify $(2^{-q} \cdot \infty)/\infty = (2^{-q} \cdot 0)/0 = 2^{-q}$.

Notice that (103) does indeed determine the value of c_λ^L on all state descriptions, and in turn by (34) determines a unique probability function, since for $\lambda > 0$ it gives that

$$\begin{aligned} c_\lambda^L\Big(\bigwedge_{i=1}^n \alpha_{h_i}(a_i)\Big) &= \prod_{j=1}^n c_\lambda^L\Big(\alpha_{h_j}(a_j) \mid \bigwedge_{i=1}^{j-1} \alpha_{h_i}(a_i)\Big) \\ &= \prod_{j=1}^n \left(\frac{r_j + \lambda 2^{-|L|}}{j - 1 + \lambda}\right) \\ &= \frac{\prod_{k=1}^{2^q} \prod_{j=0}^{m_k - 1}(j + \lambda 2^{-|L|})}{\prod_{j=0}^{n-1}(j + \lambda)} \end{aligned} \tag{104}$$

where r_j is the number of times that h_j occurs amongst $h_1, h_2, \dots, h_{j-1}$ and m_k is the number of times k occurs amongst $h_1, h_2, \dots, h_n$, whilst for $\lambda = 0$ it gives for $n > 0$ that

$$c_0^L\Big(\bigwedge_{i=1}^n \alpha_{h_i}(a_i)\Big) = \begin{cases} 2^{-q} & \text{if } h_1 = h_2 = \cdots = h_n, \\ 0 & \text{otherwise.} \end{cases} \tag{105}$$

From (104) and (105) we can see that the values of the c_λ^L are invariant under permutations of the constants and the atoms and hence that the c_λ^L satisfy Ex and Ax. Consequently then (103) holds for arbitrary (distinct) constants $b_1, b_2, \dots, b_{n+1}$ in place of $a_1, a_2, \dots, a_{n+1}$, a fact which we shall use without further mention in what follows. Clearly also the c_λ^L satisfy Reg for $\lambda > 0$.

The c_λ^L for $0 \leq \lambda \leq \infty$ are referred to as *Carnap's Continuum of Inductive Methods* and, for reasons that will become clear in the next chapter, have played a central role in 'Applied' Inductive Logic. Amongst their attractive properties (as rational choices) is Unary Language Invariance, a principle already proposed by Carnap as Axiom A11 of his *Axiom System for Inductive Logic*, see [130, page 974]:

Unary Language Invariance, ULi.

A probability function w for a unary language L satisfies Unary Language Invariance if there is a family of probability functions $w^{\mathcal{L}}$, one for each (finite) unary language $\mathcal{L}$, satisfying Px (and Ex) such that $w^L = w$ and whenever $\mathcal{L} \subseteq \mathcal{L}'$, $w^{\mathcal{L}} = w^{\mathcal{L}'} \restriction S\mathcal{L}$ (i.e. $w^{\mathcal{L}'}$ restricted to $S\mathcal{L}$).

We say that w satisfies Unary Language Invariance with $\mathcal{P}$, where $\mathcal{P}$ is some property, if the members $w^{\mathcal{L}}$ of this family also all satisfy the property $\mathcal{P}$.

Notice, of course, that for such a family we must have $w^{\mathcal{L}} = w \restriction S\mathcal{L}$ for $\mathcal{L} \subseteq L$.[59]

Given that w is our rational choice of probability function on SL it seems reasonable to suppose that were we to enlarge the language w could be extended to this larger language, after all it would appear unreasonable to assume from the start that L was all the language there could ever be (a point made by Kemeny already in [64]). Similarly if we are assuming the position that $\mathcal{P}$ is a property our rational choice of probability function w on SL should possess then we should equally demand this property of our chosen extension to larger languages and smaller languages (and hence to all (finite) languages since they obviously can all be reached by an extension and a subsequent contraction). This then provides a rational justification for Language Invariance.

The reason for requiring here that the members of this family satisfy Px is to ensure that $w^{\mathcal{L}}$ depends only on the number of relation (aka predicate) symbols in $\mathcal{L}$ and not, perversely, on which actual symbols they are.

Notice that if w on $L = \{R_1, R_2, \dots, R_q\}$ is a member of a language invariant family then there is a probability function w^+ for the *infinite* language $L^+ = \{R_1, R_2, R_3, \dots\}$ satisfying Px and extending w, simply take the union over q of the family members on the $\{R_1, R_2, \dots, R_q\}$. Conversely, given such a w^+ on L^+ we can define a language invariant family containing w by setting $w^{\mathcal{L}}(\theta) = w^+(\sigma\theta)$ for any unary language $\mathcal{L} = \{P_1, P_2, \dots, P_q\}$ where σ is an injection from $\mathcal{L}$ to L^+ and $\sigma\theta$ is the

[59] A much more detailed investigation of ULi than we have time for here is given in [68]. In particular that account gives a representation theorem for probability functions satisfying ULi.

result of replacing each (unary) relation symbol P in θ by $\sigma(P)$. Because of this the study of Inductive Logic in the presence of Language Invariance could equally well be carried out in L^+ rather than in our default finite L, an approach which was taken, for example, in [95]. Language Invariance for the c_λ^L will follow as a simple corollary of the next result which is itself of some independent interest.

PROPOSITION 16.1. *Let $\theta_1(x), \theta_2(x), \dots, \theta_k(x)$ be disjoint quantifier free formulae of L. Then for $0 < \lambda \leq \infty$*

$$c_\lambda^L\Big(\theta_j(a_{n+1}) \mid \bigwedge_{i=1}^n \theta_{h_i}(a_i)\Big) = \frac{m_j + t_j\lambda 2^{-q}}{n+\lambda}$$

and for $n > 0$,

$$c_0^L\Big(\bigwedge_{i=1}^n \theta_{h_i}(a_j)\Big) = \begin{cases} t_j 2^{-q} & \text{if } h_1 = h_2 = \dots = h_n, \\ 0 & \text{otherwise,} \end{cases}$$

where $m_j = |\{i \mid h_i = j\}|$ and $t_j = |\{r \mid \alpha_r(x) \vDash \theta_j(x)\}|$.

PROOF. The result for $\lambda = 0$ is straightforward so assume that $\lambda > 0$. For $s = 1, 2, \dots, k$ let

$$\Gamma_s = \{\alpha_r(x) \mid \alpha_r(x) \vDash \theta_s(x)\},$$

so $t_s = |\Gamma_s|$. Let $\psi(a_1, a_2, \dots, a_n)$ run over all state descriptions

$$\alpha_{g_1}(a_1) \wedge \alpha_{g_2}(a_2) \wedge \dots \wedge \alpha_{g_n}(a_n)$$

where $\alpha_{g_i}(x) \in \Gamma_{h_i}$ for $i = 1, 2, \dots, n$.

Then

$$\begin{aligned} &c_\lambda^L\Big(\theta_j(a_{n+1}) \mid \bigwedge_{i=1}^n \theta_{h_i}(a_i)\Big) \\ &\quad = \sum_{\alpha_r(x)\in\Gamma_j} c_\lambda^L\Big(\alpha_r(a_{n+1}) \mid \bigwedge_{i=1}^n \bigvee_{\alpha_{g_i}(x)\in\Gamma_{h_i}} \alpha_{g_i}(a_i)\Big) \\ &\quad = \sum_{\alpha_r(x)\in\Gamma_j} c_\lambda^L\Big(\alpha_r(a_{n+1}) \mid \bigvee_{\psi} \psi\Big) \\ &\quad = \sum_{\alpha_r(x)\in\Gamma_j} \sum_{\psi} \frac{c_\lambda^L(\alpha_r(a_{n+1}) \mid \psi) \cdot c_\lambda^L(\psi)}{c_\lambda^L(\bigvee_\psi \psi)}. \end{aligned} \tag{106}$$

But for any particular such ψ

$$\begin{aligned} \sum_{\alpha_r(x)\in\Gamma_j} c_\lambda^L(\alpha_r(a_{n+1}) \mid \psi) &= \sum_{\alpha_r(x)\in\Gamma_j} \frac{s_r + \lambda 2^{-q}}{n+\lambda} \\ &= \frac{m_j + \lambda 2^{-q}|\Gamma_j|}{n+\lambda}, \end{aligned}$$

where s_r is the number of times that $\alpha_r(x)$ is instantiated in ψ. Substituting this into (106) gives

$$c_\lambda^L\Big(\theta_j(a_{n+1}) \mid \bigwedge_{i=1}^n \theta_{h_i}(a_i)\Big) = \sum_\psi \frac{m_j + \lambda 2^{-q}|\Gamma_j|}{n+\lambda} \cdot \frac{c_\lambda^L(\psi)}{c_\lambda^L(\bigvee_\psi \psi)}$$
$$= \frac{m_j + \lambda 2^{-q}|\Gamma_j|}{n+\lambda},$$

as required. ⊣

COROLLARY 16.2. *For a fixed λ the c_λ^L form a ULi family.*

PROOF. Let (as usual) $L = \{R_1, R_2, \dots, R_q\}$ and $L' = \{R_1, R_2, \dots, R_{q+1}\}$. By iterating the argument it is enough to prove that $c_\lambda^{L'} \restriction SL = c_\lambda^L$. This follows from the previous proposition by applying it to $c_\lambda^{L'}$, taking $\theta_1(x), \dots, \theta_{2^q}(x) \in QFFL'$ to be the atoms of L, say $\theta_s(x) = \alpha_s(x)$. ⊣

A further, arguably desirable, property that the c_λ^L for $0 < \lambda < \infty$ satisfy is Reichenbach's Axiom, RA, since in this case, in the notation of (90),

$$c_\lambda^L\Big(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^n \alpha_{h_i}(a_i)\Big) - \frac{u_j(n)}{n}$$
$$= \frac{u_j(n) + 2^{-q}\lambda}{n+\lambda} - \frac{u_j(n)}{n}$$
$$= \frac{2^{-q}\lambda}{n+\lambda} - \frac{\lambda u_j(n)}{n(n+\lambda)} \to 0 \qquad \text{as } n \to \infty.$$

The next chapter sheds some light on why Carnap's Continuum and its variants occupy such a prominent place in this subject, in addition to satisfying the several attractive properties described above.

Chapter 17

IRRELEVANCE

In the very first example in this monograph, tossing a coin to decide ends at the start of a cricket match, we already met the idea of disregarding 'irrelevant knowledge' in forming beliefs. Several principles have been proposed within Inductive Logic aimed at capturing this idea, or at least aspects of it. The most historically important of these, as we shall see shortly, was what became known (following a suggestion by I.J.Good) as Johnson's Sufficientness Postulate, or Principle. This principle was a key assumption of Johnson's in his 1932 paper [58] and later in Carnap's development of Inductive Logic.

JOHNSON'S SUFFICIENTNESS POSTULATE, JSP.[60]

$$w\Big(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\Big) \tag{107}$$

depends only on n and $r = |\{\, i \mid h_i = j \,\}|$ *i.e. the number of times that* α_j *occurs amongst the* α_{h_i} *for* $i = 1, 2, \ldots, n$.

In other words, knowing (just) which atoms are satisfied by $a_1, a_2, \ldots, a_n$ the probability of a_{n+1} satisfying a particular atom α_j depends only on the sample size n and the number of these which have already satisfied α_j (and similarly for any distinct constants $b_1, b_2, \ldots, b_{n+1}$ by our standing assumption of Ex). Within the framework used by Carnap, where we may have families of properties, such as colours, in place of these atoms, this might be motivated by imagining we have an urn containing balls coloured blue, green, red and yellow, say, and we are picking from this urn with replacement. Suppose we have made n previous picks and out of them r have been red. In that case it does indeed seem intuitively clear that the probability assigned to the $n + 1$'st pick being red will only depend on n and r and not on the distribution of the colours blue, green and yellow amongst the remaining $n - r$ balls.

Of course this particular motivation for JSP would require our agent to first assume that the ambient world had been, or was being, decided

[60] As usual we apply the convention agreed on page 23 in the case when this conditional probability is not defined because the denominator is zero.

by picking copies of atoms from some urn. In the situation of 'zero knowledge' that seems quite an assumption, though of course one could certainly entertain JSP as a reasonable, rational, principle without making any such assumption about the world.

Our next theorem, proved originally by Johnson in [58] and later Kemeny [64] and Carnap & Stegmüller [18] (see [130, pages 74–75, 979–980] for the history), shows why JSP has held such an esteemed position in Inductive Logic. Before that however we prove a useful lemma.

LEMMA 17.1. *JSP implies Ax.*

PROOF. Since

$$w\Big(\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\Big) = \prod_{j=1}^{n} w\Big(\alpha_{h_j}(a_j) \mid \bigwedge_{i=1}^{j-1} \alpha_{h_i}(a_i)\Big)$$

(with both sides zero if not all the conditional probabilities are defined) JSP gives that this right hand side is invariant under permutations of atoms. Hence so is the left hand side and this yields the result.[61] ⊣

THEOREM 17.2. *Suppose that the unary language L has at least two relation symbols, i.e. $q \geq 2$.*[62] *Then the probability function w on SL satisfies JSP if and only if $w = c_\lambda^L$ for some $0 \leq \lambda \leq \infty$.*[63]

PROOF. It is clear from their defining equations that the c_λ^L satisfy JSP.

For the other direction assume that w satisfies JSP. Then by the previous lemma w satisfies Ax, a property we will use henceforth without further mention. So, since

$$1 = w\Big(\bigvee_{i=1}^{2^q} \alpha_i(a_1)\Big) = \sum_{i=1}^{2^q} w(\alpha_i(a_1)),$$

$$w(\alpha_i(a_1)) = 2^{-q}, \text{ for } i = 1, 2, \ldots, 2^q. \tag{108}$$

Now suppose that

$$w\Big(\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\Big) = 0$$

for some such state description. We may assume that n here is minimal. By (108), $n > 1$. If, say, $h_1 = h_2$ then by PIR

$$0 = w\Big(\alpha_{h_1}(a_1) \mid \bigwedge_{i=2}^{n} \alpha_{h_i}(a_i)\Big) \geq w\Big(\alpha_{h_1}(a_1) \mid \bigwedge_{i=3}^{n} \alpha_{h_i}(a_i)\Big)$$

---

[61] Recall from page 42 that in the case $n = 0$, $\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)$ is taken to be $\top$, a tautology.

[62] Carnap, [17, p101], has proposed an additional 'linearity assumption' to JSP which constrains the solutions to be of the form c_λ^L even when $q = 1$ (see also [146]). However the condition seems to us rather too ad hoc, and marginal given our enthusiasm for Language Invariance, to be included here.

[63] For $q = 1$ Carnap gives and alternative derivation (see [9]) based on symmetry considerations for the probability function c_2^L, commonly referred to as the 'Rule of Succession'.

(with all terms well defined) so

$$w\Big(\bigwedge_{i=2}^{n}\alpha_{h_i}(a_i)\Big)=0$$

contradicting the minimality of n.

Hence all the h_i must be different. So by JSP

$$0=w\Big(\alpha_{h_1}(a_1)\mid\bigwedge_{i=2}^{n}\alpha_{h_i}(a_i)\Big)=w\Big(\alpha_1(a_1)\mid\bigwedge_{i=2}^{n}\alpha_2(a_i)\Big)$$

and we must have

$$w\Big(\alpha_1(a_1)\wedge\bigwedge_{i=2}^{n}\alpha_2(a_i)\Big)=0.$$

Given the previous conclusion there cannot be any repeated atoms in $\bigwedge_{i=2}^{n}\alpha_2(a_i)$ so we must have $n=2$.

This means that for all n

$$w\Big(\bigwedge_{i=1}^{n}\alpha_{h_i}(a_i)\Big)=0$$

whenever the h_i are not all equal. So in view of (108) we must have that

$$w\Big(\bigwedge_{i=1}^{n}\alpha_j(a_i)\Big)=2^{-q}\qquad\text{for } n\geq 1,\ j=1,2,\dots,2^q,$$

in other words $w=c_0^L$.

So now assume that w is non-zero on all state descriptions, i.e. satisfies Reg, and the conditional probabilities in JSP are well defined and non-zero. Since (107) depends only on n and r denote it by $g(r,n)$. Then

$$1=w\Big(\bigvee_{i=1}^{2^q}\alpha_i(a_2)\mid\alpha_j(a_1)\Big)=\sum_{i=1}^{2^q}w(\alpha_i(a_2)\mid\alpha_j(a_1))=$$
$$g(1,1)+(2^q-1)g(0,1),$$

so

$$g(1,1)+(2^q-1)g(0,1)=1.\tag{109}$$

By PIR, $g(1,1)\geq g(0,0)$, so by Ax,

$$1\geq g(1,1)\geq 2^{-q}.$$

Hence for some $0\leq\lambda\leq\infty$,

$$g(1,1)=\frac{1+2^{-q}\lambda}{1+\lambda},\quad g(0,1)=\frac{2^{-q}\lambda}{1+\lambda},$$

the second of these following from the first and (109). Indeed we must have $\lambda>0$ since $g(0,1)>0$.

We now show by induction on $n \in \mathbb{N}$ that for this same λ

$$g(r,n) = \frac{r + \lambda 2^{-q}}{n+\lambda} \quad (r = 0, 1, \ldots, n), \tag{110}$$

which forces w to satisfy the equations defining c^L_λ.

We have already proved (110) for $n = 0, 1$ so assume that $n \geq 1$ and (110) holds for n. The idea is to use JSP to derive a suitable set of equations which force (110) to also hold for $n+1$.

Firstly for $r + s = n + 1$, and distinct $1 \leq m, k \leq 2^q$ we have that

$$\begin{aligned} 1 &= w\Big(\bigvee_{h=1}^{2^q} \alpha_h(a_{n+1}) \mid \bigwedge_{i=1}^{r} \alpha_m(a_i) \wedge \bigwedge_{i=r+1}^{n+1} \alpha_k(a_i)\Big) \\ &= \sum_{h=1}^{2^q} w\Big(\alpha_h(a_{n+1}) \mid \bigwedge_{i=1}^{r} \alpha_m(a_i) \wedge \bigwedge_{i=r+1}^{n+1} \alpha_k(a_i)\Big) \\ &= g(r, n+1) + g(s, n+1) + (2^q - 2)g(0, n+1). \end{aligned} \tag{111}$$

Secondly, adopting the convenient notation of writing $\alpha_{h_1}\alpha_{h_2}\ldots\alpha_{h_n}$ etc. for

$$\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)$$

we have that for $r + s + t = n$ and distinct $1 \leq m, j, k \leq 2^q$ (these exist since $2^q \geq 4$),

$$\begin{aligned} w(\alpha_m \mid \alpha_j \alpha_m^r \alpha_j^s \alpha_k^t) \cdot w(\alpha_j \mid \alpha_m^r \alpha_j^s \alpha_k^t) &= w(\alpha_m \alpha_j \mid \alpha_m^r \alpha_j^s \alpha_k^t) \\ &= w(\alpha_j \mid \alpha_m \alpha_m^r \alpha_j^s \alpha_k^t) \cdot w(\alpha_m \mid \alpha_m^r \alpha_j^s \alpha_k^t). \end{aligned}$$

Hence

$$g(r, n+1)g(s, n) = g(s, n+1)g(r, n). \tag{112}$$

In particular taking $s = 0$ here and using the inductive hypothesis (110) for n gives,

$$g(r, n+1) = (r\lambda^{-1}2^q + 1)g(0, n+1). \tag{113}$$

Taking $r = 1, s = n$, in (111) and substituting for $g(1, n+1)$, $g(n, n+1)$ from (113) gives

$$(\lambda^{-1}2^q + 1)g(0, n+1) + (n\lambda^{-1}2^q + 1)g(0, n+1) + (2^q - 2)g(0, n+1) = 1$$

and hence

$$g(0, n+1) = \frac{\lambda 2^{-q}}{n + 1 + \lambda}.$$

Substituting in (113) now gives (110) too for $n+1$ and $r = 1, 2, \ldots, n$, and finally also for $r = n+1$ using (111) with $r = 0, s = n+1$. $\dashv$

Studying the above proof it is easy to see that we only really needed 3 atoms here instead of the 4 given us by the assumption $q > 1$, an observation that we will need later. Notice that in the case $q = 1$ *any* probability function satisfying SN (equivalent to Ax when $q = 1$) will satisfy JSP so there will be many probability functions with this property.

There is a weakened version of JSP, see [149] for a nice account, in which the value of the conditional in (107),

$$w\Big(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\Big),$$

is assumed to depend not only n and $r = |\{\, i \mid h_i = j \,\}|$ but also on the particular atom α_j. We shall refer to this as the 'Weak Johnson's Sufficientness Postulate', WJSP.

From the standpoint of this monograph to introduce this extra dependency appears rather perverse. For in the situation of zero knowledge there is arguably complete symmetry between any two individual atoms yet this weakening denies it, for no apparent reason since there would seem to be no knowledge on which it could based. More precisely if we assume that w satisfies SN then for any two atoms α_j, α_i there is a permutation which transposes α_j and α_i and under which w is invariant, so the dependence on the particular atom is illusory and w actually must satisfy the full JSP (and conversely). For the record it seems worth stating this as a proposition.

PROPOSITION 17.3. *In the presence of SN, WJSP and JSP are equivalent.*

Despite the apparent perversity of WSJP (without full JSP) in the context of PIL, from the more philosophical viewpoint which sees Carnap's Inductive Logic as providing prescriptions or models, the greater flexibility this weakening provides can be, indeed is, very welcome.

Because of the stated dependence in the WJSP on the particular atom we no longer have Ax as a consequence. Nevertheless we are still able to prove a characterization analogous to Theorem 17.2 for the probability functions satisfying WJSP though for simplicity we shall first restrict ourselves to probability functions also satisfying Reg. The following result is due to Carnap [17, p90], see also [72].

THEOREM 17.4. *Suppose the unary language L has at least two relation symbols, i.e. $q \geq 2$. Then the probability function w on SL satisfies WJSP and Reg just if there is $0 < \lambda \leq \infty$ and $\gamma_1, \gamma_2, \dots, \gamma_{2^q} > 0$ with sum 1 such that for $1 \leq j \leq 2^q$*

$$w\Big(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\Big) = \frac{m_j + \gamma_j \lambda}{n + \lambda},$$

where $m_j = |\{\, i \mid h_i = j \,\}|$.

PROOF. Since we are assuming regularity we only need to rerun that part of the proof of Theorem 17.2 when we had $\lambda \neq 0$. We shall just sketch the required changes.

By WJSP let

$$w\left(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right) = g_j(m_j, n).$$

As with (109) we obtain for each $j = 1, 2, \ldots, 2^q$,

$$g_j(1,1) + \sum_{i \neq j} g_i(0,1) = 1. \tag{114}$$

and summing these gives

$$\sum_{i=1}^{2^q} g_i(1,1) + (2^q - 1) \sum_{i=1}^{2^q} g_i(0,1) = 2^q. \tag{115}$$

By PIR and Reg, $1 > g_j(1,1) \geq g_j(0,1) > 0$, so as in the previous proof

$$2^{-q} \sum_{i=1}^{2^q} g_i(1,1) = \frac{1 + 2^{-q}\lambda}{1+\lambda}, \quad 2^{-q} \sum_{i=1}^{2^q} g_i(0,1) = \frac{2^{-q}\lambda}{1+\lambda} \tag{116}$$

for some $0 < \lambda \leq \infty$. Let $\gamma_j > 0$ be such that

$$\gamma_j \sum_{i=1}^{2^q} g_i(0,1) = g_j(0,1). \tag{117}$$

Then $\sum_{i=1}^{2^q} \gamma_i = 1$ and

$$g_j(0,1) = \frac{\gamma_j \lambda}{1+\lambda}, \quad g_j(1,1) = \frac{1 + \gamma_j \lambda}{1+\lambda}, \tag{118}$$

this second identity coming from the fact that from (114),

$$g_j(1,1) - g_j(0,1) + \sum_{i=1}^{2^q} g_i(0,1) = 1.$$

We now show for all $n \geq 0$ that

$$g_j(r,n) = \frac{r + \gamma_j \lambda}{n + \lambda}. \tag{119}$$

We first need two identities. As in (111) we have that for $r + s = n + 1$

$$g_m(r, n+1) + g_k(s, n+1) + \sum_{i \neq m,k} g_i(0, n+1) = 1 \tag{120}$$

and as in (112), for $m \neq j$

$$g_m(r, n+1) g_j(s,n) = g_j(s, n+1) g_m(r,n). \tag{121}$$

Putting $s = r = n = 0$ in (121) and substituting (118) gives

$$\frac{\gamma_m \lambda}{1+\lambda} \cdot g_j(0,0) = \frac{\gamma_j \lambda}{1+\lambda} \cdot g_m(0,0)$$

so since the γ_m and $g_m(0,0)$ both sum to 1 this gives that $g_j(0,0) = \gamma_j$ and (119) holds for $n = 0$. We already have that (119) holds for $n = 1$ and we now prove it for larger n by induction.

Assume it holds for n. Then taking $s = 0$ in (121) and using the inductive hypothesis (119) for n gives,

$$g_m(r, n+1) = \frac{(r + \gamma_m \lambda)}{\gamma_j \lambda} g_j(0, n+1). \tag{122}$$

even for $m = j$ (by using (122) with m, j transposed, and $r = 0$). Taking $r = 1, s = n$ in (120) and substituting for $g_m(1, n+1), g_k(n, n+1), g_i(0, n+1)$ from (122) gives

$$\left(\frac{(1 + \gamma_m \lambda)}{\gamma_j \lambda} + \frac{(n + \gamma_k \lambda)}{\gamma_j \lambda} + \sum_{i \neq m,k} \frac{\gamma_i \lambda}{\gamma_j \lambda} \right) g_j(0, n+1) = 1$$

and hence

$$g_j(0, n+1) = \frac{\gamma_j \lambda}{n + 1 + \lambda}.$$

Substituting in (122) now gives (119) too for $n+1$ and $r = 1, 2, \ldots, n$, and finally also for $r = n+1$ by using (120) with $r = n+1, s = 0$. $\dashv$

If we drop the requirement of Reg in this last theorem then, by using a similar argument, we can show that the situation depends on the number, k say, of atoms α_i such that $w(\alpha_i(a_1)) \neq 0$. If $k \neq 2$ then we have the same theorem but allowing also the possibility that $\lambda = 0$ or some $\gamma_i = 0$ (where $(\gamma_i \cdot 0)/0 = (\gamma_i \cdot \infty)/\infty = \gamma_i$). If $k = 2$ then there will be many additional solutions corresponding to the solutions to JSP when $q = 1$.[64]

Several other principles have been proposed in the literature[65] which turn out to be equivalent, or at least closely related, to JSP. We mention two such (and later Theorem 38.1) which again can be said to be based on the idea of irrelevance. The first of these, due to de Cooman, Miranda and Quaeghebeur (in the more general context of previsions), [19], runs as follows in our formulation:

THE DCMQ PRINCIPLE.
For disjoint quantifier free formulae $\theta_1(x), \ldots, \theta_k(x)$

$$w\Big(\theta_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \theta_{h_i}(a_i)\Big) = w\Big(\theta_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{r_i}(a_i)\Big) \tag{123}$$

whenever $\alpha_{r_i}(x) \vDash \theta_{h_i}(x)$ for $i = 1, 2, \ldots, n$.

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[64] The convention introduced on page 23 regarding identities involving conditional probabilities is highly relevant here.

[65] See for example [73].

PROPOSITION 17.5. *The dCMQ Principle is equivalent to WJSP.*

PROOF. To show that the dCMQ Principle implies WJSP take $\theta_1(x) = \alpha_1(x)$, $\theta_2(x) = \bigvee_{i=2}^{2^q} \alpha_i(x)$. Then for $h_1, h_2, \dots, h_n \in \{1,2\}$ and any $r_1, r_2, \dots, r_n$ such that $r_i = 1$ if $h_i = 1$, $r_i \in \{2,3,\dots,2^q\}$ if $h_i = 2$ we have by this principle that

$$w\Big(\theta_1(a_{n+1}) \mid \bigwedge_{i=1}^{n} \theta_{h_i}(a_i)\Big) = w\Big(\alpha_1(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{r_i}(a_i)\Big). \tag{124}$$

Hence this right hand side must be the same for any choices of the $r_i \neq 1$. Its value therefore depends only on n, the particular atom α_1, and the number of times it occurs in the conditional, giving WJSP.

In the other direction let w satisfy WJSP and without loss of generality let $\alpha_1, \dots, \alpha_k$ list those atoms α_i such that $w(\alpha_i(a_1)) \neq 0$. Then

$$w\Big(\theta_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \theta_{h_i}(a_i)\Big) = w\Big(\phi_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \phi_{h_i}(a_i)\Big)$$

where

$$\phi_j(x) = \bigvee_{\substack{\alpha_i(x) \models \theta_j(x) \\ 1 \leq i \leq k}} \alpha_i(x).$$

In the case $k = 2$ the required conclusion that

$$w\Big(\phi_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \phi_{h_i}(a_i)\Big) = w\Big(\phi_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{r_i}(a_i)\Big)$$

holds trivially, by convention, if some $r_i > 2$, and otherwise is easily checked for the remaining cases when $\phi_j(x)$ is one of $\alpha_1(x), \alpha_2(x), \alpha_1(x) \vee \alpha_2(x), \bot$. Similarly for the case $q = 1$.

So assume that $k > 2$ and by Theorem 17.4 and the note following it let w be the solution to WJSP with parameters $\lambda, \gamma_1, \gamma_2, \dots, \gamma_{2^q}$, where $\gamma_j = 0$ for $k < j \leq 2^q$. Clearly the identity (123) holds trivially if some $r_i > k$ so assume from now on that that is not the case. Then the right hand side of (123) is

$$\sum_{\alpha_m(x) \models \theta_j(x)} w\Big(\alpha_m(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{r_i}(a_i)\Big) = \frac{\sum_{\alpha_m(x) \models \theta_j(x)} (n_m + \gamma_m \lambda)}{n + \lambda}$$

$$= \frac{(t_j + A_j \lambda)}{n + \lambda}$$

where n_m is the number of times that $r_i = m$, t_j is the number of times that $h_i = j$ and

$$A_j = \sum_{\alpha_m(x) \models \theta_j(x)} \gamma_m.$$

This last does not depend on the particular choices of $\alpha_{r_i}(x) \vDash \theta_{h_i}(x)$ (for $r_i \leq k$) so the value is the same for all of them. The left hand side of (123) is

$$w\Big(\theta_j(a_{n+1}) \mid \bigvee \Gamma\Big)$$

where Γ is the set of all possible

$$\bigwedge_{i=1}^{n} \alpha_{s_i}(a_i)$$

with $\alpha_{s_i}(x) \vDash \theta_{h_i}(x)$ for $i = 1, 2, \ldots, n$. Since for any one of those, ψ say,

$$w\Big(\theta_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{r_i}(a_i)\Big) = w(\theta_j(a_{n+1}) \mid \psi)$$

we have that

$$\begin{aligned} w\Big(\theta_j(a_{n+1}) \mid \bigvee \Gamma\Big) &= \frac{\sum_{\psi \in \Gamma} w(\theta_j(a_{n+1}) \wedge \psi)}{\sum_{\psi \in \Gamma} w(\psi)} \\ &= \frac{\sum_{\psi \in \Gamma} w(\theta_j(a_{n+1}) \mid \psi) \cdot w(\psi)}{\sum_{\psi \in \Gamma} w(\psi)} \\ &= \frac{\sum_{\psi \in \Gamma} w(\theta_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{r_i}(a_i)) \cdot w(\psi)}{\sum_{\psi \in \Gamma} w(\psi)} \\ &= w\Big(\theta_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{r_i}(a_i)\Big), \end{aligned}$$

as required. $\dashv$

A second principle closely related to JSP is the 'State Description Conditional Irrelevance Principle', one of several similar such principles in [44].

THE STATE DESCRIPTION CONDITIONAL IRRELEVANCE PRINCIPLE, SDCIP. *Let $L_1, L_2 \subseteq L$ with L_1, L_2 disjoint languages, let $\Phi(a_1, a_2, \ldots, a_n)$, $\Theta(a_{n+1}, a_{n+2}, \ldots, a_{n+r})$ be state descriptions in SL_1 and let $\Psi(a_1, a_2, \ldots, a_n)$ be a state description in SL_2. Then*

$$\begin{aligned} &w(\Theta(a_{n+1}, a_{n+2}, \ldots, a_{n+r}) \mid \Phi(a_1, a_2, \ldots, a_n) \wedge \Psi(a_1, a_2, \ldots, a_n)) \\ &\qquad = w\Big(\Theta(a_{n+1}, a_{n+2}, \ldots, a_{n+r}) \mid \Phi(a_1, a_2, \ldots, a_n)\Big). \quad (125) \end{aligned}$$

In other words the conditioning evidence $\Psi(a_1, a_2, \ldots, a_n)$, which has no constant nor predicate symbols in common with $\Theta(a_{n+1}, a_{n+2}, \ldots, a_{n+r})$, should be irrelevant, or at least its relevance should all be accounted for in $\Phi(a_1, a_2, \ldots, a_n)$.

PROPOSITION 17.6. (i) *JSP implies SDCIP.*
(ii) *If w satisfies Ax and SDCIP on a language $L \cup \{R\}$ with $|L| \geq 2$ then $w \restriction SL$ satisfies JSP.*
(iii) *JSP is equivalent to ULi with Ax and SDCIP.*

PROOF. (i) Assume JSP, so $w = c_\lambda^L$ for some λ. If $\lambda = 0$ the required identity is straightforward to check[66] so assume that $\lambda > 0$.

The proof is by induction on $|L_2|$ for *any* $\emptyset \neq L_1 \subseteq L - L_2$. Let $R \in L_2$, say

$$\Psi(\vec{a}) = \chi(\vec{a}) \wedge \bigwedge_{i=1}^{n} R^{\delta_i}(a_i)$$

where $\chi(\vec{a})$ is a state description for $L_2 - \{R\}$ (or simply a tautology if $L_2 = \{R\}$) and $\vec{a} = a_1, a_2, \dots, a_n$. Then from Proposition 16.1 and using the notation of (125),

$$\begin{aligned}
&c_\lambda^L(\Theta(a_{n+1}, a_{n+2}, \dots, a_{n+r}) \mid \Phi(\vec{a}) \wedge \Psi(\vec{a}))\\
&= c_\lambda^L(\Theta(a_{n+1}, a_{n+2}, \dots, a_{n+r}) \mid \Phi(\vec{a}) \wedge \chi(\vec{a}) \wedge \bigwedge_{i=1}^{n} R^{\delta_i}(a_i))\\
&= \sum_{\vec{v}} c_\lambda^L\Big(\Theta(a_{n+1}, a_{n+2}, \dots, a_{n+r}) \wedge\\
&\qquad\qquad \bigwedge_{i=n+1}^{n+r} R^{v_i}(a_i) \mid \Phi(\vec{a}) \wedge \bigwedge_{i=1}^{n} R^{\delta_i}(a_i) \wedge \chi(\vec{a})\Big)
\end{aligned}$$

where $\vec{v} = v_{n+1}, v_{n+2}, \dots, v_{n+r} \in \{0, 1\}$. Since $\chi(\vec{a})$ is a state description for the smaller language $L_2 - \{R\}$ (or missing altogether), by applying the inductive hypothesis this equals

$$\begin{aligned}
&\sum_{\vec{v}} c_\lambda^L\Big(\Theta(a_{n+1}, a_{n+2}, \dots, a_{n+r}) \wedge \bigwedge_{i=n+1}^{n+r} R^{v_i}(a_i) \mid \Phi(\vec{a}) \wedge \bigwedge_{i=1}^{n} R^{\delta_i}(a_i)\Big)\\
&\qquad = c_\lambda^L\Big(\Theta(a_{n+1}, a_{n+2}, \dots, a_{n+r}) \mid \Phi(\vec{a}) \wedge \bigwedge_{i=1}^{n} R^{\delta_i}(a_i)\Big). \qquad (126)
\end{aligned}$$

It remains to remove the terms in R from (126). We prove this by induction on r. Let $L_1 = \{R_1, R_2, \dots, R_s\}$ and

$$\Theta(a_{n+1}, a_{n+2}, \dots, a_{n+r}) = \Upsilon(a_{n+2}, a_{n+3}, \dots, a_{n+r}) \wedge \bigwedge_{j=1}^{s} R_j^{\varepsilon_j}(a_{n+1}).$$

[66] Recalling yet again the agreed convention given on page 23 when the conditioning sentence has zero probability.

Then the right hand side of (126) can be written as the product of

$$c_\lambda^L\Big(\bigwedge_{j=1}^{s} R_j^{\varepsilon_j}(a_{n+1}) \mid \Upsilon(a_{n+2}, a_{n+3}, \dots, a_{n+r}) \wedge \Phi(\vec{a}) \wedge \bigwedge_{i=1}^{n} R^{\delta_i}(a_i)\Big) \quad (127)$$

and

$$c_\lambda^L\Big(\Upsilon(a_{n+2}, a_{n+3}, \dots, a_{n+r}) \mid \Phi(\vec{a}) \wedge \bigwedge_{i=1}^{n} R^{\delta_i}(a_i)\Big). \quad (128)$$

But (127) is the sum of

$$c_\lambda^L\Big(R(a_{n+1}) \wedge \bigwedge_{j=1}^{s} R_j^{\varepsilon_j}(a_{n+1}) \mid \Upsilon(a_{n+2}, a_{n+3}, \dots, a_{n+r}) \wedge \Phi(\vec{a}) \wedge \bigwedge_{i=1}^{n} R^{\delta_i}(a_i)\Big),$$

$$c_\lambda^L\Big(\neg R(a_{n+1}) \wedge \bigwedge_{j=1}^{s} R_j^{\varepsilon_j}(a_{n+1}) \mid \Upsilon(a_{n+2}, a_{n+3}, \dots, a_{n+r}) \wedge \Phi(\vec{a}) \wedge \bigwedge_{i=1}^{n} R^{\delta_i}(a_i)\Big)$$

which in turn, by Corollary 16.2, is

$$\frac{t_1 + \lambda 2^{-(s+1)}}{n + r - 1 + \lambda} + \frac{t_2 + \lambda 2^{-(s+1)}}{n + r - 1 + \lambda}, \quad (129)$$

where t_1, t_2 are respectively the number of times that the atoms

$$R(x) \wedge \bigwedge_{j=1}^{s} R_j^{\varepsilon_j}(x), \quad \neg R(x) \wedge \bigwedge_{j=1}^{s} R_j^{\varepsilon_j}(x)$$

are instantiated in

$$\Upsilon(a_{n+2}, \dots, a_{n+r}) \wedge \Phi(\vec{a}) \wedge \bigwedge_{i=1}^{n} R^{\delta_i}(a_i)).$$

But clearly $t_1 + t_2$ is the number of times that the atom $\bigwedge_{j=1}^{s} R_j^{\varepsilon_j}(x)$ is instantiated in

$$\Upsilon(a_{n+2}, a_{n+3}, \dots, a_{n+r}) \wedge \Phi(a_1, a_2, \dots, a_n)$$

so (127) and (129) equal

$$c_\lambda^L\Big(\bigwedge_{j=1}^{s} R_j^{\varepsilon_j}(a_{n+1}) \mid \Upsilon(a_{n+2}, \dots, a_{n+r}) \wedge \Phi(\vec{a})\Big). \quad (130)$$

If $r=1$ then (128) will just be 1 whilst if $r>1$ we have by the (implicit) induction hypothesis that

$$c_\lambda^L\Big(\Upsilon(a_{n+2},\dots,a_{n+r})\,|\,\Phi(\vec{a})\wedge\bigwedge_{i=1}^n R^{\delta_i}(a_i)\Big)=$$
$$c_\lambda^L\left(\Upsilon(a_{n+2},\dots,a_{n+r})\,|\,\Phi(\vec{a})\right). \quad (131)$$

Hence (126) is the product of (130) and the right hand side of (131), i.e.

$$c_\lambda^L\Big(\bigwedge_{j=1}^s R_j^{\varepsilon_j}(a_{n+1})\,|\,\Upsilon(a_{n+2},\dots,a_{n+r})\wedge\Phi(\vec{a})\Big)\cdot$$
$$c_\lambda^L\Big(\Upsilon(a_{n+2},\dots,a_{n+r})\,|\,\Phi(\vec{a})\Big)=$$
$$c_\lambda^L\left(\Theta(a_{n+1},a_{n+2},\dots,a_{n+r})\,|\,\Phi(a_1,a_2,\dots,a_n)\right)$$

as required.

(ii) Assume that the probability function w for $L\cup\{R\}$ satisfies SDCIP and Ax, where R is a unary relation symbol not in L and $|L|\geq 2$. As usual let $\alpha_1(x),\alpha_2(x),\dots,\alpha_{2^q}(x)$ enumerate the atoms of L.

Consider

$$w\Big(\alpha_1(a_{n+1})\,|\,\bigwedge_{i=1}^n\alpha_{h_i}(a_i)\Big) \quad (132)$$

where $\alpha_1(x)$ is instantiated s times in $\bigwedge_{i=1}^n\alpha_{h_i}(a_i)$. By SDCIP this is equal to

$$w\Big(\alpha_1(a_{n+1})\,|\,\bigwedge_{i=1}^n\alpha_{h_i}(a_i)\wedge\bigwedge_{i=1}^n R(a_i)\Big), \quad (133)$$

so s copies of $\alpha_1(x)\wedge R(x)$ are instantiated in this conditioning sentence (and no copies of $\alpha_1(x)\wedge\neg R(x)$ are). Suppose that $\alpha_{h_i}(x),\alpha_{h_j}(x)$ and $\alpha_1(x)$ are distinct atoms. Then we can find a permutation of the atoms of $L\cup\{R\}$ so that $\alpha_1(x)\wedge R(x)$, $\alpha_1(x)\wedge\neg R(x)$ and $\alpha_{h_i}(x)\wedge R(x)$ are fixed and $\alpha_{h_j}(x)\wedge R(x)$ becomes $\alpha_{h_i}(x)\wedge\neg R(x)$, say this permutation has sent

$$\bigwedge_{i=1}^n\alpha_{h_i}(a_i)\wedge\bigwedge_{i=1}^n R(a_i)$$

to

$$\bigwedge_{i=1}^n\alpha_{g_i}(a_i)\wedge\bigwedge_{i=1}^n R^{\delta_i}(a_i).$$

Then by Ax (133) equals

$$w\Big(\alpha_1(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{g_i}(a_i) \wedge \bigwedge_{i=1}^{n} R^{\delta_i}(a_i)\Big),$$

which by SDCIP equals

$$w\Big(\alpha_1(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{g_i}(a_i)\Big).$$

But now in $\bigwedge_{i=1}^{n} \alpha_{g_i}(a_i)$ fewer distinct atoms are instantiated than in $\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)$, whilst the number of instantiations of $\alpha_1(x)$ has remained at s.

Continuing in this fashion then we see that (132) must, after applying Ex, equal

$$w\Big(\alpha_1(a_{n+1}) \mid \bigwedge_{i=1}^{s} \alpha_1(a_i) \wedge \bigwedge_{i=s+1}^{n} \alpha_2(a_i)\Big)$$

(and with any other distinct subscripts in place of 1,2). In conclusion then (132) only depends on s and n, giving JSP for $w \restriction SL$.

(iii) is now immediate.[67] ⊣

For our final result in this chapter we consider the following variation on JSP, the c-Principle, which was proposed by Hintikka & Niiniluoto in 1974, see too [50], [51], Niiniluoto [92], and also Kuipers, [71], [72], where a number of related principles are also considered.

THE c-PRINCIPLE.

$$w\Big(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\Big) \tag{134}$$

depends only n and $r = |\{\, i \mid h_i = j \,\}|$ and $s = |\{\, h_i \mid 1 \le i \le n \,\}|$, the number of different atoms so far instantiated.

This principle does address a criticism that might be leveled against JSP. Namely, taking the above mentioned urn interpretation, if we knew that the only possible colours of balls in the urn were black, pink, blue, brown, green and yellow then the probability we should give to the sixth draw being a yellow would arguably be less if all the previous five draws had uniformly produced blues than if those draws had been more of a mixture, say black, green, pink, blue, brown. In both cases no previous draw had produced yellow but in the second case it would seem that there was a good mix of colours in the urn and the non-appearance of a yellow was 'just bad luck' whereas in the first case one might start to suspect that perhaps there was no mix at all, that all the balls were blue. The c-Principle seems to offer a haven for this sentiment whereas JSP says that

[67] We do not know if JSP and SDCIP are actually equivalent in the presence of Ax for a *fixed* language.

the probability given to the next ball selected being yellow should be the same in both cases. (See [93] for a detailed discussion of this issue.)

Before we give a characterization of the probability functions satisfying the c-Principle (for $q > 1$, clearly when $q = 1$ any probability function satisfying SN satisfies the principle) it will be useful to recall, and introduce, some notation. Throughout let T stand for a non-empty subset of $\{1, 2, \dots, 2^q\}$ and let $\eta_T \in SL$ be as in (63). By Corollary 10.2 we have that any probability function w can be expressed in the form

$$w = \sum_T w(\eta_T) w_T$$

where $w_T(\theta) = w(\theta \mid \eta_T)$ when $w(\eta_T) \neq 0$, so in this case $w_T(\eta_T) = 1$.

For $0 < \rho \leq \infty$ define the probability function c_ρ^T by[68]

$$c_\rho^T\Big(\alpha_j \mid \bigwedge_{i \in T} \alpha_i^{n_i}\Big) = \begin{cases} \dfrac{n_j + \rho}{n + |T|\rho} & \text{if } j \in T, \\ 0 & \text{if } j \notin T, \end{cases}$$

where $n = \sum_i n_i$.

We are now set up to show the following theorem which is adapted from results proved by Kuipers in [71].[69,70]

THEOREM 17.7. *Let L be a language with $q > 1$ and let w be a probability function on L satisfying Reg*[71] *and the c-Principle. Then w satisfies Ax and there is a $\rho \in (0, \infty]$ such that $w_T = c_\rho^T$ for $\emptyset \neq T \subseteq \{1, 2, \dots, 2^q\}$, so*

$$w = \sum_T w(\eta_T) c_\rho^T. \tag{135}$$

Furthermore if $m_i > 0$ for $i \in T$ with $m = \sum_i m_i$ then

$$w\Big(\alpha_j \mid \bigwedge_{i \in T} \alpha_i^{m_i}\Big) = \frac{m_j + \rho}{m + \rho|T|} \cdot w\Big(\bigvee_{i \in T} \alpha_i(a_1) \mid \bigwedge_{i \in T} \alpha_i^{m_i}\Big). \tag{136}$$

Conversely if w has the form (135) for such c_ρ^T with $\rho > 0$ then w satisfies the c-Principle.

PROOF. In the forward direction let w satisfy Reg and the c-Principle, say h is the function such that, with the notation of (134),

$$w\Big(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\Big) = h(r, n, s) \tag{137}$$

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[68] Recall the convention introduced on page 53 of dropping instantiating constants.

[69] In this paper Kuipers does not dismiss the possibility of negative ρ.

[70] For much more on the c-Principle and its kin see for example [50], [51], [71], [73], [92].

[71] Without this assumption essentially the same proof works provided that the maximal r such that $w(\bigwedge_{i=1}^{r} \alpha_i(a_i)) \neq 0$ is at least 3 and we limit ourselves to T with cardinality at most this maximal r.

depends only on n and $r = |\{ i \mid h_i = j \}|$ and $s = |\{ h_i \mid 1 \leq i \leq n \}|$. Notice that just as for JSP w must satisfy Ax.

Given $\emptyset \neq T \subseteq \{1, 2, \dots, 2^q\}$ consider the function w'_T defined by

$$w'_T\Big(\bigwedge_{i=1}^{2^q} \alpha_i^{n_i}\Big) = \frac{w\left(\bigwedge_{i=1}^{2^q} \alpha_i^{n_i} \mid \bigwedge_{i \in T} \alpha_i\right)}{\prod_{i=0}^{n-1}(1 - (2^q - |T|)h(0, |T| + i, |T|))}$$

if $n_i = 0$ for $i \notin T$, and zero otherwise, where $n = \sum_i n_i$. Using the fact from (134) that

$$w\Big(\Big(\bigvee_{i \in T} \alpha_i(a_1)\Big) \wedge \Big(\bigwedge_{i \in T} \alpha_i^{n_i}\Big) \wedge \Big(\bigwedge_{i \in T} \alpha_i\Big)\Big) =$$
$$(1 - (2^q - |T|)h(0, |T| + n, |T|))\, w\Big(\Big(\bigwedge_{i \in T} \alpha_i^{n_i}\Big) \wedge \Big(\bigwedge_{i \in T} \alpha_i\Big)\Big)$$

it can be seen that w'_T is a probability function satisfying Ex and invariant under permutation of atoms α_i for $i \in T$.

Noticing that for $n = \sum_{i \in T} n_i$, by the c-Principle,

$$w\Big(\alpha_j \mid \bigwedge_{i \in T} \alpha_i^{n_i+1}\Big) = h(n_j + 1, n + |T|, |T|)$$

for $j \in T$ we see that the value of $w'_T(\alpha_j \mid \bigwedge_{i \in T} \alpha_i^{n_i})$ when $j \in T$ depends only on n, n_j and $|T|$. Just as in the proof of Theorem 17.2 we can now show that provided $|T| > 2$ there must be some $0 \leq \lambda_T \leq \infty$ such that

$$w'_T\Big(\alpha_j \mid \bigwedge_{i \in T} \alpha_i^{n_i}\Big) = \frac{n_j + \lambda_T |T|^{-1}}{n + \lambda_T},$$

and in fact we must have $\lambda_T > 0$ by Reg. Rewriting this with $\lambda_T = |T|(1 + \rho_T)$, $n + |T| = m$ and $n_i + 1 = m_i$ we now obtain that for $j \in T$

$$w\Big(\alpha_j \mid \bigwedge_{i \in T} \alpha_i^{m_i}\Big) = \frac{m_j + \rho_T}{m + \rho_T |T|} \cdot (1 - (2^q - |T|)h(0, m, |T|)), \qquad (138)$$

provided each $m_i > 0$, where ρ_T is in the range $(-1, \infty]$.

We now show that for $T, S \subseteq \{1, 2, \dots, 2^q\}$ with $2 < |T|, |S|$ we must have $\rho_T = \rho_S$. To see this it is enough to check it in the case $T = \{1, 2, \dots, r\}$, $S = \{1, 2, \dots, r + 1\}$, since by Ax ρ_T only depends on $|T|$. By Ex, for $i \in T$

$$w\Big(\alpha_i \alpha_{r+1} \mid \bigwedge_{j \in T} \alpha_j^{n_j}\Big) = w\Big(\alpha_{r+1} \alpha_i \mid \bigwedge_{j \in T} \alpha_j^{n_j}\Big),$$

so by the c-Principle and (138), when each $n_j > 0$ and $\sum_{j\in T} n_j = n$,

$$h(0,n,r)\cdot\frac{n_i+\rho_S}{n+1+(r+1)\rho_S}\cdot(1-(2^q-(r+1))h(0,n+1,r+1)) =$$
$$(1-(2^q-r)h(0,n,r))\cdot\frac{n_i+\rho_T}{n+r\rho_T}\cdot h(0,n+1,r). \quad (139)$$

Rearranging we see that this gives

$$\frac{n_i+\rho_T}{n_i+\rho_S}$$

to be independent of $n_i > 0$ (provided $n \geq r$), which is only possible if $\rho_T = \rho_S$. Let ρ denote this common value.

In the case $T = \{1,2\}$, $i = 1$ say, the equation corresponding to (139) has $h(n_1, n, 2)$ in place of

$$(1-(2^q-2)h(0,n,2))\cdot\frac{n_1+\rho_T}{n+2\rho_T}$$

which rearranges to give that

$$h(n_1,n,2) = \kappa_n\left(\frac{n_1+\rho_S}{n+1+3\rho_S}\right)$$

for some constant κ_n only depending on n. Since we have a similar expression for $h(n_2, n, 2)$ and, in this case,

$$\frac{h(n_1,n,2)+h(n_2,n,2)}{1-(2^q-2)h(0,n,2)} = 1,$$

we can eliminate κ_n to give

$$w(\alpha_1 \mid \alpha_1^{n_1}\alpha_2^{n_2}) = h(n_1,n,2) = \frac{n_1+\rho}{n+2\rho}\cdot(1-(2^q-2)h(0,n,2)).$$

Since the corresponding identity holds trivially for $|T| = 1$ we can now generalize (138) to all non-empty T as:

$$w\Big(\alpha_j \mid \bigwedge_{i\in T}\alpha_i^{m_i}\Big) = \frac{m_j+\rho}{m+\rho|T|}\cdot(1-(2^q-|T|)h(0,m,|T|)), \quad (140)$$

for $j \in T$, provided each $m_i > 0$.

Currently we have that $\rho \in (-1, \infty]$. We now show that $-1 < \rho \leq 0$ cannot happen.

Assume on the contrary that $-1 < \rho \leq 0$. By considering the sum of all probabilities of state descriptions for $a_1, a_2, \ldots, a_n$ ($n \geq 2^q$) which mention all of the atoms $\alpha_1, \alpha_2, \ldots, \alpha_{2^q}$ we see that with (140) and

$h(0, n, 2^q) = 0$,

$$1 \geq \sum_{\substack{n_1,\ldots,n_{2^q}>0\\ n_1+\cdots+n_{2^q}=n}} \binom{n}{n_1,\ldots,n_{2^q}} w\Big(\bigwedge_{i=1}^{2^q} \alpha_i^{n_i}\Big) \tag{141}$$

$$= \sum_{\substack{n_1,\ldots,n_{2^q}>0\\ n_1+\cdots+n_{2^q}=n}} \binom{n}{n_1,\ldots,n_{2^q}} w\Big(\bigwedge_{i=1}^{2^q} \alpha_i^{n_i-1} \mid \bigwedge_{i=1}^{2^q} \alpha_i\Big) w\Big(\bigwedge_{i=1}^{2^q} \alpha_i\Big)$$

$$= w\Big(\bigwedge_{i=1}^{2^q} \alpha_i\Big) \cdot \sum_{\substack{n_1,\ldots,n_{2^q}>0\\ n_1+\cdots+n_{2^q}=n}} \binom{n}{n_1,\ldots,n_{2^q}} \frac{\prod_{i=1}^{2^q}\prod_{j=1}^{n_i-1}(j+\rho)}{\prod_{j=2^q}^{n-1}(j+2^q\rho)}$$

$$= w\Big(\bigwedge_{i=1}^{2^q} \alpha_i\Big) \cdot \sum_{\substack{n_1,\ldots,n_{2^q}>0\\ n_1+\cdots+n_{2^q}=n}} \frac{\prod_{i=1}^{2^q}\prod_{j=1}^{n_i-1}(1+\rho/j)}{\prod_{j=2^q}^{n-1}(1+2^q\rho/j)} \cdot \frac{n(2^q-1)!}{n_1 n_2 \ldots n_{2^q}}. \tag{142}$$

We now make three observations. Firstly, since

$$\prod_{j>|\gamma|}^{n} (1+\gamma/j) = O(n^\gamma),$$

and for $\rho \leq 0$, n, n_i as in (141),

$$\prod_{i=1}^{2^q} (n_i - 1)^\rho \geq \Big(\sum_{i=1}^{2^q} (n_i - 1)\Big)^{2^q\rho} \geq (n-1)^{2^q\rho}.$$

we see that there is a fixed $\kappa > 0$ such that

$$\frac{\prod_{i=1}^{2^q}\prod_{j=1}^{n_i-1}(1+\rho/j)}{\prod_{j=2^q}^{n-1}(1+2^q\rho/j)} \geq \kappa.$$

Secondly one can show that there is a constant $v > 0$ such that for large n

$$\sum_{\substack{n_1,\ldots,n_{2^q}>0\\ n_1+\cdots+n_{2^q}=n}} (n_1 n_2 \ldots n_{2^q})^{-1} \geq \frac{v \log(n)}{n}.$$

Finally, by Reg, $w\big(\bigwedge_{i=1}^{2^q} \alpha_i\big) > 0$. Putting these three observations together we can now see that the right hand side of (142) tends to ∞ as $n \to \infty$, which provides the required contradiction since from (141) it must be bounded by 1.

So from now on we can take it that $\rho > 0$.

Notice that with this in place summing over the $j \in T$ in (140) and eliminating $(1 - (2^q - |T|)h(0, m, |T|))$ now gives (136).

Let v be the probability function

$$v = \sum_T w(\eta_T) c_\rho^T. \tag{143}$$

What we want to show then is that $w = v$.[72] Notice that since w satisfies Ax the $w(\eta_T)$ depend only on $|T|$ and hence v also satisfies Ax. Also since w satisfies Reg, $w\big(\bigwedge_{i=1}^{2^q} \alpha_i\big) > 0$, so

$$w\Big(\bigwedge_{i=1}^{2^q} \exists x\, \alpha_i(x)\Big) = w(\eta_{\{1,\dots,2^q\}}) > 0$$

and v also satisfies Reg.

Furthermore, as the notation would suggest, $v(\eta_T) = w(\eta_T)$ and $v(\theta \,|\, \eta_T) = c_\rho^T(\theta)$. To show this let Υ_T^m, as on page 66, be the disjunction of all state descriptions

$$\bigwedge_{i=1}^{m} \alpha_{h_i}(a_i)$$

for which $\{h_1, h_2, \dots, h_m\} = T$. Then for $S \subseteq \{1, 2, \dots, 2^q\}$, if $T \nsubseteq S$, $c_\rho^S(\Upsilon_T^m) = 0$ for $m > 0$, whilst if $T \subset S$ then $|T| < |S|$ and

$$c_\rho^S(\Upsilon_T^m) \leq c_\rho^S\Big(\bigwedge_{k=1}^{m} \bigvee_{i\in T} \alpha_i(a_k)\Big) = \prod_{n=1}^{m-1} \left(\frac{n + |T|\rho}{n + |S|\rho}\right)$$

so in both cases

$$\lim_{m\to\infty} c_\rho^S(\Upsilon_T^m) = 0.$$

Hence we must have by Lemma 10.7(i) that $c_\rho^T(\eta_T) = \lim_{m\to\infty} c_\rho^T(\Upsilon_T^m) = 1$. That $v(\eta_T) = w(\eta_T)$ and $c_\rho^T(\theta) = v(\theta \,|\, \eta_T)$ now follow from (143).

For v defined in this way a straightforward calculation shows that if $n_i > 0$ for $i \in T$ and $\sum_{i\in T} n_i = n$ then

$$v\Big(\alpha_j \mid \bigwedge_{i\in T} \alpha_i^{n_i}\Big) = (n_j + \rho)\, \beta(n, |T|, j) \tag{144}$$

where $\beta(n, |T|, j)$ only depends on n, $|T|$ and whether or nor $j \in T$. Thus v also satisfies the c-Principle.

Now let T be such that $w(\eta_T) > 0$. For a state description Θ which is a disjunct of Υ_T^m (140) and (144) give that

$$w\Big(\bigwedge_{i\in T} \alpha_i^{n_i} \,|\, \Theta\Big) = \xi(n, m, |T|)\; v\Big(\bigwedge_{i\in T} \alpha_i^{n_i} \,|\, \Theta\Big)$$

for some $\xi(n, m, |T|)$ depending only on $n, m, |T|$.

[72] This stratagem is adopted from the corresponding proof of Kuiper's on page 270 of [71].

Since we can write such a Θ as $\bigwedge_{i\in T} \alpha_i^{k_i} \wedge \bigwedge_{i\in T} \alpha_i$ it follows that

$$\begin{aligned}
w(\Theta) &= w\Big(\bigwedge_{i\in T} \alpha_i^{k_i} \mid \bigwedge_{i\in T} \alpha_i\Big) w\Big(\bigwedge_{i\in T} \alpha_i\Big) \\
&= \tau(m, |T|)\, v\Big(\bigwedge_{i\in T} \alpha_i^{k_i} \mid \bigwedge_{i\in T} \alpha_i\Big) v\Big(\bigwedge_{i\in T} \alpha_i\Big) = \tau(m, |T|)\, v(\Theta)
\end{aligned}$$

for some $\tau(m, |T|)$ depending only on m and $|T|$. Hence, for Θ ranging over the disjuncts of Υ_T^m, and assuming that $w(\eta_T) > 0$,

$$\begin{aligned}
w\Big(\bigwedge_{i\in T} \alpha_i^{n_i} \mid \Upsilon_T^m\Big) &= \frac{\sum_\Theta w\left(\bigwedge_{i\in T} \alpha_i^{n_i} \mid \Theta\right) w(\Theta)}{\sum_\Theta w(\Theta)} \\
&= \xi(n, m, |T|) \cdot \frac{\sum_\Theta v\left(\bigwedge_{i\in T} \alpha_i^{n_i} \mid \Theta\right) v(\Theta)}{\sum_\Theta v(\Theta)} \\
&= \xi(n, m, |T|)\, v\Big(\bigwedge_{i\in T} \alpha_i^{n_i} \mid \Upsilon_T^m\Big). \qquad (145)
\end{aligned}$$

Taking the limit as $m \to \infty$ of (145) now gives by Lemma 10.7 that

$$w_T\Big(\bigwedge_{i\in T} \alpha_i^{n_i}\Big) = w\Big(\bigwedge_{i\in T} \alpha_i^{n_i} \mid \eta_T\Big) = Kv\Big(\bigwedge_{i\in T} \alpha_i^{n_i} \mid \eta_T\Big) = Kc_\rho^T\Big(\bigwedge_{i\in T} \alpha_i^{n_i}\Big) \qquad (146)$$

where

$$K = \lim_{m\to\infty} \xi(n, m, |T|).$$

Indeed we must have that $K = 1$ since summing over all choices of n_i in (146) with sum n will give $1 = K \cdot 1$.

Finally then we obtain, as required, that

$$w = \sum_T w(\eta_T)\, w_T = \sum_T w(\eta_T)\, c_\rho^T = v.$$

Conversely if w satisfies Ax and has the form (143) then, as we have already seen for v, it satisfies the c-Principle. $\dashv$

Notice then that the c-Principle, somewhat exceptionally in this subject, allows solutions which give non-zero probability to non-tautologous universal sentences such as

$$\bigwedge_{j\notin T} \neg\exists x\, \alpha_j(x)$$

for $\emptyset \neq T \subseteq \{1, 2, \dots, 2^q\}$, though of course it says nothing about their probability except that it only depends on $|T|$ rather than T itself. Indeed if we want to justify these being non-zero then we have to justify the $w(\eta_T)$ being non-zero, which only leads us back to where we started.

Returning again to the derivation of Carnap's Continuum from JSP, this must be seen as something of a triumph for Carnap's programme.

For on the basis of this ostensibly rational postulate the whole problem of assigning probabilities in the absence of any knowledge has been reduced to the problem of selecting a single parameter, λ. Carnap himself spent some effort on this last step without coming to any final firm conclusion. Referring back to (103) what we can say is that for an agent selecting c^L_λ, λ is a measure of the agent's reluctance to be swayed away from ambivalence by new information. If λ is large then even if the agent only sees $a_1, a_2, \ldots, a_n$ which satisfy α_j, n will have to be correspondingly large before

$$c^L_\lambda\Big(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_j(a_i)\Big) = \frac{n+\lambda 2^{-q}}{n+\lambda} \tag{147}$$

becomes significantly different from 2^{-q}. Indeed when $\lambda = \infty$ no amount of such evidence can make any difference. Conversely if $\lambda > 0$ is small then even $n = 1$ will effect a dramatic change from 2^{-q} in the left hand side of (147), and a corresponding diminution in the probabilities of $\alpha_r(a_2)$ when $r \neq j$. Notice that the probability function c^L_0 aptly represents the natural limit of this process, in that case $\alpha_j(a_2)$ is given probability 1 on receipt of learning $\alpha_j(a_1)$ whilst all the other $\alpha_r(a_2)$ get probability 0. In this case then the agent irrevocably jumps to the conclusion that all the a_n must share the same properties as the first one seen.

One place that we might hope to find guidance on this choice of λ is the de Finetti prior of c^L_λ. It is easy to show, simply by comparing their values on state descriptions, that

$$c^L_\lambda = \kappa_q \int_{\mathbb{D}_{2^q}} w_{\vec{x}} \prod_{i=1}^{2^q} x_i^{\lambda 2^{-q}-1}\, d\mu_q(\vec{x}), \tag{148}$$

where μ_q is the standard Lebesgue measure on $\mathbb{D}_{2^q}$ and κ_q is a normalizing constant.[73] This possibly suggests that we should take $\lambda = 2^q$, giving equal likelihood to each of the $w_{\vec{x}}$ for $\vec{x} \in \mathbb{D}_{2^q}$.

Attractive as this might at first appear if we consider the restriction of c^L_λ to the language L^- with just the $q-1$ unary relation symbols $R_1, R_2, \ldots, R_{q-1}$ then we obtain (as already observed when discussing ULi) $c^{L^-}_\lambda$ and now

$$c^{L^-}_\lambda = \kappa_{q-1} \int_{\mathbb{D}_{2^{q-1}}} w_{\vec{x}} \prod_{i=1}^{2^{q-1}} x_i\, d\mu_{q-1}(\vec{x})$$

since $2^q 2^{-(q-1)} - 1 = 1$. In other words we are no longer giving the $\vec{x} \in \mathbb{D}_{2^{q-1}}$ equal likelihood[74], and of course the distortion becomes more

[73]In fact $\kappa_q = (\Gamma(\lambda 2^{-q}))^{2^q}\Gamma(\lambda)^{-1}$. The measures $\kappa_q \prod_{i=1}^{2^q} x_i^{\lambda 2^{-q}-1}\, d\mu_q$ are commonly referred to as *Dirichlet priors*.

[74]A point remarked upon by Kemeny already in [62].

pronounced the further we move away from the 'preferred' language L. But why should any one language be favoured in this way?

A second approach that has been employed to determine λ is to argue that by 'symmetry' we should have that for any possible signature $\vec{m} = \langle m_1, m_2, \dots, m_{2^q} \rangle$ with $\sum_i m_i = n$,

$$w\Big(\bigvee_{sig(\Theta(a_1,\dots,a_n))=\vec{m}} \Theta(a_1, \dots, a_n) \Big)$$

is the same independent of $\vec{m}$, where $sig(\Theta(a_1, \dots, a_n))$ is the signature of the state description $\Theta(a_1, \dots, a_n)$.

The number of possible $\vec{m}$ here is

$$\binom{n + 2^q - 1}{2^q - 1}$$

and for a particular such $\vec{m} = \langle m_1, m_2, \dots, m_{2^q} \rangle$ the number of state descriptions with that signature is

$$\binom{n}{m_1, \dots, m_{2^q}}.$$

Hence according to this recipe a state description Θ with such a signature $\vec{m}$ should get probability

$$\binom{n + 2^q - 1}{2^q - 1}^{-1} \times \binom{n}{m_1, \dots, m_{2^q}}^{-1},$$

which turns out to equal $c^L_{2^q}(\Theta)$, thus providing (another) argument for taking $\lambda = 2^q$ in the case where the language has q unary relation symbols.

Clearly though the version of 'symmetry' being applied here seems to be of a different nature from that being used in other contexts (e.g. Ex, Px, Ax) throughout this monograph, a point which will become very apparent in the forthcoming Chapter 23 when we give a precise account of what we mean by a symmetry.

We conclude this chapter by mentioning a property of the c^L_λ for $\lambda > 0$ which is sometimes seen as a failing according to the aspirations of 'Applied Inductive Logic'. Namely, because their de Finetti priors give zero measure to all the N_T for $\emptyset \neq T \subset \{1, 2, \dots, 2^q\}$ by Corollary 10.4 all non-tautologous universal sentences $\forall x\, \theta(x)$, with $\theta(x) \in QFFL$, will get probability zero. In particular then not only does SReg fail but no amount of 'evidence' $\bigwedge_{i=1}^n \theta(a_i)$ can cause

$$c^L_\lambda\Big(\forall x\, \theta(x) \mid \bigwedge_{i=1}^{n} \theta(a_i) \Big)$$

to move away from zero, despite the intuition that a sufficient number of positive instances of a law, and no negative instances, should somehow justify one suspecting that the general law does actually hold.

On the other hand, by Proposition 16.1, for $0 < \lambda < \infty$ and consistent $\theta(x) \in QFFL$,

$$\lim_{n\to\infty} c_\lambda^L\Big(\theta(a_{n+1}) \mid \bigwedge_{i=1}^{n} \theta(a_i)\Big) = 1$$

so that the more 'evidence' $\bigwedge_{i=1}^{n} \theta(a_i)$ one has the more probable, according to c_λ^L, it is that the *next* individual a_{n+1} will satisfy θ. It has been argued, by Carnap amongst others, that this instance confirmation is all we really need in practice, rather than any confirmation of the universal law that *every* a_j satisfies $\theta(x)$.

Chapter 18

ANOTHER CONTINUUM OF INDUCTIVE METHODS

We have seen in the previous chapters that if w satisfies JSP (and Ex) and $q \geq 2$ then w is a member of Carnap's Continuum of Inductive Methods, i.e. $w = c_\lambda^L$ for some $0 \leq \lambda \leq \infty$. Since Ex and JSP appear fairly natural and attractive principles it would seem unlikely that there could be other appealing principles around which would lead to an essentially different family of probability functions. However as we shall now see that is in fact the case.

Informally the Extended Principle of Instantial Relevance tells us that in the presence of information, or knowledge, $\psi(a_1, a_2, \dots, a_n)$, learning that $\theta(a_{n+1})$ holds should enhance our belief that $\theta(a_{n+2})$ will hold. But what if instead of learning $\theta(a_{n+1})$ we had learnt only some consequence $\phi(a_{n+1})$ of $\theta(a_{n+1})$, should this not also act as positive support for $\theta(a_{n+2})$ holding and so also enhance, or at least not diminish, belief in $\theta(a_{n+2})$?

For example, waking up at 6.00 am in a new country and observing rain outside might, in the spirit of PIR, cause me to increase my belief that it will also be raining at 6.00 am tomorrow morning. But suppose instead I overslept until 7.00 am and only saw that the streets were wet though the sky was clear. Then that might still cause me to think that it was probably raining an hour ago and that quite likely I would see rain at 6.00 tomorrow if I managed to get up that early.

This intuition is summed up in the following principle:[75]

The Generalized Principle of Instantial Relevance, GPIR.
For $\theta(a_1), \phi(a_1), \psi(a_1, a_2, \dots, a_n) \in QFSL$, *if* $\theta(x_1) \vDash \phi(x_1)$ *then*

$$w(\theta(a_{n+2}) \mid \phi(a_{n+1}) \wedge \psi(a_1, a_2, \dots, a_n)) \geq w(\theta(a_{n+2}) \mid \psi(a_1, a_2, \dots, a_n)). \tag{149}$$

At this point one might wonder if there should not also be such a principle for when $\phi(x_1) \vDash \theta(x_1)$. In fact it would be equivalent to GPIR.

[75] Although this principle is stated for $\psi(\vec{a}) \in QFSL$ that actually forces it to hold too for $\psi(\vec{a}) \in SL$ not necessarily quantifier free. This can be proved by induction on the quantifier complexity of ψ, for fixed $\theta(x_1), \phi(x_1)$, using the evident extension of Lemma 6.1 to blocks of like quantifiers as in Lemma 3.8.

To see this notice that

$$\begin{aligned} w(\theta(a_{n+2}) \mid \psi(\vec{a})) = {} & w(\theta(a_{n+2}) \mid \phi(a_{n+1}) \wedge \psi(\vec{a})) \cdot w(\phi(a_{n+1}) \mid \psi(\vec{a})) \\ & + w(\theta(a_{n+2}) \mid \neg\phi(a_{n+1}) \wedge \psi(\vec{a})) \cdot w(\neg\phi(a_{n+1}) \mid \psi(\vec{a})) \end{aligned}$$

and

$$w(\phi(a_{n+1}) \mid \psi(\vec{a})) + w(\neg\phi(a_{n+1}) | \psi(\vec{a})) = 1,$$

so

$$\begin{aligned} & w(\theta(a_{n+2}) \mid \phi(a_{n+1}) \wedge \psi(\vec{a})) \geq w(\theta(a_{n+2}) \mid \psi(\vec{a})) \\ & \quad \Longleftrightarrow \; w(\theta(a_{n+2}) \mid \neg\phi(a_{n+1}) \wedge \psi(\vec{a})) \leq w(\theta(a_{n+2}) \mid \psi(\vec{a})) \\ & \quad \Longleftrightarrow \; w(\neg\theta(a_{n+2}) \mid \neg\phi(a_{n+1}) \wedge \psi(\vec{a})) \geq w(\neg\theta(a_{n+2}) \mid \psi(\vec{a})), \end{aligned}$$

from which the result follows since $\phi(x_1) \vDash \theta(x_1)$ just if $\neg\theta(x_1) \vDash \neg\phi(x_1)$.

Notice then that in this case

$$w(\neg\theta(a_{n+2}) \mid \phi(a_{n+1}) \wedge \psi(\vec{a})) \leq w(\neg\theta(a_{n+2}) \mid \psi(\vec{a})) \tag{150}$$

so the inequality goes the other way round when $\phi(a_{n+1})$ provides negative support for $\neg\theta(a_{n+1})$ (in the sense of providing positive support for $\theta(a_{n+1})$).

Our next proposition shows that GPIR is simply a consequence of Ax when we drop the additional 'evidence' $\psi(a_1, a_2, \ldots, a_n)$. From this point of view then GPIR might be interpreted as asserting that as far as the basic inequality is concerned this additional evidence can be ignored.

PROPOSITION 18.1. *If w satisfies Ax then for $\theta(x), \phi(x) \in QFFL$ with $\theta(x) \vDash \phi(x)$,*

$$w(\theta(a_2) \mid \phi(a_1)) \geq w(\theta(a_2)). \tag{151}$$

PROOF. Without loss of generality let

$$\theta(a_2) \equiv \bigvee_{i=1}^{n} \alpha_{h_i}(a_2) \equiv \bigvee_{j=1}^{2^q} \bigvee_{i=1}^{n} \alpha_j(a_1) \wedge \alpha_{h_i}(a_2), \quad \phi(a_1) \equiv \bigvee_{i=1}^{m} \alpha_{h_i}(a_1), \tag{152}$$

where $n \leq m$. Clearly the result holds if $n = 0$ (i.e. $\theta(x)$ is contradictory) so we may suppose that $m > 0$. By Ax $w(\alpha_i(a_2)) = 2^{-q}$ and

$$w(\alpha_i(a_1) \wedge \alpha_j(a_2)) = \begin{cases} b & \text{if } i = j, \\ c & \text{if } i \neq j. \end{cases}$$

for some c, b such that

$$1 = w\Big(\bigvee_{j,k=1}^{2^q} \alpha_j(a_1) \wedge \alpha_k(a_2)\Big) = 2^q b + (2^{2q} - 2^q)c. \tag{153}$$

From (152), (153),

$w(\theta(a_2)) = n2^{-q},\ w(\phi(a_1)) = m2^{-q},\ w(\theta(a_2) \wedge \phi(a_1)) = nb + n(m-1)c,$

so we are required to show that

$$w(\theta(a_2) \mid \phi(a_1)) = m^{-1}2^q(nb + n(m-1)c) \geq n2^{-q} = w(\theta(a_2)).$$

Substituting for b using (153) reduces this inequality to

$$2^q(2^{-2q} - c) \geq m(2^{-2q} - c). \tag{154}$$

Since by PIR,

$$b \geq w(\alpha_1(a_1)) \cdot w(\alpha_1(a_2)) = 2^{-2q},$$

so from (153), $c \leq 2^{-2q}$ and together with $m \leq 2^q$ this gives (154). $\dashv$

Given the above, and earlier, considerations we might now venture to propose that our rational choice of w on SL should satisfy GPIR, Reg and Ax. As we shall show over the course of this chapter, that restricts w to be a member of another 'continuum of inductive methods'.[76]

First some notation. For $-(2^q - 1)^{-1} \leq \delta \leq 1$ set

$$w_L^\delta = 2^{-q} \sum_{i=1}^{2^q} w_{\vec{e_i}(\delta)} \tag{155}$$

where

$$\vec{e_i}(\delta) = \langle \gamma, \gamma, \ldots, \gamma, \gamma + \delta, \gamma, \ldots, \gamma \rangle \in \mathbb{D}_{2^q}$$

with the $\gamma + \delta$ in the ith place and (necessarily) $\gamma = 2^{-q}(1 - \delta)$.

In particular then

$$w_L^\delta\Big(\bigwedge_{i=1}^m \alpha_{h_i}(a_i)\Big) = 2^{-q} \sum_{j=1}^{2^q} \gamma^{m-m_j}(\gamma + \delta)^{m_j}$$

where as usual $m_j = |\{i \mid h_i = j\}|$. In the case $\delta \neq 1$ (so $\gamma \neq 0$) this can be rewritten as

$$w_L^\delta\Big(\bigwedge_{i=1}^m \alpha_{h_i}(a_i)\Big) = 2^{-q}\gamma^m \sum_{j=1}^{2^q} d^{m_j} \tag{156}$$

where $d = (1 + \delta/\gamma)$. Notice that $w_L^\delta = c_0^L$ when $\delta = 1$ and $w_L^\delta = c_\infty^L$ when $\delta = 0$. As will become apparent as we go along these are the only points at which the w_L^δ and c_λ^L agree.

The next theorem, and the later Theorem 18.5, are essentially rephrasings of results in [94], [95].

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[76] However the special case of GPIR

$$w(\forall x\, \phi(x) \mid \phi(a_i) \wedge \psi) \geq w(\forall x\, \phi(x) \mid \psi),$$

is easily seen to hold for all probability functions w, and to give a strict inequality when $w(\forall x\, \phi(x) \wedge \psi) > 0$ and $w(\phi(a_i) \mid \psi) < 1$ (an observation referred to by G.H. von Wright in [143] as the *Principal Theorem of Confirmation*).

THEOREM 18.2. *Let* $-(2^q-1)^{-1} < \delta < 1$. *Then* w_L^δ *satisfies GPIR, Reg, Ax, Ex.*[77]

PROOF. That w_L^δ satisfies Reg, Ex and Ax for $-(2^q-1)^{-1} < \delta < 1$ is clear from (156) so all the work is in showing that w_L^δ satisfies GPIR.

Suppose that $\phi(x) \vDash \theta(x)$. Then, (using the equivalent version of GPIR) after multiplying out the denominators, it will be enough to show that

$$w_L^\delta(\theta(a_{m+2}) \wedge \phi(a_{m+1}) \wedge \psi(\vec{a})) \cdot w_L^\delta(\psi(\vec{a})) - \\ w_L^\delta(\theta(a_{m+2}) \wedge \psi(\vec{a})) \cdot w_L^\delta(\phi(a_{m+1}) \wedge \psi(\vec{a})) \geq 0,$$

equivalently

$$\sum_{\alpha_i \vDash \theta} \Big(w_L^\delta(\alpha_i(a_{m+2}) \wedge \phi(a_{m+1}) \wedge \psi(\vec{a})) \cdot w_L^\delta(\psi(\vec{a})) - \\ w_L^\delta(\alpha_i(a_{m+2}) \wedge \psi(\vec{a})) \cdot w_L^\delta(\phi(a_{m+1}) \wedge \psi(\vec{a})) \Big) \geq 0. \quad (157)$$

In this expression, given what we are expecting to prove, the $\phi(a_{m+1})$ will support those $\alpha_i(a_{m+2})$ when $\alpha_i(x) \vDash \phi(x)$ and seemingly not do so when $\alpha_i(x) \vDash \neg\phi(x)$. This suggests first considering the terms in (157) in these two cases.

Let

$$\psi(\vec{a}) \equiv \bigvee_{h=1}^{p} \bigwedge_{r=1}^{m} \beta_{hr}(a_r)$$

(where the $\beta_{hr} \in \{\alpha_1, \dots, \alpha_{2^q}\}$), and for $1 \leq h \leq p$, $1 \leq k \leq 2^q$ let $m_{hk} = |\{\, r \mid \beta_{hr} = \alpha_k \}|$. Then for $d = (1 + \delta/\gamma)$, $\alpha_i(x) \vDash \neg\phi(x)$, and k ranging between 1 and 2^q,

$$w_L^\delta(\psi(\vec{a})) = 2^{-q}\gamma^m \sum_{s=1}^{p} \sum_{k} d^{m_{sk}}, \quad (158)$$

$$w_L^\delta(\phi(a_{n+1}) \wedge \psi(\vec{a})) \\ = 2^{-q}\gamma^{m+1} \sum_{\alpha_j \vDash \phi} \sum_{s=1}^{p} (d^{m_{sj}+1} + \sum_{k \neq j} d^{m_{sk}}) \\ = 2^{-q}\gamma^{m+1} \sum_{\alpha_j \vDash \phi} \sum_{s=1}^{p} (d^{m_{sj}+1} - d^{m_{sj}} + \sum_{k} d^{m_{sk}}), \quad (159)$$

[77] Actually it follows from Proposition 10.6 that these w_L^δ satisfy GPIR even when we drop the requirement that θ, ϕ, ψ are quantifier free.

$$w_L^\delta(\alpha_i(a_{n+2}) \wedge \psi(\vec{a}))$$
$$= 2^{-q}\gamma^{m+1}\sum_{h=1}^{p}(d^{m_{hi}+1} + \sum_{k\neq i} d^{m_{hk}})$$
$$= 2^{-q}\gamma^{m+1}\sum_{h=1}^{p}(d^{m_{hi}+1} - d^{m_{hi}} + \sum_{k} d^{m_{hk}}), \tag{160}$$

$$w_L^\delta(\alpha_i(a_{n+2}) \wedge \phi(a_{n+1}) \wedge \psi(\vec{a}))$$
$$= 2^{-q}\gamma^{m+2}\sum_{\alpha_j\vDash\phi}\sum_{h=1}^{p}(d^{m_{hj}+1} + d^{m_{hi}+1} + \sum_{k\neq i,j} d^{m_{hk}})$$
$$= 2^{-q}\gamma^{m+2}\sum_{\alpha_j\vDash\phi}\sum_{h=1}^{p}(d^{m_{hj}+1} - d^{m_{hj}} + d^{m_{hi}+1} - d^{m_{hi}} + \sum_{k} d^{m_{hk}}), \tag{161}$$

noticing that since $\alpha_j \vDash \phi$ and $\alpha_i \vDash \neg\phi$ in (161) we cannot have $i = j$.

