Lecture 1: Overview

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We begin with a very quick review of first-order logic (we will give a more leisurely review in the next lecture).

Recall that a *linearly ordered set* is a set X equipped with a binary relation $R(x, y)$ (usually written as $x \leq y$) which satisfies the following axioms:

$(A1)$ $(\forall x)[R(x, x)]$

$(A2)$ $(\forall x, y, z)[(R(x, y) \text{ and } R(y, z)) \Rightarrow R(x, z)]$

$(A3)$ $(\forall x, y)[(R(x, y) \text{ and } R(y, x)) \Rightarrow (x = x)]$

$(A4)$ $(\forall x, y)[R(x, y) \text{ or } R(y, x)]$

Axioms $(A1)$ through $(A4)$ are examples of *first-order sentences*: they are statements that can be built in finitely many steps from the relation symbol R and the binary relation $=$ using basic logical operations ("and", "or", "implies", "not") and quantification ("for all", "there exists") over the domain X.

Definition 1. A *language* L consists of a set of *predicate symbols* $\{P_i\}_{i \in I}$, each of which has some specified *arity* $n_i \geq 0$.

A *first-order theory* T consists of a language L together with a collection of first-order sentences which are constructed using the symbols of L; we will refer to those sentences as the *axioms of* T.

Example 2. Let L be the language consisting of a single predicate symbol R of arity 2. Then L, together with the collection of first-order sentences $(A1)$ through $(A4)$ above, comprise a first-order theory T. We will refer to T as the *theory of linear orderings*.

Definition 3. Let L be a language consisting of predicate symbols $\{P_i\}_{i \in I}$ of arity n_i. An L*-structure* is a set M together with a subset $M[P_i] \subseteq M^{n_i}$ for each predicate symbol P_i; we think of $M[P_i]$ as the set of all tuples $(c_1, c_2, \ldots, c_{n_i}) \in M^{n_i}$ which satisfy the predicate P_i.

Given an L-structure M and a first-order sentence φ in the language L, we can ask if φ is *true in* M; in this case, we write $M \vDash \varphi$.

Let T be a first-order theory with underlying language L. A *model of* T is an L-structure M such that $M \vDash \varphi$ for each axiom φ of T.

Example 4. Let T be the theory of linear orderings. Then a model of T is a set M together with a binary relation R satisfying axioms $(A1)$ through $(A4)$: that is, a set with a linear ordering.

Example 5 (Abelian Groups)**.** An abelian group can be understood as a set A with a ternary relation P, where $P(x, y, z) \Leftrightarrow (x + y = z)$. This relation must be required to satisfy the following (first-order) axioms:

$(B1)$

$$(\forall x, y)(\exists z)[P(x, y, z)]$$

$(B2)$

$$(\forall x, y, z', z')[(P(x, y, z) \text{ and } P(x, y, z')) \Rightarrow (z = z')]$$

$(B3)$

$$(\forall x, y, z)[P(x, y, z) \Rightarrow P(y, x, z)]$$

$(B4)$

$$(\exists e)[((\forall x)P(x, e, x)) \text{ and } (\forall x)(\exists y)P(x, y, e)]$$

$(B5)$

$$(\forall x, y, z, u, v, w)[P(x, y, u) \text{ and } P(u, z, w) \text{ and } P(y, z, v) \Rightarrow P(x, v, w)].$$

Here $(B1)$ and $(B2)$ assert that $P(x, y, z)$ determines an addition $+ : A \times A \to A$, $(B3)$ guarantees that addition is commutative, $(B4)$ guarantees that there is an identity element and additive inverses, and $(B5)$ encodes the associative law.

We can therefore view abelian groups as the models of a first-order theory T, whose underlying language has a single 3-ary relation symbol P, and whose axioms are $(B1)$ through $(B5)$. We refer to T as *the theory of abelian groups*.

One can give endless variants of Example 5: groups, rings, fields, ordered fields, and so forth can all be understood as models of a suitably chosen first-order theory. However, not all mathematical structures of interest can be described this way. For example, there is no first order theory of torsion groups, or of finite fields, or of well-ordered sets. The expressive power of first-order logic is deliberately limited, in part because we want to be able to prove results like this one:

Theorem 6 (Gödel's Completeness Theorem)**.** *Let T be a first-order theory and let φ be a sentence in the language of T. The following conditions are equivalent:*

(1) *There is a proof of φ from the axioms of T (written as $T \vdash \varphi$).*

(2) *The sentence φ is true in every model of T (written as $T \vDash \varphi$).*

Of course, to make Theorem 6 precise, we need to define what it means to give a proof of φ from the axioms of T. We will return to this point later. For the moment, let us just note that any reasonable notion of proof should be something that is *algorithmically verifiable*. So, for example, if T is the theory of abelian groups (Example 5), then it should be possible to program a computer to generate the (infinite) list of all first-order sentences (in the language $L = \{P\}$) which are provable from T. By virtue of Theorem 6, this is the same as the list of all first-order sentences (in the language L) which are true in all abelian groups. It is not at all obvious that this latter list should be computably enumerable (even very fast computers will have trouble checking something in every abelian group).

Warning 7. In the setting of Theorem 6, it is essential that we restrict our attention to *first-order* sentences. There is no counterpart of Theorem 6 for more expressive languages, like second-order logic. For example, we can characterize $\mathbf{Z}$ as the unique linearly ordered ring in which the set of positive elements $\{x : x \geq 0\}$ is well-ordered. The axiom that $\{x : x \geq 0\}$ is well-ordered is a *second-order* statement: it involves quantification over subsets of $\mathbf{Z}$, rather than over elements of $\mathbf{Z}$. If there was a reasonable analogue of Theorem 6 for second-order theories, then it would be possible to program a computer to generate a list of all statements which are true about $\mathbf{Z}$ (as a linearly ordered ring). However, it is known that such a computer program cannot exist (even if we restrict our attention to first order sentences).

Theorem 6 establishes a relationship between *syntax* and *semantics*. The relation $T \vdash \varphi$ is syntactic, in the sense that it is defined in terms of formal manipulation of symbols. On the other hand, the relation $T \vDash \varphi$ is semantic: it involves thinking about the models of T, and how φ is interpreted in those models. To bring the contrast into sharper relief, it will be convenient to state an equivalent form of Theorem 6. First, we need a bit of terminology: given a language L, a first-order *formula* is a statement $\varphi(x_1, \ldots, x_n)$ that can be formulated in the language L, but makes reference to certain "free" variables. For example, if $L = \{R\}$ is the language of linear orderings, then $(\forall y)[R(x, y)]$ is a formula with a single free variable x, which asserts that x is the least element of the linear order in question.

Given a formula $\varphi(x_1, \ldots, x_n)$ with n free variables and a model M of T, we can define a set

$$M[\varphi] = \{(c_1, c_2, \ldots, c_n) \in M^n : \ \varphi(c_1, \ldots, c_n) \text{ is true in } M \ \} \subseteq M^n.$$

Theorem 8 (Gödel's Completeness Theorem, Equivalent Form)**.** *Let T be a first-order theory and suppose we are given a pair of formulas $\varphi(x_1, \ldots, x_n)$, $\varphi'(x_1, \ldots, x_n)$ in the language of T. The following conditions are equivalent:*

(1) *There is a proof, from the axioms of T, that $\varphi(x_1, \ldots, x_n)$ and $\varphi'(x_1, \ldots, x_n)$ are equivalent.*

(2) *For every model M of T, the sets $M[\varphi]$ and $M[\varphi']$ coincide (as subsets of M^n).*

Once again, Theorem 8 is about the relationship between syntax and semantics. We can define an equivalence relation on the set of formulas in some fixed number of variables, where we declare $\varphi(x_1, \ldots, x_n)$ and $\varphi'(x_1, \ldots, x_n)$ to be equivalent if their equivalence can be proved in T. Theorem 8 asserts that the equivalence class of a formula φ is determined by its semantics: that is, by how it is interpreted in the various models of M. Motivated by this observation, we ask the following:

Question 9. Let T be a first-order theory. Suppose that, for every model M of T, we specify a subset $\widetilde{M} \subseteq M$. Under what conditions does there exist a formula $\varphi(x)$ in the language of T such that $\widetilde{M} = M[\varphi]$ for every model M?

Of course, one should not expect such a formula to exist if the subsets $\widetilde{M} \subseteq M$ are chosen at random. At the very least, we need to demand that the subsets $\widetilde{M}$ have some relation to one another as the model M varies. To formulate this more precisely, we recall the following:

Definition 10. Let T be a first-order theory and let M and M' be models of T. A function $f : M \to M'$ is called an *elementary embedding* if for every first-order formula $\varphi(x_1, \ldots, x_n)$ in the language of T and every n-tuple of elements $a_1, a_2, \ldots, a_n$, we have

$$(M \vDash \varphi(a_1, \ldots, a_n)) \Leftrightarrow (M' \vDash \varphi(f(a_1), \ldots, f(a_n)).$$

Armed with this definition, we can state an obvious necessary condition in the setting of Question 9:

($*$) For every elementary embedding $f : M \to M'$ of models of T, we have $\widetilde{M} = f^{-1}(\widetilde{M'})$.

It follows immediately from the definition that condition ($*$) is *necessary* for the collection of subsets $\widetilde{M} \subseteq M$ to be defined by some first-order formula $\varphi(x)$. However, it is not sufficient:

Example 11. Let T be the first-order theory of fields, so that the models of T are fields. For every field F, define

$$\widetilde{F} = \{x \in F : \ x \text{ is a root of unity } \} \subseteq F.$$

Then this collection of subsets satisfies condition ($*$): in fact, for *any* field extension $F \to F'$, an element $a \in F$ is a root of unity in F if and only if its image is a root of unity in F'. However, there does not exist a single first-order formula $\varphi(x)$ such that $\widetilde{F} = \{a \in F : \varphi(a) \text{ is true in } F \ \}$.

We can prove this by appealing to the language of *ultraproducts*. Let $\mathbf{C}$ denote the field of complex numbers, let $\mathcal{U}$ be some nonprincipal ultrafilter on the set of positive integers, and let

$$F = (\prod_{n>0} \mathbf{C})/\,\mathcal{U}$$

be the ultraproduct of countably many copies of $\mathbf{C}$ determined by $\mathcal{U}$. The elements of F can be identified with equivalence classes of sequences of complex numbers $(z_1, z_2, z_3, \ldots)$, where we consider two sequences to be equivalent if they agree almost everywhere (in the sense of the ultrafilter $\mathcal{U}$).

A basic foundational result in model theory asserts that, for any first-order formula $\varphi(x)$, we have

$$(F \vDash \varphi(\{z_n\})) \Leftrightarrow (\mathbf{C} \vDash \varphi(z_n) \text{ for almost all } n\ ).$$

Suppose that $\varphi(x)$ was a formula that was satisfied precisely by the roots of unity in any field. Let $z \in F$ be the equivalence class of the sequence $(e^{2\pi i/1}, e^{2\pi i/2}, e^{2\pi i/3}, \ldots)$. Then $\mathbf{C} \vDash \varphi(e^{2\pi i/n})$ for every n, so $F \vDash \varphi(z)$. It follows that z is a root of unity in F: that is, $z^k = 1$ for some fixed integer k. Applying the preceding statement again, this time for the first order statement $x^k = 1$, we conclude that $e^{2\pi i/n}$ is a kth root of unity for almost all n. This contradicts our assumption that $\mathcal{U}$ is nonprincipal.

Makkai proved that this is essentially the only obstruction:

Theorem 12 (Makkai Definability Theorem)**.** *Let T be a first-order theory and suppose we are given a subset $\widetilde{M}$ for each model of M. Assume that:*

- $(*)$ *For every elementary embedding $f : M \to M'$ of models of T, we have $\widetilde{M} = f^{-1}(\widetilde{M'})$.*
- $(*')$ *For every collection of models $\{M_j\}_{j \in J}$ and every ultrafilter $\mathcal{U}$ on J with ultraproduct $M = (\prod_{j \in J} M_j)/\mathcal{U}$, we have*
$$(\{a_j\}_{j \in J} \in \widetilde{M}) \Leftrightarrow (a_j \in \widetilde{M}_j \text{ for almost all } j \in J).$$

Then there exists a formula $\varphi(x)$ in the language of T such that $\widetilde{M} = M[\varphi]$ for each model M of T.

Theorem 12 can be regarded as a natural companion to Gödel's completeness theorem: it describes the conditions under which there *exists* a formula $\varphi(x)$ with a certain semantic interpretation, while Gödel's theorem guarantees that $\varphi(x)$ is unique up to provable equivalence. Together, they represent a further step towards recovering the syntax of a theory T from its semantics. Our ultimate goal in this course will be to prove a further refinement of Theorem 12, also due to Makkai, which can be stated in a rough form as follows:

Theorem 13 (Strong Conceptual Completeness)**.** *Let T be a first-order theory and let* $\mathrm{Mod}(T)$ *be the category of models of T (where the morphisms are given by elementary extensions). Then T can be recovered, up to equivalence, from the structure of* $\mathrm{Mod}(T)$ *as a "category with ultraproducts."*

To formulate Theorem 13 precisely, we need to say what it should mean for two first-order theories to be considered equivalent. There are some obvious situations in which we might want to say this. For example, if T and T' share the same language, each axiom of T can be proven from the axioms of T', and each axiom of T' can be proven from the axioms of T, then we should consider T and T' to be equivalent. However, there are also more subtle forms that equivalence can take.

Example 14 (Projective Geometry)**.** Let k be a field and let $\mathbf{P}^2(k)$ be the projective plane over k. The *points* of $\mathbf{P}^2(k)$ are 1-dimensional subspaces $V \subseteq k^3$. We can also speak of *lines* in $\mathbf{P}^2(k)$, which are determined by two-dimensional subspaces $W \subseteq k^3$. We say that a point (given by a 1-dimensional subspace V) is incidence to a line (given by a 2-dimensional subspace W) if $V \subseteq W$.

More generally, an *abstract projective plane* consists of a set P of *points*, a set L of *lines*, and a binary relation between points and lines called *incidence*, which we write as $p \in \ell$. This relation is required to satisfy the flllowing axioms:

- $(A1)$ For every pair of distinct points $p, p' \in P$, there is a unique line $\ell \in L$ with $p \in L$ and $p' \in L$; we denote ℓ by $\overline{pp'}$.
- $(A2)$ For every pair of distinct lines $\ell, \ell' \in L$, there is a unique point $p \in P$ such that $p \in \ell$ and $p \in \ell'$; we denote p by $\ell \cap \ell'$.

In the case where $(P, L, \in)$ comes from a field k, we can recover k. More precisely, we first need to choose four points $a, b, c, d \in P$ which are *in general position* (no three of which are collinear), and we can then try to "coordinatize the plane" to recover a field k. This construction does not always work: for a general abstract projective plane $(P, L, \in)$, we need to assume that it satisfies *Pappus's hexagon theorem*:

($A3$) Given two triples of collinear points $p, q, r \in P$ and $p', q', r' \in P$, the intersection points $\overline{pq'} \cap \overline{p'q}$, $\overline{pr'} \cap \overline{rp'}$, $\overline{qr'} \cap \overline{rq'}$ are also collinear.

However, this is all one needs. More precisely, there are mutually inverse constructions which pass back and forth between the datum of a field k, and the datum of an abstract projective plane $(P, L, \in)$ with four distinguished points in general position, satisfying Pappus's theorem.

Each of these data can be understood as a model for a suitably chosen first-order theory, but these theories look *a priori* very different. The theory of fields is usually formulated in terms of arithmetic notions (like addition and multiplication), while the theory of abstract projective planes is formulated in terms of points, lines, and incidence.

The focus of this course will be on the "categorical" approach to first-order logic, which emerges naturally from the problem of formulating a notion of equivalence between first-order theories that can accommodate situations like Example 14. One of our first goals will be to associate to each first-order theory T a category $\mathrm{Syn}(T)$ which we will refer to as *the syntactic category* of T. We will say that two first-order theories T and T' are equivalent if $\mathrm{Syn}(T)$ and $\mathrm{Syn}(T')$ are equivalent as categories. Theorem 13 can be stated more precisely as follows:

Theorem 15 (Makkai)**.** *Let T be a first-order theory. Then the syntactic category* $\mathrm{Syn}(T)$ *can be recovered from the category of models* $\mathrm{Mod}(T)$*. More precisely, there is a canonical equivalence*

$$\mathrm{Syn}(T) \simeq \mathrm{UFun}(\mathrm{Mod}(T), \mathcal{S}\mathrm{et})$$

where $\mathrm{UFun}(\mathrm{Mod}(T), \mathcal{S}\mathrm{et})$ *denotes the category of "ultrafunctors" from* $\mathrm{Mod}(T)$ *to* $\mathcal{S}\mathrm{et}$*: that is, functors which are compatible with ultraproducts.*

Remark 16. One can also go the other way: the category $\mathrm{Mod}(T)$ can be realized as a full subcategory of the category $\mathrm{Fun}(\mathrm{Syn}(T), \mathcal{S}\mathrm{et})$ of functors from $\mathrm{Syn}(T)$ to the category of sets. However, this is closer to being a definition than a theorem.

The syntactic category $\mathrm{Syn}(T)$ of a first-order theory T is an example of a very special kind of category, called a *pretopos*. Moreover, one can show that every pretopos arises in this way, provided that one adopts a sufficiently liberal interpretation of the phrase "first-order theory." More precisely, one needs to replace work with a nonclassical variant of first order logic, where one abandons the law of the excluded middle. The motivation for considering such generalizations need not come from commitment to any particular philosophical position: they can be motivated by pragmatic mathematical concerns.

Example 17. Let T be the theory of abelian groups described in Example 5. Then $\mathrm{Mod}(T)$ is a category whose objects are abelian groups, and whose morphisms $f : M \to N$ are *elementary embeddings* of abelian groups. Every elementary embedding is a group homomorphism, but not conversely. An elementary embedding $f : M \to N$ is required to preserve the truth of *any* statement which can be formulated in first-order logic. So, for example, if x is an element of M which is not divisible by 2, then $f(x)$ cannot be divisible by 2 in N. An arbitrary group homomorphism need not have this property.

On the other hand, there is a "nonclassical" theory T' whose category of models $\mathrm{Mod}(T')$ is the category whose objects are abelian groups and whose morphisms are group homomorphisms $f : M \to N$. These homomorphisms are required to preserve the truth of certain kinds of statements (such as equations $x+y = z$), but not others (such as inequations $x + y \neq z$). Passing from the setting of classical first order-logic to the setting of pretopoi accommodates such examples in a natural way.

As the terminology suggests, the notion of a pretopos is quite closely related to the notion of a *topos*, which is the other main subject of this course.

Remark 18. Throughout this course, we will always use the word *topos* to refer to what is known as a *Grothendieck topos*. There is also a more general notion of *elementary topos*, which we will not discuss.

Topos theory was introduced by Grothendieck and his students, motivated by problems in algebraic geometry. If X is an algebraic variety defined over the field $\mathbf{C}$ of complex numbers, then the set $X(\mathbf{C})$ of $\mathbf{C}$-valued points of X has a natural topology, which is induced by the usual topology on $\mathbf{C}$. This topological space is reasonably well-behaved (for example, if the algebraic variety X is smooth, then $X(\mathbf{C})$ is a manifold), and one can use it to define invariants (like homology and cohomology) which are very useful in the study of X.

Grothendieck and his students were interested in defining these sorts of invariants for algebraic varieties over other fields which have no interesting topology of their own (such as finite fields). In this case, one can still extract a topological space by endowing X with the *Zariski topology*: that is, by declaring a subset of X to be closed if it can be defined by polynomial equations. This topology is useful for some purposes, but not for extracting algebro-topological invariants like cohomology. As a remedy, Grothendieck introduced the notion of a *topos* and showed that the theory of (sheaf) cohomology could be extended from the setting of topological spaces to the setting of topoi. Moreover, he associated to each algebraic variety a particular topos (the *étale topos*) for which the resulting invariants turned out to be very powerful.

Topos theory is essentially a generalization of point-set topology. More precisely, there is a functor which associates a topos to every topological space, and that functor is fully faithful when restricted to the class of *sober* topological spaces (which includes all Hausdorff spaces). However, it is far from being essentially surjective: there are many interesting examples of topoi which do not come from topology. For example, every first-order theory T has a *classifying topos*, from which we can recover T up to equivalence (in the sense described above). So one can also think of topos theory as a generalization of first-order logic.

Remark 19. One can think of point-set topology and first-order logic as the study of two special classes of topoi. Moreover, these classes have nonempty intersection: the theory of Boolean algebras has an incarnation both in mathematical logic (as the theory of propositional logic) and in point-set topology (as the theory of *Stone spaces*: that is, compact Hausdorff spaces which are totally disconnected, like the Cantor set).

The relevant circle of ideas can be roughly pictured as follows:

$$\begin{tikzcd}
\{\text{Propositional Theories}\} \arrow[r, equal] \arrow[d, hook] & \{\text{Boolean Algebras}\} \arrow[r, equal] & \{\text{Stone Spaces}\} \arrow[d, hook] \\
\{\text{(Classical) First-Order Theories}\} \arrow[d, hook] & & \{\text{Hausdorff Spaces}\} \arrow[d, hook] \\
\{\text{Pretopoi}\} \arrow[d, equal] & & \{\text{Sober Topological Spaces}\} \arrow[d, equal] \\
\{\text{Coherent Topoi}\} \arrow[rr, hook] & & \{\text{Topoi}\} & \{\text{Spatial Topoi}\}. \arrow[l, hook']
\end{tikzcd}$$

The trajectory of this course will take us counterclockwise around this diagram, beginning with first-order logic and ending with a proof of Theorem 15.

Lecture 2: Syntax

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We now review the basic definitions of first-order logic in more detail. Recall that a *language* consists of a collection of symbols $\{P_i\}$, each of which has some specified *arity* n_i (which is a nonnegative integer).

Fix, once and for all, an infinite set V of *variables*. Given subset $V_0 \subseteq V$, a *formula of L with free variables in V_0* consists of one of the following five things:

(1) An expression $x = y$, for some pair of elements $x, y \in V_0$ (which might be the same).

(2) An expression $P_i(x_1, \ldots, x_{n_i})$, where P_i is a predicate of the language L having arity n_i, and each x_i is an element of V_0 (the x_i need not be distinct).