Taking the difference of the products (161)×(158) − (160)×(159) now gives

$$-2^{-2q}\gamma^{2m+2}\sum_{\alpha_j\vDash\phi}\sum_{h=1}^{p}\sum_{s=1}^{p}(d-1)^2 d^{m_{hj}+m_{si}}. \tag{162}$$

In the case $\alpha_i \vDash \phi$ the expressions corresponding to (158), (159), (160) are the same as before whilst (161) becomes

$$2^{-q}\gamma^{m+2}\sum_{\substack{\alpha_j\vDash\phi\\ j\neq i}}\sum_{h=1}^{p}(d^{m_{hj}+1} - d^{m_{hj}} + d^{m_{hi}+1} - d^{m_{hi}} + \sum_{k} d^{m_{hk}})$$
$$+ 2^{-q}\gamma^{m+2}\sum_{h=1}^{p}(d^{m_{hi}+2} - d^{m_{hi}} + \sum_{k} d^{m_{hk}})$$

which equals

$$2^{-q}\gamma^{m+2}\sum_{\alpha_j\vDash\phi}\sum_{h=1}^{p}(d^{m_{hj}+1} - d^{m_{hj}} + d^{m_{hi}+1} - d^{m_{hi}} + \sum_{k} d^{m_{hk}})$$
$$+ 2^{-q}\gamma^{m+2}\sum_{h=1}^{p}(d^{m_{hi}+2} - 2d^{m_{hi}+1} + d^{m_{hi}}). \tag{163}$$

Taking the difference of products $(163)\times(158) - (160)\times(159)$ and changing the variable k to j yields

$$-2^{-2q}\gamma^{2m+2}\sum_{\alpha_j\vDash\phi}\sum_{h=1}^{p}\sum_{s=1}^{p}(d-1)^2 d^{m_{hj}+m_{si}}+$$
$$2^{-2q}\gamma^{2m+2}\sum_{h=1}^{p}\sum_{s=1}^{p}\Big(d^{m_{hi}}(d-1)^2\sum_j d^{m_{sj}}\Big). \quad (164)$$

Taking the sum of the (162), (164) for all $\alpha_i \vDash \theta$ now gives that the sum in (157) is

$$2^{-2q}\gamma^{2m+2}(d-1)^2\sum_{h=1}^{p}\sum_{s=1}^{p}\Big(\sum_{\alpha_i\vDash\theta\wedge\phi}\sum_{\alpha_j} d^{m_{sj}+m_{hi}} - \sum_{\alpha_i\vDash\theta}\sum_{\alpha_j\vDash\phi} d^{m_{hj}+m_{si}}\Big). \quad (165)$$

Transposing the variables i, j in the second term in (165) gives

$$2^{-2q}\gamma^{2m+2}(d-1)^2\sum_{h=1}^{p}\sum_{s=1}^{p}\Big(\sum_{\alpha_i\vDash\theta\wedge\phi}\sum_{\alpha_j} d^{m_{sj}+m_{hi}} - \sum_{\alpha_j\vDash\theta}\sum_{\alpha_i\vDash\phi} d^{m_{hi}+m_{sj}}\Big). \quad (166)$$

But clearly since any $\alpha_i \vDash \phi$ is also such that $\alpha_i \vDash \theta\wedge\phi$ (since $\phi\vDash\theta$) the first term in the sum in (166) is at least as large as the second so the positivity of (157) is established. ⊣

Notice that $0 < d \neq 1$ for $-(2^q-1)^{-1} < \delta < 1$ so in fact for δ in this range (166) will *always* be strictly positive provided there is an atom α_j such that $\alpha_j \vDash \neg\theta$, i.e. provided θ is not a tautology, and there is an atom α_i such that $\alpha_i \vDash \phi$, i.e. ϕ is not a contradiction. Clearly if one of these conditions fails (166), and hence (157), will be zero.

We now turn to proving a converse to Theorem 18.2. First we need a lemma.

LEMMA 18.3. *Let the probability function w on SL satisfy GPIR, Ex, Ax and Reg. Let μ be the de Finetti prior of w and let*

$$\vec{b} = \langle b_1, b_2, \ldots, b_{2^q}\rangle \in \mathbb{D}_{2^q}$$

be in the support of μ. Then for any permutation $\sigma \in \mathrm{S}_{2^q}$ and $1 \le i < j \le 2^q$,

$$(b_i - b_{\sigma(i)})(b_j - b_{\sigma(j)}) \le 0.$$

PROOF. To simplify the notation here, and hereafter, we adopt the convention already introduced on page 106 of letting

$$\bigwedge_{r=1}^{2^q}\alpha_r^{k_r}$$

denote

$$\bigwedge_{r=1}^{2^q} \bigwedge_{i=1}^{k_r} \alpha_r(a_{p_r+i}),$$

where $p_r = \sum_{j=1}^{r-1} k_j$, and write α_j, α_i for $\alpha_j(a_{k+1}), \alpha_i(a_{k+2})$, respectively where $k = \sum_r k_r$. (Since w satisfies Ex the particular choice of instantiating constants will not matter here.)

For $1 \leq i, j \leq 2^q$, $i \neq j$, GPIR gives that for $\sigma \in \mathrm{S}_{2^q}$

$$w\Big(\alpha_i \mid \bigwedge_{r=1}^{2^q} \alpha_r^{k_r} \vee \bigwedge_{r=1}^{2^q} \alpha_{\sigma^{-1}(r)}^{k_r}\Big) \geq w\Big(\alpha_i \mid \alpha_j \wedge \Big(\bigwedge_{r=1}^{2^q} \alpha_r^{k_r} \vee \bigwedge_{r=1}^{2^q} \alpha_{\sigma^{-1}(r)}^{k_r}\Big)\Big) \tag{167}$$

by (150) since $\alpha_j(x) \models \neg\alpha_i(x)$.

Consider the left hand side of (167). Assuming that σ is not the identity (if it is the required conclusion is immediate) the sentences $\bigwedge_{r=1}^{2^q} \alpha_r^{k_r}$ and $\bigwedge_{r=1}^{2^q} \alpha_{\sigma^{-1}(r)}^{k_r}$ are disjoint, so

$$w\Big(\alpha_i | \bigwedge_{r=1}^{2^q} \alpha_r^{k_r} \vee \bigwedge_{r=1}^{2^q} \alpha_{\sigma^{-1}(r)}^{k_r}\Big) = \frac{w\big(\alpha_i \wedge \bigwedge_{r=1}^{2^q} \alpha_r^{k_r}\big) + w\big(\alpha_i \wedge \bigwedge_{r=1}^{2^q} \alpha_{\sigma^{-1}(r)}^{k_r}\big)}{w\big(\bigwedge_{r=1}^{2^q} \alpha_r^{k_r}\big) + w\big(\bigwedge_{r=1}^{2^q} \alpha_{\sigma^{-1}(r)}^{k_r}\big)}. \tag{168}$$

By Ax,

$$w\Big(\bigwedge_{r=1}^{2^q} \alpha_{\sigma^{-1}(r)}^{k_r}\Big) = w\Big(\bigwedge_{r=1}^{2^q} \alpha_r^{k_r}\Big) \tag{169}$$

$$w\Big(\alpha_i \wedge \bigwedge_{r=1}^{2^q} \alpha_{\sigma^{-1}(r)}^{k_r}\Big) = w\Big(\alpha_{\sigma(i)} \wedge \bigwedge_{r=1}^{2^q} \alpha_r^{k_r}\Big) \tag{170}$$

so by (168), (169) and (170) we have that

$$\begin{aligned} &w\Big(\alpha_i \mid \bigwedge_{r=1}^{2^q} \alpha_r^{k_r} \vee \bigwedge_{r=1}^{2^q} \alpha_{\sigma^{-1}(r)}^{k_r}\Big) \\ &\qquad = \frac{w\big(\alpha_i \wedge \bigwedge_{r=1}^{2^q} \alpha_r^{k_r}\big)}{2w\big(\bigwedge_{r=1}^{2^q} \alpha_r^{k_r}\big)} + \frac{w\big(\alpha_{\sigma(i)} \wedge \bigwedge_{r=1}^{2^q} \alpha_r^{k_r}\big)}{2w\big(\bigwedge_{r=1}^{2^q} \alpha_r^{k_r}\big)} \\ &\qquad = \frac{\int_{\mathbb{D}_{2^q}} x_i \prod_{r=1}^{2^q} x_r^{k_r}\, d\mu}{2\int_{\mathbb{D}_{2^q}} \prod_{r=1}^{2^q} x_r^{k_r}\, d\mu} + \frac{\int_{\mathbb{D}_{2^q}} x_{\sigma(i)} \prod_{r=1}^{2^q} x_r^{k_r}\, d\mu}{2\int_{\mathbb{D}_n} \prod_{r=1}^{2^q} x_r^{k_r}\, d\mu}. \end{aligned} \tag{171}$$

Putting $k_r = [mb_r]$ and letting $m \to \infty$ Lemma 12.1 gives us that the limiting value of the left hand side of (167) as $m \to \infty$ is

$$\frac{b_i + b_{\sigma(i)}}{2}.$$

An analogous argument gives a limiting value to the right hand side of (167) of

$$\frac{b_i b_j + b_{\sigma(i)} b_{\sigma(j)}}{b_j + b_{\sigma(j)}}.$$

The inequality in (167) must be preserved in the limit so

$$\frac{b_i + b_{\sigma(i)}}{2} \geq \frac{b_i b_j + b_{\sigma(i)} b_{\sigma(j)}}{b_j + b_{\sigma(j)}},$$

from which we obtain that

$$(b_i - b_{\sigma(i)})(b_j - b_{\sigma(j)}) \leq 0. \qquad \dashv$$

COROLLARY 18.4. *For w, μ as in Lemma 18.3, if $\langle b_1, b_2, \dots, b_{2^q} \rangle$ is in the support of μ then all the b_i are equal except for possibly one of them.*

PROOF. If there were 3 different values amongst the b_i, say $b_1 > b_2 > b_3$ then for $\sigma \in S_{2^q}$ with $\sigma(1) = 2$, $\sigma(2) = 3$ we have

$$(b_1 - b_{\sigma(1)})(b_2 - b_{\sigma(2)}) > 0$$

contradicting Lemma 18.3. If amongst the b_i two different values were repeated, say $b_1 = b_2 > b_3 = b_4$, then we can similarly obtain a contradiction with $\sigma(1) = 3$, $\sigma(2) = 4$. The result now follows. $\dashv$

We are now ready to prove the converse of Theorem 18.2, at least for $q \geq 2$.

THEOREM 18.5. *Let the probability function w on SL satisfy GPIR, Ex, Ax and Reg and $q \geq 2$. Then $w = w_L^\delta$ for some $-(2^q - 1)^{-1} < \delta < 1$.*

PROOF. Let μ be the de Finetti prior of w. By Lemma 18.4 we already know that points in the support of μ can have at most one coordinate different from all the others. Suppose that $\langle \gamma + \delta, \gamma, \gamma, \dots, \gamma \rangle$, $\langle \zeta + \beta, \zeta, \zeta, \dots, \zeta \rangle$ are distinct points in the support of μ – since w satisfies Ax μ is invariant under permutations of the coordinates (recall the convention on page 90) and we may assume that the 'odd coordinate' is the first one. We shall derive a contradiction from the assumption that $\beta \neq \delta$.

First suppose that $-(2^q - 1)^{-1} \leq \beta < \delta < 1$. Then by Corollary 12.4 there exists $0 < \eta < 1$ and state descriptions ϕ_n, ψ_n such that for any $r_1, r_2, r_3 \in \mathbb{N}$,

$$\lim_{n \to \infty} w(\alpha_1^{r_1} \alpha_2^{r_2} \alpha_3^{r_3} \mid \phi_n(a_1, \dots, a_{s_n}) \vee \psi_n(a_1, \dots, a_{t_n}))$$
$$= \eta(\gamma + \delta)^{r_1} \gamma^{r_2} \gamma^{r_3} + (1 - \eta)(\zeta + \beta)^{r_1} \zeta^{r_2} \zeta^{r_3}. \quad (172)$$

Notice that such $\alpha_1, \alpha_2, \alpha_3$ exist since $q \geq 2$.[78] Since $\alpha_2 \models \neg\alpha_3$, by GPIR we have (with the obvious shorthand notation) that

$$w(\alpha_3 \mid \alpha_2 \wedge (\phi_n \vee \psi_n)) \leq w(\alpha_3 \mid \phi_n \vee \psi_n)$$

[78] A situation similar to that in the proof of Theorem 17.2,

Using (172) and taking the limit as $n \to \infty$ now gives that

$$\frac{\eta\gamma^2 + (1-\eta)\zeta^2}{\eta\gamma + (1-\eta)\zeta} \leq \frac{\eta\gamma + (1-\eta)\zeta}{\eta + (1-\eta)}.$$

But multiplying this out gives $\gamma^2 + \zeta^2 \leq 2\gamma\zeta$, which is false since $\gamma \neq \zeta$ (as $\beta \neq \delta$).

Considering the possible remaining points of support of μ we now see that for some $0 \leq c \leq 1$ and $-(2^q - 1)^{-1} \leq \delta < 1$, $w = cw_L^\delta + (1-c)w_L^1$. We cannot have that $c = 0$, since by assumption w satisfies Reg. The aim now is to show that $c = 1$.

From GPIR we have that

$$w(\alpha_2 \mid \alpha_3\alpha_1^n) \leq w(\alpha_2 \mid \alpha_1^n),$$

so, canceling out the common factors of 2^{-q},

$$\frac{c((\gamma+\delta)^n\gamma^2 + 2(\gamma+\delta)\gamma^{n+1} + (2^q-3)\gamma^{n+2})}{c((\gamma+\delta)^n\gamma + (\gamma+\delta)\gamma^n + (2^q-2)\gamma^{n+1})} \\ \leq \frac{c((\gamma+\delta)^n\gamma + \gamma^n(\gamma+\delta) + (2^q-2)\gamma^{n+1})}{c((\gamma+\delta)^n + (2^q-1)\gamma^n) + (1-c)}.$$

But if $c < 1$ the right hand side here tends to 0 as $n \to \infty$ whereas the left hand side tends to γ if $\delta \geq 0$ and to $\gamma(2\delta + (2^q-1)\gamma)(\delta + (2^q-1)\gamma)^{-1}$ if $\delta < 0$.

Finally the value $-(2^q-1)^{-1}$ for δ is disallowed because of the assumption that our probability function satisfies Reg. $\dashv$

In Theorem 18.5 we require that $q \geq 2$, in other words that the language L has at least two relation symbols. In the case that L has just one relation symbol, R_1, GPIR holds for any probability function on SL. The reason is that in that case there are, up to logical equivalence, only 4 choices of $\theta(x)$ (and $\phi(x)$), namely $\top, \bot, R_1(x), \neg R_1(x)$. Thus if $\theta(x) \vDash \phi(x)$ either $\theta(x) = \phi(x)$ or $\theta(x) = \bot$ or $\phi(x) = \top$ and it is easy to check in each of these cases that (149) must hold.

We finally remark that, as in the derivation in [95], we can weaken the assumption of Ax in Theorem 18.5 to Px and SN. However since the w_L^δ satisfy Ax there now seems little point in doing so.

In the next chapter we will investigate some of the other properties of the w_L^δ.

Chapter 19

THE NP-CONTINUUM

The first notable property of the w_L^δ, at least for $0 \le \delta \le 1$, that we shall show is:

PROPOSITION 19.1. *For a fixed $0 \le \delta \le 1$ the w_L^δ form a Unary Language Invariant family with GPIR.*

PROOF. As usual let $L = \{R_1, \dots, R_q\}$. It is enough to show that the restriction of $w_{L'}^\delta$ to SL *is* w_L^δ where $L' = \{R_1, \dots, R_q, \dots, R_r\}$. In this case notice that whilst the support points of the de Finetti prior of w_L^δ are of the form

$$\langle \gamma, \gamma, \dots, \gamma, \gamma + \delta, \gamma, \dots, \gamma, \gamma \rangle$$

those of $w_{L'}^\delta$ are of the form

$$\langle 2^{q-r}\gamma,\ 2^{q-r}\gamma,\ \dots, 2^{q-r}\gamma,\ 2^{q-r}\gamma + \delta,\ 2^{q-r}\gamma, \dots, 2^{q-r}\gamma,\ 2^{q-r}\gamma \rangle$$

and that, in general, we require here that $\delta \ge 0$ since for $\delta < 0$, $2^{q-r}\gamma + \delta$ will be negative for r large enough.

By repeated application it is enough to suppose here that $r = q+1$ and to show that $w_{L'}^\delta$ agrees with w_L^δ on state descriptions of L. Letting the $\alpha_i(x)$ be the atoms of L and with the obvious shorthand of writing for example $\alpha_i \wedge (R_{q+1} \vee \neg R_{q+1})$ for instantiations of $\alpha_i(x) \wedge (R_{q+1}(x) \vee \neg R_{q+1}(x))$ etc. we have that

$$\begin{aligned} w_{L'}^\delta\Big(\bigwedge_{i=1}^{2^q} \alpha_i^{m_i}\Big) &= w_{L'}^\delta\Big(\bigwedge_{i=1}^{2^q} (\alpha_i \wedge (R_{q+1} \vee \neg R_{q+1}))^{m_i}\Big) \\ &= \sum_{\vec{v}^1, \dots, \vec{v}^{2^q}} w_{L'}^\delta\Big(\bigwedge_{i=1}^{2^q} \bigwedge_{j=1}^{m_i} (\alpha_i \wedge R_{q+1}^{v_j^i})\Big) \end{aligned}$$

where the $v_j^i \in \{0, 1\}$ and as usual $R_{q+1}^1 = R_{q+1}$ and $R_{q+1}^0 = \neg R_{q+1}$.

Appealing to Ex this sum equals

$$\sum_{s_1=0}^{m_1}\cdots\sum_{s_{2^q}=0}^{m_{2^q}} w_{L'}^{\delta}\Big(\bigwedge_{i=1}^{2^q}((\alpha_i\wedge R_{q+1})^{s_i}\wedge(\alpha_i\wedge\neg R_{q+1})^{m_i-s_i})\Big)\prod_{k=1}^{2^q}\binom{m_k}{s_k} =$$

$$2^{-q-1}\sum_{s_1=0}^{m_1}\cdots\sum_{s_{2^q}=0}^{m_{2^q}}\Big\{\sum_{i=1}^{2^q}(\zeta^{m-s_i}(\zeta+\delta)^{s_i}+\zeta^{m-(m_i-s_i)}(\zeta+\delta)^{m_i-s_i})\Big\}\prod_{k=1}^{2^q}\binom{m_k}{s_k}$$

where $m=\sum_{j=1}^{2^q} m_j$ and $\zeta=2^{-1}\gamma$. Moving the summation over i in this expression to the front the resulting ith summand is

$$2^{-q-1}\sum_{s_1=0}^{m_1}\cdots\sum_{s_{2^q}=0}^{m_{2^q}}\Big\{\zeta^{m-s_i}(\zeta+\delta)^{s_i}+\zeta^{m-(m_i-s_i)}(\zeta+\delta)^{m_i-s_i}\Big\}\prod_{k=1}^{2^q}\binom{m_k}{s_k}$$

$$= 2^{-q-1}\sum_{s_i=0}^{m_i}\binom{m_i}{s_i}\Big\{\zeta^{m-s_i}(\zeta+\delta)^{s_i}+$$
$$\zeta^{m-(m_i-s_i)}(\zeta+\delta)^{m_i-s_i}\Big\}\prod_{k\neq i}\Big(\sum_{s_k=0}^{m_k}\binom{m_k}{s_k}\Big)$$

$$= 2^{-q-1}2\zeta^{m-m_i}((\zeta+\delta)+\zeta)^{m_i}\Big(\prod_{k\neq i}2^{m_k}\Big)=2^{-q}\gamma^{m-m_i}(\gamma+\delta)^{m_i}.$$

Reintroducing the summation over i now gives $w_L^{\delta}\big(\bigwedge_{i=1}^{2^q}\alpha_i^{m_i}\big)$, as required. ⊣

We remark here that for $-(2^q-1)^{-1}\le\delta<0$ the probability function w_L^{δ} does not satisfy ULi with GPIR. This is already suggested by the initial remarks in the above proof and will become transparent from the forthcoming proof of Theorem 19.3.

In view of the desirability of Language Invariance it seems natural to focus on the w_L^{δ} for $0\le\delta\le1$ where they are referred to as the Nix-Paris or NP-continuum. The members of this continuum then satisfy ULi with GPIR, Ax and Reg for $0\le\delta<1$. It differs from Carnap's Continuum (as can be seen immediately by comparing de Finetti priors) except at the end points where $w_L^0=c_\infty^L$ and $w_L^1=c_0^L$. It also differs from Carnap's Continuum, for $0<\delta<1$, in failing to satisfy Reichenbach's Axiom.

To see this consider the sequence of atoms $\alpha_{h_i}(x)$ where $h_i=1$ if $i=0 \bmod 3$ and $h_i=2$ otherwise. Then with $j=1$ and the notation of the statement of RA on page 93, $u(n)/n\to1/3$ as $n\to\infty$. However

$$w_L^{\delta}\Big(\alpha_1(a_{3n})\mid\bigwedge_{i=1}^{3n-1}\alpha_{h_i}(a_i)\Big)=$$
$$\frac{\gamma^n(\gamma+\delta)^{2n}+\gamma^{2n}(\gamma+\delta)^n+(2^q-2)\gamma^{3n}}{\gamma^{n-1}(\gamma+\delta)^{2n}+\gamma^{2n}(\gamma+\delta)^{n-1}+(2^q-2)\gamma^{3n-1}}$$

which tends to γ as $n \to \infty$, so RA fails. (Of course if $\gamma = 1/3$ we can just change the frequency of the α_1.)

We now give a second characterization of the w_L^δ which is surely interesting though possibly rather perverse.

Recovery.

A probability function w on SL satisfies Recovery, *or is* Recoverable, *if whenever $\Psi(a_1, a_2, \ldots, a_n)$ is a state description then there is another state description $\Phi(a_{n+1}, a_{n+2}, \ldots, a_h)$ such that $w(\Phi \wedge \Psi) \neq 0$ and for any quantifier free sentence $\theta(a_{h+1}, a_{h+2}, \ldots, a_{h+g})$,*

$$w(\theta(a_{h+1}, a_{h+2}, \ldots, a_{h+g}) \mid \Phi \wedge \Psi) = w(\theta(a_{h+1}, a_{h+2}, \ldots, a_{h+g})). \tag{173}$$

In other words w is Recoverable if given any 'past history' as such a state description Ψ there is a possible 'future' state description Φ which will take us right back to where we started, at least as far as the quantifier free properties of the currently unobserved constants $a_{h+1}, a_{h+2}, \ldots$ are concerned.

Somewhat surprisingly it turns out that the notion of Recoverable remains the same if instead of requiring θ to be quantifier free we allow it to be any sentence of L. The reason for this is that if θ is not quantifier free then we may assume that it is in Prenex Normal Form and prove the result by induction on quantifier complexity using Lemma 6.1 to ensure there is no clash of constants with those in $\Phi \wedge \Psi$. (On this point see [102].)

As we shall now show, following [117], the w_L^δ satisfy Recovery provided $0 \leq \delta < 1$. Indeed these w_L^δ are characterized by satisfying Reg and ULi with Ax and *a single non-trivial instance* of Recovery. Clearly then the c_λ^L can never satisfy Recovery for $0 < \lambda < \infty$, not even a single instance of it.

Proposition 19.2. *For $0 \leq \delta < 1$ the w_L^δ satisfy Recovery.*

Proof. With the usual abbreviation let

$$\Psi(a_1, a_2 \ldots, a_n) = \alpha_1^{m_1} \alpha_2^{m_2} \ldots \alpha_{2^q}^{m_{2^q}}.$$

Then we have $n \geq m_1, \ldots, m_{2^q}$ and for $h = n2^q$,

$$\Phi(a_{n+1}, a_{n+2}, \ldots, a_h) = \alpha_1^{n-m_1} \alpha_2^{n-m_2} \ldots \alpha_{2^q}^{n-m_{2^q}},$$

we have

$$\Psi(a_1, a_2 \ldots, a_n) \wedge \Phi(a_{n+1}, a_{n+2}, \ldots, a_h) = \alpha_1^n \alpha_2^n \ldots \alpha_{2^q}^n.$$

It is now easy to check that

$$w_L^\delta(\theta(a_{h+1}, a_{h+2}, \ldots, a_{h+g}) \mid \Phi \wedge \Psi) = w_L^\delta(\theta(a_{h+1}, a_{h+2}, \ldots, a_{h+g}))$$

firstly for θ a state description and in turn for arbitrary $\theta \in QFSL$. $\dashv$

THEOREM 19.3. *A probability function w on SL satisfying Reg and ULi with Ax has the property that for some state description $\Phi(a_1, a_2, \ldots, a_n)$ with $n > 0$, and for all $\theta(a_{n+1}, a_{n+2}, \ldots, a_{n+g}) \in QFSL$,*

$$w(\theta(a_{n+1}, a_{n+2}, \ldots, a_{n+g}) \mid \Phi(a_1, a_2, \ldots, a_n)) = w(\theta(a_{n+1}, a_{n+2}, \ldots, a_{n+g}))$$

just if $w = w_L^\delta$ for some $0 \le \delta < 1$.

PROOF. The result from right to left follows by the previous proposition (by taking $\Phi \wedge \Psi$ with $\Psi = \alpha_1(a_1)$, say).

In the other direction let w satisfy Language Invariance with Ax and let $\Phi(a_1, a_2, \ldots, a_n)$ be as in the statement of the theorem, say atom $\alpha_i(x)$ is instantiated m_i times in Φ. Let μ be the symmetric[79] de Finetti prior of w, so, by assumption, for *any* $s_1, s_2, \ldots, s_{2^q} \in \mathbb{N}$,

$$\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{s_i} \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu = \Big(\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{s_i} \, d\mu \Big) \cdot \Big(\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{m_i} \, d\mu \Big). \tag{174}$$

Since μ is symmetric, if $\sigma \in \mathsf{S}_{2^q}$ then (174) also holds for any $s_1, s_2, \ldots, s_{2^q} \in \mathbb{N}$ with $\prod_{i=1}^{2^q} x_{\sigma(i)}^{m_i}$ in place of $\prod_{i=1}^{2^q} x_i^{m_i}$. In particular then we can obtain that

$$\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{s_i} \prod_{i=1}^{2^q} x_i^{m_i} \prod_{i=1}^{2^q} x_{\sigma(i)}^{m_i} \, d\mu = \Big(\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{s_i} \, d\mu \Big) \cdot \Big(\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{m_i} \prod_{i=1}^{2^q} x_{\sigma(i)}^{m_i} \, d\mu \Big). \tag{175}$$

Continuing in this way by taking the product over all permutations σ yields some single $t \in \mathbb{N}$ such that (174) holds with each m_i replaced by this same t, and of course similarly for any multiple of t.

Hence we have that for $\theta(a_1, \ldots, a_s)$ a state description,

$$\frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^t \; w_{\vec{x}}(\theta(a_1, \ldots, a_s)) \, d\mu}{\int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\theta(a_1, \ldots, a_s)) \, d\mu} = \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^t \, d\mu. \tag{176}$$

Replacing $\theta(a_1, \ldots, a_s)$ by $\theta_m(a_1, \ldots, a_{s_m})$ as in Corollary 12.2 and taking the limit of the right hand side of (176) as $m \to \infty$ we obtain that

$$\prod_{i=1}^{2^q} b_i^t = \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^t \, d\mu$$

[79] As in (89).

for *any* support point $\vec{b}$ of μ. We conclude then that for

$$c = \Big(\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^t \, d\mu\Big)^{1/t},$$

$$\mu\{\langle x_1, \dots, x_{2^q}\rangle \in \mathbb{D}_{2^q} \mid \prod_{i=1}^{2^q} x_i = c\,\} = 1, \tag{177}$$

and indeed $c > 0$ since w is regular.

Now using Language Invariance let w' satisfy Ax and extend w on $L' = L \cup \{R_{q+1}(x)\}$, so again we can take the atoms of L' to be

$$\alpha_1(x) \wedge R_{q+1}(x), \alpha_1(x) \wedge \neg R_{q+1}(x), \alpha_2(x) \wedge R_{q+1}(x),$$
$$\alpha_2(x) \wedge \neg R_{q+1}(x), \dots, \alpha_{2^q}(x) \wedge R_{q+1}(x), \alpha_{2^q}(x) \wedge \neg R_{q+1}(x).$$

Let μ' be the de Finetti prior of w'. Then by (55),

$$\mu'\{\vec{x} \in \mathbb{D}_{2^{q+1}} \mid (x_1 + x_2)(x_3 + x_4) \prod_{i=3}^{2^q} (x_{2i-1} + x_{2i}) = c\,\} = 1 \tag{178}$$

and by Ax also

$$\mu'\{\vec{x} \in \mathbb{D}_{2^{q+1}} \mid (x_1 + x_3)(x_2 + x_4) \prod_{i=3}^{2^q} (x_{2i-1} + x_{2i}) = c\,\} = 1,$$

$$\mu'\{\vec{x} \in \mathbb{D}_{2^{q+1}} \mid (x_1 + x_4)(x_2 + x_3) \prod_{i=3}^{2^q} (x_{2i-1} + x_{2i}) = c\,\} = 1.$$

Hence

$$\mu'\{\vec{x} \in \mathbb{D}_{2^{q+1}} \mid$$
$$(x_1+x_2)(x_3+x_4) = (x_1+x_3)(x_2+x_4) = (x_1+x_4)(x_2+x_3)\,\} = 1$$

and in consequence

$$\mu'\{\vec{x} \in \mathbb{D}_{2^{q+1}} \mid$$
$$(x_1-x_4)(x_2-x_3) = (x_1-x_3)(x_2-x_4) = (x_1-x_2)(x_3-x_4) = 0\,\} = 1.$$

From this it follows that the set of $\langle x_1, x_2, \dots, x_{2^{q+1}}\rangle$ for which all except possibly one of the x_i are equal has μ'-measure 1.

Any such vector must then be of the form

$$\langle x, x, \dots, x, 1 - (2^{q+1} - 1)x, x, \dots, x, x\rangle,$$

where $0 \leq x \leq (2^{q+1} - 1)^{-1}$ since the coordinates cannot be negative. Also from (177)

$$(2x)^{2^q - 1}(1 - (2^{q+1} - 2)x) = c. \tag{179}$$

By considering the function $F(x) = (2x)^{2^q-1}(1-(2^{q+1}-2)x)$ on $[0,(2^{q+1}-1)^{-1}]$ we can show that there can be at most two possible solutions to (179). If there is only one solution then we already have our result. So suppose that there are two, ζ, ξ say, with $\xi > \zeta$. Then again by considering $F(x)$ we must have $\zeta \leq 2^{-q-1}$, $\xi > 2^{-q-1}$. Since μ' satisfies Ax then it must spread some measure v uniformly over the coordinate permutations of the point

$$\langle 1-(2^{q+1}-1)\zeta, \zeta, \zeta, \ldots, \zeta \rangle,$$

and similarly measure $1-v$ uniformly over coordinate permutations of the point

$$\langle 1-(2^{q+1}-1)\xi, \xi, \xi, \ldots, \xi \rangle.$$

We shall show that we must have $v = 1$. For suppose otherwise. Then by a similar argument to the one given above we could show that the corresponding μ'' on $L \cup \{R_{q+1}, R_{q+2}\}$ in the language invariant family must concentrate all measure on points of this same form, in particular spread measure $1-v$ uniformly over the coordinate permutations of the point

$$\langle 1-2^{-1}(2^{q+2}-1)\xi, 2^{-1}\xi, 2^{-1}\xi, \ldots, 2^{-1}\xi \rangle.$$

Similarly adding s additional relation symbols to L' would lead us to a measure spreading measure $1-v$ uniformly over the coordinate permutations of the point

$$\langle 1-2^{-s}(2^{q+s+1}-1)\xi, 2^{-s}\xi, 2^{-s}\xi, \ldots, 2^{-s}\xi \rangle.$$

But

$$1-2^{-s}(2^{q+s+1}-1)\xi = -2^{q+1}(\xi - 2^{-q-1}) + 2^{-s}\xi,$$

which, since $\xi > 2^{-q-1}$, will make this first coordinate negative for large enough s, contradiction.

Having dismissed this possibility then it follows that μ' spreads measure 1 uniformly over the coordinate permutations of the point

$$\langle 1-(2^{q+1}-1)\zeta, \zeta, \zeta, \ldots, \zeta \rangle.$$

Hence by marginalizing the same is true of μ, that is all μ's measure is distributed (uniformly) over the coordinate permutations of the point

$$\langle 1-(2^{q+1}-1)\zeta + \zeta, 2\zeta, 2\zeta, \ldots, 2\zeta \rangle.$$

which makes $w = w_L^\delta$ with $\delta = 1 - 2^{q+1}\zeta \geq 0$, as required. ⊣

It is easy to see that this proof also shows that:

COROLLARY 19.4. *For $\vec{c} \in \mathbb{D}_{2^q}$, $v_{\vec{c}}$ has an extension to a probability function for the unary language $L \cup \{R_{q+1}\}$ which satisfies Ax just if $v_{\vec{c}} = w_L^\delta$ for some $\delta \in [-(2^{q+1}-1)^{-1}, 1]$.*

At this point the reader might be as intrigued as the authors by the persistence with which the two continua, the c_λ^L and the w_L^δ, seem to divide between them most of the rational principles one comes up with. Are they perhaps manifestations of different interpretations of 'rationality'? A suggestion made in [115] is that the principles underlying the c_λ^L require the agent to picture his/her ambient structure M as the result of some regular statistical process, for example the picking of balls from an urn in the case of JSP. So the notion of rationality being taken here is of regularity, predictability, pattern. On the other hand the version of rationality being exploited, and subsequently manifest, in the case of the w_L^δ might be seen as simplicity or economy. For example the consideration that information should be ignored, or have the potential to be ignored, where possible leads to GPIR and Recoverability, and, as we shall see in the next chapter, the Weak Irrelevance Principle.

Of course since these continua intersect when $\lambda = 0, \delta = 1$ and again when $\lambda = \infty, \delta = 0$ we could have the best of both worlds by having the agent take one of these as his/her rational choice. However c_∞^L is unaltered by any amount of evidence whilst c_0^L goes completely the other way and on the basis of observing $\alpha(a_1)$ immediately jumps to the unalterable conclusion that all the other a_i will also satisfy this α, so neither of these options seems to tally with our intuitions about induction, though they certainly have been mooted. For example in his early works Carnap showed some sympathy for c_∞^L whilst an argument for c_0^L in terms of uniquely satisfying a very general (unary) symmetry principle is given in [112] and referred to later on page 174.

Chapter 20

THE WEAK IRRELEVANCE PRINCIPLE

In this chapter we shall consider a second, possibly more acceptable, property that the w_L^δ satisfy. In this case it does not provide a characterization of the NP-continuum, there are other probability functions satisfying this principle. However, as we shall subsequently see the probability functions in the NP-continuum do in a strong sense form the building blocks for all such probability functions.

Weak Irrelevance Principle, WIP.

Suppose that $\theta, \phi \in QFSL$ are such that they have no constant nor relation symbols in common. Then

$$w(\theta \wedge \phi) = w(\theta) \cdot w(\phi).$$

Notice that according to our convention (page 23) this condition is equivalent to

$$w(\theta \mid \phi) = w(\theta),$$

i.e. the *stochastic independence* of θ and ϕ. The argument for WIP is that if θ and ϕ come from entirely disjoint languages then being given ϕ tells us nothing about θ and so belief in θ should not change when conditioning on ϕ. Indeed this seems the most patent example one can imagine of 'independence' between sentences.[80]

Notice that by Corollary 6.2 WIP implies this same principle even for $\theta, \phi \in SL$. On the other hand, if $\theta, \phi \in QFSL$ are as in the statement of WIP then we may assume that $\theta \equiv \bigvee_i \Theta_i$ and $\phi \equiv \bigvee_j \Phi_j$ for state descriptions Θ_i, Φ_j in the same (sub)languages as θ and ϕ respectively, and in that case

$$w(\theta \wedge \phi) = w\Big(\bigvee_i \Theta_i \wedge \bigvee_j \Phi_j\Big) = \sum_{i,j} w(\Theta_i \wedge \Phi_j)$$

$$w(\theta) \cdot w(\phi) = w\Big(\bigvee_i \Theta_i\Big) \cdot w\Big(\bigvee_j \Phi_j\Big) = \sum_{i,j} w(\Theta_i) \cdot w(\Phi_j)$$

[80] Several other, stronger, formulations of independence are given in [44] amongst them the principle SDCIP already considered on page 111 in connection with Johnson's Sufficientness Postulate. Interestingly, as we shall shortly see, the apparent similarity between WIP and SDCIP is illusory, they have very different families of solutions.

so WIP follows from *WIP with θ, ϕ state descriptions* in disjoint (sub)languages L_1, L_2 of L (and for different constants).

In view of Proposition 8.1, see page 52, we obviously have

PROPOSITION 20.1. *For $\vec{b} \in \mathbb{D}_{2^q}$, $w_{\vec{b}}$ satisfies WIP.*

The $w_{\vec{b}}$ of course do not satisfy Ax unless each $b_i = 2^{-q}$. A key example of probability functions which satisfy both WIP and Ax is the NP-continuum of w_L^δ,[81] as the following result from [95] shows.

THEOREM 20.2. *The w_L^δ for $0 \leq \delta \leq 1$ satisfy WIP.*

PROOF. Let L_1, L_2 be disjoint sublanguages of L and let the atoms of L_1 be $\beta_i(x)$ and those of L_2, $\eta_j(x)$. Since the w_L^δ satisfy ULi we may take it that $L = L_1 \cup L_2$ and that we need show that

$$w_L^\delta\Big(\bigwedge_{i=1}^n \beta_{h_i}(a_i) \wedge \bigwedge_{j=1}^m \eta_{g_j}(a_{n+j})\Big) = w_{L_1}^\delta\Big(\bigwedge_{i=1}^n \beta_{h_i}(a_i)\Big) \cdot w_{L_2}^\delta\Big(\bigwedge_{j=1}^m \eta_{g_j}(a_{n+j})\Big),$$

equivalently that

$$\sum_{\vec{e},\vec{f}} w_L^\delta\Big(\bigwedge_{i=1}^n (\beta_{h_i}(a_i) \wedge \eta_{e_i}(a_i)) \wedge \bigwedge_{j=1}^m (\beta_{f_j}(a_{n+j}) \wedge \eta_{g_j}(a_{n+j}))\Big) \tag{180}$$

$$\sum_{\vec{e},\vec{f}} w_L^\delta\Big(\bigwedge_{i=1}^n (\beta_{h_i}(a_i) \wedge \eta_{e_i}(a_i))\Big) \cdot w_L^\delta\Big(\bigwedge_{j=1}^m (\beta_{f_j}(a_{n+j}) \wedge \eta_{g_j}(a_{n+j}))\Big), \tag{181}$$

are equal, where the e_i range over $\{1, 2, \dots, 2^{|L_2|}\}$ and the f_j range over $\{1, 2, \dots, 2^{|L_1|}\}$.

For $1 \leq k \leq 2^{|L_1|}$, $1 \leq r \leq 2^{|L_2|}$ let $v_{k,r}$ be that $w_{\vec{x}}$ in the definition (155) of w_L^δ which would give $\beta_k(a_1) \wedge \eta_r(a_1)$ value $\gamma + \delta$. Then we can rewrite (180), (181) as

$$\sum_{\vec{e},\vec{f}} \sum_{k=1}^{2^{|L_1|}} \sum_{r=1}^{2^{|L_2|}} 2^{-|L|} v_{k,r}\Big(\bigwedge_{i=1}^n (\beta_{h_i}(a_i) \wedge \eta_{e_i}(a_i)) \wedge \bigwedge_{j=1}^m (\beta_{f_j}(a_{n+j}) \wedge \eta_{g_j}(a_{n+j}))\Big) \tag{182}$$

---

[81] We remark that all the theorems in the rest of this chapter follow more easily when we start from the definition of $u^{\bar{p},L}$ as given for general languages on page 206 and note that:

- the w_L^δ are identical with the $u^{\langle 1-\delta, \delta, 0, 0, 0, \dots \rangle, L}$;
- the $u_n^{\bar{p},L}$ defined below are special cases of the $u^{\bar{p},L}$, cf. (194);
- the definition (193) of the $u^{\bar{p},L}$ is equivalent to the definition of $u^{\bar{p},L}$ from page 206 applied to unary languages (the equivalence is demonstrated on pages 208ff).

For these results see Theorem 29.5. The unary proofs are included here to keep this part of the exposition self contained, and for their own independent interest.

$$\sum_{\vec{e},\vec{f}} \left[\sum_{k=1}^{2^{|L_1|}} \sum_{u=1}^{2^{|L_2|}} 2^{-|L|} v_{k,u} \Big(\bigwedge_{i=1}^{n} (\beta_{h_i}(a_i) \wedge \eta_{e_i}(a_i)) \Big) \right.$$
$$\left. \times \sum_{r=1}^{2^{|L_2|}} \sum_{s=1}^{2^{|L_1|}} 2^{-|L|} v_{s,r} \Big(\bigwedge_{j=1}^{m} (\beta_{f_j}(a_{n+j}) \wedge \eta_{g_j}(a_{n+j})) \Big) \right], \tag{183}$$

where again the e_i range over $\{1, 2, \ldots, 2^{|L_2|}\}$ and the f_j range over $\{1, 2, \ldots, 2^{|L_1|}\}$.

Now for the moment fix $\vec{e}$, $\vec{f}$ and the summands for particular k, r as given in (182), (183). Let

$$J(\langle \vec{e}, r \rangle) = \{\langle \vec{e}[r/u], r, u \rangle \mid 1 \leq u \leq 2^{|L_2|}\}$$

where $\vec{e}[r/u]$ is the same as $\vec{e}$ except that when $e_i = r, u$ then $e[r/u]_i = u, r$ respectively. Notice that as the $\vec{e}$ and r vary the $J(\langle \vec{e}, r \rangle)$ partition the Cartesian product of the set of $\vec{e}\,'$, where the e_i' range over $\{1, 2, \ldots, 2^{|L_2|}\}$, and the set $\{1, 2, \ldots, 2^{|L_2|}\}^2$.

Similarly let

$$K(\langle \vec{f}, k \rangle) = \{\langle \vec{f}[k/s], k, s \rangle \mid 1 \leq s \leq 2^{|L_1|}\}$$

where $\vec{f}[k/s]$ is the same as $\vec{f}$ except that when $f_j = k, s$ then $f[k/s]_j = s, k$ respectively. Again as the $\vec{f}$ and k vary the $K(\langle \vec{f}, k \rangle)$ partition the Cartesian product of the set of $\vec{f}\,'$, where the f_j' range over $\{1, 2, \ldots, 2^{|L_1|}\}$, and the set $\{1, 2, \ldots, 2^{|L_1|}\}^2$.

Hence we can now pair off

$$\Big\langle \langle \vec{e}, r \rangle, \langle \vec{f}, k \rangle \Big\rangle \leftrightarrow J(\langle \vec{e}, r \rangle) \times K(\langle \vec{f}, k \rangle). \tag{184}$$

For the contribution to (182) from $\vec{e}$, $\vec{f}$ and $v_{k,r}$ we get

$$2^{-|L|} \gamma^{n+m-p-q} (\gamma + \delta)^{p+q} \tag{185}$$

where $p = |\{i \mid h_i = k \text{ and } e_i = r\}|$ and $q = |\{j \mid g_j = r \text{ and } f_j = k\}|$.

Similarly, for $\langle \vec{e}[r/u], r, u \rangle \in J(\langle \vec{e}, r \rangle)$ and $v_{k,u}$ we get a contribution $2^{-|L|} \gamma^{n-p} (\gamma + \delta)^p$ to the first factor in (183), where again $p = |\{i \mid h_i = k \text{ and } e_i = r\}|$. Analogously for $\langle \vec{f}[k/s], k, s \rangle \in K(\langle \vec{f}, k \rangle)$ and $v_{s,r}$ we get a contribution to the second factor in (183) of $2^{-|L|} \gamma^{m-q} (\gamma + \delta)^q$ for this same $q = |\{j \mid g_j = r \text{ and } f_j = k\}|$.

Summing over the s, u (183) now gives

$$\begin{aligned} &\sum_{u=1}^{2^{|L_2|}} \sum_{s=1}^{2^{|L_1|}} (2^{-|L|} \gamma^{n-p} (\gamma + \delta)^p) \times (2^{-|L|} \gamma^{m-q} (\gamma + \delta)^q) \\ &\qquad = 2^{|L_2|} 2^{|L_1|} 2^{-2|L|} \gamma^{n+m-p-q} (\gamma + \delta)^{p+q} \\ &\qquad = 2^{-|L|} \gamma^{n+m-p-q} (\gamma + \delta)^{p+q} \end{aligned}$$

which agrees with (185). Summing over the $\vec{e}, \vec{f}, k, r$, now gives the equality of (182) and (183) as required. ⊣

Provided $q > 1$ (obviously WIP is vacuous if $q = 1$) the c_λ^L do not satisfy WIP for $0 < \lambda < \infty$, a suitable counter-example taken from [44] being

$$c_\lambda^L(R_2(a_3) \wedge R_2(a_4) \mid R_1(a_1) \wedge R_1(a_2)) > c_\lambda^L(R_2(a_3) \wedge R_2(a_4)), \quad (186)$$

as can be verified by evaluating both sides. In a way however the failure of WIP in (186) can be seen as a plus for the c_λ^L. For it implies that the c_λ^L reflect a kind of higher order relevance or induction: That a_1, a_2 both satisfying the relation $R_1(x)$ 'argues' that the constants tend to be similar to each other and hence that a_3, a_4 should both satisfy (or both not satisfy) $R_2(x)$. Of course this interpretation is somewhat speculative. However it gains objective support when we look closely at the above evaluation. For having already $R_1(a_1) \wedge R_1(a_2)$ allows a possible extension to $\bigwedge_{i=1}^4 R_1(a_i) \wedge R_2(a_i)$ which, with its four repetitions of a single atom, will gain a higher probability than any state description for a_1, a_2, a_3, a_4 with a mixture of atoms.

We shall return to this notion of similarity later when we consider principles based on analogy.

The plan now is to give a characterization of all the probability functions satisfying Ex, Ax and WIP. We first introduce a construction for building probability functions satisfying WIP.

Given (discrete) probability functions

$$w = \sum_{i=1}^k e_i w_{\vec{c}^i}, \qquad v = \sum_{j=1}^m f_j w_{\vec{d}^j}, \quad (187)$$

where the $e_i, f_j > 0$ and $\sum_i e_i = \sum_j f_j = 1$, and $0 \le \lambda \le 1$ let $w \oplus_\lambda v$ be the probability function

$$\sum_{i=1}^k \sum_{j=1}^m e_i f_j w_{(\lambda \vec{c}^i + (1-\lambda)\vec{d}^j)}.$$

Clearly here we may assume that the $\vec{c}^i$ are distinct, and similarly the $\vec{d}^j$, though it will not necessarily be the case that the $(\lambda \vec{c}^i + (1-\lambda)\vec{d}^j)$ are distinct.

The following result appears in [117].

LEMMA 20.3. *For w, v as in (187), if w, v satisfy Ex, Ax and WIP then so does $w \oplus_\lambda v$.*

PROOF. Clearly this probability function satisfies Ex since the $w_{\vec{x}}$ do.

If $\vec{c}^i = \langle c_1^i, c_2^i, \dots, c_{2^q}^i \rangle$, $\vec{d}^j = \langle d_1^j, d_2^j, \dots, d_{2^q}^j \rangle$, then

$$\lambda \vec{c}^i + (1-\lambda)\vec{d}^j = \langle \lambda c_1^i + (1-\lambda) d_1^j, \lambda c_2^i + (1-\lambda) d_2^j, \dots, \lambda c_{2^q}^i + (1-\lambda) d_{2^q}^j \rangle.$$

Since w, v satisfy Ax, for $\sigma \in \mathrm{S}_{2^q}$,

$$w = \sum_{i=1}^{k} e_i w_{\sigma\vec{c}^i}, \quad v = \sum_{j=1}^{m} f_j w_{\sigma\vec{d}^j},$$

where $\sigma\vec{c}^i = \langle c^i_{\sigma(1)}, c^i_{\sigma(2)}, \dots, c^i_{\sigma(2^q)}\rangle, \sigma\vec{d}^j = \langle d^j_{\sigma(1)}, d^j_{\sigma(2)}, \dots, d^j_{\sigma(2^q)}\rangle$. Hence

$$\begin{aligned} w \oplus_\lambda v &= \sum_{i=1}^{k}\sum_{j=1}^{m} e_i f_j w_{(\lambda\sigma\vec{c}^i + (1-\lambda)\sigma\vec{d}^j)} \\ &= \sum_{i=1}^{k}\sum_{j=1}^{m} e_i f_j w_{\sigma(\lambda\vec{c}^i + (1-\lambda)\vec{d}^j)} \end{aligned}$$

from which it follows that $w \oplus_\lambda v$ also satisfies Ax.

To show WIP we need to demonstrate the equality of

$$w \oplus_\lambda v\Big(\bigwedge_{x=1}^{2^{|L_1|}} \beta_x^{n_x} \wedge \bigwedge_{y=1}^{2^{|L_2|}} \eta_y^{m_y}\Big) \tag{188}$$

and

$$w \oplus_\lambda v\Big(\bigwedge_{x=1}^{2^{|L_1|}} \beta_x^{n_x}\Big) \cdot w \oplus_\lambda v\Big(\bigwedge_{y=1}^{2^{|L_2|}} \eta_y^{m_y}\Big) \tag{189}$$

where L_1, L_2 are disjoint sublanguages of L, the β_x, η_y are, respectively, the atoms of L_1 and L_2 and as usual we are leaving the instantiating constants implicit. Using the definition of $w \oplus_\lambda v$ and expanding (188), (189) we obtain, respectively,

$$\sum_{i=1}^{k}\sum_{j=1}^{m} e_i f_j w_{(\lambda\vec{c}^i + (1-\lambda)\vec{d}^j)}\Big(\bigwedge_{x=1}^{2^{|L_1|}} \beta_x^{n_x} \wedge \bigwedge_{y=1}^{2^{|L_2|}} \eta_y^{m_y}\Big), \tag{190}$$

$$\sum_{i,i'=1}^{k}\sum_{j,j'=1}^{m} e_i e_{i'} f_j f_{j'} w_{(\lambda\vec{c}^i + (1-\lambda)\vec{d}^j)}\Big(\bigwedge_{x=1}^{2^{|L_1|}} \beta_x^{n_x}\Big) \cdot w_{(\lambda\vec{c}^{i'} + (1-\lambda)\vec{d}^{j'})}\Big(\bigwedge_{y=1}^{2^{|L_2|}} \eta_y^{m_y}\Big). \tag{191}$$

By Proposition 20.1 the sum in (190) is

$$\sum_{i=1}^{k}\sum_{j=1}^{m} e_i f_j \prod_{x=1}^{2^{|L_1|}} w_{(\lambda\vec{c}^i + (1-\lambda)\vec{d}^j)}(\beta_x^{n_x}) \prod_{y=1}^{2^{|L_2|}} w_{(\lambda\vec{c}^i + (1-\lambda)\vec{d}^j)}(\eta_y^{m_y}).$$

Using Proposition 8.1 and the fact that for $\theta = \bigvee_k \alpha_k(a_s)$ (with a fixed s), $w_{(\lambda\vec{c}^i+(1-\lambda)\vec{d}^j)}(\theta) = \lambda w_{\vec{c}^i}(\theta) + (1-\lambda)w_{\vec{d}^j}(\theta)$, this expands to

$$\sum_{i=1}^{k}\sum_{j=1}^{m} e_i f_j \left[\prod_{x=1}^{2^{|L_1|}} \sum_{k_x \le n_x} \binom{n_x}{k_x} \lambda^{k_x} w_{\vec{c}^i}(\beta_x^{k_x})(1-\lambda)^{n_x-k_x} w_{\vec{d}^j}(\beta_x^{n_x-k_x})\right] \times$$
$$\left[\prod_{y=1}^{2^{|L_2|}} \sum_{g_y \le m_y} \binom{m_y}{g_y} \lambda^{g_y} w_{\vec{c}^i}(\eta_y^{g_y})(1-\lambda)^{m_y-g_y} w_{\vec{d}^j}(\eta_y^{m_y-g_y})\right].$$

Using Proposition 20.1 again this equals

$$\sum_{i=1}^{k}\sum_{j=1}^{m} e_i f_j \prod_{x=1}^{2^{|L_1|}} \prod_{y=1}^{2^{|L_2|}} \sum_{k_x \le n_x} \sum_{g_y \le m_y} \binom{n_x}{k_x}\binom{m_y}{g_y}$$
$$[\lambda^{k_x+g_y}(1-\lambda)^{n_x+m_y-k_x-g_y} w_{\vec{c}^i}(\beta_x^{k_x} \wedge \eta_y^{g_y}) w_{\vec{d}^j}(\beta_x^{n_x-k_x} \wedge \eta_y^{m_y-g_y})].$$

Moving the summations over i and j inside now gives

$$\prod_{x=1}^{2^{|L_1|}} \prod_{y=1}^{2^{|L_2|}} \sum_{k_x \le n_x} \sum_{g_y \le m_y} \binom{n_x}{k_x}\binom{m_y}{g_y}$$
$$[\lambda^{k_x+g_y}(1-\lambda)^{n_x+m_y-k_x-g_y} w(\beta_x^{k_x} \wedge \eta_y^{g_y}) v(\beta_x^{n_x-k_x} \wedge \eta_y^{m_y-g_y})].$$

Since w and v satisfy WIP

$$w(\beta_x^{k_x} \wedge \eta_y^{g_y}) = w(\beta_x^{k_x}) \cdot w(\eta_y^{g_y})$$
$$v(\beta_x^{n_x-k_x} \wedge \eta_y^{m_y-g_y}) = v(\beta_x^{n_x-k_x}) \cdot v(\eta_y^{m_y-g_y})$$

so making this substitution and replacing w and v by their summations in terms of the $w_{\vec{c}^i}, w_{\vec{d}^j}$ yields

$$\sum_{i,i'=1}^{k}\sum_{j,j'=1}^{m} e_i e_{i'} f_j f_{j'} \prod_{x=1}^{2^{|L_1|}} \prod_{y=1}^{2^{|L_2|}} \sum_{k_x \le n_x} \sum_{g_y \le m_y} \binom{n_x}{k_x}\binom{m_y}{g_y}$$
$$[\lambda^{k_x+g_y}(1-\lambda)^{n_x+m_y-k_x-g_y} w_{\vec{c}^i}(\beta_x^{k_x}) w_{\vec{c}^{i'}}(\eta_y^{g_y}) w_{\vec{d}^j}(\beta_x^{n_x-k_x}) w_{\vec{d}^{j'}}(\eta_y^{m_y-g_y})].$$

Since

$$w_{(\lambda\vec{c}^i+(1-\lambda)\vec{d}^j)}\left(\bigwedge_{x=1}^{2^{|L_1|}} \beta_x^{n_x}\right) =$$
$$\prod_{x=1}^{2^{|L_1|}} \sum_{k_x \le n_x} \binom{n_x}{k_x} \lambda^{k_x}(1-\lambda)^{n_x-k_x} w_{\vec{c}^i}(\beta_x^{k_x}) w_{\vec{d}^j}(\beta_x^{n_x-k_x})$$

etc. the required equality with (191) now follows. ⊣

We now introduce a new family of probability functions which as we shall see can be built from the w_L^δ, and which we will later find to be the unary versions of functions important in Polyadic Inductive Logic.

Let $\mathbb{B}$ be the set of infinite sequences

$$\bar{p} = \langle p_0, p_1, p_2, p_3, \dots \rangle$$

of non-negative reals such that $p_1 \geq p_2 \geq p_3 \geq \cdots$ and $\sum_{i=0}^{\infty} p_i = 1$. For future reference we endow $\mathbb{B}$ with the evident topology which has as open sets the

$$\{\langle p_0, p_1, p_2, p_3, \dots \rangle \in \mathbb{B} \mid \langle p_1, p_2, p_3, \dots \rangle \in S\}$$

where $S \subseteq [0,1]^\infty$ is open in the standard (weak) topology on $[0,1]^\infty$.

For $\bar{p} \in \mathbb{B}$ and $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, 2^q\}$ let $R_{\bar{p},n} = 1 - \sum_{j=1}^{n} p_j$ and

$$f(\bar{p}) =$$
$$\Big\langle 2^{-q} R_{\bar{p},n} + \sum_{f(j)=1} p_j,\ 2^{-q} R_{\bar{p},n} + \sum_{f(j)=2} p_j, \dots,\ 2^{-q} R_{\bar{p},n} + \sum_{f(j)=2^q} p_j \Big\rangle,$$

so $f(\bar{p}) \in \mathbb{D}_{2^q}$.

Now let

$$u_n^{\bar{p},L} = 2^{-nq} \sum_f w_{f(\bar{p})} \tag{192}$$

where the f range over all functions $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, 2^q\}$.

So, for example, in the case when L has just 2 relation symbols (i.e. $q = 2$), $n = 2$, $\bar{p} = \langle \frac{1}{4}, \frac{1}{2}, \frac{1}{6}, \frac{1}{12}, 0, 0, 0, \dots \rangle$ and $f(1) = 3, f(2) = 2$ we have that $R_{\bar{p},2} = \frac{1}{4} + \frac{1}{12} = \frac{1}{3}$,

$$\begin{aligned} f(\bar{p}) &= \Big\langle (\frac{1}{4} \times \frac{1}{3}) + 0,\ (\frac{1}{4} \times \frac{1}{3}) + \frac{1}{6},\ (\frac{1}{4} \times \frac{1}{3}) + \frac{1}{2},\ (\frac{1}{4} \times \frac{1}{3}) + 0 \Big\rangle \\ &= \Big\langle \frac{1}{12}, \frac{1}{4}, \frac{7}{12}, \frac{1}{12} \Big\rangle \end{aligned}$$

and, by considering each of the 2^4 functions from $\{1,2\}$ to $\{1,2,3,4\}$, $u_2^{\bar{p},L}$ equals

$$\begin{aligned} &2^{-4} w_{\langle \frac{3}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12} \rangle} + 2^{-4} w_{\langle \frac{7}{12}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12} \rangle} + 2^{-4} w_{\langle \frac{7}{12}, \frac{1}{12}, \frac{1}{4}, \frac{1}{12} \rangle} + 2^{-4} w_{\langle \frac{7}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{4} \rangle} + \\ &2^{-4} w_{\langle \frac{1}{4}, \frac{7}{12}, \frac{1}{12}, \frac{1}{12} \rangle} + 2^{-4} w_{\langle \frac{1}{12}, \frac{3}{4}, \frac{1}{12}, \frac{1}{12} \rangle} + 2^{-4} w_{\langle \frac{1}{12}, \frac{7}{12}, \frac{1}{4}, \frac{1}{12} \rangle} + 2^{-4} w_{\langle \frac{1}{12}, \frac{7}{12}, \frac{1}{12}, \frac{1}{4} \rangle} + \\ &2^{-4} w_{\langle \frac{1}{4}, \frac{1}{12}, \frac{7}{12}, \frac{1}{12} \rangle} + 2^{-4} w_{\langle \frac{1}{12}, \frac{1}{4}, \frac{7}{12}, \frac{1}{12} \rangle} + 2^{-4} w_{\langle \frac{1}{12}, \frac{1}{12}, \frac{3}{4}, \frac{1}{12} \rangle} + 2^{-4} w_{\langle \frac{1}{12}, \frac{1}{12}, \frac{7}{12}, \frac{1}{4} \rangle} + \\ &2^{-4} w_{\langle \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{7}{12} \rangle} + 2^{-4} w_{\langle \frac{1}{12}, \frac{1}{4}, \frac{1}{12}, \frac{7}{12} \rangle} + 2^{-4} w_{\langle \frac{1}{12}, \frac{1}{12}, \frac{1}{4}, \frac{7}{12} \rangle} + 2^{-4} w_{\langle \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{3}{4} \rangle}. \end{aligned}$$

Notice that the w_L^δ for $0 \leq \delta \leq 1$ are of this form, indeed $w_L^\delta = u_1^{\bar{p},L}$ where $\bar{p} = \langle 1 - \delta, \delta, 0, 0, \dots \rangle$.

The next two theorems appear in [111].

THEOREM 20.4. *The probability functions $u_n^{\bar{p},L}$ satisfy Ex, Ax and WIP.*

PROOF. We prove this by induction on n. When $n = 1$ the $u_n^{\bar{p},L}$ coincide with the w_L^δ for $0 \leq \delta \leq 1$ so the result holds by Theorem 20.2. Assume the result for n. Then since

$$u_{n+1}^{\bar{p},L} = w_L^1 \oplus_{p_{n+1}} u_n^{\bar{q},L},$$

where

$$\bar{q} = \left\langle 1 - \frac{\sum_{i=1}^n p_i}{1 - p_{n+1}}, \frac{p_1}{1 - p_{n+1}}, \dots, \frac{p_n}{1 - p_{n+1}}, 0, 0, \dots \right\rangle,$$

the inductive step follows from Lemma 20.3. ⊣

As $n \to \infty$ the weak limit probability function, $u^{\bar{p},L}$, of the probability functions $u_n^{\bar{p},L}$ exists. That is, for $\theta \in QFSL$:

$$u^{\bar{p},L}(\theta) = \lim_{n \to \infty} u_n^{\bar{p},L}(\theta) \tag{193}$$

and $u^{\bar{p},L}$ then extends to SL by Theorem 7.1.[82] It is enough to show this in the case when θ is a state description, say $\theta = \alpha_1^{m_1} \alpha_2^{m_2} \dots \alpha_{2^q}^{m_{2^q}}$ where $\sum_j m_j = m$.

So let $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, 2^q\}$ and let g_i for $i = 1, 2, \dots, 2^q$ enumerate those $g : \{1, 2, \dots, n, n+1\} \to \{1, 2, \dots, 2^q\}$ which extend f, say $g_i(n+1) = i$ for $i = 1, 2, \dots, 2^q$. Then

$$\begin{aligned}
&\Big| w_{f(\bar{p})}(\theta) - 2^{-q} \sum_{i=1}^{2^q} w_{g_i(\bar{p})}(\theta) \Big| \\
&= \Big| \prod_{j=1}^{2^q} (2^{-q} R_{\bar{p},n} + \sum_{f(r)=j} p_r)^{m_j} \; - 2^{-q} \sum_{i=1}^{2^q} \prod_{j=1}^{2^q} (2^{-q} R_{\bar{p},n+1} + \sum_{g_i(r)=j} p_r)^{m_j} \Big| \\
&= 2^{-q} \Big| \sum_{i=1}^{2^q} \Big(\prod_{j=1}^{2^q} (2^{-q} R_{\bar{p},n} + \sum_{f(r)=j} p_r)^{m_j} - \prod_{j=1}^{2^q} (2^{-q} R_{\bar{p},n+1} + \sum_{g_i(r)=j} p_r)^{m_j} \Big) \Big| \\
&\leq 2^{-q} \sum_{i=1}^{2^q} \Big(\Big| \prod_{j=1}^{2^q} (2^{-q} R_{\bar{p},n} + \sum_{f(r)=j} p_r)^{m_j} - \prod_{j=1}^{2^q} (2^{-q} R_{\bar{p},n+1} + \sum_{g_i(r)=j} p_r)^{m_j} \Big| \Big) \\
&\leq 2^{-q} \sum_{i=1}^{2^q} (m(2^{-q} R_{\bar{p},n} - 2^{-q} R_{\bar{p},n+1}) + m_i p_{n+1}) = m p_{n+1} 2^{1-q},
\end{aligned}$$

[82]We shall later, at (244), meet an alternative definition of this $u^{\bar{p},L}$.

since if $0 \leq x_i, y_i \leq 1$ for $i = 1, 2, \ldots, m$ then

$$\left|\prod_{i=1}^m x_i - \prod_{i=1}^m y_i\right| = \left|\sum_{j=1}^m x_1 x_2 \cdots x_{j-1}(x_j - y_j) y_{j+1} y_{j+2} \cdots y_m\right|$$

$$\leq \sum_{j=1}^m |x_j - y_j|.$$

Hence, since there are 2^{nq} choices here for f,

$$|u_n^{\bar{p},L}(\theta) - u_{n+1}^{\bar{p},L}(\theta)| \leq 2^{-nq}(2^{nq} m p_{n+1} 2^{1-q}) = m p_{n+1} 2^{1-q}$$

so for $k > n$ we have

$$|u_n^{\bar{p},L}(\theta) - u_k^{\bar{p},L}(\theta)| \leq m 2^{1-q} \sum_{i=n+1}^{\infty} p_i.$$

Since $\sum_{i=n+1}^{\infty} p_i \to 0$ as $n \to \infty$ so the $u_n^{\bar{p},L}(\theta)$ form a Cauchy sequence and the identity (193) follows.

Notice that

$$u_n^{\bar{p},L} = u^{\bar{q},L} \quad \text{where} \quad \bar{q} = \langle R_{\bar{p},n}, p_1, p_2, \ldots, p_{n-1}, p_n, 0, 0, \ldots \rangle \tag{194}$$

so the $u_n^{\bar{p},L}$ form a subset of the $u^{\bar{p},L}$.

Being the limit of the $u_n^{\bar{p},L}$ the $u^{\bar{p},L}$ inherit Ex, Ax and WIP. Indeed a stronger result holds,

THEOREM 20.5. *The $u^{\bar{p},L}$ satisfy ULi with Ex, Ax and WIP.*

PROOF. As in the case of the c_λ^L and the w_L^δ only the language L will vary, the parameter, $\bar{p}$, will stay fixed.

Let L' have one more relation symbol, R_{q+1}, than L, so

$$u_n^{\bar{p},L'} = 2^{-n(q+1)} \sum_g w_{g(\bar{p})}$$

where the $g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, 2^{q+1}\}$ and

$$g(\bar{p}) =$$
$$\langle 2^{-q-1} R_{\bar{p},n} + \sum_{g(j)=1} p_j, 2^{-q-1} R_{\bar{p},n} + \sum_{g(j)=2} p_j, \ldots, 2^{-q-1} R_{\bar{p},n} + \sum_{g(j)=2^{q+1}} p_j \rangle.$$

As usual we assume that the atoms $\beta_1, \beta_2, \ldots, \beta_{2^{q+1}}$ of L' are enumerated as

$$\alpha_1(x) \wedge R_{q+1}(x), \alpha_1(x) \wedge \neg R_{q+1}(x), \alpha_2(x) \wedge R_{q+1}(x),$$
$$\alpha_2(x) \wedge \neg R_{q+1}(x), \ldots, \alpha_{2^q}(x) \wedge R_{q+1}(x), \alpha_{2^q}(x) \wedge \neg R_{q+1}(x)$$

where the atoms of L are $\alpha_1(x), \alpha_2(x), \ldots, \alpha_{2^q}(x)$.

In that case for a state description, say $\bigwedge_{i=1}^m \alpha_{h_i}$, and g as above ranging over functions from $\{1, 2, \dots, 2^{q+1}\}$ to $\{1, 2, \dots, n\}$,

$$u_n^{\bar{p},L'}\Big(\bigwedge_{i=1}^m \alpha_{h_i}\Big) \tag{195}$$

$$= u_n^{\bar{p},L'}\Big(\bigwedge_{i=1}^m \beta_{2h_i-1} \vee \beta_{2h_i}\Big)$$

$$= 2^{-n(q+1)}\sum_g w_{g(\bar{p})}\Big(\bigwedge_{i=1}^m \beta_{2h_i-1} \vee \beta_{2h_i}\Big) \tag{196}$$

$$= 2^{-n(q+1)}\sum_g \prod_{i=1}^m \Big(2^{-q-1}R_{\bar{p},n} + \sum_{g(j)=2h_i-1} p_j + 2^{-q-1}R_{\bar{p},n} + \sum_{g(j)=2h_i} p_j\Big)$$

$$= 2^{-n(q+1)}\sum_g \prod_{i=1}^m \Big(2^{-q}R_{\bar{p},n} + \sum_{g(j)=2h_i-1,2h_i} p_j\Big). \tag{197}$$

Given $f : \{1, 2, \dots, 2^q\} \to \{1, 2, \dots, n\}$ the products in (197) will take the same value,

$$\prod_{i=1}^m \Big(2^{-q}R_{\bar{p},n} + \sum_{f(j)=h_i} p_j\Big),$$

for all g such that for $j = 1, 2, \dots, n$,

$$f(j) = h \iff g(j) = 2h-1 \text{ or } g(j) = 2h.$$

Since there are 2^n such g for each f we can re-write (197) as

$$2^{-n(q+1)}2^n \sum_f \prod_{i=1}^m \Big(2^{-q}R_{\bar{p},n} + \sum_{f(j)=h_i} p_j\Big),$$

which, after replacing $2^{-n(q+1)}2^n$ by 2^{-nq}, is just $u_n^{\bar{p},L}\left(\bigwedge_{i=1}^m \alpha_{h_i}\right)$, as required.

ULi now follows for the $u^{\bar{p},L}$ by taking limits. ⊣

We now give a converse to Theorem 20.5, this property of satisfying ULi with Ex, Ax and WIP actually *characterizes* the family of $u^{\bar{p},L}$. This proof will require us to use a result (Theorem 32.3) from the later chapter on Polyadic Inductive Logic. Needless to say the proof of that theorem does not depend on this converse to Theorem 20.5 !

Theorem 20.6. *The $u^{\bar{p},L}$ are the only probability functions on SL which satisfy ULi with Ex, Ax and WIP.*

Proof. Suppose that w is a probability function for the language $L = \{R_1, R_2, \dots, R_q\}$ which satisfies ULi with Ex, Ax and WIP. Let L' be the language with (unary) relation symbols $R_1, R_2, \dots, R_{2q}$. Let $\theta(a_1, a_2, \dots, a_n) \in QFSL$ and let $\theta'(a_{n+1}, a_{n+2}, \dots, a_{2n}) \in QFSL'$ be the result of replacing each R_i in θ by R_{q+i} and each a_i in θ by a_{n+i}. Since w satisfies ULi with Ex, Ax and WIP let w' be an extension of w to L' in such a language invariant family satisfying these principles.

By Theorem 32.3 there is a countably additive measure μ' on the Borel subsets of $\mathbb{B}$ (in the topology given on page 149) such that

$$w' = \int_{\mathbb{B}} u^{\bar{p},L'} \, d\mu'(\bar{p}).$$

Then since w' satisfies WIP and the $u^{\bar{p},L'}$ satisfy Ex, Ax and WIP,

$$\begin{aligned}
0 &= 2(w'(\theta \wedge \theta') - w'(\theta) \cdot w'(\theta')) \\
&= \int_{\mathbb{B}} u^{\bar{p},L'}(\theta \wedge \theta') \, d\mu'(\bar{p}) + \int_{\mathbb{B}} u^{\bar{q},L'}(\theta \wedge \theta') \, d\mu'(\bar{q}) \\
&\quad - 2\Big(\int_{\mathbb{B}} u^{\bar{p},L'}(\theta) \, d\mu'(\bar{p}) \Big) \cdot \Big(\int_{\mathbb{B}} u^{\bar{q},L'}(\theta') \, d\mu'(\bar{q}) \Big) \\
&= \int_{\mathbb{B}} u^{\bar{p},L'}(\theta) \cdot u^{\bar{p},L'}(\theta') \, d\mu'(\bar{p}) + \int_{\mathbb{B}} u^{\bar{q},L'}(\theta) \cdot u^{\bar{q},L'}(\theta') \, d\mu'(\bar{q}) \\
&\quad - 2\Big(\int_{\mathbb{B}} u^{\bar{p},L'}(\theta) \, d\mu'(\bar{p}) \Big) \cdot \Big(\int_{\mathbb{B}} u^{\bar{q},L'}(\theta') \, d\mu'(\bar{q}) \Big) \\
&= \int_{\mathbb{B}} \int_{\mathbb{B}} (u^{\bar{p},L'}(\theta) - u^{\bar{q},L'}(\theta))^2 \, d\mu'(\bar{p}) d\mu'(\bar{q}).
\end{aligned}$$

Using the countable additivity of μ', we can see that there must be a subset A of $\mathbb{B}$ with μ' measure 1 such that for *each* $\theta \in QFSL$, $u^{\bar{p},L'}(\theta)$, as a function of $\bar{p}$, is constant on A. Hence w must equal $u^{\bar{p},L}$ for any $\bar{p} \in A$. ⊣

Chapter 21

EQUALITIES AND INEQUALITIES

In Propositions 3.1, 3.2 and Lemmas 3.6, 3.7 we have already seen some equalities and inequalities which hold for *all* probability functions w on SL. Given our interest in principles Ex, Ax etc. it is natural to ask if requiring w to satisfy these entails more such equalities and inequalities holding. We start by looking at equalities of the form $w(\theta) = cw(\phi)$ for some constant c, where $\theta, \phi \in QFSL$.[83]

For $\theta(a_1, \dots, a_m) \in QFSL$, say

$$\theta(a_1, \dots, a_m) \equiv \bigvee_{j=1}^{n} \Theta_j(a_1, a_2, \dots, a_m) \tag{198}$$

as a disjunction of (disjoint) state descriptions, let $g_\theta(\vec{m})$ be the number of state descriptions amongst the Θ_j with signature (see page 50) $\vec{m} = \langle m_1, m_2, \dots, m_{2^q} \rangle$. Then, as explained on that page, for w satisfying Ex

$$w(\theta) = \sum_{\vec{m}} g_\theta(\vec{m}) w(\vec{m}) \tag{199}$$

where, for signature $\vec{m} = \langle m_1, m_2, \dots, m_{2^q} \rangle$, $w(\vec{m})$ (or even $w(\{m_1, m_2, \dots, m_{2^q}\})$ for w satisfying Ax) is short for $w(\alpha_1^{m_1} \alpha_2^{m_2} \dots \alpha_{2^q}^{m_{2^q}})$.

Proposition 21.1. *For $\theta(a_1, \dots, a_m), \phi(a_1, \dots, a_m) \in QFSL$ and $0 \leq c$, $w(\theta) = cw(\phi)$ holds for all probability functions w on SL satisfying Ex just if for all signatures $\vec{m}$, $g_\theta(\vec{m}) = cg_\phi(\vec{m})$.*

In particular $w(\theta) = c$ for all probability functions w on SL satisfying Ex just if $c = 1$ and θ is a tautology or $c = 0$ and θ is a contradiction.

Proof. The result from right to left is immediate from (199). In the other direction suppose that $w(\theta) = cw(\phi)$ holds for all probability functions w on SL satisfying Ex. Let $0 < s_1, s_2, \dots, s_{2^q} \in \mathbb{R}$ be algebraically independent and let $t_i = s_i(s_1 + s_2 + \cdots + s_{2^q})^{-1}$. Then taking $w = w_{\vec{t}}$,

[83] In view of the earlier remarks on page 63 concerning Super Regularity, aka 'Universal Certainty', this question seems to be of little interest for quantified sentences.

$w(\theta) = cw(\phi)$ gives by (199) that

$$\sum_{\substack{\langle m_1, m_2, \dots, m_{2^q} \rangle \\ \sum_i m_i = m}} s_1^{m_1} s_2^{m_2} \dots s_{2^q}^{m_{2^q}} (g_\theta(\vec{m}) - cg_\phi(\vec{m})) = 0$$

which is only possible if $g_\theta(\vec{m}) = cg_\phi(\vec{m})$ for all such signatures $\vec{m}$.

Finally suppose that $w(\theta) = c$ for all probability functions w on SL satisfying Ex. Then since $c = cw(\top)$, from the first part $g_\theta(\vec{m}) = cg_\top(\vec{m})$ for all signatures $\vec{m}$. Since the left hand side here must be an integer and

$$g_\top(\langle m, 0, 0, \dots, 0 \rangle) = 1$$

it follows that $c = 0, 1$. Hence, since there are certainly probability functions satisfying Reg and Ex, this can only happen if either θ is a contradiction or θ is a tautology. $\dashv$

The corresponding result for probability functions satisfying Ax (plus as usual Ex) is completely similar. In this case for θ etc. as in (198) let $f_\theta(\tilde{m})$ be the number of state descriptions amongst the Θ_j with *spectrum* $\tilde{m} = \{m_1, m_2, \dots, m_{2^q}\}$. So, for w satisfying Ax

$$w(\theta) = \sum_{\tilde{m}} f_\theta(\tilde{m}) w(\tilde{m}). \tag{200}$$

The following proposition (from [101]) is proved as for Proposition 21.1 except that we take instead the probability function

$$w = (2^q!)^{-1} \sum_{\sigma \in S_{2^q}} w_{\langle t_{\sigma(1)}, t_{\sigma(2)}, \dots, t_{\sigma(2^q)} \rangle}.$$

PROPOSITION 21.2. *For $\theta(a_1, \dots, a_m), \phi(a_1, \dots, a_m) \in QFSL$ and $0 \leq c$, $w(\theta) = cw(\phi)$ holds for all probability functions w on SL satisfying Ax just if for all spectra $\tilde{m}$, $f_\theta(\tilde{m}) = cf_\phi(\tilde{m})$.*

In particular $w(\theta) = c$ for all probability functions w on SL satisfying Ax just if $f_\theta(\tilde{m}) = cf_\top(\tilde{m})$ for all spectra $\tilde{m}$.

Notice that by this proposition the only values of c for which this latter can hold are the $2^{-q}k$ for $k \in \mathbb{N}, k \leq 2^q$ since the value of $c_0^L(\theta)$ must be of this form (and a little arithmetic shows that all of these can be achieved).

We now show that this 'constancy property' for θ is independent of the overlying language.

PROPOSITION 21.3. *Let $\theta \in SL$ and $L \subset L'$. Then $w(\theta) = c$ for all probability functions w on SL satisfying Ax just if $w'(\theta) = c$ for all probability functions w' on SL' satisfying Ax.*

PROOF. The direction from left to right is immediate since the restriction to SL of a probability function on SL' satisfying Ax on this larger language also satisfies it on the restricted language.

In the other direction it is simplest to use Theorem 34.1 (whose proof does not depend on this proposition!). That theorem says that for w a probability function on SL satisfying Ax there are probability functions w_1, w_2 on SL satisfying ULi with Ax and a constant $\lambda > 0$ such that

$$w = (1+\lambda)w_1 - \lambda w_2.$$

With this result available let w_1', w_2' be extensions of w_1, w_2 to SL' satisfying Ax. Then if $w'(\theta) = c$ for all probability functions w' on SL' satisfying Ax we must have that

$$w(\theta) = (1+\lambda)w_1(\theta) - \lambda w_2(\theta) = (1+\lambda)w_1'(\theta) - \lambda w_2'(\theta) = (1+\lambda)c - \lambda c = c,$$

as required. $\dashv$

Clearly the additional requirement that w satisfies not just Ex but also Ax has given us many new identities. That raises the question of whether we might obtain further identities by, say, requiring also JSP. In other words restricting ourselves to Carnap's Continuum. It turns out that the answer is yes, for example (see [101]),

$$c_\lambda^L\Big((R_2(a_1) \leftrightarrow \neg R_2(a_2)) \wedge R_2(a_3) \wedge \neg R_2(a_4) \wedge \bigwedge_{i=1}^{4} R_1(a_i)\Big)$$
$$= c_\lambda^L\Big(R_2(a_1) \wedge R_2(a_2) \wedge \neg R_1(a_4) \wedge \neg R_2(a_4) \wedge \bigwedge_{i=1}^{3} R_1(a_i)\Big)$$

holds for all the c_λ^L (with R_1, R_2 predicate symbols of L) though this is not simply a consequence of Ax. However at the present time there is no illuminating explanation as to why JSP begets such equalities – except that the algebra simply works out that way!

Although the analysis above has concentrated on quantifier free sentences the representation (64) enables us to extend these results to $\theta(a_1, \dots, a_m) \in SL$. Briefly, we have that for some state descriptions $\Theta_i^T(a_1, \dots, a_m)$, $i = 1, \dots, k_T$, involving only atoms α_j with $j \in T$,

$$\theta(\vec{a}) \equiv \bigvee_{t=1}^{2^q} \bigvee_{|T|=t} \Big(\eta_T \wedge \bigvee_{i=1}^{k_T} \Theta_i^T(\vec{a})\Big), \tag{201}$$

where η_T is as in (63) and the T range over non-empty subsets of $\{1, 2, \dots, 2^q\}$. For w satisfying Ax and w_T as in (64) we will have that $w(\eta_T)$ depends only on $|T|$ so from (201)

$$w(\theta(\vec{a})) = \sum_{t=1}^{2^q} \lambda_t \sum_{|T|=t} w_T\Big(\bigvee_{i=1}^{k_T} \Theta_i^T(\vec{a})\Big)$$

where $\lambda_t = w(\eta_T)$ for any T with $|T| = t$. For each t this right hand sum will depend only on the numbers $h_{\tilde{n}}^t(\theta)$, for spectra $\tilde{n}$ of length at most t, of these $\Theta_i^T(\vec{a})$ for $1 \leq i \leq k_T$, $|T| = t$ with spectrum $\tilde{n}$. From this observation and the tricks introduced earlier in the quantifier free case it can now be seen that $w(\theta(\vec{a}))$ will have constant value c for all probability functions satisfying Ax just if $h^t_{\tilde{n}}(\theta) = ch^t_{\tilde{n}}(\top)$ for all t and $\tilde{n}$, and similarly for $w(\theta(\vec{a})) = cw(\phi(\vec{a}))$.

Whilst we are on the subject of these 'constant sentences' we shall show that they have a perhaps rather surprising property. The following result for $\theta, \phi \in QFSL$ (as we give here) appears in [42], [43] and can be directly generalized to $\theta, \phi \in SL$ using the observations above.

Theorem 21.4. *Let L_1, L_2 be disjoint languages with $L = L_1 \cup L_2$. Let $\theta \in QFSL_1$ be such that $w_1(\theta) = c$ for all probability functions w_1 on SL_1 satisfying Ax. Then for any $\phi \in QFSL_2$ which has no constant symbols in common with θ and probability function w on SL satisfying Ax,*

$$w(\theta \wedge \phi) = w(\theta) \cdot w(\phi).$$

Proof. We may assume that $\theta = \theta(a_1, \dots, a_m)$ with none of the $a_1, \dots, a_m$ occurring in ϕ. We may also assume that $c > 0$, otherwise θ is a contradiction and the result is immediate.

As remarked earlier c must be rational, say $c = r/s$ with r, s relatively prime. By Proposition 21.2 for each spectrum $\tilde{n}$ of L_1 with $\sum_i n_i = m$,

$$f_\theta(\tilde{n}) = \left(\frac{r}{s}\right) f_\top(\tilde{n}),$$

so in L_1 we can list the state descriptions of spectrum $\tilde{n}$ appearing in the Disjunctive Normal Form equivalent of θ as $\Psi^{\tilde{n}}_1, \dots, \Psi^{\tilde{n}}_{rj}$ and those appearing in the Disjunctive Normal Form equivalent of $\top$ as $\Psi^{\tilde{n}}_1, \dots, \Psi^{\tilde{n}}_{sj}$ where $j = s^{-1} f_\top(\tilde{n})$.

Since L_1, L_2 are disjoint for each $1 \leq t, k \leq sj$ it is possible to find a permutation of the atoms of L which sends $\Psi^{\tilde{n}}_t$ to $\Psi^{\tilde{n}}_k$ and leaves ϕ (more precisely the Disjunctive Normal Form of ϕ in L) fixed. Hence for w a probability function on SL satisfying Ax,

$$w(\Psi^{\tilde{n}}_t \wedge \phi) = w(\Psi^{\tilde{n}}_k \wedge \phi).$$

and

$$w\Big(\phi \wedge \bigvee_{t=1}^{sj} \Psi^{\tilde{n}}_t\Big) = sjw(\phi \wedge \Psi^{\tilde{n}}_1) = \left(\frac{s}{r}\right) rjw(\phi \wedge \Psi^{\tilde{n}}_1) =$$
$$\left(\frac{s}{r}\right) w\Big(\phi \wedge \bigvee_{t=1}^{rj} \Psi^{\tilde{n}}_t\Big) \quad (202)$$

Summing both sides of (202) over the spectra $\tilde{n}$ of L_1 now gives

$$w(\phi \wedge \top) = w(\phi) = \left(\frac{s}{r}\right) w(\phi \wedge \theta),$$

as required, since, because $w \restriction SL_1$ continues to satisfy Ax, $w(\theta) = c = r/s$. ⊣

Turning now to inequalities it would be nice if we could in some *meaningful* way say exactly when it was that for $\theta(a_1, \ldots, a_m), \phi(a_1, \ldots, a_m) \in QFSL$, $w(\theta) \leq w(\phi)$ for all probability functions w satisfying, say Ex. Obviously this does hold (for Ex) if for some state descriptions $\Psi_j(a_1, \ldots, a_m)$ and $\psi \in QFSL$,

$$\phi(a_1, \ldots, a_m) \equiv \psi \vee \bigvee_{j=1}^{n} \Psi_j(a_1, \ldots, a_m),$$

$$\theta(a_1, \ldots, a_m) \equiv \bigvee_{j=1}^{n} \Psi_j(a_{\sigma_j(1)}, \ldots, a_{\sigma_j(m)})$$

for some $\sigma_j \in \mathrm{S}_m$, $j = 1, 2, \ldots, n$. However it is known that even for just Ex there are other such inequalities, for example from (54) with $b = c = 1$,

$$w((\alpha_1(a_1) \wedge \alpha_1(a_2)) \vee (\alpha_2(a_1) \wedge \alpha_2(a_2))) \geq \\ w((\alpha_1(a_1) \wedge \alpha_2(a_2)) \vee (\alpha_2(a_1) \wedge \alpha_1(a_2))),$$

which do not fall under this template and currently a meaningful[84] complete classification (if such even exists) remains to be found.

A more promising pursuit is to consider which inequalities between probabilities of state descriptions, i.e.

$$w\Big(\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\Big) \leq w\Big(\bigwedge_{j=1}^{m} \alpha_{g_j}(a_j)\Big), \tag{203}$$

always hold for probability functions satisfying, for example, Ex, Ax etc.. Starting with the case of Ex we can apply our now customary abbreviation to rewrite (203) as

$$w(\alpha_1^{n_1}\alpha_2^{n_2}\ldots\alpha_{2^q}^{n_{2^q}}) \leq w(\alpha_1^{m_1}\alpha_2^{m_2}\ldots\alpha_{2^q}^{m_{2^q}}) \tag{204}$$

where n_j is the number of times that α_j appears amongst the $\alpha_{h_1}, \alpha_{h_2}, \ldots, \alpha_{h_n}$ and m_j is the number of times that α_j appears amongst the $\alpha_{g_1}, \alpha_{g_2}, \ldots, \alpha_{g_m}$.

Hardly surprisingly:

PROPOSITION 21.5. (204), *equivalently* (203), *holds for all probability functions w on SL satisfying Ex if and only if $m_j \leq n_j$ for each* $1 \leq j \leq 2^q$.

PROOF. From right to left is clear by Ex and Proposition 3.1(c) since in this case there will be a permutation $\sigma \in \mathrm{S}_n$ such that

$$\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i) \vDash \bigwedge_{j=1}^{m} \alpha_{g_j}(a_{\sigma(j)}).$$

[84]We can use de Finetti's Representation Theorem together with, for example, Krivine's Positivstellensatz, see [70], [90], to provide *a* classification but by no stretch of the imagination could it be said to be meaningful.

In the other direction suppose on the contrary that, say, $n_1 < m_1$. Then for small enough $\varepsilon > 0$ and

$$\vec{e} = \langle \varepsilon, (2^q-1)^{-1}(1-\varepsilon), (2^q-1)^{-1}(1-\varepsilon), \ldots, (2^q-1)^{-1}(1-\varepsilon)\rangle$$

$$\begin{aligned} w_{\vec{e}}(\alpha_1^{n_1}\alpha_2^{n_2}\ldots\alpha_{2^q}^{n_{2^q}}) &= ((2^q-1)^{-1}(1-\varepsilon))^{n-n_1}\varepsilon^{n_1} \\ &> ((2^q-1)^{-1}(1-\varepsilon))^{m-m_1}\varepsilon^{m_1} \\ &= w_{\vec{e}}(\alpha_1^{m_1}\alpha_2^{m_2}\ldots\alpha_{2^q}^{m_{2^q}}). \end{aligned} \qquad \dashv$$

For w satisfying Ax however the answer is somewhat more interesting. Since we can permute atoms we may assume that $n_1 \geq n_2 \geq n_3 \geq \cdots \geq n_{2^q}$ and $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_{2^q}$. In this case we have the following theorem from [117]:

THEOREM 21.6. *For $n_1 \geq n_2 \geq n_3 \geq \cdots \geq n_{2^q}$ and $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_{2^q}$, a necessary and sufficient condition for*

$$w(\alpha_1^{n_1}\alpha_2^{n_2}\ldots\alpha_{2^q}^{n_{2^q}}) \leq w(\alpha_1^{m_1}\alpha_2^{m_2}\ldots\alpha_{2^q}^{m_{2^q}}) \tag{205}$$

to hold for all w satisfying Ax is that

$$\sum_{j\geq i} n_j \geq \sum_{j\geq i} m_j \quad \text{for } i = 1, 2, \ldots, 2^q. \tag{206}$$

PROOF. If (206) fails because $\sum_{j\geq 1} n_j < \sum_{j\geq 1} m_j$ then c_∞^L, which satisfies Ax, provides a counter-example to (205). On the other hand if it first fails for some $i > 1$ then let $\vec{c} \in \mathbb{D}_{2^q}$ have the first $i-1$ coordinates $(1-(2^q-i+1)\varepsilon)/(i-1)$, where $\varepsilon > 0$ is small, and the remaining coordinates ε. Then, recalling the definition of $v_{\vec{c}}$ from (86),

$$\begin{aligned} v_{\vec{c}}(\alpha_1^{m_1}\alpha_2^{m_2}\ldots\alpha_{2^q}^{m_{2^q}}) &= O\big(\varepsilon^{\sum_{j\geq i} m_j}\big) \\ v_{\vec{a}}(\alpha_1^{n_1}\alpha_2^{n_2}\ldots\alpha_{2^q}^{n_{2^q}}) &= O\big(\varepsilon^{\sum_{j\geq i} n_j}\big) \end{aligned}$$

from which the required falsification of (205) follows.

Now suppose that (206) holds. We may assume that $\sum_j m_j = \sum_j n_j$, otherwise we can iteratively remove 1 from n_r, where r is maximal such that $\sum_{j\geq i} n_j > \sum_{j\geq i} m_j$ for all $i \leq r$, to eventually obtain $n_1', n_2', \ldots, n_{2^q}'$ satisfying (206) and $\sum_j m_j = \sum_j n_j'$ (and of course $w(\alpha_1^{n_1'}\alpha_2^{n_2'}\ldots\alpha_{2^q}^{n_{2^q}'}) \geq w(\alpha_1^{n_1}\alpha_2^{n_2}\ldots\alpha_{2^q}^{2^q})$ for any w).

By Muirhead's Inequality, see [39, p44], for such $\vec{n}, \vec{m}$ and real numbers $x_1, x_2, \ldots, x_{2^q} \geq 0$,

$$\sum_{\sigma\in \mathsf{S}_{2^q}} x_{\sigma(1)}^{m_1}x_{\sigma(2)}^{m_2}\ldots x_{\sigma(2^q)}^{m_{2^q}} \geq \sum_{\sigma\in \mathsf{S}_{2^q}} x_{\sigma(1)}^{n_1}x_{\sigma(2)}^{n_2}\ldots x_{\sigma(2^q)}^{n_{2^q}}, \tag{207}$$

with equality just if

$$\vec{n} = \vec{m} \text{ or the } x_i \neq 0 \text{ are all equal.} \tag{208}$$

Let μ be the symmetric de Finetti prior for w. Then from (207),

$$\int_{\mathbb{D}_{2^q}} \sum_\sigma x_{\sigma(1)}^{m_1} x_{\sigma(2)}^{m_2} \dots x_{\sigma(2^q)}^{m_{2^q}} \, d\mu \geq \int_{\mathbb{D}_{2^q}} \sum_\sigma x_{\sigma(1)}^{n_1} x_{\sigma(2)}^{n_2} \dots x_{\sigma(2^q)}^{n_{2^q}} \, d\mu.$$

Since μ is symmetric for each of these summations the integrals of each of the summands are equal so this simplifies to

$$2^q! \int_{\mathbb{D}_{2^q}} x_1^{m_1} x_2^{m_2} \dots x_{2^q}^{m_{2^q}} \, d\mu \geq 2^q! \int_{\mathbb{D}_{2^q}} x_1^{n_1} x_2^{n_2} \dots x_{2^q}^{n_{2^q}} \, d\mu,$$

and hence from (43),

$$w(\alpha_1^{m_1} \alpha_2^{m_2} \dots \alpha_{2^q}^{m_{2^q}}) \geq w(\alpha_1^{n_1} \alpha_2^{n_2} \dots \alpha_{2^q}^{n_{2^q}}),$$

as required. ⊣

Theorem 21.6 is often referred to (by us at least) as the 'Only Rule' since, being an equivalence, it is the only such rule which necessarily applies for all probability functions satisfying Ax. Assuming JSP, which of course extends Ax, gives some further such inequalities which do not follow from the Only Rule alone, see for example [101], but currently even giving an intuitive explanation for why these hold is a challenge, let alone formally demarcating the full extent of them.

As mentioned earlier on page 70 there are several ways of formally capturing the idea that 'the more times I have seen something in the past the more times I should expect to see it in the future' as a principle within the PIL. At that point we considered one such, the Principle of Instantial Relevance, PIR. An alternative suggested by the above Only Rule is:

The Unary Principle of Induction, UPI.

$$w(\alpha_i \mid \alpha_1^{m_1} \alpha_2^{m_2} \dots \alpha_{2^q}^{m_{2^q}}) \geq w(\alpha_j \mid \alpha_1^{m_1} \alpha_2^{m_2} \dots \alpha_{2^q}^{m_{2^q}}),$$

whenever $m_i \geq m_j$.

Informally, if in the past one has seen at least as many instances of $\alpha_i(x)$ as of $\alpha_j(x)$ then one should give at least as high a probability to the next a_{m+1} satisfying $\alpha_i(x)$ as to it satisfying $\alpha_j(x)$. Theorem 21.6 now gives us:

Corollary 21.7. *Ax implies UPI.*

Apart from this formalization UPI of the intuitive idea of induction PIR itself also suggests several more alternatives which we will now briefly consider. The following principles, and some of the subsequent results we mention about them, appear in [117] and [144]:

THE NEGATIVE PRINCIPLE OF INSTANTIAL RELEVANCE, NPIR.
For $j \neq k$,

$$w\Big(\alpha_j(a_{n+2}) \mid \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\Big) \geq w\Big(\alpha_j(a_{n+2}) \mid \alpha_k(a_{n+1}) \wedge \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\Big). \tag{209}$$

THE STRONG PRINCIPLE OF INSTANTIAL RELEVANCE, SPIR.
For $j \neq k$,

$$w\Big(\alpha_j(a_{n+2}) \mid \alpha_j(a_{n+1}) \wedge \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\Big) \geq \\ w\Big(\alpha_j(a_{n+2}) \mid \alpha_k(a_{n+1}) \wedge \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\Big). \tag{210}$$

Both of these then express the idea that seeing an extra instantiation $\alpha_k(a_{n+1})$ of an atom *different* from $\alpha_j(x)$ should *decrease* one's belief in $\alpha_j(a_{n+2})$, either from what it would have been had not $\alpha_k(a_{n+1})$ been observed in the case of NPIR or from what it would have been had $\alpha_j(a_{n+1})$ been observed in the case of SPIR. Both of these then complement PIR which says that observing $\alpha_j(a_{n+1})$ should *increase* one's belief in $\alpha_j(a_{n+2})$. Clearly NPIR is a special case of GPIR.

Using PIR on the left hand side of (209) it is easy to see that NPIR implies SPIR. The c_λ^L and w_L^δ are also easily seen to satisfy NPIR.

However the situation when we only assume that w satisfies Ax is (apparently) complicated and generally rather non-intuitive, see [117], [144]. If $q = 1$ then NPIR and SPIR do both hold. If $q \geq 2$ then SPIR holds if $m_j = m_k$ or $m_j + 1 = m_k$ where $m_j = |\{i \mid h_i = j\}|$, $m_k = |\{i \mid h_i = k\}|$ but otherwise can fail when $q \geq 3$ (and apart for the two currently unresolved cases $0 < m_k = m_j - 1$, $0 < m_j = m_k - 2$, also when $q = 2$). For example if $q = 3$,

$$w = (1+K)^{-1}(v_{\langle 7/28, 3/28, 3/28, \dots, 3/28\rangle} + K v_{\langle 2^{-1}, 2^{-1}, 0, 0, \dots, 0\rangle})$$

where K is large then

$$w(\alpha_1(a_5) \mid \alpha_1(a_4) \wedge \alpha_1(a_3) \wedge \alpha_1(a_2) \wedge \alpha_2(a_1)) \\ < w(\alpha_1(a_5) \mid \alpha_2(a_4) \wedge \alpha_1(a_3) \wedge \alpha_1(a_2) \wedge \alpha_2(a_1)).$$

Clearly it is something of a puzzle to explain this strange behaviour, why should having m_j, m_k close like this make a difference?

The situation with NPIR at least largely avoids these complications. When $m_j, m_k > 0$ and $q \geq 2$ there are always probability functions satisfying Ax which provide counter-examples to NPIR, indeed such a probability function satisfying Ax is

$$\begin{aligned} w &= (1+K)^{-1}(v_{\langle 2^{-1},2^{-1},0,0,\dots,0\rangle} + Kv_{\langle 2^{-q},2^{-q},\dots,2^{-q}\rangle}) \\ &= (1+K)^{-1}(v_{\langle 2^{-1},2^{-1},0,0,\dots,0\rangle} + Kc_\infty^L) \end{aligned}$$

where K is large compared with 2^q. In particular, for $q \geq 2$, we have

$$w(\alpha_1(a_4) \mid \alpha_2(a_3) \wedge \alpha_1(a_1) \wedge \alpha_2(a_2)) > w(\alpha_1(a_4) \mid \alpha_1(a_1) \wedge \alpha_2(a_2)),$$

in the presence of $\alpha_1(a_1) \wedge \alpha_2(a_2)$ the observation of $\alpha_2(a_3)$ having strictly enhanced the conditional probability of $\alpha_1(a_4)$, despite it being a distinct, and hence disjoint, atom.

Of course such w can hardly be said to be 'rational' and one's reaction to this might be that stopping at just Ax is too early and we should make further assumptions, such as JSP (or even SPIR itself), which would guarantee us SPIR and NPIR. Adopting this stratagem however does not entirely banish the underlying problem. For on the face of it the intuition behind NPIR and SPIR would seem to be no different if we were to replace the conditioning $\bigwedge_{i=1}^n \alpha_{h_i}(a_i)$ in their formulations by $\theta(a_1, \dots, a_n) \in QFSL$. However with that change we can, by the Conditionalization Theorem (Corollary 13.3), find counter-examples as above for *any* probability function w_1 satisfying the requirements of that theorem. In particular we must have counter-examples to such an intuition even for the c_λ^L when $0 < \lambda < \infty$.

Chapter 22

PRINCIPLES OF ANALOGY

The Principle of Instantial Relevance, PIR, could be interpreted as an example of 'support by analogy', that, in the notation of (67), $\alpha(a_{n+1})$ is 'analogous' to $\alpha(a_{n+2})$ and hence that learning $\alpha(a_{n+1})$ should add support to one's believing $\alpha(a_{n+2})$. Traditionally however it has been usual to extend the term 'analogous' to apply also between occurrences of relation symbols. Thus when $q = 4$ the atoms

$$\begin{aligned} \alpha(x) &= R_1(x) \wedge \neg R_2(x) \wedge R_3(x) \wedge R_4(x), \\ \alpha'(x) &= R_1(x) \wedge \neg R_2(x) \wedge \neg R_3(x) \wedge R_4(x), \end{aligned} \tag{211}$$

might be thought of as being somewhat 'analogous' since they agree on the signs (i.e. negated or unnegated) of R_1, R_2, R_4 and only disagree in sign on the single relation symbol R_3. In turn then we might argue that because of this analogy, or nearness, learning $\alpha(a_{n+1})$ should add some support to one's believing $\alpha'(a_{n+2})$.

This idea of such support by analogy was considered quite extensively by Carnap, for example in [17], Carnap & Stegmüller [18] and later by Festa [24], Hesse [41], Maher [87], [88], Maio [89], Romeijn [123], Skyrms [136], to name but a few. Generally the aim in these accounts was to propose probability functions which manifest some such analogical support. In particular in [136] Skyrms highlighted the key test case when $q = 2$, so the atoms are

$$\begin{aligned} \alpha_1(x) &= R_1(x) \wedge R_2(x), & \alpha_2(x) &= R_1(x) \wedge \neg R_2(x), \\ \alpha_3(x) &= \neg R_1(x) \wedge R_2(x), & \alpha_4(x) &= \neg R_1(x) \wedge \neg R_2(x), \end{aligned} \tag{212}$$

and the idea is that in the presence of some background knowledge $\psi(a_1, \ldots, a_n)$ learning $\alpha_1(a_{n+1})$ should provide more support for $\alpha_1(a_{n+2})$ than learning instead $\alpha_2(a_{n+1})$ (or $\alpha_3(a_{n+1})$), and more still than learning $\alpha_4(a_{n+1})$.

Skyrms [136] likens the situation to a *Wheel of Fortune*, a rotating wheel divided into 4 quadrants, <u>N</u>orth, <u>E</u>ast, <u>W</u>est, <u>S</u>outh, where a roll of the wheel landing North up should provide more support for a bias in favour of the North quadrant that an East or West would do and yet more still than a South uppermost. Clearly here we could identify North with α_1, East with α_2. West with α_3 and South with α_4.

The formal principle in our context expressing this analogical influence appears to be the following:

The Strict Analogy Principle, SAP.

For $\alpha_i(x), \alpha_j(x), \alpha_k(x)$ atoms of L, $j \neq k$ and consistent $\psi(a_1, a_2, \ldots, a_n) \in QFSL$, if

$$\|\alpha_i(x) - \alpha_j(x)\| < \|\alpha_i(x) - \alpha_k(x)\|$$

then

$$w(\alpha_i(a_{n+2}) \mid \alpha_j(a_{n+1}) \wedge \psi(\vec{a})) > w(\alpha_i(a_{n+2}) \mid \alpha_k(a_{n+1}) \wedge \psi(\vec{a})). \quad (213)$$

Here $\|\alpha_i(x) - \alpha_j(x)\|$ is the Hamming distance between $\alpha_i(x)$ and $\alpha_j(x)$, i.e. the number of R_m which differ in sign between $\alpha_i(x)$ and $\alpha_j(x)$.[85]

The first point to note here is that SAP implies Reg, otherwise we could arrange for (213) to fail by taking ψ such that $w(\psi) = 0$.[86] Similarly for $q > 1$ no probability function satisfying Ax can hope to satisfy SAP since if, for example, $n = 0$ and we take ψ to be a tautology then the two sides of (213) will be equal. Hamming distance is however preserved under permuting predicate symbols and transposing $R_m, \neg R_m$ (and conversely any permutation of atoms which preserves Hamming distance can be fabricated by permuting predicate symbols and transposing $R_m, \neg R_m$, see [42]) so it seems only natural to assume Px and SN (together with the standing assumption of Ex) when investigating SAP.

The primary interest here then is in classifying the probability functions which satisfy SAP, Px and SN. In the case $q = 1$ this is entirely straightforward, all we actually need to be concerned with is making the inequality in (213) strict. The following result, and the two after it, appear in [42]:

Proposition 22.1. *Suppose that $q = 1$ and the probability function w on SL satisfies SN. Then either $w = \lambda c_\infty^L + (1-\lambda)c_0^L$ for some $0 \leq \lambda \leq 1$ or w satisfies SAP.*

Proof. Note that for $q = 1$, SN implies Ax, and $\alpha_2(x) = \neg\alpha_1(x)$. The inequality (213) with $\leq$ in place of $<$ is just PIR, which holds. Looking back at the proof of PIR (Theorem 11.1), it is apparent that we will always have the strict inequality in (213) if some point $\langle c, 1-c\rangle$ with $c \in (0, 1)$, $c \neq 1/2$, is in the support of the de Finetti prior of w (recall that this prior can be assumed symmetric) and $\psi(\vec{a})$ is consistent. Hence if SAP fails the only points in the support can be $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle \frac{1}{2}, \frac{1}{2}\rangle$ and thus $w = \lambda c_\infty^L + (1-\lambda)c_0^L$. Conversely considering $\psi(a_1, a_2) = \alpha_1(a_1) \wedge \alpha_2(a_2)$ shows that the strict inequality in (213) can then indeed fail for these w. ⊣

[85] So for example the Hamming distance between the atoms in (211) is 1.

[86] Recall the convention on page 23.

For $q = 2$ however it turns out that SAP, Px and SN amount to very restrictive conditions indeed. In the statement of this theorem the atoms of L are assumed to be ordered as in (212) and $y_{\langle a,b,c,d\rangle}$, with $\langle a, b, c, d\rangle \in \mathbb{D}_4$, is the probability function

$$8^{-1}(w_{\langle a,b,c,d\rangle} + w_{\langle a,c,b,d\rangle} + w_{\langle b,a,d,c\rangle} + w_{\langle c,a,d,b\rangle} \\ + w_{\langle b,d,a,c\rangle} + w_{\langle c,d,a,b\rangle} + w_{\langle d,b,c,a\rangle} + w_{\langle d,c,b,a\rangle}).$$

THEOREM 22.2. *Let $q = 2$. Then the probability function w on SL satisfies SAP, Px and SN just if one of the following hold*:

(1) $w = y_{\langle a,b,b,c\rangle}$ *for some* $\langle a, b, b, c\rangle \in \mathbb{D}_4$ *with* $a > b > c > 0$, $ac = b^2$.
(2) $w = \lambda y_{\langle a,a,b,b\rangle} + (1-\lambda)c_\infty^L$ *for some* $\langle a, a, b, b\rangle \in \mathbb{D}_4$ *with* $a > b > 0$ *and* $0 < \lambda \leq 1$.

A full proof of this theorem (and the next) appears in [42] and we will not replicate it here since it seems something of a dead end in view of:[87,88]

THEOREM 22.3. *For $q > 2$ there are no probability functions on SL satisfying SAP, Px and SN.*

On a somewhat more optimistic note the proofs of these last two theorems use the fact that we can take the ambient knowledge $\psi(a_1, \dots, a_n)$ to be *any* quantifier free sentence of L. If instead we restrict $\psi(\vec{a})$ to be a state description, as is done in most of the papers on the Wheel of Fortune (i.e. we have *complete* knowledge of the previous n turns) then there are actually many other solutions.

In more detail let w be a probability function for the (unary) language $\{R\}$ satisfying SN, $w \neq c_0^{\{R\}}, c_\infty^{\{R\}}$, and for an atom α_h of our default language L with q predicate symbols let $\rho_j(h) = 1$ if R_j appears positively in α_h and 0 otherwise. Now set

$$w^{(q)}\Big(\bigwedge_{i=1}^{n} \alpha_{h_i}\Big) = \prod_{j=1}^{q} w\big(R^{\sum_i \rho_j(h_i)}(\neg R)^{n-\sum_i \rho_j(h_i)}\big).$$

So in particular in the case $q = 2$ and the $\alpha_i(x)$ are as in (212)

$$w^{(2)}(\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\alpha_4^{n_4}) = w(R^{n_1+n_2}(\neg R)^{n_3+n_4}) \cdot w(R^{n_1+n_3}(\neg R)^{n_2+n_4}).$$

[87] In [42] a version of SAP with $\geq$ in place of $>$ is also considered but even then the outcome is hardly more appealing. Notice that in this case the analogy principle becomes an extension of SPIR. Notice too that by Lemma 6.1 with this version we can equivalently drop the restriction on ψ that it be quantifier free.

[88] In a footnote on page 46 of Carnap's [17] the editor (R.C.Jeffrey), essentially following Carnap, presents an 'Analogy Principle' which in our notation says that if $w(\alpha_i(a_2) \mid \alpha_j(a_1)) \geq w(\alpha_i(a_2) \mid \alpha_k(a_1))$ then we should have that

$$w(\alpha_i(a_{n+2}) \mid \alpha_j(a_{n+1}) \wedge \Theta(a_1, \dots, a_n)) \geq w(\alpha_i(a_{n+2}) \mid \alpha_k(a_{n+1}) \wedge \Theta(a_1, \dots, a_n))$$

for any state description $\Theta(a_1, \dots, a_n)$. To our knowledge a characterization of the probability functions satisfying this principle is lacking at this time. However, in view of Theorem 22.2 one might suspect that even for $q = 2$ it is unlikely to prove to be particularly attractive.

Then for $q = 2$, $w^{(q)}$ is a probability function on SL satisfying Px, SN *and* SAP when we restrict the $\psi(\vec{a})$ to be a state description (but not for $q > 2$). However not all probability functions satisfying these principles are of this form, for example the probability function given in (2) of Theorem 22.2 is not for $0 < \lambda \leq 1$, and currently a complete characterization is lacking, see [42].[89]

Despite the apparently limited scope of this interpretation of 'analogy' there is another way of looking at the notion which we have already met in (186) on page 146 where it was noted that for $0 < \lambda < \infty$,

$$c_\lambda^L(R_2(a_3) \wedge R_2(a_4) \mid R_1(a_1) \wedge R_1(a_2)) > c_\lambda^L(R_2(a_3) \wedge R_2(a_4)).$$

In this case we could say that $R_1(a_1) \wedge R_1(a_2)$ provided 'analogical evidence' for $R_2(a_3) \wedge R_2(a_4)$ in the sense that they both have the *same form*.

This form of support by analogy seems to be fairly common in everyday life, for example the feeling that life on Mars gains credibility by comparison with its counterpart our own planet. Indeed such support by analogy frequently serves as a heuristic in science for generating conjectures, see for example [120], [121]. Arguably it might be formalized by:

THE COUNTERPART PRINCIPLE, CP.

Let $\theta, \theta' \in SL$ be such that θ' is the result of replacing some constant/relation symbols in θ by new constant/relation symbols not occurring in θ. Then

$$w(\theta \mid \theta') \geq w(\theta). \tag{214}$$

Essentially the following theorem appears in [43].

THEOREM 22.4. *If w satisfies ULi then w satisfies the CP.*

PROOF. Assume that w satisfies ULi and let w^+ be a probability function for the infinite (unary) language $L^+ = \{R_1, R_2, R_3, \dots\}$ extending w and satisfying Px and Ex. Let θ, θ' be as in the statement of the theorem, without loss of generality assume that the constant symbols appearing in θ are amongst $a_1, a_2, \dots, a_t, a_{t+1}, \dots, a_{t+k}$, all the relation symbols appearing in θ are amongst $R_1, R_2, \dots, R_s, R_{s+1}, \dots, R_{s+j}$ and to form θ' they are replaced by $a_1, a_2, \dots, a_t$, $a_{t+k+1}, a_{t+k+2}, \dots, a_{t+2k}$, and $R_1, R_2, \dots, R_s, R_{s+j+1}$, $R_{s+j+2}, \dots, R_{s+2j}$ respectively. So with the obvious notation we can write

$$\theta = \theta(a_1, \dots, a_t, a_{t+1}, \dots, a_{t+k}, R_1, \dots, R_s, R_{s+1}, \dots, R_{s+j}),$$
$$\theta' = \theta(a_1, \dots, a_t, a_{t+k+1}, \dots, a_{t+2k}, R_1, \dots, R_s, R_{s+j+1}, \dots, R_{s+2j}).$$

With this notation let θ_{i+1} be

$$\theta(a_1, \dots, a_t, a_{t+ik+1}, \dots, a_{t+(i+1)k}, R_1, \dots, R_s, R_{s+ij+1}, \dots, R_{s+(i+1)j}),$$

[89]It is worth noting that the version of this principle with $\geq$ in place of $>$ together with Ax characterizes the c_λ^L for $q \geq 2$.

so $\theta_1 = \theta$, $\theta_2 = \theta'$. Let $\mathcal{L}$ be the unary language with a single unary relation symbol R and define $\tau : QFS\mathcal{L} \to QFSL^+$ by

$$\tau(R(a_i)) = \theta_i, \quad \tau(\neg\phi) = \neg\tau(\phi), \quad \tau(\phi \wedge \psi) = \tau(\phi) \wedge \tau(\psi), \quad \text{etc.}$$

for $\phi, \psi \in QFS\mathcal{L}$.

Now define $v : QFS\mathcal{L} \to [0, 1]$ by

$$v(\phi) = w^+(\tau(\phi)).$$

Then since w^+ satisfies (P1–2) (on SL) so does v (on $QFS\mathcal{L}$). Also since w^+ satisfies Ex and Px, for $\phi \in QFS\mathcal{L}$, permuting the θ_i in $w(\tau(\phi))$ will leave this value unchanged so permuting the a_i in ϕ will leave $v(\phi)$ unchanged. Hence v satisfies Ex.

By Gaifman's Theorem 7.1 v has an extension to a probability function on $S\mathcal{L}$ satisfying Ex and hence satisfying PIR by Theorem 11.1. In particular then

$$v(R(a_1) \mid R(a_2)) \geq v(R(a_1)). \tag{215}$$

But since $\tau(R(a_1)) = \theta$, $\tau(R(a_2)) = \theta'$ this amounts to just the Counterpart Principle

$$w(\theta \mid \theta') \geq w(\theta). \qquad \dashv$$

Of course the Counterpart Principle would be much more pleasing if we could make the inequality strict for θ non-contradictory, non-tautologous. Unfortunately we cannot hope to get this if, for example, w also satisfies SN, since in that case we can easily show that

$$w(R_2(a_2)) \mid R_1(a_1)) = 1/2 = w(R_1(a_1)).$$

Indeed by Theorem 21.4 and Proposition 21.3 if θ is a 'constant sentence', i.e. $w(\theta) = c$ for all w satisfying Ax, then we will always have $w(\theta \mid \theta') = w(\theta)$ for all w satisfying Ax. Accounts of some conditions on w and θ for which such a strict Counterpart Principle does hold for all non-constant sentences are given in [43]. In particular the c_λ^L for $0 < \lambda < \infty$ satisfy the strict Counterpart Principle for all non-constant sentences.

Reviewing CP one might expect, in the notation of Theorem 22.4, that the additional support that conditioning on θ' provides for θ would depends on the number of constants/predicates which have been replaced. For example if

$$\theta = \theta(\vec{a}, R_1, R_2, \vec{R}), \quad \theta' = \theta(\vec{a}, R_1, R_4, \vec{R}), \quad \theta'' = \theta(\vec{a}, R_3, R_4, \vec{R}),$$

where $\vec{R} = R_5, R_6, \ldots, R_k$, then

$$w(\theta \mid \theta') \geq w(\theta \mid \theta'').$$

In Chapter 25, page 186, we shall have the means to show that such inequalities do indeed hold, that as we enlarge the set of difference between θ and θ' so we lessen, or at least do not increase, the support θ' gives to θ.

We finally mention that these same results on CP can be shown, with identical proofs in fact, in the polyadic if we replace ULi with Li, see page 210.

Chapter 23

UNARY SYMMETRY

The idea that it is rational to respect symmetry when assigning beliefs led us in the previous chapters to formulate the Principles of Constant and Predicate Exchangeability, Strong Negation and Atom Exchangeability. Since these have proved rather fruitful it is natural to ask if there are other symmetries we might similarly exploit, and in turn this begs the question as to what we actually mean by a 'symmetry'. In this chapter we will suggest an answer to this question, and then consider some of its consequences.

First recall the context in which we are proposing our 'rational principles of belief assignment': Namely we imagine an agent inhabiting some world or structure M in $\mathcal{T}L$ who is required to assign probabilities $w(\theta)$ to the $\theta \in SL$ in an arguably rational way despite knowing nothing about which particular structure M from $\mathcal{T}L$ s/he is inhabiting. Given this framework it seems (to us at least) clear that the agent should act the same in this framework as s/he would in any isomorphic copy of it, on the grounds that with zero knowledge the agent should have no way of differentiating between his/her framework and this isomorphic copy.

To make sense of this idea we need an appropriate formulation of an 'automorphism' of the framework. Arguing that all the agent knows is L, $\mathcal{T}L$ and for each $\theta \in SL$ the conditions under which θ holds, equivalently the set of structures in $\mathcal{T}L$ in which θ is true, suggests that what we mean by an 'automorphism' is an automorphism σ of the two sorted structure BL with universe $\mathcal{T}L$ together with all the subsets of $\mathcal{T}L$ of the form

$$[\theta] = \{\, M \in \mathcal{T}L \mid M \vDash \theta \,\}$$

for $\theta \in SL$, and the binary relation $\in$ between elements of $\mathcal{T}L$ and the sets $[\theta]$.

Notice here that if $\theta \equiv \phi$ then $[\theta] = [\phi]$ and that $\cap, \cup, \subseteq, =, \mathcal{T}L, \emptyset$ and complement in $\mathcal{T}L$ are all definable in BL. Furthermore since

$$[\theta \wedge \phi] = [\theta] \cap [\phi], \quad [\neg\theta] = \mathcal{T}L - [\theta], \quad \text{etc.}$$

our agent is aware of how the $[\theta]$ relate to each other.[90]

[90]One could argue too that the agent should know that $[\theta]$ is the set of structures in $\mathcal{T}L$ in which θ is true. We shall return to this point later.

With this definition let σ be an automorphism of BL. Then σ gives a bijection of $\mathcal{T}L$ and since for $\theta \in SL$,

$$M \in [\theta] \iff \sigma(M) \in \sigma[\theta],$$

σ on BL is actually determined by its action on $\mathcal{T}L$ alone. Of course not every bijection σ of $\mathcal{T}L$ yields an automorphism of BL, we require in addition that for each $\theta \in SL$,

$$\sigma[\theta] = \{\sigma(M) \in \mathcal{T}L \mid M \vDash \theta\} = \{M \in \mathcal{T}L \mid M \vDash \psi\} \tag{216}$$

for some $\psi \in SL$ and conversely for every $\psi \in SL$ there is a $\theta \in SL$ satisfying (216). In this case we may write $\sigma(\theta)$, or just $\sigma\theta$, for such a ψ. This is not strictly correct since θ, ψ are only determined up to logical equivalence but since we have throughout been identifying logically equivalent sentences it should cause no additional confusion. Notice that since

$$\sigma[\neg\theta] = \mathcal{T}L - \sigma[\theta], \quad \sigma[\theta \wedge \phi] = \sigma[\theta] \cap \sigma[\phi], \quad \sigma[\theta \vee \phi] = \sigma[\theta] \cup \sigma[\phi],$$

by this convention, up to logical equivalence,

$$\sigma(\neg\theta) = \neg\sigma(\theta), \quad \sigma(\theta \wedge \phi) = \sigma(\theta) \wedge \sigma(\phi), \quad \sigma(\theta \vee \phi) = \sigma(\theta) \vee \sigma(\phi).$$

Indeed, more generally we have:

PROPOSITION 23.1. *Let σ be an automorphism of BL and w a probability function on SL. Then $w_\sigma : SL \to [0,1]$ defined by*

$$w_\sigma(\theta) = w(\sigma(\theta))$$

is a probability function on SL.

PROOF. From the above discussion it is clear that (P1–2) hold for w_σ. For (P3) let $\exists x\, \psi(x) \in SL$ and let μ_w be the countable additive measure on $\mathcal{T}L$ as in (7.2), so

$$w = \int_{\mathcal{T}L} V_M \, d\mu_w(M).$$

Then since $[\exists x\, \psi(x)] = \bigcup_{i=1}^\infty [\psi(a_i)]$,

$$\begin{aligned} w_\sigma(\exists x\, \psi(x)) &= w(\sigma(\exists x\, \psi(x))) = \mu_w(\sigma[\exists x\, \psi(x)]) \\ &= \mu_w\Big(\sigma\Big(\bigcup_{i=1}^\infty [\psi(a_i)]\Big)\Big) = \mu_w\Big(\bigcup_{i=1}^\infty \sigma[\psi(a_i)]\Big) \\ &= \mu_w\Big(\bigcup_{i=1}^\infty [\sigma\psi(a_i)]\Big) = \lim_{n\to\infty} \mu_w\Big(\bigcup_{i=1}^n [\sigma\psi(a_i)]\Big) \\ &= \lim_{n\to\infty} \mu_w\Big(\sigma\Big(\bigcup_{i=1}^n [\psi(a_i)]\Big)\Big) = \lim_{n\to\infty} \mu_w\Big(\sigma\Big[\bigvee_{i=1}^n \psi(a_i)\Big]\Big) \\ &= \lim_{n\to\infty} w\Big(\sigma\Big(\bigvee_{i=1}^n \psi(a_i)\Big)\Big) = \lim_{n\to\infty} w_\sigma\Big(\bigvee_{i=1}^n \psi(a_i)\Big), \end{aligned}$$

as required. ⊣

Given that our agent only knows BL the earlier discussion leads us to the following grand symmetry principle:[91]

THE UNARY INVARIANCE PRINCIPLE, UINV.
If σ is an automorphism of BL then $w(\theta) = w(\sigma(\theta))$ for $\theta \in SL$.

In other words rationally assigned probabilities should respect automorphisms of the overlying framework (qua BL). We use the adjective 'grand' here because as we shall now show *UINV* implies all the symmetry principles we have considered up to this point.

PROPOSITION 23.2. *UINV implies Ex.*

PROOF. Let σ be any permutation of $1, 2, 3, \ldots$ and extend σ to $\mathcal{T}L$ by setting $\sigma(M)$, for $M \in \mathcal{T}L$, to be the structure with (as always) universe $\{a_i \mid i \in \mathbb{N}^+\}$ and

$$\sigma(M) \vDash R_j(a_i) \iff M \vDash R_j(a_{\sigma(i)}).$$

Then for any $\theta(a_1, \ldots, a_n) \in SL$ mentioning at most the constants $a_1, \ldots, a_n$,

$$\sigma(M) \vDash \theta(a_1, \ldots, a_n) \iff M \vDash \theta(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

and clearly σ extends to an automorphism of BL.

Ex is now just an application of UINV with this automorphism σ. ⊣

PROPOSITION 23.3. *UINV implies Px.*

PROOF. Let $\sigma \in \mathrm{S}_q$ and extend σ to $\mathcal{T}L$ by setting $\sigma(M)$, for $M \in \mathcal{T}L$, to be the structure with (as always) universe $\{a_i \mid i \in \mathbb{N}^+\}$ and

$$\sigma(M) \vDash R_j(a_i) \iff M \vDash R_{\sigma(j)}(a_i).$$

Then displaying the relation symbols in $\theta \in SL$ by $\theta(R_1, R_2, \ldots, R_q)$,

$$\sigma(M) \vDash \theta(R_1, R_2, \ldots, R_q) \iff M \vDash \theta(R_{\sigma(1)}, R_{\sigma(2)}, \ldots, R_{\sigma(q)})$$

and clearly σ extends to an automorphism of BL.

Px is now just an application of UINV with this automorphism σ. ⊣

PROPOSITION 23.4. *UINV implies SN.*

PROOF. Define $\sigma : \mathcal{T}L \to \mathcal{T}L$ by

$$\sigma(M) \vDash R_i(a_n) \iff \begin{cases} M \vDash R_i(a_n) & \text{if } i \neq 1, \\ M \vDash \neg R_i(a_n) & \text{if } i = 1. \end{cases}$$

[91] In the literature the 'principle of symmetry' is frequently confused with 'the principle of insufficient reason (or indifference)', the idea, for example, that if all that is known about a random variable is that it lies in the interval $[0, 1]$, then the probability that it lies in a subinterval $[a, b]$ should only depend on its length $b - a$. The principle of insufficient reason gives rise to a number of well known paradoxes, for example von Kries' wine-water puzzle, see [140, chapter 12]. In these cases however there is either an underlying symmetry which resolves the 'paradox', as in Jaynes' treatment of Bertrand's Paradox in [56], or there simply is no such symmetry as we will understand it in this chapter upon which to justify the 'indifference' in the first place (see for example [112]). Either way no blame can be attached to the 'principle of symmetry'.

Then in the notation of the previous proposition,

$$\sigma(M) \vDash \theta(R_1, R_2, R_3, \dots, R_q) \iff M \vDash \theta(\neg R_1, R_2, R_3, \dots, R_q).$$

Clearly σ determines an automorphism of BL and in this case UINV yields SN. ⊣

PROPOSITION 23.5. *UINV implies Ax.*

PROOF. Let $\sigma \in \mathsf{S}_{2^q}$ and for $M \in \mathcal{T}L$ let $\sigma(M)$ be such that

$$\sigma(M) \vDash \alpha_h(a_k) \iff M \vDash \alpha_{\sigma(h)}(a_k) \tag{217}$$

for each atom $\alpha_h(x)$ of L. Notice that since there is a unique $\sigma(h)$ for which this right hand side holds h will similarly be unique on the left hand side.

Now by the discussion around (59) for this simple unary language any sentence θ mentioning only constants included amongst $a_{k_1}, a_{k_2}, \dots, a_{k_m}$ is logically equivalent to a unique (up to order) disjunction of the form

$$\bigvee_{r=1}^{n} \Big(\bigwedge_{i=1}^{m} \alpha_{h_{r,i}}(a_{k_i}) \wedge \bigwedge_{j=1}^{2^q} (\exists x\, \alpha_j(x))^{\varepsilon_{r,j}} \Big) \tag{218}$$

where the $\varepsilon_{r,j} \in \{0, 1\}$, and ϕ^ε stands, as usual, for ϕ if $\varepsilon = 1$ and $\neg\phi$ if $\varepsilon = 0$, and the disjuncts are distinct and consistent.

From this we see that σ extends to an automorphism of BL since if θ as above is logically equivalent to (218) then

$$\begin{aligned}\sigma(M) \vDash \theta &\iff \sigma(M) \vDash \bigvee_{r=1}^{n} \Big(\bigwedge_{i=1}^{m} \alpha_{h_{r,i}}(a_{k_i}) \wedge \bigwedge_{j=1}^{2^q} (\exists x\, \alpha_j(x))^{\varepsilon_{r,j}} \Big) \\ &\iff M \vDash \bigvee_{r=1}^{n} \Big(\bigwedge_{i=1}^{m} \alpha_{\sigma(h_{r,i})}(a_{k_i}) \wedge \bigwedge_{j=1}^{2^q} (\exists x\, \alpha_{\sigma(j)}(x))^{\varepsilon_{r,j}} \Big)\end{aligned}$$

by (217), so

$$\sigma[\theta] = \Big[\bigvee_{r=1}^{n} \Big(\bigwedge_{i=1}^{m} \alpha_{\sigma(h_{r,i})}(a_{k_i}) \wedge \bigwedge_{j=1}^{2^q} (\exists x\, \alpha_{\sigma(j)}(x))^{\varepsilon_{r,j}} \Big) \Big].$$

Appealing to UINV now yields Ax. ⊣

Pleasingly then UINV has given us all the early symmetry principles that we have considered. The obvious question to ask now is whether or not it gives us anything more.

Unfortunately the answer is yes! The reason this is unfortunate, rather than a cause for some celebration, is that as shown in [112] (where the earlier results in this chapter also appear), and [113], the wholesale application of UINV in these cases reduces the possible choices down to one solitary probability function, Carnap's c_0^L. So, surprisingly, UINV even dispatches the intolerably ambivalent c_∞^L. On the positive side it is shown in [112] that things cannot get any worse, c_0^L does satisfy UINV so this grand principle is at least consistent.

Full details, with further discussion, of how it is that observing UINV leaves c_0^L as our only option are given in [112] but for completeness we shall now sketch the main points.

We firstly derive another, and already questionable, symmetry principle from UINV, namely:

RANGE EXCHANGEABILITY, Rx.

$$w\Big(\bigwedge_{i=1}^{s}\alpha_{g_i}(a_i)\Big) = w\Big(\bigwedge_{i=1}^{s}\alpha_{h_i}(a_i)\Big)$$

whenever $|\{g_i \mid 1 \leq i \leq s\}| = |\{h_i \mid 1 \leq i \leq s\}|$, *i.e. whenever the number of distinct atoms amongst the* $\alpha_{g_i}(x)$ *is the same as the number amongst the* $\alpha_{h_i}(x)$.

To this end let $s \in \mathbb{N}^+$ and let $\imath$ first permute the state descriptions for $a_1, a_2, \ldots, a_s$ in such a way that if

$$\imath\Big(\bigwedge_{i=1}^{s}\alpha_{g_i}(a_i)\Big) = \bigwedge_{i=1}^{s}\alpha_{h_i}(a_i)$$

then

$$\{g_i \mid 1 \leq i \leq s\} = \{h_i \mid 1 \leq i \leq s\}$$

(i.e. the distinct atoms amongst the $\alpha_{g_i}(x)$ are the same as those amongst the $\alpha_{h_i}(x)$). For $M \in \mathcal{T}L$ define $\imath(M) \in \mathcal{T}L$ by

$$\begin{aligned}\imath(M) \vDash \imath\Big(\bigwedge_{i=1}^{s}\alpha_{g_i}(a_i)\Big) &\iff M \vDash \bigwedge_{i=1}^{s}\alpha_{g_i}(a_i),\\ \imath(M) \vDash R_j(a_m) &\iff M \vDash R_j(a_m), \quad \text{for } m > s,\ j = 1, 2, \ldots, q.\end{aligned}$$

Then $\imath$ gives an automorphism of BL, indeed for $m \geq s$,

$$\begin{aligned}&\imath\Big[\bigvee_{r=1}^{n}\Big(\bigwedge_{i=1}^{m}\alpha_{g_{r,i}}(a_i) \wedge \bigwedge_{j=1}^{2^q}(\exists x\, \alpha_j(x))^{\varepsilon_{r,j}}\Big)\Big]\\ &\qquad = \Big[\bigvee_{r=1}^{n}\Big(\imath\Big(\bigwedge_{i=1}^{s}\alpha_{g_{r,i}}(a_i)\Big) \wedge \bigwedge_{i=s+1}^{m}\alpha_{g_{r,i}}(a_i) \wedge \bigwedge_{j=1}^{2^q}(\exists x\, \alpha_j(x))^{\varepsilon_{r,j}}\Big)\Big].\end{aligned}$$

This now gives Rx in the case where $\{g_i \mid 1 \leq i \leq s\} = \{h_i \mid 1 \leq i \leq s\}$ and the full version follows from Ax, which as we have already seen also follows from UINV, and the fact that we are free to choose s as large as we wish.

Range Exchangeability is itself a rather unpalatable principle, for example it implies that

$$w\Big(\alpha_1(a_{n+1}) \mid \alpha_1(a_1) \wedge \bigwedge_{i=2}^{n} \alpha_2(a_i)\Big) = w\Big(\alpha_2(a_{n+1}) \mid \alpha_1(a_1) \wedge \bigwedge_{i=2}^{n} \alpha_2(a_i)\Big)$$

whereas one might have felt that for large n, $\alpha_1(a_1) \wedge \bigwedge_{i=2}^{n} \alpha_2(a_i)$ should provide much more support for $\alpha_2(a_{n+1})$ than for $\alpha_1(a_{n+1})$.

At this point, to simplify matters, assume that $q = 1$, in other words that the language has just one unary relation symbol R_1. [The essential details of the proof for arbitrary q are given in [113].]

For $f : \mathbb{N}^+ \to \{0, 1\}$, which we think of as an infinite sequence of zeros and ones,

$$f = f(1), f(2), f(3), \ldots,$$

define $\lambda(f) : \mathbb{N}^+ \to \{0, 1\}$ as follows:

$$\lambda(f) = \begin{cases} 1, 1, f(2), f(3), f(4), \ldots & \text{if } f(1) = 1, \\ 0, f(3), f(4), f(5), \ldots & \text{if } f(1) = f(2) = 0, \\ 1, 0, f(3), f(4), f(5), \ldots & \text{if } f(1) = 0,\ f(2) = 1. \end{cases}$$

Notice that if $X \subseteq \{f \mid f : \mathbb{N}^+ \to \{0, 1\}\}$ is a finite union of sets of the form

$$\{f \mid \langle f(1), f(2), \ldots, f(n)\rangle = \vec{\sigma}\}$$

then the image of X under λ is also such a finite union, and conversely. For future reference call this property $\star$.

Now given $M \in \mathcal{T}L$ let $f_M : \mathbb{N}^+ \to \{0, 1\}$ be the function defined by

$$f_M(n) = 1 \iff M \vDash R_1(a_n).$$

Clearly f_M determines M. Define $\gamma : \mathcal{T}L \to \mathcal{T}L$ to be such that for $M \in \mathcal{T}L$,

$$f_{\gamma M} = \lambda(f_M).$$

In particular then

$$M \vDash R_1(a_1) \wedge \neg R_1(a_2) \iff \gamma M \vDash R_1(a_1) \wedge R_1(a_2) \wedge \neg R_1(a_3), \quad (219)$$
$$M \vDash \neg R_1(a_1) \wedge \neg R_1(a_2) \wedge R_1(a_3) \iff \gamma M \vDash \neg R_1(a_1) \wedge R_1(a_2),$$
$$M \vDash \neg R_1(a_1) \wedge R_1(a_2) \iff \gamma M \vDash R_1(a_1) \wedge \neg R_1(a_2).$$

Using the property $\star$ and the fact that λ is the identity on the two constant maps $f_M : \mathbb{N}^+ \to \{0, 1\}$, equivalently γ is the identity on the two structures $M \in \mathcal{T}L$ satisfying $\forall x\, R_1(x) \vee \forall x\, \neg R_1(x)$, it can be shown that γ extends to an automorphism of BL. In particular, from (219),

$$\gamma[R_1(a_1) \wedge \neg R_1(a_2)] = [R_1(a_1) \wedge R_1(a_2) \wedge \neg R_1(a_3)],$$

so for w satisfying UINV we must have

$$w(R_1(a_1) \wedge \neg R_1(a_2)) = w(R_1(a_1) \wedge R_1(a_2) \wedge \neg R_1(a_3)). \quad (220)$$

But since w satisfying UINV must satisfy Rx we also have

$$\begin{aligned} &w(R_1(a_1) \wedge \neg R_1(a_2)) \\ &\quad = w(R_1(a_1) \wedge \neg R_1(a_2) \wedge R_1(a_3)) + w(R_1(a_1) \wedge \neg R_1(a_2) \wedge \neg R_1(a_3)) \\ &\quad = 2w(R_1(a_1) \wedge R_1(a_2) \wedge \neg R_1(a_3)) \end{aligned}$$

and together with (220) this forces

$$w(R_1(a_1) \wedge \neg R_1(a_2)) = 0.$$

Given that w must also satisfy Ex the only option left open here is that $w = c_0^L$.

As a choice for a 'rational probability function'[92] however c_0^L is not particularly popular on account of its jumping to the conclusion that every a_i will be like the first one observed, more precisely that

$$c_0^L(\alpha_{i_2}(a_2) \wedge \cdots \wedge \alpha_{i_m}(a_m) \mid \alpha_j(a_1)) = \begin{cases} 1 & \text{if } i_2 = \cdots = i_m = j, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly if we were to simply accept this conclusion then we would be drawing a curtain over (the coffin of) PIL. So, where did we go wrong? One's first thought may be that *UINV* is just too permissive, it allows in too many automorphisms. In that case perhaps we should include some extra information into BL, maybe we should introduce SL and the $\theta \in SL$ as another type in the universe of BL together with a relation linking θ and $[\theta]$ to allow that the agent has specific knowledge of the representatives of the equivalence classes. That alone however would not help, we could just arrange that the σ paired off the elements of $\{\phi \in SL \mid \phi \equiv \theta\}$ with the elements of $\{\phi \in SL \mid \phi \equiv \sigma(\theta)\}$. What we might further attempt would be to require that $\sigma(\theta)$ somehow preserved the structure of θ. However this could appear rather artificial if we still want SN and Ax to fall under UINV.

Fortunately there is another way round this problem which does seem entirely natural. It is to consider Unary PIL as merely a restriction of Polyadic PIL and it is in this direction that we now move.

[92]This probability function is often confused with the so called 'straight rule', SR, which advocates setting

$$SR\Big(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\Big) = m_j/n,$$

where as usual $m_j = |\{i \mid h_i = j\}|$. Naively appealing as this might seem (see for example [5]) it is easy to see that SR has no extension to a probability function.

Part 3

POLYADIC PURE INDUCTIVE LOGIC

Chapter 24

INTRODUCTION TO POLYADIC PURE INDUCTIVE LOGIC

In Part 1 we placed no conditions on the arity of the relation symbols in L, the restriction to unary only happened in Part 2. In this third part we shall again allow into our language binary, ternary etc. relation symbols. As we have seen, despite the logical simplicity of unary languages, for example every formula becomes equivalent to a boolean combination of Π_1 and Σ_1 formulae, Unary PIL has still a rather rich theory. For this reason it is hardly surprising that with very few exceptions (for example Gaifman [30], Gaifman & Snir [32], Scott & Krauss [132], Krauss [69], Hilpinen [46] and Hoover [52]) 'Inductive Logic' meant 'Unary Inductive Logic' up to the end of the 20th century.

Of course there was an awareness of this further challenge, Carnap [12, p123-4] and Kemeny [61], [64] both made this point. There were at least two other reasons why the move to the polyadic was so delayed. The first is that simple, everyday examples of induction with non-unary relations are rather scarce. However they do exist and we do seem to have some intuitions about them. For example suppose that you are planting an orchard and you read that apples of variety $\mathcal{A}$ are good pollinators and apples of variety $\mathcal{B}$ are readily pollinated. Then you might expect that if you plant an $\mathcal{A}$ apple next to a $\mathcal{B}$ apple you will be rewarded with an abundant harvest, at least from the latter tree. In this case one might conclude that you had applied some sort of polyadic induction to reach this conclusion, and that maybe it has a logical structure worthy of further investigation.

Having said that it is still far from clear what probability functions should be proposed here (and possibly this is a third reason for the delay). Extending the unary notation we continue to have the 'completely independent probability function' on SL, for which we will continue to use the name c_∞^L, which just treats the $\pm R(a_{i_1}, a_{i_2}, \ldots, a_{i_r})$, where r is the arity of the relation symbol R of L and the a_{i_j} need not be distinct, as stochastically independent and each with probability $1/2$. We also continue to have the polyadic version of c_0^L, (again we will adopt the same notation) which for L as usual with relations symbols R_i of arity r_i, $i = 1, 2, \dots, q$, satisfies

$$c_0^L\Big(\bigwedge_{i=1}^{q} \forall x_1, \dots, x_{r_i} \ \pm R_i(x_1, \dots, x_{r_i})\Big) = 2^{-q}$$

for any of the 2^q assignments of $\pm$ here. However neither of these seem to appropriately capture our intuitions about induction.

A second reason for a certain reluctance to venture into the polyadic is that the notation and mathematical complication increases significantly, at least when compared with the unary case. The key reason for this is that a state description, for $a_1, a_2, \dots, a_n$ say, no longer tells us everything there is to know about these constants in the way it did in the unary situation. For example for a binary relation symbol R of L it leaves unanswered the question of whether or not $R(a_1, a_{n+1})$ holds. Thus we can never really 'nail down' the constants as we could before.

As far as putatively rational principles for the polyadic are concerned these seem, at this stage, also to be far less prominent than before, as the slightly contrived orchard example above suggests. We do still have the principles Ex, *which we will continue to assume of all our probability functions*, SN and Px (transposing relation symbols of the same arity) though Ax does not immediately generalize mutatis mutandis.

Just having Ex in fact allows us to cash in on results from the unary using the same idea as in the demonstration of the Counterpart Principle (see page 168) to prove some other simple principles that might first come to mind. As an example suppose that L has a ternary relation symbol R and w is a probability function on SL (satisfying Ex). Then our earlier unary results allow us to show, for example, the 'instance of relevance'

$$\begin{aligned} w(R(a_1, a_8, a_9) \mid R(a_1, a_2, a_3) \wedge \neg R(a_1, a_4, a_5) \wedge R(a_1, a_6, a_7)) \\ \geq w(R(a_1, a_8, a_9) \mid R(a_1, a_2, a_3) \wedge \neg R(a_1, a_4, a_5)). \quad (221) \end{aligned}$$

To see this let $\mathcal{L}$ be the language with a single unary relation symbol P and define a probability function v on $S\mathcal{L}$ by

$$v\Big(\bigwedge_{i=1}^{n} P^{\varepsilon_i}(a_{k_i})\Big) = w\Big(\bigwedge_{i=1}^{n} R^{\varepsilon_i}(a_1, a_{2k_i}, a_{2k_i+1})\Big)$$

where the $\varepsilon_i \in \{0, 1\}$. Then the fact that w satisfies Ex means that v also satisfies Ex and (221) now becomes

$$v(P(a_4) \mid P(a_1) \wedge \neg P(a_2) \wedge P(a_3)) \geq v(P(a_4) \mid P(a_1) \wedge \neg P(a_2)),$$

which follows from PIR in $\mathcal{L}$.

Nevertheless some genuine new principles for Polyadic PIL have already emerged and in the forthcoming chapters we shall introduce some of them and consider their consequences. First however we look closer at Ex.

Chapter 25

POLYADIC CONSTANT EXCHANGEABILITY

In the unary case de Finetti's Theorem 9.1 provides an illuminating, and very useful, representation of the unary probability functions satisfying Ex. This raises the question of whether there is a generalization to the polyadic. In this chapter we shall give a relatively simple (provided one has some familiarity with Nonstandard Analysis) proof of a particularly elementary result along these lines due originally to Krauss, [69] (proved there for a language with a single binary relation symbol *and identity*). There are many other rather sophisticated results, see for example, [3], [52], [60], on 'exchangeable arrays' within Probability Theory, which take matters much further than we have time, or expertise, to take them in this monograph. This reluctance is furthermore undiminished by the current lack of applications of such results in our context, in contrast to the unary case.

Rather than state a result as a theorem we shall talk our way through it, introducing the notation as we go along. The proof uses methods from Nonstandard Analysis, in particular from Loeb Measure Theory, [85].[93]

Let w be a probability function on SL satisfying Ex and take a nonstandard ω_1-saturated elementary extension, U^* say, of a sufficiently large portion U of the set theoretic universe containing w and anything else that we might need as we go along. As usual we shall use c^* to denote the image in U^* of $c \in U$ where this differs from c. Let $n \in \mathbb{N}$, let $\nu \in \mathbb{N}^*$ be nonstandard and let $\Theta(a_1, a_2, \dots, a_n)$ be a state description. Then $\Theta(a_1, a_2, \dots, a_n)$ is still a state description in U^* and just as in U, it is true in U^* that

$$w^*(\Theta(a_1, \dots, a_n)) = \sum_{\Phi(a_1,\dots,a_\nu) \models \Theta(a_1,\dots,a_n)} w^*(\Phi(a_1, \dots, a_\nu)). \tag{222}$$

Since w^* satisfies Ex in U^*, for $\sigma \in U^*$ a permutation of $\{1, 2, \dots, \nu\}$ and $\Phi(a_1, a_2, \dots, a_\nu)$ a state description in U^*,

$$w^*(\Phi(a_1, a_2, \dots, a_\nu)) = w^*(\Phi(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(\nu)})).$$

[93] In fact just pages 1–7 and 17–20 of [20] suffice for this application.

Working in U^* pick a representative $\Psi(a_1, \dots, a_v)$ from each of the classes

$$\{\Phi(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(v)}) \mid \sigma \text{ a permutation of } \{1, 2, \dots, v\}\}$$

and denote the corresponding class $\mathcal{H}_\Psi$. w^* is constant on $\mathcal{H}_\Psi$ so from (222),

$$w^*(\Theta(a_1, \dots, a_n)) = \sum_{\Psi(a_1,\dots,a_v)} \omega^\Psi(\Theta(a_1, \dots, a_n))\, w^*\big(\bigvee \mathcal{H}_\Psi\big) \tag{223}$$

where $\omega^\Psi(\Theta(a_1, a_2, \dots, a_n))$ is the proportion of $\Phi(a_1, \dots, a_v) \in \mathcal{H}_\Psi$ which logically imply $\Theta(a_1, \dots, a_n)$ (i.e. *are* $\Theta(a_1, \dots, a_n)$ when restricted just to $a_1, a_2, \dots, a_n$). Note that for any $\Theta(a_1, a_2, \dots, a_n)$, $\omega^\Psi(\Theta(a_1, a_2, \dots, a_n))$, as a function of Ψ, is internal (i.e. is in U^*), and that $\mu_0\{\Psi\} = w^*\left(\bigvee \mathcal{H}_\Psi\right)$ determines a finitely additive measure on the algebra of (internal) subsets of the set A of the chosen representatives Ψ. So we can rewrite (223) as

$$w^*(\Theta(a_1, \dots, a_n)) = \int_A \omega^\Psi(\Theta(a_1, \dots, a_n))\, d\mu_0(\Psi). \tag{224}$$

There is another way to picture $\omega^\Psi(\Theta(a_1, a_2, \dots, a_n))$. Namely imagine randomly picking, in U^*, *without replacement* and according to the uniform distribution, $a_{h_1}, a_{h_2}, \dots, a_{h_n}$ from $\{a_1, \dots, a_v\}$. Then $\omega^\Psi(\Theta(a_1, a_2, \dots, a_n))$ is the probability that $\Psi(a_1, \dots, a_v)$ restricted to $a_{h_1}, \dots, a_{h_n}$ is $\Theta(a_{h_1}, a_{h_2}, \dots, a_{h_n})$, equivalently that

$$\Psi(a_1, \dots, a_v) \vDash \Theta(a_{h_1}, a_{h_2}, \dots, a_{h_n}).$$

Viewed in this way we see that as a function of standard state descriptions $\Theta(a_1, \dots, a_n)$ (so $n \in \mathbb{N}$), ω^Ψ satisfies (34) and extends to satisfy (P1), (P2) and Ex on *QFSL*, *except* that it takes values in $[0, 1]^*$ rather than just $[0, 1]$.

By well known results from Loeb Integration Theory (see [85] or [20, p17-20]) we can take standard parts of (224), denoted by $\circ$, to obtain

$$w(\Theta(a_1, \dots, a_n)) = \int_A {}^\circ\omega^\Psi(\Theta(a_1, \dots, a_n))\, d\mu(\Psi) \tag{225}$$

where μ is a countably additive measure on a σ-algebra of subsets of the set A of $\Psi(a_1, \dots, a_v)$. The functions ${}^\circ\omega^\Psi$ now satisfy (P1), (P2) and Ex with standard values so extend uniquely by Theorem 7.1 to probability functions satisfying Ex on SL. Observe that ${}^\circ\omega^\Psi(\Theta(a_1, \dots, a_n))$ would be the same if the picking of the $a_{h_1}, a_{h_2}, \dots, a_{h_n}$ in the definition of ω^Ψ had been *with replacement* since the difference in the probability of picking $a_{h_1}, a_{h_2}, \dots, a_{h_n}$ with and without replacement is

$$\prod_{i=0}^{n-1} (v - i)^{-1} - v^{-n},$$

if there are no repeats in the $a_{h_1}, a_{h_2}, \dots, a_{h_n}$ (and hence the difference is of order $\nu^{-(n+1)}$) or ν^{-n} if there are repeats. There are ν^n n-tuples $a_{h_1}, a_{h_2}, \dots, a_{h_n}$ altogether and less than $\binom{n}{2}\nu^{n-1}$ of them are with repeats so this could only produce an infinitesimal change in $\omega^\Psi(\Theta(a_1, a_2, \dots, a_n))$, and hence no change at all once we take standard parts.

Apart from Ex the ${}^\circ\omega^\Psi$ satisfy the polyadic version of the *Constant Irrelevance Principle, IP* given on page 52, namely that if $\theta, \phi \in QFSL$ have no constants in common then

$$ {}^\circ\omega^\Psi(\theta \wedge \phi) = {}^\circ\omega^\Psi(\theta) \cdot {}^\circ\omega^\Psi(\phi). $$

To see this it is enough to show, as in the proof of Proposition 8.1, that for $n, m \in \mathbb{N}$ and state descriptions $\Theta(a_1, a_2, \dots, a_n)$, $\Phi(a_{n+1}, a_{n+2}, \dots, a_m)$

$$ {}^\circ\omega^\Psi(\Theta(a_1, a_2, \dots, a_n) \wedge \Phi(a_{n+1}, a_{n+2}, \dots, a_m)) = \\ {}^\circ\omega^\Psi(\Theta(a_1, a_2, \dots, a_n)) \cdot {}^\circ\omega^\Psi(\Phi(a_{n+1}, a_{n+2}, \dots, a_m)). $$

But this clearly holds by the above observation since if the picking of the $a_{h_1}, a_{h_2}, \dots, a_{h_n}$ in the definition of ω^Ψ had been *with replacement* all these choices would have been independent of each other and hence the sum of $\omega^\Psi(\Delta(a_1, \dots, a_m))$ for the (finitely many) $\Delta(a_1, \dots, a_m)$ extending $\Theta(a_1, \dots, a_n) \wedge \Psi(a_{n+1} \dots, a_m)$ would be equal to the product

$$ \omega^\Psi(\Theta(a_1, a_2, \dots, a_n)) \cdot \omega^\Psi(\Phi(a_{n+1}, a_{n+2}, \dots, a_m)). $$

In summary then we have arrived at a result due to Krauss, [69] (by an entirely different proof):

Theorem 25.1. *If the probability function w on SL satisfies Ex then w can be represented in the form*

$$ w = \int_A {}^\circ\omega^\Psi \, d\mu(\Psi) \tag{226} $$

for some countably additive measure μ on an algebra of subsets of A and probability functions ${}^\circ\omega^\Psi$ on SL satisfying IP (and Ex).

Conversely if w is of this form then it satisfies Ex.

To further illuminate this theorem we note (in line with Krauss' results) that these ${}^\circ\omega^\Psi$ are precisely the probability functions satisfying IP (and Ex).

Proposition 25.2. *If the probability function w on SL satisfies IP (and Ex) then $w = {}^\circ\omega^\Psi$ for some Ψ (within U^*), and conversely.*

Proof. The method of proof is analogous to that used for Theorem 20.6. Suppose that w is a probability function on the sentences of the language $L = \{R_1, R_2, \dots, R_q\}$ which satisfies IP (and Ex). By Theorem 25.1 w can be represented in the form (226). Let $\theta = \theta(a_1, \dots, a_n) \in$

QFSL and let $\theta' = \theta(a_{n+1}, \dots, a_{2n})$. Then by IP,

$$\begin{aligned}
0 &= 2(w'(\theta \wedge \theta') - w(\theta) \cdot w(\theta')) \\
&= \int_A {}^{\circ}\omega^{\Psi}(\theta \wedge \theta')\, d\mu(\Psi) + \int_A {}^{\circ}\omega^{\Lambda}(\theta \wedge \theta')\, d\mu(\Lambda) \\
&\qquad\qquad - 2\Big(\int_A {}^{\circ}\omega^{\Psi}(\theta)\, d\mu(\Psi)\Big) \cdot \Big(\int_A {}^{\circ}\omega^{\Lambda}(\theta')\, d\mu(\Lambda)\Big) \\
&= \int_A {}^{\circ}\omega^{\Psi}(\theta) \cdot {}^{\circ}\omega^{\Psi}(\theta')\, d\mu(\Psi) + \int_A {}^{\circ}\omega^{\Lambda}(\theta) \cdot {}^{\circ}\omega^{\Lambda}(\theta')\, d\mu(\Lambda) \\
&\qquad\qquad - 2\Big(\int_A {}^{\circ}\omega^{\Psi}(\theta)\, d\mu(\Psi)\Big) \cdot \Big(\int_A {}^{\circ}\omega^{\Lambda}(\theta')\, d\mu(\Lambda)\Big) \\
&= \int_A \int_A ({}^{\circ}\omega^{\Psi}(\theta) - {}^{\circ}\omega^{\Lambda}(\theta))^2\, d\mu(\Psi) d\mu(\Lambda)
\end{aligned}$$

since by Ex, ${}^{\circ}\omega^{\Psi}(\theta) = {}^{\circ}\omega^{\Psi}(\theta')$ etc..

Using the countable additivity of μ we can see that there must be a subset C of A with μ measure 1 such that ${}^{\circ}\omega^{\Psi}(\theta)$ is constant on C for *each* $\theta \in QFSL$. Picking a particular $\Lambda \in C$ then we have that

$$w(\theta) = \int_A {}^{\circ}\omega^{\Psi}(\theta)\, d\mu(\Psi) = {}^{\circ}\omega^{\Lambda}(\theta)$$

for $\theta \in QFSL$, so $w = {}^{\circ}\omega^{\Lambda}$. ⊣

Note that using Proposition 8.1 in the case of unary L gives de Finetti's Representation Theorem, at least after taking a suitable coarsening of the measure μ under the map

$$\Psi = \bigwedge_{i=1}^{\nu} \alpha_{h_i}(a_i) \mapsto \langle {}^{\circ}(\nu_1/\nu), {}^{\circ}(\nu_2/\nu), \dots, {}^{\circ}(\nu_{2^q}/\nu)\rangle$$

where (in U^*) $\nu_j = |\{i \mid h_i = j\}|$.

Recall by Corollary 6.2 that in the presence of Ex having IP for quantifier free sentences also gives it for all sentences. Thus the ${}^{\circ}\omega^{\Psi}$ actually have this ostensibly stronger property.

The use of Loeb Measure Theory to derive representation theorems, such as the one above, seems to us particularly straightforward and illuminating and it will be employed again shortly to good effect.

We end this chapter with an application which was promised on page 169 in the early Unary Part 2. With the notation from that page let

$$\theta = \theta(\vec{a}, R_1, R_2, \vec{R}), \quad \theta' = \theta(\vec{a}, R_1, R_4, \vec{R}), \quad \theta'' = \theta(\vec{a}, R_3, R_4, \vec{R}),$$

where $\vec{R} = R_5, R_6, \dots, R_{5+k}$. We shall show that assuming w satisfies ULi,

$$w(\theta \mid \theta') \geq w(\theta \mid \theta''). \tag{227}$$

To this end let $\mathcal{L}$ be the language with a single binary relation symbol R and define a probability function v on $S\mathcal{L}$ by

$$v\Big(\bigwedge_{i,j=1}^{n} R^{\varepsilon_{i,j}}(a_i, a_j)\Big) = w\Big(\bigwedge_{i,j=1}^{n} \theta^{\varepsilon_{i,j}}(\vec{a}, R_i, R_j, R_{n+1}, R_{n+2}, \dots, R_{n+k})\Big),$$

where, as in the proof of Theorem 22.4, we are taking w to be extended to these larger unary languages on the right hand side. With this definition v satisfies Ex and (227) becomes

$$v(R(a_1, a_2) \mid R(a_1, a_4)) \geq v(R(a_1, a_2) \mid R(a_3, a_4)). \tag{228}$$

In the notation above (with v in place of w) $v^{\Psi}(R(a_1, a_2) \wedge R(a_1, a_4))$ is the probability that

$$\Psi(a_1, \dots, a_\nu) \vDash R(a_{h_1}, a_{h_2}) \wedge R(a_{h_1}, a_{h_4})$$

for a random choice of distinct $a_{h_1}, a_{h_2}, a_{h_4}$ from $\{1, 2, \dots, \nu\}$. In other words, if

$$e_\tau = |\{\delta \mid \Psi(a_1, \dots, a_\nu) \vDash R(a_\tau, a_\delta)\}|$$

then in U^*,

$$v^{\Psi}(R(a_1, a_2) \wedge R(a_1, a_4)) = \sum_{\tau=1}^{\nu} e_\tau^2/\nu^3 + O(\nu^{-1}),$$

since allowing also the choice of possibly non-distinct $a_{h_1}, a_{h_2}, a_{h_4}$ only adds a factor of order ν^{-1}.

Similarly

$$v^{\Psi}(R(a_1, a_2) \wedge R(a_3, a_4)) = \sum_{\tau=1}^{\nu}\sum_{\delta=1}^{\nu} e_\tau e_\delta/\nu^4 + O(\nu^{-1}).$$

By the Cauchy-Schwarz Inequality

$$\sum_{\tau=1}^{\nu} e_\tau^2/\nu^3 = \Big(\sum_{\tau=1}^{\nu}(e_\tau/\nu)^2\Big)\Big(\sum_{\tau=1}^{\nu}(1/\nu)^2\Big) \geq$$
$$\Big(\sum_{\tau=1}^{\nu}(e_\tau/\nu)(1/\nu)\Big)^2 = \sum_{\tau=1}^{\nu}\sum_{\delta=1}^{\nu} e_\tau e_\delta/\nu^4$$

which in turn gives in the above notation that

$$\int_A {}^{\circ}v^{\Psi}(R(a_1, a_2) \wedge R(a_1, a_4))\, d\mu(\Psi) \geq$$
$$\int_A {}^{\circ}v^{\Psi}(R(a_1, a_2) \wedge R(a_3, a_4))\, d\mu(\Psi)$$

and the required conclusion (228) follows.

Clearly this result can be generalized to more relation symbols and include also changes in constants, we leave the reader to fill in the details.

Chapter 26

POLYADIC REGULARITY

In the unary case we have seen that whilst we can find probability functions w which are *Super Regular* in the sense that they simultaneously satisfy $w(\theta) > 0$ for all consistent $\theta \in SL$ such functions (currently) seem rather ad hoc. One would not therefore expect anything better in the more general polyadic case. Indeed even the method of construction of such functions given earlier depended on the special representation (59) we have for sentences of a unary language so no longer works in the polyadic. Nevertheless it is easy to construct such functions by an even more ad hoc method: Enumerate as θ_n, $n \in \mathbb{N}$, the consistent sentences of L and for each pick $M_n \in \mathcal{T}L$ such that $M_n \vDash \theta_n$. Now define the required super regular w by

$$w(\theta) = \sum_{n=0}^{\infty} 2^{-(n+1)} V_{M_n}(\theta)$$

where V_{M_n} is given by (1).

Defined in this way w may not satisfy any of the basic symmetry principles, Px, SN, Ex. The first of these can be readily arranged. For σ a permutation of the relation symbols of L which preserves arity and $\theta \in SL$ let $\sigma\theta$ be the sentence formed by replacing each predicate symbol R in θ by $\sigma(R)$. Then it is straightforward to see that if $\theta \in SL$ is a tautology then so is $\sigma\theta$ and hence that

$$v(\theta) = K^{-1} \sum_{\sigma} w(\sigma\theta),$$

where K is the cardinality of the set of such σ, is a probability function on SL which satisfies Px.

We can similarly arrange that SN also holds. This ‘averaging’ method however will not immediately work to ensure Ex because there are now infinitely many (indeed uncountably many) permutations of the a_i. Instead we need a slightly more sophisticated, and seemingly potentially useful, averaging process, employed already for example by Gaifman in a similar context in [30].

Let H be the set of maps f from $\mathbb{N}^+$ to $\mathbb{N}^+$ and for (distinct) $i_1, i_2, \dots, i_n \in \mathbb{N}^+$ and $\sigma : \{i_1, i_2, \dots, i_n\} \to \mathbb{N}^+$ let

$$[\sigma] = \{f \in H \mid f \restriction \{i_1, i_2, \dots, i_n\} = \sigma\}.$$

Let μ_0 be the (normalized) measure on $\mathbb{N}^+$ given by $\mu_0\{j\} = 2^{-j}$ and let μ be the standard product measure extension of μ_0 to H. So for σ as above

$$\mu[\sigma] = \prod_{j=1}^{n} 2^{-\sigma(i_j)}$$

and μ is a countably additive (normalized) measure on the σ-algebra, i.e. Borel sets, generated from these $[\sigma]$.

Given a probability function w on SL and $\theta = \theta(a_{i_1}, a_{i_2}, \dots, a_{i_n}) \in SL$ let

$$v(\theta) = \sum_{\sigma:\{i_1,i_2,\dots,i_n\}\to\mathbb{N}^+} \mu[\sigma]\, w(\theta(a_{\sigma(i_1)}, a_{\sigma(i_2)}, \dots, a_{\sigma(i_n)})). \tag{229}$$

Notice that this is unambiguous since even if we write θ here as $\theta(a_{i_1}, a_{i_2}, \dots, a_{i_n}, a_{i_{n+1}})$ we will still obtain the same value for $v(\theta)$ because the additional factors $2^{-\sigma(i_{n+1})}$ will sum out to 1.

From this observation and the fact that if $\theta(a_{i_1}, a_{i_2}, \dots, a_{i_n}) \in SL$ is a tautology then so is $\theta(a_{\sigma(i_1)}, a_{\sigma(i_2)}, \dots, a_{\sigma(i_n)})$ it follows that v satisfies (P1) and (P2). Concerning (P3), let $\exists x\, \psi(x, a_{i_1}, \dots, a_{i_n}) \in SL$ and $\varepsilon > 0$. From (229) pick m such that

$$\sum_{\sigma:\{i_1,i_2,\dots,i_n\}\to\{1,2,\dots,m\}} \mu[\sigma] > 1 - \varepsilon, \tag{230}$$

so

$$\sum_{\sigma:\{i_1,i_2,\dots,i_n\}\to\{1,2,\dots,m\}} \mu[\sigma]\, w(\exists x\, \psi(x, a_{\sigma(i_1)}, a_{\sigma(i_2)}, \dots, a_{\sigma(i_n)})) \tag{231}$$

is within ε of $v(\exists x\, \psi(x, a_{i_1}, \dots, a_{i_n}))$. Now pick q such that for each of the m^n maps σ from $\{i_1, i_2, \dots, i_n\}$ to $\{1, 2, \dots, m\}$,

$$w\Big(\bigvee_{j=1}^{q} \psi(a_j, a_{\sigma(i_1)}, \dots, a_{\sigma(i_n)})\Big) + \varepsilon m^{-n} \geq w(\exists x\, \psi(x, a_{\sigma(i_1)}, \dots, a_{\sigma(i_n)})) \tag{232}$$

and in turn pick $r \geq q, i_1, \dots, i_n$ such that for each of the maps $\sigma : \{i_1, i_2, \dots, i_n\} \to \{1, 2, \dots, m\}$,

$$\sum_{\tau \in A_\sigma} \mu[\tau] \geq \mu[\sigma] - \varepsilon m^{-n}, \tag{233}$$

where A_σ is the set of $\tau : \{1, 2, \dots, r\} \to \mathbb{N}^+$ such that $\tau \restriction \{i_1, \dots, i_n\} = \sigma$ and $\{1, 2, \dots, q\} \subseteq \{\tau(1), \tau(2), \dots, \tau(r)\}$.