(3) An expression $\varphi \vee \varphi'$, where φ and φ' are previously constructed formulae with free variables in V_0.

(4) An expression $\neg\varphi$, where φ is a previously constructed formula with free variables in V_0.

(5) An expression $(\exists x)[\varphi]$, where $x \in V \backslash V_0$, where φ is a previously constructed formula with free variables in $V_0 \cup \{x\}$.

Definition 1. Let L be a language. An *L-structure* is a set M together with a subset $M[P_i] \subseteq M^{n_i}$ for each predicate symbol P_i of L.

Let M be an L-structure, let φ be a formula of L with free variables $\{x_1, \ldots, x_n\}$, and suppose we're given an n-tuple of elements $\vec{c} = (c_1, \ldots, c_n)$ in M. The relation $M \vDash \varphi(\vec{c})$ is defined by recursion as follows:

(1) If φ is the formula $x_i = x_j$, then $M \vDash \varphi(\vec{c})$ if and only if $c_i = c_j$.

(2) If φ is the formula $P_i(x_{j_1}, \ldots, x_{j_{n_i}})$, then $M \vDash \varphi(\vec{c})$ if and only if and only if the tuple $(x_{j_1}, \ldots, x_{j_{n_j}})$ belongs to $M[P_i]$.

(3) $M \vDash (\varphi \vee \varphi')(\vec{c})$ if and only if either $M \vDash \varphi(\vec{c})$ or $M \vDash \varphi'(\vec{c})$.

(4) $M \vDash (\neg\varphi)(\vec{c})$ if and only if it is not true that $M \vDash \varphi(\vec{c})$.

(5) If φ has the form $(\exists y)\psi$ for some formula ψ with free variable $\{x_1, \ldots, x_n, y\}$, the $M \vDash \varphi(\vec{c})$ if and only if there exists some element $a \in M$ such that $M \vDash \psi(\vec{c}, a)$.

We let $M[\varphi]$ denote the set $\{(c_1, \ldots, c_n) \in M^n : M \vDash \varphi(c_1, \ldots, c_n)\}$.

Definition 2. A *sentence* in a language L is a formula φ with no free variables.

A *theory* T consists of a language L, together with a collection of sentences in the language L (which we refer to as the *axioms of* T.

A *model* of a theory T is an L-structure M such that $M \vDash \varphi$ for each axiom φ of T. In this case, we write $M \vDash T$.

Notation 3 (Substitutions of Variables)**.** We will typically denote a formula by $\varphi(\vec{x})$, where $\vec{x} = (x_1, \ldots, x_n)$ is an enumeration of the set of free variables in φ. However, we will constantly engage in a certain abuse of this notation. Suppose that $\varphi(\vec{x})$ is a formula with free variables in a set of variables $V_0 = \{x_1, \ldots, x_n\}$, and that we are given another finite set of free variables V_1, together with a map

$$V_0 \to V_1 \qquad (x_i \in V_0) \mapsto (y_i \in V_1).$$

We can then form a new formula with free variables in V_1 by replacing each occurrence of the variable x_i by y_i; we will denote this formula by $\varphi(y_1, \ldots, y_n)$. Note that we need not assume that the map $V_0 \to V_1$ is bijective. For example, if $\varphi(x, y)$ is a formula with free variables x and y, then we can consider $\varphi(z, z)$ as a formula with a single free variable z. We can also consider the case where V_0 is a subset of V_1 and each y_i is equal to x_i; in this case, we can regard $\varphi(\vec{x})$ as a formula with free variables in V_1, where it just happens that some of the variables have never been mentioned.

The preceding discussion comes with a caveat: for the substitution procedure $\varphi(x_1, \ldots, x_n) \mapsto \varphi(y_1, \ldots, y_n)$ to be sensible, we must assume that the variables of V_1 do not appear as *bound variables* in the formula φ. For example, if $\varphi(x)$ is a formula of the form $(\exists y)R(y, x)$, then it is sensible to write $\varphi(z)$ for the formula $(\exists y)R(y, z)$ for any variable z distinct from y, but we don't want to think of the sentence $(\exists y)R(y, y)$ as an instance of the formula φ.

Remark 4. We have opted for some economy here in the interest of keeping our definition compact. Using the preceding operations we can define several others. For example:

- We define $\varphi(\vec{x}) \wedge \varphi'(\vec{x})$ to be $\neg((\neg\varphi(\vec{x})) \vee (\neg\varphi'(\vec{x})))$,
- We define $\varphi(\vec{x}) \Rightarrow \varphi'(\vec{x})$ to be $(\neg\varphi(\vec{x})) \vee \varphi'(\vec{x})$.
- We define $\varphi(\vec{x}) \Leftrightarrow \varphi'(\vec{x})$ to be $(\varphi(\vec{x}) \Rightarrow \varphi'(\vec{x})) \wedge (\varphi'(\vec{x}) \Rightarrow \varphi(\vec{x}))$.
- Given a sequence of variables $\vec{y} = (y_1, \ldots, y_m)$ and a formula $\varphi(\vec{x}, \vec{y})$, we define $(\exists\vec{y})[\varphi(\vec{x}, \vec{y})]$ to be $(\exists y)(\exists y_2) \cdots (\exists y_m)[\varphi(\vec{x}, \vec{y})]$.
- Given a sequence of variables $\vec{y} = (y_1, \ldots, y_m)$ and a formula $\varphi(\vec{x}, \vec{y})$, we define $(\forall\vec{y})[\varphi(\vec{x}, \vec{y})]$ to be $\neg(\exists\vec{y})[\neg\varphi(\vec{x}, \vec{y})]$.
- Given a sequence of variables $\vec{y} = (y_1, \ldots, y_m)$ and a formula $\varphi(\vec{x}, \vec{y})$, we define $(\exists!\vec{y})[\varphi(\vec{x}, \vec{y})]$ to be

  $$(\exists\vec{y})[\varphi(\vec{x}, \vec{y})] \wedge (\forall\vec{y})(\forall\vec{z})[(\varphi(\vec{x}, \vec{y}) \wedge \varphi(\vec{x}, \vec{z})) \Rightarrow (y_1 = z_1) \wedge \cdots \wedge (y_m = z_m)]$$

  where $\vec{z} = (z_1, \ldots, z_m)$ is some list of auxiliary variables which are not bound in φ.

Let T be a first order theory. Our next goal is to introduce a category $\mathrm{Syn}_0(T)$, which we will refer to as the *weak syntactic category of* T. The objects of $\mathrm{Syn}_0(T)$ are just formulas in the language of T. To avoid confusion, if $\varphi(\vec{x})$ if a formula in the language of T, we will write $[\varphi(\vec{x})]$ for the corresponding object of $\mathrm{Syn}_0(T)$. Roughly speaking, we want to think of $[\varphi(\vec{x})]$ as the "collection of all tuples $\vec{x}$ satisfying φ". However, we view this as an abstract entity, existing without reference to any fixed model of T.

Definition 5 (Morphisms in $\mathrm{Syn}_0(T)$)**.** Let $X = [\varphi(\vec{x}]$ and $Y = [\psi(\vec{y})]$ be objects of $\mathrm{Syn}_0(T)$. A *morphism from X to Y* in $\mathrm{Syn}_0(T)$ consists of a collection of maps

$$f_M : M[\varphi] \to M[\psi],$$

defined for every model M of T, which are *definable* in the following sense:

($*$) After renaming the variables to arrange that the variables $\vec{x} = (x_1, \ldots, x_m)$ to not appear in ψ and the variables $\vec{y} = (y_1, \ldots, y_n)$ do not appear in φ, there exists a formula $\theta(\vec{x}, \vec{y})$ such that, for each model $M \vDash T$, we have

$$M[\theta] = \Gamma(f_M) := \{(c_1, \ldots, c_m, d_1, \ldots, d_n) : M \vDash \varphi(c_1, \ldots, c_m) \text{ and } f_M(c_1, \ldots, c_m) = (d_1, \ldots, d_n)\}$$

Proposition 6. *Let T be a first-order theory. Then:*

(*a*) *Let $X = [\varphi(\vec{x})]$ be an object of $\mathrm{Syn}_0(T)$. Then the collection of identity maps*

$$\{\mathrm{id}_{M[\varphi]} : M[\varphi] \to M[\varphi]\}_{M \vDash T}$$

is a morphism $\mathrm{id}_X : X \to X$ of $\mathrm{Syn}_0(T)$.

(*b*) *Let $X = [\varphi(\vec{x})]$, $Y = [\varphi'(\vec{y})]$, and $Z = [\varphi''(\vec{z})]$ be objects of $\mathrm{Syn}_0(T)$, and suppose we are given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ given by maps $f_M : M[\varphi] \to M[\varphi']$ and $g_M : M[\varphi'] \to M[\varphi'']$ for each model $M \vDash T$. Then the collection of composite maps $g_M \circ f_M$ determines a morphism from X to Z in $\mathrm{Syn}_0(T)$, which we will denote by $g \circ f$.*

Proof. To prove (*a*), we first choose some auxiliary variables $(y_1, \ldots, y_n)$ not appearing in $\varphi(\vec{x})$. We then observe that graph of the identity map $\mathrm{id} : M[\varphi] \to M[\varphi]$ is given by $M[\theta]$, where $\theta(\vec{x}, \vec{y})$ is the formula

$$\varphi(\vec{x}) \wedge (x_1 = y_1) \wedge \cdots \wedge (x_n = y_n).$$

To prove (*b*), we note that after suitable renaming of variables we may assume that there are formulas $\theta(\vec{x}, \vec{y})$ and $\theta'(\vec{y}, \vec{z})$ such that $M[\theta] = \Gamma(f_M)$ and $M[\theta'] = \Gamma(g_M)$ for each model $M \vDash T$. Then we have $\Gamma(g_M \circ f_M) = M[\rho]$, where $\rho(\vec{x}, \vec{z})$ is the formula $(\exists \vec{y})[\theta(\vec{x}, \vec{y}) \wedge \theta'(\vec{y}, \vec{z})]$. □

Corollary 7. *For every theory T, $\mathrm{Syn}_0(T)$ is a category (with the composition law described in Proposition 6). We will refer to $\mathrm{Syn}_0(T)$ as the weak syntactic category of T.*

Remark 8. Let T be a (typed) first-order theory and let φ be a sentence in the language of T. We will write $T \vDash \varphi$ if $M \vDash \varphi$ for every model M of T.

Let $X = [\varphi(\vec{x})]$ and $Y = [\psi(\vec{y})]$ be objects of $\mathrm{Syn}_0(T)$. Assume that the variables of $\vec{x}$ to not appear in ψ and the variables of $\vec{y}$ do not appear in φ (which we can always arrange by renaming variables). Unwinding the definitions, we see that the morphisms from X to Y in $\mathrm{Syn}_0(T)$ are given by equivalence classes of formulas $\theta(\vec{x}, \vec{y})$ such that

$$T \vDash (\forall \vec{x}, \vec{y})[\theta(\vec{x}, \vec{y}) \to \varphi(\vec{x}) \wedge \psi(\vec{y})] \wedge (\forall \vec{x})[\varphi(\vec{x}) \Rightarrow (\exists! \vec{y})\theta(\vec{x}, \vec{y})]$$

(More informally, this statement can be read as "T implies that θ defines the graph of a function from tuples $\vec{x}$ satisfying φ to tuples $\vec{y}$ satisfying ψ.") Here we consider two formulas $\theta(\vec{x}, \vec{y})$ and $\theta'(\vec{x}, \vec{y})$ to be equivalent if

$$T \vDash (\forall \vec{x}, \vec{y})[\theta(\vec{x}, \vec{y}) \Leftrightarrow \theta'(\vec{x}, \vec{y})].$$

More informally: θ and θ' are equivalent if T implies that they define the graph of the same function.

Warning 9. At this point, the reader might reasonably object that $\mathrm{Syn}_0(T)$ is defined by reference to the models of T, and is therefore not really "syntactic" in nature. To obtain a fully "syntactic" definition, we should define the objects of $\mathrm{Syn}_0(T)$ to be formulas in the language of T, with morphisms from $X = [\varphi(\vec{x})]$ and $Y = [\psi(\vec{y})]$ given by equivalence classes of formulas $\theta(\vec{x}, \vec{y})$ satisfying

$$T \vdash (\forall \vec{x}, \vec{y})[\theta(\vec{x}, \vec{y}) \to \varphi(\vec{x}) \wedge \psi(\vec{y})] \wedge (\forall \vec{x})[\varphi(\vec{x}) \Rightarrow (\exists! \vec{y})\theta(\vec{x}, \vec{y})],$$

where we consider $\theta(\vec{x}, \vec{y})$ and $\theta'(\vec{x}, \vec{y})$ to be equivalent if

$$T \vdash (\forall \vec{x}, \vec{y})[\theta(\vec{x}, \vec{y}) \Leftrightarrow \theta'(\vec{x}, \vec{y})].$$

To make this definition precise, we need to say what it means for a sentence φ to be provable in a theory T. After doing so, we ultimately discover that the result is equivalent to the definition we have given above, by virtue of Gödel's completeness theorem. However, this is in some sense "cheating": for example, later in this course we would like to apply these ideas to recover a proof of Gödel's theorem. We will return to this point later.

Construction 10. Let T be a (typed) first-order theory and let M be a model of T. For every object $X = [\varphi(\vec{x})]$ in the weak syntactic category $\mathrm{Syn}_0(T)$, we let $M[X]$ denote the set $M[\varphi] = \{(c_1, \ldots, c_n) : M \vDash \varphi(c_1, \ldots, c_n)\}$. It follows immediately from the definition of morphism in $\mathrm{Syn}_0(T)$ that we can regard the construction $X \mapsto M[X]$ as a functor from $\mathrm{Syn}_0(T)$ to the category $\mathcal{S}\mathrm{et}$ of sets.

Proposition 11. *Let T be a first-order theory and let M and N be models of T. The following data are equivalent:*

(1) *The datum of an elementary embedding $f_0 : M \to N$*

(2) *The datum of a natural transformation f from $M[\bullet]$ to $N[\bullet]$ (in the category of functors from $\mathrm{Syn}_0(T)$ to $\mathcal{S}\mathrm{et}$).*

Proof. Assume first that $f_0 : M \to N$ is an elementary embedding. For every formula $\varphi(x_1, \ldots, x_n)$ in the language of T and every $c_1, \ldots, c_n \in M$, we have

$$(M \vDash \varphi(c_1, \ldots, c_n)) \Leftrightarrow (N \vDash \varphi(f_0(c_1), \ldots, f_0(c_n))).$$

It follows that the construction $(c_1, \ldots, c_n) \mapsto (f_0(c_1), \ldots, f_0(c_n))$ determines a map of sets $f_\varphi : M[\varphi] \to N[\varphi]$. We claim that these maps determine a natural transformation of functors. In other words, for every morphism $g : [\varphi(x_1, \ldots, x_n)] \to [\psi(y_1, \ldots, y_m)]$ in the weak syntactic category, the diagram

$$\begin{tikzcd} M[\varphi] \arrow[r, "f_\varphi"] \arrow[d, "g_M"] & N[\varphi] \arrow[d, "g_N"] \\ M[\psi] \arrow[r, "f_\psi"] & N[\psi] \end{tikzcd}$$

commutes. Equivalently, we wish to show that the map $f_0^{n+m} : M^{n+m} \to N^{n+m}$ carries the graph $\Gamma(g_M)$ into the graph $\Gamma(g_N)$. Since g is a morphism in the syntactic category, we can choose a formula $\theta(\vec{x}, \vec{y})$ such that $\Gamma(g_m) = M[\theta]$ and $\Gamma(g_N) = N[\theta]$. We are therefore reduced to showing the implication

$$M \vDash \theta(c_1, \ldots, c_n, d_1, \ldots, d_m) \Rightarrow N \vDash \theta(f_0(c_1), \ldots f_0(c_n), f_0(d_1), \ldots f_0(d_m)),$$

which follows from our assumption that f_0 is an elementary embedding.

Now suppose that we are given a natural transformation of functors $f : M[\bullet] \to N[\bullet]$, given by a collection of maps $f_\varphi : M[\varphi] \to N[\varphi]$. Taking $\varphi(x)$ to be the formula $x = x$, we obtain a map $f_0 : M \to N$. We claim that f_0 is an elementary embedding. Let $\varphi(x_1, \ldots, x_n)$ be any formula in the language of T. It follows from the naturality of f that we have a commutative diagram

$$\begin{tikzcd} M[\varphi] \arrow[r, "f_\varphi"] \arrow[d] & N[\varphi] \arrow[d] \\ M^n \arrow[r, "f_0^n"] & N^n, \end{tikzcd}$$

where the vertical maps are inclusions. From the existence of such a diagram, we deduce the implication

$$(M \vDash \varphi(c_1, \ldots, c_n)) \Rightarrow (N \vDash \varphi(f_0(c_1), \ldots, f_0(c_n))).$$

Applying the same argument to the formula $\neg\varphi(\vec{x})$, we deduce the reverse implication, so that f_0 is an elementary embedding.

We leave it to the reader to verify that these constructions are inverse to one another. $\square$

The conclusion of Proposition 11 is that it is not really necessary to distinguish between a model M of T and the associated functor $M[\bullet] : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$; they are essentially the same data. One can think of the functor $M[\bullet]$ as an "unbiased" presentation of the datum of a model of T: one does not think of a model as something that has a single underlying set: instead, one considers *all* the sets which can be constructed from the model by means of a first-order formula.

Of course, if we want to think about models as a special kind of functor, we will need to decide what kind of functors that we want to allow:

Question 12. Let T be a first-order theory and let $F : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ be a functor. When does there exist a model M of T and an isomorphism of functors $F \simeq M[\bullet]$?

We will take up Question 12 in the next lecture.

We close by describing a mild generalization of the formalism described above. Recall that in the previous lecture, we introduced the notion of a *projective plane*. A projective plane is naturally described as a pair of sets P and L (the "points" and "lines", respectively), together with a binary relation between elements of P and elements of L. This is most naturally described by a slight variation on the preceding definitions.

Variant 13 (Typed First-Order Logic)**.** A *typed language* L consists of a set $\mathfrak{T}$ of *types* together with a set $\{P_i\}$ of predicate symbols. Each predicate symbol has an *arity* which is no longer a nonnegative integer, but a finite sequence $\vec{t} = (t_1, \ldots, t_n)$ of types.

We assume that each variable name is assigned a type, in such a way that there are infinitely many variable names of each type.

The notion of formula is defined as before, with two caveats:

(0) We add a new formula $\perp$ (for any number of free variables), to be understood simply as "false." This is necessary only in the case where there are no types (hence no variables); otherwise, we could just pick a variable x and replace $\perp$ by a contradictory assertion like $(\exists x)[\neg(x = x)]$.

(1′) We allow formulas of the form $x = y$ only when x and y are variables names of the same type.

(2′) If P_i is a predicate of arity $(t_1, \ldots, t_n)$, then we allow formulas of the form $P_i(x_1, x_2, \ldots, x_n)$ only when each x_j has type t_j.

An L-structure consists of a set $M[t]$ for every type $t \in \mathfrak{T}$, together with a subset $M[P_i] \subseteq M[t_1] \times \cdots \times M[t_n]$ for each predicate symbol P_i of arity $(t_1, \ldots, t_n)$. More generally, for any formula $\varphi(x_1, \ldots, x_n)$ with variables x_i of types t_i, we define the notion $M \vDash \varphi(c_1, \ldots, c_n)$ for $(c_1, \ldots, c_n) \in M[t_1] \times \cdots \times M[t_n]$ by induction as before, with the added clause that $M \vDash \perp (c_1, \ldots, c_n)$ is always false. We set

$$M[\varphi] = \{(c_1, \ldots, c_n) \in M[t_1] \times \cdots \times M[t_n] : M \vDash \varphi(c_1, \ldots, c_n)\}.$$

Example 14. In the special case where there is only one type, Variant 13 reduces to the definitions given above it.

Remark 15. In the case where there are only finitely many types, Variant 13 represents no real gain in generality. For example, instead of axiomatizing the theory of projective planes in terms of two sets P and L (together with the binary incidence relation $\subseteq P \times L$), one could consider the structure given by the disjoint union $P \amalg L$, augmented by two new unary relations

$$U(x) = \text{“}x \text{ is a point”} \qquad V(x) = \text{“}x \text{ is a line”}$$

$$(\forall x)(U(x) \vee V(x)) \qquad (\forall x)\neg[U(x) \wedge V(x)].$$

However, this sort of trick does not work if we want to allow infinitely many types.

Example 16 (Propositional Logic)**.** In the situation of Variant 13, suppose that the set $\mathfrak{T}$ of types is empty. Then the language L consists of a collection of symbols $\{P_i\}$, all of which are forced to have arity 0. The set V of variable names is empty (note that we still have infinitely many variables of each type, vacuously). All formulas are sentences, which can be built by combining the symbols P_i and $\perp$ by means of the negation operation $\neg$ and the disjunction operation $\vee$. An L-structure is equivalent to the datum of a map

$$\{P_i\} \to \{\mathrm{True}, \mathrm{False}\}$$

which assigns a truth value to each P_i. This determines, more generally, a truth value for each sentence φ built from the P_i.

Example 17 (The Syntactic Category of a Propositional Theory)**.** Let T be a propositional theory (that is, a typed first order theory with no types). Then every formula in the language of T is a sentence. Moreover, for every pair of sentences φ and ψ, we have

$$\mathrm{Hom}_{\mathrm{Syn}_0(T)}([\varphi],[\psi]) = \begin{cases} * & \text{if } T \vDash (\varphi \Rightarrow \psi) \\ \emptyset & \text{otherwise.} \end{cases}$$

In other words, the category $\mathrm{Syn}_0(T)$ is equivalent to the partially ordered set of equivalence classes of sentences. This partially ordered set is a Boolean algebra, with join given by $(\varphi, \psi) \mapsto \varphi \vee \psi$ and complements given by $\varphi \mapsto \neg\varphi$.

Lecture 3: The Structure of $\mathrm{Syn}_0(T)$

January 26, 2018

Throughout this lecture, we fix a first-order theory T. We will assume for simplicity of exposition that T is untyped, but all of our considerations extend to the case of typed theories without essential change. We begin with the following question from the previous lecture:

Question 1. Let T be a first-order theory and let $F : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ be a functor. When does there exist a model M of T and an isomorphism of functors $F \simeq M[\bullet]$?

A satisfying answer to Question 1 would characterize the models of T as those functors $F : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ which preserve some sort of structure which is present on both $\mathrm{Syn}_0(T)$ and $\mathcal{S}\mathrm{et}$. We therefore begin with the following:

Question 2. What are the important structural features of the category $\mathrm{Syn}_0(T)$, and how do they interact with the functors $M[\bullet] : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ determined by models $M \vDash T$?

Proposition 3. *The category* $\mathrm{Syn}_0(T)$ *admits fiber products. Moreover, for every model* $M \vDash T$*, the functor*

$$\mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et} \qquad X \mapsto M[X]$$

preserves fiber products.

Proof. Suppose we are given a pair of morphisms $f : X \to Z$ and $g : Y \to Z$ in $\mathrm{Syn}_0(T)$; we would like to construct a fiber product $X \times_Z Y$. Write $X = [\alpha(\vec{x})]$, $Y = [\beta(\vec{y})$, and $Z = [\gamma(\vec{z})]$, and represent the morphisms f and g by formulae $\theta(\vec{x}, \vec{z})$ and $\theta'(\vec{y}, \vec{z})$. Introduce new sequences of variables $\vec{x}'$ and $\vec{y}'$ of the same length as $\vec{x}$ and $\vec{y}$, respectively, and let $\rho(\vec{x}', \vec{y}')$ be the formula $(\exists \vec{z})[\theta(\vec{x}', \vec{z}) \wedge \theta'(\vec{y}', \vec{z})]$. Note that for any model $M \vDash T$, there is a pullback square of sets

$$\begin{tikzcd} M[\rho] \arrow[r, "\pi_M"] \arrow[d, "\pi'_M"] & M[X] \arrow[d, "f_M"] \\ M[Y] \arrow[r, "g_M"] & M[Z]. \end{tikzcd}$$

Moreover, the collections of functions $\{\pi_M\}_{M \vDash T}$ and $\{\pi'_M\}_{M \vDash T}$ comprise morphisms $\pi : [\rho] \to X$ and $\pi' : [\rho] \to Y$ in the category $\mathrm{Syn}_0(T)$: their graphs are defined by the formulae $(\vec{x} = \vec{x'})$ and $(\vec{y} = \vec{y}')$, respectively. We claim that π and π' exhibit ρ as a fiber product of X and Y over Z (this fiber product is then obviously preserved by the functor $M[\bullet]$ for every $M \vDash T$).

Suppose that we are given some other object $W = [\varphi(\vec{w})]$ in $\mathrm{Syn}_0(T)$, together with maps $u : W \to X$ and $v : W \to Y$ satisfying $f \circ u = g \circ v$. Then, for each model $M \vDash T$, there is a unique map $h_M : M[W] \to M[\rho]$ satisfying $u_M = \pi_M \circ h_M$ and $v_M = \pi'_M \circ h_M$. To complete the proof, it will suffice to show that the functions $\{h_M\}_{M \vDash T}$ comprise a morphism $h : W \to [\rho]$ in the category $\mathrm{Syn}_0(T)$. To prove this, choose formulas $\psi(\vec{w}, \vec{x})$ and $\psi'(\vec{w}, \vec{y})$ satisfying

$$\Gamma(u_M) = M[\psi(\vec{w}, \vec{x})] \qquad \Gamma(v_M) = M[\psi(\vec{w}, \vec{y})]$$

for each $M \vDash T$. Then we have $\Gamma(h_M) = M[\mu(\vec{w}, \vec{x}', \vec{y}')]$, where $\mu(\vec{w}, \vec{x}', \vec{y}')$ is the formula $\psi(\vec{w}, \vec{x}') \wedge \psi(\vec{w}, \vec{y}')$. $\square$

Corollary 4. *The category* $\mathrm{Syn}_0(T)$ *admits finite limits. Moreover, for every model* M *of* T*, the functor* $M[\bullet] : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ *preserves finite limits.*

Proof. Since all finite limits can be built out of fiber products and final objects, it will suffice to show that $\mathrm{Syn}_0(T)$ has a final object which is preserved by each of the functors $M[\bullet]$. Fix a sentence φ such that $T \vDash \varphi$ (for example, φ could be an axiom of T, or it could be the sentence $(\forall x)[x = x]$), and set $\mathbf{1} = [\varphi] \in \mathrm{Syn}_0(T)$. Then, for every model M, the set $M[\mathbf{1}]$ has one element. It follows that, for any object $X = [\psi(\vec{x})] \in \mathrm{Syn}_0(T)$, there is a unique map $f_M : M[X] \to M[\mathbf{1}]$ for each $M \vDash T$. The functions $\{f_M\}_{M \vDash T}$ comprise a morphism f of $\mathrm{Syn}_0(T)$ (their graphs are defined by the formula $\psi(\vec{x})$ itself), which is evidently the unique map from X to $\mathbf{1}$ in $\mathrm{Syn}_0(T)$. It follows that $\mathbf{1}$ is a final object of $\mathrm{Syn}_0(T)$. □

Lemma 5. *Let* $f : X \to Y$ *be a morphism in* $\mathrm{Syn}_0(T)$*. Then* f *is an isomorphism if and only if, for every model* M *of* T*, the induced map* $f_M : M[X] \to M[Y]$ *is an isomorphism.*

Proof. The "only if" direction is obvious. To prove the converse, we must show that if each f_M is an isomorphism, then the collection of functions $\{f_M^{-1}\}_{M \vDash T}$ determines a morphism from Y to X in $\mathrm{Syn}_0(T)$. This is clear: any first-order definition of the graphs of each f_M is also a first-order definition of the graphs of f_M^{-1} (with the variables read in reverse order). □

Corollary 6. *Let* $f : X \to Y$ *be a morphism in* $\mathrm{Syn}_0(T)$*. Then* f *is a monomorphism if and only if, for every model* M *of* T*, the induced map* $f_M : M[X] \to M[Y]$ *is a monomorphism.*

Proof. Apply Lemma 5 to the diagonal map $X \to X \times_Y X$. □

Construction 7. Let $\varphi_0(\vec{x})$ and $\varphi(\vec{x})$ be two formulas in the same free variables such that $T \vDash (\forall \vec{x})[\varphi_0(\vec{x}) \Rightarrow \varphi(\vec{x})]$. Then, for every model M of T, we can identify $M[\varphi_0]$ with a subset of $M[\varphi]$. This identification determines a monomorphism from $X_0 = [\varphi_0(\vec{x})]$ to $X = [\varphi(\vec{x})]$ in the category $\mathrm{Syn}_0(T)$.

Let us call a monomorphism in $\mathrm{Syn}_0(T)$ *special* if it arises from Construction 7. We next show, up to isomorphism, every monomorphism in $\mathrm{Syn}_0(T)$ is special.

Proposition 8. *Let* $f : X \to Z$ *be a morphism in* $\mathrm{Syn}_0(T)$*. Then* f *admits a factorization* $X \xrightarrow{g} Y \xrightarrow{h} Z$*, where* g *is an epimorphism in* $\mathrm{Syn}_0(T)$ *and* h *is a special monomorphism in* $\mathrm{Syn}_0(T)$*. Moreover, we can arrange that this factorization is preserved by the functor* $M[\bullet]$ *for every model* $M \vDash T$ *(that is, each* g_M *is a surjection of sets, and each* h_M *is an injection of sets).*

Proof. Write $X = [\varphi(\vec{x})]$ and $Z = [\psi(\vec{z})]$, so that f is given by a formula $\theta(\vec{x}, \vec{z})$. Set $Y = [(\exists \vec{X})[\theta(\vec{x}, \vec{z})]]$. Note that, for every model $M \vDash T$, we can identify $M[Y]$ with the image of the map $f_M : M[X] \to M[Z]$. In particular, the map f_M factors canonically as a composition

$$M[X] \xrightarrow{g_M} M[Y] \xrightarrow{h_M} M[Z].$$

Note that $T \vDash (\forall \vec{z})[(\exists \vec{x} \theta(\vec{x}, \vec{z})) \Rightarrow \psi(\vec{z})]$, so that the collection of maps $\{h_M\}_{M \vDash T}$ define a special monomorphism $h : Y \to Z$ in $\mathrm{Syn}_0(T)$. Moreover, the graphs of the maps $\{g_M\}_{M \vDash T}$ are defined by the formula $\theta(\vec{x}, \vec{z})$, and therefore determine a morphism $g : X \to Y$ in $\mathrm{Syn}_0(T)$. To complete the proof, it will suffice to show that g is an epimorphism; we will prove a stronger assertion below. □

Warning 9. The factorization of Proposition 8 is not completely determined (even up to isomorphism) by saying that g is an epimorphism and h is a monomorphism. For example, if T is a propositional theory, then the category $\mathrm{Syn}_0(T)$ is equivalent to a poset: in this case, *every* morphism of $\mathrm{Syn}_0(T)$ is both an epimorphism and a monomorphism.

To address the uniqueness of the factorization appearing in Proposition 8, it is convenient to introduce some category-theoretic terminology.

Definition 10. Let $\mathcal{C}$ be a category which admits fiber products, and suppose we are given a morphism $g : X \to Y$ in $\mathcal{C}$. Let $X \times_Y X$ denote the fiber product of X with itself over Y, and let $\pi, \pi' : X \times_Y X \to X$ denote the projection maps onto the two factors. We will say that g is an *effective epimorphism* if g exhibits Y as a coequalizer of the maps $\pi, \pi' : X \times_Y X \to X$. In other words, g is an effective epimorphism if, for every object $W \in \mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(Y, W) \simeq \{u \in \mathrm{Hom}_{\mathcal{C}}(X, W) : u \circ \pi = u \circ \pi'\}.$$

Remark 11. Let $\mathcal{C}$ be a category which admits fiber products. Then every effective epimorphism is an epimorphism. In the category of sets, the converse is true: if $g : X \to Y$ is a surjective map of sets, then we can recover Y as the quotient of X by the equivalence relation $R = X \times_Y X = \{(x, x') : g(x) = g(x')\}$. However, this is not true in a general category.

Example 12. Let $\mathcal{C}$ be the category of commutative rings. Then a ring homomorphism $f : R \to S$ is an effective epimorphism in $\mathcal{C}$ if and only if f is surjective. However, there are plenty of non-surjective ring homomorphisms which are epimorphisms in $\mathcal{C}$, such as localization maps $R \mapsto R[1/t]$.

Example 13. The map $g : X \to Y$ constructed in the proof of Proposition 8 is actually an effective epimorphism. To prove this, suppose we are given an object W and a map $u : X \to W$ in $\mathrm{Syn}_0(T)$ satisfying $u \circ \pi = u \circ \pi'$, where $\pi, \pi' : X \times_Y X \to X$ are the projection maps. For every model $M \vDash T$, we have $(u \circ \pi)_M = (u \circ \pi')_M$. Since g_M is a surjection of sets, it is an effective epimorphism; it follows that there is a unique map $\overline{u}_M : M[Y] \to M[W]$ such that $u_M = \overline{u}_M \circ g_M$. We claim that $\{\overline{u}_M\}_{M \in \vDash T}$ determines a morphism $\overline{u} : Y \to W$ in $\mathrm{Syn}_0(T)$ (which is automatically the unique solution to $u = g \circ \overline{u}$). Writing $W = [\alpha(\vec{w})]$ (and retaining the notation of Proposition 8), we see that u can be described by the formula $\overline{\beta}(\vec{z}, \vec{w})$ given by

$$(\exists \vec{x})[\theta(\vec{x}, \vec{z}) \wedge \beta(\vec{z}, \vec{w})],$$

where $\beta(\vec{x}, \vec{w})$ is any formula defining the morphism u.

Proposition 14. *Let $\mathcal{C}$ be a category which admits fiber products and let $f : X \to Z$ be a morphism in $\mathcal{C}$. If f can be factored as a composition $X \xrightarrow{g} Y \xrightarrow{h} Z$ where h is a monomorphism and g is an effective epimorphism, then that factorization is unique (up to unique isomorphism).*

Proof. Suppose we are given another factorization $X \xrightarrow{g'} Y' \xrightarrow{h'} Z$, where h' is a monomorphism and g' is an effective epimorphism. We claim that there is a unique morphism $u : Y \to Y'$ for which the diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\mathrm{id}"] & Y \arrow[r, "g"] \arrow[d, "u"] & Z \arrow[d, "\mathrm{id}"] \\ X \arrow[r, "g'"] & Y' \arrow[r, "h'"] & Z \end{tikzcd}$$

commutes. Since g is an effective epimorphism, it will suffice to show that $g' \circ \pi = g' \circ \pi'$, where $\pi, \pi' : X \times_Y X \to X$ are the projection maps. Since h' is a monomorphism, we are reduced to proving that $h' \circ g' \circ \pi = h' \circ g' \circ \pi'$. But we can rewrite this equality as $h \circ g \circ \pi = h \circ g \circ \pi'$, which follows from the identity $g \circ \pi = g \circ \pi'$.

Applying the same argument with the roles of Y and Y' reversed, we will obtain a morphism $v : Y' \to Y$; it follows from the uniqueness of the factorization above that u and v are mutually inverse isomorphisms. $\square$

Corollary 15. *Let $f : X \to Z$ be a morphism in the syntactic category $\mathrm{Syn}_0(T)$. Then the factorization $X \xrightarrow{g} Y \xrightarrow{h} Z$ is characterized uniquely (up to unique isomorphism) by the fact that g is an effective epimorphism and h is a monomorphism.*

Corollary 16. *Let $X = [\varphi(\vec{x})]$ be an object of $\mathrm{Syn}_0(T)$, and let $f : X_0 \hookrightarrow X$ be a monomorphism in $\mathrm{Syn}_0(T)$. Then f is isomorphic to a special monomorphism $[\varphi_0(\vec{x})] \hookrightarrow [\varphi(\vec{x})]$ (by an isomorphism which is the identity on X).*

Proof. Proposition 8 supplies a factorization of f as a composition $X_0 \xrightarrow{g} Y \xrightarrow{h} X$ where g is an effective epimorphism and h is a special monomorphism. However, if f is already a monomorphism, then we also have the factorization $X_0 \xrightarrow{\mathrm{id}} X_0 \xrightarrow{f} X$. Invoking the uniqueness of Proposition 14, we deduce that there is a commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X_0 \arrow[r, "f"] \arrow[d] & X \arrow[d, "\mathrm{id}"] \\ X_0 \arrow[r, "g"] & Y \arrow[r, "h"] & X \end{tikzcd}$$

where the vertical maps are isomorphisms. □

Notation 17. Let $\mathcal{C}$ be any category and let X be an object of $\mathcal{C}$. We let $\mathrm{Sub}(X)$ denote the set of equivalence classes of monomorphisms $i_0 : X_0 \to X$, where two monomorphisms $i_0 : X_0 \to X$ and $i_1 : X_1 \to X$ are considered to be equivalent if there is an isomorphism $e : X_0 \simeq X_1$ for which the diagram

$$\begin{tikzcd} X_0 \arrow[rr, "e"] \arrow[dr, "i_0"] & & X_1 \arrow[dl, "i_1"'] \\ & X & \end{tikzcd}$$

commutes. We will refer to $\mathrm{Sub}(X)$ as the *set of subobjects of* X.

We will generally abuse notation by simply identifying elements of $\mathrm{Sub}(X)$ with the objects X_0 representing them (in this case, we implicitly assume that a monomorphism $X_0 \hookrightarrow X$ has been supplied). Given a pair of subobjects $X_0, X_1 \in \mathrm{Sub}(X)$, we write $X_0 \subseteq X_1$ if there exists a commutative diagram

$$\begin{tikzcd} X_0 \arrow[rr, "e"] \arrow[dr, "i_0"] & & X_1 \arrow[dl, "i_1"'] \\ & X; & \end{tikzcd}$$

in this case e is automatically unique (and is also a monomorphism).

Proposition 18. *Let $X = [\varphi(\vec{x})]$ be an object of $\mathrm{Syn}_0(T)$. Then:*

(1) *Every subobject of X is has the form $[\varphi_0(\vec{x})]$, where $\varphi_0(\vec{x})$ satisfies $T \vDash (\forall \vec{z})[\varphi_0(\vec{x}) \Rightarrow \varphi(\vec{x})]$ (equipped with the special monomorphism to X described in Construction 7.*

(2) *Given a pair of subobjects $X_0 = [\varphi_0(\vec{x})]$ and $X_1 = [\varphi_1(\vec{x})]$, we have $X_0 \subseteq X_1$ if and only if $T \vDash (\forall \vec{x})[\varphi_0(\vec{x}) \Rightarrow \varphi_1(\vec{x})]$.*

(3) *Given a pair of subobjects $X_0 = [\varphi_0(\vec{x})]$ and $X_1 = [\varphi_1(\vec{x})]$, we have $X_0 = X_1$ (in $\mathrm{Sub}(X)$) if and only if $T \vDash (\forall \vec{x})[\varphi_0(\vec{x}) \Leftrightarrow \varphi_1(\vec{x})]$*

Proof. Assertion (1) is Corollary 16 and (3) follows from (2). For assertion (2), it is clear that if $X_0 \subseteq X_1$, then for every model $M \vDash T$ we must have $M[X_0] \subseteq M[X_1]$ (where we identify both with subsets of $M[X]$), so that $T \vDash (\forall \vec{x})[\varphi_0(\vec{x}) \Rightarrow \varphi_1(\vec{x})]$. Conversely, if each $M[X_0]$ is a subset of $M[X_1]$, then the inclusion maps $e_M : M[X_0] \hookrightarrow M[X_1]$ define a morphism $e : X_0 \to X_1$ (as in Construction 7), for which the diagram

$$\begin{tikzcd} X_0 \arrow[rr, "e"] \arrow[dr] & & X_1 \arrow[dl] \\ & X & \end{tikzcd}$$

commutes in $\mathrm{Syn}_0(T)$. □

Corollary 19. *Let Z be an object of $\mathrm{Syn}_0(T)$. Then the partially ordered set $\mathrm{Sub}(Z)$ is a Boolean algebra. Moreover, for every model M of T, the construction $(M[Z_0] \in \mathrm{Sub}(Z)) \mapsto (M[Z_0] \subseteq M[Z])$ induces a Boolean algebra homomorphism from $\mathrm{Sub}(Z)$ to the lattice of all subsets of $M[Z]$.*

Proof. Write $Z = [\varphi(\vec{z})]$, and let us identify $\mathrm{Sub}(Z)$ with the collection of equivalence classes of formulae $\varphi_0(\vec{z})$ in the free variables $\vec{z}$ satisfying $T \vDash (\forall \vec{z})[\varphi_0(\vec{z}) \Rightarrow \varphi(\vec{z})]$. This poset has a largest element (given by $\varphi(\vec{z})$), a smallest element (given by $(\exists x)\neg[x = x]$), meets (given by $\varphi_0(\vec{z}) \wedge \varphi_1(\vec{z})$), joins (given by $\varphi_0(\vec{z}) \vee \varphi_1(\vec{z})$), and complements (given by $\varphi_0(\vec{z}) \mapsto (\neg\varphi_0(\vec{z}) \wedge \varphi(\vec{z}))$). $\square$

We are now ready to answer Question 1:

Theorem 20. *Let $F : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ be a functor. Then F arises from a model $M \vDash T$ if and only if it satisfies the following three conditions:*

(*a*) *The functor F preserves finite limits.*

(*b*) *The functor F carries effective epimorphisms in $\mathrm{Syn}_0(T)$ to surjections of sets.*

(*c*) *For every object $X \in \mathrm{Syn}_0(T)$, the induced map $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is a homomorphism of upper semilattices: that is, it preserves least upper bounds of finite subsets.*

We will prove Theorem 20 in the next lecture.

Lecture 4: Coherent Categories

January 31, 2018

Let T be a first-order theory and let $\mathrm{Syn}_0(T)$ denote the weak syntactic category of T. In the previous lecture, we proved that $\mathrm{Syn}_0(T)$ has the following properties:

$(A1)$ The category $\mathrm{Syn}_0(T)$ admits finite limits. In particular, it admits fiber products.

$(A2)$ Every morphism $f : X \to Z$ in $\mathrm{Syn}_0(T)$ admits a factorization $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g is an effective epimorphism and h is a monomorphism.

$(A3)$ For every object $X \in \mathrm{Syn}_0(T)$, the partially ordered set $\mathrm{Sub}(X)$ is an upper semilattice: that is, it has a least element, and every pair of subobjects $X_0, X_1 \subseteq X$ have a least upper bound $X_0 \vee X_1$.

Remark 1. In fact, we actually proved the following stronger version of $(A3)$:

$(A3')$ For every object $X \in \mathrm{Syn}_0(T)$, the partially ordered set $\mathrm{Sub}(X)$ is a Boolean algebra.

We begin with a proof of the result promised in Lecture 3:

Theorem 2. *Let T be a first-order theory and let $F : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ be a functor. Then F arises from a model $M \vDash T$ if and only if it satisfies the following three conditions:*

(1) *The functor F preserves finite limits.*

(2) *The functor F carries effective epimorphisms in $\mathrm{Syn}_0(T)$ to surjections of sets.*

(3) *For every object $X \in \mathrm{Syn}_0(T)$, the induced map $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is a homomorphism of upper semilattices: that is, it carries the least element of $\mathrm{Sub}(X)$ to the empty set, and carries joins $X_0 \vee X_1$ to unions of subsets of $F(X)$.*

Remark 3. In the statement of Theorem 2, we can replace (c) by the following *a priori* stronger statement:

$(3')$ For every object $X \in \mathrm{Syn}_0(T)$, the induced map $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is a homomorphism of Boolean algebras.

Note that condition (1) already guarantees that the map $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is a homomorphism of *lower* semilattices: that is, it carries X to $F(X)$, and carries meets $X_0 \wedge X_1$ to intersections of the corresponding subsets. It follows that it also preserves complements (if U is an element of a Boolean algebra with greatest element $\top$ and least element $\bot$, then the complement U^c is characterized by the identities $U \wedge U^c = \bot$ and $U \vee U^c = \top$, and is therefore preserved by all lattice homomorphisms).