Then

$$
\begin{aligned}
v&\Big(\bigvee_{j=1}^{r} \psi(a_j, a_{i_1}, \dots, a_{i_n})\Big)\\
&= \sum_{\tau:\{1,\dots,r\}\to\mathbb{N}^+} \mu[\tau]\, w\Big(\bigvee_{j=1}^{r} \psi(a_{\tau(j)}, a_{\tau(i_1)}, \dots, a_{\tau(i_n)})\Big)\\
&\geq \sum_{\sigma:\{i_1,\dots,i_n\}\to\{1,\dots,m\}} \sum_{\tau\in A_\sigma} \mu[\tau]\, w\Big(\bigvee_{j=1}^{r} \psi(a_{\tau(j)}, a_{\tau(i_1)}, \dots, a_{\tau(i_n)})\Big)\\
&\geq \sum_{\sigma:\{i_1,\dots,i_n\}\to\{1,\dots,m\}} \sum_{\tau\in A_\sigma} \mu[\tau]\,(w(\exists x\, \psi(x, a_{\tau(i_1)}, \dots, a_{\tau(i_n)})) - \varepsilon m^{-n}),\\
&\hspace{20em}\text{by (232)},\\
&\geq \sum_{\sigma:\{i_1,\dots,i_n\}\to\{1,\dots,m\}} (\mu[\sigma] - \varepsilon m^{-n})\,(w(\exists x\, \psi(x, a_{\sigma(i_1)}, \dots, a_{\sigma(i_n)})) - \varepsilon m^{-n}),\\
&\hspace{20em}\text{by (233)},\\
&\geq \sum_{\sigma:\{i_1,\dots,i_n\}\to\{1,\dots,m\}} \mu[\sigma]\, w(\exists x\, \psi(x, a_{\sigma(i_1)}, \dots, a_{\sigma(i_n)})) - 2\varepsilon\\
&\geq \sum_{\sigma:\{i_1,\dots,i_n\}\to\mathbb{N}^+} \mu[\sigma]\, w(\exists x\, \psi(x, a_{\sigma(i_1)}, \dots, a_{\sigma(i_n)})) - 3\varepsilon, \qquad \text{by (230)},\\
&\geq v(\exists x\, \psi(x, a_{i_1}, \dots, a_{i_n})) - 3\varepsilon, \qquad \text{by (231)},
\end{aligned}
$$

and (P3) follows by using (5) for the reverse inequality.

To show that v satisfies Ex let $\nu : \{1, \dots, n\} \to \mathbb{N}^+$ be injective. Then

$$
\begin{aligned}
v&(\theta(a_1, \dots, a_n))\\
&= \sum_{\sigma:\{1,\dots n\}\to\mathbb{N}^+} \mu[\sigma]\, w(\theta(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}))\\
&= \sum_{\sigma\nu^{-1}\nu:\{1,\dots n\}\to\mathbb{N}^+} \mu[\sigma\nu^{-1}\nu]\, w(\theta(a_{\sigma\nu^{-1}\nu(1)}, a_{\sigma\nu^{-1}\nu(2)}, \dots, a_{\sigma\nu^{-1}\nu(n)}))\\
&= \sum_{\sigma\nu^{-1}:\{\nu(1),\dots \nu(n)\}\to\mathbb{N}^+} \mu[\sigma\nu^{-1}]\, w(\theta(a_{\sigma\nu^{-1}(\nu(1))}, a_{\sigma\nu^{-1}(\nu(2))}, \dots, a_{\sigma\nu^{-1}(\nu(n))})),\\
&\hspace{16em}\text{since } \mu[\sigma\nu^{-1}\nu] = \mu[\sigma\nu^{-1}],\\
&= v(\theta(a_{\nu(1)}, a_{\nu(2)}, \dots, a_{\nu(n)}))
\end{aligned}
$$

since the $\sigma\nu^{-1}$ here are just the $\tau : \{\nu(1), \nu(2), \dots, \nu(n)\} \to \mathbb{N}^+$.

Finally, v inherits Px, SN and SReg if these hold for w, this last from (229) when σ is the identity.

Again however, as in the unary case, we know of no 'justified' probability function which satisfies SReg, just such ad hoc examples as above. Indeed, just as with JSP in the unary, we know that the forthcoming polyadic principles of Spectrum Exchangeability, Sx, and Invariance, INV, are actually inconsistent with SReg, see respectively [53] and page 314, or [125].

Henceforth we will continue to assume, unless otherwise stated, that all our probability functions satisfy Ex. Our next step is to introduce what to date has proved to be one of the key notions in PIL.

Chapter 27

SPECTRUM EXCHANGEABILITY

By way of motivation consider the alternative formulation of (unary) Atom Exchangeability, Ax, as given on page 87:

$w\Big(\bigwedge_{i=1}^{m} \alpha_{h_i}(b_i)\Big)$ *depends only on the multiset* $\{m_1, m_2, \dots, m_{2^q}\}$ *where* $m_j = |\{i \mid h_i = j\}|$.

For brevity let Θ denote this state description. Then since the atom that a b_i satisfies exactly determines its properties, b_i, b_j will be *indistinguishable* with respect to Θ, denoted $b_i \sim_\Theta b_j$, just if they satisfy the same atom. Equivalently

$$b_i \sim_\Theta b_j \iff \Theta(b_1, b_2, \dots, b_m) \wedge b_i = b_j \text{ is consistent,} \tag{234}$$

that is consistent when we add equality to the language L (and the axioms of equality to $\models$).

Clearly from (234) $\sim_\Theta$ is an equivalence relation on the set $\{b_1, b_2, \dots, b_m\}$ and the m_i are just the sizes of its equivalence classes. If we now define the *Spectrum* of $\Theta(b_1, b_2, \dots, b_m)$, $\mathcal{S}(\Theta)$, to be the multiset[94] of sizes of the *(non-empty) equivalence classes* with respect to $\sim_\Theta$ we can reformulate Ax as:

For state descriptions $\Theta(b_1, b_2, \dots, b_m), \Phi(b_1, b_2, \dots, b_m)$, *if* $\mathcal{S}(\Theta) = \mathcal{S}(\Phi)$ *then* $w(\Theta) = w(\Phi)$.

The value of expressing Ax in this form is that it immediately generalizes to polyadic L. For a state description $\Theta(b_1, b_2, \dots, b_m)$ of L the equivalence relation $b_i \sim_\Theta b_j$ on $\{b_1, b_2, \dots, b_m\}$ is again defined by (234), equivalently by the requirement that for any r-ary relation symbol from L and not necessarily distinct $b_{k_1}, \dots, b_{k_u}, b_{k_{u+2}}, \dots, b_{k_r}$ from $\{b_1, b_2, \dots, b_m\}$,

$$\Theta(b_1, b_2, \dots, b_m) \models R(b_{k_1}, \dots, b_{k_u}, b_i, b_{k_{u+2}}, \dots, b_{k_r}) \leftrightarrow R(b_{k_1}, \dots, b_{k_u}, b_j, b_{k_{u+2}}, \dots, b_{k_r}).$$

[94] In some earlier accounts the spectrum was the corresponding vector $\langle m_{k_1}, m_{k_2}, \dots, m_{k_m}\rangle$ where the m_{k_i} are the m_i in non-increasing order.

Let $\mathcal{E}(\Theta)$ denote the set of equivalence classes of Θ.[95] Again we can define the *Spectrum of* Θ, $\mathcal{S}(\Theta)$, as the multiset of sizes of the equivalence classes in $\mathcal{E}(\Theta)$ and the corresponding generalization of Ax to polyadic L now becomes:

The Spectrum Exchangeability Principle, Sx.
For state descriptions $\Theta(b_1, b_2, \dots, b_m)$, $\Phi(b_1, b_2, \dots, b_m) \in SL$, *if* $\mathcal{S}(\Theta) = \mathcal{S}(\Phi)$ *then* $w(\Theta) = w(\Phi)$.

To illustrate these notions suppose that L has just a single binary relation symbol R. Then the conjunction, $\Theta(a_1, a_2, a_3, a_4)$ say, of

$$\begin{array}{cccc}
R(a_1,a_1) & \neg R(a_1,a_2) & R(a_1,a_3) & R(a_1,a_4) \\
R(a_2,a_1) & \neg R(a_2,a_2) & R(a_2,a_3) & \neg R(a_2,a_4) \\
R(a_3,a_1) & \neg R(a_3,a_2) & R(a_3,a_3) & R(a_3,a_4) \\
R(a_4,a_1) & R(a_4,a_2) & R(a_4,a_3) & R(a_4,a_4)
\end{array}$$

is a state description, the equivalence classes of $\sim_\Theta$ are $\{a_1, a_3\}$, $\{a_2\}$, $\{a_4\}$, so

$$\mathcal{E}(\Theta) = \{\{a_1, a_3\}, \{a_2\}, \{a_4\}\},$$

and the spectrum $\mathcal{S}(\Theta)$ is $\{2, 1, 1\}$.

Similarly if the state description $\Phi(a_1, a_2, a_3, a_4)$ is the conjunction of

$$\begin{array}{cccc}
\neg R(a_1,a_1) & \neg R(a_1,a_2) & R(a_1,a_3) & R(a_1,a_4) \\
\neg R(a_2,a_1) & \neg R(a_2,a_2) & \neg R(a_2,a_3) & \neg R(a_2,a_4) \\
R(a_3,a_1) & R(a_3,a_2) & R(a_3,a_3) & R(a_3,a_4) \\
R(a_4,a_1) & R(a_4,a_2) & R(a_4,a_3) & R(a_4,a_4)
\end{array}$$

then the equivalence classes of $\sim_\Phi$ are $\{a_1\}$, $\{a_2\}$, $\{a_3, a_4\}$, the spectrum is again $\{2, 1, 1\}$ and Sx prescribes that $w(\Theta) = w(\Phi)$. (Notice that by our standing assumption Ex the particular instantiating constants here are not important.)

The spectrum $\{1, 1, 1, \dots, 1\}$ of state descriptions with t constants where all the constants are distinguishable will be of special importance and it will be denoted 1_t.

The polyadic probability functions c_0^L, c_∞^L introduced earlier both satisfy Sx. In the case of c_∞^L this is clear since it gives the same value to all state descriptions with the same number of constants. As for c_0^L, a moment's thought shows that it only gives non-zero probability to state descriptions $\Theta(a_1, \dots, a_n)$ for which $\mathcal{E}(\Theta) = \{\{a_1, \dots, a_n\}\}$, and these all get the same probability. In fact as we shall see later these two are (again) extreme members of a natural family of probability functions satisfying Sx.

[95]More precisely 'of $\sim_\Theta$', though we shall frequently adopt the less pedantic usage.

To date Sx has figured large in the study of PIL, not least because it sheds light on many of the obvious questions one might ask, as we shall see in future chapters. For the present however it will be useful to derive some simple properties of spectra.

First suppose that L is the disjoint union of two languages L_1, L_2. Let $\Theta(a_1, a_2, \dots, a_m)$ be a state description in L. Then

$$\Theta(a_1, a_2, \dots, a_m) = \Theta_1(a_1, a_2, \dots, a_m) \wedge \Theta_2(a_1, a_2, \dots, a_m)$$

for some state descriptions Θ_1, Θ_2 of L_1, L_2 respectively. Furthermore

$$a_i \sim_\Theta a_j \iff a_i \sim_{\Theta_1} a_j \text{ and } a_i \sim_{\Theta_2} a_j.$$

Thus we can see that if $S_1, S_2, \dots, S_r$ and $T_1, T_2, \dots, T_k$ are the equivalence classes of $\sim_{\Theta_1}, \sim_{\Theta_2}$ respectively then the equivalences classes of $\sim_\Theta$ are the non-empty sets amongst the $S_i \cap T_j$, $i = 1, 2, \dots, r$, $j = 1, 2, \dots, k$. In this way then we can sometimes reduce problems about spectra for large languages down to the more easily visualized case of a language with a single relation symbol.

As a simple consequence of the above we have the following obviously desirable property of probability functions satisfying Sx:

Proposition 27.1. *Let w be a probability function on SL satisfying Sx and let L' be a sublanguage of L. Then w restricted to L', $w \restriction L'$, also satisfies Sx.*

Proof. Recalling the convention on page 9 let L'' be the language $L - L'$ (clearly we may assume that L'' has some relation symbols). Let $\Theta(a_1, \dots, a_m)$, $\Phi(a_1, \dots, a_m)$ be state descriptions in L' with the same spectrum. Assume for the present that $\sim_\Theta$ and $\sim_\Phi$ have exactly the same equivalence classes, i.e. $\mathcal{E}(\Theta) = \mathcal{E}(\Phi)$.

Any state description for $a_1, \dots, a_m$ in L which extends Θ can be expressed as

$$\Theta(a_1, \dots, a_m) \wedge \Psi(a_1, \dots, a_m) \tag{235}$$

where Ψ is a state description in L'' and can be paired with the state description

$$\Phi(a_1, \dots, a_m) \wedge \Psi(a_1, \dots, a_m). \tag{236}$$

Furthermore, from the earlier discussion, since $\sim_\Theta$ and $\sim_\Phi$ have exactly the same equivalence classes (235) and (236) will also have the same equivalence classes and spectrum. Hence, since w satisfies Sx,

$$w(\Theta(a_1, \dots, a_m) \wedge \Psi(a_1, \dots, a_m)) = w(\Phi(a_1, \dots, a_m) \wedge \Psi(a_1, \dots, a_m)).$$

Summing this over all choices of $\Psi(a_1, \dots, a_m)$ gives as required that $w(\Theta) = w(\Phi)$.

Finally, if Θ and Φ have the same spectrum but $\mathcal{E}(\Theta) \neq \mathcal{E}(\Phi)$ then we can find a permutation $\sigma \in \mathsf{S}_m$ such that $\Theta(a_1, a_2, \dots, a_m)$ and

$\Phi(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(m)})$ do have the same equivalence classes and so, as above,

$$\begin{aligned} w(\Theta(a_1, a_2, \ldots, a_m)) &= w(\Phi(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(m)})) \\ &= w(\Phi(a_1, a_2, \ldots, a_m)) \end{aligned}$$

by Ex. (In what follows we shall use this 'trick' without further explanation.) $\dashv$

A second observation about spectra concerns their possible extensions to more constants. Suppose that $\Theta(a_1, a_2, \ldots, a_m), \Phi(a_1, a_2, \ldots, a_m, a_{m+1})$ are state descriptions and Φ extends Θ i.e. $\Phi \models \Theta$. Let the equivalence classes of $\sim_\Theta$, $\sim_\Phi$ be, respectively,

$$\begin{aligned} \mathcal{E}(\Theta) &= \{I_1, I_2, \ldots, I_r\}, \\ \mathcal{E}(\Phi) &= \{J_1, J_2, \ldots, J_k\}. \end{aligned}$$

Clearly if $i, j \leq m$ and $a_i \sim_\Phi a_j$ then, because $\Phi \models \Theta$, $a_i \sim_\Theta a_j$. So ignoring the new element a_{m+1} the equivalence classes for Φ are the result of splitting the classes for Θ. Whether or not there is such a proper splitting depends on the properties of a_{m+1}.

Without loss of generality suppose that $a_{m+1} \in J_1$. There are now 2 possible cases.

If a_{m+1} is not the sole element in J_1, say $a_{m+1} \neq a_n \in J_1$, then a_{m+1} is indistinguishable from a_n in $\Phi(a_1, \ldots, a_m, a_{m+1})$. Thus for any r-ary relation symbol R of L, $a_i \sim_\Theta a_j$ and not necessarily distinct

$$a_{k_1}, \ldots, a_{k_u}, a_{k_{u+2}}, \ldots, a_{k_r} \in \{a_1, a_2, \ldots, a_m, a_{m+1}\},$$

$$\begin{aligned} &\Phi \models R(a_{k_1}, \ldots, a_{k_u}, a_i, a_{k_{u+2}}, \ldots, a_{k_r}) \\ &\iff \Phi \models R(a_{p_1}, \ldots, a_{p_u}, a_i, a_{p_{u+2}}, \ldots, a_{p_r}), \\ &\qquad\qquad \text{where } a_{p_l} = a_{k_l} \text{ if } k_l \neq m+1 \text{ and } a_{p_l} = a_n \text{ if } k_l = m+1, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{since } a_n \sim_\Phi a_{m+1}, \\ &\iff \Theta \models R(a_{p_1}, \ldots, a_{p_u}, a_i, a_{p_{u+2}}, \ldots, a_{p_r}), \quad \text{since } \Phi \models \Theta, \\ &\iff \Theta \models R(a_{p_1}, \ldots, a_{p_u}, a_j, a_{p_{u+2}}, \ldots, a_{p_r}), \quad \text{since } a_i \sim_\Theta a_j, \\ &\iff \Phi \models R(a_{p_1}, \ldots, a_{p_u}, a_j, a_{p_{u+2}}, \ldots, a_{p_r}), \quad \text{since } \Phi \models \Theta, \\ &\iff \Phi \models R(a_{k_1}, \ldots, a_{k_u}, a_j, a_{k_{u+2}}, \ldots, a_{k_r}), \quad \text{since } a_n \sim_\Phi a_{m+1}. \end{aligned}$$

From this we see that the equivalence classes for $\sim_\Phi$ are exactly the same as those for $\sim_\Theta$ except that a_{m+1} has joined that class I_t which already contains a_n. So in this case $\sim_\Phi, \sim_\Theta$ agree on $\{a_1, a_2, \ldots, a_m\}$.

The other possibility is that a_{m+1} is the sole element of its equivalence class J_1 according to $\sim_\Phi$. In that case the equivalence classes of $\sim_\Theta$ may or may not split in the passage to $\sim_\Phi$.[96]

Armed with the above observations we can now prove a lemma which will have important consequences in later chapters. This lemma, by a somewhat different proof, was first proved for binary languages in [94], [96], and then extended to general polyadic languages in Landes [74].

LEMMA 27.2. *Suppose that the state descriptions* $\Theta(a_1, \dots, a_m)$, $\Phi(a_1, \dots, a_m)$ *of* L *have the same spectrum and let* $\{n_1, n_2, \dots, n_s\}$ *be a multiset with* $n_1, \dots, n_s \in \mathbb{N}^+$ *and* $n = \sum_{i=1}^s n_i > m$. *Then the number of state descriptions for* $a_1, a_2, \dots, a_n$ *with spectrum* $\{n_1, n_2, \dots, n_s\}$ *extending* Θ *is the same as the number with this spectrum extending* Φ.

PROOF. The result is straightforward for L purely unary so assume otherwise. Clearly it is enough to show this result for $n = m + 1$. By permuting the constants in Φ we may clearly assume that

$$\mathcal{E}(\Theta) = \mathcal{E}(\Phi) = \{I_1, I_2, \dots, I_k\} \text{ say.}$$

For each r-ary relation symbol R of L with $r > 1$ (by assumption there are some) and non-empty strict subset $S = \{i_1, i_2, \dots, i_p\}$ of $\{1, 2, \dots, r\}$, with $i_1 < i_2 < \dots < i_p$, let R_S be the formula $R(z_1, z_2, \dots, z_r)$ of L where $z_j = x_h$ if $j = i_h$ and for the remaining $j \notin S$, $z_j = a_{m+1}$. So for example if R is 4-ary and $S = \{1, 3\} \subset \{1, 2, 3, 4\}$, $R_S = R(x_1, a_{m+1}, x_2, a_{m+1})$.

For each such R and S introduce a new relation symbol $\overline{R}_S$ with arity $|S|$ and let $\overline{L}$ be the language with these new relation symbols. Then any state description for $a_1, a_2, \dots, a_{m+1}$ extending Θ can be written in the form

$$\Theta(a_1, \dots, a_m) \wedge \bigwedge_{R_S} \bigwedge_{i_1, \dots, i_{|S|}=1}^{m} \pm R_S(a_{i_1}, \dots, a_{i_{|S|}}) \wedge \bigwedge_R \pm R(a_{m+1}, a_{m+1}, \dots, a_{m+1}) \quad (237)$$

where in R_S we have suppressed mention of the a_{m+1}.

Dropping the last conjunct, $\bigwedge_R \pm R(a_{m+1}, a_{m+1}, \dots, a_{m+1})$, (which, notice, is only instrumental in determining the equivalence class of a_{m+1}) we see that there is a one to one correspondence with the state description

$$\Theta(a_1, a_2, \dots, a_m) \wedge \bigwedge_{R_S} \bigwedge_{i_1, \dots, i_{|S|}=1}^{m} \pm \overline{R}_S(a_{i_1}, \dots, a_{i_{|S|}}). \quad (238)$$

[96]In the case that all the relation symbols of L are unary or binary there is a fixed bound on the number of classes a single class of $\sim_\Theta$ can split into. Once L has a relation symbol of arity greater than 2 there is no such bound.

Since L and $\overline{L}$ are disjoint languages the spectrum of this state description is determined by the equivalence classes for $\Theta(a_1, a_2, \ldots, a_m)$ and the equivalence classes for

$$\bigwedge_{R_S} \bigwedge_{i_1,\ldots,i_{|S|}=1}^{m} \pm\overline{R}_S(a_{i_1}, \ldots, a_{i_{|S|}}), \tag{239}$$

and so will be the same for

$$\Phi(a_1, a_2, \ldots, a_m) \wedge \bigwedge_{R_S} \bigwedge_{i_1,\ldots,i_{|S|}=1}^{m} \pm\overline{R}_S(a_{i_1}, \ldots, a_{i_{|S|}}). \tag{240}$$

Now consider the case where the equivalence classes for (238) are a splitting of those for Θ. In that case a_{m+1} must be in a class by itself in the spectrum for (237) so we can simply pair off (238) with the corresponding state description (240) with Φ in place of Θ to give the required equipollence.

Finally consider the case when the spectrum of (238) equals that of Θ (and similarly for (240) and Φ). In that case if $e_1, e_2, \ldots, e_k$ are chosen representatives from each of the classes $I_1, I_2, \ldots, I_k$ then the only free choices in (237) to achieve this will be of

$$\bigwedge_{R_S} \bigwedge_{i_1,\ldots,i_{|S|}=1}^{k} \pm R_S(e_{i_1}, \ldots, e_{i_{|S|}}) \wedge \bigwedge_{R} \pm R(a_{m+1}, a_{m+1}, \ldots, a_{m+1}),$$

independently of Θ and Φ. For both Θ and Φ exactly k of these choices (making a_{m+1} indistinguishable from one of $e_1, \ldots, e_k$) will result in a_{m+1} joining one of the $I_1, I_2, \ldots, I_k$, so the number for which a_{m+1} forms a class of its own will have to be the same for both Θ and Φ. $\dashv$

Looking back over the above proof it becomes evident that it actually shows that there is a stronger connection between the extensions of $\Theta(a_1, \ldots, a_m)$ and $\Phi(a_1, \ldots, a_m)$ than simply having the same number for the same spectrum. Namely, when $\mathcal{E}(\Theta) = \mathcal{E}(\Phi)$ then they have the same number of extensions to state descriptions $\Psi(a_1, \ldots, a_n)$ with a particular set of equivalence classes. Precisely:

PROPOSITION 27.3. *If $\Theta(a_1, \ldots, a_m)$, $\Phi(a_1, \ldots, a_m)$ are state descriptions with $\mathcal{E}(\Theta) = \mathcal{E}(\Phi)$ then for any partition E of $\{a_1, \ldots, a_n\}$ the number of state descriptions $\Psi(a_1, \ldots, a_n)$ extending $\Theta(a_1, \ldots, a_m)$ with $\mathcal{E}(\Psi) = E$ is the same as the number extending $\Phi(a_1, \ldots, a_m)$ with $\mathcal{E}(\Psi) = E$.*

Chapter 28

CONFORMITY

As already remarked one's lack of familiarity with polyadic relations, as opposed to purely unary ones, seems to make the explication of putatively rational principles involving them all the more difficult. However there is one rather evident such principle which arises simply because we have more variables: Namely that if we permute the order of the variables in a relation then this should not alter our assigned probabilities. Precisely:

The Variable Exchangeability Principle, Vx.

Let R be an r-ary relation symbol of L, $\sigma \in \mathsf{S}_r$ and for $\theta \in SL$ let θ' be the result of replacing each $R(t_1, t_2, \dots, t_r)$ appearing in θ by $R(t_{\sigma(1)}, t_{\sigma(2)}, \dots, t_{\sigma(r)})$. Then $w(\theta) = w(\theta')$.

So, as an example of this principle, it gives that for binary R we should have

$$w(R(a_1, a_1) \wedge \neg R(a_2, a_2) \wedge R(a_1, a_2)) = \\ w(R(a_1, a_1) \wedge \neg R(a_2, a_2) \wedge R(a_2, a_1)).$$

Vx does not follow from Ex. To see this let L be the language with a single binary relation symbol R and let L_1 the language with a single unary relation symbol P. For $\phi \in SL$ let ϕ^* be the result of replacing each occurrence of $R(t_1, t_2)$ in ϕ, where t_1, t_2 are terms of L, by $P(t_1)$ and define $w : SL \to [0, 1]$ by $w(\phi) = c_\infty^{L_1}(\phi^*)$. Then it is a straightforward exercise to check that w is a probability function on SL satisfying Ex. However w does not satisfy Vx since, for example,

$$w(R(a_1, a_2) \wedge \neg R(a_1, a_3)) = c_\infty^{L_1}(P(a_1) \wedge \neg P(a_1)) = 0,$$
$$w(R(a_2, a_1) \wedge \neg R(a_3, a_1)) = c_\infty^{L_1}(P(a_2) \wedge \neg P(a_3)) = 1/4.$$

Clearly in the above definition of Vx we could restrict the θ to be simply a state description and hence, since θ' will then have the same spectrum as θ, we have:

Proposition 28.1. *Sx implies Variable Exchangeability, Vx.*

The obvious argument in favour of Vx as a rational principle is that there is no reason why $R(a_1, a_2, \dots, a_r)$ should be treated in any way differently from $R(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(r)})$. But this argument seems to equally justify a much more general principle where we allow constant symbols and repeated variables in the arguments of R. To motivate this principle suppose that L has just a single 4-ary predicate symbol R and let L_P be the language with a single 3-ary predicate symbol P. For $\theta \in SL_P$ let θ_1 be the result of replacing each occurrence of $P(t_1, t_2, t_3)$ (where the t_i are terms of L) in θ by $R(t_1, t_2, t_3, a_1)$. Similarly let θ_2 be the result of replacing each occurrence of $P(t_1, t_2, t_3)$ in θ by $R(t_1, t_2, t_3, t_3)$. So now θ_1, θ_2 are sentences of L.

Then for an arguably rational probability function w can we justify saying that $w(\theta_1)$ should be greater than $w(\theta_2)$, or vice-versa? We would claim that we cannot, that there is no rational reason for supposing that $w(\theta_1)$ should exceed $w(\theta_2)$ nor any reason for supposing that $w(\theta_2)$ should exceed $w(\theta_1)$. In turn then the adoption of any probability w which did give them different values would seem to be making an unjustified, even unjustifiable, assumption. The only way then that a choice of probability function w would be immune to this criticism is if *it gave them both the same value*.

We now show in this special case that w satisfying Sx is enough to ensure such equality. We shall then go on to explain the general *Principle of Conformity*, which again follows from Sx and of which this is but a particular example.

PROPOSITION 28.2. *For $\theta, \theta_1, \theta_2$ as above and w satisfying Sx,*

$$w(\theta_1) = w(\theta_2).$$

PROOF. First notice that it is enough to take θ to be a state description,

$$\bigwedge_{i,j,k=1}^{m} P^{\varepsilon_{ijk}}(a_i, a_j, a_k)$$

where as usual the $\varepsilon_{ijk} \in \{0, 1\}$, $P^1 = P$ and $P^0 = \neg P$ etc.. In this case then

$$\theta_1 = \bigwedge_{i,j,k=1}^{m} R^{\varepsilon_{ijk}}(a_i, a_j, a_k, a_1),$$

$$\theta_2 = \bigwedge_{i,j,k=1}^{m} R^{\varepsilon_{ijk}}(a_i, a_j, a_k, a_k).$$

The θ_1, θ_2 are not themselves state descriptions but since w satisfies Sx it is enough to show that for any spectrum $\tilde{s}$ the number of state descriptions for $a_1, a_2, \dots, a_m$ extending θ_1 with spectrum $\tilde{s}$ is the same as the number extending θ_2 with spectrum $\tilde{s}$.

In fact we claim something slightly stronger than this, namely that for any partition E of $\{a_1, \dots, a_m\}$ the number of state descriptions

$\Psi(a_1, \dots, a_m)$ extending (equivalently consistent with) θ_1 with $\mathcal{E}(\Psi) = E$ is the same as the number extending θ_2 with $\mathcal{E}(\Psi) = E$. (Notice that if $a_i \not\sim_\theta a_j$ then for any $\Psi(a_1, \dots, a_m)$ extending θ_1 (or θ_2) $a_i \not\sim_\Psi a_j$ so $\mathcal{E}(\Psi)$ must be a *refinement* of $\mathcal{E}(\theta)$, denoted $\mathcal{E}(\Psi) \trianglerighteq \mathcal{E}(\theta)$ and meaning that every class in $\mathcal{E}(\Psi)$ is a subset of a class in $\mathcal{E}(\theta)$.)

The proof of the claim proceeds by induction on $|E|$. Assume that the result is true for partitions of $\{a_1, a_2, \dots, a_m\}$ containing fewer than $|E|$ classes. Clearly we can assume that $E \trianglerighteq \mathcal{E}(\theta)$, so $|E| \geq |\mathcal{E}(\theta)|$, otherwise the number of Ψ will be zero for both θ_1 and θ_2.

If $|E| = |\mathcal{E}(\theta)|$ then perforce $E = \mathcal{E}(\theta)$, since $E \trianglerighteq \mathcal{E}(\theta)$. Let $e_1, e_2, \dots, e_{|E|}$ be representatives of the $|E|$ classes in E, with $e_1 = a_1$. Then to form $\Psi(a_1, a_2, \dots, a_m)$ extending θ_1 with $\mathcal{E}(\Psi) = E$ we have a free choice of $\pm R(e_{g_1}, e_{g_2}, e_{g_3}, e_{g_4})$ when $e_{g_4} \neq a_1$, so $2^{|E|^3(|E|-1)}$ choices altogether. After that all other choices are determined by θ_1 and the fact that $\mathcal{E}(\Psi) = E$. Similarly for $\Psi(a_1, a_2, \dots, a_m)$ extending θ_2 with $\mathcal{E}(\Psi) = E$ we have a free choice of $\pm R(e_{g_1}, e_{g_2}, e_{g_3}, e_{g_4})$ when $e_{g_3} \neq e_{g_4}$, so again $2^{|E|^3(|E|-1)}$ choices altogether and the result is proven in this base case.

For the induction step let $E \triangleright \mathcal{E}(\theta)$, i.e. E is a *proper* refinement of $\mathcal{E}(\theta)$. We shall show the equality of

$$\sum_{\mathcal{E}(\theta) \trianglelefteq \tau \trianglelefteq E} |\{\Psi(a_1, \dots, a_m) \mid \Psi \vDash \theta_1 \text{ and } \mathcal{E}(\Psi) = \tau\}|, \tag{241}$$

$$\sum_{\mathcal{E}(\theta) \trianglelefteq \tau \trianglelefteq E} |\{\Psi(a_1, \dots, a_m) \mid \Psi \vDash \theta_2 \text{ and } \mathcal{E}(\Psi) = \tau\}|. \tag{242}$$

from which the required result follows via the inductive hypothesis.

Again let $e_1, e_2, \dots, e_{|E|}$ be representatives of the $|E|$ classes in E with $e_1 = a_1$. Let $\Psi(a_1, \dots, a_m)$ be a state description extending θ_1 with $\mathcal{E}(\Psi) \trianglelefteq E$. Then since the $e_1, \dots, e_{|E|}$ contain representatives (not necessarily unique) for each of the equivalence classes in $\mathcal{E}(\Psi)$, Ψ is *determined* by the state description $\Omega(e_1, e_2, \dots, e_{|E|})$ for $e_1, e_2, \dots, e_{|E|}$ which it extends. Clearly also $\Omega \wedge \theta_1$ is consistent.

Conversely given a state description $\Omega(e_1, e_2, \dots, e_{|E|})$ for $e_1, e_2, \dots, e_{|E|}$ which is consistent with θ_1 we can define a unique state description $\Psi(a_1, a_2, \dots, a_m)$ *determined* by Ω as above such that $\mathcal{E}(\Psi) \trianglelefteq E$ (since then the $e_1, \dots, e_{|E|}$ must contain representatives, not necessarily unique, for each of the equivalence classes in $\mathcal{E}(\Psi)$). Furthermore Ψ will have to extend θ_1. To see this notice that if $R^{\varepsilon_{ijk}}(a_i, a_j, a_k, a_1)$ is a conjunct in Ψ then $R^{\varepsilon_{ijk}}(e_{s_i}, e_{s_j}, e_{s_k}, a_1)$ also appears in Ψ, where a_i, e_{s_i} are in the same part of the partition E etc.. So this also appears in Ω and hence in θ_1. But then $P^{\varepsilon_{ijk}}(e_{s_i}, e_{s_j}, e_{s_k})$ appears in θ so since $\mathcal{E}(\theta) \trianglelefteq E$, $P^{\varepsilon_{ijk}}(a_i, a_j, a_k)$ appears in θ and $R^{\varepsilon_{ijk}}(a_i, a_j, a_k, a_1)$ appears in θ_1.

It follows that (241) is equal to

$$|\{\Omega(e_1, e_2, \dots, e_{|E|}) \mid \Omega \wedge \theta_1 \text{ is consistent}\}| = 2^{|E|^3(|E|-1)}$$

by observing that we need to fix $|E|^3$ entries in the array for these Ω to accommodate the consistency of $\Omega \wedge \theta_1$.

An exactly similar argument applies in the case of θ_2 and hence the required equality of (241), (242) is proven. $\dashv$

Notice that having obtained this equality in the case of $R(x_1, x_2, x_3, a_1)$ and $R(x_1, x_2, x_3, x_3)$ we can by analogous transpositions derive equality for the cases of

$$R(x_1, x_2, a_3, x_3), R(x_1, x_2, x_1, x_3), R(a_2, x_2, x_1, x_3), \dots, R(t_1, t_2, t_3, t_4),$$

provided the variables amongst the terms t_1, t_2, t_3, t_4 are exactly x_1, x_2, x_3. Similarly Sx will imply for all fixed $n, l \in \mathbb{N}$

$$w\Big(\bigwedge_{i,j=1}^{m} R^{\varepsilon_{ij}}(a_i, a_j, a_n, a_l)\Big) = w\Big(\bigwedge_{i,j=1}^{m} R^{\varepsilon_{ij}}(a_i, a_i, a_j, a_l)\Big) =$$
$$w\Big(\bigwedge_{i,j=1}^{m} R^{\varepsilon_{ij}}(a_i, a_i, a_i, a_j)\Big) = w\Big(\bigwedge_{i,j=1}^{m} R^{\varepsilon_{ij}}(a_i, a_j, a_i, a_j)\Big) \quad \text{etc.,} \tag{243}$$

where, by summing over a suitable disjunction of state descriptions for $a_1, a_2, \dots, a_h$ with $h \geq m, n, l$, we can even take $n, l > m$.

For notational simplicity Proposition 28.2 was stated for the case of a language L with a single 4-ary relation symbol R. It should be clear that the proof generalizes to n-ary relation symbols and, by applying the observation made just prior to Proposition 27.1, to languages with more than one relation symbol.

In order to state our 'Principle of Conformity' to this level of generality[97] we need some notation. Suppose as usual that our language L has relation symbols $R_1, R_2, \dots, R_q$ of arities $r_1, r_2, \dots, r_q$ respectively. Let L^- be a language with relation symbols $P_1, P_2, \dots, P_q$ with arities $k_1, k_2, \dots, k_q$ such that $k_j \leq r_j$ for $j = 1, 2, \dots, q$. For each $i = 1, 2, \dots q$ let $u^i_1, u^i_2, \dots, u^i_{r_i}$ and $s^i_1, s^i_2, \dots, s^i_{r_i}$ be terms of L such that the variables appearing in each of these lists are exactly $x_1, x_2, \dots, x_{k_i}$ and let

$$R_i^{\vec{u}}(x_1, x_2, \dots, x_{k_i}) = R_i(u^i_1, u^i_2, \dots, u^i_{r_i}),$$
$$R_i^{\vec{s}}(x_1, x_2, \dots, x_{k_i}) = R_i(s^i_1, s^i_2, \dots, s^i_{r_i}).$$

Finally for $\theta \in SL^-$ let $\theta_1 \in SL$ be the result of replacing each $P_i(t_1, t_2, \dots, t_{k_i})$ in θ by $R_i^{\vec{u}}(t_1, t_2, \dots, t_{k_i})$ and let $\theta_2 \in SL$ be the result of replacing each $P_i(t_1, t_2, \dots, t_{k_i})$ in θ by $R_i^{\vec{s}}(t_1, t_2, \dots, t_{k_i})$.

THE PRINCIPLE OF CONFORMITY, PC.
Whenever the $R_i, P_i, u^i_j, s^i_j, \theta$, etc. are as above, $w(\theta_1) = w(\theta_2)$.

[97] It would seem that even this is not the most general version attainable!

The intuition here is that in this context of assigning probabilities there is no reason for w to treat the $R_i^{\vec{u}}(t_1, t_2, \dots, t_{k_i})$ and $R_i^{\vec{s}}(t_1, t_2, \dots, t_{k_i})$ in any way differently, and so in turn no reason for w to treat θ_1 and θ_2 differently.

Notice that Vx is actually a special case of PC. In view of that and Proposition 28.1 the following theorem might have been anticipated.

Theorem 28.3. *Sx implies the Principle of Conformity, PC.*

All the key ideas for the proof of this theorem appear already in the proof of Proposition 28.2. For the full details see [74]. In particular (243) is a special case of this theorem when $q = 1, r_1 = 4, k_1 = 2$, the variables are (for instance) x_1, x_2, the terms for the first and last expressions are $u_1^1 = x_1$, $u_2^1 = x_2$, $u_3^1 = a_n$, $u_4^1 = a_l$ and $s_1^1 = x_1$, $s_2^1 = x_2$, $s_3^1 = x_1$, $s_4^1 = x_2$.

We finally point out that (to our knowledge) neither PC nor Sx can be justified as 'symmetry principles', at least in as far as we later suggest a formalization of that notion in Chapter 39. Variable Exchangeability on the other hand patently is a symmetry principle according to this formulation (see page 296).

Chapter 29

THE PROBABILITY FUNCTIONS $u^{\bar{p},L}$

Up to now we have only exhibited two probability functions for a polyadic language L, namely c_0^L, c_∞^L, and both satisfied Sx (and Ex). We shall now introduce a family of probability functions[98], of which these two are 'extreme' members, satisfying Sx (and Ex). As we shall see later the probability functions in this family form the building blocks for all probability functions satisfying Sx and in this sense parallel the role of the unary probability functions $w_{\vec{c}}$ within the probability functions satisfying Ex. By the end of this chapter we will show that the members of this family actually extend the unary probability functions $u^{\bar{p},L}$ introduced on page 149 and because of that we will adopt the same notation for polyadic L.

Let $\mathbb{B}$ be, as previously on page 149, the set of infinite sequences

$$\bar{p} = \langle p_0, p_1, p_2, p_3, \dots \rangle$$

of non-negative reals such that $p_1 \geq p_2 \geq p_3 \geq \cdots$ and $\sum_{i=0}^{\infty} p_i = 1$. To help with visualization we shall think of the subscripts $0, 1, 2, \dots$ as *colours* and p_i as the probability of picking colour i. Amongst these colours 0 will stand for 'black' and will have a special status.

Let $c_1, c_2, \dots, c_m$ be a sequence of colours, so $\vec{c} = \langle c_1, c_2, \dots, c_m \rangle \in \mathbb{N}^m$. We say that a state description $\Theta(a_{i_1}, a_{i_2}, \dots, a_{i_m})$ is *consistent with* $\vec{c}$ if whenever $c_s = c_t \neq 0$ then $a_{i_s} \sim_\Theta a_{i_t}$. Note that $c_s = c_t = 0$ imposes no requirement on a_{i_s} being equivalent or inequivalent to a_{i_t}.

Notice that $\Theta(a_{i_1}, a_{i_2}, \dots, a_{i_m})$ is *consistent with* $\vec{c}$ just if $\Theta(a_1, a_2, \dots, a_m)$ is *consistent with* $\vec{c}$. Also if $n < m$ and $\Phi(a_{k_1}, a_{k_2}, \dots, a_{k_n})$ is the state description such that $\Theta(a_1, a_2, \dots, a_m) \vDash \Phi(a_{k_1}, a_{k_2}, \dots, a_{k_n})$, denoted henceforth by $\Theta(a_1, a_2, \dots, a_m)[a_{k_1}, a_{k_2}, \dots, a_{k_n}]$, then Θ being consistent with $\vec{c}$ forces Φ to be consistent with $\langle c_{k_1}, c_{k_2}, \dots, c_{k_n} \rangle$.

Let $\mathcal{C}(\vec{c}, \vec{a})$ be the set of all state descriptions for $\vec{a} = \langle a_{i_1}, a_{i_2}, \dots, a_{i_m} \rangle$ consistent with $\vec{c}$, in particular $\mathcal{C}(\emptyset, \emptyset)$ is just $\{\top\}$. When considering $\Theta(\vec{a}) \in \mathcal{C}(\vec{c}, \vec{a})$ we say that a_{i_s} has colour c_s. By the above remarks there is an obvious one-to-one correspondence between the elements of $\mathcal{C}(\vec{c}, \langle a_{i_1}, \dots, a_{i_m} \rangle)$ and those of $\mathcal{C}(\vec{c}, \langle a_1, \dots, a_m \rangle)$.

[98]Most early papers on Sx follow [96] in giving an alternative construction to the one given here.

We shall need the following observation concerning the size of these sets. In this lemma let (distinct) $i_1, i_2, \dots i_m \leq k$, $\vec{a} = \langle a_{i_1}, \dots, a_{i_m} \rangle$, $\vec{a}^+ = \langle a_1, \dots, a_k \rangle$, $\vec{c} = \langle c_{i_1}, c_{i_2}, \dots, c_{i_m} \rangle$, $\vec{c}^+ = \langle c_1, c_2, \dots, c_k \rangle$.

LEMMA 29.1. *Let $\Theta(\vec{a}) \in \mathcal{C}(\vec{c}, \vec{a})$. Then the number of $\Phi(\vec{a}^+) \in \mathcal{C}(\vec{c}^+, \vec{a}^+)$ extending Θ equals $|\mathcal{C}(\vec{c}^+, \vec{a}^+)| \cdot |\mathcal{C}(\vec{c}, \vec{a})|^{-1}$ (and as such it is independent of the choice of Θ).*

PROOF. Suppose for the moment that $k = m + 1$, $a_{i_1}, \dots, a_{i_m}$ contains all the constant symbols in $a_1, a_2, \dots, a_k$ except a_{m+1}, for notational simplicity say $\vec{a} = \langle a_{i_1}, \dots, a_{i_m} \rangle = \langle a_1, \dots, a_m \rangle$.

If $c_{m+1} = c_s \neq 0$ for some $s \leq m$ then the result is clear since $|\mathcal{C}(\vec{c}^+, \vec{a}^+)| = |\mathcal{C}(\vec{c}, \vec{a})|$ and there is exactly one extension of $\Theta(\vec{a})$ in $\mathcal{C}(\vec{c}^+, \vec{a}^+)$, namely the one in which a_{m+1} joins the same equivalence class as a_s. Otherwise suppose that c_{m+1} is black or a new colour and let $a_{g_1}, a_{g_2}, \dots, a_{g_t}$ comprise of one representative for each of the previous non-black colours together with all the previous black a_s. In that case in forming an extension of Θ in $\mathcal{C}(\vec{c}^+, \vec{a}^+)$, we will be permitted, for each r-ary relation symbol of L, a free choice of any $\pm R(a_{j_1}, a_{j_2}, \dots, a_{j_r})$ where a_{m+1} does appear in $a_{j_1}, a_{j_2}, \dots, a_{j_r}$ and otherwise only the $a_{g_1}, a_{g_2}, \dots, a_{g_t}$ are allowed to appear. After that everything is fixed by the requirement to maintain indistinguishability for a_s, a_t with the same non-black colour.

From this it follows that in the case of this $\vec{a}, \vec{a}^+, \vec{c}, \vec{c}^+$ the number of such extensions is $|\mathcal{C}(\vec{c}^+, \vec{a}^+)| \cdot |\mathcal{C}(\vec{c}, \vec{a})|^{-1}$. The general version stated in the lemma now follows by repeated application. ⊣

Now define for a state description $\Theta(a_1, a_2, \dots, a_m)$,

$$u^{\bar{p},L}(\Theta) = \sum_{\substack{\vec{c} \in \mathbb{N}^m \\ \Theta \in \mathcal{C}(\vec{c}, \vec{a})}} |\mathcal{C}(\vec{c}, \vec{a})|^{-1} \prod_{s=1}^{m} p_{c_s}. \tag{244}$$

LEMMA 29.2. *$u^{\bar{p},L}$ as defined by (244) extends uniquely to a probability function on SL.*

PROOF. It is enough to show that with the definition (244) conditions (i),(ii),(iii) from (34) hold. Of these (i) and (ii) are obvious. For (iii) notice that for $\vec{c} = \langle c_1, c_2, \dots, c_m \rangle$, $\vec{c}^+ = \langle c_1, c_2, \dots, c_{m+1} \rangle$, $\vec{a} = \langle a_1, \dots, a_m \rangle$, $\vec{a}^+ = \langle a_1, \dots, a_{m+1} \rangle$,

$$\sum_{\substack{\Phi(\vec{a}^+) \models \Theta(\vec{a}) \\ \Phi \in \mathcal{C}(\vec{c}^+, \vec{a}^+)}} |\mathcal{C}(\vec{c}^+, \vec{a}^+)|^{-1} p_{c_{m+1}} \prod_{s=1}^{m} p_{c_s}$$

is equal to 0 if $\Theta(\vec{a}) \notin \mathcal{C}(\vec{c}, \vec{a})$ and

$$|\mathcal{C}(\vec{c}, \vec{a})|^{-1} p_{c_{m+1}} \prod_{s=1}^{m} p_{c_s}$$

otherwise, by Lemma 29.1. Hence

$$\sum_{\vec{c}^+\in\mathbb{N}^{m+1}} \sum_{\substack{\Phi(\vec{a}^+)\vDash\Theta(\vec{a})\\ \Phi\in\mathcal{C}(\vec{c}^+,\vec{a}^+)}} |\mathcal{C}(\vec{c}^+,\vec{a}^+)|^{-1} p_{c_{m+1}} \prod_{s=1}^m p_{c_s}$$
$$= \sum_{\substack{\vec{c}\in\mathbb{N}^m\\ \Theta\in\mathcal{C}(\vec{c},\vec{a})}} \sum_{c_{m+1}\in\mathbb{N}} |\mathcal{C}(\vec{c},\vec{a})|^{-1} p_{c_{m+1}} \prod_{s=1}^m p_{c_s}$$

which by summing out the c_{m+1} gives the required result. ⊣

Notice that, just as in the unary case in fact, $u^{\bar{p},L}$ equals c_0^L when $\bar{p} = \langle 0,1,0,0,\dots\rangle$ and equals c_∞^L when $\bar{p} = \langle 1,0,0,\dots\rangle$ since in both cases they give the same answers on state descriptions $\Theta(a_1,\dots,a_m)$.

We now show that the $u^{\bar{p},L}$ satisfy the key properties we are currently interested in, Ex and Sx.

LEMMA 29.3. *$u^{\bar{p},L}$ satisfies Ex and Sx.*

PROOF. That $u^{\bar{p},L}$ satisfies Ex follows directly by Lemma 7.3. However, for future applications it will be worthwhile giving an alternative proof by deriving a useful identity from which this follows: Namely that (244) generalizes to

$$u^{\bar{p},L}(\Theta(\vec{a})) = \sum_{\substack{\vec{c}\in\mathbb{N}^m\\ \Theta\in\mathcal{C}(\vec{c},\vec{a})}} |\mathcal{C}(\vec{c},\vec{a})|^{-1} \prod_{s=1}^m p_{c_s}. \tag{245}$$

where now $\vec{a} = \langle a_{i_1}, a_{i_2},\dots,a_{i_m}\rangle$, $\vec{c} = \langle c_{i_1}, c_{i_2},\dots,c_{i_m}\rangle$. That is, the instantiating constants no longer need to be simply $a_1,a_2,\dots,a_m$. Let $k \geq i_1,i_2,\dots,i_m$, $\vec{a}^+ = \langle a_1,a_2,\dots,a_k\rangle$, $\vec{c} = \langle c_{i_1},c_{i_2},\dots,c_{i_m}\rangle$, $\vec{c}^+ = \langle c_1,c_2,\dots,c_k\rangle$. Then

$$\begin{aligned}
u^{\bar{p},L}(\Theta(\vec{a})) &= \sum_{\Phi(\vec{a}^+)\vDash\Theta(\vec{a})} u^{\bar{p},L}(\Phi(\vec{a}^+))\\
&= \sum_{\Phi(\vec{a}^+)\vDash\Theta(\vec{a})} \sum_{\substack{\vec{c}^+\in\mathbb{N}^k\\ \Phi\in\mathcal{C}(\vec{c}^+,\vec{a}^+)}} |\mathcal{C}(\vec{c}^+,\vec{a}^+)|^{-1} \prod_{s=1}^k p_{c_s}\\
&= \sum_{\vec{c}^+\in\mathbb{N}^k} \sum_{\substack{\Phi(\vec{a}^+)\vDash\Theta(\vec{a})\\ \Phi\in\mathcal{C}(\vec{c}^+,\vec{a}^+)}} |\mathcal{C}(\vec{c}^+,\vec{a}^+)|^{-1} \prod_{s=1}^k p_{c_s}\\
&= \sum_{\substack{\vec{c}^+\in\mathbb{N}^k\\ \Theta\in\mathcal{C}(\vec{c},\vec{a})}} |\mathcal{C}(\vec{c},\vec{a})|^{-1} \prod_{s=1}^k p_{c_s}, \qquad \text{by Lemma 29.1,}\\
&= \sum_{\substack{\vec{c}\in\mathbb{N}^m\\ \Theta\in\mathcal{C}(\vec{c},\vec{a})}} |\mathcal{C}(\vec{c},\vec{a})|^{-1} \prod_{s=1}^m p_{c_{i_s}},
\end{aligned}$$

as required.

It remains to show that $u^{\bar{p},L}$ satisfies Sx. Since we have established Ex it is enough to show for state descriptions $\Theta(a_1,\dots,a_m), \Phi(a_1,\dots,a_m)$ with

$$\mathcal{E}(\Theta(a_1,\dots,a_m)) = \mathcal{E}(\Phi(a_1,\dots,a_m))$$

that

$$u^{\bar{p},L}(\Theta(a_1,\dots,a_m)) = u^{\bar{p},L}(\Phi(a_1,\dots,a_m)).$$

But clearly in this case for any $\vec{c} \in \mathbb{N}^m$ we have $\Theta \in \mathcal{C}(\vec{c},\vec{a})$ just if $\Phi \in \mathcal{C}(\vec{c},\vec{a})$, so Sx follows. ⊣

Notice that by a similar proof to that for Proposition 21.1 but using the probability functions $u^{\bar{p},L}$ where $\bar{p}$ is

$$\langle 0, s_1(s_1+\dots+s_n)^{-1}, s_2(s_1+\dots+s_n)^{-1}, \dots, s_n(s_1+\dots+s_n)^{-1}, 0, 0, \dots\rangle$$

and the $s_i > 0$ are algebraically independent we can show that for state descriptions $\Theta(a_1,\dots,a_n), \Phi(a_1,\dots,a_n)$, $w(\Theta) = w(\Phi)$ for all probability functions w satisfying Sx just if $\mathcal{S}(\Theta) = \mathcal{S}(\Phi)$. This is an observation which one might have taken for granted. However it could have been that the requirement of satisfying Sx actually forced more equalities to hold than simply those it prescribed directly. Fortunately this does not happen in this case (nor in the cases of a number of similar principles which we shall meet later).

As remarked at the start of this chapter in the case of a purely unary language these same probability functions $u^{\bar{p},L}$ have already been introduced on page 149 via an alternative definition. We shall soon justify this remark by showing the equivalence of the two definitions when the language L is purely unary. First however it will be useful to have the following approximation lemma for the $u^{\bar{p},L}$ as defined by (244), where (as earlier) for $\bar{p} \in \mathbb{B}$, $R_{\bar{p},n} = 1 - \sum_{i=1}^n p_i$ and

$$\bar{p}\restriction n = \langle R_{\bar{p},n}, p_1, p_2, \dots, p_n, 0, 0, \dots\rangle.$$

LEMMA 29.4. *For each $\theta \in QFSL$*

$$\lim_{n\to\infty} u^{\bar{p}\restriction n,L}(\theta) = u^{\bar{p},L}(\theta).$$

PROOF. It is enough to prove this for a state description $\Theta(a_1,\dots,a_m)$. In this case, from (244),

$$u^{\bar{p}\restriction n,L}(\Theta) = \sum_{\substack{\vec{c}\in\mathbb{N}^m \\ \Theta\in\mathcal{C}(\vec{c},\vec{a})}} |\mathcal{C}(\vec{c},\vec{a})|^{-1} \prod_{s=1}^m q_{c_s} \tag{246}$$

where

$$q_s = \begin{cases} R_{\bar{p},n} & \text{if } s = 0, \\ p_s & \text{if } 1 \le s \le n, \\ 0 & \text{if } s > n. \end{cases}$$

For $\vec{c}$ with $0 < c_s \le n$ for each $s = 1,\dots,m$ the summands in (244) and (246) are the same. In the case where some $c_s > n$ the contribution to

(246) is 0 whilst the total contribution to (244) for all $\vec{c}$ with some $c_i > n$ must tend to zero as $n \to \infty$ (because the limit (244) is finite).

Finally for those $\vec{c}$ such that all the $c_s \leq n$ and some $k \geq 1$ of the c_i are 0,

$$\begin{aligned}
0 &\leq |\mathcal{C}(\vec{c}, \vec{a})|^{-1} \prod_{s=1}^{m} q_{c_s} - |\mathcal{C}(\vec{c}, \vec{a})|^{-1} \prod_{s=1}^{m} p_{c_s} \\
&= (q_0^k - p_0^k) |\mathcal{C}(\vec{c}, \vec{a})|^{-1} \prod_{c_s > 0} q_{c_s} \\
&\leq k(q_0 - p_0) \prod_{c_s > 0} q_{c_s},
\end{aligned}$$

so the sum of the left hand side over all such $\vec{c}$ *for a particular choice of k zeros* is bounded by

$$k(q_0 - p_0) \Big(\sum_{i=1}^{n} q_i \Big)^{m-k} \leq k(q_0 - p_0) \leq m(q_0 - p_0).$$

Putting all these $2^m - 1$ parts together then, since $q_0 = R_{\bar{p},n} \to p_0$ as $n \to \infty$, the total difference between (244) and (246) tends to zero as $n \to \infty$. ⊣

Now let L be purely unary, say with q relation symbols as usual. In view of the above lemma and the definition of $u^{\bar{p},L}$ given on page 150 for a unary language, it is enough to show that for a state description $\Theta(a_1, \dots, a_m) = \bigwedge_{i=1}^{2^q} \alpha_i^{r_i}$ (with the α_i instantiated in that order) and $\bar{p} = \langle p_0, p_1, p_2, \dots, p_n, 0, 0, \dots \rangle$, $u^{\bar{p},L}(\Theta)$ as defined on page 149 equals $u^{\bar{p},L}(\Theta)$ as defined by (244).

In other words that

$$\sum_{\substack{\vec{c} \in \{0,1,\dots,n\}^m \\ \Theta \in \mathcal{C}(\vec{c}, \vec{a})}} |\mathcal{C}(\vec{c}, \vec{a})|^{-1} \prod_{s=1}^{m} p_{c_s} \tag{247}$$

is equal to

$$2^{-nq} \sum_{f} \prod_{i=1}^{2^q} \Big(p_0 2^{-q} + \sum_{f(j)=i} p_j \Big)^{r_i}, \tag{248}$$

where the f range over all functions $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, 2^q\}$.

Considering first (247), let $\vec{c} \in \{0, 1, \dots, n\}^m$. Then the requirement that $\Theta \in \mathcal{C}(\vec{c}, \vec{a})$ is simply that if $\alpha_g(a_s), \alpha_h(a_t)$ are conjuncts in Θ and $c_s = c_t \neq 0$ then $g = h$. Similarly $|\mathcal{C}(\vec{c}, \vec{a})|$ is the number of state descriptions $\bigwedge_{i=1}^{m} \alpha_{h_i}(a_i)$ such that whenever $c_s = c_t \neq 0$ then $h_s = h_t$, and the number of these state descriptions will be the number of ways of picking the h_i to satisfy that condition. So if there are b many of the c_s equal to zero then we have 2^{bq} 'free' choices for these h_s and as for the,

e many say, distinct non-zero values of c_s appearing in $\vec{c}$ we must make h_s constant on all s with the same value of c_s. This contributes then a further 2^{eq} choices, so altogether the coefficient of $\prod_{s=1}^m p_{c_s}$ in (247) is

$$|\mathcal{C}(\vec{c},\vec{a})|^{-1} = 2^{-q(b+e)}.$$

Turning now to (248), consider a term in $\prod_{s=1}^m p_{c_s}$ formed from multiplying out $\prod_{i=1}^{2^q}(p_0 2^{-q} + \sum_{f(j)=i} p_j)^{r_i}$. For this term to appear it must be the case that if $c_s = c_t \neq 0$ then a_s, a_t satisfy the same atom $\alpha_i(x)$ in Θ, as was the case in the preceding paragraph. Furthermore in that case each occurrence of this term will carry with it a coefficient $2^{-nq}2^{-bq}$ where b is the number of times that $c_s = 0$. It will also appear 2^{kq} times where

$$k = |\{1, 2, \dots, n\} - \{c_s \mid 1 \leq s \leq m\}|$$

since there are that many possible f here which could give this same $\vec{c}$. Summing over all such f then gives the term $\prod_{s=1}^m p_{c_s}$ a coefficient $2^{q(k-n-b)}$. But since $n - k = e$ this agrees with the coefficient in (247) and the required identity follows.

We conclude this chapter by showing that the $u^{\bar{p},L}$ satisfy, in the polyadic[99] case, two important properties which we have already considered for unary languages:

LANGUAGE INVARIANCE, Li.

A probability function w for a language L satisfies Language Invariance, Li, if there is a family of probability functions $w^{\mathcal{L}}$, one on each (finite, possibly polyadic) language $\mathcal{L}$, satisfying Px (and Ex) such that $w^L = w$ and whenever $\mathcal{L} \subseteq \mathcal{L}'$, $w^{\mathcal{L}} = w^{\mathcal{L}'} \restriction S\mathcal{L}$ (i.e. $w^{\mathcal{L}'}$ restricted to $S\mathcal{L}$).

Again we say that w satisfies Language Invariance with $\mathcal{P}$, where $\mathcal{P}$ is some property, if the members $w^{\mathcal{L}}$ of this family also all satisfy the property $\mathcal{P}$.

WEAK IRRELEVANCE PRINCIPLE, WIP.

Suppose that $\theta, \phi \in QFSL$, where L is polyadic, are such that they have no constant nor relation symbols in common. Then

$$w(\theta \wedge \phi) = w(\theta) \cdot w(\phi).$$

THEOREM 29.5. *The $u^{\bar{p},L}$ satisfy Li with WIP and Sx.*

PROOF. That the $u^{\bar{p},L}$ satisfy Sx was proved in Lemma 29.3.

To show that the $u^{\bar{p},L}$ satisfy Li let $\mathcal{L}$ be a language properly extending L. Just as in the unary case we shall show that $u^{\bar{p},\mathcal{L}}$ agrees with $u^{\bar{p},L}$ on SL. It is enough to show this for a state description $\Theta(a_1, \dots, a_m)$ of L. In this case from (244),

$$u^{\bar{p},L}(\Theta) = \sum_{\substack{\vec{c} \in \mathbb{N}^m \\ \Theta \in \mathcal{C}^L(\vec{c},\vec{a})}} |\mathcal{C}^L(\vec{c},\vec{a})|^{-1} \prod_{s=1}^m p_{c_s},$$

[99] Unless otherwise stated we take it that 'polyadic' also includes the 'unary' as a special case.

and

$$u^{\bar{p},\mathcal{L}}(\Theta) = \sum_{\Phi} \sum_{\substack{\vec{c}\in\mathbb{N}^m \\ \Phi\in\mathcal{C}^{\mathcal{L}}(\vec{c},\vec{a})}} |\,\mathcal{C}^{\mathcal{L}}(\vec{c},\vec{a})|^{-1} \prod_{s=1}^{m} p_{c_s}, \tag{249}$$

where the $\Phi(a_1,\dots,a_m)$ are the state descriptions of $\mathcal{L}$ which logically imply $\Theta(a_1,\dots,a_m)$ and we have added the appropriate superscripts to the $\mathcal{C}(\vec{c},\vec{a})$.

Notice that for such a Φ we can express it as

$$\Theta(a_1,\dots,a_m)\wedge\Phi'(a_1,\dots,a_m)$$

where $\Phi'(a_1,\dots,a_m)$ is a state description of the language $\mathcal{L}-L$. Furthermore

$$\Phi\in\mathcal{C}^{\mathcal{L}}(\vec{c},\vec{a}) \iff [\Theta\in\mathcal{C}^{L}(\vec{c},\vec{a}) \text{ and } \Phi'\in\mathcal{C}^{\mathcal{L}-L}(\vec{c},\vec{a})].$$

Hence, by thinking for a moment of the Θ,Φ' as simply any two state descriptions for L and $\mathcal{L}-L$ respectively,

$$|\mathcal{C}^{\mathcal{L}}(\vec{c},\vec{a})| = |\mathcal{C}^{L}(\vec{c},\vec{a})|\cdot|\mathcal{C}^{\mathcal{L}-L}(\vec{c},\vec{a})|$$

and from (249) we obtain

$$\begin{aligned}
u^{\bar{p},\mathcal{L}}(\Theta) &= \sum_{\Phi'} \sum_{\substack{\vec{c}\in\mathbb{N}^m,\,\Theta\in\mathcal{C}^{L}(\vec{c},\vec{a}) \\ \Phi'\in\mathcal{C}^{\mathcal{L}-L}(\vec{c},\vec{a})}} |\,\mathcal{C}^{L}(\vec{c},\vec{a})|^{-1}|\,\mathcal{C}^{\mathcal{L}-L}(\vec{c},\vec{a})|^{-1} \prod_{s=1}^{m} p_{c_s},\\
&= \sum_{\substack{\vec{c}\in\mathbb{N}^m \\ \Theta\in\mathcal{C}^{L}(\vec{c},\vec{a})}} |\,\mathcal{C}^{L}(\vec{c},\vec{a})|^{-1}\Bigg(\sum_{\Phi'\in\mathcal{C}^{\mathcal{L}-L}(\vec{c},\vec{a})} |\mathcal{C}^{\mathcal{L}-L}(\vec{c},\vec{a})|^{-1}\Bigg) \prod_{s=1}^{m} p_{c_s},\\
&= \sum_{\substack{\vec{c}\in\mathbb{N}^m \\ \Theta\in\mathcal{C}^{L}(\vec{c},\vec{a})}} |\,\mathcal{C}^{L}(\vec{c},\vec{a})|^{-1} \prod_{s=1}^{m} p_{c_s} = u^{\bar{p},L}(\Theta),
\end{aligned}$$

as required.

It only remains to show that $u^{\bar{p},L}$ satisfies WIP. Suppose that L_1, L_2 are disjoint sublanguages of L, $L = L_1\cup L_2$, and $\Theta(a_1,\dots,a_m), \Phi(a_{m+1},\dots,a_{m+n})$ are state descriptions of L_1, L_2 respectively. Then, setting $\vec{a} = a_1,\dots,a_m$ and $\vec{b} = a_{m+1},\dots,a_{m+n}$, by Li,

$$\begin{aligned}
&u^{\bar{p},L}(\Theta)\cdot u^{\bar{p},L}(\Phi)\\
&\quad = u^{\bar{p},L_1}(\Theta)\cdot u^{\bar{p},L_2}(\Phi) &(250)\\
&\quad = \sum_{\substack{\vec{c}\in\mathbb{N}^m \\ \Theta\in\mathcal{C}^{L_1}(\vec{c},\vec{a})}} |\mathcal{C}^{L_1}(\vec{c},\vec{a})|^{-1}\prod_{s=1}^{m} p_{c_s} \times \sum_{\substack{\vec{d}\in\mathbb{N}^n \\ \Phi\in\mathcal{C}^{L_2}(\vec{d},\vec{b})}} |\mathcal{C}^{L_2}(\vec{d},\vec{b})|^{-1}\prod_{s=1}^{n} p_{d_s}\\
&\quad = \sum_{\substack{\vec{c}\in\mathbb{N}^m \\ \Theta\in\mathcal{C}^{L_1}(\vec{c},\vec{a})}} \sum_{\substack{\vec{d}\in\mathbb{N}^n \\ \Phi\in\mathcal{C}^{L_2}(\vec{d},\vec{b})}} |\mathcal{C}^{L_1}(\vec{c},\vec{a})|^{-1}|\mathcal{C}^{L_2}(\vec{d},\vec{b})|^{-1}\prod_{s=1}^{m+n} p_{e_s} &(251)
\end{aligned}$$

where $\vec{e} = \vec{c}\,{}^\frown\vec{d} = c_1,\dots,c_m,d_1,\dots,d_n$, etc..

Now suppose that $\Theta(\vec{a}) \in \mathcal{C}^{L_1}(\vec{c}, \vec{a})$ and $\Phi(\vec{b}) \in \mathcal{C}^{L_2}(\vec{d}, \vec{b})$ and consider forming a state description $\Psi(a_1, \dots, a_{m+n}) \in \mathcal{C}^L(\vec{e}, \vec{a}\,\hat{}\,\vec{b})$ extending $\Theta(a_1, \dots, a_m) \wedge \Phi(a_{m+1}, \dots, a_{m+n})$. The only constraints on Ψ are that it must be consistent with the colouring $\vec{e}$ and with the 'choices' already determined by Θ and Φ, for example if a_1, a_2, a_{m+1} all get the same colour, R is a binary relation symbol of L_1 and $\Theta \vDash R(a_1, a_2)$ then we must have that $\Psi \vDash R(a_1, a_{m+1})$. But clearly the free choices in forming Ψ do not depend on these particular $\Theta(\vec{a}) \in \mathcal{C}^{L_1}(\vec{c}, \vec{a})$ and $\Phi(\vec{b}) \in \mathcal{C}^{L_2}(\vec{d}, \vec{b})$ only on $\vec{e}$. In other words the number, N say, of $\Psi(a_1, \dots, a_{m+n}) \in \mathcal{C}^L(\vec{e}, \vec{a}\,\hat{}\,\vec{b})$ extending $\Theta(a_1, \dots, a_m) \wedge \Phi(a_{m+1}, \dots, a_{m+n})$ will be the same no matter which $\Theta(\vec{a}) \in \mathcal{C}^{L_1}(\vec{c}, \vec{a})$ and $\Phi(\vec{b}) \in \mathcal{C}^{L_2}(\vec{d}, \vec{b})$ we started from. That is,

$$|\mathcal{C}^L(\vec{e}, \vec{a}\,\hat{}\,\vec{b})| = N \cdot |\mathcal{C}^{L_1}(\vec{c}, \vec{a})| \cdot |\mathcal{C}^{L_2}(\vec{d}, \vec{b})|. \tag{252}$$

Consequently, with the $\Psi(a_1, \dots, a_{m+n})$ ranging over state descriptions of L extending $\Theta \wedge \Phi$, if $\Theta \in \mathcal{C}^{L_1}(\vec{c}, \vec{a})$ and $\Phi \in \mathcal{C}^{L_2}(\vec{d}, \vec{b})$,

$$\begin{aligned}
&\sum_{\Psi \in \mathcal{C}^L(\vec{e}, \vec{a}\,\hat{}\,\vec{b})} |\mathcal{C}^L(\vec{e}, \vec{a}\,\hat{}\,\vec{b})|^{-1} \\
&\quad = \sum_{\Psi \in \mathcal{C}^L(\vec{e}, \vec{a}\,\hat{}\,\vec{b})} (N \cdot |\mathcal{C}^{L_1}(\vec{c}, \vec{a})| \cdot |\mathcal{C}^{L_2}(\vec{d}, \vec{b})|)^{-1} \qquad \text{by (252)} \\
&\quad = |\mathcal{C}^{L_1}(\vec{c}, \vec{a})|^{-1} |\mathcal{C}^{L_2}(\vec{d}, \vec{b})|^{-1},
\end{aligned} \tag{253}$$

whilst if not both $\Theta \in \mathcal{C}^{L_1}(\vec{c}, \vec{a})$ and $\Phi \in \mathcal{C}^{L_2}(\vec{d}, \vec{b})$ then the right hand side of (253) is zero. Hence, with the $\Psi(a_1, \dots, a_{m+n})$ again state descriptions of L extending $\Theta \wedge \Phi$,

$$\begin{aligned}
u^{\bar{p},L}(\Theta \wedge \Phi) &= \sum_{\Psi} \sum_{\substack{\vec{e} \in \mathbb{N}^{m+n} \\ \Psi \in \mathcal{C}^L(\vec{e}, \vec{a}\,\hat{}\,\vec{b})}} |\mathcal{C}^L(\vec{e}, \vec{a}\,\hat{}\,\vec{b})|^{-1} \prod_{s=1}^{m+n} p_{e_s} \\
&= \sum_{\substack{\vec{c} \in \mathbb{N}^m \\ \Theta \in \mathcal{C}^{L_1}(\vec{c}, \vec{a})}} \sum_{\substack{\vec{d} \in \mathbb{N}^n \\ \Phi \in \mathcal{C}^{L_2}(\vec{d}, \vec{b})}} |\mathcal{C}^{L_1}(\vec{c}, \vec{a})|^{-1} |\mathcal{C}^{L_2}(\vec{d}, \vec{b})|^{-1} \prod_{s=1}^{m+n} p_{e_s} \\
&\qquad\qquad \text{by (253)} \\
&= u^{\bar{p},L}(\Theta) \cdot u^{\bar{p},L}(\Phi) \qquad \text{by (251), as required.} \quad \dashv
\end{aligned}$$

As in the unary case, with Theorem 20.6, by using Theorem 32.1 we will see that the converse to Theorem 29.5 also holds: Even when extended to polyadic languages the $u^{\bar{p},L}$ are still the only probability functions satisfying Li with WIP and Sx.

Chapter 30

THE HOMOGENEOUS/HETEROGENEOUS DIVIDE

It turns out that there are two basic types of probability functions satisfying Sx. In this chapter we shall explain what they are and the sense in which every probability function satisfying Sx is a mixture of these types.

A probability function w on SL satisfying Sx is said to be *homogeneous* if for all t

$$\lim_{m\to\infty} \sum_{|\mathcal{S}(\Theta(a_1,a_2,\ldots,a_m))|=t} w(\Theta(a_1,a_2,\ldots,a_m)) = 0 \tag{254}$$

where the $\Theta(a_1,a_2,\ldots,a_m)$ range over the possible state descriptions of $a_1,a_2,\ldots,a_m$ in L. In other words the probability that all the a_i will fall in some fixed finite number of equivalence classes with respect to indistinguishability is zero.

As opposed to homogeneous w is t-*heterogeneous* if

$$\lim_{m\to\infty} \sum_{|\mathcal{S}(\Theta(a_1,a_2,\ldots,a_m))|=t} w(\Theta(a_1,a_2,\ldots,a_m)) = 1. \tag{255}$$

In other words the probability that all the a_i will fall in some t (non-empty) equivalence classes with respect to indistinguishability is 1. Henceforth if we say that a probability function is homogeneous or heterogeneous then it will be implicit that it satisfies Sx. Notice that if L is purely unary then no probability function on SL satisfying Sx (i.e. Ax in this case) can be homogeneous. Indeed, as we shall shortly see, if in this case L has q predicate symbols then such a w will be a mixture of t-heterogeneous probability functions on SL for $t \leq 2^q$.

The next proposition provides us with a large family of homogeneous probability functions. Let

$$\mathbb{B}_\infty = \{\langle p_0, p_1, p_2, p_3, \ldots\rangle \in \mathbb{B} \mid p_0 > 0 \text{ or } p_j > 0 \text{ for all } j > 0\}.$$

Proposition 30.1. *If L is not purely unary and*

$$\bar{p} = \langle p_0, p_1, p_2, \ldots\rangle \in \mathbb{B}_\infty,$$

then $u^{\bar{p},L}$ is homogeneous.

Proof. We start by deriving an alternative definition of $u^{\bar{p},L}$ which is particularly appropriate for proving this proposition.

Consider the following random process for constructing state descriptions $\Theta(a_1, a_2, \dots, a_m)$ and sequences $\langle c_1, c_2, \dots, c_m\rangle \in \mathbb{N}^m$. At stage $m = 0$ we pick Θ_0 with probability 1 to be $\top$ and $\vec{c}$ to be the empty sequence. Suppose that by stage m we have produced with probability $j^{\bar{p},L}(\Theta_m(\vec{a}), \vec{c})$ a state description $\Theta_m(a_1, a_2, \dots, a_m)$ and sequence $\langle c_1, c_2, \dots, c_m\rangle \in \mathbb{N}^m$ so that $\Theta_m(\vec{a}) \in \mathcal{C}(\vec{c}, \vec{a})$.

Pick c_{m+1}, with probability $p_{c_{m+1}}$, and a state description $\Theta_{m+1}(a_1, \dots, a_m, a_{m+1})$ randomly (i.e. according to the uniform distribution) from amongst all those state descriptions in $\mathcal{C}(\langle c_1, \dots, c_m, c_{m+1}\rangle, \langle a_1, \dots, a_m, a_{m+1}\rangle)$ which extend $\Theta_m(\vec{a})$. Set $j^{\bar{p},L}(\Theta_{m+1}(\vec{a}^+), \vec{c}^+)$, with the obvious notation, to be the probability that the process produces $\Theta_{m+1}(\vec{a}^+)$ and $\vec{c}^+$. That is, by Lemma 29.1,

$$j^{\bar{p},L}(\Theta_{m+1}(\vec{a}^+), \vec{c}^+) = p_{c_{m+1}} \frac{|\,\mathcal{C}(\vec{c}, \vec{a})|}{|\,\mathcal{C}(\vec{c}^+, \vec{a}^+)|} j^{\bar{p},L}(\Theta_m(\vec{a}), \vec{c}). \tag{256}$$

From (256) it follows that for $\Theta(\vec{a}) \in \mathcal{C}(\vec{c}, \vec{a})$,

$$j^{\bar{p},L}(\Theta(\vec{a}), \vec{c}) = \frac{1}{|\,\mathcal{C}(\vec{c}, \vec{a})|} \prod_{s=1}^{m} p_{c_s}$$

(whilst for $\Theta(\vec{a}) \notin \mathcal{C}(\vec{c}, \vec{a})$, $j^{\bar{p},L}(\Theta(\vec{a}), \vec{c}) = 0$) and

$$u^{\bar{p},L}(\Theta(\vec{a})) = \sum_{\vec{c} \in \mathbb{N}^m} j^{\bar{p},L}(\Theta(\vec{a}), \vec{c}). \tag{257}$$

The importance of (257) is that it tells us that $u^{\bar{p},L}(\Theta(\vec{a}))$ is the *probability that in m steps this process yields* $\Theta(\vec{a})$.

Returning to the proof of the proposition we show (254) by induction on t. Assume the result holds below t (note that it holds trivially for $t = 0$) and let $\varepsilon > 0$. Pick m large so that

$$\sum_{|\mathcal{S}(\Theta(\vec{a}))| < t} u^{\bar{p},L}(\Theta(\vec{a})) < \varepsilon. \tag{258}$$

where $\vec{a} = \langle a_1, \dots, a_m\rangle$.

We will first consider the harder case when $p_0 = 0$ so $p_j > 0$ for all $j \in \mathbb{N}^+$. Let T be a *finite* set of pairs $\langle\Theta(\vec{a}), \vec{c}\rangle$, where $\vec{c} = \langle c_1, \dots, c_m\rangle \in \mathbb{N}^m$, with $|\mathcal{S}(\Theta(\vec{a}))| = t$, $\Theta(\vec{a}) \in \mathcal{C}(\vec{c}, \vec{a})$ such that

$$\sum_{\substack{|\mathcal{S}(\Theta(\vec{a}))| = t \\ \langle\Theta(\vec{a}), \vec{c}\rangle \notin T}} j^{\bar{p},L}(\Theta(\vec{a}), \vec{c}) < \varepsilon. \tag{259}$$

Pick r, z, v, in that order, such that r is greater than any c_i from a $\vec{c}$ appearing in a pair $\langle \Theta(\vec{a}), \vec{c} \rangle$ from T and

$$(2^{-(2t+1)}t)^z < \varepsilon \quad \text{and} \quad \sum_{j=r+1}^{r+z} (1-p_j)^v < \varepsilon. \tag{260}$$

Let A be the set of $\vec{c}^{\,*} = \langle c_1, \dots, c_m, \dots, c_{m+v} \rangle \in \mathbb{N}^{m+v}$ such that each of $r+1, \dots, r+z$ appears among the $c_{m+1}, \dots, c_{m+v}$. Note that

$$\sum_{\vec{c}^{\,*} \in (\mathbb{N}^{m+v} - A)} \prod_{s=1}^{m+v} p_{c_s} \le \sum_{j=r+1}^{r+z} (1-p_j)^v < \varepsilon$$

and hence

$$\sum_{\substack{\vec{c}^{\,*} \in (\mathbb{N}^{m+v} - A) \\ \Phi \in \mathcal{C}(\vec{c}^{\,*}, \vec{a}^{\,*})}} j^{\bar{p},L}(\Phi(\vec{a}^{\,*}), \vec{c}^{\,*}) < \varepsilon. \tag{261}$$

By (257) we have

$$\sum_{|\mathcal{S}(\Phi(\vec{a}^{\,*}))| \le t} u^{\bar{p},L}(\Phi(\vec{a}^{\,*})) = \sum_{\substack{|\mathcal{S}(\Phi(\vec{a}^{\,*}))| \le t \\ \vec{c}^{\,*} \in \mathbb{N}^{m+v}}} j^{\bar{p},L}(\Phi(\vec{a}^{\,*}), \vec{c}^{\,*}).$$

The last sum is over the set

$$S = \{ \langle \Phi(\vec{a}^{\,*}), \vec{c}^{\,*} \rangle \mid |\mathcal{S}(\Phi(\vec{a}^{\,*}))| \le t \text{ and } \vec{c}^{\,*} \in \mathbb{N}^{m+v} \}.$$

We now specify four subsets of S, with union S, and show that summing the $j^{\bar{p},L}(\Phi(\vec{a}^{\,*}), \vec{c}^{\,*})$ over any of them gives at most ε, which suffices to demonstrate the induction step. Recalling the notation introduced on page 205 let the sets be $S_1, S_2, S_3, S_4 \subseteq S$ with the additional defining conditions as follows:

S_1: $|\mathcal{S}(\Phi(\vec{a}^{\,*})[a_1, \dots, a_m])| < t$,
S_2: $|\mathcal{S}(\Phi(\vec{a}^{\,*})[a_1, \dots, a_m])| = t$, $\langle \Phi(\vec{a}^{\,*})[a_1, \dots, a_m], \langle c_1, \dots, c_m \rangle \rangle \notin T$,
S_3: $\vec{c}^{\,*} \notin A$,
S_4: $|\mathcal{S}(\Phi(\vec{a}^{\,*})[a_1, \dots, a_m])| = t$, $\langle \Phi(\vec{a}^{\,*})[a_1, \dots, a_m], \langle c_1, \dots, c_m \rangle \rangle \in T$, $\vec{c}^{\,*} \in A$.

The sums over S_1, S_2, S_3 are each less than ε by virtue of (258), (259) and (261) respectively.

Turning now to S_4, this set can be partitioned into subsets of pairs with identical fixed $\vec{c}^{\,*} \in A$ and identical $\Phi(\vec{a}^{\,*})[a_1, \dots, a_m] = \Theta(a_1, \dots, a_m)$, which satisfies

$$|\mathcal{S}(\Theta(a_1, \dots, a_m))| = t, \langle \Theta(a_1, \dots, a_m), \langle c_1, \dots, c_m \rangle \rangle \in T\,.$$

Consider some fixed $\vec{c}^{\,*}$ and Θ. Let z' be the number of new colours amongst the $c_{m+1}, \dots, c_{m+v}$, that is, $|\{c_1, \dots, c_{m+v}\} - \{c_1, \dots, c_m\}|$. Since $\vec{c}^{\,*} \in A$, all of $r+1, \dots, r+z$ appear amongst the $c_{m+1}, \dots c_{m+v}$ so $z' \ge z$. Consider the process generating state descriptions used to define $u^{\bar{p},L}$ as it produces $\vec{c}^{\,*}$ and corresponding state descriptions extending Θ. Every time one of the new colours is encountered, there are at least 2^{2t+1} choices (corresponding to the minimal case of L having one binary relation symbol, and ignoring the previously encountered new colours) for extending the current state description, because there must be at least t different colours already amongst $c_1, \dots, c_m$. Only t of these choices however can possibly keep the length of the spectrum to be t, namely choices which make the individual join one of the t existing classes, if there still are only t classes. Hence there are at least $2^{z'(2t+1)}$ extensions of $\Theta(a_1, \dots, a_m)$ in $\mathcal{C}(\vec{c}^*, \vec{a}^*)$ but only $t^{z'}$ extensions of $\Theta(a_1, \dots, a_m)$ in $\mathcal{C}(\vec{c}^*, \vec{a}^*)$ with spectrum of length t. Since $j^{\bar{p},L}(\Phi(\vec{a}^*), \vec{c}^*)$ has the same value for every $\Phi(\vec{a}^*) \in \mathcal{C}(\vec{c}^*, \vec{a}^*)$, we obtain that

$$\sum_{\substack{\Phi(\vec{a}^*)[a_1,\dots,a_m] = \Theta(a_1,\dots,a_m) \\ \langle \Phi(\vec{a}^*), \vec{c}^* \rangle \in S_4}} j^{\bar{p},L}(\Phi(\vec{a}^*), \vec{c}^*)$$

is less or equal to

$$(t2^{-(2t+1)})^{z'} \sum_{\Phi(\vec{a}^*)[a_1,\dots,a_m] = \Theta(a_1,\dots,a_m)} j^{\bar{p},L}(\Phi(\vec{a}^*), \vec{c}^*)$$

and summing over the $\vec{c}^*$ and Θ we obtain that the sum over S_4 is less or equal to

$$(t2^{-(2t+1)})^{z} \sum_{\substack{\vec{c}^* \in \mathbb{N}^{(m+v)} \\ \Phi(\vec{a}^*)}} j^{\bar{p},L}(\Phi(\vec{a}^*), \vec{c}^*) \leq \varepsilon\,,$$

as required.

The case when $p_0 > 0$ is similar. We do not need to introduce T and r; we define z as before and we set $v = zv_0$ so that $z(1-p_0)^{v_0} < \varepsilon$. Then we take A as the set of those $\vec{c}^*$ for which p_0 occurs at least z times amongst $c_{m+1}, \dots, c_{m+v}$ and use an analogous argument. ⊣

When $\bar{p} \in \mathbb{B}$ is of the form

$$\langle 0, p_1, p_2, \dots, p_t, 0, 0, \dots \rangle$$

with $p_t > 0$ then $u^{\bar{p},L}(\Theta) = 0$ whenever the state description Θ has spectrum of length greater than t. However unless $t = 1$, $u^{\bar{p},L}$ will not be 'purely' t-heterogeneous but, as we shall shortly see, a proper mixture of s-heterogeneous probability functions for $s \leq t$. We can however produce t-heterogeneous probability functions by a variation of the construction of the $u^{\bar{p},L}$.

The idea is to use an alternative to the weighting $|\mathcal{C}(\vec{c}, \vec{a})|^{-1}$. Let

$$\bar{p} \in \mathbb{B}_t = \{\langle 0, p_1, p_2, p_3, \dots, \rangle \in \mathbb{B} \mid p_t > 0 = p_{t+1}\}.$$

Given a state description $\Theta(a_{i_1}, a_{i_2}, \dots, a_{i_m})$ and $\vec{c} \in \{1, 2 \dots, t\}^m$ let $\mathcal{G}(\vec{c}, \Theta)$ be zero if Θ is not consistent with $\vec{c}$, that is if it is not the case that $a_{i_s} \sim_\Theta a_{i_h}$ whenever $c_s = c_h$.

Otherwise, list the first representative of each distinct colour amongst the $c_1, c_2, \dots, c_m$ as $c_{g_1}, c_{g_2}, \dots, c_{g_r}$, so certainly $r \leq t$. Let

$$\Theta' = \Theta(a_{i_1}, a_{i_2}, \dots, a_{i_m})[a_{i_{g_1}}, a_{i_{g_2}}, \dots, a_{i_{g_r}}].$$

Notice then that if the equivalence classes of Θ are $I_1, I_2, \dots, I_h$ then the equivalence classes of Θ' are the

$$I_j \cap \{a_{i_{g_1}}, a_{i_{g_2}}, \dots, a_{i_{g_r}}\}$$

and these are all non-empty since amongst the $a_{i_{g_j}}$ there must be at least one representative from each equivalence class (and possibly more than one because two or more colours may be contributing to the same equivalence class).

Now let $\vec{b}$ be any vector of t (distinct) constant symbols starting with $a_{i_{g_1}}, a_{i_{g_2}}, \dots, a_{i_{g_r}}$ and set $\mathcal{G}(\vec{c}, \Theta)$ to be the fraction

$$\frac{|\{\Phi(\vec{b}) \mid \Phi(\vec{b}) \vDash \Theta' \text{ and } \mathcal{S}(\Phi) = 1_t\}|}{|\{\Phi(\vec{b}) \mid \mathcal{S}(\Phi) = 1_t\}|} \tag{262}$$

where, following the convention, the $\Phi(\vec{b})$ are state descriptions for $\vec{b}$. In other words $\mathcal{G}(\vec{c}, \Theta)$ is the proportion of all state descriptions for $\vec{b}$ with spectrum 1_t which extend Θ'.

We are now ready to specify our t-heterogeneous probability functions by:

$$v^{\bar{p},L}(\Theta(a_1, a_2, \dots, a_m)) = \sum_{\vec{c} \in \{1,2,\dots,t\}^m} \mathcal{G}(\vec{c}, \Theta) \prod_{s=1}^{m} p_{c_s}. \tag{263}$$

Reviewing the construction of the $u^{\bar{p},L}$ from the previous chapter it can be checked that similarly (263) consistently and unambiguously determines a probability function $v^{\bar{p},L}$ satisfying Ex and Sx: For (34)(iii) notice that for $\vec{c} = \langle c_1, c_2, \dots, c_m \rangle$, $\vec{c}^+ = \langle c_1, c_2, \dots, c_{m+1} \rangle$, $\vec{a} = \langle a_1, \dots, a_m \rangle$, $\vec{a}^+ = \langle a_1, \dots, a_{m+1} \rangle$ and a state description $\Theta(\vec{a})$ we have

$$\begin{aligned}
\sum_{\Theta^+(\vec{a}^+) \vDash \Theta(\vec{a})} v^{\bar{p},L}(\Theta^+) &= \sum_{\Theta^+(\vec{a}^+) \vDash \Theta(\vec{a})} \sum_{\vec{c}^+ \in \{1,\dots,t\}^{m+1}} \mathcal{G}(\vec{c}^+, \Theta^+) p_{c_{m+1}} \prod_{s=1}^{m} p_{c_s} \\
&= \sum_{\vec{c}^+ \in \{1,\dots,t\}^{m+1}} \sum_{\Theta^+(\vec{a}^+) \vDash \Theta(\vec{a})} \mathcal{G}(\vec{c}^+, \Theta^+) p_{c_{m+1}} \prod_{s=1}^{m} p_{c_s}
\end{aligned}$$

and consider a given $\vec{c}^+ \in \{1, \dots, t\}^{m+1}$. If $c_{m+1} = c_s$ for some $s \leq m$ then for (the unique) $\Theta^+ \vDash \Theta$ consistent with $\vec{c}^+$ we have $\mathcal{G}(\vec{c}^+, \Theta^+) = \mathcal{G}(\vec{c}, \Theta)$ whilst for $\Theta^+ \vDash \Theta$ inconsistent with $\vec{c}^+$ we have $\mathcal{G}(\vec{c}^+, \Theta^+) = 0$. If c_{m+1} is new then

$$\mathcal{G}(\vec{c}, \Theta) = \sum_{\Theta^+ \vDash \Theta} \mathcal{G}(\vec{c}^+, \Theta^+).$$

Hence

$$v^{\bar{p},L}(\Theta) = \sum_{\Theta^+(\vec{a}^+) \vDash \Theta(\vec{a})} v^{\bar{p},L}(\Theta^+)\,.$$

Ex again follows directly from Lemma 7.3 and for Sx it is enough to show that for state descriptions Θ_1, Θ_2 with $\mathcal{E}(\Theta_1) = \mathcal{E}(\Theta_2)$ we have $v^{\bar{p},L}(\Theta_1) = v^{\bar{p},L}(\Theta_2)$. This follows from the definition of $v^{\bar{p},L}$ upon noticing that for such Θ_1, Θ_2 and a given $\vec{c}$ they are either both inconsistent with $\vec{c}$ or both consistent with it and in that case the state descriptions Θ_1' and Θ_2' as in (262) have the same spectrum so $\mathcal{G}(\vec{c}, \Theta_1) = \mathcal{G}(\vec{c}, \Theta_2)$ by Lemma 27.2. Furthermore we can prove an analogous lemma to Lemma 29.1 and in turn show for $v^{\bar{p},L}$, just as we did for $u^{\bar{p},L}$, that (263), generalizes to

$$v^{\bar{p},L}(\Theta(a_{i_1}, a_{i_2}, \ldots, a_{i_m})) = \sum_{\vec{c} \in \{1,2,\ldots,t\}^m} \mathcal{G}(\vec{c}, \Theta) \prod_{s=1}^{m} p_{c_s}\,.$$

Again we have a definition of the $v^{\bar{p},L}$ in terms of a probabilistic process similar to that which gave the $j^{\bar{p},L}$ for $u^{\bar{p},L}$ (cf the proof of Proposition 30.1); the only difference is that the factor $|\mathcal{C}(\vec{c}, \vec{a})| \cdot |\mathcal{C}(\vec{c}^+, \vec{a}^+)|^{-1}$ in (256) is replaced by $|\mathcal{G}(\vec{c}^+, \Theta_{m+1})| \cdot |\mathcal{G}(\vec{c}, \Theta_m)|^{-1}$, with $\mathcal{G}(\emptyset, \top) = 1$. It is an easy exercise to use this to show that indeed $v^{\bar{p},L}$ is t-heterogeneous.

Unlike the $u^{\bar{p},L}$ the $v^{\bar{p},L}$ do not satisfy Li, indeed as we shall see on page 280 they fail it in the most spectacular way possible for L not purely unary.

Notice that in the case of a purely unary language L (with q predicate symbols), the probability function $v_{\vec{c}}$ introduced in (86) is actually the same as $v^{\bar{c},L}$ where

$$\bar{c} = \langle 0, c_{i_1}, c_{i_2}, \ldots, c_{i_{2^q}}, 0, 0, \ldots \rangle$$

and $c_{i_1}, c_{i_2}, \ldots, c_{i_{2^q}}$ list $c_1, c_2, \ldots, c_{2^q}$ in non-increasing order of magnitude.

We now turn our attention to the so called *Ladder Theorem* (see [77], [94], [96]) which shows that every probability function satisfying Sx is a mixture of a homogeneous and t-heterogeneous ($t = 1, 2, 3, \ldots$) probability functions.

THEOREM 30.2. *Every probability function w satisfying Sx can be written as a sum of probability functions*

$$w = \eta_0 w^{[0]} + \sum_{t=1}^{\infty} \eta_t w^{[t]} \tag{264}$$

where the $\eta_i \geq 0$, $\sum_i \eta_i = 1$, the $w^{[t]}$ are t-heterogeneous and $w^{[0]}$ is homogeneous. Furthermore this representation is unique up to a free choice of the $w^{[t]}$ when $\eta_t = 0$.

Conversely any probability function given by such a sum will satisfy Sx.

PROOF. For $t \geq 1$ and a state description $\Theta(a_1, \dots, a_m)$ define

$$z^{[t]}(\Theta(a_1, \dots, a_m)) = \lim_{n \to \infty} w\Big(\bigvee_{\substack{\Phi(a_1,\dots,a_n) \vDash \Theta(a_1,\dots,a_m) \\ |\mathcal{S}(\Phi(a_1,\dots,a_n))| = t}} \Phi(a_1, \dots, a_n) \Big). \tag{265}$$

To see that this limit is well defined notice that the sequence

$$w\Big(\bigvee_{\substack{\Phi(a_1,\dots,a_n) \vDash \Theta(a_1,\dots,a_m) \\ |\mathcal{S}(\Phi(a_1,\dots,a_n))| \leq t}} \Phi(a_1, \dots, a_n) \Big)$$

is, for $n \geq m$, non-increasing since if $m \leq k \leq n$ and $\Phi(a_1, \dots, a_n) \vDash \Theta(a_1, \dots, a_m)$ and $|\mathcal{S}(\Phi(a_1, \dots, a_n))| \leq t$ then

$$\Phi(a_1, \dots, a_n)[a_1, \dots, a_k] \vDash \Theta(a_1, \dots, a_m) \text{ and}$$
$$|\mathcal{S}(\Phi(a_1, \dots, a_n)[a_1, \dots, a_k])| \leq t$$

so

$$\bigvee_{\substack{\Phi(a_1,\dots,a_n) \vDash \Theta(a_1,\dots,a_m) \\ |\mathcal{S}(\Phi(a_1,\dots,a_n))| \leq t}} \Phi(a_1, \dots, a_n) \vDash \bigvee_{\substack{\Psi(a_1,\dots,a_k) \vDash \Theta(a_1,\dots,a_m) \\ |\mathcal{S}(\Psi(a_1,\dots,a_k))| \leq t}} \Psi(a_1, \dots, a_k).$$

Hence the limits corresponding to (265) with $\leq t$ in place of $= t$ exists and consequently (by taking differences) so do the limits in (265).

Let $\eta_t = z^{[t]}(\top)$. Assuming for the present that $\eta_t \neq 0$ set $w^{[t]} = \eta_t^{-1} z^{[t]}$. Clearly $z^{[t]}$ satisfies (i), (iii) of (34) so under our assumption that $\eta_t > 0$ $w^{[t]}$ satisfies (i)–(iii) of (34), and hence extends to a probability function on SL. Also $w^{[t]}$ satisfies Ex by Lemma 7.3 since clearly for $\sigma \in \mathrm{S}_m$,

$$\Phi(a_1, \dots, a_n) \vDash \Theta(a_1, \dots, a_m)$$
$$\iff \Phi(a_{\sigma(1)}, \dots, a_{\sigma(m)}, a_{m+1}, \dots, a_n) \vDash \Theta(a_{\sigma(1)}, \dots, a_{\sigma(m)}),$$
$$w(\Phi(a_1, \dots, a_n)) = w(\Phi(a_{\sigma(1)}, \dots, a_{\sigma(m)}, a_{m+1}, \dots, a_n)),$$

and

$$|\mathcal{S}(\Phi(a_1, \dots, a_n))| = |\mathcal{S}(\Phi(a_{\sigma(1)}, \dots, a_{\sigma(m)}, a_{m+1}, \dots, a_n))|,$$

so

$$w\Big(\bigvee_{\substack{\Phi(a_1,\dots,a_n) \vDash \Theta(a_1,\dots,a_m) \\ |\mathcal{S}(\Phi(a_1,\dots,a_n))| = t}} \Phi(a_1, \dots, a_n) \Big)$$
$$= w\Big(\bigvee_{\substack{\Phi(a_{\sigma(1)},\dots,a_{\sigma(m)},a_{m+1},\dots,a_n) \vDash \Theta(a_{\sigma(1)},\dots,a_{\sigma(m)}) \\ |\mathcal{S}(\Phi(a_{\sigma(1)},\dots,a_{\sigma(m)},a_{m+1},\dots,a_n))| = t}} \Phi(a_{\sigma(1)}, \dots, a_{\sigma(m)}, a_{m+1}, \dots, a_n) \Big)$$
$$= w\Big(\bigvee_{\substack{\Phi(a_1,\dots,a_n) \vDash \Theta(a_{\sigma(1)},\dots,a_{\sigma(m)}) \\ |\mathcal{S}(\Phi(a_1,\dots,a_n))| = t}} \Phi(a_1, \dots, a_n) \Big)$$

and (265) gives that

$$z^{[t]}(\Theta(a_1, \dots, a_m)) = z^{[t]}(\Theta(a_{\sigma(1)}, \dots, a_{\sigma(m)})).$$

By Lemma 27.2 $w^{[t]}$ satisfies Sx since if state descriptions $\Theta(a_1, \dots, a_m)$, $\Phi(a_1, \dots, a_m)$ have the same spectra then they will have the same range and multiplicity of spectra in their extensions to $a_1, a_2, \dots, a_n$, so (265) will only depend on the spectrum of Θ.

Let $\varepsilon > 0$. To show that $w^{[t]}$ is t-heterogeneous we need to show that for large m

$$z^{[t]}\Big(\bigvee_{|\mathcal{S}(\Phi(a_1,\dots,a_m))|=t} \Phi(a_1, \dots, a_m)\Big) \tag{266}$$

is within ε of η_t. For this it suffices to show that for $n > m$

$$w\Big(\bigvee_{\substack{|\mathcal{S}(\Phi(a_1,\dots,a_n))|=t \\ |\mathcal{S}(\Phi(a_1,\dots,a_n)[a_1,\dots,a_m])|=t}} \Phi(a_1, \dots, a_n)\Big) \tag{267}$$

is within ε of

$$w\Big(\bigvee_{|\mathcal{S}(\Phi(a_1,\dots,a_n))|=t} \Phi(a_1, \dots, a_n)\Big), \tag{268}$$

since (268) tends to η_t as $n \to \infty$.

Consider a state description $\Phi(a_1, \dots, a_n)$ which appears in the disjunction in (268) but not (267), so

$$|\mathcal{S}(\Phi(a_1, \dots, a_n))| = t, \quad |\mathcal{S}(\Phi(a_1, \dots, a_n)[a_1, \dots, a_m])| < t.$$

This means that $w(\Phi(a_1, \dots, a_n))$ has been lost in passing from

$$w\Big(\bigvee_{|\mathcal{S}(\Psi(a_1,\dots,a_m))|<t} \Psi(a_1, \dots, a_m)\Big) \quad \text{to} \quad w\Big(\bigvee_{|\mathcal{S}(\Psi(a_1,\dots,a_n))|<t} \Psi(a_1, \dots, a_n)\Big).$$

These terms are decreasing (to limit $\sum_{s<t} \eta_s$) so taking m large enough the sum of all such losses must be less that ε. It follows that (267), (268) are also within ε of each other, as required.

Finally consider

$$z^{[0]} = w - \sum_{t>0} \eta_t w^{[t]} = w - \sum_{t>0} z^{[t]}$$

where $w^{[t]}$ is an arbitrary probability function on SL if $\eta_t = 0$. For a state description $\Theta(a_1, \dots, a_m)$ and $n \geq m$,

$$w(\Theta(a_1, \dots, a_m)) = \sum_{t>0} w\Big(\bigvee_{\substack{\Phi(a_1,\dots,a_n) \vDash \Theta(a_1,\dots,a_m) \\ |\mathcal{S}(\Phi(a_1,\dots,a_m))|=t}} \Phi(a_1, \dots, a_n)\Big)$$

so for $k \in \mathbb{N}$

$$\begin{aligned}
w(\Theta(a_1,\dots,a_m)) &= \lim_{n\to\infty} \sum_{t>0} w\Big(\bigvee_{\substack{\Phi(a_1,\dots,a_n)\vDash\Theta(a_1,\dots,a_m)\\ |\mathcal{S}(\Phi(a_1,\dots,a_m))|=t}} \Phi(a_1,\dots,a_n)\Big)\\
&\geq \lim_{n\to\infty} \sum_{t=1}^{k} w\Big(\bigvee_{\substack{\Phi(a_1,\dots,a_n)\vDash\Theta(a_1,\dots,a_m)\\ |\mathcal{S}(\Phi(a_1,\dots,a_m))|=t}} \Phi(a_1,\dots,a_n)\Big)\\
&= \sum_{t=1}^{k} \lim_{n\to\infty} w\Big(\bigvee_{\substack{\Phi(a_1,\dots,a_n)\vDash\Theta(a_1,\dots,a_m)\\ |\mathcal{S}(\Phi(a_1,\dots,a_m))|=t}} \Phi(a_1,\dots,a_n)\Big)\\
&= \sum_{t=1}^{k} z^{[t]}(\Theta(a_1,\dots,a_m))\\
&= \sum_{t=1}^{k} \eta_t w^{[t]}(\Theta(a_1,\dots,a_m)). \qquad (269)
\end{aligned}$$

Hence $z^{[0]}$ takes non-negative values on state descriptions, in particular then taking the state description $\top$,

$$\eta_0 = 1 - \sum_{i=1}^{\infty} \eta_i \geq 0.$$

If $\eta_0 = 0$ then it follows from (269) that $w = \sum_{t>0} \eta_t w^{[t]}$ from which the required representation follows.

If $\eta_0 > 0$ then because of the linear form of (P1–3) it is easy to see that $w^{[0]} = \eta_0^{-1} z^{[0]}$ is a probability function and from (269)

$$w = \eta_0 w^{[0]} + \sum_{t>0} \eta_t w^{[t]}, \qquad (270)$$

this identity also ensuring that $w^{[0]}$ satisfies Ex and Sx.

Finally in this case of $\eta_0 > 0$, $w^{[0]}$ is homogeneous. To see this notice that for $t > 0$

$$\begin{aligned}
\lim_{m\to\infty} \sum_{|\mathcal{S}(\Theta(a_1,\dots,a_m))|=t} w(\Theta(a_1,\dots,a_m)) &= \eta_t\\
&= \eta_t \lim_{m\to\infty} \sum_{|\mathcal{S}(\Theta(a_1,\dots,a_m))|=t} w^{[t]}(\Theta(a_1,\dots,a_m))
\end{aligned}$$

since $w^{[t]}$ is t-heterogeneous when $\eta_t > 0$. Hence from (270),

$$\begin{aligned} 0 \leq \eta_0 \sum_{|\mathcal{S}(\Theta(a_1,\dots,a_m))|=t} & w^{[0]}(\Theta(a_1,\dots,a_m)) \\ \leq \sum_{|\mathcal{S}(\Theta(a_1,\dots,a_m))|=t} & w(\Theta(a_1,\dots,a_m)) - \\ & \eta_t \sum_{|\mathcal{S}(\Theta(a_1,\dots,a_m))|=t} w^{[t]}(\Theta(a_1,\dots,a_m)) \end{aligned}$$

and taking the limit of both sides as $m \to \infty$ shows as required that

$$\lim_{m\to\infty} \sum_{|\mathcal{S}(\Theta(a_1,\dots,a_m))|=t} w^{[0]}(\Theta(a_1,\dots,a_m)) = 0.$$

The uniqueness of the η_t, and thence $w^{[t]}$ when $\eta_t > 0$, follows directly since these can be captured from a representation of the form (264) by applying the above procedure. $\dashv$

The Ladder Theorem 30.2 allows us to represent a probability function satisfying Sx as a convex sum of a homogeneous and heterogeneous probability functions. What we shall do in the next chapters is go further down this path by giving representation theorems for homogeneous and heterogeneous probability functions, and for this it will turn out that the $u^{\bar{p},L}$, $v^{\bar{p},L}$ will provide the key building blocks.

Chapter 31

REPRESENTATION THEOREMS FOR Sx

The aim of this chapter is to prove Representation Theorems for homogeneous probability functions (Theorem 31.10), heterogeneous probability functions (Theorem 31.11) and general probability functions satisfying Sx (Theorem 31.12) on a not purely unary language L. These results were originally given in [77], [110] by direct proofs not involving the elegant methods of nonstandard analysis which, as in Chapter 25, we shall again adopt here.[100]

We first need to introduce some notation. For $n \in \mathbb{N}$ we define $Spec(n)$ to be the set of spectra on n, that is $Spec(n)$ contains multisets $\{n_1, \dots, n_s\}$ with the $n_i \in \mathbb{N}^+$ and $\sum_{i=1}^{s} n_i = n$. We shall use $\tilde{n}, \tilde{m}, \dots$ for spectra. Let $n, m \in \mathbb{N}$, $m < n$, $\tilde{m} \in Spec(m)$ and $\tilde{n} \in Spec(n)$. By virtue of Lemma 27.2 we can unambiguously define[101] $\mathcal{N}(\tilde{m}, \tilde{n})$ to be the number of state descriptions for $a_1, a_2, \dots, a_n$ with spectrum $\tilde{n}$ that extend some/any fixed state description $\Theta(a_1, \dots, a_m)$ with spectrum $\tilde{m}$.

Let w be a probability function on SL satisfying Sx, and let $\Theta(a_1, \dots, a_m)$ be a state description. For $n > m$ we have

$$w(\Theta(a_1, \dots, a_m)) = \sum_{\Psi(a_1,\dots,a_n) \vDash \Theta(a_1,\dots a_m)} w(\Psi(a_1, \dots, a_n)).$$

We can group the Ψ and make the above

$$w(\Theta(a_1, \dots, a_m)) = \sum_{\tilde{n} \in Spec(n)} \sum_{\substack{\Psi \vDash \Theta \\ \mathcal{S}(\Psi) = \tilde{n}}} w(\Psi(a_1, \dots, a_n)).$$

Writing $w(\tilde{n})$ for $w(\Psi(a_1, \dots, a_n))$ where $\Psi(a_1, \dots, a_n)$ is some/any state description with $\mathcal{S}(\Psi) = \tilde{n}$, and assuming that $\mathcal{S}(\Theta) = \tilde{m}$ we get

$$w(\Theta(a_1, \dots, a_m)) = w(\tilde{m}) = \sum_{\tilde{n} \in Spec(n)} \mathcal{N}(\tilde{m}, \tilde{n}) w(\tilde{n})$$

---

[100] In Chapter 33 we give a direct, elementary, proof of a restricted version of the representation result for t-heterogeneous probability functions.

[101] More precisely, we should indicate that these are state descriptions in SL but since *in this chapter* the language L is fixed, we omit it here.

$$= \sum_{\tilde{n} \in Spec(n)} \frac{\mathcal{N}(\tilde{m}, \tilde{n})}{\mathcal{N}(\emptyset, \tilde{n})} \mathcal{N}(\emptyset, \tilde{n}) w(\tilde{n})$$

(where $\mathcal{N}(\emptyset, \tilde{n})$ is the number of all state descriptions with spectrum $\tilde{n}$).

Working again in a nonstandard extension U^* (see page 183), take $\nu \in \mathbb{N}^*$ nonstandard. We have

$$w^*(\Theta(a_1, \dots, a_m)) = w^*(\tilde{m}) = \sum_{\tilde{\nu} \in Spec^*(\nu)} \frac{\mathcal{N}^*(\tilde{m}, \tilde{\nu})}{\mathcal{N}^*(\emptyset, \tilde{\nu})} \mathcal{N}^*(\emptyset, \tilde{\nu}) w^*(\tilde{\nu}) . \tag{271}$$

Let $\tilde{\nu} \in Spec^*(\nu)$ and for $m \leq \nu$ and a state description $\Theta(a_1, \dots, a_m)$ with spectrum $\tilde{m} \in Spec^*(m)$ define

$$U_{\tilde{\nu}}(\Theta(a_1, \dots, a_m)) \ = \ \frac{\mathcal{N}^*(\tilde{m}, \tilde{\nu})}{\mathcal{N}^*(\emptyset, \tilde{\nu})} .$$

These values are non-negative, $U_{\tilde{\nu}}(\top) = 1$, and for any state description $\Theta(a_1, \dots, a_m)$ with $m < \nu$ we have

$$U_{\tilde{\nu}}(\Theta(a_1, \dots, a_m)) = \sum_{\Psi(a_1, \dots, a_{m+1}) \vDash \Theta(a_1, \dots, a_m)} U_{\tilde{\nu}}(\Psi(a_1, \dots, a_{m+1}))$$

since

$$\begin{aligned} \sum_{\Psi(a_1, \dots, a_{m+1}) \vDash \Theta(a_1, \dots, a_m)} & U_{\tilde{\nu}}(\Psi(a_1, \dots, a_{m+1})) \\ &= \sum_{\tilde{k} \in Spec(m+1)} \mathcal{N}^*(\tilde{m}, \tilde{k}) \frac{\mathcal{N}^*(\tilde{k}, \tilde{\nu})}{\mathcal{N}^*(\emptyset, \tilde{\nu})} \\ &= \frac{\sum_{\tilde{k} \in Spec(m+1)} \mathcal{N}^*(\tilde{m}, \tilde{k}) \mathcal{N}^*(\tilde{k}, \tilde{\nu})}{\mathcal{N}^*(\emptyset, \tilde{\nu})} = \frac{\mathcal{N}^*(\tilde{m}, \tilde{\nu})}{\mathcal{N}^*(\emptyset, \tilde{\nu})} . \end{aligned}$$

It follows that (34) holds for ${}^\circ U_{\tilde{\nu}}$ and hence it extends to a probability function on SL. It is easily checked that ${}^\circ U_{\tilde{\nu}}$ satisfies Ex and Sx.

Loeb measure theory allows us to conclude from (271) that for some σ-additive measure τ on $Spec^*(\nu)$ we have

$$w(\Theta(a_1, \dots, a_m)) = w(\tilde{m}) = \int_{Spec^*(\nu)} {}^\circ U_{\tilde{\nu}}(\Theta(a_1, \dots, a_m)) \, d\tau(\tilde{\nu}). \tag{272}$$

It will be convenient at this point to adopt a convention by which we also treat spectra as vectors, with the entries listed in decreasing order. Hence writing $\tilde{\nu} = \langle \nu_1, \dots, \nu_t \rangle$ means that $\tilde{\nu}$ is the multiset $\{\nu_1, \dots, \nu_t\}$ and $\nu_1 \geq \nu_2 \geq \dots \geq \nu_t$. We continue writing 1_t for the spectrum consisting of t 1's.

PROPOSITION 31.1. *Let $v \in \mathbb{N}^*$ be nonstandard and let $\tilde{v} = \langle v_1, \dots, v_t \rangle \in Spec^*(v)$. Let $\bar{p} = \langle p_0, p_1, p_2, \dots \rangle$ with $p_i = {}^\circ(v_i/v)$ for $i \in \mathbb{N} \cap \{1, 2, \dots, t\}$, $p_i = 0$ for $i \in \mathbb{N}$, $i > t$, and let $p_0 = 1 - \sum_{i \in \mathbb{N}^+} p_i$ in the case that t is nonstandard, $p_0 = 0$ otherwise. Then*

(i) *If t is nonstandard then for all standard state descriptions $\Theta(a_1, \dots, a_m)$ we have*

$$^\circ U_{\tilde{v}}(\Theta(a_1, \dots, a_m)) = u^{\bar{p},L}(\Theta(a_1, \dots, a_m)).$$

(ii) *If $t \in \mathbb{N}$ and $p_t \neq 0$ then for all standard state descriptions $\Theta(a_1, \dots, a_m)$ we have*

$$^\circ U_{\tilde{v}}(\Theta(a_1, \dots, a_m)) = v^{\bar{p},L}(\Theta(a_1, \dots, a_m)).$$

(iii) *If $t \in \mathbb{N}$ and $p_t = 0$ then for any $m \in \mathbb{N}$ and $\Theta(a_1, \dots, a_m)$ such that $|\mathcal{S}(\Theta(a_1, \dots, a_m))| \geq t$,*

$$^\circ U_{\tilde{v}}(\Theta(a_1, \dots, a_m)) = 0.$$

Before proving this proposition, we need some lemmas. For the moment we return to working in the standard universe, though of course we also have the U^* versions of the results we derive. First of all we derive convenient expressions for $u^{\bar{p},L}(\Theta(a_1, \dots, a_m))$ and $v^{\bar{p},L}(\Theta(a_1, \dots, a_m))$ (and prove a useful approximation lemma). Then we turn to $\mathcal{N}(\tilde{m}, \tilde{n})/\mathcal{N}(\emptyset, \tilde{n})$ and express it in a similar fashion in order to see, upon returning to U^*, that the proposition holds.

Let $\bar{p} = \langle p_0, p_1, p_2, \dots \rangle \in \mathbb{B}$ and let $\Theta(a_1, \dots, a_m)$ be a state description. Recall that $\mathcal{E}(\Theta)$ denotes the set of equivalence classes of Θ and $E \trianglerighteq \mathcal{E}(\Theta)$ means that E is a partition of $\{a_1, \dots, a_m\}$ which refines $\mathcal{E}(\Theta)$.

For $E \trianglerighteq \mathcal{E}(\Theta)$, $E = \{E_1, \dots, E_{|E|}\}$ fix representatives $e_1, \dots, e_{|E|}$ of the $|E|$ classes in E, let $Z(E)$ be the set of mappings ρ from $\{1, \dots, |E|\}$ to $\{0, 1, 2, \dots\}$ that satisfy

$$\text{if } \rho(i) = \rho(j) \text{ then } i = j \text{ or } \rho(i) = \rho(j) = 0, \tag{273}$$

$$\text{if } \rho(i) = 0 \text{ then } |E_i| = 1, \tag{274}$$

and let $Z^t(E)$ be the set of injective mappings from $\{1, \dots, |E|\}$ to $\{1, 2, \dots, t\}$.

Let $SD^L(k)$ stand for the number of state descriptions of L on k constants (up to logical equivalence).

LEMMA 31.2. *With the above notation we have*

$$u^{\bar{p},L}(\Theta) = \sum_{E \trianglerighteq \mathcal{E}(\Theta)} \sum_{\rho \in Z(E)} \frac{1}{SD^L(|E|)} \prod_{i=1}^{|E|} p_{\rho(i)}^{|E_i|}. \tag{275}$$

We remark that if $p_0 = 0$ *and* $p_{t+1} = 0$ *then* $Z(E)$ *in the above formula can be replaced by* $Z^t(E)$. *For such* $\bar{p}$, $v^{\bar{p},L}$ *is also defined and we have*

$$v^{\bar{p},L}(\Theta) = \sum_{E \trianglerighteq \mathcal{E}(\Theta)} \frac{\mathcal{N}(\tilde{r}_E, 1_t)}{\mathcal{N}(\emptyset, 1_t)} \sum_{\rho \in Z^t(E)} \prod_{i=1}^{|E|} p_{\rho(i)}^{|E_i|} \tag{276}$$

where $\tilde{r}_E$ *is the multiset* $\{r_1, \dots, r_{|\mathcal{S}(\Theta)|}\}$ *with the* r_i *being the numbers of classes into which* E *partitions the classes of* $\mathcal{E}(\Theta)$.

PROOF. Recall from (244) that $u^{\bar{p},L}(\Theta)$ is defined by

$$u^{\bar{p},L}(\Theta) = \sum_{\substack{\vec{c} \in \mathbb{N}^m \\ \Theta \in \mathcal{C}(\vec{c}, \vec{a})}} |\mathcal{C}(\vec{c}, \vec{a})|^{-1} \prod_{s=1}^{m} p_{c_s}.$$

For $\vec{c} \in \mathbb{N}^m$ such that $\Theta \in \mathcal{C}(\vec{c}, \vec{a})$ define the equivalence relation $\sim_{\vec{c}}$ on $\{ 1, 2, \dots, m \}$ by

$$i \sim_{\vec{c}} j \iff c_i = c_j \neq 0 \text{ or } c_i = 0 \text{ and } i = j\,.$$

Each such equivalence $\sim_{\vec{c}}$ corresponds to a partition of $\{a_1, \dots, a_m\}$ that is a refinement of $\mathcal{E}(\Theta)$. Collecting all the $\vec{c}$ that correspond to the same $E \trianglerighteq \mathcal{E}(\Theta)$ produces (275) since for each of them $|\mathcal{C}(\vec{c}, \vec{a})| = SD^L(|E|)$, and $\prod_{s=1}^m p_{c_s}$ is $\prod_{i=1}^{|E|} p_{\rho(i)}^{|E_i|}$ for the mapping ρ that sends i to the c_{s_i} where $e_i = a_{s_i}$.

The last part follows similarly from

$$v^{\bar{p},L}(\Theta(a_1, a_2, \dots, a_m)) = \sum_{\vec{c} \in \{1,2,\dots,t\}^m} \mathcal{G}(\vec{c}, \Theta) \prod_{s=1}^{m} p_{c_s}.$$

(see (263)) upon noticing that $\mathcal{G}(\vec{c}, \Theta)$ equals $\mathcal{N}(\tilde{r}_E, 1_t)/\mathcal{N}(\emptyset, 1_t)$ for the corresponding E. ⊣

Continuing with the same notation,

LEMMA 31.3. *Let* $E \trianglerighteq \mathcal{E}(\Theta)$. *Let* $g > 0$ *and* $z \in \mathbb{N}$ *be such that* $p_j \leq g$ *for* $j > z$. *Let* $Z_z(E)$ *stand for the set of those* ρ *from* $Z(E)$ *that moreover satisfy*

$$\text{if } \rho(i) > z \text{ then } |E_i| = 1. \tag{277}$$

Then

$$\Big| \sum_{\rho \in Z(E)} \prod_{i=1}^{|E|} p_{\rho(i)}^{|E_i|} - \sum_{\rho \in Z_z(E)} \prod_{i=1}^{|E|} p_{\rho(i)}^{|E_i|} \Big| \leq |E| g.$$

Similarly when we replace $p_{\rho(i)}^{|E_i|}$ *by some* $f(p_{\rho(i)}, |E_i|)$ *with* f *satisfying*

$$f(p, d) \leq p^d.$$

Proof. Assume that i_0 is such that $|E_{i_0}| > 1$. Then the sum, over all distinct choices of the ρ with $\rho(i_0) > z$, of

$$\prod_{i=1}^{|E|} p_{\rho(i)}^{|E_i|}$$

is less than g since it is bounded by

$$\Big(\sum_{j>z} p_j^{|E_{i_0}|}\Big) \prod_{i\in\{1,2,\dots,|E|\}-\{i_0\}} \Big(\sum_{j=0}^{\infty} p_j\Big)^{|E_i|}$$

and

$$\sum_{j=0}^{\infty} p_j = 1 \text{ and } \Big(\sum_{j>z} p_j^{|E_{i_0}|}\Big) \le \Big(\sum_{j>z} p_j^2\Big) \le g\Big(\sum_{j>z} p_j\Big) \le g.$$

The lemma follows since there are at most $|E|$ choices of i_0. ⊣

We now derive two, rather technical, expressions for $\mathcal{N}(\emptyset,\tilde{n})$, $\mathcal{N}(\tilde{m},\tilde{n})$ where $n, m \in \mathbb{N}$ and $\tilde{m} \in Spec(m)$, $\tilde{n} \in Spec(n)$.

Lemma 31.4. *Let $m < n$ and let $\tilde{m} = \langle m_1,\dots,m_s\rangle \in Spec(m)$ and $\tilde{n} = \langle n_1,\dots,n_t\rangle \in Spec(n)$. Let q_i be the number entries in $\tilde{n}$ equal to i. Assume that the state description $\Theta(a_1,\dots,a_m)$ has spectrum $\tilde{m}$. Then*

$$\mathcal{N}(\emptyset,\tilde{n}) = \frac{1}{\prod_{i=1}^{n} q_i!}\binom{n}{\tilde{n}}\mathcal{N}(\emptyset, 1_t), \tag{278}$$

$$\mathcal{N}(\tilde{m},\tilde{n}) = \frac{1}{\prod_{i=1}^{n} q_i!}\sum_{E \trianglerighteq \mathcal{E}(\Theta)}\ \sum_{\rho\in Z^t(E)} \binom{n-m}{\tilde{n}_\rho - E, \tilde{n}_{rest}}\mathcal{N}(\tilde{r}_E, 1_t), \tag{279}$$

and hence

$$\frac{\mathcal{N}(\tilde{m},\tilde{n})}{\mathcal{N}(\emptyset,\tilde{n})} = \sum_{E \trianglerighteq \mathcal{E}(\Theta)} \frac{\mathcal{N}(\tilde{r}_E, 1_t)}{\mathcal{N}(\emptyset, 1_t)} \sum_{\rho\in Z^t(E)} \binom{n-m}{\tilde{n}_\rho - E, \tilde{n}_{rest}}\binom{n}{\tilde{n}}^{-1}, \tag{280}$$

where

- $\binom{n}{\tilde{n}}$ *stands for* $\dfrac{n!}{\prod_{i=1}^{t} n_i!}$.
- $\tilde{r}_E$*, as earlier, is the multiset* $\{r_1,\dots,r_{|\mathcal{S}(\Theta)|}\}$ *where the r_i are numbers of classes into which E partitions the classes of $\mathcal{E}(\Theta)$.*
- *the expression* $\binom{n-m}{\tilde{n}_\rho - E, \tilde{n}_{rest}}$ *stands for*

$$\frac{(n-m)!}{\prod_{i=1}^{|E|}(n_{\rho(i)} - |E_i|)!\prod_{k\notin Im(\rho)} n_k!}$$

with the convention that this is 0 if some $n_{\rho(i)} < |E_i|$.[102]

[102] Here $Im(\rho)$ is the image of ρ.

PROOF. Any state description $\Phi(a_1, \dots, a_n)$ with spectrum $\tilde{n}$ is uniquely characterized by $\mathcal{E}(\Phi) = \{B_1, \dots, B_t\}$ together with the state description $\Phi[b_1, b_2, \dots, b_t]$ (with spectrum 1_t) where $b_i = \min B_i$. There are $\left(\prod_{i=1}^n q_i!\right)^{-1} \binom{n}{\tilde{n}}$ different $\{B_1, \dots, B_t\}$. Hence formula (278).

For the formula (279) assume first that all the n_j are different and consider a state description $\Phi(a_1, \dots, a_n)$ with spectrum $\tilde{n}$, extending Θ. Let $\mathcal{E}(\Phi) = \{B_1, \dots, B_t\}$. The set of non-empty $B_i \cap \{a_1, \dots, a_m\}$ must be a refinement E of $\mathcal{E}(\Theta)$. Note that state descriptions corresponding to different E's must be different. We continue with the aim of counting those state descriptions corresponding to a fixed E. Each of them is *uniquely* characterized by a state description with spectrum 1_t such that its restriction to the $|E|$ representatives of the classes B_i that have nonempty intersection with $\{a_1, \dots, a_m\}$ is Θ restricted to these representatives, together with an injective mapping

$$\rho : \{1, 2, \dots, |E|\} \to \{1, 2, \dots, t\}$$

and a partition of $\{m+1, \dots, n\}$ into t classes (some possibly empty) of sizes $n_1^-, \dots, n_t^-$ where

$$n_k^- = \begin{cases} n_{\rho(i)} - |E_i| & \text{if } k = \rho(i), \\ n_k & \text{otherwise.} \end{cases}$$

The number of such mappings and partitions is

$$\sum_{\rho \in Z^t(E)} \binom{n-m}{\tilde{n}_\rho - E, \tilde{n}_{rest}}.$$

This gives the required result in the case that all the n_j are different. Otherwise let q_i be the number of n_j equal to i. Then any permutation σ of the j such that $n_j = i$ (and the identity on the remaining j) will give $\sigma\rho$ and instances of $\tilde{n}_{rest}$ which result in the same state descriptions with spectrum $\tilde{n}$ extending Θ as the original ρ. In other words the expression (279) without the factor $(\prod_{i=1}^n q_i!)^{-1}$ would be counting a particular state description with spectrum $\tilde{n}$ extending Θ to multiplicity $\prod_{i=1}^n q_i!$ times. Hence this divisor in (279). ⊣

LEMMA 31.5. *Continuing to use the notation of Lemma* 31.4, *let* $E \trianglerighteq \mathcal{E}(\Theta)$ *and assume that* $2m^2 < n$. *Then*

$$\left| \sum_{\rho \in Z^t(E)} \binom{n-m}{\tilde{n}_\rho - E, \tilde{n}_{rest}} \binom{n}{\tilde{n}}^{-1} - \sum_{\rho \in Z^t(E)} \prod_{i=1}^{|E|} \left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|} \right| \le 8m^2 n^{-\frac{1}{2}}.$$

PROOF. Let E be fixed. We have

$$\sum_{\rho\in Z^t(E)} \binom{n-m}{\tilde n_\rho - E, \tilde n_{rest}} \binom{n}{\tilde n}^{-1}$$
$$= \frac{(n-m)!}{n!} \sum_{\rho\in Z^t(E)} \prod_{i=1}^{|E|} n_{\rho(i)}(n_{\rho(i)}-1)\dots(n_{\rho(i)}-|E_i|+1)$$
$$= \frac{n^m(n-m)!}{n!} \sum_{\rho\in Z^t(E)} \prod_{i=1}^{|E|} \frac{n_{\rho(i)}}{n}\left(\frac{n_{\rho(i)}}{n}-\frac{1}{n}\right)\dots\left(\frac{n_{\rho(i)}}{n}-\frac{|E_i|-1}{n}\right). \tag{281}$$

Let z be such that

$$n_j \geq n^{\frac{1}{2}} \text{ for } 1 \leq j \leq z \text{ and } n_j < n^{\frac{1}{2}} \text{ for } j > z.$$

If $\rho \in Z_z^t(E)$ then for $1 \leq i \leq |E|$ we have $|E_i| = 1$ or $n_{\rho(i)} \geq n^{\frac{1}{2}}$ so

$$(1-|E_i|\,n^{-\frac{1}{2}})^{|E_i|}\left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|} \leq \frac{n_{\rho(i)}}{n}\left(\frac{n_{\rho(i)}}{n}-\frac{1}{n}\right)\dots\left(\frac{n_{\rho(i)}}{n}-\frac{|E_i|-1}{n}\right)$$
$$\leq \left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|}$$

and hence (using the facts that $|E_i| \leq m$ and $\sum_{i=1}^{|E|} |E_i| = m$)

$$(1-m^2n^{-\frac{1}{2}}) \prod_{i=1}^{|E|}\left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|}$$
$$\leq (1-mn^{-\frac{1}{2}})^m \prod_{i=1}^{|E|}\left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|}$$
$$\leq \prod_{i=1}^{|E|} \frac{n_{\rho(i)}}{n}\left(\frac{n_{\rho(i)}}{n}-\frac{1}{n}\right)\dots\left(\frac{n_{\rho(i)}}{n}-\frac{|E_i|-1}{n}\right)$$
$$\leq \prod_{i=1}^{|E|}\left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|}.$$

Summing over $\rho \in Z_z^t(E)$ we find that

$$\Big| \sum_{\rho\in Z_z^t(E)} \prod_{i=1}^{|E|} \frac{n_{\rho(i)}}{n}\left(\frac{n_{\rho(i)}}{n}-\frac{1}{n}\right)\dots\left(\frac{n_{\rho(i)}}{n}-\frac{|E_i|-1}{n}\right) - \sum_{\rho\in Z_z^t(E)} \prod_{i=1}^{|E|}\left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|} \Big| \tag{282}$$

is less than

$$m^2 n^{-\frac{1}{2}} \sum_{\rho \in Z_z^t(E)} \prod_{i=1}^{|E|} \left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|}$$

and since

$$\sum_{\rho \in Z_z^t(E)} \prod_{i=1}^{|E|} \left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|} \leq \sum_{\rho \in Z^t(E)} \prod_{i=1}^{|E|} \left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|} \leq \left(\sum_{i=1}^{t} \frac{n_i}{n}\right)^m = 1\,, \tag{283}$$

it is less than $m^2 n^{-\frac{1}{2}}$. By Lemma 31.3 both sums in (282) change by at most $|E|n^{-\frac{1}{2}}$ when $Z_z^t(E)$ is replaced by $Z^t(E)$ so since $|E| \leq m$

$$\Bigg| \sum_{\rho \in Z^t(E)} \prod_{i=1}^{|E|} \frac{n_{\rho(i)}}{n} \left(\frac{n_{\rho(i)}}{n} - \frac{1}{n}\right) \ldots \left(\frac{n_{\rho(i)}}{n} - \frac{|E_i| - 1}{n}\right) - \sum_{\rho \in Z^t(E)} \prod_{i=1}^{|E|} \left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|} \Bigg|$$

is less than $(2m + m^2)n^{-\frac{1}{2}}$. Finally, multiplying the above difference by

$$\frac{n^m (n-m)!}{n!} = \frac{n^m}{n(n-1)\ldots(n-m+1)},$$

which is within $2m^2/n$ of 1 (and so less than 2), and noting that

$$\left| \frac{n^m (n-m)!}{n!} \sum_{\rho \in Z^t(E)} \prod_{i=1}^{|E|} \left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|} - \sum_{\rho \in Z^t(E)} \prod_{i=1}^{|E|} \left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|} \right|$$

is consequently less than $2m^2/n$, we obtain the lemma. ⊣

COROLLARY 31.6. *With the above notation and* $2m^2 < n$,

$$\left| \frac{\mathcal{N}(\tilde{m}, \tilde{n})}{\mathcal{N}(\emptyset, \tilde{n})} - \sum_{E \trianglerighteq \mathcal{E}(\Theta)} \frac{\mathcal{N}(\tilde{r}_E, 1_t)}{\mathcal{N}(\emptyset, 1_t)} \sum_{\rho \in Z^t(E)} \prod_{i=1}^{|E|} \left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|} \right| < 8 h_\Theta m^2 n^{-1/2}$$

where $h_\Theta = |\{E \mid E \trianglerighteq \mathcal{E}(\Theta)\}|$.

We remark that with $\bar{p}(\tilde{n}) = \langle 0, \frac{n_1}{n}, \ldots, \frac{n_t}{n}, 0, 0, \ldots \rangle$, by virtue of Lemma 31.2, the above amounts to

$$\left| \frac{\mathcal{N}(\tilde{m}, \tilde{n})}{\mathcal{N}(\emptyset, \tilde{n})} - v^{\bar{p}(\tilde{n}), L}(\Theta) \right| < 8 h_\Theta m^2 n^{-1/2}. \tag{284}$$

Lemma 31.7. *Assume that the language L is not purely unary. Let* $k, t \in \mathbb{N}$, $7, k < t$ *and* $\tilde{r} \in Spec(k)$. *Then*

$$\left| \frac{\mathcal{N}(\tilde{r}, 1_t)}{\mathcal{N}(\emptyset, 1_t)} - \frac{1}{SD(k)} \right| \leq t^2 2^{k-t}.$$

Proof. Let $\Psi(a_1, \dots, a_k)$ be a state description with spectrum $\tilde{r}$. For given $i < j \leq t$ the proportion of extensions $\Phi(a_1, \dots, a_t)$ of Ψ such that $a_i \sim_\Phi a_j$ is at most 2^{k-t} (since the language is not purely unary) and hence the proportion of extensions having spectrum other than 1_t is less than $t^2 2^{k-t}$. Denoting the number of state descriptions for $a_1, \dots, a_t$ extending Ψ by $Ext_t(\Psi)$, we have

$$1 \geq \frac{\mathcal{N}(\tilde{r}, 1_t)}{Ext_t(\Psi)} \geq 1 - t^2 2^{k-t}.$$

Similarly, the probability that a state description $\Phi(a_1, \dots, a_t)$ has spectrum other than 1_t is less or equal to $t^2 2^{-t}$ so

$$1 \geq \frac{\mathcal{N}(\emptyset, 1_t)}{SD(t)} \geq 1 - t^2 2^{-t}$$

and hence (since $t > 7$, $t^2 2^{-t} < \frac{1}{2}$)

$$1 + 2t^2 2^{-t} \geq \frac{SD(t)}{\mathcal{N}(\emptyset, 1_t)} \geq 1.$$

Also notice that for any other state description $\Psi'(a_1, \dots, a_k)$ we must have $Ext_t(\Psi') = Ext_t(\Psi)$, since there will be the same number of free choices in each case, so, independently of the choice of Ψ, we must have

$$Ext_t(\Psi) = \frac{SD(t)}{SD(k)}.$$

The result follows from the above and

$$\frac{\mathcal{N}(\tilde{r}, 1_t)}{\mathcal{N}(\emptyset, 1_t)} = \frac{\mathcal{N}(\tilde{r}, 1_t)}{Ext_t(\Psi)} \frac{Ext_t(\Psi)}{SD(t)} \frac{SD(t)}{\mathcal{N}(\emptyset, 1_t)}. \qquad \dashv$$

Using the second inequality in (283), Corollary 31.6 and Lemma 31.7 we now obtain:

Corollary 31.8. *With the above notation and assumption on L and* $2m^2 < n$,

$$\left| \frac{\mathcal{N}(\tilde{m}, \tilde{n})}{\mathcal{N}(\emptyset, \tilde{n})} - \sum_{E \trianglerighteq \mathcal{E}(\Theta)} \sum_{\rho \in Z^t(E)} \frac{1}{SD^L(|E|)} \prod_{i=1}^{|E|} \left(\frac{n_{\rho(i)}}{n} \right)^{|E_i|} \right| < h_\Theta \left(8m^2 n^{-1/2} + t^2 2^{m-t} \right)$$

where $h_\Theta = |\{E \mid E \trianglerighteq \mathcal{E}(\Theta)\}|$.

Again, using Lemma 31.2, we see that with $\bar{p}(\tilde{n}) = \langle 0, \frac{n_1}{n}, \dots, \frac{n_t}{n}, 0, 0, \dots \rangle$ the above gives

$$\left| \frac{\mathcal{N}(\tilde{m}, \tilde{n})}{\mathcal{N}(\emptyset, \tilde{n})} - u^{\bar{p}(\tilde{n}),L}(\Theta) \right| < h_\Theta (8m^2 n^{-1/2} + t^2 2^{m-t}) . \tag{285}$$

For the next lemma we return again to the nonstandard universe U^*.

LEMMA 31.9. *Let $v \in \mathbb{N}^*$ be nonstandard, $\tilde{v} = \langle v_1, \dots, v_t \rangle \in Spec^*(v)$ and $\bar{p}(\tilde{v}) = \bar{s} = \langle 0, \frac{v_1}{v}, \dots, \frac{v_t}{v}, 0, 0, \dots \rangle$. Define $\bar{p} = \langle p_0, p_1, \dots \rangle \in \mathbb{B}$ by $p_i = {}^\circ\left(\frac{v_i}{v}\right)$ for $i \in \mathbb{N} \cap \{1, 2, \dots, t\}$, $p_i = 0$ for $i \in \mathbb{N}$, $i > t$, and $p_0 = 1 - \sum_{i \in \mathbb{N}^+} p_i$ in the case that t is nonstandard, $p_0 = 0$ otherwise.*

(i) *If t is nonstandard then for standard state descriptions we have*

$${}^\circ((u^*)^{\bar{s},L}(\Theta(a_1, \dots, a_m))) = u^{\bar{p},L}(\Theta(a_1, \dots, a_m)) .$$

(ii) *If $t \in \mathbb{N}$ and $p_t \neq 0$ then*

$${}^\circ((v^*)^{\bar{s},L}(\Theta(a_1, \dots, a_m))) = v^{\bar{p},L}(\Theta(a_1, \dots, a_m)) .$$

PROOF. (i) Let t be nonstandard and let $z \in \mathbb{N}$, so $z < t$. By Lemma 31.3, the expression (275) for $(u^*)^{\bar{s},L}(\theta(a_1, \dots, a_m))$ changes only by a standard multiple (which is dependent on Θ but not on z) of v_z/v if the $Z(E)$ are replaced by $Z_z(E)$. Let

$$\bar{s}^z = \left\langle \left(\sum_{i=z+1}^{t} \frac{v_i}{v} \right), \frac{v_1}{v}, \frac{v_2}{v}, \dots, \frac{v_z}{v}, 0, 0, \dots \right\rangle .$$

Consider a fixed E and write $\langle s_0, s_1, s_2, \dots \rangle$ for $\bar{s}$ and $\langle s_0^z, s_1^z, s_2^z, \dots, \rangle$ for $\bar{s}^z$. Then $\sum_{\rho \in Z_z(E)} \prod s_{\rho(i)}$ differs only by a standard multiple of n_z/n from $\sum_{\rho \in Z(E)} \prod s^z_{\rho(i)}$ since terms in $\sum_{\rho \in Z(E)} \prod s^z_{\rho(i)}$ where $\rho(i) = 0$ for some i are matched with terms in $\sum_{\rho \in Z_z(E)} \prod s_{\rho(i)}$ where the same $\rho(i) > z$, *except that* terms in which $\rho(i) = \rho(j) > z$ for some $i \neq j$ are missing. But as in the proof of Lemma 31.3 it is easily seen that the overall contribution thus outstanding in $\sum_{\rho \in Z_z(E)} \prod s_{\rho(i)}$ is less than a standard multiple of n_z/n independent of z.

Finally, ${}^\circ((u^*)^{\bar{s}^z,L}(\Theta(a_1, \dots, a_m))$ is equal to $u^{\langle {}^\circ s_0^z, {}^\circ s_1^z, \dots \rangle, L}(\Theta(a_1, \dots, a_m))$ and this in turn converges to $u^{\bar{p},L}(\Theta(a_1, \dots, a_m))$ by Lemma 29.4. The result follows.

(ii) In this case from Lemma 31.2 we get that

$$(v^*)^{\bar{s},L}(\Theta) = \sum_{E \trianglerighteq \mathcal{E}(\Theta)} \frac{\mathcal{N}(\tilde{r}_E, 1_t)}{\mathcal{N}(\emptyset, 1_t)} \sum_{\rho \in Z^t(E)} \prod_{i=1}^{|E|} s_{\rho(i)}^{|E_i|} , \tag{286}$$

$$v^{\bar{p},L}(\Theta) = \sum_{E \trianglerighteq \mathcal{E}(\Theta)} \frac{\mathcal{N}(\tilde{r}_E, 1_t)}{\mathcal{N}(\emptyset, 1_t)} \sum_{\rho \in Z^t(E)} \prod_{i=1}^{|E|} p_{\rho(i)}^{|E_i|}$$

These clearly have the same standard part. ⊣

Proof of Proposition 31.1. (i) Notice that in this case since t is nonstandard L cannot be purely unary. Hence we have (285) (which holds in U^* too of course with $n = \nu$) and using this we see that the standard parts of $(u^*)^{\bar{s},L}(\Theta(a_1, \ldots, a_m))$ and $U_{\tilde{\nu}}(\Theta(a_1, \ldots, a_m))$ are equal. The result follows by Lemma 31.9(i).

(ii) This follows analogously using (284) (in U^*) and Lemma 31.9(ii).

(iii) Using (284) (in U^*) and then taking standard parts we get (for a state description $\Theta(a_1, \ldots, a_m)$ with spectrum $\tilde{m}$)

$$ {}^\circ U_{\tilde{\nu}}(\Theta) \;=\; {}^\circ\left(\frac{\mathcal{N}^*(\tilde{m}, \tilde{\nu})}{\mathcal{N}^*(\emptyset, \tilde{\nu})}\right) \;=\; \sum_{E \trianglerighteq \mathcal{E}(\Theta)} \frac{\mathcal{N}(\tilde{r}_E, 1_t)}{\mathcal{N}(\emptyset, 1_t)} \sum_{\rho \in Z^t(E)} \prod_{i=1}^{|E|} p_{\rho(i)}^{|E_i|}. $$

If $\tilde{m}$ has t or more classes then so does any E with $E \trianglerighteq \mathcal{E}(\Theta)$ and there is either no $\rho \in Z^t(E)$ at all (if $|E| > t$) or $\rho(i) = t$ for some $i \in \{1, \ldots, |E|\}$ so ${}^\circ U_{\tilde{\nu}}(\Theta(a_1, \ldots, a_m)) = 0$. ⊣

We are now ready to give Representation Theorems for homogeneous, and for heterogeneous, probability functions. Note again that by the remark on page 58 we only need prove them for state description arguments.

Theorem 31.10. *Let w be a homogeneous*[103] *probability function on SL. Then there is a measure μ on the Borel subsets of*

$$ \mathbb{B}_\infty = \{\langle p_0, p_1, p_2, p_3, \ldots \rangle \in \mathbb{B} \mid p_0 > 0 \text{ or } p_j > 0 \text{ for all } j > 0\} $$

such that

$$ w = \int_{\mathbb{B}_\infty} u^{\bar{p},L} \, d\mu(\bar{p}). \tag{287} $$

Conversely given such a measure μ, w defined by (287) *is a homogeneous probability function on SL.*

Proof. Let w be homogeneous and $k \in \mathbb{N}^*$ nonstandard. Then, by homogeneity in U^*, for sufficiently large $\nu \in \mathbb{N}^*$,

$$ w^*\Big(\bigvee_{|\mathcal{S}(\Theta(a_1,\ldots,a_\nu))| \le k} \Theta(a_1, \ldots, a_\nu) \Big) \tag{288} $$

is infinitesimal. Hence in the derivation of (272) taking standard parts cuts out the contribution from spectra $\tilde{\nu}$ with $|\tilde{\nu}| \le k$ and we can restrict the range of integration to give instead

$$ w(\Theta(a_1, \ldots, a_m)) = \int_{\substack{\tilde{\nu} \in Spec^*(\nu) \\ |\tilde{\nu}| > k}} {}^\circ U_{\tilde{\nu}}(\Theta(a_1, \ldots, a_m)) \, d\tau(\tilde{\nu}). \tag{289} $$

For $\tilde{\nu} \in Spec(\nu)$, $|\tilde{\nu}| > k$ we have from Proposition 31.1(i) that

$$ {}^\circ U_{\tilde{\nu}}(\Theta(a_1, \ldots, a_m)) = u^{\bar{p},L}(\Theta(a_1, \ldots, a_m)) $$

with $\bar{p} \in \mathbb{B}$. Defining μ on the Borel subsets A of $\mathbb{B}$ by

[103] Recall that in this case L cannot be purely unary.

$$\mu(A) = \tau\{\tilde{\nu} \mid \langle 1 - \textstyle\sum_{i \in \mathbb{N}^+} {}^\circ\!\left(\frac{\nu_i}{\nu}\right), {}^\circ\!\left(\frac{\nu_1}{\nu}\right), {}^\circ\!\left(\frac{\nu_2}{\nu}\right), \dots \rangle \in A\},$$

(289) now clearly becomes,[104]

$$w = \int_{\mathbb{B}} u^{\bar{p},L} \, d\mu(\bar{p}). \tag{290}$$

For $0 < t \in \mathbb{N}$ let

$$\mathbb{B}_t = \{\langle p_0, p_1, p_2, p_3, \dots \rangle \in \mathbb{B} \mid p_0 = 0 \text{ and } p_t > 0 = p_{t+1}\}.$$

For $\bar{p} \in \mathbb{B}_t$ and for each $n \in \mathbb{N}$ we have

$$u^{\bar{p},L}\Big(\bigvee_{|\mathcal{S}(\Theta(a_1,\dots,a_n))| \le t} \Theta(a_1, \dots, a_n) \Big) = 1$$

so

$$w\Big(\bigvee_{|\mathcal{S}(\Theta(a_1,\dots,a_n))| \le t} \Theta(a_1, \dots, a_n) \Big) \ge \mu(\mathbb{B}_t)$$

for each $n \in \mathbb{N}$ and hence by homogeneity of w we must have $\mu(\mathbb{B}_t) = 0$. By countable additivity also $\mu(\mathbb{B} - \mathbb{B}_\infty) = 0$ and (287) follows.

The result in the other direction is immediate since for $\bar{p} \in \mathbb{B}_\infty$, $u^{\bar{p},L}$ is, by Proposition 30.1, homogeneous. ⊣

THEOREM 31.11. *Let w be a t-heterogeneous probability function on SL. Then there is a measure μ on the Borel subsets of*

$$\mathbb{B}_t = \{\langle p_0, p_1, p_2, p_3, \dots \rangle \in \mathbb{B} \mid p_0 = 0 \textit{ and } p_t > 0 = p_{t+1}\}$$

such that

$$w = \int_{\mathbb{B}_t} v^{\bar{p},L} \, d\mu(\bar{p}). \tag{291}$$

Conversely given such a measure μ, w defined by (291) *is a t-heterogeneous probability function on SL.*

PROOF. First assume that L is not purely unary. By t-heterogeneity in U^*, for nonstandard $\nu \in \mathbb{N}^*$,

$$w^*\Big(\bigvee_{|\mathcal{S}(\Theta(a_1,\dots,a_\nu))| \ne t} \Theta(a_1, \dots, a_\nu) \Big) \tag{292}$$

is infinitesimal. Hence as before

$$w(\Theta(a_1, \dots, a_m)) = \int_{\substack{\tilde{\nu} \in Spec^*(\nu) \\ |\tilde{\nu}| = t}} {}^\circ U_{\tilde{\nu}}(\Theta(a_1, \dots, a_m)) \, d\tau(\tilde{\nu}). \tag{293}$$

Now consider (293) for a state description $\Psi(a_1, \dots, a_k)$ in place of $\Theta(a_1, \dots, a_m)$. By Proposition 31.1 (iii), ${}^\circ U_{\tilde{\nu}}(\Psi(a_1, \dots, a_k)) = 0$ when $\frac{\nu_t}{\nu}$ is infinitesimal and $|\mathcal{S}(\Psi(a_1, \dots, a_k))| = t$. Hence for such state descriptions the integration can be restricted to the set of these $\tilde{\nu} \in Spec^*(\nu)$ for

[104] For example by Proposition 1, Chapter 15 of [126].

which $|\tilde{\nu}| = t$ and $\frac{\nu_t}{\nu}$ is not infinitesimal. By Proposition 31.1(ii) all such $\tilde{\nu}$ which upon taking standard parts give the same $\bar{p}$ satisfy

$$^\circ U_{\tilde{\nu}}(\Psi(a_1, \dots, a_k)) = v^{\bar{p},L}(\Psi(a_1, \dots, a_k)) .$$

Hence, as in the corresponding point in the proof of the previous theorem, for $\Psi(a_1, \dots, a_k)$ with $|\mathcal{S}(\Psi(a_1, \dots, a_k))| = t$,

$$w(\Psi(a_1, \dots, a_k)) = \int_{\mathbb{B}_t} v^{\bar{p},L}(\Psi(a_1, \dots, a_k))\, d\mu(\bar{p}) .$$

By virtue of the t-heterogeneity of w it follows that for any state description $\Theta(a_1, \dots, a_m)$ we have

$$\begin{aligned} w(\Theta(a_1, \dots, a_m)) &= \lim_{k \to \infty} w\Big(\bigvee_{\substack{\Psi(a_1,\dots,a_k) \vDash \Theta(a_1,\dots,a_m) \\ |\mathcal{S}(\Psi(a_1,\dots,a_k))| = t}} \Psi(a_1, \dots, a_k) \Big) \\ &= \lim_{k \to \infty} \int_{\mathbb{B}_t} v^{\bar{p},L}\Big(\bigvee_{\substack{\Psi(a_1,\dots,a_k) \vDash \Theta(a_1,\dots,a_m) \\ |\mathcal{S}(\Psi(a_1,\dots,a_k))| = t}} \Psi(a_1, \dots, a_k) \Big)\, d\mu(\bar{p}). \end{aligned}$$

Using Lebesgue's Dominated Convergence Theorem we can swap the limit and integral signs here and the t-heterogeneity of $v^{\bar{p},L}$ then yields the required result.

The converse result is straightforward. $\dashv$

Notice that in the case that L is purely unary this result is a special case of Theorem 14.1 using the observation from page 218 that the probability function $v_{\vec{c}}$ introduced in (86) is actually the same as $v^{\bar{c},L}$, where

$$\bar{c} = \langle 0, c_{i_1}, c_{i_2}, \dots, c_{i_{2^q}}, 0, 0, \dots \rangle$$

and $c_{i_1}, c_{i_2}, \dots, c_{i_{2^q}}$ list $c_1, c_2, \dots, c_{2^q}$ in non-increasing order of magnitude.

Putting these last two results together with the Ladder Theorem 30.2 we have:

Theorem 31.12. *Let w be a probability function on SL satisfying Sx. Then there is a measure μ on $\mathbb{B}$ such that*

$$w = \int_{\mathbb{B}_\infty} u^{\bar{p},L}\, d\mu(\bar{p}) + \int_{\mathbb{B} - \mathbb{B}_\infty} v^{\bar{p},L}\, d\mu(\bar{p}). \tag{294}$$

Conversely given such a measure μ, w defined by (294) *is a probability function on SL satisfying Sx.*

In the next three chapters we shall derive some further representation theorems for probability functions satisfying Sx. For the present however we shall give a corollary to Theorem 31.12 which amounts to saying that if our agent's chosen probability function satisfies Sx then for $1 < m \in \mathbb{N}$ s/he should give zero probability to $a_1, a_2, \dots, a_m$ all being indistinguishable from each other but distinguishable from $a_{m+1}, a_{m+2}, \dots$.[105]

[105]This result is related to classical results by Kingman, [66], [67], on partition structures and genetic diversity which lead to the conclusion that for $1 < m \in \mathbb{N}$ there are never exactly

COROLLARY 31.13. *Let w satisfy Sx and let $m > 1$. Then*

$$\lim_{n\to\infty} w\Big(\bigvee_{\Theta\in A_n} \Theta(a_1,\dots,a_n)\Big) = 0 \tag{295}$$

where A_n is the set of state descriptions $\Theta(a_1,\dots,a_n)$ for which $a_i \sim_\Theta a_j$ for $1 \le i,j \le m$ and $a_i \nsim_\Theta a_j$ for $1 \le i \le m < j \le n$.

PROOF. The idea of the proof is a typical application of a Representation Theorem such as Theorem 31.12: We first show the result for the building blocks $u^{\bar{p},L}$ and $v^{\bar{p},L}$ and then apply the theorem to obtain it for a general w satisfying Sx.

We start then by deriving the result for $u^{\bar{p},L}$. Assume for the moment that L is not purely unary. Fix $\varepsilon > 0$ and let $j > 0$ be such that

$$\sum_{i=j}^{\infty} p_i < \frac{\varepsilon}{4m}\,. \tag{296}$$

Let k be such that

$$\sum_{\substack{0<i<j\\ p_i>0}} (1-p_i)^k < \frac{\varepsilon}{4} \quad\text{and}\quad 2^{-k} < \frac{\varepsilon}{4} \tag{297}$$

and if $p_0 > 0$ then

$$k(1-p_0)^k < \varepsilon/4. \tag{298}$$

Consider $\langle c_1, c_2, \dots, c_n\rangle \in \mathbb{N}^n$ with $n \ge k^2 + m$. If some Θ is to be in $A_n \cap \mathcal{C}(\vec{c},\vec{a})$, where $\vec{a} = \langle a_1, a_2, \dots, a_n\rangle$ etc., then it must be that either all the $c_i = 0$ for $i = 1, 2, \dots, m$ or for some i in this range $c_i \neq 0$ and c_i does not also appear amongst $c_{m+1}, c_{m+2}, \dots, c_{m+k}$. From (296) and (297) the probability of picking $c_1, c_2, \dots, c_n$ such that this latter holds is at most $\varepsilon/2$.

So suppose that $c_i = 0$ for $i = 1, 2, \dots, m$ (note that this can only happen when $p_0 \neq 0$). Then the probability that some block

$$c_{m+ik+1}, c_{m+ik+2}, \dots, c_{m+(i+1)k},$$

where $0 \le i < k$, does not contain a 0 is at most $\varepsilon/4$ by (298). On the other hand if every such block does contain at least one 0 then the proportion of $\Phi(\vec{a}) \in \mathcal{C}(\vec{c},\vec{a})$ for which $a_i \sim_\Phi a_j$ for $1 \le i,j \le m$ is at most 2^{-k}. (It is at this point that we require $m > 1$.) Hence splitting up the sum

$$\sum_{\Theta\in A_n} \sum_{\substack{\vec{c}\in\mathbb{N}^n\\ \Theta\in\mathcal{C}(\vec{c},\vec{a})}} |\mathcal{C}(\vec{c},\vec{a})|^{-1} \prod_{i=1}^{n} p_{c_i} = \sum_{\vec{c}\in\mathbb{N}^n} \sum_{\substack{\Theta\in A_n\\ \Theta\in\mathcal{C}(\vec{c},\vec{a})}} |\mathcal{C}(\vec{c},\vec{a})|^{-1} \prod_{i=1}^{n} p_{c_i}$$

---

m examples of a plant species in an idealized infinite jungle, either there is only one or there are infinitely many.

into these cases shows that

$$u^{\bar{p},L}\Big(\bigvee_{\Theta\in A_n}\Theta(a_1,\dots,a_n)\Big)\le \varepsilon/2+\varepsilon/4+\sum_{\vec{c}\in\mathbb{N}^n}2^{-k}\prod_{i=1}^{n}p_{c_i}\le\varepsilon. \quad (299)$$

The cases for $u^{\bar{p},L}$ when L is purely unary and the case for $v^{\bar{p},L}$ clearly follow by similar, somewhat easier in fact, arguments.

Now let μ be as in Theorem 31.12. By appealing to countable additivity, for any $\varepsilon>0$ there is a $n\in\mathbb{N}^+$ such that the measure of the set P of $\bar{p}\in\mathbb{B}$ for which (299) holds for $u^{\bar{p},L}$ (if $\bar{p}\in\mathbb{B}_\infty$) or $v^{\bar{p},L}$ (if $\bar{p}\in\mathbb{B}-\mathbb{B}_\infty$) is at least $1-\varepsilon$. For this n then

$$\begin{aligned} w\Big(\bigvee_{\Theta\in A_n}\Theta(\vec{a})\Big) &\le \int_{\mathbb{B}_\infty\cap P} u^{\bar{p},L}\Big(\bigvee_{\Theta\in A_n}\Theta(\vec{a})\Big)\,d\mu(\bar{p})+ \\ &\quad + \int_{\mathbb{B}\cap P-\mathbb{B}_\infty} v^{\bar{p},L}\Big(\bigvee_{\Theta\in A_n}\Theta(\vec{a})\Big)\,d\mu(\bar{p})+\int_{\mathbb{B}-P}1\,d\mu(\bar{p}) \\ &\le \varepsilon+\varepsilon+\varepsilon=3\varepsilon, \end{aligned}$$

as required. ⊣

Notice that it certainly is possible for a probability function w to satisfy Sx and have, with probability 1, a_1 distinguishable from all the other constants, indeed the completely independent probability function c_∞^L ($=u^{\langle 1,0,0,\dots\rangle,L}$) achieves this. In fact it also follows by a similar argument to that for the above corollary that according to c_∞^L, with probability 1, any two constants will be distinguishable and moreover c_∞^L is the only probability function satisfying Sx with this property.

We actually have a somewhat stronger result:

Theorem 31.14. *Let $w\neq c_\infty^L$ satisfy Sx. Then there are $\rho,\gamma>0$ such that for all m eventually*

$$w\Big(\bigvee_{\substack{\Theta(a_1,\dots,a_m)\\ |\{a_i\,|\,a_i\sim_\Theta a_1\}|>\rho m}}\Theta(a_1,\dots,a_m)\Big)\ge\gamma.$$

Proof. Given the representation of w as in (294) there must be a $\delta>0$ such that $\mu(K_\delta)>0$ where

$$K_\delta=\{\bar{p}\in\mathbb{B}\mid p_1\ge\delta\},$$

otherwise $w=c_\infty^L$.

Suppose for the present that $\mu(K_\delta \cap \mathbb{B}_\infty) > 0$. Then for $\bar{p} \in K_\delta \cap \mathbb{B}_\infty$, $\rho = \delta/2$,

$$
\begin{aligned}
u^{\bar{p},L}\Big(\bigvee_{\substack{\Theta(a_1,\dots,a_m)\\ |\{a_i \mid a_i \sim_\Theta a_1\}|>\rho m}} \Theta(a_1,\dots,a_m)\Big) \\
&= \sum_{\substack{\Theta(a_1,\dots,a_m)\\ |\{a_i \mid a_i \sim_\Theta a_1\}|>\rho m}} \sum_{\substack{\vec{c}\in\mathbb{N}^m\\ \Theta\in\mathcal{C}(\vec{c},\vec{a})}} |\mathcal{C}(\vec{c},\vec{a})|^{-1} \prod_{s=1}^m p_{c_s} \\
&\geq \sum_{\substack{\vec{c}\in\mathbb{N}^m\\ |\{j \mid c_j=c_1=1\}|\geq\rho m}} \prod_{s=1}^m p_{c_s}
\end{aligned}
$$

since when $|\{j \mid c_j = c_1 = 1\}| \geq \rho m$ we must have that for any $\Theta(a_1,\dots, a_m) \in \mathcal{C}(\vec{c},\vec{a})$, $|\{a_i \mid a_i \sim_\Theta a_1\}| > \rho m$.

This last is just the sum of all $\prod_{s=1}^m p_{c_s}$ where $c_1 = 1$ and at least a proportion ρ of the c_s, $s = 1, 2, \dots, m$ are equal to 1. But this is $p_1(1 - X)$, where X is the probability of picking a sequence $c_2, c_3, \dots, c_m$ with at most $\rho(m-1)$ 1's in it. By Chebyshev's Inequality (see for example [7, Theorem 22.4])

$$X \leq \frac{p_1(1-p_1)}{(m-1)(p_1-\rho)^2} \leq \frac{4}{(m-1)\delta^2}$$

so there is a $\nu > 0$ such that for $\bar{p} \in K_\delta \cap \mathbb{B}_\infty$ and suitably large m, $p_1(1 - X) > \nu$.

Hence

$$
\begin{aligned}
w\Big(\bigvee_{\substack{\Theta(a_1,\dots,a_m)\\ |\{a_i \mid a_i \sim_\Theta a_1\}|>\rho m}} \Theta(a_1,\dots,a_m)\Big) \geq \\
&\geq \int_{K_\delta\cap\mathbb{B}_\infty} u^{\bar{p},L}\Big(\bigvee_{\substack{\Theta(a_1,\dots,a_m)\\ |\{a_i \mid a_i \sim_\Theta a_1\}|>\rho m}} \Theta(a_1,\dots,a_m)\Big) d\mu(\bar{p}) \\
&\geq \int_{K_\delta\cap\mathbb{B}_\infty} \nu \, d\mu(\bar{p}) = \nu\mu(K_\delta \cap \mathbb{B}_\infty) = \gamma
\end{aligned}
$$

as required.

Finally in the case that there is no such $\mu(K_\delta \cap \mathbb{B}_\infty) > 0$ then we can repeat the above argument with $\mathbb{B} - \mathbb{B}_\infty$ in place of $\mathbb{B}_\infty$ and $v^{\bar{p},L}$ in place of $u^{\bar{p},L}$. ⊣

Notice that in fact we have a slightly stronger result than the one stated by the theorem since the $a_i \sim_\Theta a_1$ that we count actually have the same colour as a_1 and so will be in a sense be 'forever after' indistinguishable from a_1.

Chapter 32

LANGUAGE INVARIANCE WITH Sx

The methods developed in the previous chapter can be used to yield another representation theorem, namely for probability functions satisfying Sx and Li. The appeal of Language Invariance as a most desirable principle, and the relative simplicity of the theorem make it a pleasing result.

THEOREM 32.1. *Let w^L be a probability function on SL satisfying Li with Sx. Then there is a measure μ on $\mathbb{B}$ such that*

$$w^L = \int_{\mathbb{B}} u^{\bar{p},L} \, d\mu(\bar{p}). \tag{300}$$

Conversely given such a measure μ, w^L defined by (300) *is a probability function on SL satisfying Li with Sx.*

PROOF. For the proof in the converse direction it suffices to recall that by Theorem 29.5 for any fixed $\bar{p}$ the $u^{\bar{p},\mathcal{L}}$ form a language invariant family, so $\int_{\mathbb{B}} u^{\bar{p},\mathcal{L}} \, d\mu(\bar{p})$ is a language invariant family containing w.

The proof in the forward direction closely resembles that of Theorem 31.10. We do not however need the assumption that L is not a purely unary language. Let w^L be a probability function on SL satisfying Li with Sx, and denote other members of a fixed language invariant family satisfying Sx and containing w^L accordingly by $w^{\mathcal{L}}$ (so $w^{\mathcal{L}}$ is on $S\mathcal{L}$).

Using the observation that spectral equivalences of a state description in a language $\mathcal{L} \supseteq L$ is the intersection of the two spectral equivalences pertaining to the restrictions of the state description to L and $\mathcal{L} - L$ respectively, and Lemma 27.2, we see that we can unambiguously define $\mathcal{N}^{L,\mathcal{L}}(\tilde{m}, \tilde{n})$ to be the number of state descriptions *in the language* $\mathcal{L}$ for $a_1, a_2, \dots, a_n$ with spectrum $\tilde{n}$ that extend some/any fixed state description $\Theta(a_1, \dots, a_m)$ in the language L with spectrum $\tilde{m}$.

For a state description $\Theta(a_1, \dots, a_m)$ in the language L, $n > m$ and $L \subseteq \mathcal{L}$ we have

$$w^L(\Theta(a_1, \dots, a_m)) = \sum_{\Psi(a_1,\dots,a_n) \vDash \Theta(a_1,\dots a_m)} w^{\mathcal{L}}(\Psi(a_1, \dots, a_n))$$

where the Ψ are state descriptions in the language $\mathcal{L}$. Just as before we can group them and make the above

$$w^L(\Theta(a_1,\ldots,a_m)) = \sum_{\tilde{n}\in Spec(n)} \sum_{\substack{\Psi\vDash\Theta \\ \mathcal{S}(\Psi)=\tilde{n}}} w^{\mathcal{L}}(\Psi(a_1,\ldots,a_n)).$$

Consequently, with $\tilde{m}$ the spectrum of $\Theta(a_1,\ldots,a_m)$ and writing $w^{\mathcal{L}}(\tilde{n})$ for $w^{\mathcal{L}}(\Psi(a_1,\ldots,a_n))$ where $\Psi(a_1,\ldots,a_n)$ is some/any state description in the language $\mathcal{L}$ with $\mathcal{S}(\Psi)=\tilde{n}$, we have

$$\begin{aligned} w^L(\Theta(a_1,\ldots,a_m)) = w^L(\tilde{m}) &= \sum_{\tilde{n}\in Spec(n)} \mathcal{N}^{L,\mathcal{L}}(\tilde{m},\tilde{n})w^{\mathcal{L}}(\tilde{n}) \\ &= \sum_{\tilde{n}\in Spec(n)} \frac{\mathcal{N}^{L,\mathcal{L}}(\tilde{m},\tilde{n})}{\mathcal{N}^{\mathcal{L}}(\emptyset,\tilde{n})}\mathcal{N}^{\mathcal{L}}(\emptyset,\tilde{n})w^{\mathcal{L}}(\tilde{n})\,. \end{aligned}$$

(where $\mathcal{N}^{\mathcal{L}}(\emptyset,\tilde{n})$ is the number of all state descriptions in the language $\mathcal{L}$ with spectrum $\tilde{n}$).