The necessity of conditions (1) and (3) was noted in the previous lecture. The necessity of (2) is a consequence of the following:

Lemma 4. *Let $f : X \to Z$ be a morphism in $\mathrm{Syn}_0(T)$. Then f is an effective epimorphism if and only if, for every model $M \vDash T$, the induced map $f_M : M[X] \to M[Z]$ is surjective.*

Proof. Let $X \xrightarrow{g} Y \xrightarrow{h} Z$ be the canonical factorization of produced in the previous lecture. Since the factorization of f as an effective epimorphism followed by a monomorphism is unique (up to unique isomorphism), it follows that f is an effective epimorphism if and only if h is an isomorphism. We saw in the previous lecture that this is equivalent to the requirement that $h_M : M[Y] \to M[Z]$ is bijective for each $M \vDash T$, which is equivalent to the surjectivity of $g_M : M[X] \to M[Y]$. $\square$

Proof Sketch of Theorem 2. The necessity of conditions (1), (2), and (3) has now been established; let us show the sufficiency. Let $F : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ be a functor satisfying (1), (2), and (3); we wish to construct a model M of T and a collection of bijections $F(X) \simeq M[X]$ depending functorially on $X \in \mathrm{Syn}_0(T)$.

Fix a variable e and let $E \in \mathrm{Syn}_0(T)$ be the object corresponding to some tautology having e as a free variable (such as the formula $e = e$). For every finite set of variables $V_0 = \{x_1, \ldots, x_n\}$, let E^{V_0} denote the object of $\mathrm{Syn}_0(T)$ given by the formula $(x_1 = x_1) \wedge \cdots \wedge (x_n = x_n)$ (regarded as a formula with free variables in V_0). For every model $N \vDash T$, we have canonical bijections

$$N[E] \simeq N \qquad N[E^{V_0}] \simeq N^{V_0},$$

which exhibit E^{V_0} as a product $\prod_{x \in V_0} E$ in the category $\mathrm{Syn}_0(T)$.

Set $M = F(E)$. Since the functor F preserves products, we have canonical isomorphisms $F(E^{V_0}) \simeq M^{V_0}$. For every formula $\varphi(\vec{x})$ having free variables $V_0 = \{x_1, \ldots, x_n\}$, we have a special monomorphism $[\varphi(\vec{x})] \hookrightarrow E^{V_0}$ in $\mathrm{Syn}_0(T)$. Condition (1) guarantees that F preserves monomorphisms, so that we obtain an injective map of sets

$$\iota_{\varphi(\vec{x})} : F([\varphi(\vec{x})]) \hookrightarrow F(E^{V_0}) \simeq M^{V_0}.$$

Suppose we are given a map between finite sets of variables $V_0 \to V_1$, carrying each x_i to some element $y_i \in V_1$. Assuming that none of the variables in V_1 are bound in φ, we can then consider the formula $\varphi(\vec{y})$ with variables in V_1. In the category $\mathrm{Syn}_0(T)$, we have a commutative diagram

$$\begin{tikzcd} {[\varphi(\vec{y})]} \arrow[r] \arrow[d] & E^{V_1} \arrow[d] \\ {[\varphi(\vec{x})]} \arrow[r] & E^{V_0} \end{tikzcd}$$

where the horizontal maps are special monomorphisms. This diagram becomes a pullback square in every model of T, and is therefore a pullback square in $\mathrm{Syn}_0(T)$. Invoking assumption (1), we deduce:

($*$) In the situation above, we have a pullback square of sets

$$\begin{tikzcd} F([\varphi(\vec{y})]) \arrow[r, "\iota_{\varphi(\vec{y})}"] \arrow[d] & M^{V_1} \arrow[d] \\ F([\varphi(\vec{x})] \arrow[r, "\iota_{\varphi(\vec{x})}"] & M^{V_0}. \end{tikzcd}$$

Let P_i be a predicate of the language of T having arity n_i, so that we can regard $P_i(x_1, \ldots, x_{n_i})$ as a formula with n_i free variables. We regard M as a structure for the language of T by taking $M[P_i]$ to be the subset

$$F([P_i(x_1, \ldots, x_{n_i})]) \subseteq M^{\{x_1, \ldots, x_{n_i}\}} \simeq M^{n_i}.$$

We now prove the following:

($*'$) For every formula $\varphi(x_1, \ldots, x_n)$, the map $\iota_{\varphi(\vec{x})}$ induces a bijection $F([\varphi(\vec{x})]) \simeq M[\varphi(\vec{x})]$.

The proof proceeds by induction on the construction of φ; there are five cases to consider:

(i) Suppose first that $\varphi(\vec{x})$ is a formula of the form $x = y$. By virtue of $(*)$, we can assume that x and y are the only free variables of φ (and that they are distinct). In this case, we note that $[\varphi]$ is equivalent, as a subobject of $E^{\{x,y\}} \simeq E \times E$, to the subobject given by the diagonal embedding $E \hookrightarrow E \times E$ (since this is true in every model of T). It follows that F carries $[\varphi]$ to the image of the diagonal map $M \simeq F(E) \to F(E) \times F(E) = M \times M$, which is $M[\varphi]$.

(ii) Suppose that $\varphi(\vec{x}) = P_i(x_{j_1}, \ldots, x_{j_{n_i}})$. In this case, the desired result follows from the definition of $M[P_i]$ (together with $(*)$).

(iii) Suppose that $\varphi(\vec{x})$ has the form $\varphi_0(\vec{x}) \vee \varphi_1(\vec{x})$. In this case, the desired result follows from our inductive hypothesis together with (3).

(iv) Suppose that $\varphi(\vec{x})$ has the form $\neg\psi(\vec{x})$. In this case, the desired result follows from our inductive hypothesis together condition ($3'$) of Remark 3.

(v) Suppose that $\varphi(\vec{x})$ has the form $(\exists y)[\psi(\vec{x}, y)]$. In this case, we have a natural map $f : [\psi(\vec{x}, y)] \to [\varphi(\vec{x})]$ in $\mathrm{Syn}_0(T)$. The realization of f in every model is surjective, so f is an effective epimorphism (Lemma 4). Applying (2), we conclude that the induced map $F([\psi(\vec{x}, y)]) \to F([\varphi(\vec{x})])$ is surjective, so that $\iota_{\varphi(\vec{x})}(F([\varphi(\vec{x})]))$ can be identified with the image of the composite map

$$F([\psi(\vec{x}, y)]) \xrightarrow{\iota_{\psi(\vec{x},y)}} M^n \times M \to M^n.$$

Applying our inductive hypothesis, we conclude that this is the set

$$\{(c_1, \ldots, c_n) \in M^n : (\exists d \in M)[M \vDash \psi(c_1, \ldots, c_n, d)]\} = \{(c_1, \ldots, c_n) \in M^n : M \vDash \varphi(c_1, \ldots, c_n)\}.$$

Applying $(*')$ in the case where φ is an axiom of T, we deduce that

$$(M \vDash \varphi) \Leftrightarrow (F([\varphi]) \neq \emptyset).$$

However, $[\varphi]$ is a final object of $\mathrm{Syn}_0(T)$ and F preserves final objects by (a), so that $F([\varphi])$ is a singleton and therefore φ is true in M. It follows that M is a model of T, and assertion $(*')$ supplies bijections $F([\varphi(\vec{x})]) \simeq M[\varphi(\vec{x})]$ for each formula $\varphi(\vec{x})$. We leave it to the reader to verify that these bijections are natural in $[\varphi(\vec{x})]$ (as an object of $\mathrm{Syn}_0(T)$). $\square$

To complete the transition from the language of first-order logic to the language of category theory, we need to address the following:

Question 5. Let $\mathcal{C}$ be a small category. Under what conditions does there exist a typed first-order theory T such that $\mathcal{C}$ is equivalent to $\mathrm{Syn}_0(T)$?

We have already noted that $\mathcal{C}$ must satisfy conditions ($A1$) through ($A3$) above. Roughly speaking, we can think of ($A1$) through ($A3$) as describing *operations* that can be performed in the category $\mathcal{C}$: that is, procedures for combining various objects of $\mathcal{C}$ to produce new objects. We now formulate two additional axioms which describe compatibilities among these operations.

Proposition 6. *Let T be a typed first order theory and let $\mathrm{Syn}_0(T)$ be the syntactic category of T. Then weak syntactic category $\mathrm{Syn}_0(T)$ satisfies the following:*

($A4$) *For every pullback diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

in $\mathrm{Syn}_0(T)$, if f is an effective epimorphism, then f' is also an effective epimorphism.

Proof. Let M be a model of T. Then the diagram

$$\begin{array}{ccc} M[X'] & \longrightarrow & M[X] \\ \downarrow{\scriptstyle f'_M} & & \downarrow{\scriptstyle f_M} \\ M[Y'] & \longrightarrow & M[Y] \end{array}$$

is a pullback square of sets. Since f is an effective epimorphism, the map f_M is surjective (Lemma 4). It follows that f'_M is also surjective. Since this is true for every model M of T, Lemma 4 implies that f'_M is an effective epimorphism. □

Note that if $\mathcal{C}$ is any category which admits fiber products, then any morphism $f : X \to Y$ in $\mathcal{C}$ induces a map of posets $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$, given by $f^{-1}(Y_0) = X \times_Y Y_0$.

Proposition 7. *For every typed first-order theory T, the weak syntactic category $\mathrm{Syn}_0(T)$ satisfies the following:*

($A5$) *For every morphism $f : X \to Y$ in $\mathrm{Syn}_0(T)$, the map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is a homomorphism of upper semilattices. That is, it preserves smallest elements, and we have $f^{-1}(Y_0 \vee Y_1) = f^{-1}(Y_0) \vee f^{-1}(Y_1)$ for $Y_0, Y_1 \subseteq Y$.*

Proof. Let $f : X \to Y$ be a morphism in $\mathrm{Syn}_0(T)$ and suppose we are given a pair of subobjects $Y_0, Y_1 \subseteq Y$. Then Y_0 and Y_1 are contained in $Y_0 \vee Y_1$, so $f^{-1}(Y_0)$ and $f^{-1}(Y_1)$ are contained in $f^{-1}(Y_0 \vee Y_1)$. It follows that we have $f^{-1}(Y_0) \vee f^{-1}(Y_1) \subseteq f^{-1}(Y_0 \vee Y_1)$ (as subobjects of X), To show that this inclusion is an equality, it will suffice to show that in every model $M \vDash T$, we have $M[f^{-1}(Y_0) \vee f^{-1}(Y_1)] = M[f^{-1}(Y_0 \vee Y_1)]$ (as subsets of $M[X]$). Since M is compatible with pullbacks and with joins of subobjects, this reduces to the equality $f_M^{-1} M[Y_0] \cup f_M^{-1} M[Y_1] = f_M^{-1}(M[Y_0] \cup M[Y_1])$. □

Definition 8. Let $\mathcal{C}$ be a category. We will say that $\mathcal{C}$ is *coherent* if it satisfies the following axioms:

($A1$) The category $\mathcal{C}$ admits finite limits.

($A2$) Every morphism $f : X \to Z$ in $\mathcal{C}$ admits a factorization $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g is an effective epimorphism and h is a monomorphism.

($A3$) For every object $X \in \mathcal{C}$, the partially ordered set $\mathrm{Sub}(X)$ is an upper semilattice: that is, it has a least element, and every pair of subobjects $X_0, X_1 \subseteq X$ have a least upper bound $X_0 \vee X_1$.

($A4$) The collection of effective epimorphisms in $\mathcal{C}$ is stable under pullback.

($A5$) For every morphism $f : X \to Y$ in $\mathcal{C}$, the map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is a homomorphism of upper semilattices.

Example 9. For every typed first-order theory T, the weak syntactic category $\mathrm{Syn}_0(T)$ is a coherent category.

Example 10. The category $\mathcal{S}\mathrm{et}$ of sets is a coherent category.

Example 11. Let P be a partially ordered set, considered as a category. Then axioms ($A1$) through ($A5$) can be restated as follows:

($A1$) The partially ordered set P is a lower semilattice: that is, it has a largest element $\top$ and pairwise meets $p \wedge q$.

($A2$) Automatically satisfied; for each $p \leq q$ in P, we associate the factorization $p \leq p \leq q$.

($A3$) The partially ordered set P is also an upper semilattice: that is, it has a smallest element $\bot$ and pairwise joins $p \vee q$.

($A4$) Automatically satisfied, since the effective epimorphisms in P are isomorphisms.

($A5$) P satisfies the distributive law $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$.

A partially ordered set satisfying these conditions is called a *distributive lattice*.

Remark 12. Let $\mathcal{C}$ be any coherent category and let X be an object of $\mathcal{C}$. Then the poset of subobjects $\mathrm{Sub}(X)$ is a distributive lattice.

Remark 13. Let $\mathcal{C}$ be any category which admits fiber products. Then, for every object $X \in \mathcal{C}$, the poset $\mathrm{Sub}(X)$ is automatically a lower semilattice: it has a largest element given by X itself, and every pair of subobjects $X_0, X_1 \subseteq X$ has a greatest lower bound given by the fiber product $X_0 \times_X X_1$. Moreover, for any map $f : X \to Y$ in $\mathcal{C}$, the pullback map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is automatically a homomorphism of lower semilattices: that is, it preserves largest elements and intersections.

Remark 14. Let $\mathcal{C}$ be a category satisfying ($A2$). We saw in the previous lecture that if a morphism $f : X \to Z$ admits a factorization $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g is an effective epimorphism and h is a monomorphism, then the factorization is unique (up to unique isomorphism). We will emphasize that uniqueness by writing $Y = \mathrm{Im}(f)$ and referring to it as the *image* of f.

Remark 15. Let $\mathcal{C}$ be a category satisfying ($A1$) and ($A2$). Then ($A4$) is equivalent to the following:

($A4'$) For any pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Z' \arrow[r] & Z \end{tikzcd}$$

in $\mathcal{C}$, we have $\mathrm{Im}(f') = \mathrm{Im}(f) \times_Z Z'$ (as subobjects of Z').

To see that ($A4'$) $\Rightarrow$ ($A4$), consider as a diagram as above and suppose that f is an effective epimorphism. Then $\mathrm{Im}(f) = Z$, so ($A4'$) implies that $\mathrm{Im}(f') = Z'$. It follows that f' is also an effective epimorphism. Conversely, suppose that ($A4$) is satisfied and consider a diagram as above, which we extend to a commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "g'"] & X \arrow[d, "g"] \\ \mathrm{Im}(f) \times_{Z'} Z \arrow[r] \arrow[d, "h'"] & \mathrm{Im}(f) \arrow[d, "h"] \\ Z' \arrow[r] & Z \end{tikzcd}$$

where g is an effective epimorphism, h is a monomorphism, and the lower square is a pullback. Since the outer rectangle is also a pullback, the upper square is a pullback. Applying ($A4$), we deduce that g' is an effective epimorphism. Moreover, h' is a pullback of h, and therefore a monomorphism. It follows that $\mathrm{Im}(f) \times_{Z'} Z = \mathrm{Im}(f')$ as subobjects of Z', as desired.

We now return to Question 5.

Definition 16. Let $\mathcal{C}$ be a coherent category. We will say that $\mathcal{C}$ is *Boolean* if it satisfies the following stronger version of ($A3$):

($A3'$) For every object $X \in \mathcal{C}$, the partially ordered set $\mathrm{Sub}(X)$ is a Boolean algebra.

Theorem 17. *Let $\mathcal{C}$ be a small category. The following conditions are equivalent:*

(a) *The category $\mathcal{C}$ is a Boolean coherent category.*

(b) *There exists a typed first-order theory T and an equivalence of categories $\mathcal{C} \simeq \mathrm{Syn}_0(T)$.*

We will discuss Theorem 17 in the next lecture.

Remark 18. In the situation of Theorem 17, suppose that we want to guarantee that $\mathcal{C}$ is equivalent to $\mathrm{Syn}_0(T)$, where T is an *untyped* first-order theory. In this case, we must add the following axiom to our list:

($A6$) There exists a single object $X \in \mathcal{C}$ with the property that every object of $\mathcal{C}$ can be realized as a subobject of X^n for some $n \gg 0$.

Lecture 5: Booleanization

January 31, 2018

Recall the following definition from the previous lecture:

Definition 1. Let $\mathcal{C}$ be a category. We will say that $\mathcal{C}$ is a *coherent category* if it satisfies the following axioms:

($A1$) The category $\mathcal{C}$ admits finite limits.

($A2$) Every morphism $f : X \to Z$ in $\mathcal{C}$ admits a factorization $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g is an effective epimorphism and h is a monomorphism.

($A3$) For every object $X \in \mathcal{C}$, the partially ordered set $\mathrm{Sub}(X)$ is an upper semilattice: that is, it has a least element, and every pair of subobjects $X_0, X_1 \subseteq X$ have a least upper bound $X_0 \vee X_1$.

($A4$) The collection of effective epimorphisms in $\mathcal{C}$ is stable under pullback.

($A5$) For every morphism $f : X \to Y$ in $\mathcal{C}$, the map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is a homomorphism of upper semilattices.

Definition 2. Let $\mathcal{C}$ and $\mathcal{C}'$ be coherent categories. A *morphism of coherent categories* from $\mathcal{C}$ to $\mathcal{C}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ with the following properties:

(1) The functor F is left exact: that is, it preserves finite limits.

(2) The functor F carries effective epimorphisms to effective epimorphisms.

(3) For every object $X \in \mathcal{C}$, the induced map $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is a homomorphism of upper semilattices: that is, it preserves smallest elements and joins.

Remark 3. Let $F : \mathcal{C} \to \mathcal{C}'$ satisfy condition (1) of Definition 2. Then F preserves monomorphisms. Consequently, condition (2) is equivalent to the requirement that for every morphism $f : X \to Z$, the functor F carries the canonical factorization $X \to \mathrm{Im}(f) \hookrightarrow Z$ to the factorization $F(X) \to \mathrm{Im}(F(f)) \hookrightarrow F(Z)$. Note also that in the situation of (3), the map $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is automatically a homomorphism of lower semilattices (that is, it preserves largest elements and meets).

Example 4. Let P and P' be distributive lattices. Then, when viewed as categories, P and P' are coherent categories. A morphism of coherent categories from P to P' is just a lattice homomorphism: that is, a map of partially ordered sets that preserves least upper bounds and greatest lower bounds for finite subsets.

Example 5. Let $\mathcal{C}$ be a category containing an object X. We can then form a new category $\mathcal{C}_{/X}$, whose objects are pairs (U, f), where $U \in \mathcal{C}$ is an object and $f : U \to X$ is a morphism. A morphism from (U, f) to (V, g) in $\mathcal{C}$ is a morphism $h : U \to V$ in $\mathcal{C}$ such that $f = g \circ h$. The construction $(U, f) \mapsto U$ determines a forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}$, and we will generally abuse notation by identifying an object of $\mathcal{C}_{/X}$ with its image under this forgetful functor.

Exercise 6. Show that if $\mathcal{C}$ is a coherent category, then so is $\mathcal{C}_{/X}$. Moreover, the formation of fiber products, images, and unions of subobjects in $\mathcal{C}_{/X}$ can be computed in the underlying category $\mathcal{C}$.

Beware that the forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}$ is not a morphism of coherent categories, because it does not preserve final objects. However, the forgetful functor has a right adjoint which *is* a morphism of coherent categories. More generally, suppose that $f : X \to Y$ is any morphism in $\mathcal{C}$. Then f determines a functor $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$, given by the construction $U \mapsto U \times_X Y$.

Exercise 7. Show that if $f : X \to Y$ is a morphism in a coherent category $\mathcal{C}$, then the functor $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ is a morphism of coherent categories. In fact, this is precisely the content of axioms $(A4)$ and $(A5)$.

Definition 8. Let $\mathcal{C}$ be a coherent category. A *model* of $\mathcal{C}$ is a morphism of coherent categories $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$. We let $\mathrm{Mod}(\mathcal{C})$ denote the full subcategory of the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ spanned by the models of $\mathcal{C}$; we refer to $\mathrm{Mod}(\mathcal{C})$ as the *category of models of* $\mathcal{C}$.

Example 9. Let $\mathcal{C} = \mathrm{Syn}_0(T)$ be the weak syntactic category of a (typed) first-order theory T. Then we can identify $\mathrm{Mod}(\mathcal{C})$ with the category of models of T (with morphisms given by elementary embeddings): that is the content of the theorem from the previous lecture.

We now return to a question from the previous lecture: given a category $\mathcal{C}$, can we construct a first-order theory T whose weak syntactic category is $\mathcal{C}$?

Construction 10. Let $\mathcal{C}$ be a small coherent category. We define a typed first-order theory $T(\mathcal{C})$ as follows:

- The types of $T(\mathcal{C})$ are the objects of $\mathcal{C}$. We use uppercase letters like X and Y to denote these types, and the corresponding lowercase letters x, y, etcetera to denote variables of those types.
- For every morphism $f : X \to Y$ in $\mathcal{C}$, the language of $T(\mathcal{C})$ has a single predicate P_f, of arity (X, Y).

By definition, a structure for the language $T(\mathcal{C})$ is a rule which associates to each object $X \in \mathcal{C}$ a set $M[X]$, and to each morphism $f : X \to Y$ a relation $M[P_f] \subseteq M[X] \times M[Y]$. We now list the axioms of $T(\mathcal{C})$, along with the constraints they place on a structure M:

- For every $f : X \to Y$, we have an axiom $(\forall x)(\exists! y)[P_f(x, y)]$. (So that $M[P_f]$ is the graph of a function $f_M : M[X] \to M[Y]$.)
- If $i : X \to X$ is the identity morphism, we have an axiom $(\forall x)[P_i(x, x)]$. (So that $i_M : M[X] \to M[X]$ is the identity map.)
- Given a pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$, we have an axiom $(\forall x, y, z)[(P_f(x, y) \wedge P_g(y, z)) \Rightarrow P_{gf}(x, z)]$. (So that $g_M \circ f_M = (g \circ f)_M$.)

These first axioms guarantee that a model M of $T(\mathcal{C})$ can be viewed as a functor from $\mathcal{C}$ to the category of sets. We now add additional axioms to guarantee that this functor has nice properties:

- If $\mathbf{1}$ is a final object of $\mathcal{C}$ and e is a variable of type $\mathbf{1}$, we have an axiom $(\exists! e)[e = e]$ (So that M preserves final objects.)
- For every pullback square
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$
in $\mathcal{C}$, we have an axiom $(\forall x, y, y')[(P_f(x, y) \wedge P_g(y', y)) \Rightarrow (\exists! x')[P_{f'}(x', x) \wedge P_{g'}(x', y')]]$ (So that M preserves pullback squares.)
- If $f : X \to Y$ is an effective epimorphism in $\mathcal{C}$, we have an axiom $(\forall y)(\exists x)[P_f(x, y)]$ (so that M carries effective epimorphisms to surjections of sets).

- If X is an object of $\mathcal{C}$ and $f : X_0 \hookrightarrow X$ is a monomorphism which exhibits X_0 as the smallest element of $\mathrm{Sub}(X)$, then we have an axiom $\neg(\exists x_0)[x_0 = x_0]$ (so that M carries the smallest element of $\mathrm{Sub}(X)$ to the empty set).
- If X is an object of $\mathcal{C}$ which is given as the join of subobjects $f : Y \hookrightarrow X$ and $g : Z \hookrightarrow X$, then we have an axiom $(\forall x)[(\exists y)[P_f(y, x)] \vee (\exists z)[P_g(z, x)]]$ (so that M carries joins in $\mathrm{Sub}(W)$ to unions of subsets of $M[W]$).

The theory $T(\mathcal{C})$ has the property that models of $T(\mathcal{C})$ are the same as models of $\mathcal{C}$. Let us now make that idea more precise:

Construction 11. We define a functor $\lambda : \mathcal{C} \to \mathrm{Syn}_0(T(\mathcal{C}))$ as follows:

- For each object $X \in \mathcal{C}$, we set $\lambda(X) = [x = x]$, where x is some variable of the type X.
- For every morphism $f : X \to Y$ in $\mathcal{C}$, we let $\lambda(f) : \lambda(X) \to \lambda(Y)$ denote the morphism in $\mathcal{C}$ defined by the formula $P_f(x, y)$.

By construction, a model M of $T(\mathcal{C})$ can be viewed as a morphism of coherent categories $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$, so that $M[P_f(x, y)]$ is the graph of a function f_M from $M[X]$ to $M[Y]$. Moreover, since we have $(g \circ f)_M = g_M \circ f_M$ for each M, it follows that $\lambda(g \circ f) = \lambda(g) \circ \lambda(f)$. Similarly, $\lambda(\mathrm{id}_X) = \mathrm{id}_{\lambda(X)}$, so that λ is a functor from $\mathcal{C}$ to $\mathrm{Syn}_0(T(\mathcal{C}))$. Moreover, this functor preserves finite limits, effective epimorphisms, and joins of subobjects (since these properties can be tested in every model of $T(\mathcal{C})$). In other words, λ is a morphism of coherent categories.

Note that the identification

$$\{\text{Models of } T(\mathcal{C})\} \simeq \{ \text{ Models of } \mathcal{C} \ \}$$

is simply given by composition with the functor F of Construction 11. This composition determines a functor

$$\mathrm{Mod}(T(\mathcal{C})) \simeq \mathrm{Mod}(\mathrm{Syn}_0(T(\mathcal{C}))) \to \mathrm{Mod}(\mathcal{C}).$$

By construction, the composite functor is *bijective* on objects. Beware that it is not necessarily an equivalence of categories. Our next goal is to discuss the following:

Theorem 12. *Let $\mathcal{C}$ be a small coherent category. Then the functor $\lambda : \mathcal{C} \to \mathrm{Syn}_0(T(\mathcal{C}))$ of Construction 11 is an equivalence of categories if and only if $\mathcal{C}$ is Boolean.*

Remark 13. The "only if" direction is obvious, since the weak syntactic category $\mathrm{Syn}_0(T(\mathcal{C}))$ is Boolean.

Remark 14. In the situation of Theorem 12, we can think of the weak syntactic category $\mathrm{Syn}_0(T(\mathcal{C}))$ as a "Booleanization" of $\mathrm{Syn}_0(\mathcal{C})$. If $f : \mathcal{C} \to \mathcal{D}$ is any morphism of coherent categories, then f can be completed to a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\lambda_{\mathcal{C}}"] & \mathcal{D} \arrow[d, "\lambda_{\mathcal{D}}"] \\ \mathrm{Syn}_0(T(\mathcal{C})) \arrow[r, "\mathrm{Syn}_0(T(f))"] & \mathrm{Syn}_0(T(\mathcal{D})). \end{tikzcd}$$

which commutes up to canonical isomorphism. If $\mathcal{D}$ is Boolean, then Theorem 12 guarantees that $\lambda_{\mathcal{D}}$ is an equivalence, so that f is isomorphic to the composition $\mathcal{C} \xrightarrow{\lambda_{\mathcal{C}}} \mathrm{Syn}_0(T(\mathcal{C}))\, \mathrm{Syn}_0(T(\mathcal{C})) \xrightarrow{g} \mathcal{D}$ where g is the composition $\lambda_{\mathcal{D}}^{-1} \circ \mathrm{Syn}_0(T(f))$. It is possible to show that this factorization is essentially unique (but this requires additional input).

Example 15 (The Theory of Groups)**.** We will see later that there is a natural way to choose a coherent category $\mathcal{C}$ for which the category $\mathrm{Mod}(\mathcal{C})$ is equivalent to the category whose objects are groups and whose morphisms are group homomorphisms. In this case, $T(\mathcal{C})$ would be equivalent to the category whose objects are groups and whose morphisms are elementary embeddings of groups. Here we could replace groups by other mathematical structures of a similar flavor (abelian groups, rings, Lie algebras, etcetera).

Example 16. A topological space X is said to be *spectral* if it satisfies the following conditions:

- The quasi-compact open subsets of X form a basis for the topology of X.
- The space X is quasi-compact, and the intersection $U \cap V$ is quasi-compact whenever $U, V \subseteq X$ are quasi-compact open sets.
- Every irreducible closed subset of X has a unique generic point.

For example, the underlying topological space of any quasi-compact and quasi-separated scheme is spectral (and conversely, due to a theorem of Hochster).

Let X be a spectral space. We say that a subset $K \subseteq X$ is *constructible* if it belongs to the Boolean algebra generated by the quasi-compact open subsets of X. We can then equip X with a new topology, called the *constructible topology*, by taking the constructible subsets of X as a basis. Let us denote the resulting topological space by X^c; one can show that it is a *Stone space* (that is, it is compact, Hausdorff, and totally disconnected).

The construction $X \mapsto X^c$ can be regarded as a special case of the Booleanization procedure of Construction 11. If X is a spectral space, then the collection of quasi-compact open subsets of X forms a distributive lattice P, which we can regard as a coherent category. Then the Booleanization $\mathrm{Syn}_0(T(P))$ is a Boolean coherent category in which every object admits a monomorphism to the final object, and is therefore equivalent to a Boolean algebra. This turns out to be the Boolean algebra of constructible subsets of X, or equivalently of quasi-compact open subsets of X^c.

In the situation above, we can identify X with the set of equivalence classes of models of P, and X^c with the set of equivalence classes of models of $\mathrm{Syn}_0(T(P))$. The fact that the topological spaces X and X^c have the same points is an illustration of the general fact that a coherent category $\mathcal{C}$ and its Booleanization $\mathrm{Syn}_0(T(\mathcal{C}))$ have "the same" models. However, the *categories* of models need not be equivalent. In the example of a spectral space X, this corresponds to the observation that in general there can be closure relations between points of X (that is, it is possible for a point $x \in X$ to lie in the closure of a different point $y \in X$), but not in X^c (since X^c is a Hausdorff space).

We now begin the proof of Theorem 12 (we will continue in the next lecture).

Proposition 17. *Let $\mathcal{C}$ be a small Boolean coherent category. Let X be an object of $\mathcal{C}$ which is given as a product $\prod_{1 \leq i \leq n} X_i$, and suppose that $\varphi(x_1, \ldots, x_n)$ is a formula in the language of $T(\mathcal{C})$ whose variables x_i have type X_i. Then there exists a subobject $Y \subseteq X$ such that $\lambda(Y)$ and $[\varphi(x_1, \ldots, x_n)]$ coincide as subobjects of $\lambda(X_1) \times \cdots \times \lambda(X_n) \simeq \lambda(X)$.*

Proof. We proceed by induction on the construction of the formula φ. There are five cases:

(*i*) Suppose $\varphi(\vec{x})$ has the form $x_i = x_j$, for some pair i, j with $X_i = X_j$. In this case, we can take Y to be the fiber product $X \times_{X_i \times X_j} X_i$.

(*ii*) Suppose that $\varphi(\vec{x})$ has the form $P_f(x_i, x_j)$, where $f : X_i \to X_j$ is a morphism in $\mathcal{C}$. In this case, we take Y to be the fiber product $X \times_{X_i \times X_j} X_i$ (where X_i is embedded in the product $X_i \times X_j$ as the graph of f).

(*iii*) Suppose that $\varphi(\vec{x})$ has the form $\varphi_0(\vec{x}) \vee \varphi_1(\vec{x})$. By our inductive hypothesis we can assume that there are subobjects $Y_0, Y_1 \subseteq X$ satisfying $\lambda(Y_0) = [\varphi_0(\vec{x})]$ and $\lambda(Y_1) = [\varphi_1(\vec{x})]$ (as subobjects of $\lambda(X)$). We then take $Y = Y_0 \vee Y_1$.

(*iv*) Suppose that $\varphi(\vec{x})$ has the form $\neg\psi(\vec{x})$. By the inductive hypothesis, we can choose a subobject $Y' \subseteq X$ such that $\lambda(Y') = [\psi(\vec{x})]$ as subobjects of $\lambda(X)$. Our assumption that $\mathcal{C}$ is Boolean guarantees that Y' has a complement $Y \in \mathrm{Sub}(X)$. Since F induces a Boolean algebra homomorphism $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$, it follows that $\lambda(Y) = [\varphi(\vec{x})]$ (as subobjects of $\lambda(X)$).

(*v*) Suppose that $\varphi(\vec{x})$ has the form $(\exists z)[\psi(\vec{x}, z)]$, where z is a variable of type Z. In this case, our inductive hypothesis guarantees that there exists a subobject $\overline{Y} \subseteq X \times Z$ such that $\lambda(\overline{Y}) = [\psi(\vec{x}, z)]$ as subobjects of $\lambda(X \times Z) \simeq \lambda(X) \times \lambda(Z)$. Let Y denote the image of the composite map $\overline{Y} \hookrightarrow X \times Z \to X$. Since F preserves images, it follows that $\lambda(Y)$ is the image of the map $[\psi(\vec{x}, z)] \to F(X)$, which coincides with $[\varphi(\vec{x})]$.

$\square$

Lecture 6: Completeness Theorems

February 1, 2018

We begin with the following classical statement:

Theorem 1 (Gödel's Completeness Theorem, Version 0)**.** *Let T be a first-order theory. If T is consistent, then there exists a model of T.*

To precisely formulate Theorem 1, we need a definition for what it means for a theory to be consistent, which we have not yet given. However, we can precisely formulate an analogous statement in the setting of categorical logic.

Definition 2. Let $\mathcal{C}$ be a coherent category with final object $\mathbf{1}$. We will say that $\mathcal{C}$ is *inconsistent* if $\mathrm{Sub}(\mathbf{1})$ has only a single element. Otherwise, we will say that $\mathcal{C}$ is *consistent*.

Exercise 3. Show that if a coherent category $\mathcal{C}$ is inconsistent, then $\mathcal{C}$ is equivalent to the category $\{*\}$ having only a single object and a single (identity) morphism.

Theorem 4 (Deligne's Completeness Theorem, Version 0)**.** *Let $\mathcal{C}$ be a small coherent category. If $\mathcal{C}$ is consistent, then there exists a model of $\mathcal{C}$.*

For our purposes, it will be useful to consider some variations on Theorems 1 and 4.

Theorem 5 (Gödel's Completeness Theorem, Version 1)**.** *Let T be a first order theory and let φ be a sentence in the language of T. If φ is true in every model of T, then φ is provable from the axioms of T.*

Equivalently, if φ is not *provable from T, then we can find a model of T in which φ is false.*

Remark 6. Theorem 1 follows immediately from Theorem 5 by taking φ to be a contradictory sentence, like $(\exists x)[\neg(x = x)]$. Conversely, we can deduce Theorem 5 by applying Theorem 1 to the theory $T \cup \{\neg\varphi\}$.

To translate Theorem 5 into the setting of categorical logic, we replace the notion of *sentence* in a first-order language by the notion of a *subobject of* $\mathbf{1}$ in a coherent category $\mathcal{C}$.

Theorem 7 (Deligne's Completeness Theorem, Version 1)**.** *Let $\mathcal{C}$ be a small coherent category and let $U \subseteq \mathbf{1}$ be a subobject of the final object. If $U \neq \mathbf{1}$, then there exists a model $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ such that $M[U]$ is empty.*

Remark 8. Theorem 7 immediately implies Theorem 4 by taking U to be the least element of $\mathrm{Sub}(\mathbf{1})$ (which is carried to the empty set by any model of $\mathcal{C}$). In the non-Boolean case, it is a little bit harder (though still possible) to go in the other direction. We'll return to this point later.

We will prove Theorem 7 much later in this course.

Here is another equivalent form of Gödel's completeness theorem, which we mentioned in the first lecture:

Theorem 9 (Gödel's Completeness Theorem, Version 2)**.** *Let T be a first order theory and let $\varphi(\vec{x})$, $\psi(\vec{x})$ be two formulas of the language of T, using the same free variables. If $M[\psi(\vec{x})] \subseteq M[\varphi(\vec{x})]$ for every model M of T, then $T \vdash (\forall \vec{x})[\psi(\vec{x}) \Rightarrow \varphi(\vec{x})]$*

Equivalently, if the implication $\psi(\vec{x}) \Rightarrow \varphi(\vec{x})$ is not *provable from T, then we can find a model M such that $M[\psi(\vec{x})] \nsubseteq M[\varphi(\vec{x})]$.*

To prove Theorem 5 for a sentence φ, we can simply apply Theorem 9 (in the case of no free variables) by choosing ψ to be a tautological sentence, like $(\forall x)[x = x]$. Conversely, Theorem 9 can be deduced by applying Theorem 5 to a theory T' obtained by "adding new constants" $c_1, c_2, \ldots, c_n$, and adding an axiom demanding that they satisfy the formula ψ. We now translate this idea into categorical logic.

Theorem 10 (Deligne's Completeness Theorem, Version 2)**.** *Let $\mathcal{C}$ be a small coherent category, let $X \in \mathcal{C}$ be an object, and let $U \subseteq X$ be a subobject. If $U \neq X$, then there exists a model $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ such that $M[U] \neq M[X]$.*

Theorem 7 is just a special case of Theorem 10, where we take X to be the final object. But we can go in the reverse direction as well:

Proof of Theorem 10 from Theorem 7. We saw in the previous lecture that $\mathcal{C}_{/X}$ is also a coherent category. Moreover, we can regard X as the final object of $\mathcal{C}_{/X}$, and U as a subobject of the final object of $\mathcal{C}_{/X}$. Consequently, if $U \neq X$, then we can apply Theorem 7 to construct a model $N : \mathcal{C}_{/X} \to \mathcal{S}\mathrm{et}$ such that $N[U] = \emptyset$. Define $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ by the formula $M[Y] = N[X \times Y]$. From an exercise of the previous lecture, the functor

$$\mathcal{C} \to \mathcal{C}_{/X} \qquad Y \mapsto X \times Y$$

is a morphism of coherent categories, so that M is a model of $\mathcal{C}$. We will complete the proof by showing that the inclusion $M[U] \hookrightarrow M[X]$ is not bijective. Equivalently, we wish to show that $N[X \times U] \hookrightarrow N[X \times X]$ is not bijective (where we regard $X \times U$ and $X \times X$ as objects of $\mathcal{C}_{/X}$ via projection onto the first factor). Suppose otherwise. Then the pullback diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ X \times U \arrow[r] & X \times X \end{tikzcd}$$

in $\mathcal{C}_{/X}$ would yield a pullback diagram of sets

$$\begin{tikzcd} N[U] \arrow[r] \arrow[d] & N[X] \arrow[d] \\ N[X \times U] \arrow[r] & N[X \times X] \end{tikzcd}$$

where the bottom horizontal map is bijective, so the upper horizontal map would be bijective as well. This contradicts our choice of N (since $N[U] = \emptyset$ and $N[X]$ is a singleton). $\square$

Exercise 11. Let $\mathcal{C}$ be a coherent category and let $X \in \mathcal{C}$ be an object. Show that the following data are equivalent:

- Models of the coherent category $\mathcal{C}_{/X}$.
- Models M of the coherent category $\mathcal{C}$, together with a chosen element of the set $M[X]$.

In the case where $\mathcal{C} = \mathrm{Syn}_0(T)$ is the weak syntactic category of a first-order theory T and $X = [\varphi(x_1, \ldots, x_n)]$, this supports the idea that $\mathcal{C}_{/X}$ corresponds to a theory obtained by "adding constants" $c_1, c_2, \ldots, c_n$ that are required to satisfy the formula $\varphi(c_1, \ldots, c_n)$.

In the statement of Theorem 10, we do not need to assume that U is a subobject of X:

Theorem 12 (Deligne's Completeness Theorem, Version 3)**.** *Let $\mathcal{C}$ be a small coherent category, and let $f : U \to X$ be a morphism in $\mathcal{C}$. If f is not an isomorphism, then there exists a model $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ such that the induced map $M[U] \to M[X]$ is not bijective.*

Proof. Assume that, for every model M, the induced map $f_M : M[U] \to M[X]$ is bijective. Factor f as a composition $U \xrightarrow{g} \mathrm{Im}(f) \xrightarrow{h} X$, where g is an effective epimorphism and h is a monomorphism. In every model M, the induced factorization $M[U] \xrightarrow{g_M} M[\mathrm{Im}(f)] \xrightarrow{h_M} M[X]$ exhibits $M[\mathrm{Im}(f)]$ as the image of f_M. Since f_M is bijective, it follows that g_M and h_M are bijective. Applying Theorem 10 to the morphism h, we deduce that h is an isomorphism.

Let $R = U \times_{\mathrm{Im}(f)} U$ be the equivalence relation determined by g, which we regard as a subobject of $U \times U$. Let us also regard U as a subobject of $U \times U$ via the diagonal inclusion, so that U is a subobject of R. For every model M, the assumption that g_M is bijective guarantees that $M[U] = M[R]$ (as subsets of $M[U \times U]$. Applying Theorem 10, we conclude that $U = R$ as subobjects of $U \times U$. Since g is an effective epimorphism, we conclude $\mathrm{Im}(f)$ can be identified with the coequalizer of the projection maps $\pi_0, \pi_1 : U \times_{\mathrm{Im}(f)} U \to U$, which is equivalent to the coequalizer of the identity maps $\mathrm{id}_U, \mathrm{id}_U : U \to U$. It follows that g is also an isomorphism, so that $f = h \circ g$ is an isomorphism. □

Corollary 13. *Let $\lambda : \mathcal{C} \to \mathcal{D}$ be a morphism of small coherent categories. Assume that f satisfies the following conditions:*

(1) *As a functor, λ is essentially surjective: that is, every object of $\mathcal{D}$ is isomorphic to $\lambda(X)$, for some $X \in \mathcal{C}$.*

(2) *For every object $X \in \mathcal{C}$, the functor λ induces a surjection $\mathrm{Sub}(X) \to \mathrm{Sub}(\lambda(X))$.*

(3) *Every model of $\mathcal{C}$ can be extended to a model of $\mathcal{D}$.*

Then λ is an equivalence of categories.

Proof. By virtue of (1), it will suffice to show that the functor λ is fully faithful. We first show that it is faithful. Let $f, g : X \to Y$ be two morphisms in $\mathcal{C}$ having the same source and target, and suppose that $\lambda(f) = \lambda(g)$. We wish to show that $f = g$. Let $E \subseteq X$ be the equalizer of f and g; we wish to show that $E = X$. If not, we can apply Theorem 10 to choose a model $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ such that $M[E] \neq M[X]$. Using (3), we can assume that $M = N \circ \lambda$, where $N : \mathcal{D} \to \mathcal{S}\mathrm{et}$ is a model of $\mathcal{D}$. Then $N[\lambda(E)] \neq N[\lambda(X)]$, so that $\lambda(E) \neq \lambda(X)$ (as a subobject of $\lambda(X)$). Since λ preserves finite limits, we can identify $\lambda(E)$ with the equalizer of $\lambda(f)$ and $\lambda(g)$. It follows that $\lambda(f) \neq \lambda(g)$, contradicting our assumption.