Checking through the steps in the previous chapter and using the same notation we find analogously to Corollary 31.6 that

$$\left| \frac{\mathcal{N}^{L,\mathcal{L}}(\tilde{m},\tilde{n})}{\mathcal{N}^{\mathcal{L}}(\emptyset,\tilde{n})} - \sum_{E\trianglerighteq\mathcal{E}(\Theta)} \frac{\mathcal{N}^{L,\mathcal{L}}(\tilde{r}_E,1_t)}{\mathcal{N}^{\mathcal{L}}(\emptyset,1_t)} \sum_{\rho\in Z^t(E)} \prod_{i=1}^{|E|}\left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|} \right| < 8h_\Theta m^2 n^{-1/2}\,,$$

and if there are s unary predicate symbols in $\mathcal{L}-L$, analogously as in Lemma 31.7, but without any need to assume that there is a non-unary predicate symbol in $\mathcal{L}$, that

$$\left|\frac{\mathcal{N}^{L,\mathcal{L}}(\tilde{r},1_t)}{\mathcal{N}^{\mathcal{L}}(\emptyset,1_t)} - \frac{1}{SD^L(k)}\right| \le t^2 2^{-s}\,, \tag{301}$$

where $\tilde{r}\in Spec(k)$, which yields

$$\left| \frac{\mathcal{N}^{L,\mathcal{L}}(\tilde{m},\tilde{n})}{\mathcal{N}^{\mathcal{L}}(\emptyset,\tilde{n})} - \sum_{E\trianglerighteq\mathcal{E}(\Theta)} \sum_{\rho\in Z^t(E)} \frac{1}{SD^L(|E|)} \prod_{i=1}^{|E|}\left(\frac{n_{\rho(i)}}{n}\right)^{|E_i|} \right| < h_\Theta\left(8m^2n^{-1/2}+t^2 2^{-s}\right). \tag{302}$$

For our further arguments, we move again to a nonstandard extension U^*. Take $\nu,\xi\in\mathbb{N}^*$ nonstandard, $\nu<\xi$. In U^* the Language Invariant family includes members $w^{\mathcal{L}}$ with $\mathcal{L}-L$ containing ξ unary predicate symbols. Fix one such $w^{\mathcal{L}}$. Omitting stars from w^L and $\mathcal{N}$ we have

$$w^L(\Theta(a_1,\ldots,a_m)) = w^L(\tilde{m}) = \sum_{\tilde{\nu}\in Spec^*(\nu)} \frac{\mathcal{N}^{L,\mathcal{L}}(\tilde{m},\tilde{\nu})}{\mathcal{N}^{\mathcal{L}}(\emptyset,\tilde{\nu})}\mathcal{N}^{\mathcal{L}}(\emptyset,\tilde{\nu})w^{\mathcal{L}}(\tilde{\nu})\,. \tag{303}$$

Let $\tilde{v} \in Spec^*(v)$ and define, for $m \leq v$ and a state description $\Theta(a_1, \ldots, a_m)$ with spectrum $\tilde{m} \in Spec^*(m)$,

$$U_{\tilde{v}}(\Theta(a_1, \ldots, a_m)) = \frac{\mathcal{N}^{L,\mathcal{L}}(\tilde{m}, \tilde{v})}{\mathcal{N}^{\mathcal{L}}(\emptyset, \tilde{v})}.$$

As before we find that ${}^\circ U_{\tilde{v}}$ extends to a probability function on SL satisfying Ex and Sx and we conclude from (303) that for some σ-additive measure τ on $Spec^*(v)$ we have

$$w^L(\Theta(a_1, \ldots, a_m)) = w^L(\tilde{m}) = \int_{Spec^*(v)} {}^\circ U_{\tilde{v}}(\Theta(a_1, \ldots, a_m))\, d\tau(\tilde{v}). \tag{304}$$

Furthermore, with $\bar{p} = \langle p_0, p_1, p_2, \ldots \rangle$ defined by $p_i = {}^\circ(v_i/v)$ for $i \in \mathbb{N} \cap \{1, 2, \ldots, t\}$, $p_i = 0$ for $i \in \mathbb{N}$, $i > t$ and $p_0 = 1 - \sum_{i \in \mathbb{N}^+} p_i$, we have, by virtue of (301), and regardless of the values of t and p_t, that

$${}^\circ U_{\tilde{v}}(\Theta(a_1, \ldots, a_m)) = u^{\bar{p},L}(\Theta(a_1, \ldots, a_m)).$$

For standard t this is immediate from Lemma 31.2 and from (302), and for nonstandard t it follows from Lemma 31.2, (302) and Lemma 31.9(i).

The result now follows easily by the methods of the previous chapter. ⊣

As a simple corollary to this theorem we have:

Corollary 32.2. *Every probability function w satisfying Ax on the language L with just a single unary predicate symbol satisfies Li with Sx.*

Proof. By (87) w has a representation

$$w = \int_{\mathbb{D}_2} v_{\vec{x}}\, d\mu(\vec{x}).$$

Since on this language, for $0 \leq x \leq 1/2$,

$$v_{\langle x, 1-x \rangle} = v_{\langle 1-x, x \rangle} = u^{h(x),L}$$

where

$$h(x) = \langle 2x, 1-2x, 0, 0, \ldots \rangle,$$

this representation can be written as

$$\begin{aligned} w &= v_{\langle 1/2, 1/2 \rangle}\mu(\{\langle 1/2, 1/2 \rangle\}) + 2\int_{0 \leq x < 1/2} v_{\langle x, 1-x \rangle}\, d\mu(\langle x, 1-x \rangle) \\ &= u^{\langle 1,0,0,0,\ldots \rangle, L}\mu(\{\langle 1/2, 1/2 \rangle\}) + 2\int_{0 \leq x < 1/2} u^{h(x),L}\, d\mu(\langle x, 1-x \rangle) \\ &= \int_{\mathbb{B}} u^{\bar{p},L} dv(\bar{p}) \end{aligned}$$

where $v(\{\langle 1, 0, 0, 0, \ldots \rangle\}) = \mu(\{\langle 1/2, 1/2 \rangle\})$ and for A a Borel subset of $\mathbb{B} - \{\langle 1, 0, 0, 0, \ldots \rangle\}$,

$$v(A) = 2\mu\{\langle x, 1-x \rangle \mid 0 \leq x < 1/2 \text{ and } \langle 2x, 1-2x, 0, 0, \ldots \rangle \in A\}$$

The result now follows from the Theorem 32.1. ⊣

A remarkable feature of the proof of Theorem 32.1 is that it works even when L is unary and we only have ULi with Sx, equivalently ULi with Ax, because in the key Lemma 31.7 we can add in nonstandardly many new unary relation symbols to provide the required inequality. In consequence we have the following result.

Theorem 32.3. *Let w be a probability function on a unary language L satisfying ULi with Ax. Then there is a measure μ on $\mathbb{B}$ such that*

$$w = \int_{\mathbb{B}} u^{\bar{p},L}\, d\mu(\bar{p}) \tag{305}$$

and hence w satisfies Li with Sx.

Rather surprisingly, as is apparent from the next lemma and the discussion following it, the language invariant family in the above theorem which contains a given w on a unary language need not be unique, although if different, any two such families have to differ already on some unary language. On the other hand, if w is a probability function on SL that contains a non-unary relation symbol and w satisfies Li with Sx then the language invariant family containing w *is* unique.

Lemma 32.4. *Let $\{U_L\}$ and $\{V_L\}$ be two language invariant families satisfying Sx which agree on unary languages. Then they agree on all languages.*

Proof. Let L be a language, let $n \in \mathbb{N}$ and fix L' to be a unary language disjoint from L with more than n atoms α_i.

We will first show, by *downward* induction on $\mathcal{E}(\Phi)$ with respect to the ordering $\trianglelefteq$, that $U_{L\cup L'}(\Phi(\vec{a})) = V_{L\cup L'}(\Phi(\vec{a}))$ for all state descriptions $\Phi(\vec{a})$ in the language $L \cup L'$ on n individuals.[106]

Let $\Phi(\vec{a})$ be such that $\mathcal{E}(\Phi)$ consist of singletons (so $\mathcal{E}(\Phi)$ is the top element w.r.t.$\trianglelefteq$) and let $\Psi(\vec{a})$ be any state description in the language $L \cup L'$ extending

$$\alpha_1(a_1) \wedge \alpha_2(a_2) \wedge \cdots \wedge \alpha_n(a_n).$$

Then $\mathcal{E}(\Psi(\vec{a}))$ must again be this top element and $U_{L\cup L'}$ must take the same value on all such extensions as on $\Phi(\vec{a})$ by Sx. Hence, since

$$\begin{aligned} U_{L'}\Big(\bigwedge_{i=1}^{n} \alpha_i(a_i)\Big) &= U_{L\cup L'}\Big(\bigwedge_{i=1}^{n} \alpha_i(a_i)\Big) \\ &= \sum_{\Psi(\vec{a})} U_{L\cup L'}(\Psi(\vec{a})), \end{aligned}$$

where the summation is over state descriptions $\Psi(\vec{a})$ in $L \cup L'$ extending $\bigwedge_{i=1}^{n} \alpha_i(a_i)$, we see that if M is the number of such $\Psi(\vec{a})$ then for any one

[106] Recall that for state descriptions $\Psi(a_1, \dots, a_n)$ and $\Theta(a_1, \dots, a_n)$, $\mathcal{E}(\Theta) \trianglelefteq \mathcal{E}(\Psi)$ means that $\mathcal{E}(\Psi)$ is a refinement of $\mathcal{E}(\Theta)$.

of them

$$U_{L\cup L'}(\Psi(\vec{a})) = M^{-1} U_{L'}\Big(\bigwedge_{i=1}^{n} \alpha_i(a_i)\Big) \tag{306}$$

and this is also the value of $U_{L\cup L'}(\Phi(\vec{a}))$. Noting that this reasoning also applies to the language invariant family $\{V_L\}$, that $U_{L'} = V_{L'}$ and that M depends only on L' and L, we conclude that

$$U_{L\cup L'}(\Phi(\vec{a})) = V_{L\cup L'}(\Phi(\vec{a}))$$

holds in this base case.

Now suppose that $U_{L\cup L'}(\Psi(\vec{a})) = V_{L\cup L'}(\Psi(\vec{a}))$ holds for all $\Psi(\vec{a})$ with $\mathcal{E}(\Phi) \lhd \mathcal{E}(\Psi)$, where $\Psi(\vec{a}), \Phi(\vec{a})$ are state descriptions in the language $L \cup L'$. Let $\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)$ have the same spectrum as $\Phi(\vec{a})$. Then again,

$$U_{L'}\Big(\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\Big) = \sum_{\Psi(\vec{a})} U_{L\cup L'}(\Psi(\vec{a})) \tag{307}$$

where the $\Psi(\vec{a})$ range over state descriptions in $L \cup L'$ extending $\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)$. Now for all of these $\Psi(\vec{a})$, $\mathcal{E}(\Psi)$ is greater or equal $\mathcal{E}(\Phi)$ in the ordering $\trianglelefteq$, and some $\Psi(\vec{a})$ with $\mathcal{E}(\Psi) = \mathcal{E}(\Phi)$ do appear on the right hand side of this expression. Furthermore the identity (307) also holds with V in place of U, and by the inductive hypothesis the terms on the right hand side are the same except possibly for those involving the $\Psi(\vec{a})$ with $\mathcal{E}(\Psi) = \mathcal{E}(\Phi)$. But since the left hand sides are the same (as $U_{L'} = V_{L'}$), the right hand sides must also be the same. Hence by Sx, $U_{L\cup L'}(\Phi) = V_{L\cup L'}(\Phi)$ as required.

Since for $\Theta(\vec{a})$ a state description in the language L we have

$$U_L(\Theta(\vec{a})) = \sum_{\Psi(\vec{a})} U_{L\cup L'}(\Psi(\vec{a}))$$

where the $\Psi(\vec{a})$ range over state descriptions in $L \cup L'$ extending $\Theta(\vec{a})$, and similarly for V, the lemma now follows. ⊣

THEOREM 32.5. *Let w be a probability function for a language L with a non-unary relation symbol and let w satisfy Li with Sx. Then the language invariant family containing w is unique.*

PROOF. The proof is analogous to the proof of the above lemma. A non-unary relation symbol in L guarantees the existence of state descriptions with spectrum 1_n for every $n \in \mathbb{N}^+$ so other members of the family are not needed. ⊣

We have previously considered two parameterized families of purely unary probability functions satisfying ULi with Ax, namely the c_λ^L of Carnap's continuum (Chapter 16) and the NP-continuum w_L^δ (Chapters 18,19). Each of these families must be part of a language invariant family extending over all (finite) languages, which raises the question as to what these families are once we move outside the purely unary, equivalently what are the de Finetti priors on $\mathbb{B}$?

Recalling that the w_L^δ are defined by

$$w_L^\delta\Big(\bigwedge_{i=1}^m \alpha_{h_i}(a_i)\Big) = 2^{-q}\sum_{j=1}^{2^q}(\delta+\gamma)^{m_j}\gamma^{m-m_j}$$

where $2^q\gamma+\delta = 1$, $0 \le \delta \le 1$ and $m_j = |\{i\,|h_i = j\}|$, it is easily seen that the corresponding de Finetti prior on $\mathbb{B}$ is given by the discrete measure which puts all the measure on the single point

$$\langle 1-\delta, \delta, 0, 0, 0, \dots\rangle \in \mathbb{B}.$$

Turning to the case of Carnap's c_λ^L, this agrees with w_L^1 when $\lambda = 0$ and with w_L^0 when $\lambda = \infty$. For $0 < \lambda < \infty$ the de Finetti prior on $\mathbb{B}$ giving the family of c_λ^L has been elucidated by Kingman, the corresponding distribution being the Poisson-Dirichlet distribution for parameter λ, see [67], and for a complementary discussion [148, chapter 10].

An interesting question arises here, since as we have seen in the unary case the c_λ^L for $0 \le \lambda \le \infty$ are characterized as the only probability functions additionally satisfying Johnson's Sufficientness Postulate. However, as shown in [141], beyond the purely unary the *natural generalization* of Johnson's Sufficientness Postulate has only two solutions, corresponding to $\lambda = \infty$, the 'completely independent' probability function, and $\lambda = 0$, the sole 1-heterogeneous probability function. This leads to the question of whether there is *some* version of Johnson's Sufficientness Postulate which does characterize these extensions of the c_λ^L also above the purely unary. Unfortunately the Poisson-Dirichlet distributions appear somewhat inaccessible (see [67, section 3.3.6]) and we are currently not aware of any satisfactory answer to this problem in terms of the observance of a particular principle within Inductive Logic.

Concerning the case of $\lambda = 2$ we remark that the probability function $c_2^{L_1}$ for the language L_1 with a single unary relation symbol[107] has, on $\mathbb{B}$, beside the Poisson-Dirichlet prior with parameter 2 (which yields the whole language invariant family given by c_2^L on the finite unary languages) also another prior that yields $c_2^{L_1}$ on L_1 but gives other probability functions for other unary languages. It is the point measure on $\mathbb{B}$ that assigns measure 1 to the single point

$$\overline{d} = \langle 0, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \dots\rangle\,.$$

Hence $c_2^{L_1}$ is also a member of the language invariant family $u^{\overline{d},L}$.

[107] Notice from (148) that $c_2^{L_1}$ has the uniform measure as its de Finetti prior on $\mathbb{D}_2$.

To see this[108] note that (writing d_i for $\frac{1}{2^i}$) we have by (192) and (193)

$$u^{\overline{d},L_1}\Big(\bigwedge_{j=1}^{m_1} \alpha_1(a_j) \wedge \bigwedge_{j=m_1+1}^{m_1+m_2} \alpha_2(a_j)\Big) =$$

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{\substack{A_1,A_2 \text{ a} \\ \text{partition of} \\ \{1,\dots,n\}}} \left(\frac{d_n}{2} + \sum_{i\in A_1} d_i\right)^{m_1} \left(\frac{d_n}{2} + \sum_{i\in A_2} d_i\right)^{m_2}, \qquad (308)$$

Since there is a one to one correspondence between the subsets of $\{\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}\}$ and numbers in $\{\frac{i}{2^n} \mid i = 0, 1, 2, 3, \dots, 2^n - 1\}$ and since $\{\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}\}$ sums to $1 - \frac{1}{2^n}$,

$$u^{\overline{d},L_1}\Big(\bigwedge_{j=1}^{m_1} \alpha_1(a_j) \wedge \bigwedge_{j=m_1+1}^{m_1+m_2} \alpha_2(a_j)\Big) =$$

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{i=0}^{2^n-1} \left(\frac{i}{2^n} + \frac{1}{2^{n+1}}\right)^{m_1} \left(1 - \frac{i}{2^n} - \frac{1}{2^{n+1}}\right)^{m_2}.$$

It is easily seen that this limit is the same as

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{i=0}^{2^n-1} \left(\frac{i}{2^n}\right)^{m_1} \left(1 - \frac{i}{2^n}\right)^{m_2}$$

$$= \int_0^1 y^{m_1}(1-y)^{m_2}\, dy = c_2^{L_1}\Big(\bigwedge_{j=1}^{m_1} \alpha_1(a_j) \wedge \bigwedge_{j=m_1+1}^{m_1+m_2} \alpha_2(a_j)\Big).$$

However it can be checked that for L_2 having 2 unary predicate symbols (so 4 atoms),

$$c_2^{L_2}(\alpha_1(a_1) \wedge \alpha_2(a_2) \wedge \alpha_3(a_3)) = 1/192$$

whereas

$$u^{\overline{d},L_2}(\alpha_1(a_1) \wedge \alpha_2(a_2) \wedge \alpha_3(a_3)) = 1/224$$

so we have here two different language invariant families but having a common member on L_1.

Together with Corollary 32.2 this shows that there are many examples of probability functions on SL_1 belonging to two, or more, Li with Sx families. At present however we do not know if there are similar examples of probability functions for the language with *two* unary relation symbols.

[108] We thank Elizabeth Howarth for pointing out a simplification to our original proof.

Chapter 33

Sx WITHOUT LANGUAGE INVARIANCE

In this chapter we give a representation theorem for t-heterogeneous probability functions which appeared in [96]. This is actually a corollary of the Representation Theorem 31.11 for t-heterogeneous probability functions given in Chapter 31 but based on *unary* t-heterogeneous probability functions rather than measures on $\mathbb{B}_t$. Our reason for including it here is that it possibly provides some new insights.

To explain the idea behind this representation theorem suppose that w is a t-heterogeneous probability function on SL and let $\Theta(a_1, \dots, a_m)$ be a state description of L with spectrum of length t. Then a state description $\Phi(a_1, \dots, a_n)$ extending $\Theta(a_1, \dots, a_m)$ will only get non-zero probability according to w if $a_{m+1}, a_{m+2}, \dots, a_n$ all join the existing equivalence classes of $\Theta(a_1, \dots, a_m)$, otherwise more than t equivalence classes will be formed and being t-heterogeneous w will have to give Φ probability 0.

From $\Theta(a_1, \dots, a_m)$ onwards $w(\Phi(a_1, \dots, a_n) \mid \Theta(a_1, \dots, a_m))$ will no longer depend on the particular language L we started from, provided it is large enough to furnish at least t equivalence classes. Furthermore since w is t-heterogeneous, in the limit w will be concentrating all probability on state descriptions with spectra of length t, so knowing what w does on these should enable us to work backwards and recover all of w.

To this end then let L, L_1 be languages having at least t atoms if they are purely unary. Recalling the notation from Chapter 31 we shall use $\tilde{n}, \tilde{m}$ etc. for spectra and $\mathcal{N}^{L_1}(\tilde{m}, \tilde{n})$ for the number of state descriptions of L_1 with spectrum $\tilde{n}$ which extend a particular state description of L_1 with spectrum $\tilde{m}$. (By Lemma 27.2 this does not depend on the particular state description chosen.) For $\tilde{n}$ a spectrum of length t set

$$d^L_{L_1}(t) = \frac{\mathcal{N}^L(\emptyset, \tilde{n})}{\mathcal{N}^{L_1}(\emptyset, \tilde{n})}.$$

Notice that $d^L_{L_1}(t)$ depends only on t, L, L_1 and not on the particular $\tilde{n}$ chosen.

Let w be a t-heterogeneous probability function on SL and for $\Theta(a_1,\dots,a_m)$ a state description of L_1 with spectrum $\tilde{m}$ define

$$w^{|L_1}(\Theta(a_1,\dots,a_m)) = \lim_{n\to\infty} \sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} d^L_{L_1}(t)\mathcal{N}^{L_1}(\tilde{m},\tilde{n})w(\tilde{n}). \qquad (309)$$

Theorem 33.1. *The limit defined in* (309) *exists and* $w^{|L_1}$ *extends to a* t*-heterogeneous probability function on* SL_1. *Furthermore* $w^{|L_1|L} = w$.

Proof. To show that the limit exists notice that for $n \le k$, $\tilde{n} \in Spec(n)$, $|\tilde{n}| = t$,

$$w(\tilde{n}) = \sum_{\substack{\tilde{k}\in Spec(k)\\ |\tilde{k}|=t}} \mathcal{N}^L(\tilde{n},\tilde{k})w(\tilde{k}).$$

Hence for $m \le n \le k$ and $\tilde{m} \in Spec(m)$,

$$\begin{aligned}
&\sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} d^L_{L_1}(t)\mathcal{N}^{L_1}(\tilde{m},\tilde{n})w(\tilde{n})\\
&\qquad = \sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} \sum_{\substack{\tilde{k}\in Spec(k)\\ |\tilde{k}|=t}} d^L_{L_1}(t)\mathcal{N}^{L_1}(\tilde{m},\tilde{n})\mathcal{N}^{L}(\tilde{n},\tilde{k})w(\tilde{k})\\
&\qquad = \sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} \sum_{\substack{\tilde{k}\in Spec(k)\\ |\tilde{k}|=t}} d^L_{L_1}(t)\mathcal{N}^{L_1}(\tilde{m},\tilde{n})\mathcal{N}^{L_1}(\tilde{n},\tilde{k})w(\tilde{k})\\
&\qquad\qquad\qquad \text{since } \mathcal{N}^L(\tilde{n},\tilde{k}) = \mathcal{N}^{L_1}(\tilde{n},\tilde{k}) \text{ when } |\tilde{n}| = |\tilde{k}|,\\
&\qquad \le \sum_{\substack{\tilde{k}\in Spec(k)\\ |\tilde{k}|=t}} d^L_{L_1}(t)\mathcal{N}^{L_1}(\tilde{m},\tilde{k})w(\tilde{k})
\end{aligned}$$

since clearly

$$\sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} \mathcal{N}^{L_1}(\tilde{m},\tilde{n})\mathcal{N}^{L_1}(\tilde{n},\tilde{k}) \le \mathcal{N}^{L_1}(\tilde{m},\tilde{k}).$$

From this it follows that the sequence defining $w^{|L_1}(\Theta(a_1,\dots,a_m))$ in (309) is increasing. It is also bounded by 1 since

$$\begin{aligned}
\sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} d^L_{L_1}(t)\mathcal{N}^{L_1}(\tilde{m},\tilde{n})w(\tilde{n}) &\le \sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} d^L_{L_1}(t)\mathcal{N}^{L_1}(\emptyset,\tilde{n})w(\tilde{n})\\
&= \sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} \mathcal{N}^{L}(\emptyset,\tilde{n})w(\tilde{n})\\
&\le 1.
\end{aligned}$$

To show that $w^{|L_1}$ extends to a probability function it is enough to check (i)–(iii) from (34). Of these (i) is immediate, as is (ii) since

$$\begin{aligned} w^{|L_1}(\top) &= \lim_{n\to\infty} \sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} d^L_{L_1}(t)\mathcal{N}^{L_1}(\emptyset,\tilde{n})w(\tilde{n}) \\ &= \lim_{n\to\infty} \sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} \mathcal{N}^{L}(\emptyset,\tilde{n})w(\tilde{n}) = 1 \end{aligned}$$

since w is t-heterogeneous. Also (iii) follows directly since for $\Theta(a_1, \dots, a_m)$ a state description of L_1 and the Φ ranging over state descriptions of L_1 for $a_1, \dots, a_{m+1}$,

$$\begin{aligned} \sum_{\Phi\vDash\Theta} w^{|L_1}(\Phi(a_1,\dots,a_{m+1})) &= \lim_{n\to\infty} \sum_{\Phi\vDash\Theta} \sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} d^L_{L_1}(t)\mathcal{N}^{L_1}(\mathcal{S}(\Phi),\tilde{n})w(\tilde{n}) \\ &= \lim_{n\to\infty} \sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} d^L_{L_1}(t)\mathcal{N}^{L_1}(\mathcal{S}(\Theta),\tilde{n})w(\tilde{n}) \\ &= w^{|L_1}(\Theta(a_1,\dots,a_m)), \end{aligned}$$

since for $m < n$,

$$\sum_{\Phi\vDash\Theta} \mathcal{N}^{L_1}(\mathcal{S}(\Phi),\tilde{n}) = \mathcal{N}^{L_1}(\mathcal{S}(\Theta),\tilde{n}).$$

In addition $w^{|L_1}$ satisfies Ex and Sx since the right hand side of (309) is invariant under permutations of the $a_1, \dots, a_m$ and depends only on the spectrum of $\Theta(a_1, \dots, a_m)$. Indeed $w^{|L_1}$ is t-heterogeneous. To see this notice that because the terms on the right hand side of (309) are increasing, for Θ ranging over state descriptions of L_1 for $a_1, \dots, a_m$,

$$\begin{aligned} \lim_{m\to\infty} \sum_{\substack{\Theta(a_1,\dots,a_m)\\ |\mathcal{S}(\Theta)|=t}} w^{|L_1}(\Theta) &\geq \lim_{m\to\infty} \sum_{\substack{\Theta(a_1,\dots,a_m)\\ |\mathcal{S}(\Theta)|=t}} d^L_{L_1}(t)\mathcal{N}^{L_1}(\mathcal{S}(\Theta),\mathcal{S}(\Theta))w(\mathcal{S}(\Theta)) \\ &= \lim_{m\to\infty} \sum_{\substack{|\tilde{m}|=t\\ \tilde{m}\in Spec(m)}} d^L_{L_1}(t)\mathcal{N}^{L_1}(\emptyset,\tilde{m})w(\tilde{m}) \\ &= \lim_{m\to\infty} \sum_{\substack{|\tilde{m}|=t\\ \tilde{m}\in Spec(m)}} \mathcal{N}^{L}(\emptyset,\tilde{m})w(\tilde{m}) \\ &= 1, \end{aligned}$$

since w is t-heterogeneous.

It now only remains to show that $w^{|L_1|L} = w$. Directly from (309) we have that for a state description $\Theta(a_1, \dots, a_m)$ of L with spectrum $\tilde{m}$,

$$
\begin{aligned}
w^{|L_1|L}&(\Theta(a_1,\dots,a_m)) \\
&= \lim_{n\to\infty} \sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} d_L^{L_1}(t)\mathcal{N}^L(\tilde{m},\tilde{n})w^{|L_1}(\tilde{n}) \\
&= \lim_{n\to\infty}\lim_{k\to\infty} \sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} \sum_{\substack{\tilde{k}\in Spec(k)\\ |\tilde{k}|=t}} d_L^{L_1}(t)d_{L_1}^{L}(t)\mathcal{N}^L(\tilde{m},\tilde{n})\mathcal{N}^{L_1}(\tilde{n},\tilde{k})w(\tilde{k}) \\
&= \lim_{n\to\infty}\lim_{k\to\infty} \sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} \sum_{\substack{\tilde{k}\in Spec(k)\\ |\tilde{k}|=t}} \mathcal{N}^L(\tilde{m},\tilde{n})\mathcal{N}^{L}(\tilde{n},\tilde{k})w(\tilde{k}), \\
&\qquad\qquad \text{since } \mathcal{N}^L(\tilde{n},\tilde{k}) = \mathcal{N}^{L_1}(\tilde{n},\tilde{k}) \text{ when } |\tilde{n}| = |\tilde{k}|, \\
&= \lim_{n\to\infty} \sum_{\substack{\tilde{n}\in Spec(n)\\ |\tilde{n}|=t}} \mathcal{N}^L(\tilde{m},\tilde{n})w(\tilde{n}) \qquad (310)
\end{aligned}
$$

since for $|\tilde{n}| = t$, the t-heterogeneity of w gives

$$\sum_{\substack{\tilde{k}\in Spec(k)\\ |\tilde{k}|=t}} \mathcal{N}^L(\tilde{n},\tilde{k})w(\tilde{k}) = w(\tilde{n}).$$

But (310) is just $w(\Theta(a_1, \dots, a_m))$, so, as required,

$$w^{|L_1|L}(\Theta(a_1,\dots,a_m)) = w(\Theta(a_1,\dots,a_m)). \qquad \dashv$$

This result is interesting in that it shows that each t-heterogeneous probability function w actually belongs to a family of t-heterogeneous probability functions $w^{|\mathcal{L}}$ for each language $\mathcal{L}$, provided in the unary case that $\mathcal{L}$ has at least t atoms. However, by considering the case of $v^{\bar{p},L}$ where $\bar{p}$ has exactly t non-zero entries, it is clear that this does not constitute a language invariant family in general.

For not purely unary languages Theorem 33.1 can be extended to probability functions satisfying Sx whose Ladder Theorem representation (page 218) has $\eta_0 = 0$. To see this first notice that if we had replaced t in the definition (309) by $s \neq t$ then the resulting limit would have come out to be zero (for t-heterogeneous w). Indeed we could have replaced this definition of $w^{|L_1}$ by

$$w^{|L_1}(\Theta(a_1,\dots,a_m)) = \lim_{n\to\infty} \sum_{\tilde{n}\in Spec(n)} d_{L_1}^{L}(|\tilde{n}|)\mathcal{N}^{L_1}(\tilde{m},\tilde{n})w(\tilde{n}). \qquad (311)$$

Now suppose that instead of being t-heterogeneous as above w just has a Ladder Theorem representation

$$w = \sum_{j\geq 1} \eta_j w^{[j]}.$$

Then applying definition (311) to w we obtain

$$w^{|L_1} = \sum_{j \geq 1} \eta_j w^{[j]|L_1}.$$

In this way then we can successfully extend the operation $w \mapsto w^{|L_1}$ to such probability functions for not purely unary languages. In particular the operation extends to *$\leq t$-heterogeneous probability functions*, that is probability functions which are convex sums of s-heterogeneous probability functions for $s \leq t$.

As mentioned at the start of this chapter Theorem 33.1 is actually a corollary of Theorem 31.11, a study of the proof of that latter showing that if w is a t-heterogeneous probability function on SL (with at least t atoms if unary) and

$$w = \int_{\mathbb{B}_t} v^{\bar{p},L} \, d\mu(\bar{p})$$

then

$$w^{|L_1} = \int_{\mathbb{B}_t} v^{\bar{p},L_1} \, d\mu(\bar{p}).$$

We shall conclude this chapter by giving a further result (from [95]) demonstrating how t-heterogeneous probability functions are closely related to their unary counterparts. As motivation for why we consider this result of some interest we recall an early philosophical issue which concerned us in this subject: In polyadic relations what is the nature of the connection between the individual arguments? In particular to what extent might binary and higher arity relations between individuals be understood merely in terms of the separate unary properties satisfied by these individuals? More dramatically, could it be that *any* polyadic relation can be understood, or explained, in this way?

To take a simple example consider the binary relation of 'is a good pollinator of' between apple trees. An eighteenth century gardener might well have seen this as a relationship which could only be tested by experiment in the field, by planting two trees in close proximity. Nowadays however we are led to believe that this is all a matter of the chromosome structures of the two individuals. Whether or not one will prove to be a good pollinator of the other can be predicted from their *individual properties alone*.

For a more controversial example consider the relationship of 'would like' between people. In practice we rather often feel that we can predict whether or not people as yet unknown to each other[109] will get on. It seems reasonable (to us at least) to suppose that our prediction would converge to certainty in the limiting case of knowing everything about the two parties *as individuals*.

[109] Of course if they are already known to each other it certainly is determined by them individually, we just have to ask A if she likes B!

Whether or not one gives this suggestion any credence it still seems interesting to consider, within the context of PIL, the extent to which binary and higher arity relations might be represented as merely combinations of unary properties.[110]

In order to provide a possible route to explore this idea let L be, as usual, our default polyadic language with relation symbols $R_1, \dots, R_q$ of arities $r_1, \dots, r_q$ respectively, and let L_0 be a unary language. Let $\rho^s(x_1, x_2, \dots, x_{r_s})$, for $s = 1, \dots, q$, be formulae of L_0, possibly mentioning some constant symbols. Given $\psi(R_1, \dots, R_q, \vec{a}) \in SL$, where we have explicitly indicated the relation symbols, let $\psi(\rho^1, \dots, \rho^q, \vec{a})$ be the sentence of L_0 formed by replacing each occurrence of $R_s(t_1, \dots, t_{r_s})$ in $\psi(R_1, \dots, R_q, \vec{a})$ by $\rho^s(t_1, \dots, t_{r_s})$, for $s = 1, \dots, q$, where the t_j range over the constant symbols and variables. Notice that by elementary predicate logic

$$\models \psi(R_1, \dots, R_q, \vec{a}) \Rightarrow \models \psi(\rho^1, \dots, \rho^q, \vec{a}). \tag{312}$$

Now let v be a probability function on SL_0 and define $v^{\vec{\rho}} : SL \to [0, 1]$ by

$$v^{\vec{\rho}}(\psi(R_1, \dots, R_q, \vec{a})) = v(\psi(\rho^1, \dots, \rho^q, \vec{a})).$$

Using (312) it is a straightforward exercise to show that $v^{\vec{\rho}}$ satisfies (P1–3) and so is a probability function on SL. In a clear sense $v^{\vec{\rho}}$ is *reducible to*, or *represented by*, the unary probability function v and so provides a possible formalization of the above idea. We will use a slightly more general concept[111] and say that a probability function w on SL has a *reduction to* SL_0 if there are probability functions $v_1, v_2, \dots, v_k$ on SL_0, $\vec{\rho_1}, \vec{\rho_2}, \dots, \vec{\rho_k} \in FL_0$[112] and $\lambda_1, \dots, \lambda_k \geq 0$ with $\sum_i \lambda_i = 1$ such that

$$w = \sum_{i=1}^{k} \lambda_i v_i^{\vec{\rho_i}}.$$

At this time of writing we know rather little about which probability functions on SL have a reduction to a unary language. However, generalizing and simplifying slightly Theorem 8 of [96] we do have:

Theorem 33.2. *If w is a $\leq t$-heterogeneous probability function w on SL and L_0 is a unary language with m relation symbols, where $2^m \geq t$, then w has a reduction to SL_0.*

Proof. We shall prove this result for w t-heterogeneous, the theorem then following directly by the Ladder Theorem 30.2.

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[110] An earlier research theme based around this idea led to the so called 'natural' probability functions J_n of [21], [99], [116], [118].

[111] There are clearly several other variations possible here. The one we have chosen is appropriate for what we are able to prove in the theorem that follows!

[112] Using (59) and (64) it can be shown that we can without loss replace FL_0 here by $QFFL_0$.

Let $\alpha_1(x), \alpha_2(x), \dots, \alpha_{2^m}(x)$ denote the atoms of L_0. The formulae $\rho^s(x_1, x_2, \dots, x_{r_s})$ will be of the form

$$\bigvee_{\langle i_1,\dots,i_{r_s}\rangle \in X_s} (\alpha_{i_1}(x_1) \wedge \alpha_{i_2}(x_2) \wedge \cdots \wedge \alpha_{i_{r_s}}(x_{r_s}))$$

where $X_s \subseteq \{1, 2, \dots, 2^m\}^{r_s}$. So $\rho^1, \dots, \rho^q$ are to be thought of as functions of the $X_1, X_2, \dots, X_q$ respectively, and w will be represented as a linear combination of these $v^{\vec{\rho}}$.

Let $\Theta(\vec{R}, a_1, a_2, \dots, a_n)$ be a state description of L with spectrum of length t and, for some choice of $\vec{X}$, suppose that

$$\alpha_{i_1}(a_1) \wedge \alpha_{i_2}(a_2) \wedge \cdots \wedge \alpha_{i_n}(a_n) \vDash \Theta(\vec{\rho}, a_1, a_2, \dots, a_n). \tag{313}$$

This means that

$$\begin{aligned}
&\Theta(\vec{R}, a_1, \dots, a_n) \vDash R_s(a_{e_1}, a_{e_2}, \dots, a_{e_{r_s}}) \\
&\quad \Rightarrow \Theta(\vec{\rho}, a_1, \dots, a_n) \vDash \rho_s(a_{e_1}, a_{e_2}, \dots, a_{e_{r_s}}), \qquad \text{by (312)}, \\
&\quad \Rightarrow \alpha_{i_1}(a_1) \wedge \alpha_{i_2}(a_2) \wedge \cdots \wedge \alpha_{i_n}(a_n) \vDash \rho_s(a_{e_1}, a_{e_2}, \dots, a_{e_{r_s}}) \\
&\quad \Rightarrow \alpha_{i_{e_1}}(a_{e_1}) \wedge \alpha_{i_{e_2}}(a_{e_2}) \wedge \cdots \wedge \alpha_{i_{e_{r_s}}}(a_{e_{r_s}}) \vDash \rho_s(a_{e_1}, a_{e_2}, \dots, a_{e_{r_s}}) \\
&\quad \Rightarrow \langle i_{e_1}, i_{e_2}, \dots, i_{e_{r_s}} \rangle \in X_s,
\end{aligned}$$

and similarly with $\neg R_s, \neg X_s$ in place of R_s, X_s respectively. So

$$\Theta(\vec{R}, a_1, \dots, a_n) \vDash R_s(a_{e_1}, a_{e_2}, \dots, a_{e_{r_s}}) \iff \langle i_{e_1}, i_{e_2}, \dots, i_{e_{r_s}} \rangle \in X_s. \tag{314}$$

Suppose that $a_j \sim_\Theta a_h$. Then $R_s(a_{e_1}, a_{e_2}, \dots, a_{e_{r_s}})$ is a conjunct of Θ just if $R_s(a_{f_1}, a_{f_2}, \dots, a_{f_{r_s}})$ is a conjunct of Θ whenever $\langle f_1, f_2, \dots, f_{r_s}\rangle$ is the result of replacing any number of occurrences of j in $\langle e_1, e_2, \dots, e_{r_s}\rangle$ by h, or visa-versa. Hence (314) gives that

$$\langle i_{e_1}, i_{e_2}, \dots, i_{e_{r_s}} \rangle \in X_s \iff \langle i_{f_1}, i_{f_2}, \dots, i_{f_{r_s}} \rangle \in X_s$$

whenever $\langle i_{f_1}, i_{f_2}, \dots, i_{f_{r_s}} \rangle$ is the result of replacing any number of occurrences of i_j in $\langle i_{e_1}, i_{e_2}, \dots, i_{e_{r_s}} \rangle$ by i_h, or visa-versa. So also

$$\alpha_{m_1}(a_1) \wedge \alpha_{m_2}(a_2) \wedge \cdots \wedge \alpha_{m_n}(a_n) \vDash \Theta(\vec{\rho}, a_1, a_2, \dots, a_n) \tag{315}$$

whenever $\langle m_1, m_2, \dots, m_n\rangle$ results from $\langle i_1, i_2, \dots, i_n\rangle$ by replacing any number of occurrences of i_j by i_h, or visa-versa.

Conversely if, for $\langle i_1, i_2, \dots, i_n\rangle, \langle m_1, m_2, \dots, m_n\rangle \in \{1, 2, \dots, 2^m\}^n$ it is the case that (315) holds whenever $\langle m_1, m_2, \dots, m_n\rangle$ results from $\langle i_1, i_2, \dots, i_n\rangle$ by replacing any number of occurrences of i_j by i_h, or visa-versa, then by reversing the above argument we see that we must have $a_j \sim_\Theta a_h$. In particular

$$i_j = i_h \Longrightarrow a_j \sim_\Theta a_h. \tag{316}$$

Now let $v = w^{|L_0}$, let $\vec{X}$, $\alpha_{i_1}(a_1) \wedge \alpha_{i_2}(a_2) \wedge \cdots \wedge \alpha_{i_n}(a_n)$ be as in (313) and suppose that

$$v(\alpha_{i_1}(a_1) \wedge \alpha_{i_2}(a_2) \wedge \cdots \wedge \alpha_{i_n}(a_n)) > 0. \tag{317}$$

Then, since v is t-heterogeneous, there can be at most t different i_j, otherwise this value would have been zero. On the other hand for Θ as above in (313) there must be at least t different i_j since otherwise by (316) $\sim_\Theta$ would have less than t equivalence classes.

Indeed for the $i_1, i_2, \ldots, i_n$ as in (313), (317) it follows that the equivalence classes of $\sim_\Theta$ must be the non-empty $\{a_j \mid i_j = k\}$ for $k = 1, 2, \ldots, 2^m$ and

$$v(\alpha_{i_1}(a_1) \wedge \alpha_{i_2}(a_2) \wedge \cdots \wedge \alpha_{i_n}(a_n)) = v(\tilde{n})$$

where $\tilde{n} = \mathcal{S}(\Theta)$.

Conversely suppose that $\alpha_{i_1}(a_1) \wedge \alpha_{i_2}(a_2) \wedge \cdots \wedge \alpha_{i_n}(a_n)$ is a state description of L_0 such that the non-empty $\{a_j \mid i_j = k\}$ for $k = 1, 2, \ldots, 2^m$ are precisely the equivalence classes of $\sim_\Theta$. Then there will be a choice of $X_1, \ldots, X_q$ such that

$$\langle i_{e_1}, i_{e_2}, \ldots, i_{e_{r_s}} \rangle \in X_s \iff \Theta \vDash R_s(a_{e_1}, a_{e_2}, \ldots, a_{e_{r_s}})$$

and for any such choice we will have (313) – the reason for multiple choices here being that tuples with some i's not occurring amongst the $i_1, \ldots, i_n$ can be allocated arbitrarily.

In summary then for any of the $2^m!/(2^m - t)!$ state descriptions $\alpha_{i_1}(a_1) \wedge \alpha_{i_2}(a_2) \wedge \cdots \wedge \alpha_{i_n}(a_n)$ such that the non-empty $\{a_j \mid i_j = k\}$ for $k = 1, 2, \ldots, 2^m$ are precisely the equivalence classes of $\sim_\Theta$ there will be the same number, K say, of $\vec{X}$ for which (313) holds. These will be the only state descriptions and $\vec{X}$ for which (313) and (317) hold and furthermore since v satisfies Ax they will all get the same value $v(\tilde{n})$ under v.

Setting $\lambda = K2^m!((2^m - t)!)^{-1} d^L_{L_0}(t)$ we now obtain that

$$\begin{aligned}
\sum_{\vec{X}} v(\Theta(\vec{\rho}, \vec{a})) &= K2^m!((2^m - t)!)^{-1} v(\tilde{n}) \\
&= \lambda\, w(\tilde{n}), && \text{by (309)} \\
&= \lambda\, w(\Theta(\vec{R}, \vec{a})). && (318)
\end{aligned}$$

Since w is t-heterogeneous, for $\phi(\vec{R}, a_1, \ldots, a_r) \in QFSL$,

$$w(\phi(\vec{R}, a_1, \ldots, a_r)) = \lim_{n \to \infty} \sum_{\substack{\Theta(\vec{R}, a_1, \ldots, a_n) \vDash \phi(\vec{R}, a_1, \ldots, a_r) \\ |\mathcal{S}(\Theta(\vec{R}, a_1, \ldots, a_n))| = t}} w(\Theta(\vec{R}, a_1, \ldots, a_n)).$$

Hence, with (312), for $k \geq r$,

$$\begin{aligned}
\sum_{\vec{X}} v(\phi(\vec{\rho}, a_1, \dots, a_r)) \\
&= \sum_{\vec{X}} v\Big(\bigvee_{\Theta(\vec{R}, a_1, \dots, a_k) \vDash \phi(\vec{R}, a_1, \dots, a_r)} \Theta(\vec{\rho}, a_1, \dots, a_k) \Big) \\
&= \sum_{\vec{X}} \sum_{\Theta(\vec{R}, a_1, \dots, a_k) \vDash \phi(\vec{R}, a_1, \dots, a_r)} v(\Theta(\vec{\rho}, a_1, \dots, a_k)) \\
&= \lim_{n \to \infty} \sum_{\vec{X}} \sum_{\Theta(\vec{R}, a_1, \dots, a_n) \vDash \phi(\vec{R}, a_1, \dots, a_r)} v(\Theta(\vec{\rho}, a_1, \dots, a_n)) \\
&\geq \limsup_{n \to \infty} \sum_{\vec{X}} \sum_{\substack{\Theta(\vec{R}, a_1, \dots, a_n) \vDash \phi(\vec{R}, a_1, \dots, a_r) \\ |\mathcal{S}(\Theta(\vec{R}, a_1, \dots, a_n))| = t}} v(\Theta(\vec{\rho}, a_1, \dots, a_n)) \\
&= \limsup_{n \to \infty} \sum_{\substack{\Theta(\vec{R}, a_1, \dots, a_n) \vDash \phi(\vec{R}, a_1, \dots, a_r) \\ |\mathcal{S}(\Theta(\vec{R}, a_1, \dots, a_n))| = t}} \lambda\, w(\Theta(\vec{R}, a_1, \dots, a_n)) \\
&= \lambda\, w(\phi(\vec{R}, a_1, \dots, a_r)).
\end{aligned}$$

Repeating this with $\neg\phi(\vec{R}, a_1, \dots, a_r)$ in place of $\phi(\vec{R}, a_1, \dots, a_r)$ shows that we must have

$$\sum_{\vec{X}} v(\phi(\vec{\rho}, a_1, \dots, a_r)) = \lambda w(\phi(\vec{R}, a_1, \dots, a_r))$$

and this now straightforwardly extends to $\phi \in SL$, as required. ⊣

Notice that in this theorem v satisfies Ax. In fact if we take any t-heterogeneous probability function v on L_0 satisfying Ax, where again $2^m \geq t$, then it can be shown that the probability function w on SL defined by

$$w(\phi(\vec{R}, \vec{a})) = \lambda^{-1} \sum_{\vec{X}} v(\phi(\vec{\rho}, \vec{a}))$$

is t-heterogeneous. This again then demonstrates a close connection between t-heterogeneous probability functions on polyadic languages and their counterparts on unary languages.

From the footnote on page 63 we now have the following corollary.

Corollary 33.3. *For w a t-heterogeneous probability function and fixed n the set of values*

$$\{w(\psi(a_1, \dots, a_n)) \mid \psi(a_1, \dots, a_n) \in SL\}$$

is finite.

This result also holds for homogeneous probability functions but does not hold in general for probability functions satisfying Sx (see [53]).

We cannot hope for a similar result to Theorem 33.2 for a homogeneous probability function w on SL since the $v^{\vec{\rho}}$ will always have the property of giving zero probability to state descriptions with spectrum length greater than 2^m.

By approximating $u^{\bar{p},L}$, where $\bar{p} = \langle p_0, p_1, p_2, \dots \rangle$, by the $\leq 2t$-heterogeneous $w_t = u^{\bar{q},L}$, where

$$\bar{q} = \langle 0, p_1, p_2, \dots, p_s, q_{s+1}, q_{s+2}, \dots, q_{s+t}, 0, 0, \dots \rangle$$

and s is maximal such that $p_s \geq t^{-1}$ and $q_{s+i} = t^{-1}(1 - \sum_{j=1}^{s} p_j)$ for $s = 1, 2, \dots, t$ we can show that any probability function on SL satisfying Sx is the (pointwise) limit on $QFSL$ as $t \to \infty$ of $\leq t$-heterogeneous probability functions and in turn then is the limit on $QFSL$ of probability functions having a reduction to a unary language (which depends on t). However the family of probability functions which are such limits is clearly much broader than this and currently still awaits classification.

Chapter 34

A GENERAL REPRESENTATION THEOREM FOR Sx

In this chapter we shall utilize representation theorems from previous chapters to prove a general representation theorem for all probability functions satisfying Sx which sheds more light on just why some probability functions satisfying Sx do not also satisfy Li with Sx. To this end we will prove two results, one for when the language L is purely unary and a second for when it is not. We start with the purely unary L, in which case of course Sx reduces to Ax and by Theorem 32.3 Li with Sx is equivalent to ULi with Ax.

Theorem 34.1. *Let the probability function w for the unary language L satisfy Ax. Then there are probability functions v_1, v_2 on SL satisfying ULi with Ax and a constant $\lambda \geq 0$ such that $w = (1+\lambda)v_1 - \lambda v_2$.*

Proof. First suppose that, in the notation of Theorem 14.1,

$$w = v_{\vec{c}} = |\mathsf{S}_{2^q}|^{-1} \sum_{\sigma \in \mathsf{S}_{2^q}} w_{\sigma\vec{c}}$$

where $\vec{c} = \langle c_1, c_2, \ldots, c_{2^q} \rangle \in \mathbb{D}_{2^q}$ and $\sigma\vec{c} = \langle c_{\sigma(1)}, c_{\sigma(2)}, \ldots, c_{\sigma(2^q)} \rangle$. We prove the result by induction on $k = |\{i \mid c_i \neq 0\}|$ with a uniform value λ_k of λ.

If $k = 1$ then we have $w = w_L^1$ so we can take $\lambda_1 = 0$, $v_1 = w$. Now suppose that the result is proved below k, $1 < k \leq 2^q$, without loss of generality say

$$c_1 \geq c_2 \geq \cdots \geq c_k > 0 = c_{k+1} = c_{k+2} = \cdots = c_{2^q}.$$

Then for $\bar{c} = \langle 0, c_1, c_2, \ldots, c_k, 0, 0, \ldots \rangle \in \mathbb{B}$, in the notation of (192),

$$w = |\mathsf{S}_{2^q}|^{-1}(2^q - k)!\Big(2^{kq} u^{\bar{c},L} - \sum_f v_{f(\bar{c})}\Big) \tag{319}$$

where the f range over *non-injective* functions from $\{1, 2, \ldots, k\}$ to $\{1, 2, \ldots, 2^q\}$ (note that there are $2^{kq} - (2q) \times \cdots \times (2^q - k + 1)$ of them). Each of these $f(\bar{c}) \in \mathbb{D}_{2^q}$ have some $m_f \leq k - 1$ non-zero entries so by the inductive assumption

$$v_{f(\bar{c})} = (1 + \lambda_{m_f})v_1^f - \lambda_{m_f} v_2^f$$

for some v_1^f, v_2^f probability functions on SL satisfying ULi with Ax. Hence from (319),

$$w = |\mathsf{S}_{2^q}|^{-1}(2^q - k)!\left(2^{kq}u^{\bar{c},L} + \sum_f \lambda_{m_f} v_2^f - \sum_f (1+\lambda_{m_f})v_1^f\right)$$

and taking

$$\lambda_k = |\mathsf{S}_{2^q}|^{-1}(2^q - k)!\sum_f (1+\lambda_{m_f}) = |\mathsf{S}_{2^q}|^{-1}(2^q - k)!\Big(2^{kq} + \sum_f \lambda_{m_f}\Big) - 1,$$

$$v_1 = \Big(2^{kq} + \sum_f \lambda_{m_f}\Big)^{-1}\Big(2^{kq}u^{\bar{c},L} + \sum_f \lambda_{m_f} v_2^f\Big),$$

$$v_2 = \Big(\sum_f (1+\lambda_{m_f})\Big)^{-1}\sum_f (1+\lambda_{m_f})v_1^f,$$

gives the required result. Notice that the v_1^f, v_2^f here are uniformly (in $\vec{c}$) linear functions of probability functions of the form $v_{g(\vec{c})}$ for some continuous functions $g : \mathbb{D}_{2^q} \to \mathbb{D}_{2^q}$.

Now let w be any probability function on SL satisfying Ax. By Theorem 14.1, for some measure μ,

$$w = \int_{\mathbb{D}_{2^q}} v_{\vec{c}}\; d\mu(\vec{c}).$$

Splitting $\mathbb{D}_{2^q}$ into those (Borel) sets

$$D_k = \{\vec{c} \in \mathbb{D}_{2^q} \mid |\{i \mid c_i \neq 0\}| = k\}$$

we have that

$$\begin{aligned} w &= \sum_{k=1}^{2^q} \int_{D_k} v_{\vec{c}}\; d\mu(\vec{c}) \\ &= \sum_{k=1}^{2^q} \int_{D_k} ((1+\lambda_k)v_{1,\vec{c}} - \lambda_k v_{2,\vec{c}})\; d\mu(\vec{c}) \\ &= \sum_{k=1}^{2^q} \int_{D_k} (1+\lambda_k)v_{1,\vec{c}}\; d\mu(\vec{c}) - \sum_{k=1}^{2^q} \int_{D_k} \lambda_k v_{2,\vec{c}}\; d\mu(\vec{c}) \end{aligned} \tag{320}$$

where, for $\vec{c} \in D_k$,

$$v_{\vec{c}} = (1+\lambda_k)v_{1,\vec{c}} - \lambda_k v_{2,\vec{c}}$$

in the required form. Putting

$$\lambda = \sum_{k=1}^{2^q} \lambda_k \mu(D_k)$$

in (320) now gives the required representation for w. ⊣

A (unary) result similar in form to the above theorem can also be proved even if we drop Ax, when the ULi families of probability functions just satisfy the standing requirements of Px and Ex. Namely the following is shown in [68]:

THEOREM 34.2. *Let w be a unary probability function on SL satisfying Px. Then there exist $\lambda \geq 0$ and probability functions w_1, w_2 satisfying ULi such that*

$$w = (1 + \lambda)w_1 - \lambda w_2.$$

We now embark on the somewhat more torturous proof in the case that L is not purely unary. The following theorem appears in [108].

THEOREM 34.3. *Suppose that L is not purely unary and let w be a probability function on SL satisfying Sx. Then*

$$w = (1 + \lambda)v_1 - \lambda v_2$$

for some $0 \leq \lambda \leq 1$ and probability functions v_1, v_2 satisfying Li with Sx.

The proof goes via a discussion and several lemmas.

Let

$$\bar{p} \in \mathbb{B}_t = \{\langle p_0, p_1, p_2, p_3, \dots \rangle \in \mathbb{B} \mid p_0 = 0 \text{ and } p_t > 0 = p_{t+1}\}.$$

Let $\mathbb{E}^t$ be the set of all partitions of the set of colours $\{1, 2, \dots, t\}$ and let $F = \{F_1, F_2, \dots, F_k\} \in \mathbb{E}^t$, so $k = |F|$. Recall that for $F, G \in \mathbb{E}^t$ we write $F \trianglelefteq G$ if G is a refinement of F. We take I to be the maximal partition in $\mathbb{E}^t$ consisting of all singletons.

Let $F(\bar{p})$ be the element of $\mathbb{B}_{|F|}$ formed by ordering the numbers

$$\sum_{j \in F_s} p_j, \ \ s = 1, 2, \dots, |F|$$

and setting the remaining entries to be 0. Note that $I(\bar{p}) = \bar{p}$.

To reduce the notation we will drop the superscript L in what follows. The following lemma was originally proved by Landes in [74, Theorem 12].

LEMMA 34.4.

$$u^{F(\bar{p})} = \sum_{G \trianglelefteq F} \frac{\mathcal{N}(\emptyset, 1_{|G|})}{SD(|F|)} v^{G(\bar{p})}.$$

PROOF. It is clearly enough to prove this in the case $F = I$, i.e.

$$u^{\bar{p}} = \sum_{G \trianglelefteq I} \frac{\mathcal{N}(\emptyset, 1_{|G|})}{SD(t)} v^{G(\bar{p})}. \tag{321}$$

First note that

$$\sum_{G \trianglelefteq I} \frac{\mathcal{N}(\emptyset, 1_{|G|})}{SD(t)} = 1$$

so the right hand side of (321) defines a probability function and the lemma will follow once we show that

$$u^{\bar{p}}\left(\Phi(a_1,\dots,a_k)\right) \leq \sum_{G \trianglelefteq I} \frac{\mathcal{N}(\emptyset, 1_{|G|})}{SD(t)}\, v^{G(\bar{p})}\left(\Phi(a_1,\dots,a_k)\right)$$

for all state descriptions $\Phi(a_1,\dots,a_k)$, $k \in \mathbb{N}$.

Let $\Phi(a_1,\dots,a_k)$ be a state description, let n be large compared with k, $\vec{a} = \langle a_1,\dots,a_n\rangle$, and let C be the set of $\vec{c} = \langle c_1,\dots,c_n\rangle$ such that all the colours $1,\dots,t$ appear amongst $\{c_1,\dots,c_n\}$. Using the $j^{\bar{p}}$ as defined on page 214 we clearly have

$$\lim_{n\to\infty} \sum_{\substack{\vec{c}\in C\\ \Theta(\vec{a})\vDash\Phi}} j^{\bar{p}}(\Theta(a_1,\dots,a_n),\vec{c}) = u^{\bar{p}}(\Phi(a_1,\dots,a_k))$$

since the probability of some colour not being picked amongst $c_1,\dots,c_n$ tends to 0 as $n \to \infty$.

Hence it suffices to show that

$$\sum_{\substack{\vec{c}\in C\\ \Theta(\vec{a})\vDash\Phi}} j^{\bar{p}}(\Theta(a_1,\dots,a_n),\vec{c}) \leq \sum_{G \trianglelefteq I} \frac{\mathcal{N}(\emptyset, 1_{|G|})}{SD(t)}\, v^{G(\bar{p})}\left(\Phi(a_1,\dots,a_k)\right).$$

Consider a state description $\Theta(\vec{a}), \Theta \vDash \Phi$. Let $\vec{d} = \langle d_1, d_2,\dots,d_n\rangle \in C$ be such that $j^{\bar{p}}(\Theta,\vec{d}) \neq 0$. Then we must have

$$j^{\bar{p}}(\Theta,\vec{d}) = |\mathcal{C}(\vec{a},\vec{d})|^{-1}\prod_{i=1}^{n} p_{d_i} = SD(t)^{-1}\prod_{i=1}^{n} p_{d_i}. \tag{322}$$

Let $\vec{c} \in C$ and let $\sim_{\vec{c}}$ be the equivalence relation on $\{1,2,\dots,t\}$ such that

$$c_i \sim_{\vec{c}} c_j \iff a_i \sim_\Theta a_j.$$

Note that this is a correct definition of an equivalence relation since $c_i = c_j$ implies $a_i \sim_\Theta a_j$ (the colours here are $1,\dots,t$ so 0 does not occur) and $\sim_\Theta$ is an equivalence relation. Let $[c_i]$ be the equivalence class with respect to $\sim_{\vec{c}}$ containing c_i. Note that $\sim_{\vec{c}}$ has the same number of classes as $\mathcal{E}(\Theta)$. For $\vec{d} \in C$ let $\vec{d} \equiv \vec{c}$ if $d_i \sim_{\vec{c}} c_i$ for $i = 1,2,\dots,n$. In this case notice that $\sim_{\vec{d}} = \sim_{\vec{c}}$.

Using (322) we have

$$\sum_{\substack{\vec{d}\in C\\ \vec{d}\equiv\vec{c}}} j^{\bar{p}}(\Theta(a_1,\dots,a_n),\vec{d}) = SD(t)^{-1}\prod_{i=1}^{\square n}\Big(\sum_{d\in[c_i]} p_d\Big) \leq$$

$$SD(t)^{-1}\prod_{i=1}^{n}\Big(\sum_{d\in[c_i]} p_d\Big) \tag{323}$$

where the $\Box$ means that only the terms in the formal expansion of this product which contain each of the p_i at least once are to be counted.

Writing G for $\sim_{\vec{c}}$ it is now easy to see that $G(\bar{p})$ has $|\mathcal{E}(\Theta)|$ non-zero entries $\sum_{d\in[c_i]} p_d$ and

$$\mathcal{N}(\emptyset, 1_{|G|})^{-1}\prod_{i=1}^{n}\Big(\sum_{d\in[c_i]} p_d\Big)$$

is the contribution as in (263) given to $v^{G(\bar{p})}(\Theta(a_1,\dots,a_n))$ by the vector of probabilities

$$\sum_{d\in[c_1]} p_d,\ \sum_{d\in[c_2]} p_d,\ \dots,\ \sum_{d\in[c_n]} p_d.$$

Consequently,

$$\sum_{\substack{\vec{d}\in C\\ \sim_{\vec{d}}\,=\,G}} j^{\bar{p}}(\Theta(a_1,\dots,a_n),\vec{d}) \le \frac{\mathcal{N}(\emptyset, 1_{|G|})}{SD(t)}\, v^{G(\bar{p})}\left(\Theta(a_1,\dots,a_n)\right)$$

and summing over state descriptions $\Theta(\vec{a})$, $\Theta \vDash \Phi$ and over $G \trianglelefteq I$ yields the required result. $\dashv$

Thinking of the rows and columns to be indexed by elements of $\mathbb{E}^t$ let A be the $|\mathbb{E}^t| \times |\mathbb{E}^t|$ matrix with entry on the Fth row, Gth column $\mathcal{N}(\emptyset, 1_{|G|})/SD(|F|)$ for $G \trianglelefteq F$ and 0 otherwise. Since any partial ordering can be embedded in a linear (total) ordering we can assume that the Fth row/column is below/right of the Gth row/column if $G \triangleleft F$. In this case A is a lower triangular matrix and it is then easy to see from Lemma 34.4 that $\vec{u}^T = A\vec{v}^T$, where the Fth element of $\vec{u}$ is $u^{F(\bar{p})}$ etc..

Since the diagonal elements of A are non-zero A has a non-zero determinant and hence an inverse. For $F \in \mathbb{E}^t$ let the entry in the Ith row Fth column of A^{-1} be a_F. Then for $G \triangleleft I$,

$$a_I = \frac{SD(t)}{\mathcal{N}(\emptyset, 1_t)}, \tag{324}$$

$$\frac{a_G}{SD(|G|)} = -\sum_{G\triangleleft F} \frac{a_F}{SD(|F|)}, \tag{325}$$

$$v^{I(\bar{p})} = \sum_{F\in\mathbb{E}^t} a_F u^{F(\bar{p})}. \tag{326}$$

From (324), (325) we see that

$$\frac{a_G}{SD(|G|)} = \frac{N_G a_I}{SD(t)} = \frac{N_G}{\mathcal{N}(\emptyset, 1_t)}$$

where

$$N_G = \sum_{z:G\rightsquigarrow I} (-1)^{|z|}$$

and $z : G \rightsquigarrow I$ means that z is a path

$$G = F_0 \lhd F_1 \lhd F_2 \lhd \cdots \lhd F_k = I$$

(with $|z| = k$ in this case).

From this it follows from (326) that

$$v^{\bar{p}} = v^{I(\vec{p})} = \sum_{N_F > 0} \frac{N_F\, SD(|F|)\, u^{F(\bar{p})}}{\mathcal{N}(\emptyset, 1_t)} - \sum_{N_F < 0} \frac{(-N_F) SD(|F|) u^{F(\bar{p})}}{\mathcal{N}(\emptyset, 1_t)}. \tag{327}$$

Each of the two sums in (327) is a linear combination of language invariant probability functions satisfying Sx, hence a constant multiple of a language invariant probability function satisfying Sx. Considering their values for a tautology shows the multiplicative factors to be respectively

$$\lambda_+^t = \sum_{N_F > 0} \frac{N_F SD(|F|)}{\mathcal{N}(\emptyset, 1_t)}, \quad \lambda_-^t = \sum_{N_F < 0} \frac{(-N_F) SD(|F|)}{\mathcal{N}(\emptyset, 1_t)}$$

with $\lambda_t^+ = \lambda_t^- + 1$.

Hence (327) shows that for any $\bar{p} \in \mathbb{B}_t$ there are two language invariant probability functions w_+, w_- such that

$$v^{\bar{p}} = \lambda_+^t w_+ - \lambda_-^t w_-$$

and it also specifies these functions.

Using the representation (30.2) and taking integrals and finite sums proves the theorem for any $\leq t$-heterogeneous probability function (satisfying Sx of course).

Now consider an arbitrary probability function w satisfying Sx. By the representation (30.2) we have

$$w = \sum_{t=0}^{\infty} \eta_t w^{[t]}$$

with $w^{[0]}$ homogeneous, the $w^{[t]}$ t-heterogeneous for $t > 0$ and $\sum_{t=0}^{\infty} \eta_t = 1$. For each $t > 0$ we have

$$w^{[t]} = \lambda_+^t w_+^{[t]} - \lambda_-^t w_-^{[t]}$$

so

$$w = \eta_0 w^{[0]} + \sum_{t=1}^{\infty} \eta_t (\lambda_+^t w_+^{[t]} - \lambda_-^t w_-^{[t]}).$$

To separate these positive and negative factors each into a constant times a language invariant probability function we require that

$$\sum_{t=1}^{\infty} \eta_t \lambda_-^t < \infty. \tag{328}$$

To show this we need to prove that

$$\sum_{t>0} \sum_{\substack{F \in \mathbb{E}^t \\ N_F < 0}} \eta_t \frac{(-N_F) SD(|F|)}{\mathcal{N}(\emptyset, 1_t)} < \infty.$$

Clearly for this it would be enough if the

$$\sum_{\substack{F \in \mathbb{E}^t \\ N_F < 0}} \frac{(-N_F) SD(|F|)}{\mathcal{N}(\emptyset, 1_t)} \tag{329}$$

were bounded. Note that if this is the case then the corresponding sum with $N_F > 0$ can be at most 1 more (because $w_+^{[t]}(\top) = w_-^{[t]}(\top) = 1$).

Lemma 34.5. *For $F \in \mathbb{E}^t$, $|F| = k$ we have $|N_F| \leq 2^{(t-k)^2}$.*

Proof. For $0 < r \leq t - k$ let

$$M_F^r = |\{ z : F \rightsquigarrow I \mid |z| = r \}|.$$

We illustrate a method of coding the elements of this set, in the first case with a simple example. Suppose $r = 3$,

$$F = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}.$$

Let h be the map with domain $\{1, 2, 3, 5, 6, 7\}$ which sends 1 and 3 to 1, 6 to 2 and 2,5,7 to 3. Then h codes the $z_h : F \rightsquigarrow I$ given by successively dividing the sets in F at the numbers mapped to 1 by h, then those mapped to 2, then those mapped to 3, i.e. z_h is the path

$$\begin{aligned} F = \{ &\{1, 2, 3, 4\}, \{5, 6, 7, 8\} \}, \\ \{ &\{1\}, \{2, 3\}, \{4\}, \{5, 6, 7, 8\} \}, \\ \{ &\{1\}, \{2, 3\}, \{4\}, \{5, 6\}, \{7, 8\} \}, \\ \{ &\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\} \} = I. \end{aligned}$$

Of course we assumed here that the original partition of F and all subsequent partitionings kept the elements in the order $1, 2, 3, 4, 5, 6, 7, 8$. In general this would not be the case. However any other such path can be obtained from this one by taking a permutation of the elements in the sets in F.

Some paths are thus obtained through more than one combination of a mapping and a permutation but we do have that

$$M_F^r \leq J(t - k, r) \prod_{i=1}^{k} (|F_i|!)$$

where $J(t - k, r)$ is the number of maps from a set with $t - k$ elements onto one with r elements and $F = \{F_1, F_2, \ldots, F_k\}$.

Consequently,

$$|N_F| \leq \sum_{r=1}^{t-k} |M_F^r| \leq (t-k)(t-k)^{t-k}(t-k)! \leq 2^{(t-k)^2},$$

as required. ⊣

Let $\left\{ {t \atop k} \right\}$ denote a Stirling Number of the second kind, that is the number of partitions of $\{1, 2, \ldots, t\}$ into k non-empty sets.

Lemma 34.6.

$$\sum_{k=1}^{t-1} \left\{ {t \atop k} \right\} 4^{-k(t-k)} \leq 1/2.$$

Proof. The proof is by induction on t. If $t = 2$ then the result holds, so suppose it holds below $t \geq 2$. Then using the identity

$$\left\{ {t \atop k} \right\} = \left\{ {t-1 \atop k-1} \right\} + k \left\{ {t-1 \atop k} \right\}$$

and the inductive hypothesis, and dropping zero terms, we obtain that

$$\begin{aligned}
&\sum_{k=1}^{t-1} \left\{ {t \atop k} \right\} 4^{-k(t-k)} \\
&= \sum_{k=2}^{t-1} \left\{ {t-1 \atop k-1} \right\} 4^{-k(t-k)} + \sum_{k=1}^{t-1} \left\{ {t-1 \atop k} \right\} 4^{-k(t-k)} \\
&= \sum_{k=1}^{t-2} \left\{ {t-1 \atop k} \right\} 4^{-(k+1)(t-1-k)} + \sum_{k=1}^{t-1} k \left\{ {t-1 \atop k} \right\} 4^{-k(t-1-k)-k} \\
&= \sum_{k=1}^{t-2} \left\{ {t-1 \atop k} \right\} 4^{-k(t-1-k)} \cdot 4^{-(t-1-k)} \\
&\quad + \sum_{k=1}^{t-2} \left\{ {t-1 \atop k} \right\} 4^{-k(t-1-k)} \cdot \frac{k}{4^k + (t-1)\left\{ {t-1 \atop t-1} \right\} 4^{-(t-1)}} \\
&\leq (1/2)4^{-1} + (1/2)4^{-1} + (t-1)4^{-(t-1)} \\
&\leq 1/8 + 1/8 + 1/4 \; = \; 1/2.
\end{aligned}$$

⊣

We can now conclude the proof of the main theorem. By considering the 'worst case' where L has a single binary relation symbol (recall that L is not entirely unary) we have that the probability that a random state description $\Theta(a_1, \ldots, a_t)$ does not have spectrum 1_t is at most the sum of the probabilities that for some $1 \leq i < j \leq t$, $a_i \sim_\Theta a_j$, i.e.

$$\sum_{1 \leq i < j \leq t} 2^{-2t+1} = t(t-1)2^{-2t} \leq 1/8.$$

Hence

$$\frac{SD(t)}{\mathcal{N}(\emptyset, 1_t)} \leq 8/7 \quad \text{for all } t.$$

Again by considering this 'worst case' language, if $f \in \mathbb{E}^t$ has k classes then

$$\frac{SD(|F|)}{SD(t)} \leq \frac{2^{k^2}}{2^{t^2}}.$$

Noticing that if $F = I$ then $N_F > 0$ we consequently have that

$$\sum_{\substack{F \in \mathbb{E}^t \\ N_F < 0}} \frac{(-N_F)SD(|F|)}{\mathcal{N}(\emptyset, 1_t)} = \sum_{\substack{F \in \mathbb{E}^t \\ N_F < 0}} \frac{(-N_F)SD(|F|)}{SD(t)} \frac{SD(t)}{\mathcal{N}(\emptyset, 1_t)} \leq$$

$$\sum_{k=1}^{t-1} \left\{ \begin{matrix} t \\ k \end{matrix} \right\} 2^{(t-k)^2} \frac{2^{k^2}}{2^{t^2}} (8/7) = (8/7) \sum_{k=1}^{t-1} \left\{ \begin{matrix} t \\ k \end{matrix} \right\} 4^{-k(t-k)} \leq 1,$$

by Lemma 34.6, which concludes the proof of the Representation Theorem. ⊣

The following corollary is an immediate consequence of the Representation Theorem 32.1 for probability functions satisfying Li with Sx.

COROLLARY 34.7. *For any probability function on SL satisfying Sx there are measures μ_+ and μ_- on $\mathbb{B}$ and $0 \leq \lambda \leq 1$ such that*

$$w = (\lambda + 1) \int_{\mathbb{B}} u^{\bar{p},L} d\mu_+ - \lambda \int_{\mathbb{B}} u^{\bar{p},L} d\mu_-. \tag{330}$$

Note that a converse to Corollary 34.7 also holds: For any choice of λ, μ_+, μ_-, *if* the right hand side of (330) defines a probability function *then* it will satisfy Spectrum Exchangeability. This then gives an interesting insight into why some probability functions w satisfying Spectrum Exchangeability fail to satisfy Language Invariance. If we had $\lambda = 0$ in (330) then w would satisfy Li by Theorem 32.1 . However in the case $\lambda > 0$ this artifice can fail because the resulting function on the right hand side of (330) gives negative values when these language invariant extensions of the $u^{\bar{p},L}$ are substituted.

As a further simple consequence of Corollary 34.7 we have:

COROLLARY 34.8. *If $\theta \in SL$ and $u^{\bar{p},L}(\theta) = c$ for all $u^{\bar{p},L}$ then for any probability function w on SL satisfying Sx, $w(\theta) = c$.*

Chapter 35

THE CARNAP-STEGMÜLLER PRINCIPLE

The Representation Theorem 32.1 also throws light on an idea of Carnap and Stegmüller dating back to their [18, p226]. The authors call it *Analogieschluss* which roughly translates to English as *inference by analogy.* The intuition here is that for state descriptions $\Psi(b_1,\dots,b_m), \Phi(b_1,\dots,b_m)$ for languages L_1, L_2, respectively, with no relation symbols in common, the probability of the state description $\Psi(b_1,\dots,b_m) \wedge \Phi(b_1,\dots,b_m)$ of $L_1 \cup L_2$ should be greater the more Φ 'agrees with' Ψ in the sense of not violating the equivalences (i.e. indistinguishabilities) already determined by $\sim_\Psi$.

The following theorem, stated in [79] and proved in [74, p104], shows that for a language invariant family for Sx and a suitable formalization of 'agrees with' this does indeed hold.[113]

Theorem 35.1. *Let $\Psi(b_1,\dots,b_m)$, $\Phi(b_1,\dots,b_m)$ be state descriptions of L_1, L_2, respectively, where L_1, L_2 have no relation symbols in common, such that $\sim_\Psi \trianglerighteq \sim_\Phi$. Let $L = L_1 \cup L_2$ and let w be a probability function on SL satisfying Li with Sx. Then for all state descriptions $\Theta(b_1,\dots,b_m)$ of L_2*

$$w(\Psi \wedge \Phi) \geq w(\Psi \wedge \Theta).$$

This inequality is sharp just if $\sim_\Psi \not\trianglerighteq \sim_\Theta$ and for μ as in Theorem 32.1,

$$\int_{\mathbb{B}-\{\langle 1,0,0,\dots\rangle\}} u^{\bar{p},L}(\Psi)\, d\mu(\bar{p}) > 0,$$

equivalently w is not of the form $\lambda u^{\langle 1,0,0\dots\rangle,L} + (1-\lambda)w'$ where $0 \leq \lambda \leq 1$ and w' is a probability function on SL such that $w'(\Psi) = 0$.

Proof. If $\sim_\Psi \trianglerighteq \sim_\Theta$ then $\mathcal{S}(\Psi \wedge \Phi) = \mathcal{S}(\Psi \wedge \Theta) = \mathcal{S}(\Psi)$ (in L_1) and by Sx, $w(\Psi \wedge \Phi) = w(\Psi \wedge \Theta)$. So suppose from now on that $\sim_\Psi \not\trianglerighteq \sim_\Theta$, hence $\mathcal{S}(\Psi \wedge \Theta)$ is a strict refinement of $\mathcal{S}(\Psi \wedge \Phi)$.

[113] In a similar vein recall the Counterpart Principle, (214), and Theorem 22.4.

We first consider the special case of the probability functions $u^{\bar{p},L}$. By (245),

$$u^{\bar{p},L}(\Psi(\vec{b}) \wedge \Phi(\vec{b})) = \sum_{\substack{\vec{c} \in \mathbb{N}^m \\ \Psi \wedge \Phi \in \mathcal{C}(\vec{c},\vec{b})}} |\mathcal{C}(\vec{c},\vec{b})|^{-1} \prod_{s=1}^{m} p_{c_s},$$

$$u^{\bar{p},L}(\Psi(\vec{b}) \wedge \Theta(\vec{b})) = \sum_{\substack{\vec{c} \in \mathbb{N}^m \\ \Psi \wedge \Theta \in \mathcal{C}(\vec{c},\vec{b})}} |\mathcal{C}(\vec{c},\vec{b})|^{-1} \prod_{s=1}^{m} p_{c_s}.$$

But a moment's thought shows that whenever $\vec{c}$ is such that $\Psi \wedge \Theta \in \mathcal{C}(\vec{c},\vec{b})$ then also $\Psi \wedge \Phi \in \mathcal{C}(\vec{c},\vec{b})$. Hence

$$u^{\bar{p},L}(\Psi \wedge \Phi) \geq u^{\bar{p},L}(\Psi \wedge \Theta).$$

Furthermore this inequality will be strict if there is a $\vec{c}$ such that $\Psi \wedge \Phi \in \mathcal{C}(\vec{c},\vec{b})$ but not $\Psi \wedge \Theta \in \mathcal{C}(\vec{c},\vec{b})$ *and* the factor $\prod_{s=1}^{m} p_{c_s}$ is non-zero. In fact this will always be the case if there is some $\vec{c}$ for which $\Psi \wedge \Phi \in \mathcal{C}(\vec{c},\vec{b})$, $\prod_{s=1}^{m} p_{c_s} \neq 0$ and not all the c_s are 0 (i.e blacks), since then we can produce such a $\vec{c}$ with a non-zero contribution to $u^{\bar{p},L}(\Psi \wedge \Phi)$ which gives the same non-black colour to all the b_j in some equivalence class which is split in $\Psi \wedge \Theta$, so this $\vec{c}$ does not contribute to $u^{\bar{p},L}(\Psi \wedge \Theta)$. In other words it will always be the case if $u^{\bar{p},L}(\Psi \wedge \Phi) > 0$, equivalently $u^{\bar{p},L}(\Psi) > 0$, and $\bar{p} \neq \langle 1, 0, 0, \ldots \rangle$.

Using the Representation Theorem 32.1 for probability functions satisfying Li with Sx,

$$w = \int_{\mathbb{B}} u^{\bar{p},L} \, d\mu(\bar{p}),$$

the above discussion now gives that

$$w(\Psi \wedge \Phi) \geq w(\Psi \wedge \Theta)$$

with strict inequality if

$$\int_{\mathbb{B} - \{\langle 1,0,0,\ldots \rangle\}} u^{\bar{p},L}(\Psi) \, d\mu(\bar{p}) > 0. \qquad \dashv$$

Chapter 36

INSTANTIAL RELEVANCE AND Sx

Throughout this chapter L is a general language which can also be purely unary.

Previously we have considered how the idea that *the more I have seen something in the past the more likely I should expect to see it in the future* can be captured within Unary Inductive Logic. In Chapter 11 we discussed Carnap's Principle of Instantial Relevance, PIR, and in Chapter 21 we introduced the Unary Principle of Induction, UPI, as two possible formalizations of this intuition and we saw that PIR follows from Ex and the UPI from Ax.

It appears that PIR does not easily generalize to Polyadic Inductive Logic because it supposes that a state description involving an a_i fixes all there is to know about a_i, which is no longer true once we allow non-unary relation symbols. Extending a state description for $n+1$ constants to involve a new constant can be done in considerably more ways than extending a state description for n constants because there are more constants that relate to the new one. Consequently probabilities of extending a state description with n constants in a certain way are not easily compared with probabilities of extending state descriptions with $n+1$ constants.

However, UPI does generalize straightforwardly to the polyadic case:

The Principle of Induction, PI.

Let $\Theta(b_1, \dots, b_m)$ *be a state description and let* $\Theta_1(b_1, \dots, b_m, b_{m+1})$ *and* $\Theta_2(b_1, \dots, b_m, b_{m+1})$ *be extensions of* Θ. *Then*

$$w(\Theta_1 \mid \Theta) \;\geq\; w(\Theta_2 \mid \Theta) \tag{331}$$

whenever

$$0 \neq |\{i \mid 1 \leq i \leq m \text{ and } b_{m+1} \sim_{\Theta_1} b_i\}| \geq \\ |\{i \mid 1 \leq i \leq m \text{ and } b_{m+1} \sim_{\Theta_2} b_i\}|\,.$$

(If both the above are 0 then we may have any of $<$, $>$ or $=$ in (331).)

It is an open question if PI is satisfied by every probability function satisfying Sx (we would conjecture it is) but we can prove it for probability functions satisfying Li with Sx:

THEOREM 36.1. *Let w be a probability function satisfying Language Invariance with Sx. Then w satisfies PI.*

As in the unary case (see Chapter 21, page 161), this theorem is a consequence of a more general property reminiscent of the Only Rule (Theorem 21.6), which we will now investigate.

For spectra[114] $\tilde{n} = \langle n_1, n_2, \dots, n_r \rangle$ and $\tilde{m} = \langle m_1, m_2, \dots, m_t \rangle$ let

$$\tilde{n} \preceq \tilde{m} \iff \sum_{i \geq j} n_i \geq \sum_{i \geq j} m_i \text{ for all } j = 1, 2, \dots, \max\{r, t\}$$

where we take $n_i = 0$ for $r < i \leq \max\{r, t\}$ and $m_i = 0$ for $t < i \leq \max\{r, t\}$.

LEMMA 36.2. *Let $\Theta(b_1, \dots, b_n)$, $\Psi(b_1, \dots, b_m)$ be state descriptions. Then*

$$\mathcal{S}(\Theta) \preceq \mathcal{S}(\Psi) \iff \textit{For all } \bar{p} \in \mathbb{B}\,,\ u^{\bar{p},L}(\Theta) \leq u^{\bar{p},L}(\Psi)\,.$$

PROOF. Let $\mathcal{S}(\Theta) = \langle n_1, \dots, n_r \rangle = \tilde{n}$ and $\mathcal{S}(\Psi) = \langle m_1, \dots, m_t \rangle = \tilde{m}$. We may assume that $n = \sum_{i \geq 1} n_i = \sum_{i \geq 1} m_i = m$ (otherwise $n > m$ so we could consider $\Theta' \vDash \Theta$ with spectrum $\tilde{n}' = \langle n'_1, \dots, n'_s \rangle \preceq \tilde{m}$ obtained by restricting Θ to some constants only, making certain classes smaller as we did in the unary case, see page 160).

$\Rightarrow$: Assume that Θ, Ψ as above are such that $\mathcal{S}(\Theta) \preceq \mathcal{S}(\Psi)$. This means that $r \geq t$ but we will assume $r = t$ appending zeros to $\tilde{m}$ if necessary. Let $\bar{p} \in \mathbb{B}$.