We now show that the functor λ is full. Let X and Y be objects of $\mathcal{C}$, and suppose we are given some morphism $\overline{f} : \lambda(X) \to \lambda(Y)$. We wish to show that $\overline{f} = \lambda(f)$, for some morphism $f : X \to Y$ in $\mathcal{C}$. Let $\Gamma(\overline{f})$ denote the graph of $\overline{f}$, regarded as a subobject of $\lambda(X) \times \lambda(Y) \simeq \lambda(X \times Y)$. Using assumption (2), we can assume that $\Gamma(\overline{f}) = \lambda(U)$ for some subobject $U \subseteq X \times Y$. To complete the proof, it will suffice to show that U is the graph of a morphism from X to Y: that is, that the composite map $U \hookrightarrow X \times Y \xrightarrow{\pi_X} X$ is an isomorphism in $\mathcal{C}$. Using Theorem 12, we are reduced to proving that the map $M[U] \to M[X]$ is bijective for every model M of $\mathcal{C}$. Using assumption (3), we can assume that $M = N \circ \lambda$, where $N : \mathcal{D} \to \mathcal{S}\mathrm{et}$ is a model of $\mathcal{D}$. In this case, the result is clear, because the map $\lambda(U) \to \lambda(X)$ is an isomorphism (it is the projection from the graph $\Gamma(\overline{f}) \subseteq \lambda(X) \times \lambda(Y)$ onto the first factor). □

We can now prove the Theorem from the previous lecture:

Theorem 14. *Let $\mathcal{C}$ be a small Boolean coherent category. Then the map $\lambda : \mathcal{C} \to \mathrm{Syn}_0(T(\mathcal{C}))$ of the previous lecture is an equivalence of categories.*

Proof. The functor λ satisfies conditions (1) and (3) of Corollary 13 by construction. We saw in the previous lecture that it also satisfies condition (2) when $\mathcal{C}$ is Boolean. □

We close this lecture by describing another application of Corollary 13. Let T be a first-order theory. Suppose that we wanted to deduce Gödel's completeness theorem (in any of the three incarnations above) from Deligne's completeness theorem (in the corresponding incarnation). In this case, we cannot really proceed by applying Deligne's theorem to the weak syntactic category $\mathrm{Syn}_0(T)$ as we have defined it, because

our definition already has Gödel's theorem "built in." Recall that we can identify morphisms from $X = [\varphi(\vec{x})]$ to $Y = [\psi(\vec{y})]$ with equivalence classes of formulae

$$T \vDash (\forall \vec{x}, \vec{y})[\theta(\vec{x}, \vec{y}) \Rightarrow \varphi(\vec{x}) \wedge \psi(\vec{y})] \wedge (\forall \vec{x})[\varphi(\vec{x}) \Rightarrow (\exists! \vec{y})[\theta(\vec{x}, \vec{y})]],$$

where $\theta(\vec{x}, \vec{y})$ is equivalent to $\theta'(\vec{x}, \vec{y})$ if

$$T \vDash (\forall \vec{x}, \vec{y})[\theta(\vec{x}, \vec{y}) \Leftrightarrow \theta'(\vec{x}, \vec{y})].$$

This definition was not really syntactic in nature: it refers to the truth of formulae, rather than to their provability. What we should really do is to define morphisms from X to Y to be equivalence classes of formulae $\theta(\vec{x}, \vec{y})$ satisfying

$$T \vdash (\forall \vec{x}, \vec{y})[\theta(\vec{x}, \vec{y}) \Rightarrow \varphi(\vec{x}) \wedge \psi(\vec{y})] \wedge (\forall \vec{x})[\varphi(\vec{x}) \Rightarrow (\exists! \vec{y})[\theta(\vec{x}, \vec{y})]],$$

modulo the equivalence relation given by

$$T \vdash (\forall \vec{x}, \vec{y})[\theta(\vec{x}, \vec{y}) \Leftrightarrow \theta'(\vec{x}, \vec{y})].$$

Let us denote the category obtained from this procedure by $\mathrm{Syn}_0'(T)$. Of course, this definition depends *a priori* on what it means to be provable. However, suppose that we give a definition which is sufficiently robust that we can justify analogues for $\mathrm{Syn}_0'(T)$ for all of the main facts about $\mathrm{Syn}_0(T)$; namely:

(a) The constructions of fiber products, images, and unions of subobjects that we have given in $\mathrm{Syn}_0(T)$ also work in $\mathrm{Syn}_0'(T)$. For example, if $\theta(\vec{x}, \vec{y})$ is a formula defining a morphism from $[\varphi(\vec{x})]$ to $[\psi(\vec{y})]$, then that morphism factors as a composition

$$[\varphi(\vec{x})] \xrightarrow{\theta} [(\exists \vec{x})\theta(\vec{x}, \vec{y'})] \xrightarrow{\vec{y}=\vec{y'}} [\psi(\vec{y})],$$

where the map on the left is an effective epimorphism and the map on the right is a monomorphism.

(b) Every morphism of coherent categories $\mathrm{Syn}_0'(T) \to \mathcal{S}\mathrm{et}$ arises from a model of the theory T.

In this case, (a) will imply that the construction $[\varphi(\vec{x})] \mapsto [\varphi(\vec{x})]$ determines a morphism of coherent categories $\mathrm{Syn}_0'(T) \to \mathrm{Syn}_0(T)$. This functor will satisfy conditions (1) and (2) of Corollary 13, and assumption (b) guarantees that it will also satisfy (3). Applying Corollary 13, we deduce that $\mathrm{Syn}_0'(T)$ and $\mathrm{Syn}_0(T)$ are equivalent. We can also deduce Gödel's completeness theorem by applying Theorem 4 to the coherent category $\mathrm{Syn}_0'(T)$: if T is consistent as a first-order theory, then $\mathrm{Syn}_0'(T)$ will be consistent as a coherent category (in the sense of Definition 2), and therefore has a model $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$; ($b$) allows us to interpret this as a model of the theory T.

Lecture 7: Pretopoi

February 5, 2018

For the reader's convenience, we recall the definition of a coherent category:

Definition 1. A category $\mathcal{C}$ is *coherent* if it satisfies the following axioms:

$(A1)$ $\mathcal{C}$ admits finite limits.

$(A2)$ Every morphism $f : X \to Z$ factors as a composition $X \xrightarrow{g} Y \xrightarrow{h} Z$, where h is a monomorphism and g is an effective epimorphism.

$(A3)$ For every object $X \in \mathcal{C}$, the poset $\mathrm{Sub}(X)$ is an upper semilattice: that is, it has a least element and joins $X_0 \vee X_1$.

$(A4)$ The collection of effective epimorphisms in $\mathcal{C}$ is stable under pullbacks.

$(A5)$ For every morphism $f : X \to Y$ in $\mathcal{C}$, the map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is a homomorphism of upper semilattices: that is, it preserves least elements and joins.

Let T be a first-order theory. In Lecture 2, we associated to T a category $\mathrm{Syn}_0(T)$, which we call the *weak syntactic category of* T. However, there is a sense in which this category is not really an invariant of T. More precisely, there are examples of first-order theories that we might like to say are equivalent, despite the fact that their weak syntactic categories are not equivalent (as categories).

Example 2. We define first-order theories T and T' as follows:

- The language of T has no predicates, and T has a single axiom $(\exists! x)[x = x]$.
- The language of T' has a single 1-ary predicate P, and a pair of axioms

$$(\exists! y)[P(y)] \qquad (\exists! z)[\neg P(z)].$$

Up to isomorphism, the theory T has only one model: a set M having exactly one element. Similarly, the theory T' has only one model, consisting of a pair of distinct points (one of which satisfies the predicate P, and one of which does not). In particular, the categories $\mathrm{Mod}(T)$ and $\mathrm{Mod}(T')$ are equivalent. However, the weak syntactic categories $\mathrm{Syn}_0(T)$ and $\mathrm{Syn}_0(T')$ are not equivalent.

Exercise 3. Show that the category $\mathrm{Syn}_0(T)$ is equivalent to the poset $\{0 < 1\}$, and that $\mathrm{Syn}_0(T')$ is equivalent to the category of finite sets $\mathcal{S}\mathrm{et}^{\mathrm{fin}}$.

Our ultimate goal in this course is to prove Makkai's strong conceptual completeness theorem, which asserts roughly that one can recover the syntax of a first-order theory T from its semantics (encoded by the category of models $\mathrm{Mod}(T)$ with the structure given by ultraproducts). To have any hope of proving such a statement, we cannot interpret "the syntax of T" as being its weak syntactic category $\mathrm{Syn}_0(T)$: Example 2 shows that two theories can have essentially the same semantics, but different weak syntactic categories. We will correct this issue by replacing $\mathrm{Syn}_0(T)$ by a certain enlargement, which we will denote by $\mathrm{Syn}(T)$ and refer to as the *syntactic category of* T.

Before defining this enlargement in general, let's begin by inspecting the difference between the categories $\mathrm{Syn}_0(T)$ and $\mathrm{Syn}_0(T')$ appearing in Example 2. Both of these categories are coherent: in particular, they are categories in which we can form unions of subobjects of a fixed object. However, in the category $\mathrm{Syn}_0(T') \simeq \mathcal{S}\mathrm{et}^{\mathrm{fin}}$ we can do something a little bit better: given two finite sets S and T we can form their *disjoint union* $S \amalg T$. This is a new finite set which contains S and T as (disjoint) subobjects. Categorically, it can be described as coproduct of S and T in the category $\mathcal{S}\mathrm{et}^{\mathrm{fin}}$. This is actually an instance of a general phenomenon:

Proposition 4. *Let $\mathcal{C}$ be a coherent category (or, more generally, any category satisfying $(A1)$, $(A3)$, and $(A5)$). Let X be an object of $\mathcal{C}$ and suppose we are given a collection of subobjects $X_1, X_2, \ldots, X_n \subseteq X$, such that $X_i \wedge X_j = \emptyset$ for $i \neq j$ (here $\emptyset$ denotes the least element of $\mathrm{Sub}(X)$). Then $X_1 \vee \cdots \vee X_n$ is a coproduct of the subobjects $\{X_i\}_{1 \leq i \leq n}$ in the category $\mathcal{C}$.*

Proof. Without loss of generality, we may assume that $X = X_1 \vee \cdots \vee X_n$. Suppose we are given an object $Y \in \mathcal{C}$ and a collection of maps $f_i : X_i \to Y$. We wish to show that there is a unique map $f : X \to Y$ satisfying $f|_{X_i} = f_i$. For $1 \leq i \leq n$, regard $\Gamma(f_i)$ as a subobject of $X_i \times Y \subseteq X \times Y$, and set $Z = \Gamma(f_1) \vee \cdots \vee \Gamma(f_n)$. We will complete the proof by showing that Z is the graph of map from X to Y: that is, that the composition $Z \hookrightarrow X \times Y \to X$ is an isomorphism. Let us denote the composition by h.

We first claim that h is a monomorphism. To prove this, we note that $Z \times_X Z$ can be identified with a subobject of the product $(X \times Y) \times_X (X \times Y) \simeq X \times Y \times Y$. Let $\pi : X \times Y \times Y \to X$ be the projection map. Using axiom $(A5)$, we can identify $Z \times_X Z$ with the join of the subobjects $\Gamma(f_i) \times_X \Gamma(f_j)$. For $i \neq j$, we have $\Gamma(f_i) \times_X \Gamma(f_j) \subseteq \pi^{-1}(X_i \wedge X_j) = \pi^{-1}(\emptyset)$. Since π^{-1} preserves least elements, it follows that $\Gamma(f_i) \times_X \Gamma(f_j)$ is a smallest element of $\mathrm{Sub}(X \times Y \times Y)$. It follows that $Z \times_X Z$ is given by the join of the subobjects $\Gamma(f_i) \times_X \Gamma(f_i)$, each of which is contained in the image of the diagonal $Z \to Z \times_X Z$. We therefore have $Z \simeq Z \times_X Z$, so that h is a monomorphism as desired.

Let $\mathrm{Im}(h)$ denote the subobject of X determined by the monomorphism $h : Z \to X$. Since Z contains each $\Gamma(f_i)$, we have $X_i \subseteq \mathrm{Im}(h)$ for $1 \leq i \leq n$. The equality $X = X_1 \vee \cdots \vee X_n$ then implies that $\mathrm{Im}(h) = X$, so that h is an isomorphism as desired. $\square$

In the special case $n = 0$, we obtain the following:

Corollary 5. *Let $\mathcal{C}$ be a coherent category. For any object $X \in \mathcal{C}$, the least element of $\mathrm{Sub}(X)$ is initial when regarded as an object of $\mathcal{C}$. In particular, $\mathcal{C}$ has an initial object, which we will henceforth denote by $\emptyset$.*

Definition 6. Let $\mathcal{C}$ be a category which admits fiber products, and let $X, Y \in \mathcal{C}$ be a pair of objects which admits a coproduct $X \amalg Y$. We will say that $X \amalg Y$ is a *disjoint coproduct* of X and Y if the following pair of conditions is satisfied:

- Each of the maps $X \to (X \amalg Y) \leftarrow Y$ is a monomorphism.
- The fiber product $X \times_{X \amalg Y} Y$ is an initial object of $\mathcal{C}$.

We will that $\mathcal{C}$ *has disjoint coproducts* if it has an initial object and every pair of objects $X, Y \in \mathcal{C}$ has a disjoint coproduct $X \amalg Y$.

Corollary 7. *Let $\mathcal{C}$ be a coherent category containing an object X and let $X_0, X_1 \subseteq X$ be subobjects satisfying $X_0 \wedge X_1 = \emptyset$. Then $X_0 \vee X_1$ is a disjoint coproduct of X_0 and X_1.*

Proof. We have monomorphisms $X_0 \hookrightarrow X_0 \vee X_1 \hookleftarrow X_1$, and the fiber product

$$X_0 \times_{X_0 \vee X_1} X_1 \simeq X_0 \times_X X_1 \simeq \emptyset$$

is an initial object of $\mathcal{C}$ by assumption. Proposition 4 guarantees that these maps exhibit $X_0 \vee X_1$ as a coproduct of X_0 and X_1. $\square$

Proposition 8. *Let $\mathcal{C}$ be a category. The following conditions are equivalent:*

(1) *The category $\mathcal{C}$ is coherent and has disjoint coproducts.*

(2) *The category $\mathcal{C}$ satisfies $(A1)$, $(A2)$, and $(A4)$, along with the following modified versions of $(A3)$ and $(A5)$:*

   ($A3'$) *The category $\mathcal{C}$ has disjoint coproducts.*

   ($A5'$) *The formation of finite coproducts in $\mathcal{C}$ is preserved by pullbacks. More precisely, for every morphism $f : X \to Y$ in $\mathcal{C}$, the pullback functor*

$$f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X} \qquad f^*(U) = U \times_Y X$$

   preserves finite coproducts.

Proof. Suppose first that (1) is satisfied. Then $(A1)$, $(A2)$, $(A3')$, and $(A4)$ are automatic. We will prove $(A5')$. Fix a morphism $f : X \to Y$ in $\mathcal{C}$ and a collection of objects $\{Y_i\}_{1 \leq i \leq n}$ in $\mathcal{C}_{/Y}$. We wish to show that the canonical map

$$(X \times_Y Y_1) \amalg \cdots \amalg (X \times_Y Y_n) \to X \times_Y (Y_1 \amalg \cdots \amalg Y_n)$$

is an isomorphism. To prove this, we can replace Y by the coproduct $Y_1 \amalg \cdots \amalg Y_n$ (and X by the fiber product $X \times_Y (Y_1 \amalg \cdots \amalg Y_n)$), and thereby reduce to the case where Y is the coproduct of the objects Y_i. Since coproducts in $\mathcal{C}$ are disjoint, we can view each Y_i as a subobject of Y, and these subobjects are disjoint (that is, $Y_i \wedge Y_j = \emptyset$ for $i \neq j$). Similarly, we can view the fiber products $X \times_Y Y_i$ as disjoint subobjects of X. In this case, Proposition 4 supplies identifications

$$Y_1 \amalg \cdots \amalg Y_n \simeq Y_1 \vee \cdots \vee Y_n$$

$$(X \times_Y Y_1) \amalg \cdots \amalg (X \times_Y Y_n) \simeq (X \times_Y Y_1) \vee \cdots \vee (X \times_Y Y_n)$$

(where the joins are formed in $\mathrm{Sub}(Y)$ and $\mathrm{Sub}(X \times Y)$, respectively). We are therefore reduced to showing that the map f^{-1} preserves joins of subobjects, which is a special case of axiom $(A5)$.

We now show that (2) $\Rightarrow$ (1). Assume that the category $\mathcal{C}$ satisfies $(A1)$, $(A2)$, $(A3')$, $(A4)$, and $(A5')$. We wish to show that it is a coherent category: that is, it satisfies $(A3)$ and $(A5)$. Let X be an object of $\mathcal{C}$, and suppose we are given a collection of subobjects $X_1, \ldots, X_n \subseteq X$; we wish to show that there exists a least upper bound U for the set $\{X_1, \ldots, X_n\}$ in the poset $\mathrm{Sub}(X)$. Assumption $(A3')$ guarantees that there exists a coproduct $X_1 \amalg \cdots \amalg X_n$ in $\mathcal{C}$. The least upper bound U is then given by taking U to be the image of the map $X_1 \amalg \cdots \amalg X_n \to X$ (which exists by virtue of $(A2)$). This completes the proof of $(A3)$. To prove $(A5)$,we must show that the formation of least upper bounds is compatible with pullback along a morphism $f : Y \to X$. This follows from the construction, since coproducts are compatible with pullback $(A5')$ and images are compatible with pullback by $(A4)$. $\square$

We now discuss another sort of construction that we cannot quite carry out in a coherent category.

Definition 9. Let $\mathcal{C}$ be a category which admits finite limits and let X be an object of $\mathcal{C}$. We say that a subobject $R \subseteq X \times X$ is an *equivalence relation* on X if, for every object $Y \in \mathcal{C}$, the image of the induced map

$$\mathrm{Hom}_{\mathcal{C}}(Y, R) \to \mathrm{Hom}_{\mathcal{C}}(Y, X \times X) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, X) \times \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

is an equivalence relation on the set $\mathrm{Hom}_{\mathcal{C}}(Y, X)$.

Every morphism $f : X \to Y$ determines an equivalence relation R on X, given by the fiber product $R = X \times_Y X$. If f is an effective epimorphism, then we can recover the object Y (and the morphism f) from the equivalence relation R: namely, it is the coequalizer of the pair of projection maps $\pi, \pi' : X \times_Y X \to X$.

Definition 10. Let $\mathcal{C}$ be a category which admits finite limits, let X be an object of $\mathcal{C}$, and let $R \subseteq X \times X$ be an equivalence relation. We say that R is *effective* if there exists an effective epimorphism $f : X \to Y$ such that $R = X \times_Y X$ (as subobject of $X \times X$).

Example 11. In the category of sets, every equivalence relation is effective. This is just the statement that if $R \subseteq X \times X$ is an equivalence relation on X, then an ordered pair $(x, y) \in X \times X$ belongs to R if and only if x and y have the same image in the quotient X/R.

Proposition 12. *Let $\mathcal{C}$ be a category satisfying $(A1)$ and $(A4)$. Suppose that every equivalence relation in $\mathcal{C}$ is effective. Then $\mathcal{C}$ also satisfies $(A2)$.*

The proof will require an elementary observation:

Lemma 13. *Let $\mathcal{C}$ be a category satisfying $(A1)$ and $(A4)$ and suppose we are given a pullback diagram*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{C}$. If f is an effective epimorphism and g' is an isomorphism, then g is an isomorphism.

Proof. Using $(A4)$ we see that f' is also an effective epimorphism. We have a commutative diagram

$$\begin{tikzcd} X' \times_{Y'} X' \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X' \arrow[r] \arrow[d] & Y' \arrow[d, "g"] \\ X \times_Y X \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r] & Y \end{tikzcd}$$

where the rows are coequalizer diagrams and the left and middle vertical maps are isomorphisms. It follows that g is an isomorphism as well. $\square$

Proof of Proposition 12. Let $f : X \to Z$ be a morphism in $\mathcal{C}$. Then $R = X \times_Z X$ is an equivalence relation on X. Suppose that this equivalence relation is effective: that is, there exists an effective epimorphism $g : X \to Y$ such that $R = X \times_Y X$ (as subobjects of $X \times X$). In particular, Y is the coequalizer of the projection maps $\pi, \pi' : R \to X$. Since $f \circ \pi = f \circ \pi'$, there is a unique morphism $h : Y \to Z$ such that $f = h \circ g$. To complete the proof, it will suffice to show that h is a monomorphism: that is, that the diagonal map $Y \to Y \times_Z Y$ is an isomorphism. We have a commutative diagram of pullback squares

$$\begin{tikzcd} X \times_Y X \arrow[r, "\delta'"] \arrow[d] & X \times_Z X \arrow[r] \arrow[d] & X \times X \arrow[d, "g \times g"] \\ Y \arrow[r, "\delta"] & Y \times_Z Y \arrow[r] & Y \times Y. \end{tikzcd}$$

Since g is an effective epimorphism, the vertical maps in this diagram are effective epimorphisms. Moreover, δ' is an isomorphism by construction (since $X \times_Z X$ and $X \times_Y X$ are both equal to R as subobjects of $X \times X$). Applying Lemma 13, we see that δ is an isomorphism, as desired. $\square$

We are now ready to introduce one of the central objects of interest in this course.

Definition 14. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a *pretopos* if it satisfies the following axioms:

$(A1)$ The category $\mathcal{C}$ admits finite limits.

$(A2')$ Every equivalence relation in $\mathcal{C}$ is effective.

$(A3')$ The category $\mathcal{C}$ admits finite coproducts, and coproducts are disjoint.

$(A4)$ The collection of effective epimorphisms in $\mathcal{C}$ is closed under pullbacks.

($A5'$) The formation of finite coproducts in $\mathcal{C}$ is preserved by pullback.

We have proven:

Proposition 15. *Every pretopos is a coherent category.*

Proof. Combine Propositions 12 and 8. $\square$

Example 16. The category of sets is a pretopos.

Example 17. The category of finite sets is a pretopos.

Lecture 8: Grothendieck Topologies

February 7, 2018

In the last lecture, we introduced the notion of a *pretopos*. Recall that a pretopos is a coherent category $\mathcal{C}$ with a few additional features: the category $\mathcal{C}$ is required to admit finite coproducts (which are required to be disjoint), and every equivalence relation in $\mathcal{C}$ is required to be effective.

Exercise 1. Let $\mathcal{C}$ and $\mathcal{D}$ be pretopoi, and let $\lambda : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite limits and effective epimorphisms. Show that the following conditions are equivalent:

(1) The functor λ is a morphism of coherent categories: that is, for every object $X \in \mathcal{C}$, the induced map $\mathrm{Sub}(X) \to \mathrm{Sub}(\lambda(X))$ is a homomorphism of upper semilattices.

(2) The functor λ preserves finite coproducts.

If these conditions are satisfied, we will say that λ is a *morphism of pretopoi*.

Definition 2. Let $\mathcal{C}$ be a pretopos. A *model of* $\mathcal{C}$ is a morphism of pretopoi $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$. We let $\mathrm{Mod}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ spanned by the models of $\mathcal{C}$.

Definition 3. Let T be a (typed) first-order theory. We say that T *eliminates imaginaries* if the weak syntactic category $\mathrm{Syn}_0(T)$ is a pretopos.