First assume $p_0 = 0$. Let $\mathcal{E}(\Theta) = \{E_1, \dots, E_r\}$. In this case $u^{\bar{p}}(\Theta(b_1, \dots, b_m))$ equals

$$\sum_{k>0} \sum_{\substack{X \subset \mathbb{N}^+ \\ |X|=k}} \sum_{\substack{S_1 \cup \dots \cup S_r = X \\ S_i \cap S_j = \emptyset,\ 1 \leq i < j \leq r}} \sum_{\vec{c} \in C(X, \vec{S})} |\mathcal{C}(\vec{c}, \vec{b})|^{-1} \prod_{j=1}^{m} p_{c_j} \tag{332}$$

where $C(X, \vec{S})$ is the set of $\langle c_1, \dots, c_m \rangle \in \mathbb{N}^m$ such that $\{c_1, c_2, \dots, c_m\} = X$ (that is, the colours used are exactly those from X) and

$$c_i \in S_j \Rightarrow b_i \in E_j$$

i.e. the colours used for constants from the jth equivalence class of $\sim_\Theta$ are from S_j. Hence

$$u^{\bar{p},L}(\Theta) = \sum_{k>0} \sum_{\substack{X \subset \mathbb{N}^+ \\ |X|=k}} \sum_{\substack{S_1 \cup \dots \cup S_r = X \\ S_i \cap S_j = \emptyset,\ 1 \leq i < j \leq r}} \mathcal{N}(\emptyset, 1_k)^{-1} \prod_{j=1}^{r} \Big(\sum_{i \in S_j} p_i \Big)^{\square n_j}, \tag{333}$$

where the $\square$ in $\Big(\sum_{i \in S_j} p_i \Big)^{\square n_j}$ etc. indicates that in the expansion of this power we only count those terms which have a non-zero power of p_i

[114] Recall the convention from page 224 concerning writing spectra as vectors in non-increasing order.

for each $i \in S_j$, etc.[115] Thus employing (333) for both Θ and Ψ and considering particular X and k, $u^{\bar{p}}(\Theta) \leq u^{\bar{p}}(\Psi)$ is a consequence of the following generalization of Muirhead's Inequality shown in [109].

THEOREM 36.3. *If* $\vec{n} = \langle n_1, n_2, \dots, n_r \rangle$, $\vec{m} = \langle m_1, m_2, \dots, m_r \rangle \in \mathbb{N}^r$ *satisfy* $n_1 \geq n_2 \geq \dots, \geq n_r \geq 0$, $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$, $\sum_{i=1}^r m_i = \sum_{i=1}^r n_i$ *and* $\vec{n} \preceq \vec{m}$ *then for* $0 \leq p_1, \dots, p_k \in \mathbb{R}$,

$$\sum_{\substack{S_1 \cup \cdots \cup S_r = \{p_1, \dots, p_k\} \\ S_i \cap S_j = \emptyset,\ 1 \leq i < j \leq r}} \prod_{j=1}^{r} \Big(\sum_{i \in S_j} p_i \Big)^{\square n_j} \leq \sum_{\substack{S_1 \cup \cdots \cup S_r = \{p_1, \dots, p_k\} \\ S_i \cap S_j = \emptyset,\ 1 \leq i < j \leq r}} \prod_{j=1}^{r} \Big(\sum_{i \in S_j} p_i \Big)^{\square m_j}. \tag{334}$$

The general case when $p_0 \neq 0$ follows upon noting that $u^{\bar{p},L}$ can be approximated arbitrarily closely by a $u^{\bar{q},L}$ with $q_0 = 0$. To see this let N be large and let $\bar{q}$ be the sequence in $\mathbb{B}$ resulting from $\bar{p}$ by replacing the colour black by N shades of grey, each assigned probability p_0/N. In other words $q_0 = 0$ and the q_i for $i > 0$ are just the same probabilities $p_1, p_2, p_3, \dots$ of the old non-black colours together with N probabilities $p_0/N, p_0/N, \dots, p_0/N$ for these new greys inserted at an appropriate position. So

$$q_i = \begin{cases} 0 & \text{if } i = 0, \\ p_i & \text{if } i \leq M, \\ p_0/N & \text{if } M+1 \leq i \leq M+N, \\ p_{i-N} & \text{if } i > M+N, \end{cases}$$

where M is the last i for which $p_i > p_0/N$. For $\vec{c} = \langle c_1, \dots, c_m \rangle \in \mathbb{N}^m$ let $\mathcal{K}(\vec{c})$ be the set of all $\vec{k} = \langle k_1, \dots, k_m \rangle$ such that

$$k_j = \begin{cases} \text{one of } M, M+1, \dots, M+N & \text{if } c_j = 0, \\ c_j & \text{if } 1 \leq c_j \leq M, \\ c_j + N & \text{if } M < c_j, \end{cases}$$

so $q_{k_j} = p_{c_j}$ unless $c_j = 0$ in which case $q_{k_j} = p_0/N$. Noting that

- for every $\vec{k} \in (\mathbb{N}^+)^m$ there is a unique $\vec{c} \in \mathbb{N}^m$ such that $\vec{k} \in \mathcal{K}(\vec{c})$
- for every $\vec{c} \in \mathbb{N}^m$ we have

$$\prod_{i=1}^{m} p_{c_i} = \sum_{\vec{k} \in \mathcal{K}(\vec{c})} \prod_{i=1}^{m} q_{k_i}$$

- $\mathcal{C}(\vec{c}, \vec{b}) = \mathcal{C}(\vec{k}, \vec{b})$ unless $\vec{k}$ is such that $k_i = k_j \in \{M+1, \dots, M+N\}$ for some $i \neq j$ (in which case $\mathcal{C}(\vec{c}, \vec{b}) \supset \mathcal{C}(\vec{k}, \vec{b})$),

[115] Note that for S and l such that $l < |S|$ we have $\left(\sum_{i \in S} p_i\right)^{\square l} = 0$. As usual we take sums over the empty set to be 0, so

$$\Big(\sum_{i \in \emptyset} p_i \Big)^{\square l} = \begin{cases} 1 & \text{if } l = 0, \\ 0 & \text{if } l > 0. \end{cases}$$

we can see that

$$u^{\bar{p},L}(\Theta(b_1,\dots,b_m)) = \sum_{\vec{c}\in\mathbb{N}^m} |\mathcal{C}(\vec{c},\vec{b})|^{-1} \prod_{i=1}^{m} p_{c_i}$$

and

$$u^{\bar{q},L}(\Theta(b_1,\dots,b_m)) = \sum_{\vec{c}\in\mathbb{N}^m} \sum_{\vec{k}\in\mathcal{K}(\vec{c})} |\mathcal{C}(\vec{k},\vec{b})|^{-1} \prod_{i=1}^{m} q_{k_i}$$

differ by

$$\sum_{\vec{c}\in\mathbb{N}^m} \sum_{\vec{k}} (|\mathcal{C}(\vec{k},\vec{b})|^{-1} - |\mathcal{C}(\vec{c},\vec{b})|^{-1}) \prod_{i=1}^{m} q_{k_i}$$

where the second summation is restricted to those $\vec{k} \in \mathcal{K}(\vec{c})$ which feature some $k \in \{M+1,\dots,M+N\}$ more than once. Hence this difference is bounded from above by the sum over such $\vec{k}$ of $\prod_{i=1}^{m} q_{k_i}$ which in turn is at most

$$\binom{m}{2} \times N \times (p_0/N)^2 \leq m^2/N.$$

Similarly for $\Psi(b_1,\dots,b_m)$ and since N can be arbitrarily large, the result follows from the case already proved.

$\Leftarrow$: To show the other direction suppose that $\Theta(b_1,b_2,\dots,b_m)$ and $\Psi(b_1,b_2,\dots,b_m)$ are such that

$$\langle n_1,n_2,\dots,n_r\rangle = \mathsf{S}(\Theta) \not\preceq \mathsf{S}(\Psi) = \langle m_1,m_2,\dots,m_t\rangle,$$

with $n_r > 0$. Let $j \leq r$ be such that

$$N = \sum_{i\leq j} n_i > \sum_{i\leq j} m_i = M,$$

such a j must exist since $\sum_{i=1}^{r} n_r = \sum_{i=1}^{t} m_i$ and $\mathcal{S}(\Theta) \not\preceq \mathcal{S}(\Psi)$. Let $\bar{p}$ be such that $p_0 = 0 = p_i$ for $i > r$, $p_i = (1-\varepsilon)/j$ for $i \leq j$, and $p_i = \varepsilon/(s-j)$ for $j < i \leq s$, where $\varepsilon > 0$ is small and $s = \max\{t,r\}$. Then it is straightforward to see that

$$u^{\bar{p}}(\Theta(b_1,b_2,\dots,b_m)) \geq D_1\varepsilon^{m-N}$$
$$u^{\bar{p}}(\Psi(b_1,b_2,\dots,b_m)) \leq D_2\varepsilon^{m-M}$$

for some constants $D_1, D_2 > 0$, which gives as required that

$$u^{\bar{p}}(\Theta(b_1,b_2,\dots,b_m)) \not\leq u^{\bar{p}}(\Psi(b_1,b_2,\dots,b_m)).$$

This completes the proof of Lemma 36.2. ⊣

Corollary 36.4. *Let w be a probability function satisfying Li with Sx and let $\Theta(b_1,\dots,b_n)$, $\Psi(b_1,\dots,b_m)$ be state descriptions such that $\mathcal{S}(\Theta) \preceq \mathcal{S}(\Psi)$. Then $w(\Theta) \leq w(\Psi)$.*

Proof. This follows from Theorem 32.1 and the previous lemma. ⊣

Proof of Theorem 36.1. Let $\Theta(b_1, \ldots, b_m)$ have spectrum $\langle m_1, \ldots, m_r \rangle$. Adding b_{m+1} to the jth equivalence class of Θ produces a state description with the same spectrum as that of Θ except that m_j is increased by 1. Hence if Θ_1 is obtained by adding b_{m+1} to a larger class and Θ_2 is obtained by adding it to a smaller class (or having it form a class of its own, possibly also splitting classes of Θ) then $\mathcal{S}(\Theta_2) \preceq \mathcal{S}(\Theta_1)$ and so by the previous corollary $w(\Theta_2) \leq w(\Theta_1)$. Consequently also $w(\Theta_2|\Theta) \leq w(\Theta_1|\Theta)$, as required. ⊣

We would conjecture that Theorem 36.1 also holds for heterogeneous probability functions, and hence, by the Ladder Theorem 30.2, that Sx alone implies PI. However at present there are only some partial results in this direction concerning heterogeneous probability functions and PI, most notably:

Theorem 36.5. *If w is t-heterogeneous and $\Theta(b_1, \ldots, b_m)$, $\Psi(b_1, \ldots, b_m)$ have spectra of length t and satisfy $\mathcal{S}(\Theta) \preceq \mathcal{S}(\Psi)$ then $w(\Theta) \leq w(\Psi)$. Similarly when both Θ and Ψ have spectra of length $t - 1$.*

This follows from the Representation Theorem 31.11 and from the following lemma originally proved by Landes in [74].

Lemma 36.6. *Let $\bar{p} \in \mathbb{B}_t$ and let $\Theta(b_1, \ldots, b_m)$, $\Psi(b_1, \ldots, b_m)$ have spectra of length t and assume that $\mathcal{S}(\Theta) \preceq \mathcal{S}(\Psi)$. Then $v^{\bar{p},L}(\Theta) \leq v^{\bar{p},L}(\Psi)$. Similarly when both Θ and Ψ have spectra of length $t - 1$.*

Proof. If Θ and Ψ have spectra of length t, say $\mathcal{S}(\Theta) = \langle n_1, \ldots, n_t \rangle$ and $\mathcal{S}(\Psi) = \langle m_1, \ldots, m_t \rangle$, then the $\vec{c} \in \{1, \ldots, t\}^m$ consistent with Θ are those that feature all the t colours, exactly one for each equivalence class, and for such a $\vec{c}$ we have $\mathcal{G}(\vec{c}, \Theta) = 1/\mathcal{N}(\emptyset, 1_t)$, so

$$v^{\bar{p},L}(\Theta) = \frac{1}{\mathcal{N}(\emptyset, 1_t)} \sum_{\sigma \in \mathsf{S}_t} \prod_{i=1}^{t} p_{\sigma(i)}^{n_i}$$

and similarly for Ψ. So the result follows from the (original) Muirhead Inequality, see [39, p44], which gives us that

$$\sum_{\sigma \in \mathsf{S}_t} \prod_{i=1}^{t} p_{\sigma(i)}^{n_i} \leq \sum_{\sigma \in \mathsf{S}_t} \prod_{i=1}^{t} p_{\sigma(i)}^{m_i}.$$

Now suppose that Θ and Ψ both have spectra of length $t - 1$, say $\mathcal{S}(\Theta) = \langle n_1, \ldots, n_{t-1} \rangle$, $\mathcal{S}(\Psi) = \langle m_1, \ldots, m_{t-1} \rangle$. Then in

$$v^{\bar{p},L}(\Theta) = \sum_{\vec{c} \in \{1,\ldots,t\}^m} \mathcal{G}(\vec{c}, \Theta) \prod_{i=1}^{m} p_{c_i}$$

only those $\vec{c}$ feature with non-zero $\mathcal{G}(\vec{c}, \Theta)$ which use exactly $t - 1$ colours. To see this notice that $\vec{c}$ must use at least $t - 1$ colours because there are $t - 1$ classes, and if it used all t colours then there would be no extension to a state description with spectrum 1_t and $\mathcal{G}(\vec{c}, \Theta)$ would be zero. For each such $\vec{c}$ consistent with Θ we have

$$\mathcal{G}(\vec{c}, \Theta) = \frac{\mathcal{N}(1_{t-1}, 1_t)}{\mathcal{N}(\emptyset, 1_t)}$$

and these $\vec{c}$ must have one colour for each equivalence class of Θ so

$$v^{\bar{p},L}(\Theta) = \frac{\mathcal{N}(1_{t-1}, 1_t)}{\mathcal{N}(\emptyset, 1_t)} \sum_{1 \le k_1 < \cdots < k_{t-1} \le t} \sum_{\sigma \in \mathsf{S}_{t-1}} \prod_{i=1}^{t-1} p_{k_{\sigma(i)}}^{n_i}$$

and similarly for Ψ. Hence the result follows again from the Muirhead Inequality which gives us that for every such $k_1, \dots, k_{t-1}$,

$$\sum_{\sigma \in \mathsf{S}_{t-1}} \prod_{i=1}^{t-1} p_{k_{\sigma(i)}}^{n_i} \le \sum_{\sigma \in \mathsf{S}_{t-1}} \prod_{i=1}^{t-1} p_{k_{\sigma(i)}}^{m_i}. \qquad \dashv$$

Chapter 37

EQUALITY

In our language L we have omitted equality, which might seem strange since this is an essential feature of many applications using predicate logic and indeed appears in the seminal papers of Gaifman, Snir, Scott, Krauss on Probability Logic, see for example [30], [32], [132]. However the fact is that, at least in its present state of development, equality seems to us rather secondary in the context of PIL.[116]

To see why this is consider the experiment of picking cards from a set of playing cards with replacement, using a language that describes the suit but not individual cards. Suppose we have picked, and replaced, a heart and then again pick a heart. Does it matter whether it is the same card that we chose before or another card of the same suit? Clearly if we are only interested in assigning a probability to picking a card from a particular suit in the future this information is irrelevant, a set containing merely the four queens is the same as far as we are concerned as the full pack.

Similarly with our agent in question Q. How could our agent ever reasonably assign any rational degree of belief to a_i and a_j being equal, unless $i = j$? It would seem that to rationally make any such commitment would require knowledge which the agent is explicitly assumed not to possess.

Of course adding equality to L does increase its expressive power, for example we now have a sentence expressing that there is exactly one a_i satisfying the predicate $P(x)$. But in view of the above discussion what does that mean as far as our agent is concerned?

Formally adding equality to L poses few problems. In (P1) and (P2) we restrict logical consequence, $\models$, to structures which satisfy the axioms of equality (see for example [91, p94]), so in particular (P1) now gives that the axioms of equality must receive probability 1. In the statements of Px and SN the relation symbol R is not allowed to be $=$ and in the definition of a state description $\Theta(\vec{a})$ we now need to add the requirement that it is consistent, meaning that if $a_i = a_j$ is a conjunct of $\Theta(\vec{a})$ then

[116] An exception here is Hilpinen's [45] which specifically investigated equality in the context of Unary Inductive Logic.

$a_i \sim_\Theta a_j$. $\mathcal{T}L$ is defined as before but for structures satisfying the axioms of equality. Gaifman's Theorem 7.1 and its Corollary 7.2 now follow and beyond that Part 1, *The Basics*, develops as before.

It turns out however that whilst equality may arguably be of little concern to our agent in the context of question $\mathcal{Q}$ it is implicitly present in the principle Sx (at least in the presence of Li) and making this explicit provides us with some *mathematical* insight. This presence comes about through the basic building block probability functions $u^{\bar{p},L}$. For, informally, suppose that within the calculation of $u^{\bar{p},L}(\Theta(\vec{a}))$, a_i and a_j receive the same colour. Then thereafter they will forever remain indistinguishable and hence, in accord with Leibniz Principle [86] (or [28]), should be considered to be identical. Thus it appears that we might extend $u^{\bar{p},L}$ on a language L without equality to L plus equality, which we denote by $L_=$, by identifying constants with the same colour.

First we need to extend some of the notation. We shall say that $\Psi(b_1, \dots, b_m) \in QFSL_=$ is a state description for $b_1, \dots, b_m$, where as usual $b_1, \dots, b_m$ are distinct constants from the $a_1, a_2, a_3, \dots$, if $\Psi(\vec{b})$ is consistent with the axioms of equality and has the form

$$\Theta(b_1, \dots, b_m) \wedge \bigwedge_{1 \leq i,j \leq m} \pm\, b_i = b_j \tag{335}$$

where $\Theta(b_1, \dots, b_m)$ is a state description for L.

We shall refer to the conjunct

$$\bigwedge_{1 \leq i,j \leq m} \pm\, b_i = b_j$$

as an *equality table*, or, as in this case, *the equality table of* Ψ, usually denoting it by Υ. Notice that the requirement of consistency with the axioms of equality entails that:

If $b_i = b_j$ occurs (positively) in the equality table for Ψ then $b_i \sim_\Theta b_j$. In other words, $\sim_\Upsilon$ is a refinement of $\sim_\Theta$.

Conversely if this condition holds between an equality table $\Upsilon(b_1, \dots, b_m)$ and a state description $\Theta(b_1, \dots, b_m) \in QFSL$ then $\Theta(b_1, \dots, b_m) \wedge \Upsilon(b_1, \dots, b_m)$ will be consistent, and hence will be a state description in $L_=$.

Indistinguishability with respect to state descriptions Ψ of $L_=$ is a considerable simplification from what it was for L alone since now $b_i \sim_\Psi b_j$ just when $\Psi \vDash b_i = b_j$. The spectrum of Ψ as in (335) is just the spectrum of Υ and $b_i \sim_\Psi b_j$ implies $b_i \sim_{\Psi'} b_j$ for any state description Ψ' extending Ψ. In other words, once b_i and b_j are equivalent then they remain equivalent forever, in all extending state descriptions. Thus by adding equality we have regained a property which we had in the unary case and which relates Sx to Kingman's 'classical' investigation of Partition Structures, see [66], [67].

We are now in a position to elaborate on the above remarks concerning the implicit presence of equality in the $u^{\bar{p},L}$. Namely, revise the definition of $\mathcal{C}(\vec{c},\vec{b})$ given on page 205 to state descriptions of $L_=$ by setting $\mathcal{C}_=(\vec{c},\vec{b})$ to be the set of all state descriptions $\Psi(\vec{b})$ as in (335) such that $\Theta(b_1,\dots,b_m) \in \mathcal{C}(\vec{c},\vec{b})$ and

$$\Upsilon(\vec{b}) \vDash b_i = b_j \iff c_i = c_j > 0.$$

Notice that

$$|\,\mathcal{C}_=(\vec{c},\vec{b})| = |\,\mathcal{C}(\vec{c},\vec{b})|, \tag{336}$$

where the latter is for L, since $\vec{c}$ determines the table uniquely. Now define by analogy with $u^{\bar{p},L}$ the function $u^{\bar{p},L_=}$ on state descriptions Ψ as above by

$$u^{\bar{p},L_=}(\Psi) = \sum_{\substack{\vec{c}\in\mathbb{N}^m \\ \Psi\in\mathcal{C}_=(\vec{c},\vec{b})}} |\,\mathcal{C}_=(\vec{c},\vec{b})|^{-1} \prod_{s=1}^{m} p_{c_s}. \tag{337}$$

As for $u^{\bar{p},L}$ we can now show that $u^{\bar{p},L_=}$ extends to a probability function on $SL_=$, the only novel feature being to check that $u^{\bar{p},L_=}$ gives the axioms of equality probability 1. To take an example here, we require that for R an r-ary relation symbol of L $u^{\bar{p},L_=}$ gives probability 1 to

$$\forall x_1,\dots,x_{2r}\Big(\bigwedge_{j=1}^{r} x_j = x_{r+j} \to (R(x_1,\dots,x_r) \to R(x_{r+1},\dots,x_{2r}))\Big),$$

equivalently probability 0 to each of the

$$R(b_1,\dots,b_r) \wedge \neg R(b_{r+1},\dots,b_{2r}) \wedge \bigwedge_{j=1}^{r} b_j = b_{r+j}$$

where, contrary to the standing convention, the b_i need not be distinct. But that is clear from (337) since a state description extending this sentence could not be in any $\mathcal{C}_=(\vec{c},\vec{b})$. The other equality axioms are confirmed similarly.

Again as for $u^{\bar{p},L}$ the probability functions $u^{\bar{p},L_=}$ satisfy Ex and Sx.

It now follows that if the probability function w on SL has a representation in the form

$$w = \int_{\mathbb{B}} u^{\bar{p},L}\, d\mu(\bar{p}),$$

equivalently by Theorem 32.1 if w satisfies Li with Sx, then w has an extension to $L_=$, namely

$$w = \int_{\mathbb{B}} u^{\bar{p},L_=}\, d\mu(\bar{p}).$$

In fact as shown in [77] the result goes the other way too:

THEOREM 37.1. *Let w be a probability function on SL. Then the following are equivalent*:

(i) *w has an extension $w^{L_=}$ on $L_=$ satisfying Sx.*
(ii) *w has a representation of the form*

$$w = \int_{\mathbb{B}} u^{\bar{p},L}\, d\mu(\bar{p}).$$

(iii) *w satisfies Li with Sx.*

Furthermore when L is not purely unary the extension in (i) *is unique.*

PROOF. The equivalence of (ii) and (iii) follows from Theorem 32.1. The above discussion shows that (ii) implies (i) so for the first part it only remains to show that (i) implies (iii).

Assume that w satisfies (i) and to cut down on the notation denote this extension also by w. First notice then that $w \restriction SL$ satisfies Sx. Since suppose that $\mathcal{S}(\Theta_1(\vec{b})) = \mathcal{S}(\Theta_2(\vec{b}))$, without loss of generality that $\sim_{\Theta_1} = \sim_{\Theta_2}$ since by our standing assumption w satisfies Ex. Then for any equality table $\Upsilon(\vec{b})$, $\Theta_1(\vec{b}) \wedge \Upsilon(\vec{b})$ will be consistent just if $\Theta_2(\vec{b}) \wedge \Upsilon(\vec{b})$ is consistent and by Sx for $L_=$,

$$w(\Theta_1(\vec{b}) \wedge \Upsilon(\vec{b})) = w(\Theta_2(\vec{b}) \wedge \Upsilon(\vec{b})).$$

Taking the sum over all $\Upsilon(\vec{b})$ gives $w(\Theta_1(\vec{b})) = w(\Theta_2(\vec{b}))$ and Sx for L follows.

Now let L' be a language (without equality) extending L. In view of the previous paragraph it is enough to show that w has a unique extension to $L'_=$ satisfying Sx. By way of motivation here suppose for the moment that we did have such an extension w'. Then it would have to satisfy that if $\Upsilon(\vec{b})$ was an equality table and $\Theta(\vec{b})$, $\Phi(\vec{b})$ state descriptions of L' each separately consistent with Υ then

$$w'(\Theta \wedge \Upsilon) = w'(\Phi \wedge \Upsilon)$$

since these two state descriptions for $L'_=$ would have the same spectra (namely that of Υ). Hence, since w and w' agree on Υ, it would be forced that

$$w'(\Theta \wedge \Upsilon) = K_\Upsilon^{-1} w'(\Upsilon) = K_\Upsilon^{-1} w(\Upsilon) \tag{338}$$

where K_Υ is the number of state descriptions $\Phi(\vec{b})$ of L' such that $\Upsilon \wedge \Phi$ is consistent. It only remains to show that w' *defined* by (338) indeed is (more precisely extends to) a probability function since Sx then follows from the fact that for consistent $\Theta \wedge \Upsilon$, $w'(\Theta \wedge \Upsilon)$ only depends on Υ, and $w'(\Upsilon) = w(\Upsilon)$ depends only on the spectrum of Υ.

Conditions (i) and (ii) of (34) clearly hold for w' defined in this way. Concerning condition (iii) let $\Theta(a_1, \dots, a_m)$ be a state description of L'

and $\Upsilon(a_1, \dots, a_m)$ an equality table consistent with $\Theta(a_1, \dots, a_m)$. Then from (338) we have that for $\Upsilon^+ = \Upsilon^+(a_1, \dots, a_{m+1})$,

$$w'(\Theta \wedge \Upsilon) = K_\Upsilon^{-1} w(\Upsilon) = \sum_{\Upsilon^+ \vDash \Upsilon} \frac{w(\Upsilon^+)}{K_{\Upsilon^+}} \cdot \frac{K_{\Upsilon^+}}{K_\Upsilon}. \tag{339}$$

The first fraction in this summation is $w'(\Theta^+ \wedge \Upsilon^+)$ for any state description $\Theta^+(a_1, \dots, a_{m+1})$ of L' such that $\Theta^+ \wedge \Upsilon^+$ is consistent. The second fraction equals the number of extensions Θ^+ of Θ such that $\Theta^+ \wedge \Upsilon^+$ is consistent. Hence

$$w'(\Theta \wedge \Upsilon) = \sum_{\substack{\Theta^+ \vDash \Theta \\ \Upsilon^+ \vDash \Upsilon}} w'(\Theta^+(a_1, \dots, a_{m+1}) \wedge \Upsilon^+(a_1, \dots, a_{m+1})). \tag{340}$$

Thus w' defined using (338) on the $\Theta(a_1, \dots, a_m) \wedge \Upsilon(a_1, \dots, a_m)$ extends to a probability function on $SL'_=$ and, by Lemma 7.3, satisfies Ex. In turn then it also satisfies (338) for $\Theta(b_1, \dots, b_m) \wedge \Upsilon(b_1, \dots, b_m)$ and so, as already observed, Sx.

It remains to show the uniqueness of $w^{L_=}$ when L has at least one non-unary relation symbol. Suppose on the contrary that w_1, w_2 both satisfied Sx on $L_=$ and extended w. Recall from page 201 the partial ordering on the equivalence relations on $\{1, 2, \dots, m\}$ defined by

$$\sim_1 \lhd \sim_2 \iff \sim_2 \text{ is a proper refinement of } \sim_1 .$$

We shall show by reverse induction on the ordering $\lhd$ of the $\sim_\Upsilon$ that for $\Theta(b_1, \dots, b_m)$ a state description of L and $\Upsilon(b_1, \dots, b_m)$ an equality table consistent with Θ, $w_1(\Theta \wedge \Upsilon) = w_2(\Theta \wedge \Upsilon)$. Since w_1, w_2 both extend w, for Θ as above, for $i = 1, 2$,

$$\begin{aligned} w(\Theta) &= \sum_{\sim_\Theta \unlhd \sim_\Upsilon} w_i(\Theta \wedge \Upsilon) \\ &= \sum_{\sim_\Theta = \sim_\Upsilon} w_i(\Theta \wedge \Upsilon) + \sum_{\sim_\Theta \lhd \sim_\Upsilon} w_i(\Theta \wedge \Upsilon), \end{aligned} \tag{341}$$

where the Υ range over equality tables for $b_1, \dots, b_m$. Thus the right hand sides of (341) are equal for w_1 and w_2. Take $\sim_{\Theta_0}$ to be the top element of the ordering $\lhd$, i.e. where the equivalence classes are all singletons. Notice that there is such a state description because of our assumption that L is not purely unary. Then we obtain from (341) that

$$\sum_{\sim_{\Theta_0} = \sim_\Upsilon} w_1(\Theta_0 \wedge \Upsilon) = \sum_{\sim_{\Theta_0} = \sim_\Upsilon} w_2(\Theta_0 \wedge \Upsilon).$$

In other words $w_1(\Theta_0 \wedge \Upsilon_0) = w_2(\Theta_0 \wedge \Upsilon_0)$ for the unique equality table Υ_0 consistent with Θ_0. By Sx for both w_1 and w_2 then they must also agree on any $\Theta \wedge \Upsilon$ for which $\sim_\Upsilon$ is this top element.

Having established this base case and using (341) it immediately follows by reverse induction on the ordering $\lhd$ that w_1 and w_2 agree on all spectra and hence agree everywhere. ⊣

One might have expected to have also had uniqueness of the extension of w to $L_=$ even in the case when L was purely unary. Somewhat surprisingly however this is not the case in general: As shown earlier on page 245 for L having just a single unary predicate symbol Carnap's c_2^L is actually a member of two distinct Li families satisfying Sx. By the above theorem then c_2^L must have extensions to $L_=$ giving both of these families, and hence two distinct such extensions.

In the above we have tacitly assumed that the language L we were expanding with equality is non-empty. However the demonstration of (iii) from (i) in the proof of Theorem 37.1 works perfectly well even if the language $L_=$ has only $=$ as a relation symbol. That is, if w is a probability function on the sentences of the language L_{Eq} whose only relation symbol is equality and w satisfies Ex and Sx then w has a unique extension to any language $L_=$ which satisfies Ex and Sx. Indeed it is easy to see that for L_{Eq} simply the requirement of Ex (plus giving the axioms of equality probability 1) ensures that Sx holds. This is worth capturing as a corollary:

COROLLARY 37.2. *Let w be a probability function on SL with a representation of the form*

$$w = \int_{\mathbb{B}} u^{\bar{p},L} \, d\mu(\bar{p}).$$

Then w is uniquely determined by the restriction to L_{Eq} of its extension $w^{L_=}$ to $L_=$.

Conversely, a probability function w_{Eq} on SL_{Eq} satisfying Ex uniquely determines a probability function $w^{L_=}$ on $L_=$ satisfying Sx and in turn a probability function $w = w^{L_=} \restriction SL$ on SL satisfying Sx.

At this point one might expect that we could replicate the above procedure with $v^{\bar{p},L}$ in place of $u^{\bar{p},L}$. However, Theorem 37.1 makes it clear that that cannot in general be the case, otherwise by that theorem $v^{\bar{p},L}$, where $\bar{p} \in \mathbb{B}_t$, would have a representation in the form given in (ii), which it does not for $t > 1$.[117] In fact the situation is more dramatic than that: As detailed by Landes in [74], for $t > 1$ and w a t-heterogeneous probability function for a not purely unary language L we cannot add equality nor any new relation symbol at all to L and extend w to that language in such a way as to retain Sx.

To see why this is let w, t, L satisfy these requirements, let L' be the result of adding, say, a new unary relation symbol R to L and suppose

[117] When $t = 1$ the single t-heterogeneous probability function is c_0^L and it does have such a representation.

that w' was a probability function on SL' extending w and satisfying Sx (and Ex as usual).

First note that w' must also be t-heterogeneous. For suppose that $\Phi'(b_1, \ldots, b_{t+1})$ was a state description of L' with spectrum 1_{t+1} and $w'(\Phi'(b_1, \ldots, b_{t+1})) > 0$. Since w' satisfies Sx this would also have to hold for any other state description of L' with this same spectrum, so since L is not purely unary we may also suppose that $\Phi(b_1, \ldots, b_{t+1})$ has this same spectrum where Φ is the state description of L which Φ' logically implies. But this cannot be since

$$w'(\Phi'(\vec{b})) \leq w'(\Phi(\vec{b})) = w(\Phi(\vec{b})) = 0.$$

Turning to the second requirement for t-heterogeneity, with the analogous notation, we have that

$$\begin{aligned}
&\lim_{m\to\infty} w'\Big(\bigvee_{|\mathcal{S}(\Phi'(a_1,\ldots,a_m))|<t} \Phi'(a_1, \ldots, a_m)\Big) \\
&\quad\leq \lim_{m\to\infty} w'\Big(\bigvee_{|\mathcal{S}(\Phi(a_1,\ldots,a_m))|<t} \Phi(a_1, \ldots, a_m)\Big) \\
&\quad= \lim_{m\to\infty} w\Big(\bigvee_{|\mathcal{S}(\Phi(a_1,\ldots,a_m))|<t} \Phi(a_1, \ldots, a_m)\Big) = 0.
\end{aligned}$$

Having established that w' must also be t-heterogeneous let $\Theta(a_1, \ldots, a_t)$ be a state description of L with equivalence classes $\{a_1\}$, $\{a_2\}$, ... , $\{a_{t-2}\}$, $\{a_{t-1}, a_t\}$ and let

$$\Psi(a_1, \ldots, a_t) = \neg R(a_t) \wedge \bigwedge_{i=1}^{t-1} R(a_i)\,.$$

So $\Theta \wedge \Psi$ is a state description of L' with equivalence classes

$$\{a_1\}, \{a_2\}, \ldots, \{a_{t-2}\}, \{a_{t-1}\}, \{a_t\}\,,$$

and in consequence of t-heterogeneity $w'(\Theta \wedge \Psi) > 0$.

Now, with the analogous notation again and $m > t$, let $\Theta^+(a_1, \ldots, a_m) \wedge \Psi^+(a_1, \ldots, a_m)$ be a state description of L' extending $\Theta \wedge \Psi$ with

$$w'(\Theta^+(a_1, \ldots, a_m) \wedge \Psi^+(a_1, \ldots, a_m)) > 0. \tag{342}$$

Then Θ^+ must still have $|\mathcal{S}(\Theta^+)| = t - 1$. Since if some a_j with $t < j \leq m$ split the class $\{a_{t-1}, a_t\}$ of $\sim_\Theta$ then $\sim_{\Theta^+}$ would have at least $t + 1$ equivalence classes, contradicting (342) whilst if some a_j with $t < j \leq m$ started a new class beyond those of $\sim_\Theta$ then together with the fact that a_{t-1}, a_t are distinguished by Ψ, $\Theta^+ \wedge \Psi^+$ would again have at least $t + 1$ equivalence classes and (342) would again be contradicted.

But now this gives that

$$\sum_{\substack{|\mathcal{S}(\Theta^+)|=t-1\\ \Theta^+\vDash\Theta}} w(\Theta^+(a_1,\dots,a_m))$$
$$= \sum_{\substack{|\mathcal{S}(\Theta^+)|=t-1\\ \Theta^+\vDash\Theta}} w'(\Theta^+(a_1,\dots,a_m))$$
$$= \sum_{\substack{|\mathcal{S}(\Theta^+)|=t-1\\ \Theta^+\vDash\Theta}} \sum_{\Psi^+} w'(\Theta^+(a_1,\dots,a_m)\wedge\Psi^+(a_1,\dots,a_m))$$
$$\geq w'(\Theta(a_1,\dots,a_t)\wedge\Psi(a_1,\dots,a_t)) > 0,$$

and taking the limit as $m\to\infty$ this contradicts the t-heterogeneity of w.

The corresponding result for adding equality follows exactly similar lines, we refer the reader to Landes [74] for the details.

We conclude this chapter with a corollary to the proof of Theorem 37.1, also due to Landes [74, Theorem 16], which gives some further insight into why probability functions satisfying Li with Sx and probability functions on $SL_=$ are so closely related.

COROLLARY 37.3. *Let w^L be a probability function for a not purely unary language L which satisfies Li with Sx and let w^{Eq} be the restriction to SL_{Eq} of the unique extension $w^{L_=}$ of w^L to $SL_=$. Then for an equality table $\Upsilon(b_1,\dots,b_m)$,*

$$w^{Eq}(\Upsilon) = \lim_{|\mathcal{L}|\to\infty} \sum_{\substack{\Theta(b_1,\dots,b_m)\\ \sim_\Upsilon=\sim_\Theta}} w^{\mathcal{L}}(\Theta),$$

where the $\Theta(b_1,\dots,b_m)$ are state descriptions of $\mathcal{L}$, $w^{\mathcal{L}}$ is that probability function for the language $\mathcal{L}$ in the language invariant family containing w^L and $|\mathcal{L}|$ is the number of relation symbols in $\mathcal{L}$.

PROOF. We first show that

$$\lim_{|\mathcal{L}|\to\infty} \sum_{\substack{\Theta(b_1,\dots,b_m)\\ \sim_\Theta\neq\sim_\Upsilon}} w^{\mathcal{L}_=}(\Theta\wedge\Upsilon) = 0 \tag{343}$$

where $\Theta(b_1,\dots,b_m)$ ranges over the state descriptions for $b_1,\dots,b_m$ in $\mathcal{L}$. Since the summands in (343) for which $\sim_\Upsilon$ is not a refinement of $\sim_\Theta$ are zero it is enough to show that for $\sim_\Upsilon$ a strict refinement of an equivalence relation E on $\{b_1,b_2,\dots,b_m\}$,

$$\lim_{|\mathcal{L}|\to\infty} \sum_{\substack{\Theta(b_1,\dots,b_m)\\ \sim_\Theta=E}} w^{\mathcal{L}_=}(\Theta\wedge\Upsilon) = 0. \tag{344}$$

Since,

$$w^{Eq}(\Upsilon) = \sum_{\Theta(b_1,\dots,b_m)} w^{\mathcal{L}_=}(\Theta\wedge\Upsilon) \geq \sum_{\substack{\Theta(b_1,\dots,b_m)\\ \sim_\Theta=\sim_\Upsilon}} w^{\mathcal{L}_=}(\Theta\wedge\Upsilon)$$

to show (344) it is enough to show, with the obvious shorthand, that

$$\lim_{|\mathcal{L}|\to\infty} \frac{\sum_{\sim_\Theta = E} w^{\mathcal{L}_=}(\Theta \wedge \Upsilon)}{\sum_{\sim_\Theta = \sim_\Upsilon} w^{\mathcal{L}_=}(\Theta \wedge \Upsilon)} = 0. \tag{345}$$

From (338) all the $w^{\mathcal{L}_=}(\Theta \wedge \Upsilon)$ appearing here are equal (with value K_Υ^{-1}) so the fraction in (345) is just

$$\frac{|\{\Theta \mid \sim_\Theta = E\}|}{|\{\Theta \mid \sim_\Theta = \sim_\Upsilon\}|}.$$

But it is easy to see that as $|\mathcal{L}| \to \infty$ this ratio tends to zero because if $b_{i_1}, b_{i_2}, \dots, b_{i_{|E|}}$ are representatives from each of the $|E|$ equivalence classes of E and $b_{i_1}, b_{i_2}, \dots, b_{i_{|E|}}, \dots, b_{i_h}$ from each of the $|\sim_\Upsilon| > |E|$ classes of $\sim_\Upsilon$ then, at each increment of $|\mathcal{L}|$, for every choice of $\Theta[b_{i_1}, b_{i_2}, \dots, b_{i_{|E|}}]$ (which determines Θ when $\sim_\Theta = E$) there will be, since $h > |E|$, at least double that number of choices of $\Theta[b_{i_1}, b_{i_2}, \dots, b_{i_{|E|}}, \dots, b_{i_h}]$ (which determines Θ when $\sim_\Theta = \sim_\Upsilon$). This establishes (345) and hence (344) and (343).

Finally then

$$\lim_{|\mathcal{L}|\to\infty} \sum_{\substack{\Theta(b_1,\dots,b_m) \\ \sim_\Upsilon = \sim_\Theta}} w^{\mathcal{L}}(\Theta) = \lim_{|\mathcal{L}|\to\infty} \sum_{\Delta} \sum_{\substack{\Theta(b_1,\dots,b_m) \\ \sim_\Upsilon = \sim_\Theta}} w^{\mathcal{L}_=}(\Theta \wedge \Delta), \tag{346}$$

where Δ ranges over the equality tables for $b_1, b_2, \dots, b_m$,

$$= \lim_{|\mathcal{L}|\to\infty} \sum_{\substack{\Theta(b_1,\dots,b_m) \\ \sim_\Upsilon = \sim_\Theta}} w^{\mathcal{L}_=}(\Theta \wedge \Upsilon),$$

since by (343), with Δ in place of Υ, the summands in (346) for which $\Delta \neq \Upsilon$ go to zero in the limit,

$$= \lim_{|\mathcal{L}|\to\infty} \sum_{\Theta(b_1,\dots,b_m)} w^{\mathcal{L}_=}(\Theta \wedge \Upsilon),$$

by using (343) again. Since this sequence is constantly $w(\Upsilon)$ the result is proved. $\dashv$

We remark that this corollary also holds, by the same proof, even if L is purely unary; it is just that in that case the Li family etc. (which now may not be unique) has to be chosen to correspond to a particular representation as in Theorem 37.1(ii). Precisely, we should take the Li family for Sx to be of the form

$$w^{\mathcal{L}} = \int_{\mathbb{B}} u^{\bar{p},\mathcal{L}} \, d\mu(\bar{p})$$

and

$$w^{L_=} = \int_{\mathbb{B}} u^{\bar{p},L_=}\, d\mu(\bar{p})$$

for some fixed representation

$$w^L = \int_{\mathbb{B}} u^{\bar{p},L}\, d\mu(\bar{p})$$

of w^L.

Chapter 38

THE POLYADIC JOHNSON'S SUFFICIENTNESS POSTULATE

Historically a central principle in Unary Inductive Logic has been Johnson's Sufficientness Postulate, JSP, limiting as it does the 'rational' probability functions to those in Carnap's Continuum of Inductive Methods. In view of this it is natural to ask if the postulate might be generalized to the polyadic.

In fact there seems an obvious way of doing this since JSP for a purely unary language can be equivalently formulated as saying:

Whenever $\Theta(b_1, \dots, b_n), \Phi(b_1, \dots, b_n, b_{n+1})$ *are state descriptions for* L *with* Φ *extending* Θ *then* $w(\Phi(b_1, \dots, b_n, b_{n+1}) \,|\, \Theta(b_1, \dots, b_n))$ *depends only on* n *and* $|\{i \,|\, b_i \sim_\Phi b_{n+1}, 1 \leq i \leq n\}|$. (347)

Unfortunately however this possible generalization of JSP is really just too strong. As shown in [141] for purely binary languages and subsequently extended in [74] to non purely unary languages (see also [79]) it is satisfied by just two probability functions, c_0^L, c_∞^L, (aka $u^{\langle 0,1,0,\dots\rangle,L}$, $u^{\langle 1,0,0,\dots\rangle,L}$), neither of which seem particularly attractive on other grounds. (On a more positive note however since both of these satisfy Sx we do at least have that (347) implies Sx, just as JSP implies Ax in the unary case.)

At this time it is not clear what, if anything, is the 'right' generalization of JSP to the polyadic. However this problem nicely disappears if we replace L in (347) by $L_=$, a principle we shall denote by JSP$_=$. This might of course have been anticipated in view of our observation on page 276 of the previous chapter that restricting to $L_=$ takes us back to the same situation as in the unary where the equivalence classes of a state description do not split with further extensions.

The following theorem, which we shall later generalize to $L_=$, is a special case of a result proved by Zabell, in [148, p249] (see also [72, chapter 6], [149, p295]), which gives a complete characterization for the weaker version of JSP$_{Eq}$ (i.e. JSP$_=$ when $L_= = L_{Eq}$) where additionally in the case $|\{i \,|\, b_i \sim_\Phi b_{n+1}, 1 \leq i \leq n\}| = 0$ the probability

$w(\Phi(b_1,\dots,b_n,b_{n+1}) \mid \Theta(b_1,\dots,b_n))$ depends also on $|\mathcal{S}(\Theta)|$ (and Regularity is assumed). The proof given below is extracted from a more detailed version in [74, chapter 13.3].

THEOREM 38.1. *A probability function w on SL_{Eq} satisfies JSP_{Eq} if and only if there is a $\lambda \in [0,\infty]$ such that for all equality tables $\Upsilon(b_1,\dots,b_n)$, $\Upsilon^+(b_1,\dots,b_{n+1})$ of L_{Eq} with $\Upsilon^+(b_1,\dots,b_{n+1})$ extending $\Upsilon(b_1,\dots,b_n)$,*

$$w(\Upsilon^+(b_1,\dots,b_{n+1}) \mid \Upsilon(b_1,\dots,b_n)) = \begin{cases} \dfrac{s}{\lambda+n} & \text{if } s>0, \\ \dfrac{\lambda}{\lambda+n} & \text{if } s=0, \end{cases} \tag{348}$$

where $s = |\{\, i \mid b_{n+1} \sim_{\Upsilon^+} b_i \text{ and } i \le n \,\}| = |\{\, i \mid i \le n \text{ and } \Upsilon^+ \vDash b_i = b_{n+1} \,\}|$.[118]

We name such a probability function c_λ^{Eq} after its obvious parallel with c_λ^L.

PROOF. That c_λ^{Eq} defined by (348) determines a probability function on SL_{Eq} satisfying Ex follows as in the analogous demonstration for the unary c_λ^L on page 99, and clearly also satisfies JSP_{Eq}.

In the other direction suppose that w satisfies JSP_{Eq} and let the function g be such that

$$w(\Upsilon^+(b_1,\dots,b_n,b_{n+1})) \mid \Upsilon(b_1,\dots,b_n)) = g(s,n)$$

where again $s = |\{\, i \mid b_{n+1} \sim_{\Upsilon^+} b_i \text{ and } i \le n \,\}|$.

First notice that[119]

$$\begin{aligned} g(0,0) &= w(a_1 = a_1) = 1, \\ g(0,1) + g(1,1) &= w(a_1 \neq a_2) + w(a_1 = a_2) = 1 \end{aligned} \tag{349}$$

Suppose that $g(1,1) = 0$. Then for any table $\Upsilon(b_1,\dots,b_n)$ which logically implies some $b_i = b_j$ with $i \neq j$,

$$w(\Upsilon(\vec{b})) \le w(b_i = b_j) = g(1,1) = 0,$$

so it must be that for each n,

$$w\Big(\bigwedge_{i<j\le n} a_i \neq a_j\Big) = 1$$

and (348) holds with $\lambda = \infty$.

Similarly if $g(0,1) = 0$ then for any table $\Upsilon(b_1,\dots,b_n)$ which logically implies some $b_i \neq b_j$ with $i \neq j$,

$$w(\Upsilon(\vec{b})) \le w(b_i \neq b_j) = g(0,1) = 0,$$

---

[118] With the usual conventions in the case $\lambda = \infty$ that $n/\infty = 0$, $\infty + n = \infty$, $\frac{\infty}{\infty} = 1$.

[119] Since $a_1 \neq a_2 \equiv (a_1 \neq a_2 \wedge a_1 = a_1 \wedge a_2 = a_2 \wedge a_2 \neq a_1)$ etc. there is no need for us to write out the complete equality table here, a shortening we shall use frequently in what follows.

so it must be that for each n,

$$w\Big(\bigwedge_{i<j\leq n} a_i = a_j\Big) = 1$$

and (348) holds with $\lambda = 0$.

So we may assume that there is some $0 < \lambda < \infty$ such that

$$g(1,1) = \frac{1}{1+\lambda}, \quad g(0,1) = \frac{\lambda}{1+\lambda}. \tag{350}$$

The idea, as in the corresponding proof for JSP, is to generate enough equations in the $g(s,n)$ to uniquely determine these values. We shall show by induction on $n \geq 1$ that for λ as in (350),

$$g(s,n) = \begin{cases} \dfrac{s}{n+\lambda} & \text{if } s > 0, \\[2ex] \dfrac{\lambda}{n+\lambda} & \text{if } s = 0. \end{cases} \tag{351}$$

The result for $n = 1$ follows from (350). Assume then that the result has been established up to $n-1 \geq 1$, so in particular $w(\Upsilon(b_1,\dots,b_n)) > 0$ for any equality table $\Upsilon(b_1,\dots,b_n)$. By Ex,

$$w(a_1 = a_n \wedge \bigwedge_{\substack{i<j\leq n+1\\ \langle i,j\rangle\neq\langle 1,n\rangle}} a_i \neq a_j) = w(a_1 = a_{n+1} \wedge \bigwedge_{\substack{i<j\leq n+1\\ \langle i,j\rangle\neq\langle 1,n+1\rangle}} a_i \neq a_j)$$

so

$$g(1,n-1)g(0,n) = g(0,n-1)g(1,n). \tag{352}$$

Also for $0 \leq s < n$ and $\Upsilon \equiv \big(\bigwedge_{1\leq i<j\leq s+1} a_i \neq a_j \wedge \bigwedge_{s<i,j\leq n} a_i = a_j\big)$,

$$w\Big(\big(\bigwedge_{i=1}^{s+1} a_{n+1} \neq a_i\big) \vee \big(\bigvee_{i=1}^{s+1} a_{n+1} = a_i\big) \mid \Upsilon(\vec{a})\Big) = 1$$

which gives

$$g(0,n) + sg(1,n) + g(n-s,n) = 1. \tag{353}$$

Using the inductive hypothesis we can now solve the equations (352) and (353) to verify (351) for n. ⊣

The plan now is to generalize this theorem to JSP$_=$. First however it will be useful to have the following lemma which parallels Lemma 17.1 from the unary case.

LEMMA 38.2. *JSP$_=$ implies Sx.*

PROOF. Let $\Theta_i(b_1,\dots,b_n)$ for $i = 1,2$ be state descriptions of $L_=$ with the same spectrum, say

$$\Theta_i(b_1,\dots,b_n) = \Phi_i(b_1,\dots,b_n) \wedge \Upsilon_i(b_1,\dots,b_n)$$

where $\Phi_i(\vec{b})$ is a state description of L and $\Upsilon_i(\vec{b})$ an equality table.

Since we have Ex we may suppose that $\Upsilon_1(\vec{b}) = \Upsilon_2(\vec{b})$. Then

$$\begin{aligned} |\{i \mid b_i \sim_{\Theta_1} b_n, 1 \leq i < n\}| &= |\{i \mid b_i \sim_{\Upsilon_1} b_n, 1 \leq i < n\}| \\ &= |\{i \mid b_i \sim_{\Upsilon_2} b_n, 1 \leq i < n\}| \\ &= |\{i \mid b_i \sim_{\Theta_2} b_n, 1 \leq i < n\}|. \end{aligned}$$

Hence for w satisfying JSP$_=$, $w(\Theta_1(b_1, \dots, b_n))$ is determined by n, $w(\Theta_1(\vec{b})[b_1, \dots, b_{n-1}]$ and $|\{i \mid b_i \sim_{\Upsilon_1} b_n, 1 \leq i < n\}|$.

Since the equality table of $\Theta_1(\vec{b})[b_1, \dots, b_{n-1}]$ is $\Upsilon_1(\vec{b})[b_1, \dots, b_{n-1}]$ this gives that $w(\Theta_1(b_1, \dots, b_n))$ is determined by n, and the sequence for $j = 1, 2, \dots, n$ of $|\{i \mid b_i \sim_{\Upsilon_1[b_1,\dots,b_j]} b_j, 1 \leq i < j\}|$. But since $\Upsilon_1(\vec{b}) = \Upsilon_2(\vec{b})$, $\Theta_2(\vec{b})$ is likewise determined by this same sequence, so $w(\Theta_1(\vec{b})) = w(\Theta_2(\vec{b}))$, as required. ⊣

COROLLARY 38.3. *A probability function w on $SL_=$ satisfies JSP$_=$ if and only if there is a $\lambda \in [0, \infty]$ such that for all state descriptions $\Theta(b_1, \dots, b_n)$, $\Theta^+(b_1, \dots, b_{n+1})$ of $L_=$ with $\Theta^+(b_1, \dots, b_{n+1})$ extending $\Theta(b_1, \dots, b_n)$,*

$$w(\Theta^+(b_1, \dots, b_{n+1}) \mid \Theta(b_1, \dots, b_n)) = \begin{cases} \dfrac{s}{\lambda + n} & \text{if } s > 0, \\ \dfrac{\lambda SD^L(n)}{(\lambda + n)SD^L(n+1)} & \text{if } s = 0, \end{cases} \tag{354}$$

where $s = |\{\, i \mid b_{n+1} \sim_{\Theta^+} b_i \text{ and } i \leq n \,\}|$.

PROOF. If w satisfies JSP$_=$ then the restriction of w to SL_{Eq} satisfies JSP$_{Eq}$. To see this notice that for equality tables $\Upsilon(b_1, \dots, b_n)$, $\Upsilon^+(b_1, \dots, b_{n+1})$, with Υ^+ extending Υ,

$$w(\Upsilon^+ \mid \Upsilon) = \frac{\sum_{\Phi^+} w(\Phi^+(b_1, \dots, b_{n+1}) \wedge \Upsilon^+(b_1, \dots, b_{n+1}))}{\sum_{\Phi} w(\Phi(b_1, \dots, b_n) \wedge \Upsilon(b_1, \dots, b_n))}$$

where the Φ^+ range over state descriptions of L consistent with Υ^+ etc.. But since, by Lemma 38.2, w satisfies Sx all the terms in the numerator are equal, and similarly in the denominator, so

$$w(\Upsilon^+ \mid \Upsilon) = \frac{J(\Upsilon^+)w(\Phi^+(b_1, \dots, b_{n+1}) \wedge \Upsilon^+(b_1, \dots, b_{n+1}))}{J(\Upsilon)w(\Phi(b_1, \dots, b_n) \wedge \Upsilon(b_1, \dots, b_n))} \tag{355}$$

for any particular Φ^+ consistent with Υ^+ and extending Φ, where $J(\Upsilon)$ is the number of state descriptions $\Phi(b_1, \dots, b_n)$ of L consistent with $\Upsilon(b_1, \dots, b_n)$ etc..

If $s > 0$ then $J(\Upsilon) = J(\Upsilon^+)$ in which case (355) reduces to

$$\frac{w(\Phi^+(b_1, \dots, b_{n+1}) \wedge \Upsilon^+(b_1, \dots, b_{n+1}))}{w(\Phi(b_1, \dots, b_n) \wedge \Upsilon(b_1, \dots, b_n))} \tag{356}$$

and by JSP$_=$ this depends only on s and n.

If $s = 0$ then

$$\frac{J(\Upsilon^+)}{J(\Upsilon)} = \frac{SD^L(n+1)}{SD^L(n)}, \tag{357}$$

since there are no restrictions on b_{n+1}, so again (355) depends only on s and n. This completes the proof that w restricted to SL_{Eq} satisfies JSP_{Eq}.

From this, Theorem 38.1 and (356), we obtain that for $s > 0$,

$$\begin{aligned} w(\Phi^+(b_1,\dots,b_{n+1}) \wedge \Upsilon^+(b_1,\dots,b_{n+1})) \mid w(\Phi(b_1,\dots,b_n) \wedge \Upsilon(b_1,\dots,b_n)) \\ = w(\Upsilon^+(b_1,\dots,b_{n+1}) \mid \Upsilon(b_1,\dots,b_n) = \frac{s}{n+\lambda} \end{aligned}$$

and similarly using instead (357) in the case $s = 0$ we obtain as required

$$\begin{aligned} & w(\Phi^+(b_1,\dots,b_{n+1}) \wedge \Upsilon^+(b_1,\dots,b_{n+1})) \mid \\ & \qquad\qquad w(\Phi(b_1,\dots,b_n) \wedge \Upsilon(b_1,\dots,b_n)) \\ & = \frac{SD^L(n)}{SD^L(n+1)} \cdot w(\Upsilon^+(b_1,\dots,b_{n+1}) \mid \Upsilon(b_1,\dots,b_n)) \\ & = \frac{\lambda SD^L(n)}{(n+\lambda)SD^L(n+1)}. \end{aligned}$$

⊣

Clearly the probability function defined by (354) is the extension of c_λ^{Eq} to $SL_=$ as given by the construction in the previous chapter. In turn we could now produce the restriction of this probability function to SL with the intention of realizing some illuminating polyadic version of Johnson's Sufficientness Postulate which characterizes it. Unfortunately the formula that arises directly in this way (see [74, p126]) verges on meaninglessness and any such epiphany is still awaited.

We conclude this chapter with a result[120] (see [74] or [79]) showing that c_λ^{Eq} on L_{Eq} is in a sense the limit of Carnap's c_λ^L as the size of the *unary* language L tends to infinity. Precisely:

COROLLARY 38.4. *For an equality table $\Upsilon(b_1,\dots,b_n)$ of L_{Eq},*

$$c_\lambda^{Eq}(\Upsilon(\vec{b})) = \lim_{|L|\to\infty} \sum_{\sim_\Theta = \sim_\Upsilon} c_\lambda^L(\Theta(b_1,\dots,b_n))$$

where the limit is taken over unary languages L and the sum is over the state descriptions $\Theta(b_1,\dots,b_n)$ of L for which $\sim_\Theta = \sim_\Upsilon$.

PROOF. Simply writing out the formulae involved when $\mathcal{S}(\Upsilon) = \{s_1, s_2,\dots,s_k\}$ we have

$$c_\lambda^{Eq}(\Upsilon(b_1,\dots,b_n)) = \frac{\lambda^{k-1}\prod_{i=1}^k \prod_{j=1}^{s_i-1} j}{\prod_{j=1}^{n-1}(j+\lambda)},$$

[120] This result straightforwardly generalizes to the extension of c_λ^{Eq} to $L_=$.

$$\sum_{\sim\Theta=\sim\Upsilon} c_\lambda^L(\Theta(b_1,\dots,b_n)) = \prod_{i=0}^{k-1}(2^q - i) \times \frac{\lambda^{k-1}2^{-kq}\prod_{i=1}^{k}\prod_{j=1}^{s_i-1}(j+\lambda 2^{-q})}{\prod_{j=1}^{n-1}(j+\lambda)}$$

for $|L| = q$ (where we have in each case canceled out a factor of λ in the denominator to cover the case $\lambda = 0$). From this it is easy to see that we obtain equality in the limit as $q \to \infty$. $\dashv$

Chapter 39

POLYADIC SYMMETRY

In this chapter we shall revisit the ideas proposed in Chapter 23 on Unary Symmetry but now in the context of a polyadic language L.

To recall the notation, just as in the unary case let $\mathcal{T}L$ be the set of structures for L with universe $\{a_1, a_2, a_3, \dots\}$, with the evident convention that the constant a_i of L is interpreted in $M \in \mathcal{T}L$ by the element $a_i \in M$. Let BL be the two sorted structure with universe $\mathcal{T}L$ together with the sets

$$[\theta] = \{ M \in \mathcal{T}L \mid M \vDash \theta \} \quad \text{for } \theta \in SL$$

and the membership relation between elements of $\mathcal{T}L$ and these sets.

An automorphism η of BL amounts to just a bijection of $\mathcal{T}L$ such that for each $\theta \in SL$,

$$\eta[\theta] = \{ \eta(M) \mid M \in \mathcal{T}L,\ M \vDash \theta \} = [\psi]$$

for some $\psi \in SL$, and conversely for every $\psi \in SL$ there is a $\theta \in SL$ satisfying this. Again as in the unary case we may write $\eta\theta$, or $\eta(\theta)$, for the sentence $\phi \in SL$ (up to logical equivalence) for which $\eta[\theta] = [\phi]$. Note that with this convention we have $\eta(\neg\theta) = \neg\eta(\theta)$, $\eta(\theta \wedge \phi) = \eta(\theta) \wedge \eta(\phi)$ etc. up to logical equivalence. Indeed we again have as in the unary Proposition 23.1 that for w a probability function w_η defined by $w_\eta(\theta) = w(\eta\theta)$ is also a probability function.

With the same rational justification as in the unary case we are led to propose an 'ultimate polyadic symmetry principle':

The Invariance Principle, INV.
If η is an automorphism of BL then $w(\theta) = w(\eta\theta)$ for $\theta \in SL$.

Three obvious questions now arise. Firstly, is INV even consistent? Secondly, does INV enable us to justify Sx on the grounds of symmetry? And thirdly, what new, comprehendible, polyadic symmetry principles arise as corollaries of INV?

The answer to the first question is yes. If, as usual, the relation symbols of L are $R_1, R_2, \dots, R_q$ etc. then the atoms of the Boolean Algebra BL

are just the singleton sets

$$\left[\bigwedge_{j=1}^{q} \forall x_1, x_2, \dots, x_{r_j} \ \pm R_j(x_1, x_2, \dots, x_{r_j})\right]$$

and since any automorphism of BL must map atoms to atoms it is easy to see for the polyadic case, just as for the unary case, that the probability function giving each of the 2^q sentences

$$\bigwedge_{j=1}^{q} \forall x_1, x_2, \dots, x_{r_j} \ \pm R_j(x_1, x_2, \dots, x_{r_j})$$

probability 2^{-q} will satisfy INV. So certainly INV is consistent. Of course this is just the same argument as we had in the unary case, and in that case it this turned out to be the only solution to UINV. Whether or not that turns out to also be the case in the polyadic is currently an open question though if failed attempts to prove it are anything to go by it would seem not.

The answer to the second question is currently open. Given that Sx has been proposed as the extension of Ax to the polyadic, and Ax is justified by UINV, one might hope for a positive answer here. However, as we shall see, the present 'best try' falls short of giving us Sx and instead it yields a principle that arguably has equal or even better claims to be the polyadic version of Ax.

That brings us to the third question. Just as in the unary case, INV gives the standard 'old' symmetry principles Ex, Px, SN and Vx but in order to go much further than this we now need to give a rather technical characterization of a more extensive family of automorphisms of BL which map state descriptions to state descriptions. A version of the account which follows appeared previously in [114].

For $\Theta(b_1, b_2, \dots, b_n)$ a state description of L we shall call $\Theta(x_{i_1}, x_{i_2}, \dots, x_{i_n}) \in FL$ a *state formula*. In such expressions we take it as read that the $i_1, i_2, \dots, i_n$ are distinct unless otherwise indicated. In practice, to avoid multiple subscripts, we frequently use y_i, z_i etc. to stand for choices from the $x_1, x_2, x_3, \dots$. For a state formula $\Theta(z_1, z_2, \dots, z_n)$ we may on occasion, to avoid possible confusion, write $\Theta(b_1/z_1, b_2/z_2, \dots, b_n/z_n)$, rather than the usual $\Theta(b_1, b_2, \dots, b_n)$, for the result of simultaneously replacing each occurrence of z_i in $\Theta(z_1, z_2, \dots, z_n)$ by b_i for $i = 1, 2, \dots, n$ (and similarly for other substitutions).

We carry over notation given for state descriptions to state formulae, for example talking about the spectrum of a state formula. In particular, for distinct $z_{i_1}, z_{i_2}, \dots, z_{i_m}$ from $z_1, z_2, \dots, z_n$

$$\Theta(z_1, z_2, \dots, z_n)[z_{i_1}, z_{i_2}, \dots, z_{i_m}]$$

denotes the unique state formula (up to logical equivalence) in variables $z_{i_1}, z_{i_2}, \dots, z_{i_m}$ which is implied by $\Theta(z_1, z_2, \dots, z_n)$.

We shall say that a function $\mathcal{F}$ *permutes state formulae* if for each n and (distinct) variables $z_1, z_2, \dots, z_n$, $\mathcal{F}$ permutes the state formulae $\Theta(z_1, z_2, \dots, z_n)$ in these variables. Notice that since we are identifying formulae which are merely logically equivalent it is implicit in this definition that $\mathcal{F}$ respects this identification. So for example if $\sigma \in S_n$ and Θ, Φ are state formulae such that

$$\Theta(x_1, x_2, \dots, x_n) \equiv \Phi(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$$

then

$$\mathcal{F}(\Theta(x_1, x_2, \dots, x_n)) \equiv \mathcal{F}(\Phi(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})).$$

Analogously we say that an automorphism η of BL *permutes state formulae* if there is a function $\overline{\eta}$ which permutes state formulae such that for any (distinct) $b_1, b_2, \dots, b_n$ and state formula $\Theta(z_1, z_2, \dots, z_n)$

$$\eta[\Theta(b_1, b_2, \dots, b_n)] = [\overline{\eta}(\Theta(z_1, z_2, \dots, z_n))(b_1, b_2, \dots, b_n)]. \tag{358}$$

Clearly in this case η determines $\overline{\eta}$. Furthermore η^{-1} also permutes state descriptions since from (358),

$$\begin{aligned}
\eta^{-1}[\Theta(b_1, b_2, \dots, b_n)] &= \eta^{-1}[\overline{\eta}(\overline{\eta}^{\,-1}(\Theta(z_1, z_2, \dots, z_n)))(b_1, b_2, \dots, b_n)] \\
&= \eta^{-1}\eta[\overline{\eta}^{\,-1}(\Theta(z_1, z_2, \dots, z_n))(b_1, b_2, \dots, b_n)] \\
&= [\overline{\eta}^{\,-1}(\Theta(z_1, z_2, \dots, z_n))(b_1, b_2, \dots, b_n)].
\end{aligned}$$

In particular then

$$\overline{(\eta^{-1})} = (\overline{\eta})^{-1}. \tag{359}$$

Requiring an automorphism η of BL to permute state formulae appears quite demanding. It says that as far as state descriptions for any $b_1, b_2, \dots, b_n$ are concerned η simply and uniformly permutes them. On the other hand in this polyadic case it is currently very much an open question what other possibilities there are for η (apart from permuting constants) though we do know, see pages 300 and 315, that there are automorphisms which differ from those permuting state formulae.

The plan now is to derive some properties of a function $\mathcal{F}$ which permutes the state formulae which are equivalent to it being of the form $\overline{\eta}$ for some automorphism η of BL. First though we need even more notation.

For σ a surjection from the set of variables $\{y_1, y_2, \dots, y_n\}$ onto the set of variables $\{z_1, z_2, \dots, z_m\}$, denoted $\sigma : \{y_1, y_2, \dots, y_n\} \twoheadrightarrow \{z_1, z_2, \dots, z_m\}$, and $\Theta(z_1, z_2, \dots, z_m)$ a state formula there is a unique (up to logical equivalence) state formula $\Phi(y_1, y_2, \dots, y_n)$ such that

$$\Phi(\sigma(y_1), \sigma(y_2), \dots, \sigma(y_n)) \equiv \Theta(z_1, z_2, \dots, z_m).$$

We denote this state formula Φ by

$$(\Theta(z_1, z_2, \ldots, z_m))_\sigma(y_1, y_2, \ldots, y_n),$$

or Θ_σ if the variables are clear from the context.

So if $\sigma(y_i) = \sigma(y_j)$ then y_i, y_j are indistinguishable according to Θ_σ and if $Z_1, \ldots, Z_s$ are the equivalence classes of $\sim_\Theta$ then $\sigma^{-1}Z_1, \ldots, \sigma^{-1}Z_s$ are the equivalence classes of $\sim_{\Theta_\sigma}$.

To give an example, suppose that L has a single binary relation symbol R, so a typical state formula for two variables, $\Theta(z_1, z_2)$, might be the conjunction of

$$\begin{array}{rr} \neg R(z_1, z_1) & R(z_1, z_2) \\ R(z_2, z_1) & R(z_2, z_2) \end{array}$$

which with the obvious shorthand notation we can write as just the 2×2 $\{0, 1\}$-matrix

$$\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}$$

where 1 or 0 at the ij-place means $\Theta \vDash R(z_i, z_j)$ or $\Theta \vDash \neg R(z_i, z_j)$ respectively.

Then for $\sigma : \{y_1, y_2, y_3, y_4\} \twoheadrightarrow \{z_1, z_2\}$ mapping y_1 and y_3 to z_1, and y_2 and y_4 to z_2, Θ_σ is the state formula for four variables y_1, y_2, y_3, y_4 that can similarly be represented by the matrix

$$\begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{array}$$

THEOREM 39.1. *Let $\mathcal{F}$ be a function permuting state formulae such that:*

(A) *For each state formula $\Theta(z_1, \ldots, z_m)$ and mapping $\sigma : \{y_1, y_2, \ldots, y_n\} \twoheadrightarrow \{z_1, z_2, \ldots, z_m\}$,*

$$(\mathcal{F}(\Theta))_\sigma = \mathcal{F}(\Theta_\sigma).$$

(B) *For each state formula $\Theta(z_1, \ldots, z_m)$ and $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, m\}$*

$$\mathcal{F}(\Theta)[z_{i_1}, \ldots, z_{i_k}] = \mathcal{F}(\Theta[z_{i_1}, \ldots, z_{i_k}]).$$

Then $\mathcal{F} = \overline{\eta}$ for some automorphism η of BL which permutes state formulae.

PROOF. We start by making two observations. First notice that if $\mathcal{F}$ is a function permuting state formulae and satisfying (A) and (B) then so is $\mathcal{F}^{-1}$. To see this for (A) notice that $\mathcal{F}^{-1}$ certainly permutes state formulae and so by (A) for $\mathcal{F}$

$$(\mathcal{F}^{-1}(\Theta))_\sigma = \mathcal{F}^{-1}(\mathcal{F}(\mathcal{F}^{-1}(\Theta))_\sigma) = \mathcal{F}^{-1}((\mathcal{F}(\mathcal{F}^{-1}(\Theta)))_\sigma) = \mathcal{F}^{-1}(\Theta_\sigma).$$

As for (B), from the corresponding property for $\mathcal{F}$,

$$\begin{aligned}\mathcal{F}^{-1}(\Theta[z_{i_1},\dots,z_{i_k}]) &= \mathcal{F}^{-1}(\mathcal{F}(\mathcal{F}^{-1}(\Theta))[z_{i_1},\dots,z_{i_k}])\\ &= \mathcal{F}^{-1}(\mathcal{F}(\mathcal{F}^{-1}(\Theta)[z_{i_1},\dots,z_{i_k}]))\\ &= \mathcal{F}^{-1}(\Theta)[z_{i_1},\dots,z_{i_k}].\end{aligned}$$

The second observation is a consequence of (A) which we will use frequently and without further mention in what follows. By taking $n = m$ and $\sigma(y_i) = z_i$ for $i = 1, 2, \dots, m$ we find that for any state formula $\Theta(z_1, \dots, z_m)$ we have

$$\mathcal{F}(\Theta(y_1, y_2, \dots, y_m)) = (\mathcal{F}(\Theta))(y_1, y_2, \dots, y_m). \tag{360}$$

Turning now to the main proof, given $M \in \mathcal{T}L$ define ηM by

$$\eta M \vDash (\mathcal{F}(\Theta))(b_1, b_2, \dots, b_m) \iff M \vDash \Theta(b_1, b_2, \dots, b_m) \tag{361}$$

where $\Theta(z_1, z_2, \dots, z_m)$ is a state formula and $b_1, b_2, \dots, b_m$ are (as usual) distinct choices from the a_i. Notice that this does not over specify ηM since if $n > m$, $b_1, b_2, \dots, b_n$ are distinct choices from the a_i and $\Phi(z_1, z_2, \dots, z_n)$, $\Theta(z_1, z_2, \dots, z_m)$ are state formulae such that

$$M \vDash \Theta(b_1, b_2, \dots, b_m) \quad \text{and} \quad M \vDash \Phi(b_1, b_2, \dots, b_n)$$

then

$$\Phi \vDash \Theta$$

so by (B)

$$\mathcal{F}(\Phi) \vDash \mathcal{F}(\Theta)$$

and $\mathcal{F}(\Theta)(b_1, b_2, \dots, b_m)$ is consistent with $\mathcal{F}(\Phi)(b_1, b_2, \dots, b_n)$.

We now show that with this definition of ηM (361) holds *even if the b_i are not all distinct*. So suppose that $b_1, b_2, \dots, b_m$ are possibly not all distinct and let $\sigma : \{y_1, \dots, y_m\} \twoheadrightarrow \{z_1, \dots, z_k\}$ be such that

$$\sigma(y_i) = \sigma(y_j) \iff b_i = b_j.$$

Let $\Psi(z_1, z_2, \dots, z_k)$ be the state formula such that

$$\Theta(y_1, y_2, \dots, y_m) \equiv (\Psi(z_1, z_2, \dots, z_k))_\sigma.$$

From (A) we have

$$\mathcal{F}(\Theta) = (\mathcal{F}(\Psi))_\sigma. \tag{362}$$

Let y_{i_j} for $j = 1, 2, \dots, k$ be such that $\sigma(y_{i_j}) = z_j$. Assume that $M \vDash \Theta(b_1, b_2, \dots, b_m)$. Then also $M \vDash \Psi(b_{i_1}, b_{i_2}, \dots, b_{i_k})$ and since the $b_{i_1}, b_{i_2}, \dots, b_{i_k}$ are distinct, from (361),

$$\eta M \vDash \mathcal{F}(\Psi(z_1, z_2, \dots, z_k))(b_{i_1}/z_1, b_{i_2}/z_2, \dots, b_{i_k}/z_k),$$

so

$$\eta M \vDash ((\mathcal{F}\Psi)_\sigma(y_1, y_2, \dots, y_m))(b_1/y_1, b_2/y_2, \dots, b_m/y_m).$$

The required result now follows from (362) since the reverse direction in (361) is given (for example) by the same argument with $\digamma^{-1}$ in place of $\digamma$.

Armed with this strengthened version of (361) we are now ready to show that η extends to an automorphism of BL. Suppose that $\psi(b_1, b_2, \dots, b_t)$ is a sentence of L, say

$$\psi(b_1, b_2, \dots, b_t) \equiv \forall x_{t+1} \exists x_{t+2} \dots \forall x_s \bigvee_{j=1}^{m} \Theta_j(b_1, \dots, b_t, x_{t+1}, \dots, x_s)$$

where the $\Theta_j(x_1, \dots, x_s)$ are state formulae. From the strengthened version of (361) it follows that for any $b_{t+1}, b_{t+2}, \dots, b_s$, with $b_1, \dots, b_s$ *not necessarily all distinct*,

$$M \vDash \Theta_j(b_1, \dots, b_t, b_{t+1}, \dots, b_s) \\ \iff \eta M \vDash \digamma(\Theta_j(x_1, \dots, x_s))(b_1, \dots, b_t, b_{t+1}, \dots, b_s)$$

so

$$M \vDash \psi(b_1, b_2, \dots, b_t) \iff$$

$$\eta M \vDash \forall x_{t+1} \exists x_{t+2} \dots \forall x_s \bigvee_{j=1}^{m} \digamma(\Theta_j(x_1, \dots, x_s))(b_1, \dots, b_t, x_{t+1}, \dots, x_s)$$

as required. ⊣

Corollary 39.2. *INV implies the Permutation Invariance Principle (PIP):*

The Permutation Invariance Principle, PIP.

If $\digamma$ is a permutation of state formulae of L satisfying (A) & (B) *of Theorem* 39.1 *then for a state description* $\Theta(b_1, b_2, \dots, b_m)$,

$$w(\Theta(b_1, b_2, \dots, b_m)) = w(\digamma(\Theta(x_1, x_2, \dots, x_m))(b_1, b_2, \dots, b_m)).$$

It is easy to see that there are functions $\digamma$ which satisfy (A) and (B) and hence provide automorphisms of BL. One is when for some fixed relation symbol R, $\digamma(\Theta(z_1, z_2, \dots, z_n))$ is the result of replacing $\pm R$ everywhere in Θ by $\mp R$. With PIP this is just the Strong Negation Principle, SN. Other examples are when $\digamma$ permutes R with another relation symbol of the same arity or when $\digamma$ simply permutes the order of place markers in R. These give respectively Predicate Exchangeability, Px, and Variable Exchangeability, Vx. Thus each of these principles is a consequence of PIP (and hence of INV).

This obviously raises the question whether PIP gives anything beyond these rather well known principles. The following example shows that the answer is yes.

Suppose that L has a single binary relation symbol R. As pointed out earlier, state formulae for n variables may be represented by $n \times n$ matrices with entries 0 or 1. Let $\digamma$ effect the following transposition

$$\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \quad \Longleftrightarrow \quad \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \tag{363}$$

and leave all 1×1 and the remaining 2×2 matrices fixed. By this we mean that, for example, $\digamma$ maps a state formula represented by the left hand side of (363) to the state formula represented by the right hand side of (363) with the same variables. So

$$\neg R(z_1, z_1) \wedge R(z_1, z_2) \wedge \neg R(z_2, z_1) \wedge \neg R(z_2, z_2)$$

is mapped to

$$\neg R(z_1, z_1) \wedge \neg R(z_1, z_2) \wedge R(z_2, z_1) \wedge \neg R(z_2, z_2),$$

and so on. Now extend $\digamma$ to any state formula (with $n \geq 2$) by setting $\digamma(\Theta(z_1, \dots, z_n)) = \Phi(z_1, \dots, z_n)$ where for $1 \leq i, j \leq n$ (possibly $i = j$) the matrix associated with z_i, z_j (in that order) in $\Phi(z_1, \dots, z_n)$ is $\digamma$ of the matrix associated with z_i, z_j in $\Theta(z_1, \dots, z_n)$. Notice that $\Phi(z_1, \dots, z_n)$ is indeed a well defined state formula here.

So, for example, with the same notation, $\digamma$ effects the following map of a state formula on z_1, z_2, z_3

$$\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{matrix} \quad \longmapsto \quad \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}$$

A few scribbles now show that $\digamma$ satisfies (A) and (B) of Theorem 39.1 and hence $\digamma$ determines an automorphism of BL. However the instance of PIP that it engenders does not follow from Constant Exchangeability, Predicate Exchangeability, Strong Negation or Variable Exchangeability.

We now turn to the more difficult task of showing the converse to Theorem 39.1.

Theorem 39.3. *Let η be an automorphism of BL which permutes state formulae. Then the function $\overline{\eta}$ satisfies* (A), (B) *of Theorem* 39.1, *i.e.*

(A) *For each state formula $\Theta(z_1, \dots, z_m)$ and mapping* $\sigma : \{y_1, y_2, \dots, y_n\} \twoheadrightarrow \{z_1, z_2, \dots, z_m\}$,

$$(\overline{\eta}(\Theta))_\sigma = \overline{\eta}(\Theta_\sigma).$$

(B) *For each state formula $\Theta(z_1, \dots, z_m)$ and $i_1, i_2, \dots, i_k \in \{1, 2, \dots, m\}$*

$$\overline{\eta}(\Theta)[z_{i_1}, \dots, z_{i_k}] = \overline{\eta}(\Theta[z_{i_1}, \dots, z_{i_k}]).$$

Proof. To show (B), notice that

$$[\neg\Theta[z_{i_1}, \dots, z_{i_k}](a_{i_1}, \dots, a_{i_k}) \wedge \Theta(a_1, \dots, a_m)] = \emptyset$$

so applying η gives

$$[\neg\overline{\eta}(\Theta[z_{i_1}, \dots, z_{i_k}])(a_{i_1}, \dots, a_{i_k}) \wedge \overline{\eta}(\Theta(z_1, \dots, z_m))(a_1, \dots, a_m)] = \emptyset,$$

equivalently,

$$\models \overline{\eta}(\Theta(z_1,\dots,z_m))(a_1,\dots,a_m) \to \overline{\eta}(\Theta[z_{i_1},\dots,z_{i_k}])(a_{i_1},\dots,a_{i_k}),$$

and the result follows.

To show (A) we first show by induction on $n \geq 2$ that if $\Theta(x_1,\dots,x_n)$ is a state formula then

$$x_1 \sim_\Theta x_2 \iff x_1 \sim_{\overline{\eta}\Theta} x_2. \tag{364}$$

For the base case $n = 2$ suppose that $x_1 \sim_\Theta x_2$. Let $\Theta^-(x_1) = \Theta(x_1, x_1)$ and[121]

$$\eta(\forall x_2\, \Theta(a_1, x_2)) = \psi(\vec{a}).$$

Without loss of generality we may assume that a_2 does not appear in $\vec{a}$ (otherwise replace a_2 everywhere by such a constant). The sentence

$$\Theta^-(a_1) \wedge \forall x_2\, \Theta(a_1, x_2)$$

is consistent and hence so is

$$\overline{\eta}(\Theta^-(x_1))(a_1) \wedge \psi(\vec{a}).$$

Let $\Phi(x_1, x_2)$ be the unique state description such that $\Phi(x_1, x_1) \equiv \overline{\eta}(\Theta^-(x_1))$. Then

$$\Phi(a_1, a_2) \wedge \psi(\vec{a})$$

is consistent (since a proof of a contradiction from $\Phi(a_1, a_2) \wedge \psi(\vec{a})$ would, upon replacing a_2 by a_1, yield a proof of contradiction from $\Phi(a_1, a_1) \wedge \psi(\vec{a})$). Consequently

$$\overline{\eta}^{\,-1}(\Phi)(a_1, a_2) \wedge \forall x_2\, \Theta(a_1, x_2)$$

is consistent. But that is only possible if

$$\overline{\eta}^{\,-1}(\Phi)(a_1, a_2) = \Theta(a_1, a_2)$$

so $\overline{\eta}(\Theta) = \Phi$ and $x_1 \sim_{\overline{\eta}(\Theta)} x_2$. The proof in the other direction is immediate using η^{-1} in place of η and recalling from (359) that η^{-1} also permutes state formulae.

Now suppose that $2 < n$ and the result has been proved for less than n variables. Suppose that for $\Theta(x_1,\dots,x_n)$, $x_1 \sim_\Theta x_2$ but $x_1 \not\sim_{\overline{\eta}\Theta} x_2$. Pick representatives $y_1, y_2, \dots, y_e$ from each of the equivalence classes of $\sim_\Theta$ with $y_1 = x_1$. For $k = 1, 2, \dots, e$ let $\Phi^k(x_1, x_2, \dots, x_n)$ be the state formula such that

$$\Theta(x_1, x_2, x_3, x_4, \dots, x_n) \models \Phi^k(x_1, y_k, x_3, x_4, \dots, x_n).$$

So

$$\begin{aligned}\Theta(x_1, x_2, x_3, x_4, \dots, x_n)&[x_1, x_3, x_4, \dots, x_n]\\ &= \Phi^k(x_1, x_2, x_3, x_4, \dots, x_n)[x_1, x_3, x_4, \dots, x_n]\end{aligned}$$

[121] One would be correct in supposing that $\psi(\vec{a}) = \forall x_2\, \overline{\eta}(\Theta)(a_1, x_2)$. However a proof of that seems to be just as convoluted as the one given here.

and the equivalences classes of $\sim_{\Phi^k}$ are the same as those of $\sim_\Theta$ except that x_2 has moved from the class containing x_1 to the class containing y_k. In particular then $x_2 \sim_{\Phi^k} y_k$ and $y_i \nsim_{\Phi^k} y_j$ for $i \neq j$. Notice that $\Phi^1 = \Theta$ by the choice of y_1.

Then

$$\forall x_2 \bigvee_{k=1}^{e} \Phi^k(a_1, x_2, a_3, a_4, \ldots, a_n)$$

is consistent. Let $\psi(\vec{a})$ be its image under η, where we may assume that a_2 does not appear in $\vec{a}$. As before

$$\Theta(a_1, a_1, a_3, a_4, \ldots, a_n) \wedge \forall x_2 \bigvee_{k=1}^{e} \Phi^k(a_1, x_2, a_3, a_4, \ldots, a_n)$$

is consistent. Let

$$\begin{aligned}\Theta^-(x_1, x_3, x_4, \ldots, x_n) &= \Theta(x_1, x_1, x_3, x_4, \ldots, x_n) \\ &= \Theta(x_1, x_2, x_3, x_4, \ldots, x_n)[x_1, x_3, x_4, \ldots, x_n].\end{aligned}$$

Then

$$\overline{\eta}(\Theta^-(x_1, x_3, x_4, \ldots, x_n))(a_1, a_3, a_4, \ldots, a_n) \wedge \psi(\vec{a})$$

is consistent. Let $\Delta(x_1, x_2, x_3, \ldots, x_n)$ be the unique state formula such that

$$\Delta(x_1, x_1, x_3, x_4, \ldots, x_n) = \overline{\eta}(\Theta^-(x_1, x_3, x_4, \ldots, x_n)).$$

In other words $\Delta(x_1, x_2, x_3, \ldots, x_n)$ is formed from $\overline{\eta}(\Theta^-(x_1, x_3, x_4, \ldots, x_n))$ by simply making x_2 a clone of x_1. So the equivalence classes of $\sim_\Delta$ are the same as those for $\sim_{\overline{\eta}\Theta^-}$ but with the addition of x_2 to the class containing x_1. By the inductive hypothesis $\Delta(x_1, x_1, x_3, \ldots, x_n)$ and $\Theta^-(x_1, x_3, x_4, \ldots, x_n)$ have the same equivalence classes, from which it follows that x_2, y_k are not equivalent with respect to $\Delta(x_1, x_2, x_3, \ldots, x_n)$ for $1 < k \leq e$.

We now have that

$$\Delta(a_1, a_2, a_3, a_4, \ldots, a_n) \wedge \psi(\vec{a})$$

is consistent (since $\Delta(a_1, a_1, a_3, a_4, \ldots, a_n) \wedge \psi(\vec{a})$ is consistent) and hence so is

$$\overline{\eta}^{-1}(\Delta(x_1, x_2, x_3, \ldots, x_n))(a_1, a_2, a_3, \ldots, a_n) \wedge \\ \forall x_2 \bigvee_{k=1}^{e} \Phi^k(a_1, x_2, a_3, a_4, \ldots, a_n). \quad (365)$$

Since $x_1 \sim_\Delta x_2$ but $x_1 \nsim_{\overline{\eta}(\Theta)} x_2$,

$$\overline{\eta}^{-1}(\Delta(x_1, x_2, x_3, \ldots, x_n)) \neq \Theta(x_1, x_2, x_3, \ldots, x_n),$$

so it must be the case that

$$\overline{\eta}^{-1}(\Delta(x_1, x_2, x_3, \ldots, x_n)) = \Phi^k(x_1, x_2, x_3, \ldots, x_n) \quad (366)$$

for some $2 \leq k \leq e$. To simplify the notation assume that $k = e\,(> 1)$. Let

$$\Delta^-(x_2, x_3, \dots, x_n) = \Delta(x_1, x_2, x_3, \dots, x_n)[x_2, x_3, \dots, x_n],$$
$$\Phi^-(x_2, x_3, \dots, x_n) = \Phi^e(x_1, x_2, x_3, \dots, x_n)[x_2, x_3, \dots, x_n].$$

Then from (B) and (366),

$$\overline{\eta}(\Phi^-(x_2, x_3, \dots, x_n)) = \Delta^-(x_2, x_3, \dots, x_n).$$

Also $x_2 \sim_{\Phi^-} y_e$, since already $x_2 \sim_{\Phi^e} y_e$, whilst $x_2 \nsim_{\Delta^-} y_e$ since the equivalence classes of $\sim_{\Delta^-}$ are the same as those of $\sim_\Delta$ except that x_1 has been removed from its class (which remains non-empty since it contains x_2). But this means that (364) fails at $n-1$, contradicting the inductive hypothesis.

To complete the proof of (A) let $\Theta(z_1, \dots, z_m)$ be a state formula and σ a mapping of $\{y_1, y_2, \dots, y_n\}$ onto $\{z_1, z_2, \dots, z_m\}$. By (B), since $\Theta_\sigma(y_1, \dots, y_n)$ extends $\Theta(y_{i_1}, \dots, y_{i_m})$ whenever $\sigma(y_{i_j}) = z_j$ for all $j = 1, \dots, m$, we have that $\overline{\eta}(\Theta_\sigma)(y_1, \dots, y_n)$ extends $\overline{\eta}(\Theta)(y_{i_1}, \dots, y_{i_m})$ whenever $\sigma(y_{i_j}) = z_j$ for all $j = 1, \dots, m$. This, combined with the above observation that indistinguishability is preserved, means that

$$\overline{\eta}(\Theta_\sigma) = (\overline{\eta}(\Theta))_\sigma,$$

as required. ⊣

Theorem 39.3 tells us then that the Permutation Invariance Principle is all we can hope for if we do not go beyond automorphisms of BL which permute state formulae. Why this seems currently somewhat disheartening is that it is not clear what other alternatives there are for 'interesting' automorphisms of BL. One might initially hope that slightly relaxing what was meant by a 'permutation of state formulae' might yield further automorphisms but, as the discussion in [114] explains, to date this has not worked.

On the bright side however we are not (yet!) in the situation that prevailed in the purely unary case where we had so many automorphisms, and in consequence symmetry principles, that only one probability function (Carnap's c_0^L), and a generally undesirable one at that, satisfied them all.

The existence of other automorphisms (possibly tempered with the further, reasonable, demand of automorphisms of BL that they be extendable to BL' whenever L' extends L [122]) is a matter for further investigation. Here we merely remark that some automorphisms of BL other than those which permute state formulae do exist, for example for L having a single

[122] It is easy to check that automorphisms which permute state formulae do have this property.

binary relation symbol R define for $M \in \mathcal{T}L$,

$$\tau(M) = \begin{cases} M^- & \text{if } M = M^+, \\ M^+ & \text{if } M = M^-, \\ M & \text{otherwise,} \end{cases}$$

where $M^+, M^- \in \mathcal{T}L$ are characterized by $M^+ \vDash \forall x, y\, R(x, y)$, and $M^- \vDash \forall x, y\, \neg R(x, y)$. Then τ determines an automorphism of BL but for example the state description

$$\Theta(a_1, \dots, a_n) = \bigwedge_{i,j \in \{1,\dots,n\}} R(a_i, a_j)$$

is mapped to

$$(\Theta(a_1, \dots, a_n) \wedge \neg\forall x, y\, R(x, y)) \vee \forall x, y\, \neg R(x, y)$$

which is not (equivalent to) a state description. Invoking INV in this special case would lead us to require that

$$w(\Theta(a_1, \dots, a_n)) = w((\Theta(a_1, \dots, a_n) \wedge \neg\forall x, y\, R(x, y)) \vee \forall x, y\, \neg R(x, y))$$

and also

$$w(\forall x, y\, R(x, y)) = w(\forall x, y\, \neg R(x, y)).$$

However, whether such automorphisms can provide a fruitful source of novel and philosophically interesting symmetry principles remains to be investigated.

Seemingly, a shortcoming of the Permutation Invariance Principle is that to apply it we first need to demonstrate a state formulae permutation F and prove that it satisfies (A) and (B). It would be of much more practical value if we could simply recognize just when there was such an F mapping $\Theta(x_1, x_2, \dots, x_n)$ to $\Phi(x_1, x_2, \dots, x_n)$ without actually having to make it explicit. That is just what we will do in the next chapter.

Chapter 40

SIMILARITY

It turns out that the key property that two state formulae need to have in order to ensure that there is a state formula permutation satisfying (A) and (B) and mapping one to the other is that of similarity:

State formulae $\Theta(z_1, z_2, \dots, z_n)$, $\Phi(z_1, z_2, \dots, z_n)$ are *similar*,[123] denoted $\Theta(\vec{z}) \approx \Phi(\vec{z})$, if for all (distinct) $i_1, i_2, \dots, i_r$ and $j_1, j_2, \dots, j_s$ from $\{1, 2, \dots, n\}$ and $\sigma : \{z_{i_1}, z_{i_2}, \dots, z_{i_r}\} \twoheadrightarrow \{z_{j_1}, z_{j_2}, \dots, z_{j_s}\}$ we have

$$\Theta[z_{i_1}, z_{i_2}, \dots, z_{i_r}] = (\Theta[z_{j_1}, z_{j_2}, \dots, z_{j_s}])_\sigma \iff \Phi[z_{i_1}, z_{i_2}, \dots, z_{i_r}] = (\Phi[z_{j_1}, z_{j_2}, \dots, z_{j_s}])_\sigma.$$

Note that the above definition means that in particular, if

$$\Theta[z_{i_1}, z_{i_2}, \dots, z_{i_m}] = (\Theta[z_{j_1}, z_{j_2}, \dots, z_{j_m}])(z_{i_1}/z_{j_1}, \dots, z_{i_m}/z_{j_m})$$

then

$$\Phi[z_{i_1}, z_{i_2}, \dots, z_{i_m}] = (\Phi[z_{j_1}, z_{j_2}, \dots, z_{j_m}])(z_{i_1}/z_{j_1}, \dots, z_{i_m}/z_{j_m}).$$

We remark that in the above definition of similarity it would suffice to limit r (and s) to be less or equal to the highest arity of a relation symbol in L, a point we shall return to later in Chapter 41.

Theorem 40.1. *Let* $\Theta(z_1, z_2, \dots, z_n)$, $\Phi(z_1, z_2, \dots, z_n)$ *be state formulae. Then there is a permutation* F *of state formulae satisfying* (A) *and* (B) *of Theorem* 39.1 *such that*

$$F(\Theta(z_1, z_2, \dots, z_n)) = \Phi(z_1, z_2, \dots, z_n)$$

iff

$$\Theta(z_1, z_2, \dots, z_n) \approx \Phi(z_1, z_2, \dots, z_n). \tag{367}$$

In other words, $\Theta(z_1, z_2, \dots, z_n)$ and $\Phi(z_1, z_2, \dots, z_n)$ are similar just when there is a permutation of state formulae satisfying (A) and (B) of Theorem 39.1 mapping one to the other, and the Permutation Invariance

[123] With the obvious meaning we shall sometimes also refer to state descriptions as 'similar'.

Principle amounts to saying that similar state descriptions should be given the same probability. Precisely[124]:

For state descriptions $\Theta(b_1, b_2, \dots, b_n)$ *and* $\Phi(b_1, b_2, \dots, b_n)$ *if*

$$\Theta(z_1, z_2, \dots, z_n) \approx \Phi(z_1, z_2, \dots, z_n)$$

then

$$w(\Theta(b_1, b_2, \dots, b_n)) = w(\Phi(b_1, b_2, \dots, b_n)).$$

An advantage of this formulation is that in small examples, such as the one given earlier, it is possible to 'see' that the similarity holds. We now turn to the proof of this theorem.

Proof. We first show the $\Leftarrow$ direction of (367).

Assume that $\Theta(\vec{z}) \approx \Phi(\vec{z})$. We define $\mathcal{F}(\Psi(w_1, \dots, w_k))$ for any distinct variables $w_1, \dots, w_k$ and state formula $\Psi(w_1, \dots, w_k)$ by induction on k to satisfy the following properties:

A_k : If $q \leq k$, $\sigma : \{w_1, \dots, w_k\} \twoheadrightarrow \{y_1, \dots, y_q\}$ and $\Psi(w_1, \dots, w_k)$, $\Lambda(y_1, \dots, y_q)$ are state formulae such that $\Psi = \Lambda_\sigma$ then

$$\mathcal{F}(\Psi) = \mathcal{F}(\Lambda_\sigma) = (\mathcal{F}(\Lambda))_\sigma.$$

B_k : If $\Psi(w_1, \dots, w_k)$ is a state formula and $\{u_1, \dots, u_r\} \subseteq \{w_1, \dots, w_k\}$ then

$$\mathcal{F}(\Psi[u_1, \dots, u_r]) = (\mathcal{F}(\Psi))[u_1, \dots, u_r].$$

C_k : If $\{u_1, \dots, u_k\} \subseteq \{z_1, \dots, z_n\}$ then

$$\mathcal{F}(\Theta[u_1, \dots, u_k]) = \Phi[u_1, \dots, u_k].$$

In addition it will be clear as we proceed that for each k, $\mathcal{F}$ respects logical equivalence and (modulo logical equivalence) gives a bijection of the state formulae in k variables $w_1, \dots, w_k$ (for any choice of $w_1, \dots, w_k$).

We start with $k = 1$. There are two cases:

Case 1_1: $\Psi(w_1) = (\Theta[z_i])(w_1/z_i)$.

In this case set

$$\mathcal{F}(\Psi(w_1)) = (\Phi[z_i])(w_1/z_i).$$

Notice that this definition is unambiguous since if

$$\Psi(w_1) = (\Theta[z_i])(w_1/z_i) = (\Theta[z_j])(w_1/z_j)$$

then $\Theta[z_i] = (\Theta[z_j])(z_i/z_j)$ so by similarity $\Phi[z_i] = (\Phi[z_j])(z_i/z_j)$ and consequently

$$(\Phi[z_i])(w_1/z_i) = (\Phi[z_j])(w_1/z_j).$$

Notice also that

$$\mathcal{F}(\Psi(w_1)) = \mathcal{F}(\Psi(y_1))(w_1/y_1),$$

[124]In this form, PIP has sometimes been referred to as Nathanial's Invariance Principle, NIP.

equivalently that if $\sigma : \{w_1\} \twoheadrightarrow \{y_1\}$ then

$$(\mathcal{F}(\Psi(y_1)))_\sigma = \mathcal{F}((\Psi(y_1))_\sigma).$$

Hence A_1 holds. Clearly also B_1 and C_1 hold. Furthermore on the $\Psi(w_1)$ covered by this case $\mathcal{F}$ is injective since if

$$\mathcal{F}(\Psi(w_1)) = (\Phi[z_i])(w_1/z_i) = (\Phi[z_j])(w_1/z_j) = \mathcal{F}(\Psi'(w_1))$$

then by similarity

$$\Psi(w_1) = (\Theta[z_i])(w_1/z_i) = (\Theta[z_j])(w_1/z_j) = \Psi'(w_1).$$

Now assume that we have defined $\mathcal{F}(\Psi(w_1))$ in all case 1_1's and go on to:

Case 2_1: Not case 1_1.

In this case extend $\mathcal{F}$ to all $\Psi(x_1)$ (for a fixed x_1) so as to make $\mathcal{F}$ a bijection on the state formulae in x_1 and extend $\mathcal{F}$ to the $\Psi(w_1)$ in case 2_1 by

$$\mathcal{F}(\Psi(w_1)) = \mathcal{F}(\Psi(x_1))(w_1/x_1).$$

Clearly A_1, B_1 and (vacuously) C_1 are again satisfied.

Now suppose that we have defined $\mathcal{F}$ on state formulae with less than k variables to satisfy A_q, B_q and C_q for $q < k$. We define $\mathcal{F}(\Psi(w_1, \dots, w_k))$ by cases again:

Case 0_k: There is some $\sigma : \{w_1, \dots, w_k\} \twoheadrightarrow \{y_1, \dots, y_q\}$ with $q < k$ and some state formula $\Lambda(y_1, \dots, y_q)$ such that

$$\Psi(w_1, \dots, w_k) = (\Lambda(y_1, \dots, y_q))_\sigma.$$

In this case set

$$\mathcal{F}(\Psi(w_1, \dots, w_k)) = (\mathcal{F}\Lambda(y_1, \dots, y_q))_\sigma. \tag{368}$$

To see that this definition is unambiguous suppose as well that

$$\Psi(w_1, \dots, w_k) = (\Omega(u_1, \dots, u_r))_\tau$$

for some $\tau : \{w_1, \dots, w_k\} \twoheadrightarrow \{u_1, \dots, u_r\}$, $r < k$ and state formula $\Omega(u_1, \dots, u_r)$. In that case we may assume that r is chosen as small as possible here. Then $u_1, \dots, u_r$ must all be distinguishable (i.e. not indistinguishable) in $\Omega(u_1, \dots, u_r)$. Define $\nu : \{y_1, \dots, y_q\} \twoheadrightarrow \{u_1, \dots, u_r\}$ as follows: For y_i pick w_g such that $\sigma(w_g) = y_i$ and set $\nu(y_i) = \tau(w_g)$. This is a good definition since if $\sigma(w_g) = \sigma(w_h)$ then w_g, w_h must be indistinguishable in $\Psi(w_1, \dots, w_k)$ so $\tau(w_h), \tau(w_g)$ must be indistinguishable in $\Omega(u_1, \dots, u_r)$ and hence equal. So $\tau = \nu\sigma$. Furthermore

$$\Psi(\sigma(w_1), \dots, \sigma(w_k)) = \Lambda(y_1, \dots, y_r)$$

so

$$\Psi(\nu\sigma(w_1), \dots, \nu\sigma(w_k)) = \Lambda(\nu(y_1), \dots, \nu(y_r)),$$

and since

$$\Psi(\tau(w_1), \dots, \tau(w_k)) = \Omega(u_1, \dots, u_r),$$

we have

$$\Lambda(v(y_1), \dots, v(y_q)) = \Omega(u_1, \dots, u_r),$$

that is,

$$(\Omega(u_1, \dots, u_r))_v = \Lambda(y_1, \dots, y_q).$$

Hence by the inductive hypothesis

$$\mathcal{F}(\Lambda(y_1, \dots, y_q)) = (\mathcal{F}(\Omega(u_1, \dots, u_r)))_v$$

and consequently (since $v\sigma = \tau$)

$$(\mathcal{F}(\Lambda(y_1, \dots, y_q)))_\sigma = (\mathcal{F}(\Omega(u_1, \dots, u_r)))_\tau$$

as required.

Clearly in the definitions made here in this case A_k and, by similarity C_k, are satisfied. Concerning B_k, if $\mathcal{F}(\Psi(w_1, \dots, w_k))$ has been defined via (368) then in the notation of B_k

$$\begin{aligned}\Psi(w_1, \dots, w_k)[u_1, \dots, u_r] &= (\Lambda(y_1, \dots, y_q))_\sigma[u_1, \dots, u_r] \\ &= (\Lambda(\vec{y})[v_1, \dots, v_h])_\tau\end{aligned}$$

where $\{v_1, \dots v_h\} = \sigma\{u_1, \dots, u_r\}$ and $\tau = \sigma \restriction \{u_1, \dots u_r\}$. Since we may assume $r < k$, otherwise there is nothing to prove, by inductive hypothesis,

$$\begin{aligned}\mathcal{F}(\Psi(w_1, \dots, w_k)[u_1, \dots, u_r]) &= \mathcal{F}((\Lambda(\vec{y})[v_1, \dots, v_h])_\tau) \\ &= (\mathcal{F}(\Lambda(\vec{y})[v_1, \dots, v_h]))_\tau \qquad \text{by } A_r \\ &= (\mathcal{F}(\Lambda(\vec{y}))[v_1, \dots, v_h])_\tau \qquad \text{by } B_q.\end{aligned}$$

But clearly

$$\begin{aligned}(\mathcal{F}(\Lambda(\vec{y}))[v_1, \dots, v_h])_\tau &= (\mathcal{F}(\Lambda(\vec{y})))_\sigma[u_1, \dots, u_r] \\ &= \mathcal{F}(\Psi(w_1, \dots, w_k))[u_1, \dots, u_r] \qquad \text{by } A_k,\end{aligned}$$

as required.

Now assume that all definitions according to case 0_k have been made and proceed to case 1_k.

Case 1_k: For some $\sigma : \{w_1, \dots, w_k\} \twoheadrightarrow \{u_1, \dots, u_q\} \subseteq \{z_1, \dots, z_n\}$ and $\Psi(\vec{w}) = (\Theta(\vec{z})[\vec{u}])_\sigma$.

If $q < k$ then $\mathcal{F}(\Psi(w_1, \dots, w_k))$ has already been defined according to case 0_k, indeed defined by

$$\begin{aligned}\mathcal{F}(\Psi(\vec{w})) &= (\mathcal{F}(\Theta(\vec{z})[\vec{u}]))_\sigma \qquad (369) \\ &= (\Phi(\vec{z})[\vec{u}])_\sigma \qquad \text{by } A_q \text{ and } C_q.\end{aligned}$$

Furthermore the argument above shows that in this case where we could have chosen such a $q < k$ (369) still holds even when $q = k$.

So suppose that we cannot choose such a $q < k$. Then in this case 1_k we have $q = k$, σ is a bijection and

$$\Psi(w_1, \dots, w_k) = \Theta(\vec{z})[\vec{u}](w_1/\sigma(w_1), \dots, w_k/\sigma(w_k)).$$

Set

$$\mathcal{F}(\Psi(w_1,\ldots,w_k)) = \Phi(\vec{z})[\vec{u}](w_1/\sigma(w_1),\ldots,w_k/\sigma(w_k)).$$

Notice again that this definition is unambiguous since by the similarity of Θ and Φ any such choice of σ will give the same answer. Notice too that in this case $\Phi(\vec{z})[\vec{u}](w_1/\sigma(w_1),\ldots,w_k/\sigma(w_k))$ could not already have been in the image of $\mathcal{F}$ as defined so far.

To show A_k in this case we only have to consider the case $q = k$. This amounts to showing that when we have some bijection $\lambda : \{w_1,\ldots,w_k\} \to \{y_1,\ldots,y_k\}$ and

$$\Psi(w_1,\ldots,w_k) = (\Lambda(y_1,\ldots,y_k))_\lambda \tag{370}$$

then

$$\mathcal{F}(\Psi(w_1,\ldots,w_k)) = (\mathcal{F}(\Lambda(y_1,\ldots,y_k)))_\lambda.$$

But (370) forces that $\mathcal{F}(\Lambda(\vec{y}))$ must also be defined by case 1_k so for σ and $\vec{u}$ as in the statement of this case 1_k,

$$\begin{aligned}
\mathcal{F}(\Lambda(y_1,\ldots,y_k)) &= \mathcal{F}((\Psi(w_1,\ldots,w_k))_{\lambda^{-1}}) \\
&= \mathcal{F}((\Theta(\vec{z})[\vec{u}])_{\sigma\lambda^{-1}}) \\
&= (\Phi(\vec{z})[\vec{u}])_{\sigma\lambda^{-1}} \\
&= (\mathcal{F}(\Psi(w_1,\ldots,w_k)))_{\lambda^{-1}},
\end{aligned}$$

as required.

Having confirmed A_k notice that C_k is clearly true directly from the definition in this case so it remains only to check B_k. Let $\{v_1,\ldots,v_h\} \subset \{w_1,\ldots,w_k\}$ with $h < k$. Then with the above notation, $q = k$ and

$$\begin{aligned}
\Psi(w_1,\ldots,w_k)[v_1,\ldots,v_h] &= (\Theta(z_1,\ldots,z_n)[u_1,\ldots,u_q])_\sigma[v_1,\ldots,v_h] \\
&= (\Theta(z_1,\ldots,z_n)[y_1,\ldots,y_h])_\tau
\end{aligned}$$

where $\tau : \{v_1,\ldots,v_h\} \twoheadrightarrow \{y_1,\ldots,y_h\} \subseteq \{u_1,\ldots,u_q\}$ is the restriction of σ to $\{v_1,\ldots,v_h\}$. But then

$$\begin{aligned}
\mathcal{F}(\Psi(w_1,\ldots,w_k)[v_1,\ldots,v_h]) &= \mathcal{F}((\Theta(z_1,\ldots,z_n)[y_1,\ldots,y_h])_\tau) \\
&= (\Phi(z_1,\ldots,z_n)[y_1,\ldots,y_h])_\tau \\
&\qquad\qquad \text{by } A_h \text{ and } C_h \\
&= (\Phi(z_1,\ldots,z_n)[u_1,\ldots,u_q])_\sigma[v_1,\ldots,v_h] \\
&= \mathcal{F}(\Psi(w_1,\ldots,w_k))[v_1,\ldots,v_h],
\end{aligned}$$

as required.

Now suppose that we have made all possible definitions of $\mathcal{F}(\Psi(\vec{w}))$ falling under cases 0_k and 1_k. That now leaves us with:

Case 2_k: Not cases 0_k or 1_k.

Assume that $\Psi(\vec{w})$ falls under this case. Fix variables $x_1,\ldots,x_k$ and notice that $\Psi(x_1,\ldots,x_k)$ must also fall under case 2_k since we have already shown that A_k holds for the cases 0_k and 1_k. Let $\Delta(x_1,\ldots,x_k)$

be the conjunction of all the state formulae $\Psi(x_1, \dots, x_k)[u_1, \dots, u_{k-1}]$ with $\{u_1, \dots, u_{k-1}\} \subset \{x_1, \dots, x_k\}$ and denote by $\Omega(x_1, \dots, x_k)$ the conjunction of all the state formulae $\mathcal{F}(\Psi(x_1, \dots, x_k))[u_1, \dots, u_{k-1}]$ with $\{u_1, \dots, u_{k-1}\} \subset \{x_1, \dots, x_k\}$.

$\Omega(\vec{x})$ must be consistent since otherwise there would be some proper subsets $\{v_1, v_2, \dots, v_{k-1}\}$, $\{y_1, y_2, \dots, y_{k-1}\}$ of $\{x_1, x_2, \dots, x_k\}$, some relation symbol R of the language L and $\{z_1, z_2 \dots, z_s\}$ a subset of both $\{v_1, v_2, \dots, v_{k-1}\}$ and $\{y_1, y_2, \dots, y_{k-1}\}$ such that

$$\mathcal{F}(\Psi(x_1, \dots, x_k)[v_1, v_2, \dots, v_{k-1}]) \vDash R(z_1, z_2, \dots, z_s)$$

whilst

$$\mathcal{F}(\Psi(x_1, \dots, x_k)[y_1, y_2, \dots, y_{k-1}]) \vDash \neg R(z_1, z_2, \dots, z_s).$$

But then by B_s,

$$\mathcal{F}(\Psi(x_1, \dots, x_k)[z_1, z_2, \dots, z_s]) \vDash R(z_1, z_2 \dots, z_s) \wedge \neg R(z_1, z_2, \dots, z_s)$$

which is clearly impossible since it has already been successfully defined to be a state formula.

Now let $\Psi_t(x_1, \dots, x_k)$, $t = 1, 2, \dots, h$ list the state formulae in the fixed variables $x_1, \dots, x_k$ logically implying $\Delta(x_1, \dots, x_k)$, and let $\Gamma_t(x_1, \dots, x_k)$, $t = 1, 2, \dots, h$ list the state formulae in the fixed variables $x_1, \dots, x_k$ extending $\Omega(x_1, \dots, x_k)$. In particular then $\Psi(x_1, \dots, x_k)$ is amongst these $\Psi_t(\vec{x})$. Clearly there are the same number of such extensions in each case since $\Delta(\vec{x}), \Omega(\vec{x})$ each decide exactly the $R(u_1, \dots, u_q)$ for R a relation symbol of L and $\{u_1, \dots, u_q\}$ a subset of $\{x_1, \dots, x_k\}$, *possibly with repeats*, with at most $k - 1$ elements so both $\Delta(\vec{x})$ and $\Omega(\vec{x})$ allow the same number of free choices.

Now suppose $\mathcal{F}(\Psi_t(x_1, \dots, x_k))$ has already been defined according to cases 0_k and 1_k. Then by B_k for this case $\mathcal{F}(\Psi_t(x_1, \dots, x_k))$ will be $\Gamma_r(x_1, \dots, x_k)$ for some r. Let $\mathcal{H}$ be the set of remaining $\Psi_t(x_1, \dots, x_k)$ where $\mathcal{F}(\Psi_t(x_1, \dots, x_k))$ has not already been defined and similarly let $\mathcal{J}$ be the set of remaining $\Gamma_r(x_1, \dots, x_k)$ where $\mathcal{F}^{-1}(\Gamma_r(x_1, \dots, x_k))$ has not already been defined, so $|\mathcal{H}| = |\mathcal{J}|$.

Suppose that $\sigma : \{x_1, \dots, x_k\} \twoheadrightarrow \{x_1, \dots, x_k\}$ and $\Psi_t(x_1, \dots, x_k) \in \mathcal{H}$. Then $\Psi_t(\sigma(x_1), \dots, \sigma(x_k))$ must also be in $\mathcal{H}$, since if not then by A_k for the already proven cases $0_k, 1_k$

$$\Psi_t(x_1, \dots, x_k) = (\Psi_t(\sigma(x_1), \dots, \sigma(x_k)))_{\sigma^{-1}}$$

would not be in $\mathcal{H}$ either.

Similarly if $\Gamma_r(x_1, \dots, x_k) \in \mathcal{J}$ then $\Gamma_r(\sigma(x_1), \dots, \sigma(x_k)) \in \mathcal{J}$. Hence we can define $\mathcal{F}$ on the $\Psi_t(x_1, \dots, x_k) \in \mathcal{H}$ by simply pairing them off with the $\Gamma_r(x_1, \dots, x_k) \in \mathcal{J}$ in such a way that if $\Psi_t(x_1, \dots, x_k)$ is paired with $\Gamma_r(x_1, \dots, x_k)$ then, for σ as above,

$$\Psi_t(\sigma(x_1), \dots, \sigma(x_k)) \text{ is paired with } \Gamma_r(\sigma(x_1), \dots, \sigma(x_k)). \qquad (371)$$

Finally we extend $\mathcal{F}$ to the $\Psi_t(w_1, \dots, w_k)$ for $\Psi_t(x_1, \dots, x_k) \in \mathcal{H}$ and k-tuples $w_1, \dots, w_k$ of (distinct) variables by

$$\mathcal{F}(\Psi_t(w_1, \dots, w_k)) = \mathcal{F}(\Psi_t(x_1, \dots, x_k))(w_1/x_1, \dots, w_k/x_k).$$

We now need to check A_k, B_k and C_k for this case.

In the notation of A_k this holds vacuously for $q < k$ (since there can be no such q when $\mathcal{F}$ is defined by case 2_k) whilst if $q = k$ the property follows directly from the definition (371). B_k holds since with the above notation $\mathcal{F}(\Psi(w_1, \dots, w_k))$ extends $\Omega(w_1, \dots, w_k)$ and hence extends $\mathcal{F}(\Psi(w_1, \dots, w_k)[u_1, \dots, u_r])$ for $\{u_1, \dots, u_r\} \subseteq \{w_1, \dots, w_k\}$. Finally C_k holds vacuously.

We have shown that A_k, B_k, C_k hold for all k. From this it follows that $\mathcal{F}$ is a permutation of state formulae which satisfies conditions A and B. Finally from C_n we obtain as required that

$$\mathcal{F}(\Theta(z_1, \dots, z_n)) = \Phi(z_1, \dots, z_n).$$

This completes the proof of (367) in the $\Leftarrow$ direction. To prove the converse suppose that $\mathcal{F}$ is a permutation of state formulae satisfying (A) and (B) and

$$\mathcal{F}(\Theta(\vec{z})) = \Phi(\vec{z}).$$

Suppose that $u_1, \dots, u_r$ and $v_1, \dots, v_s$ are distinct variables from amongst $\{z_1, \dots, z_n\}$ and that $\sigma : \{u_1, \dots, u_r\} \twoheadrightarrow \{v_1, \dots, v_s\}$ is such that

$$\Theta(\vec{z})[\vec{u}] = (\Theta(\vec{z})[\vec{v}])_\sigma. \tag{372}$$

Then by (B)

$$\mathcal{F}(\Theta(\vec{z})[\vec{u}]) = (\mathcal{F}(\Theta(\vec{z})))[\vec{u}] = \Phi(\vec{z})[\vec{u}]$$

and using (A), (B)

$$\mathcal{F}((\Theta(\vec{z})[\vec{v}])_\sigma) = (\mathcal{F}(\Theta(\vec{z})[\vec{v}]))_\sigma = (\mathcal{F}(\Theta(\vec{z}))[\vec{v}])_\sigma = (\Phi(\vec{z})[\vec{v}])_\sigma.$$

Consequently

$$\Phi(\vec{z})[\vec{u}] = (\Phi(\vec{z})[\vec{v}])_\sigma$$

follows from (372). $\dashv$

As we remarked earlier, PIP contains as special cases the Principles of Strong Negation, SN, Predicate Exchangeability, Px, and Variable Exchangeability, Vx. However, it can now be seen that it does not provide a justification for Spectrum Exchangeability, Sx, since for example the state formulae

$$R(x_1, x_1) \wedge \neg R(x_1, x_2) \wedge \neg R(x_2, x_1) \wedge R(x_2, x_2),$$
$$R(x_1, x_1) \wedge R(x_1, x_2) \wedge \neg R(x_2, x_1) \wedge R(x_2, x_2)),$$

for the language with a single binary relation symbol have the same spectra but are not similar in the sense defined above. On the other hand we do have a result in the other direction:

Corollary 40.2. *Suppose that* $\Theta(x_1, \ldots, x_n) \approx \Phi(x_1, \ldots, x_n)$. *Then* $\sim_\Theta = \sim_\Phi$.

Proof. Assume that $\Theta(x_1, \ldots, x_n) \approx \Phi(x_1, \ldots, x_n)$ and let $x_{i_1}, \ldots, x_{i_r}$ be a set of representatives of the equivalence classes in $\sim_\Theta$ (one from each class). Let $\sigma : \{x_1, \ldots, x_n\} \twoheadrightarrow \{x_{i_1}, \ldots, x_{i_r}\}$ be the mapping that assigns each x_i the representative of its class (so $x_i \sim_\Theta x_j \iff \sigma(x_i) = \sigma(x_j)$). We have $\Theta(\vec{x}) = (\Theta[x_{i_1}, \ldots, x_{i_r}])_\sigma$, so by similarity $\Phi(\vec{x}) = (\Phi[x_{i_1}, \ldots, x_{i_r}])_\sigma$. This means that $\sim_\Theta$ must be a refinement of $\sim_\Phi$ and since the argument can be applied also the other way round the two equivalence relations must be equal. $\dashv$

Corollary 40.3. *Sx implies PIP.*

Proof. Immediate from Corollary 40.2 and the definitions of Sx and Theorem 40.1. $\dashv$

Chapter 41

PIP AND ATOM EXCHANGEABILITY

We have motivated the principle of Spectrum Exchangeability as the immediate generalization of one of the possible formulations (in the presence of Ex) of the unary principle Ax, see page 193. Unlike Ax, and somewhat surprisingly, to our knowledge Sx appears to have no natural justification as a symmetry principle when symmetry is understood in terms of respecting automorphisms in the spirit of previous chapters. In a sense it is PIP, that occupies the place that Sx should be in, and thus PIP is perhaps a more natural generalization of Ax, as becomes apparent when we start from other formulations of it. To explain we need to introduce the notion of *polyadic atoms*.

Throughout the chapter L is as usual a language with relation symbols $R_1, \ldots, R_q$, of arities $r_1, \ldots, r_q$, and we let $l = \max\{r_1, \ldots, r_q\}$. A state formula of L for l variables is called a polyadic (l-ary) atom. We list the polyadic atoms as $\gamma_1(x_1, \ldots, x_l), \ldots, \gamma_N(x_1, \ldots, x_l)$ in some fixed order. When we write just γ_i we mean $\gamma_i(x_1, \ldots, x_l)$, *with these variables*.

Observe that this gives the same notion of atoms when the language is unary. Furthermore, any state formula $\Theta(z_1, \ldots, z_n)$ for our general language can be expressed as a conjunction

$$\bigwedge_{\langle i_1,\ldots,i_l\rangle \in \{1,\ldots,n\}^l} \gamma_{h_{i_1,\ldots,i_l}}(z_{i_1}, \ldots, z_{i_l}) \tag{373}$$

and every such *consistent* conjunction yields a state formula. However, unless the language is unary, not all conjunctions of the above form are consistent; atoms $\gamma_{h_{i_1,\ldots,i_l}}$ for $i_1, \ldots, i_l$ where some i_j are permuted or repeated must be related or structured accordingly. Note that for $i_1, \ldots, i_l$,

$$\gamma_{h_{i_1,\ldots,i_l}}(z_{i_1}, \ldots, z_{i_l}) = \Theta[z_{i_{m_1}}, \ldots, z_{i_{m_s}}],$$

where $i_{m_1}, \ldots, i_{m_s}$ are the distinct numbers amongst the $i_1, \ldots, i_l$.

Most of the results in this chapter were originally proved in [125]. Recall that a function permutes state formulae if for each n and distinct variables $z_1, z_2, \ldots, z_n$, the function permutes the state formulae $\Theta(z_1, z_2, \ldots, z_n)$ in these variables, and an automorphism η of BL permutes state formulae if there is a function $\overline{\eta}$ which permutes state formulae and determines the action of η on state descriptions as in (358). Moreover, by Theorems 39.1 and 39.3 a function F permuting state formulae determines an automorphism of BL in this way just when it satisfies (A) and (B) of Theorem 39.1.

Lemma 41.1. *Let F be a permutation of state formulae that satisfies* (A) *and* (B) *of Theorem* 39.1. *Then it is uniquely defined by its restriction to state formulae for l variables, that is, to l-ary atoms.*

In other words, to determine uniquely an automorphism permuting state formulae we do not need to specify the action of the associated function on all state formulae but only on atoms.

Proof. Using the property (B), this is apparent from the fact that any state formula can be expressed in the form (373). ⊣

Let Γ be the set of permutations τ of $\{1, \dots, N\}$ such that the permutation ξ of l-ary atoms specified by $\xi(\gamma_k) = \gamma_{\tau(k)}$ defines, as in the above lemma, a permutation of state formulae which satisfies (A) and (B) of Theorem 39.1[125].

Corollary 41.2. *A probability function w on SL satisfies PIP just when for any state description*

$$\bigwedge_{\langle i_1,\dots,i_l\rangle \in \{1,\dots,n\}^l} \gamma_{h_{i_1,\dots,i_l}}(b_{i_1}, \dots, b_{i_l})$$

and $\tau \in \Gamma$ we have

$$w\Big(\bigwedge_{\langle i_1,\dots,i_l\rangle \in \{1,\dots,n\}^l} \gamma_{h_{i_1,\dots,i_l}}(b_{i_1}, \dots, b_{i_l}) \Big)$$
$$= w\Big(\bigwedge_{\langle i_1,\dots,i_l\rangle \in \{1,\dots,n\}^l} \gamma_{\tau(h_{i_1,\dots,i_l})}(b_{i_1}, \dots, b_{i_l}) \Big).$$

The above shows PIP to be a generalization of Ax in its original formulation, see page 87. The difference is that in the polyadic case we must restrict the permutations of atoms to the 'consistent' ones, those coming from Γ.

As we recalled at the beginning of this chapter, when we reformulate Ax as saying that the probability of a state description depends only on its spectrum and use the same form in the polyadic case, we arrive at the principle Sx. However, a more careful rephrasing of Ax along these lines without the use of Ex still yields PIP as we will explain next.

If we do not incorporate Ex in the statement, Ax says that *the probability of a state description depends only on its spectral equivalence*, that

[125] In [125] it is shown that τ is in Γ just when for each $m \le l$, distinct $1 \le i_1, \dots, i_m \le l$, $\sigma : \{x_1, \dots, x_l\} \twoheadrightarrow \{x_{i_1}, \dots, x_{i_m}\}$ and $k, s \in \{1, \dots, N\}$

$$(\gamma_k[x_{i_1}, \dots, x_{i_m}])_\sigma = \gamma_s \iff (\gamma_{\tau(k)}[x_{i_1}, \dots, x_{i_m}])_\sigma = \gamma_{\tau(s)}\,.$$

is, if $\Theta(b_1, \dots, b_n)$ and $\Phi(b_1, \dots, b_n)$ are state descriptions for a unary language,

$$\Theta(b_1, \dots, b_n) = \bigwedge_{i=1}^{n} \alpha_{k_i}(b_i), \qquad \Phi(b_1, \dots, b_n) = \bigwedge_{i=1}^{n} \alpha_{h_i}(b_i),$$

such that for $1 \leq i, j \leq n$ we have $k_i = k_j$ just when $h_i = h_j$ then $w(\Theta) = w(\Phi)$.

The straightforward polyadic version of this is:

THE POLYADIC ATOM EXCHANGEABILITY PRINCIPLE, PAx.

Let

$$\Theta(b_1, \dots, b_n) = \bigwedge_{\langle i_1, \dots, i_l \rangle \in \{1, \dots, m\}^l} \gamma_{k_{i_1, \dots, i_l}}(b_{i_1}, \dots, b_{i_l}) \tag{374}$$

and

$$\Phi(b_1, \dots, b_n) = \bigwedge_{\langle i_1, \dots, i_l \rangle \in \{1, \dots, n\}^l} \gamma_{h_{i_1, \dots, i_l}}(b_{i_1}, \dots, b_{i_l}) \tag{375}$$

be state descriptions such that for all $\langle i_1, \dots, i_l \rangle, \langle j_1, \dots, j_l \rangle \in \{1, \dots, n\}^l$

$$k_{i_1, \dots, i_l} = k_{j_1, \dots, j_l} \iff h_{i_1, \dots, i_l} = h_{j_1, \dots, j_l}\,. \tag{376}$$

Then $w(\Theta) = w(\Phi)$.

THEOREM 41.3. *The principle PAx is equivalent to PIP.*

PROOF. We need to show that (376) holds just when there exists a permutation of state formulae $\mathcal{F}$ that satisfies (A) and (B) of Theorem 39.1. which maps $\Theta(z_1, \dots, z_n)$ to $\Phi(z_1, \dots, z_n)$.

If such a permutation $\mathcal{F}$ exists then $\gamma_{h_{i_1, \dots, i_l}} = \mathcal{F}(\gamma_{k_{i_1, \dots, i_l}})$ and hence (376) holds.

Conversely, assume (376). By Theorem 40.1 it suffices to prove that $\Theta(z_1, \dots, z_n)$ and $\Phi(z_1, \dots, z_n)$ are similar. Let (distinct) $i_1, i_2, \dots, i_t$ and $j_1, j_2, \dots, j_s$ be from $\{1, 2, \dots, n\}$ and let $\sigma : \{z_{i_1}, z_{i_2}, \dots, z_{i_t}\} \twoheadrightarrow \{z_{j_1}, z_{j_2}, \dots, z_{j_s}\}$ be such that

$$\Theta[z_{i_1}, z_{i_2}, \dots, z_{i_t}] = (\Theta[z_{j_1}, z_{j_2}, \dots, z_{j_s}])_\sigma\,.$$

Note that this means that for each $m_1, \dots, m_r$ (possibly with repeats) from $\{i_1, \dots, i_t\}$ and each r-ary relation symbol R of our language

$$\Theta \vDash R(z_{m_1}, \dots, z_{m_r}) \iff \Theta \vDash R(\sigma(z_{m_1}), \dots, \sigma(z_{m_r}))$$

Somewhat abusing notation, consider σ as the function from $\{i_1, \dots, i_t\}$ onto $\{j_1, \dots, j_s\}$ and write $\sigma(i_d) = j_e$ in place of $\sigma(z_{i_d}) = z_{j_e}$. Recalling that l is the highest arity in the language the above amounts to saying that for any $m_1, \dots, m_l$ (possibly with repeats) from $\{i_1, \dots, i_t\}$, $k_{m_1, \dots, m_l} = k_{\sigma(m_1), \dots, \sigma(m_l)}$.

Now assume for contradiction that

$$\Phi[z_{i_1}, z_{i_2}, \dots, z_{i_t}] \neq (\Phi[z_{j_1}, z_{j_2}, \dots, z_{j_s}])_\sigma\,.$$

By the same arguments we find that for some $m_1, \dots, m_l$ (possibly with repeats) from $\{i_1, \dots, i_t\}$, $h_{m_1,\dots,m_l} \neq h_{\sigma(m_1),\dots,\sigma(m_l)}$. But that contradicts (376). Hence

$$\Phi[z_{i_1}, z_{i_2}, \dots, z_{i_t}] = (\Phi[z_{j_1}, z_{j_2}, \dots, z_{j_s}])_\sigma$$

which establishes the similarity of Θ and Φ since their roles are interchangeable. $\dashv$

One application of the fact that automorphisms permuting state formulae are determined by their action on polyadic l-ary atoms arises by noting that in consequence there are only finitely many such automorphisms η and hence any probability function can be turned into a probability function satisfying PIP by averaging over permuted versions of it. In more detail, since the automorphisms permuting state formulae form a finite group, if we start with a probability function w and define $v(\theta)$ for $\theta \in SL$ by

$$v(\theta) = M^{-1} \sum_{\eta} w_\eta(\theta),$$

where M is the cardinality of the set of such η, and $w_\eta(\theta) = w(\eta\theta)$, we obtain a probability function[126] v that satisfies PIP. Hence, for example, starting from a probability function w satisfying Super Regularity, SReg, and Ex (see page 189) we obtain v that satisfies PIP, Ex and SReg.

We note that this is in contrast to INV since there are no probability distributions satisfying INV and SReg. For unary languages this is apparent from Chapter 23, where it is shown that the only probability function satisfying INV for a unary language L is c_0^L. For polyadic languages it follows by modifying the unary argument in a way which we will now illustrate in the case of L containing just one binary predicate R. Let L_1 be the language with a single unary relation symbol P, and let $\phi \in SL$ be the sentence

$$\forall x(\forall y R(x, y) \vee \forall y \neg R(x, y)).$$

For $M \in \mathcal{T}L$ such that $M \vDash \phi$ define $\beta(M) \in \mathcal{T}L_1$ by

$$M \vDash R(a_i, a_1) \iff \beta(M) \vDash P(a_i).$$

Note that β is a bijection between $\{M \in \mathcal{T}L \mid M \vDash \phi\}$ and $\mathcal{T}L_1$. For $\psi \in SL$ define ψ^* to be the result of replacing each occurrence of $R(t_1, t_2)$ in ψ, where t_1, t_2 are terms of L, by $P(t_1)$. Then it is easily shown that for $M \vDash \phi$ and $\psi \in SL$

$$M \vDash \psi \iff \beta(M) \vDash \psi^*.$$

Similarly if we define ξ^+ for $\xi \in SL_1$ to be the result of replacing each occurrence of $P(t_1)$ in ξ, where t_1 is a term of L_1, by $R(t_1, a_1)$ then for

[126] Since each w_η is a probability function, see Proposition 23.1 which - as remarked on page 291 - also holds in the polyadic case.

$M \vDash \phi$ and $\xi \in SL_1$

$$M \vDash \xi^+ \iff \beta(M) \vDash \xi\,.$$

Now assume that δ is an automorphism of BL_1. Define a bijection $\tau : \mathcal{T}L \to \mathcal{T}L$ as follows:

$$\tau(M) = \begin{cases} \beta^{-1}(\delta(\beta(M))) & \text{if } M \vDash \phi, \\ M & \text{otherwise.} \end{cases}$$

Then τ is an automorphism of BL since for $\psi \in SL$

$$\tau[\psi] = \tau[(\psi \wedge \neg\phi) \vee (\psi \wedge \phi)] = [\psi \wedge \neg\phi] \cup [(\delta(\psi^*))^+ \wedge \phi].$$

Considering the same unary automorphisms for δ's as in Chapter 23, we find that the undesirable phenomena reoccur *when attention is restricted to consequences of* ϕ. In particular, for a probability function w on SL satisfying INV we must have

$$w(R(a_1, a_1) \wedge \neg R(a_2, a_1) \wedge \phi) = 0,$$

and hence SReg fails.

Returning to similarity, when the language is not purely unary it is an equivalence relation on the set of state descriptions/formulae which is much finer than that of having the same spectrum. Beyond having the same spectrum, similar state formulae must share the same *structure*. The notion of structure can be made precise but for the present we shall leave it, offering instead a theorem which, like its cousin Lemma 27.2, initially looked unlikely and messy to prove, but is in fact in this case almost trivial.

Proposition 41.4. *Suppose that the state descriptions* $\Theta(a_1, \dots, a_m)$, $\Phi(a_1, \dots, a_m)$ *of L are similar and let* $\Psi(a_1, \dots, a_n)$ *be a state description,* $n > m$. *Then the number of state descriptions for* $a_1, a_2, \dots, a_n$ *similar to* Ψ *and extending* Θ *equals the number of state descriptions for* $a_1, a_2, \dots, a_n$ *similar to* Ψ *and extending* Φ.

Proof. Since Θ and Φ are similar, there is an automorphism permuting state formulae which maps $\Theta(z_1, \dots, z_m)$ to $\Phi(z_1, \dots, z_m)$. This automorphism gives a bijection between the sets of state formulae extending Θ and Φ respectively, and it respects similarity. Hence the result. $\dashv$

Chapter 42

THE FUNCTIONS $u_{\bar{E}}^{\bar{p},L}$

In this chapter we define a family of basic probability functions $u_{\bar{E}}^{\bar{p},L}$ satisfying PIP which were originally introduced in [114]. These functions are closely related to the $u^{\bar{p},L}$ but here we merely consider them for

$$\bar{p} \in \mathbb{B}^+ = \{\bar{p} \in \mathbb{B} \mid p_0 = 0\}.$$

They provide us with a source of examples to understand properties of PIP and it is hoped that eventually in some sense they will yield a representation theorem for PIP probability functions along the lines of theorems obtained in Chapters 31 and 32 for Sx using the $u^{\bar{p},L}$. This however is a subject for further research.

For each $k \geq 1$ let $\mathbb{E}_k$ be the set of equivalence relations $\equiv_k$ on $(\mathbb{N}^+)^k$.

Let $\mathbb{E} \subseteq \mathbb{E}_1 \times \mathbb{E}_2 \times \mathbb{E}_3 \times \cdots$ contain those sequences of equivalence relations

$$\bar{E} = \langle \equiv_1^{\bar{E}}, \equiv_2^{\bar{E}}, \equiv_3^{\bar{E}}, \dots \rangle$$

such that the following condition holds:

$$\begin{aligned} &\text{If } \langle c_1, \dots, c_k \rangle \equiv_k^{\bar{E}} \langle d_1, \dots, d_k \rangle \text{ then for } s_1, \dots, s_m \in \{1, \dots, k\} \\ &\text{(not necessarily distinct) } \langle c_{s_1}, \dots, c_{s_m} \rangle \equiv_m^{\bar{E}} \langle d_{s_1}, \dots, d_{s_m} \rangle. \end{aligned} \tag{377}$$

Note that if $\bar{E} = \langle \equiv_1^{\bar{E}}, \equiv_2^{\bar{E}}, \equiv_3^{\bar{E}}, \dots \rangle$ is in $\mathbb{E}$ then the following must hold:

- If $\langle c_1, \dots, c_k \rangle \equiv_k^{\bar{E}} \langle d_1, \dots, d_k \rangle$ and σ is a permutation of $\{1, 2, \dots, k\}$ then
$$\langle c_{\sigma(1)}, \dots, c_{\sigma(k)} \rangle \equiv_k^{\bar{E}} \langle d_{\sigma(1)}, \dots, d_{\sigma(k)} \rangle.$$
- If $\langle c_1, \dots, c_k \rangle \equiv_k^{\bar{E}} \langle d_1, \dots, d_k \rangle$, $d_s = d_t$ and $s \neq t$, say $s < t$, then
$$\langle c_1, \dots, c_s, \dots, c_t, \dots, c_k \rangle \equiv_k^{\bar{E}} \langle c_1, \dots, c_s, \dots, c_s, \dots, c_k \rangle$$
(i.e. $\vec{c}$ is equivalent to $\vec{c}$ with c_t replaced by c_s).

Let $\bar{E} \in \mathbb{E}$ and $\vec{c} = \langle c_1, c_2, \dots, c_m \rangle \in (\mathbb{N}^+)^m$. We say that a state description $\Theta(b_1, b_2, \dots, b_m)$ is *consistent with $\vec{c}$ under $\bar{E}$* if for any r-ary relation symbol R of L and any $i_1, \dots, i_r$ and $j_1, \dots, j_r$ from $\{1, 2, \dots, m\}$ (not necessarily distinct) such that

$$\langle c_{i_1}, \dots, c_{i_r} \rangle \equiv_r^{\bar{E}} \langle c_{j_1}, \dots, c_{j_r} \rangle ,$$

we have that

$$\Theta \vDash R(b_{i_1}, \dots, b_{i_r}) \iff \Theta \vDash R(b_{j_1}, \dots, b_{j_r}) .$$

Notice that for a state description $\Theta(b_1, b_2, \dots, b_m)$ consistent with $\vec{c}$ under $\bar{E}$, if $c_s = c_t$ then $b_s \sim_\Theta b_t$. This is true since if $j_1, \dots, j_r$ obtains from $i_1, \dots, i_r$ by replacing some occurrences of s by t or vice versa, then $\langle c_{i_1}, \dots, c_{i_r} \rangle \equiv_r^{\bar{E}} \langle c_{j_1}, \dots, c_{j_r} \rangle$ because they are equal.

Notice also that $\Theta(b_1, \dots, b_m)$ is consistent with $\vec{c}$ under $\bar{E}$ just if $\Theta(a_1, \dots, a_m)$ is consistent with $\vec{c}$ under $\bar{E}$. In that case also if $n < m$ then for (distinct) $k_1, \dots, k_n$ from $\{1, \dots, m\}$ the state description $\Theta(a_1, a_2, \dots, a_m)[a_{k_1}, \dots, a_{k_n}]$ is consistent with $\langle c_{k_1}, c_{k_2}, \dots, c_{k_n} \rangle$ under $\bar{E}$.

Finally we remark that if l is the highest arity of the language and

$$\Theta(b_1, \dots, b_m) = \bigwedge_{\langle i_1, \dots, i_l \rangle \in \{1, \dots, m\}^l} \gamma_{h_{i_1, \dots, i_l}}(b_{i_1}, \dots, b_{i_l}),$$

where γ_i are the polyadic atoms defined in the previous chapter, then as is easily checked $\Theta(b_1, b_2, \dots, b_m)$ is consistent with $\vec{c}$ under $\bar{E}$ just if for all $\langle i_1, \dots, i_l \rangle, \langle j_1, \dots, j_l \rangle \in \{1, 2, \dots, m\}^l$

$$\langle c_{i_1}, \dots, c_{i_l} \rangle \equiv_l^{\bar{E}} \langle c_{j_1}, \dots, c_{j_l} \rangle \Longrightarrow h_{i_1, \dots, i_l} = h_{j_1, \dots, j_l} .$$

Let $\mathcal{C}_{\bar{E}}(\vec{c}, \vec{b})$ be the set of all state descriptions for $\vec{b} = \langle b_1, \dots, b_m \rangle$ consistent with $\vec{c}$ under $\bar{E}$. We can specify the number of elements in $\mathcal{C}_{\bar{E}}(\vec{c}, \vec{b})$ as follows. Define a binary relation $\sim_k^{\bar{c}, \bar{E}}$ on $\{b_1, \dots, b_m\}^k$ for each k by

$$\langle b_{i_1}, \dots, b_{i_k} \rangle \sim_k^{\bar{c}, \bar{E}} \langle b_{j_1}, \dots, b_{j_k} \rangle \iff \langle c_{i_1}, \dots, c_{i_k} \rangle \equiv_k^{\bar{E}} \langle c_{j_1}, \dots, c_{j_k} \rangle.$$

$\sim_k^{\bar{c}, \bar{E}}$ is clearly an equivalence relation. For each r-ary relation symbol R of L and each equivalence class A of $\sim_r^{\bar{c}, \bar{E}}$, a state description from $\mathcal{C}_{\bar{E}}(\vec{c}, \vec{b})$ must choose one of

$$\bigwedge_{\langle b_{i_1}, \dots, b_{i_r} \rangle \in A} R(b_{i_1}, \dots, b_{i_r}), \qquad \bigwedge_{\langle b_{i_1}, \dots, b_{i_r} \rangle \in A} \neg R(b_{i_1}, \dots, b_{i_r}) .$$

For the given sequence $\langle c_1, \dots, c_m \rangle$ there are 2^g possible state descriptions which do this, where g is the sum *for all the q relation symbols* of L of the number of equivalence classes in $\{b_1, \dots, b_n\}^r$ with respect to the equivalence $\sim_r^{\bar{c}, \bar{E}}$ (these classes are counted as many times as there are r-ary relation symbols in L). Since each state description in $\mathcal{C}_{\bar{E}}(\vec{c}, \vec{b})$ is uniquely determined by these g choices, there are 2^g elements in $\mathcal{C}_{\bar{E}}(\vec{c}, \vec{b})$.

As we did in Lemma 29.1 with $\mathcal{C}$ we can now show that for (distinct) $i_1, i_2, \dots, i_m \leq k$, $\vec{a} = \langle a_{i_1}, \dots, a_{i_m}\rangle$, $\vec{a}^+ = \langle a_1, \dots, a_k\rangle$, $\vec{c} = \langle c_{i_1}, c_{i_2}, \dots, c_{i_m}\rangle$, $\vec{c}^+ = \langle c_1, c_2, \dots, c_k\rangle$ the following holds.

Lemma 42.1. *Let $\Theta(\vec{a}) \in \mathcal{C}_{\bar{E}}(\vec{c}, \vec{a})$. Then the number of $\Phi(\vec{a}^+) \in \mathcal{C}_{\bar{E}}(\vec{c}^+, \vec{a}^+)$ extending Θ equals $|\mathcal{C}_{\bar{E}}(\vec{c}^+, \vec{a}^+)| \cdot |\mathcal{C}_{\bar{E}}(\vec{c}, \vec{a})|^{-1}$ (and as such it is independent of the choice of Θ).*

Proof. This is clear from the preceding considerations: If g^+ is the number of classes for which choices must be made to determine a state description in $\mathcal{C}_{\bar{E}}(\vec{c}^+, \vec{a}^+)$ and g the analogous number for $\mathcal{C}_{\bar{E}}(\vec{c}, \vec{a})$ then the number of $\Phi(\vec{a}^+) \in \mathcal{C}_{\bar{E}}(\vec{c}^+, \vec{a}^+)$ extending $\Theta \in \mathcal{C}_{\bar{E}}(\vec{c}, \vec{a})$ is 2^{g^+-g}. ⊣

For $\bar{p} \in \mathbb{B}^+$ and a state description $\Theta(a_1, a_2, \dots, a_m)$ define,

$$u_{\bar{E}}^{\bar{p},L}(\Theta(\vec{a})) = \sum_{\substack{\vec{c}\in(\mathbb{N}^+)^m \\ \Theta\in\mathcal{C}_{\bar{E}}(\vec{c},\vec{a})}} |\mathcal{C}_{\bar{E}}(\vec{c}, \vec{a})|^{-1} \prod_{s=1}^{m} p_{c_s}. \tag{378}$$

Lemma 42.2. *Let $\bar{p} \in \mathbb{B}^+$ and $\bar{E} \in \mathbb{E}$. Then $u_{\bar{E}}^{\bar{p},L}$ as defined by* (378) *extends uniquely to a probability function on SL satisfying Ex.*

Proof. The proof that $u_{\bar{E}}^{\bar{p},L}$ extends uniquely to a probability function on SL is similar to the proof of Lemma 29.2. Using Lemma 42.1 it follows that for this extension (378) holds also for any (distinct) $b_1, \dots, b_m$ in place of $a_1, \dots, a_m$ and hence $u_{\bar{E}}^{\bar{p},L}$ satisfies Ex. ⊣

To show that the functions $u_{\bar{E}}^{\bar{p},L}$ also satisfy PIP, we need the following lemma:

Lemma 42.3. *Let $\Theta(z_1, \dots, z_m)$ and $\Phi(z_1, \dots, z_m)$ be similar state formulae and let $\bar{E} \in \mathbb{E}$, $\vec{c} \in (\mathbb{N}^+)^m$. Then*

$$\Theta(\vec{b}) \in \mathcal{C}_{\bar{E}}(\vec{c}, \vec{b}) \iff \Phi(\vec{b}) \in \mathcal{C}_{\bar{E}}(\vec{c}, \vec{b})\,.$$

Proof. Let

$$\Theta(b_1, \dots, b_m) = \bigwedge_{\langle i_1,\dots,i_l\rangle\in\{1,\dots,m\}^l} \gamma_{k_{i_1,\dots,i_l}}(b_{i_1}, \dots, b_{i_l}),$$

$$\Phi(b_1, \dots, b_m) = \bigwedge_{\langle i_1,\dots,i_l\rangle\in\{1,\dots,m\}^l} \gamma_{h_{i_1,\dots,i_l}}(b_{i_1}, \dots, b_{i_l}).$$

In the proof of Theorem 41.3 we showed that similarity of Θ and Φ is equivalent to the condition that for all $\langle i_1, \dots, i_l\rangle, \langle j_1, \dots, j_l\rangle \in \{1, \dots, m\}^l$

$$k_{i_1,\dots,i_l} = k_{j_1,\dots,j_l} \iff h_{i_1,\dots,i_l} = h_{j_1,\dots,j_l}\,.$$

As observed above, $\Theta(b_1, b_2, \dots, b_m) \in \mathcal{C}_{\bar{E}}(\vec{c}, \vec{b})$ just if for all $\langle i_1, \dots, i_l\rangle$, $\langle j_1, \dots, j_l\rangle \in \{1, 2, \dots, m\}^l$,

$$\langle c_{i_1}, \dots, c_{i_l}\rangle \equiv_l^{\bar{E}} \langle c_{j_1}, \dots, c_{j_l}\rangle \Longrightarrow h_{i_1,\dots,i_l} = h_{j_1,\dots,j_l}\,.$$

Hence the result follows from applying the same to $\Phi(b_1, \dots, b_m)$. ⊣

COROLLARY 42.4. *Let $\bar{p} \in \mathbb{B}^+$ and $\bar{E} \in \mathbb{E}$. Then the probability function $u_{\bar{E}}^{\bar{p},L}$ satisfies PIP.*

PROOF. Suppose that $\Theta(b_1, \ldots, b_m) \approx \Phi(b_1, \ldots, b_m)$. By the above lemma, for $\vec{c} = \langle c_1, \ldots, c_m \rangle \in (\mathbb{N}^+)^m$ we have $\Theta \in \mathcal{C}_{\bar{E}}(\vec{c}, \vec{b}) \iff \Phi \in \mathcal{C}_{\bar{E}}(\vec{c}, \vec{b})$ and hence, using (378), $u_{\bar{E}}^{\bar{p},L}(\Theta) = u_{\bar{E}}^{\bar{p},L}(\Phi)$. ⊣

THEOREM 42.5. *Let $\bar{p} \in \mathbb{B}^+$ and $\bar{E} \in \mathbb{E}$. Then the probability functions $u_{\bar{E}}^{\bar{p},L}$ form a language invariant family with PIP.*

PROOF. Let $\Theta(b_1, \ldots, b_m)$ be a state description of L and let $L^+ = L \cup \{R\}$ where R is a new, r-ary relation symbol. Writing $\mathcal{C}_{\bar{E}}^{L}(\vec{c}, \vec{b})$ and $\mathcal{C}_{\bar{E}}^{L^+}(\vec{c}, \vec{b})$ for the sets of state descriptions consistent with $\vec{c}$ under $\bar{E}$ in L and L^+ respectively we have that

$$u_{\bar{E}}^{\bar{p},L}(\Theta) = \sum_{\substack{\vec{c} \in (\mathbb{N}^+)^m \\ \Theta \in \mathcal{C}_{\bar{E}}^{L}(\vec{c}, \vec{a})}} |\mathcal{C}_{\bar{E}}^{L}(\vec{c}, \vec{a})|^{-1} \prod_{s=1}^{m} p_{c_s},$$

$$u_{\bar{E}}^{\bar{p},L^+}(\Theta) = \sum_{\Theta^+ \models \Theta} \sum_{\substack{\vec{c} \in (\mathbb{N}^+)^m \\ \Theta^+ \in \mathcal{C}_{\bar{E}}^{L^+}(\vec{c}, \vec{a})}} |\mathcal{C}_{\bar{E}}^{L^+}(\vec{c}, \vec{a})|^{-1} \prod_{s=1}^{m} p_{c_s}.$$

Let $\vec{c} = \langle c_1, \ldots, c_m \rangle \in (\mathbb{N}^+)^m$. Then

$$|\mathcal{C}_{\bar{E}}^{L^+}(\vec{c}, \vec{a})| = 2^{h_{\vec{c}}} |\mathcal{C}_{\bar{E}}^{L}(\vec{c}, \vec{a})|$$

where $h_{\vec{c}}$ is the number of equivalence classes, with respect to $\sim_r^{\vec{c},\bar{E}}$, in $\{b_1, \ldots, b_m\}^r$. If Θ^+ is a state description in L^+ that extends Θ and is consistent with $\vec{c}$ under $\bar{E}$ then Θ must be consistent with $\vec{c}$ under $\bar{E}$, and for each such $\vec{c}$ there are $2^{h_{\vec{c}}}$ extensions of Θ consistent with $\vec{c}$. Hence

$$\begin{aligned} u_{\bar{E}}^{\bar{p},L^+}(\Theta) &= \sum_{\substack{\vec{c} \in (\mathbb{N}^+)^m \\ \Theta \in \mathcal{C}_{\bar{E}}^{L}(\vec{c}, \vec{a})}} 2^{h_{\vec{c}}} |\mathcal{C}_{\bar{E}}^{L^+}(\vec{c}, \vec{a})|^{-1} \prod_{s=1}^{m} p_{c_s} \\ &= \sum_{\substack{\vec{c} \in (\mathbb{N}^+)^m \\ \Theta \in \mathcal{C}_{\bar{E}}^{L}(\vec{c}, \vec{a})}} |\mathcal{C}_{\bar{E}}^{L}(\vec{c}, \vec{a})|^{-1} \prod_{s=1}^{m} p_{c_s} = u_{\bar{E}}^{\bar{p},L}(\Theta), \end{aligned}$$

as required. ⊣

We can use the above function to show that PIP does not imply Sx nor Conformity and that unlike ULi families with Sx, ULi families with PIP can have multiple extensions to language invariant families with PIP.

Consider for example[127] $\bar{p} = \langle 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, \ldots \rangle$ and L containing a single binary relation symbol.

[127]This example is from [124].

Let $\bar{E} = \langle \equiv_1^{\bar{E}}, \equiv_2^{\bar{E}}, \dots, \rangle$ be defined by: $\langle c_1, \dots, c_k \rangle \equiv_k^{\bar{E}} \langle d_1, \dots d_k \rangle$ just if one of the following conditions hold

- $\langle c_1, \dots, c_k \rangle = \langle d_1, \dots d_k \rangle$,
- For all $j \in \{1, \dots, k\}$ we have $(c_j = 1$ and $d_j = 3)$ or $(\, c_j = 2$ and $d_j = 4)$,
- For all $j \in \{1, \dots, k\}$ we have $(c_j = 3$ and $d_j = 1)$ or $(\, c_j = 4$ and $d_j = 2)$.

Then $\bar{E}$ satisfies (377) and in this case $u_{\bar{E}}^{\bar{p},L}$ as defined by (378) gives different values to state descriptions $\Theta(b_1, \dots, b_4)$, $\Phi(b_1, \dots, b_4)$ represented by matrices

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \qquad\qquad \begin{matrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}$$

respectively. The reason is that for the above $\bar{p}$, the condition $\vec{c} \in (\mathbb{N}^+)^4$ in (378) can clearly be replaced by $\vec{c} \in (\{1,2,3,4\})^4$, and since neither Θ nor Φ can be in $\mathcal{C}_{\bar{E}}^L(\vec{c}, \vec{b})$ if $\vec{c}$ contains repeats (that would require the corresponding individuals to be indistinguishable in the state description), the sum is over $\vec{c} = \langle c_1, c_2, c_3, c_4 \rangle = \langle \sigma(1), \sigma(2), \sigma(3), \sigma(4) \rangle$ where σ is a permutation of $\{1,2,3,4\}$. For such $\vec{c}$, $\langle c_{i_1}, c_{i_2} \rangle \equiv_2^{\bar{E}} \langle c_{j_1}, c_{j_2} \rangle$ implies that either $(i_1 = i_2$ and $j_1 = j_2)$ or $(i_1 \neq i_2$ and $j_1 \neq j_2)$. In that case the state description Θ (represented by the former matrix) satisfies

$$\Theta \vDash R(b_{i_1}, b_{i_2}) \iff \Theta \vDash R(b_{j_1}, b_{j_2}),$$

so Θ is in $\mathcal{C}_{\bar{E}}^L(\vec{c}, \vec{b})$ for all such $\vec{c}$. However, this is not the case for Φ, for example the fact that $\Phi \vDash \neg R(b_1, b_2)$ and $\Phi \vDash R(b_3, b_4)$ disqualifies $\langle 1,2,3,4 \rangle$ since $\langle 1,2 \rangle \equiv_2^{\bar{E}} \langle 3,4 \rangle$.[128] Hence from (378), $u_{\bar{E}}^{\bar{p},L}(\Theta) \neq u_{\bar{E}}^{\bar{p},L}(\Phi)$. However, both Θ and Φ have the same spectrum 1_4 and hence $u_{\bar{E}}^{\bar{p},L}$ does not satisfy Sx.

Furthermore, with $\bar{p}$ and $\bar{E}$ as above, for any *unary* language $\mathcal{L}$ colours $1,3$ (and similarly $2,4$) act as if they are the same colour so we have that $u_{\bar{E}}^{\bar{p},\mathcal{L}} = u^{\bar{q},\mathcal{L}}$ where $\bar{q} = \langle 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \dots \rangle$. More precisely, this can be justified as follows. From (244) and (378), upon leaving out zero summands, we get that for any $\Theta(b_1, \dots, b_m)$

$$u^{\bar{q},\mathcal{L}}(\Theta(\vec{b})) = \sum_{\substack{\vec{c} \in \{1,2\}^m \\ \Theta \in \mathcal{C}^{\mathcal{L}}(\vec{c}, \vec{b})}} |\mathcal{C}^{\mathcal{L}}(\vec{c}, \vec{b})|^{-1} \prod_{s=1}^m q_{c_s},$$

[128] In fact, a closer inspection of Φ shows that Φ is in $\mathcal{C}_{\bar{E}}^L(\vec{c}, \vec{b})$ for *no* $\vec{c}$.

$$u_{\bar{E}}^{\bar{p},\mathcal{L}}(\Theta(\vec{b})) = \sum_{\substack{\vec{d}\in\{1,2,3,4\}^m \\ \Theta\in\mathcal{C}_{\bar{E}}^{\mathcal{L}}(\vec{d},\vec{b})}} |\mathcal{C}_{\bar{E}}^{\mathcal{L}}(\vec{d},\vec{b})|^{-1}\prod_{s=1}^{m} p_{d_s}\,.$$

If $\vec{c}\in\{1,2\}^m$ and $D(\vec{c})$ is the set of all $\vec{d}\in\{1,2,3,4\}^m$ obtained from $\vec{c}$ by replacing each occurrence of 1 by 1 or 3 and each occurrence of 2 by 2 or 4 then the $D(\vec{c})$ partition $\{1,2,3,4\}^m$ and, since we only have unary relation symbols to consider, $\mathcal{C}_{\bar{E}}^{\mathcal{L}}(\vec{d},\vec{b}) = \mathcal{C}^{\mathcal{L}}(\vec{c},\vec{b})$ for $\vec{d}\in D(\vec{c})$. Also

$$\prod_{s=1}^{m} q_{c_s} = \sum_{\vec{d}\in D(\vec{c})}\prod_{s=1}^{m} p_{d_s}\,,$$

and consequently

$$u_{\bar{E}}^{\bar{p},\mathcal{L}}(\Theta(\vec{b})) = \sum_{\substack{\vec{c}\in\{1,2\}^m \\ \Theta\in\mathcal{C}^{\mathcal{L}}(\vec{c},\vec{b})}}\ \sum_{\vec{d}\in D(\vec{c})} |\mathcal{C}_{\bar{E}}^{\mathcal{L}}(\vec{d},\vec{b})|^{-1}\prod_{s=1}^{m} p_{d_s} = u^{\bar{q},\mathcal{L}}(\Theta(\vec{b}))\,.$$

It follows that the ULi family $u^{\bar{q},\mathcal{L}}$ provides an example of a unary family that has two extensions to a general language invariant family with PIP, one of them with Sx (the $u^{\bar{q},\mathcal{L}}$) and the other one failing Sx (the $u_{\bar{E}}^{\bar{p},\mathcal{L}}$).

To see that PIP does not imply Conformity, let L contain one binary relation symbol R and let $\bar{p} = \langle 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \dots\rangle$. Take $\bar{E}$ to be defined by $\langle c_1,\dots,c_k\rangle \equiv_k^{\bar{E}} \langle d_1,\dots d_k\rangle$ iff either $\langle c_1,\dots,c_k\rangle = \langle d_1,\dots d_k\rangle$ or for all $i\in\{1,2,\dots,k\}$, $c_i=1,\ d_i=2$ or for all $i\in\{1,2,\dots,k\}$, $d_i=1,\ c_i=2$.

Then the state descriptions for $\vec{b}=\langle b_1,b_2\rangle$ represented by matrices

$$\begin{matrix}1 & 1\\ 1 & 1\end{matrix}\qquad\qquad \begin{matrix}0 & 0\\ 0 & 0\end{matrix}$$

are compatible under $\bar{E}$ with any $\vec{c}\in\{1,2\}^2$, other state descriptions represented by matrices with both entries on the diagonal equal to 1 or both equal to 0 are compatible with $\vec{c}=\langle 1,2\rangle$, $\langle 2,1\rangle$ but not with $\langle 1,1\rangle$, $\langle 2,2\rangle$, and the state descriptions with one entry on the diagonal equal to 1 and the other entry equal to 0 are compatible with no $\vec{c}$. Hence $|\mathcal{C}_{\bar{E}}(\vec{c},\vec{b})|$ is 2 for $\vec{c}=\langle 1,1\rangle$ or $\vec{c}=\langle 2,2\rangle$ and 8 for $\vec{c}=\langle 1,2\rangle$ or $\vec{c}=\langle 2,1\rangle$.

Now consider the sentences $R(a_1,a_1)\wedge R(a_2,a_2)$ and $R(a_1,a_1)\wedge R(a_2,a_1)$. Using (378), and the Disjunctive Normal Form Theorem, we can see that the former is accorded the probability 1/2 by $u_{\bar{E}}^{\bar{p},L}$ but the latter a strictly smaller probability 3/8, whilst Conformity would require these probabilities to be the same.

Chapter 43

LESS WELL TRAVELLED ROADS

In the course of the previous 42 chapters we have introduced numerous more or less rational principles which our agent, dwelling in an unknown structure M for L, might choose to adopt in order to address the question

Q: *In the situation of zero knowledge, logically, or rationally, what belief should I give to a sentence* $\theta \in SL$ *being true in* M?

We have argued from the start, via the Dutch Book argument, that it is rational to identify belief with probability, in the sense that it should satisfy conditions (P1–3), At this point the facet of 'rational', or at least 'irrational', being used is that it is irrational to agree to bets which guarantee one a certain loss. In general however we have offered no definition of 'logical' or 'rational'. Instead we have embraced certain overarching meta-principles, or slogans, which we may feel are 'rational', just in the way that we may feel that something is funny without being able to define what we mean by 'funny'.

We have particularly focused on four such slogans: That it is rational to:

(i) Obey symmetries: If, in context, θ and θ' are linked by a *symmetry* then they should be assigned equal probability.
(ii) Ignore irrelevant information: If θ' is *irrelevant* to θ then conditioning θ on θ' should not change the probability assigned to θ.
(iii) Enhance your probabilities on receipt of (positively) *relevant* information: If θ' is supportive of θ then conditioning θ on θ' should increase, or at least not decrease, the probability assigned to θ.
(iv) Respect analogies: The more θ' is *like* θ the more conditioning on θ' should enhance the probability assigned to θ.

Of course these are just templates for principles. They do not tell us what we mean by symmetry, irrelevance, relevance, analogy, that is a matter for us to experiment with, and largely that is what this monograph has been doing. It is up to the reader to decide if we have done the right experiments. Certainly there are many more interpretations of these four key notions which remain to be investigated, not to mention the possibility of proposing wholly new meta-principles or slogans.

Looking back over the Unary Part of this monograph one noticeable feature is that whilst there are potentially continuum many *different* probability functions which could have a speaking part in the event only rather a limited fraction come to the fore, the $w_{\vec{c}}$, the c_λ^L, the w_L^δ and more generally the $u^{\bar{p},L}$. Somehow the story all seems to revolve around these characters, even when we cross borders between symmetry, irrelevance, relevance, analogy, and in that way all four of these notions seem to be interrelated. But they also seem to be fundamental in another way. Rather than being separate entities they are frequently combined in theorems. In particular, through the relationship we have seen in representation theorems where the probability functions satisfying some symmetry principle turn out to be convex mixtures of very simple building block probability functions satisfying some irrelevance principle.

A somewhat similar situation obtains too when we move to the polyadic with the $u^{\bar{p},L}$, $v^{\bar{p},L}$, and perhaps too the $u_{\bar{E}}^{\bar{p},L}$, hogging the limelight. In this case however it is somewhat less remarkable since, as we openly acknowledge, the Polyadic Part is seriously skewed towards the study of Sx and later PIP. One reason for this perceived imbalance is that in general the theory of these principles works out particularly elegantly; possibly future exploration of other areas of the polyadic will paint a somewhat different picture.

Two features which have become rather central in this monograph, at least compared with the early literature in the subject, are the representation theorems and the idea of language invariance. We have spent much effort in proving representation theorems because of their perceived value in this subject, which goes back to Gaifman and Humburg's applications of de Finetti's Theorem, see [31], [54]. The obvious reason they are valuable is that they often allow us to study classes of probability functions by studying a particular subclass of these functions which we can easily get a handle on. But they have another value too in that they frequently allow us to express the problems of PIL we are interested in as problems within straightforward mathematical analysis, and so make available to us a powerful range of well established tools. Whilst using such methods may seem to the more philosophically minded reader 'unfortunate', tasteless even, the evidence to date seems to show that they are simply unavoidable, and no matter how strong our purist leanings we simply have to embrace and live with some more technical mathematics.

Language Invariance too has proved to be a useful, clarifying, condition to impose, see for example Theorem 32.1, as well as having a strong case to be considered rational. Moreover there is emerging evidence in the representation theorems such as 34.1, 34.3 (see also an analogous result for Px in [68]) that the language invariant probability functions can act as simple building blocks for more general probability functions.

One area of the polyadic which has to date proved particularly impenetrable is finding significant new relevance theorems such as we had with the Principle of Instantial Relevance, PIR, in the unary. One such result we do have is Theorem 36.1 telling us that the relevance principle PI holds for probability functions satisfying Language Invariance with Sx. However, given its reliance on the background paper [109], proving that theorem seems to require much more work than PIR ever did, and it remains an open question whether or not we can relax the requirement of language invariance.

Another, largely dark, area of the polyadic concerns automorphisms and the status of INV. With automorphisms of BL defined as we have the unary version of INV, UINV, is just too strong for most palates and there would seem to be scope for considering alternative definitions of what we mean by an 'automorphism' in this case. In the genuinely polyadic there is currently no evidence that INV produces a similar collapse but then again we have very little understanding at present of automorphisms outside of those which permute state formulae.

Going beyond the narrow confines of PIL imposed by limiting ourselves to a simple relational language, there are many opportunities for further research, though as we have already remarked earlier when introducing equality such extensions may challenge our ideas of what our agent should be capable of and in turn make the concept of 'rational' all the more elusive. One such extension which is currently under some investigation is the addition of function symbols, which indeed were present in some of the early papers in Inductive Logic, for example [32]. And again PIL for second order, infinitary, modal, fuzzy, etc. languages and logics all seem possibilities, albeit perhaps stretching the motivational link with the agent to near breaking point.

A second largely unexplored area concerns given knowledge. In our account we have assumed our agent has the problem of forming beliefs on the basis of no knowledge about the ambient structure M. Thereafter, were s/he to be given some information $\theta \in SL$ then we supposed that the agent would update their chosen rational probability function by the conventional Bayesian conditioning on θ, though as explained on page 23 we feel it is preferable here to remain entirely within the framework of question Q and treat this as what the agent *imagines* they would do were they to receive any such information θ.

An alternative here is to suppose that the agent possesses some knowledge at the start, say that $M \vDash \theta$, and that they consider what probabilities it is rational to assign directly on the basis of this knowledge, that θ holds in the ambient structure, without ever pondering the question of what to do if they knew nothing. Whether or not this leads to different conclusions, remains, to our knowledge, an open issue.

A third topic concerns the question Q itself. Why not, instead of asking the agent what probabilities they would rationally assign, ask the agent straight out what their best guess is for the structure M they are inhabiting? An immediate objection here is that it is (surely) rational for the agent to accept Constant Exchangeability, Ex, but outside of a few very special, and arguably unnatural options, it would not be possible for this to be reflected in the agent's choice of M. Nevertheless, we could grant that after all the rationality considerations had been applied the agent be then permitted a random choice from equally acceptable alternatives (much as is done in [107] for a related problem in Propositional Uncertain Reasoning) and in this amended form the question reasserts itself.

Finally, after so many words and so many theorems, are there signs that, at least for the moment, one probability function or family of probability functions is coming out on top? Of course PIL is not really in the business of making recommendations, that's where Pure and Applied Inductive Logic differ, and there is no sense in which we would wish to say that you should adopt a particular probability function when going about your everyday business. However what we could argue, on a purely personal basis, is that certain families of probability functions satisfy more of the rational principles that we 'like' than other families. On that score Carnap's Continuum of Inductive Methods, the c_λ^L, for $\lambda > 0$, still seems to have the best claims *in the unary context* and hence their unique language invariant extension to the polyadic may offer good chances in the polyadic context. But, by analogy with the cricket match we started with on page 3, we are hardly past the first innings, it could all change yet.

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SYMBOLS AND ABBREVIATIONS