Every first-order theory T can be replaced by an essentially equivalent first-order theory which eliminates imaginaries. Suppose, for example, that $X = [\varphi(x)]$ is a formula in the language of T, having a single free variable x of type t, and that $R \subseteq X \times X$ is an equivalence relation defined by some formula $\psi(x, x')$. In this case, we can enlarge the language of T by adding a new type s (to be interpreted as "X modulo the equivalence relation R") and a new binary predicate P of type (t, s) (to be interpreted as the graph of the projection map $X \to X/R$), and adding the axioms

$$(\forall x, y)[P(x, y) \Rightarrow \varphi(x)]$$

$$(\forall x)[\varphi(x) \Rightarrow (\exists! y) P(x, y)]$$

$$(\forall x, x', y, y')[(P(x, y) \wedge P(x', y')) \Rightarrow (y = y' \Leftrightarrow \psi(x, x'))]$$

$$(\forall y)(\exists x)[P(x, y)].$$

Elaborating on this construction, one can produce a typed first-order theory T^{eq}, having the same models as T, for which T^{eq} eliminates imaginaries. This is a special case of a more general construction:

Theorem 4. *Let $\mathcal{C}$ be a small coherent category. Then there exists a small pretopos $\mathcal{C}^{\mathrm{eq}}$ and a morphism of coherent categories $\lambda : \mathcal{C} \to \mathcal{C}^{\mathrm{eq}}$ with the following universal property: if $\mathcal{D}$ is any pretopos, then composition with λ induces an equivalence of categories $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}^{\mathrm{eq}}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D})$. Here $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the morphisms of coherent categories, and $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}^{\mathrm{eq}}, \mathcal{D})$ is defined similarly.*

In particular, composition with λ induces an equivalence of categories $\mathrm{Mod}(\mathcal{C}^{\mathrm{eq}}) \to \mathrm{Mod}(\mathcal{C})$.

In the situation of Theorem 4, the pretopos $\mathcal{C}^{\mathrm{eq}}$ is determined (up to canonical equivalence) by the coherent category $\mathcal{C}$; we will refer to $\mathcal{C}^{\mathrm{eq}}$ as the *pretopos completion* of $\mathcal{C}$.

Definition 5. Let T be a (typed) first order theory. We let $\mathrm{Syn}(T)$ denote the pretopos completion of $\mathrm{Syn}_0(T)$. We will refer to $\mathrm{Syn}(T)$ as the *syntactic category* of T.

One of our goals over the next several lectures is construct the pretopos completion $\mathcal{C}^{\mathrm{eq}}$. To get a feeling for what we need to do, let us first suppose that we are given a triple of objects $X, Y, Z \in \mathcal{C}$. It could be that the objects Y and Z do not have a coproduct in $\mathcal{C}$ (they also might admit a coproduct in $\mathcal{C}$ which fails to be disjoint, in which case they will have a different coproduct in $\mathcal{C}^{\mathrm{eq}}$). We would like $\mathcal{C}^{\mathrm{eq}}$ to be an enlargement of the category $\mathcal{C}$ in which the coproduct $Y \amalg Z$ exists and is disjoint. Let us abuse notation by identifying objects of $\mathcal{C}$ with their image in $\mathcal{C}^{\mathrm{eq}}$ (our proof of Theorem 4 will show that the functor $\lambda : \mathcal{C} \to \mathcal{C}^{\mathrm{eq}}$ is fully faithful, so this abuse is mostly harmless). So, in order to make sense of the category $\mathcal{C}^{\mathrm{eq}}$, we must in particular define the set of maps $\mathrm{Hom}_{\mathcal{C}^{\mathrm{eq}}}(X, Y \amalg Z)$. Note that we have some obvious candidates for elements of this set: since Y and Z should be subobjects of $Y \amalg Z$, we should have monomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathcal{C}^{\mathrm{eq}}}(X, Y \amalg Z) \hookleftarrow \mathrm{Hom}_{\mathcal{C}}(X, Z).$$

However, it is not reasonable to expect to find *all* maps from X to $Y \amalg Z$ in this way. For example, if X can be written as the join of disjoint subobjects $X_0, X_1 \subseteq X$, then it is a disjoint coproduct of X_0 and X_1 in both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{eq}}$. Consequently, any pair of maps $f_0 : X_0 \to Y$ and $f_1 : X_1 \to Z$ in $\mathcal{C}$ can be uniquely amalgamated to obtain a map $f : X \simeq X_0 \amalg X_1 \to Y \amalg Z$ in the pretopos $\mathcal{C}^{\mathrm{eq}}$. Roughly speaking, we would like to arrange that all morphisms in $\mathcal{C}^{\mathrm{eq}}$ are defined by a procedure similar to this: that is, they can be built by "gluing together" morphisms that are already present in the coherent category $\mathcal{C}$. To make this precise, it will be convenient to construct $\mathcal{C}^{\mathrm{eq}}$ as a subcategory of a much larger category of *sheaves* on the coherent category $\mathcal{C}$.

Definition 6. Let $\mathcal{C}$ be a category which admits finite limits. A *Grothendieck topology on* $\mathcal{C}$ is a specification of a collection of families of maps $\{f_i : U_i \to X\}_{i \in I}$, which we refer to as *coverings*, satisfying the following axioms:

($T1$) If $\{f_i : U_i \to X\}$ is a covering and $g : Y \to X$ is any morphism in $\mathcal{C}$, then the collection of projection maps $\{U_i \times_X Y \to Y\}_{i \in I}$ is a covering.

($T2$) If $\{f_i : U_i \to X\}_{i \in I}$ is a covering and, for every $i \in I$, we specify a covering $\{g_{ij} : V_{ij} \to U_i\}$, then the collection of composite maps $\{(f_i \circ g_{ij}) : V_{ij} \to X\}$ is a covering.

($T3$) If $\{f_i : U_i \to X\}$ is a collection of morphisms such that some f_i admits a section (that is, a morphism $s : X \to U_i$ satisfying $f_i \circ s = \mathrm{id}_X$), then $\{f_i : U_i \to X\}$ is a covering.

Remark 7. One can also consider Grothendieck topologies on categories which do not admit finite limits. In this case, requirement ($T1$) must be rephrased. In this course, we will not need to consider Grothendieck topologies on such categories.

Exercise 8. Let $\mathcal{C}$ be a category which admits fiber products which is equipped with a Grothendieck topology, and suppose that $\{U_i \to X\}$ is a covering. Show that any larger collection of morphisms (with the same target X) is also a covering.

Definition 9. A *Grothendieck site* is a category $\mathcal{C}$ together with a Grothendieck topology on $\mathcal{C}$.

Example 10. Let X be a topological space and let $\mathcal{U}$ be the collection of all open subsets of X, regarded as a partially ordered set with respect to inclusions. Then, when regarded as a category, the poset $\mathcal{U}$ carries a Grothendieck topology, where a collection of maps $\{U_i \to U\}_{i \in I}$ is a covering if $\bigcup_{i \in I} U_i = U$.

The original motivation for Definition 6 came from algebraic geometry:

Example 11. Let X be a scheme and let $\mathcal{C}$ be the category of schemes U equipped with an étale map $U \to X$. Then $\mathcal{C}$ can be equipped with a Grothendieck topology, where a collection of maps $\{U_i \to U\}$ is a covering if the induced map $\amalg_i U_i \to U$ is surjective. When endowed with this topology, $\mathcal{C}$ is referred to as the *small étale site* of the scheme X.

For the moment, we will be primarily interested in Grothendieck sites which arise from coherent categories:

Proposition 12. *Let $\mathcal{C}$ be a coherent category. Then $\mathcal{C}$ admits a Grothendieck topology which can be described as follows: a collection of morphisms $\{f_i : U_i \to X\}_{i \in I}$ is a covering if there exists some finite subset $I_0 \subseteq I$ such that $\bigvee_{i \in I_0} \mathrm{Im}(f_i) = X$ (in the lattice $\mathrm{Sub}(X)$).*

Proof. Since the formation of images and joins of subobjects is compatible with pullback, it is clear that this notion of covering satisfies axiom $(T1)$.

Note that a collection of maps $\{f_i : U_i \to X\}_{i \in I}$ is a covering if, for some finite subset $I_0 \subseteq I$, there is no proper subobject $X' \subsetneq X$ such that f_i factors through X', for each $i \in I_0$. From this description, it is clear that $(T3)$ is satisfied.

To verify $(T2)$, suppose that we are given coverings $\{U_i \to X\}_{i \in I}$ and $\{V_{ij} \to U_i\}_{j \in J_i}$. We wish to show that the collection of composite maps $\{V_{ij} \to U_i \to X\}$ is a covering. Without loss of generality, we may assume that I is finite and that each J_i is finite. Suppose that there exists a subobject $X' \subseteq X$ such that each of the composite maps $V_{ij} \to U_i \to X$ factors through X'. Then each $V_{ij} \to U_i$ factors through $X' \times_X U_i$. Using our assumption that $\{V_{ij} \to U_i\}$, we deduce that $X' \times_X U_i = U_i$ (as subobjects of U_i). In other words, the maps $U_i \to X$ factor through X'. Using our assumption that $\{U_i \to X\}$ is a covering, we conclude that $X' = X$. $\square$

Definition 13. Let $\mathcal{C}$ be a category. A *presheaf* (of sets) on $\mathcal{C}$ is a functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$.

Suppose that $\mathcal{C}$ admits fiber products and is equipped with a Grothendieck topology. We will say that a presheaf $\mathscr{F}$ is a *sheaf* if, for every covering $\{U_i \to X\}_{i \in I}$, the diagram of sets

$$\mathscr{F}(X) \longrightarrow \prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(U_i \times_X U_j)$$

is an equalizer: that is, we can $\mathscr{F}(X)$ with the set of tuples $\{s_i \in \mathscr{F}(U_i)\}$ having the property that, for every pair $i, j \in I$, the elements s_i and s_j have the same image in $\mathscr{F}(U_i \times_X U_j)$. We let $\mathrm{Shv}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ spanned by the sheaves on $\mathcal{C}$.

Example 14. Let X be a topological space and let $\mathcal{U}$ be the partially ordered set of open subsets of X. A *sheaf on X* is defined to be a sheaf on the category $\mathcal{U}$, where we equip $\mathcal{U}$ with the Grothendieck topology of Example 10. We will denote the category of sheaves on X by $\mathrm{Shv}(X)$.

Example 15 (Representable Sheaves)**.** Let $\mathcal{C}$ be any category. For each object $Y \in \mathcal{C}$, let h_Y denote the functor represented by Y, given by the formula $h_Y(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$. We say that a Grothendieck topology on $\mathcal{C}$ is *subcanonical* if h_Y is a sheaf, for each $Y \in \mathcal{C}$. If this condition is satisfied, then the construction $Y \mapsto h_Y$ determines a fully faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Shv}(\mathcal{C})$.

Proposition 16. *Let $\mathcal{C}$ be a coherent category. Then the Grothendieck topology of Proposition 12 is subcanonical. In particular, we can view $\mathcal{C}$ as a full subcategory of* $\mathrm{Shv}(\mathcal{C})$.

Proof. Fix an object $Y \in \mathcal{C}$; we wish to show that the functor h_Y is a sheaf. To prove this, suppose we are given a covering $\{U_i \to X\}$. We wish to show that the diagram of sets

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(U_i, Y) \rightrightarrows \prod_{i,j \in I} \mathrm{Hom}_{\mathcal{C}}(U_i$$

$$times_X U_j, Y)$$

is an equalizer. In other words, we wish to show every collection of morphisms $f_i : U_i \to Y$ which is compatible, in the sense that f_i and f_j define the same morphism from $U_i \times_X U_j$ into Y, define an essentially unique map $f : X \to Y$. Without loss of generality, we can assume that I is finite (exercise: justify this). For each $i \in I$, let X_i denote the image of the map $U_i \to X$. Then we have an effective epimorphism $U_i \to X_i$, and the map coequalizes the projection maps

$$\pi, \pi' : U_i \times_{X_i} U_i \simeq U_i \times_X U_i \to U_i.$$

It follows that each f_i factors uniquely as a composition $U_i \to X_i \xrightarrow{\overline{f}_i} Y$. Moreover, for $i \neq j$, the maps $\overline{f}_i$ and $\overline{f}_j$ have the same restriction to $X_i \wedge X_j$ (since this can be checked after composition with the effective epimorphism

$$U_i \times_X U_j \to X_i \times_X X_j = X_i \wedge X_j.$$

It will therefore suffice to show that we can (uniquely) amalgamate the maps $\overline{f}_i$ to a single map $f : X \to Y$.

We now proceed as in the previous lecture. Each $\overline{f}_i$ has a graph $\Gamma(\overline{f}_i)$, which we can view as a subobject of $X \times Y$. Set $Z = \bigwedge \Gamma(\overline{f}_i)$. We will complete the proof by showing that Z is the graph of a morphism from X to Y. To prove this, it will suffice to show that the composition

$$Z \hookrightarrow X \times Y \to X$$

is a monomorphism (it will then necessarily be an isomorphism, since its image contains $\bigvee X_i$ which is equal to X by virtue of our assumption that $\{U_i \to X\}$ is covering). In other words, it will suffice to show that the diagonal map $Z \hookrightarrow Z \times_X Z$ is an isomorphism. Note that, as subobjects of $X \times Y \times Y$, Z can be identified with the join of the graphs of the maps $(\overline{f}_i, \overline{f}_i) : X_i \to Y \times Y$, while $Z \times_X Z$ can be identified with the join of the graphs of the morphisms

$$(\overline{f}_i|_{X_i \wedge X_j}, \overline{f}_j|_{X_i \wedge X_j}) : X_i \wedge X_j \to Y \times Y.$$

We complete the proof by observing that we have inclusions

$$\Gamma(\overline{f}_i|_{X_i \wedge X_j}, \overline{f}_j|_{X_i \wedge X_j}) \subseteq \Gamma(\overline{f}_i, \overline{f}_i),$$

since $\overline{f}_i$ and $\overline{f}_j$ have the same restriction to $X_i \wedge X_j$. □

We can now describe our approach to the proof of Theorem 4. Given a coherent category $\mathcal{C}$, we can view it as a subcategory of the sheaf category $\mathrm{Shv}(\mathcal{C})$. We will realize the pretopos completion $\mathcal{C}^{\mathrm{eq}}$ as a slightly larger subcategory of $\mathrm{Shv}(\mathcal{C})$. In order to show that this works, we will need to study the structural features of the category $\mathrm{Shv}(\mathcal{C})$, and other categories like it.

Definition 17. Let $\mathcal{X}$ be a category. We say that $\mathcal{X}$ is a *topos* if there is an equivalence of categories $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{C})$, where $\mathcal{C}$ is a small Grothendieck site which admits finite limits.

Example 18. For every topological space X, the category $\mathrm{Shv}(X)$ is a topos.

Example 19. Let T be a first-order theory and let $\mathrm{Syn}_0(T)$ be the weak syntactic category of T, equipped with the Grothendieck topology of Proposition 12. Then $\mathrm{Shv}(\mathrm{Syn}_0(T))$ is a topos. We will refer to $\mathrm{Shv}(\mathrm{Syn}_0(T))$ as the *classifying topos of* T.

A useful heuristic is that a topos is a kind of generalized topological space; the classifying topos of a first-order theory T can be viewed as a kind of "space of all models of T."

Warning 20. In the situation of Definition 17, the category $\mathcal{C}$ is not uniquely determined by $\mathcal{X}$. In general, many different Grothendieck sites can give rise to the same topos. For example, we will see that if $\mathcal{C}$ is a small coherent category and $\mathcal{C}^{\mathrm{eq}}$ is its pretopos completion, then the topoi $\mathrm{Shv}(\mathcal{C})$ and $\mathrm{Shv}(\mathcal{C}^{\mathrm{eq}})$ are equivalent.

Remark 21. In the situation of Definition 17, the requirement that $\mathcal{C}$ admits finite limits is not necessary: we impose it only because we have not defined the notion of Grothendieck topology in full generality. However, it is a harmless assumption (if a category has the form $\mathrm{Shv}(\mathcal{C})$ for an arbitrary small Grothendieck site $\mathcal{C}$, then it also has the form $\mathrm{Shv}(\mathcal{C}')$ where $\mathcal{C}'$ is a small Grothendieck site which admits finite limits).

Lecture 9: Sheaves

February 11, 2018

Recall that a category $\mathcal{X}$ is a *topos* if there exists an equivalence $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{C})$, where $\mathcal{C}$ is a small category (which can be assumed to admit finite limits) equipped with a Grothendieck topology. In this lecture, we will describe some of the important categorical properties enjoyed by topoi. Our main goal is to prove the following:

Theorem 1. *Every topos is a pretopos.*

The main ingredient we will need in the proof of Theorem 1 is the following:

Theorem 2. *Let $\mathcal{C}$ be a small category which admits finite limits and is equipped with a Grothendieck topology. The inclusion functor $\mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ admits a left adjoint $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Shv}(\mathcal{C})$. Moreover, the functor L preserves finite limits.*

For any presheaf $\mathscr{F}$, we will refer to $L\mathscr{F}$ as the *sheafification of* $\mathscr{F}$.

Let us first show that Theorem 2 implies Theorem 1. Let $\mathcal{X}$ be a topos, and write $\mathcal{X} = \mathrm{Shv}(\mathcal{C})$ where $\mathcal{C}$ is a small category (which admits finite limits). We proceed in several steps.

Lemma 3. *The category $\mathcal{X}$ admits arbitrary inverse limits.*

Proof. We first note that the category $\mathcal{S}\mathrm{et}$ admits arbitrary inverse limits. It follows that the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ of presheaves admits arbitrary inverse limits. These are just computed pointwise (by the formula $(\varprojlim \mathscr{F}_\alpha)(C) = \varprojlim(\mathscr{F}_\alpha(C))$ for $C \in \mathcal{C}$). It follows from this description that if each $\mathscr{F}_\alpha$ is a sheaf, then so is the inverse limit $\mathscr{F} = \varprojlim \mathscr{F}_\alpha$; in this case, $\mathscr{F}$ is also a limit of the diagram $\{\mathscr{F}_\alpha\}$ in the category $\mathrm{Shv}(\mathcal{C})$. $\square$

Lemma 4. *The category $\mathcal{X}$ admits arbitrary colimits (that is, direct limits).*

Proof. Since the category $\mathcal{S}\mathrm{et}$ has all colimits, it follows that $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ admits arbitrary colimits (which are again computed pointwise by the formula $(\varinjlim \mathscr{F}_\alpha)(C) = \varinjlim(\mathscr{F}_\alpha(C))$ for $C \in \mathcal{C}$). This construction generally does not carry sheaves to sheaves. However, we can obtain a sheaf $\mathscr{F}$ by applying the sheafification functor L to $\varinjlim \mathscr{F}_\alpha$. Note that, for any sheaf $\mathscr{G}$, we have canonical bijections

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(\mathscr{F}, \mathscr{G}) &= \mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(L \varinjlim \mathscr{F}_\alpha, \mathscr{G}) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(\varinjlim \mathscr{F}_\alpha, \mathscr{G}) \\ &\simeq \varprojlim \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(\mathscr{F}_\alpha, \mathscr{G}). \end{aligned}$$

It follows that if each $\mathscr{F}_\alpha$ is a sheaf, then $L \varinjlim(\mathscr{F}_\alpha)$ is a colimit of the diagram $\{\mathscr{F}_\alpha\}$ in the category $\mathrm{Shv}(\mathcal{C})$. $\square$

Lemma 5. *Every equivalence relation in $\mathcal{X}$ is effective.*

Proof. Let $\mathscr{F}$ be a sheaf and let $\mathscr{R} \subseteq \mathscr{F} \times \mathscr{F}$ be an equivalence relation on $\mathscr{F}$. Then, for every object $C \in \mathcal{C}$, we can view $\mathscr{R}(C)$ as an equivalence relation on $\mathscr{F}(C)$. Set $\mathscr{E}(C) = \mathscr{F}(C)/\mathscr{R}(C)$. The construction $C \mapsto \mathscr{E}(C)$ determines a presheaf of sets on $\mathcal{C}$, given by the coequalier of the two projection maps $\pi, \pi' : \mathscr{R} \to \mathscr{F}$ in the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$. It follows that the sheafification $L\mathscr{E}$ is a coequalizer of π and π' in the category $\mathrm{Shv}(\mathcal{C})$. To show that $\mathscr{R}$ is effective, we must show that the canonical map $\delta : \mathscr{R} \to \mathscr{F} \times_{L\mathscr{E}} \mathscr{F}$ is an isomorphism. Note that we can identify δ with the canonical map

$$L(\mathscr{F} \times_{\mathscr{E}} \mathscr{F}) \to L\mathscr{F} \times_{L\mathscr{E}} L\mathscr{F},$$

which is an isomorphism because the sheafification functor L preserves finite limits. □

Lemma 6. *Coproducts in $\mathcal{X}$ are disjoint.*

Proof. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves, and let $\mathscr{F} \amalg \mathscr{G}$ denote the coproduct of $\mathscr{F}$ and $\mathscr{G}$ *in the category of presheaves* (given by the formula $(\mathscr{F} \amalg \mathscr{G})(C) = \mathscr{F}(C) \amalg \mathscr{G}(C)$). Then the sheafification $L(\mathscr{F} \amalg \mathscr{G})$ is a coproduct of $\mathscr{F}$ and $\mathscr{G}$ in the category of sheaves. We wish to show that the fiber product

$$\mathscr{F} \times_{L(\mathscr{F} \amalg \mathscr{G})} \mathscr{G}$$

is an initial object of $\mathrm{Shv}(\mathcal{C})$. Since the functor L preserves finite limits, we can identify this fiber product with $L(\mathscr{F} \times_{\mathscr{F} \amalg \mathscr{G}} \mathscr{G}) = L(\emptyset)$, where $\emptyset$ denotes the initial object of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$. We conclude by observing that L carries the initial presheaf to the initial sheaf. □

Lemma 7. *Colimits in the category $\mathcal{X}$ are universal. That is, for every morphism $f : \mathscr{F} \to \mathscr{G}$ in the category $\mathcal{X}$ and every diagram $\{\mathscr{G}_\alpha\}$ in $\mathcal{X}_{/\mathscr{G}}$, the canonical map*

$$\varinjlim_\alpha (\mathscr{F} \times_{\mathscr{G}} \mathscr{G}_\alpha) \to \mathscr{F} \times_{\mathscr{G}} (\varinjlim \mathscr{G}_\alpha)$$

is an isomorphism (where the colimits are taken in the category of sheaves*).*

Exercise 8. Check that Lemma 7 holds when $\mathcal{X} = \mathcal{S}\mathrm{et}$ is the category of sets.

Proof of Lemma 7. Let us break with the notation used in the statement of Lemma 7 and instead write $(\varinjlim \mathscr{G}_\alpha)$ and $\varinjlim_\alpha (\mathscr{F} \times_{\mathscr{G}} \mathscr{G}_\alpha)$ for the appropriate colimits formed in the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ of *presheaves*. In this case, everything is computed pointwise, so that Exercise 8 shows that the map

$$\varinjlim_\alpha (\mathscr{F} \times_{\mathscr{G}} \mathscr{G}_\alpha) \to \mathscr{F} \times_{\mathscr{G}} (\varinjlim \mathscr{G}_\alpha)$$

is an isomorphism of *presheaves*. Applying the sheafification functor L (and using that it commutes with finite limits), we deduce that the canonical map

$$\begin{aligned} L \varinjlim_\alpha (\mathscr{F} \times_{\mathscr{G}} \mathscr{G}_\alpha) &\to L(\mathscr{F} \times_{\mathscr{G}} (\varinjlim \mathscr{G}_\alpha)) \\ &\to L\mathscr{F} \times_{L\mathscr{G}} L(\varinjlim \mathscr{G}_\alpha) \\ &\simeq \mathscr{F} \times_{\mathscr{G}} L(\varinjlim \mathscr{G}_\alpha). \end{aligned}$$

is an isomorphism, which is the content of Lemma 7. □

Lemma 9. *Suppose we are given a pullback diagram*

$$\begin{tikzcd} \mathscr{F}' \arrow[r] \arrow[d, "f'"] & \mathscr{F} \arrow[d, "f"] \\ \mathscr{G}' \arrow[r] & \mathscr{G} \end{tikzcd}$$

in the category $\mathcal{X}$. If f is an effective epimorphism, then so is f'.

Proof. Set $\mathscr{R} = \mathscr{F} \times_{\mathscr{G}} \mathscr{F}$ and set $\mathscr{R}' = \mathscr{F}' \times_{\mathscr{G}'} \mathscr{F}'$. Our assumption that f is an effective epimorphism guarantees that we have a coequalizer diagram

$$\mathscr{R} \rightrightarrows \mathscr{F} \longrightarrow \mathscr{G}$$

Pulling back along the map $\mathscr{G}' \to \mathscr{G}$ and applying Lemma 7, we obtain a coequalizer diagram

$$\mathscr{R}' \rightrightarrows \mathscr{F}' \longrightarrow \mathscr{G}'$$

so that f' is also an effective epimorphism. □

Proof of Theorem 1. Let $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{C})$ be a topos. Then $\mathcal{X}$ admits finite limits (Lemma 3), equivalence relations in $\mathcal{X}$ are effective (Lemma 5), $\mathcal{X}$ admits disjoint coproducts (Lemmas 4 and 6), pullbacks of effective epimorphisms are effective effective epimorphisms (Lemma 9), and coproducts are preserved by pullback (Lemma 7). In particular, $\mathcal{X}$ is a pretopos. □

Exercise 10. Let $\mathcal{C}$ be a pretopos and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory. Suppose that that inclusion functor $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to \mathcal{C}_0$ which preserves finite limits. Show that $\mathcal{C}_0$ is also a pretopos (when applied to the inclusion $\mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$, this shows that Theorem 2 implies Theorem 1).

We now turn to the proof of Theorem 2. For the remainder of this lecture, we fix a small category $\mathcal{C}$ with finite limits, which admits a Grothendieck topology. Recall that a functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is a *sheaf* if, for every covering $\{U_i \to X\}$ in $\mathcal{C}$, the diagram

$$\mathscr{F}(X) \longrightarrow \textstyle\prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(U_i \times_X U_j)$$

is an equalizer in the category of sets. In what follows, we will refer to an element $s \in \mathscr{F}(Y)$ as a *section of* $\mathscr{F}$ *over* Y; given a morphism $Y' \to Y$ in $\mathcal{C}$, we denote the image of s in $\mathscr{F}(Y')$ by $s|_{Y'}$. By definition, the equalizer

$$\mathrm{Eq}(\textstyle\prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(U_i \times_X U_j)\ )$$

is the set of all tuples $\{s_i \in \mathscr{F}(U_i)\}_{i \in I}$ satisfying $s_i|_{U_i \times_X U_j} = s_j|_{U_i \times_X U_j}$.

Given any tuple of elements $\{s_i \in \mathscr{F}(U_i)\}_{i \in I}$ and a map $g : V \to X$ which factors as a composition $V \xrightarrow{g_i} U_i \to X$, we can form the restriction $s_i|_V \in \mathscr{F}(V)$. In general, this will depend on the choice of i and of the map g_i. However, the condition $s_i|_{U_i \times_X U_j} = s_j|_{U_i \times_X U_j}$ is exactly what we need in order to guarantee that the restriction $s_i|_V$ is independent of i. Put another way, we have a canonical bijection

$$\mathrm{Eq}(\textstyle\prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(U_i \times_X U_j)\ ) \simeq \varprojlim_{V \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(V),$$

where $\mathcal{C}^{(0)}_{/X}$ denotes the full subcategory of $\mathcal{C}_{/X}$ spanned by those maps $V \to X$ which factor through some U_i (the factorization need not be specified). To exploit this, it will be convenient to introduce some terminology.

Definition 11. Let X be an object of $\mathcal{C}$. A *sieve on* X is a full subcategory $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$ with the property that, for each morphism $U \to V$ in $\mathcal{C}_{/X}$, if V belongs to $\mathcal{C}^{(0)}_{/X}$ then U also belongs to $\mathcal{C}^{(0)}_{/X}$. We will say that a sieve $\mathcal{C}^{(0)}_{/X}$ is *covering* if it contains a collection of maps $\{U_i \to X\}$ which form a covering of X (with respect to our chosen Grothendieck topology).

Exercise 12. Let $\{f_i : U_i \to X\}$ be a collection of morphisms in $\mathcal{C}$ having some common codomain X, and let $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$ be the full subcategory spanned by those maps $g : V \to X$ which factor through some f_i. Show that $\mathcal{C}^{(0)}_{/X}$ is a sieve on X, which is covering sieve if and only if $\{f_i : U_i \to X\}$ is a covering.

Applying the above discussion, we can reformulate the definition of sheaf as follows:

Definition 13. Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a presheaf. Then $\mathscr{F}$ is a sheaf if and only if, for every covering sieve $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$, the map $\mathscr{F}(X) \to \varinjlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(U)$ is a bijection.

We are now ready to describe the sheafification procedure explicitly.

Construction 14. Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a presheaf. We define a new presheaf $\mathscr{F}^\dagger : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ by the formula

$$\mathscr{F}^\dagger(X) = \varinjlim_{\mathcal{C}^{(0)}_{/X}} \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(U).$$

Here the colimit is taken over all covering sieves $\mathcal{C}^{(0)}_{/X}$ on X (which we regard as a partially ordered set under reverse inclusion).

Note that for any object $X \in \mathcal{C}$, the sieve $\mathcal{C}_{/X}$ is always a covering. We therefore have a canonical map

$$\alpha_{\mathscr{F}} : \mathscr{F}(X) \simeq \varprojlim_{U \in \mathcal{C}_{/X}} \mathscr{F}(U) \to \varinjlim_{\mathcal{C}^{(0)}_{/X}} \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(U) \simeq \mathscr{F}^\dagger(X)$$

(see the proof of Lemma 18 below for a more precise definition of the presheaf $\mathscr{F}^\dagger$ and of this map). We will deduce Theorem 2 from the following more precise statements:

Proposition 15. *For every presheaf $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, the composite map*

$$\mathscr{F} \xrightarrow{\alpha_{\mathscr{F}}} \mathscr{F}^\dagger \xrightarrow{\alpha_{\mathscr{F}^\dagger}} \mathscr{F}^{\dagger\dagger}$$

exhibits $\mathscr{F}^{\dagger\dagger}$ as a sheafification of $\mathscr{F}$. In other words, $\mathscr{F}^{\dagger\dagger}$ is a sheaf and, for every sheaf $\mathscr{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, the induced map

$$\mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(\mathscr{F}^{\dagger\dagger}, \mathscr{G}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(\mathscr{F}, \mathscr{G})$$

is bijective.

Proposition 16. *The functor $\mathscr{F} \mapsto \mathscr{F}^\dagger$ is left exact: that is, it preserves finite limits.*

Let us begin with the second statement.

Lemma 17. *Let X be an object of $\mathcal{C}$. Then the collection of covering sieves $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$ is closed under finite intersections.*

Proof. Let $\mathcal{C}^{(0)}_{/X}$ and $\mathcal{C}^{(1)}_{/X}$ be covering sieves on X. Then $\mathcal{C}^{(0)}_{/X}$ contains a covering $\{U_i \to X\}_{i \in I}$ and $\mathcal{C}^{(1)}_{/X}$ contains a covering $\{V_j \to X\}_{j \in J}$. Since coverings are closed under pullback, the collection of maps $\{U_i \times_X V_j \to U_i\}_{j \in J}$ is a covering for each $i \in I$. It follows that the collection of composite maps $\{U_i \times_X V_j \to X\}$ is a covering of X. We conclude by observing that each of these maps is contained in the sieve $\mathcal{C}^{(0)}_{/X} \cap \mathcal{C}^{(1)}_{/X}$. $\square$

Proof of Proposition 16. Fix an object $X \in \mathcal{C}$, and regard the construction

$$\mathscr{F} \mapsto \mathscr{F}^\dagger(X) = \varinjlim_{\mathcal{C}^{(0)}_{/X}} \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(U)$$

as a functor of the presheaf $\mathscr{F}$. It follows from Lemma 17 that the colimit appearing in this expression is filtered (it is taken over a directed poset), and therefore commutes with finite limits. Since limits in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ are computed pointwise, it follows that the construction $\mathscr{F} \mapsto \mathscr{F}^\dagger$ commutes with finite limits. $\square$

Lemma 18. *Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a presheaf and let $\mathscr{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a sheaf. Then composition with $\alpha_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^{\dagger}$ induces a bijection*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(\mathscr{F}^{\dagger}, \mathscr{G}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(\mathscr{F}, \mathscr{G}).$$

Proof. We begin by giving a more precise definition of the functor $\mathscr{F}^{\dagger}$ and of the natural map $\alpha : \mathscr{F} \to \mathscr{F}^{\dagger}$. For this, we need an auxiliary construction:

Notation 19. We define a category $\mathrm{Cov}(\mathcal{C})$ as follows:

- The objects of $\mathrm{Cov}(\mathcal{C})$ are pairs $(X, \mathcal{C}^{(0)}_{/X})$, where $X \in \mathcal{C}$ is an object and $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$ is a covering sieve.
- Given a pair of objects $(X, \mathcal{C}^{(0)}_{/X}), (Y, \mathcal{C}^{(0)}_{/Y}) \in \mathrm{Cov}(\mathcal{C})$, a morphism from $(X, \mathcal{C}^{(0)}_{/X})$ to $(Y, \mathcal{C}^{(0)}_{/Y})$ is a morphism $u : X \to Y$ in the category $\mathcal{C}$ with property that, for every morphism $U \to X$ belonging to the sieve $\mathcal{C}^{(0)}_{/X}$, the composite map $U \to X \xrightarrow{u} Y$ belongs to the sieve $\mathcal{C}^{(0)}_{/Y}$.

We have evident functors $i : \mathcal{C} \to \mathrm{Cov}(\mathcal{C})$ and $\rho : \mathrm{Cov}(\mathcal{C}) \to \mathcal{C}$, given by the formulae

$$i(X) = (X, \mathcal{C}_{/X}) \qquad \rho(X, \mathcal{C}^{(0)}_{/X}) = X$$

Composition with i and ρ determines pullback functors

$$i^* : \mathrm{Fun}(\mathrm{Cov}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \qquad \rho^* : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathrm{Cov}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}\mathrm{et}).$$

The functor i^* admits a right adjoint $i_* : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathrm{Cov}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ (given by right Kan extension along i), and the functor ρ^* admits a left adjoint $\rho_! \, \mathrm{Fun}(\mathrm{Cov}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ (given by left Kan extension along ρ). Concretely, we have

$$i_*(\mathscr{F})(X, \mathcal{C}^{(0)}_{/X}) \simeq \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(U) \qquad \rho_!(\mathscr{H})(X) = \varinjlim_{\mathcal{C}^{(0)}_{/X}} \mathscr{H}(X, \mathcal{C}^{(0)}_{/X}),$$

where the colimit is taken over the collection of all covering sieves on X. The construction $\mathscr{F} \mapsto \mathscr{F}^{\dagger}$ can now be described more precisely by the formula $\mathscr{F}^{\dagger} = \rho_! i_* \mathscr{F}$.

Note that, since i is a full faithful embedding, the natural map $i^* i_* \mathscr{F} \to \mathscr{F}$ is an equivalence for every presheaf $\mathscr{F}$. Consequently, for any pair of presheaves $\mathscr{F}, \mathscr{G} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$, we have a natural map

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(\mathscr{F}^{\dagger}, \mathscr{G}) &= \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(\rho_! i_* \mathscr{F}, \mathscr{G}) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathrm{Cov}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(i_* \mathscr{F}, \rho^* \mathscr{G}) \\ &\xrightarrow{u} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(i^* i_* \mathscr{F}, i^* \rho^* \mathscr{G}) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(\mathscr{F}, \mathscr{G}). \end{aligned}$$

These maps depends functorially on $\mathscr{G}$, and are therefore (by Yoneda's lemma) given by precomposition with some map of presheaves $\mathscr{F} \to \mathscr{F}^{\dagger}$; this gives a more precise definition of the map $\alpha_{\mathscr{F}}$ appearing in the statement of Lemma 18. To complete the proof of Lemma 18, we must show that u is a bijection when $\mathscr{G}$ is a sheaf. Note that u can be identified with the canonical map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{Cov}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(i_* \mathscr{F}, \rho^* \mathscr{G}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathrm{Cov}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(i_* \mathscr{F}, i_* i^* \rho^* \mathscr{G}).$$

We are therefore reduced to showing that if $\mathscr{G}$ is a sheaf, then the unit map $\rho^* \mathscr{G} \to i_* i^* \rho^* \mathscr{G}$ is an equivalence of presheaves on $\mathrm{Cov}(\mathcal{C})$. Unwinding the definitions, we must show that for each object $(X, \mathcal{C}^{(0)}_{/X})$, the canonical map

$$\mathscr{G}(X) \to \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{G}(U)$$

is a bijection, which is a translation of our hypothesis that $\mathscr{G}$ is a sheaf (Definition 13). $\square$

To complete the proof of Proposition 15, we must show that applying the functor $\mathscr{F} \mapsto \mathscr{F}^\dagger$ twice converts any presheaf into a sheaf. This is a consequence of a more precise assertion.

Definition 20. Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor. We will say that $\mathscr{F}$ is a *separated presheaf* if, for every covering $\{U_i \to X\}$, the natural map $\mathscr{F}(X) \to \prod \mathscr{F}(U_i)$ is injective. Equivalently, we say that $\mathscr{F}$ is a separated presheaf if the unit map $\rho^* \mathscr{F} \to i_* i^* \rho^* \mathscr{F}$ is a monomorphism of presheaves on $\mathrm{Cov}(\mathcal{C})$.

Notation 21. Given a presheaf $\mathscr{F}$, an object $X \in \mathcal{C}$, and a covering sieve $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$, we let $\mathscr{F}(X, \mathcal{C}^{(0)}_{/X})$ denote the value of the functor $(i_* \mathscr{F})$ on the pair $(X, \mathcal{C}^{(0)}_{/X})$, given by the direct limit $\mathscr{F}(X, \mathcal{C}^{(0)}_{/X}) = \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(U)$.

Lemma 22. *Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a separated presheaf. Then, for every object $X \in \mathcal{C}$ and every inclusion of covering sieves $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}^{(1)}_{/X} \subseteq \mathcal{C}_{/X}$, the restriction map $\mathscr{F}(X, \mathcal{C}^{(1)}_{/X}) \to \mathscr{F}(X, \mathcal{C}^{(0)}_{/X})$ is injective.*

Proof. For each object $V \in \mathcal{C}^{(1)}_{/X}$, let $\mathcal{C}^{(0)}_{/V}$ denote the sieve on V given by those maps $U \to V$ for which the composite map $U \to V \to X$ belongs to $\mathcal{C}^{(0)}_{/U}$. We then compute

$$\begin{aligned} \mathscr{F}(X, \mathcal{C}^{(1)}_{/X}) &= \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}(V) \\ &= \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \varprojlim_{U \in \mathcal{C}_{/V}} \mathscr{F}(U) \\ &\hookrightarrow \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \varprojlim_{U \in \mathcal{C}^{(0)}_{/V}} \mathscr{F}(U) \\ &\simeq \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(U). \end{aligned}$$

□

Lemma 23. *Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a separated presheaf. Then, for every object $X \in \mathcal{C}$ and every covering sieve $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$, the canonical map $\mathscr{F}(X, \mathcal{C}^{(0)}_{/X}) \to \mathscr{F}^\dagger(X)$ is injective.*

Proof. Apply Lemma 22 (note that a filtered colimit of injections is still an injection). □

Lemma 24. *Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a separated presheaf. Then $\mathscr{F}^\dagger$ is a sheaf.*

Proof. Let X be an object of $\mathcal{C}$ and let $\mathcal{C}^{(1)}_{/X}$ be a covering sieve on X; we wish to show that the canonical map

$$\theta : \mathscr{F}^\dagger(X) \to \mathscr{F}^\dagger(X, \mathcal{C}^{(1)}_{/X})$$

is bijective. Write $\mathscr{F}^\dagger(X)$ as a filtered colimit $\varinjlim_{\mathcal{C}^{(0)}_{/X}} \mathscr{F}(X, \mathcal{C}^{(0)}_{/X})$. Here it suffices to take the colimit over those covering sieves $\mathcal{C}^{(0)}_{/X}$ which are contained in $\mathcal{C}^{(1)}_{/X}$. Using the notation of Lemma 22, we compute

$$\mathscr{F}(X, \mathcal{C}^{(0)}_{/X}) = \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}(V, \mathcal{C}^{(0)}_{/V}).$$

We can therefore write θ as a filtered colimit of maps

$$\theta_{\mathcal{C}^{(0)}_{/X}} : \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}(V, \mathcal{C}^{(0)}_{/V}) \to \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}^\dagger(V),$$

where $\mathcal{C}^{(0)}_{/X}$ ranges over all covering sieves on X which are contained in $\mathcal{C}^{(1)}_{/X}$. Lemma 23 guarantees that each $\theta_{\mathcal{C}^{(0)}_{/}X}$ is a monomorphism, so that θ is a monomorphism.

To complete the proof, we show that for each element $s \in \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}^\dagger(V)$, we can choose some covering $\mathcal{C}^{(0)}_{/X}$ for which s belongs to the image of $\theta_{\mathcal{C}^{(0)}_{/X}}$. For each object $V \in \mathcal{C}^{(1)}_{/X}$, let s_V denote the image of s in $\mathscr{F}^\dagger(V)$. From the definition of $\mathscr{F}^\dagger$, we see that there exists a covering sieve $\mathcal{C}^{(2)}_{/V} \subseteq \mathcal{C}_{/V}$ for which s_V belongs to the image of the monomorphism

$$\mathscr{F}(V, \mathcal{C}^{(2)}_{/V}) \hookrightarrow \mathscr{F}^\dagger(V).$$

We now complete the proof by taking $\mathcal{C}^{(0)}_{/X}$ to be the smallest sieve on X which contains each of the composite maps $U \to V \to X$, where $U \to V$ belongs to $\mathcal{C}^{(2)}_{/V}$ and $V \to X$ belongs to $\mathcal{C}^{(1)}_{/X}$. □

Lemma 25. *Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be any presheaf. Then $\mathscr{F}^\dagger$ is a separated presheaf.*

Proof. We argue as in the proof of Lemma 24. Fix an object $X \in \mathcal{C}$ and let $\mathcal{C}^{(1)}_{/X}$ be a covering sieve on X; we wish to show that the canonical map

$$\theta : \mathscr{F}^\dagger(X) \to \mathscr{F}^\dagger(X, \mathcal{C}^{(1)}_{/X})$$

is injective. Write θ as as a filtered colimit of maps

$$\theta_{\mathcal{C}^{(0)}_{/X}} : \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}(V, \mathcal{C}^{(0)}_{/V}) \to \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}^\dagger(V),$$

where $\mathcal{C}^{(0)}_{/X}$ ranges over covering sieves on X that are contained in $\mathcal{C}^{(1)}_{/X}$. Suppose we are given a pair of elements $s, t \in \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}(V, \mathcal{C}^{(0)}_{/V})$ which have the same image under $\theta_{\mathcal{C}^{(0)}_{/X}}$. For each $V \in \mathcal{C}^{(1)}_{/X}$, let s_V and t_V denote the images of s and t in $\mathscr{F}(V, \mathcal{C}^{(0)}_{/V})$. Then s_V and t_V have the same image in $\mathscr{F}^\dagger(V)$. It follows that we can choose some covering sieve $\mathcal{C}^{(2)}_{/V} \subseteq \mathcal{C}^{(0)}_{/V}$ such that s_V and t_V have the same image in $\mathscr{F}(V, \mathcal{C}^{(2)}_{/V})$. Let $\mathcal{C}'_{/X}$ denote the smallest sieve on X which contains all composite maps $U \to V \to X$, where $U \to V$ belongs to $\mathcal{C}^{(2)}_{/V}$ and $V \to X$ bellongs to $\mathcal{C}^{(1)}_{/X}$. Then $\mathcal{C}'_{/X}$ is a covering sieve, and s and t have the same image in $\mathscr{F}(X, \mathcal{C}'_{/X})$, hence also in $\mathscr{F}^\dagger(X)$. □

Proof of Proposition 15. Combine Lemmas 18, 24, and 25. □

Lecture 10: Giraud's Theorem

February 12, 2018

In the previous lecture, we showed that every topos $\mathcal{X}$ is a pretopos. Our goal in this lecture is to prove a converse to this assertion, due to Giraud. Roughly speaking, Giraud's theorem asserts that a pretopos $\mathcal{X}$ is a topos if and only if it has infinite coproducts which are compatible with pullback, and satisfies a mild set-theoretic condition guaranteeing that it is not "too big" to arise as a category of sheaves on a small category.

Definition 1. Let $\mathcal{X}$ be a pretopos which admits infinite coproducts. We will say that a collection of morphisms $\{f_i : U_i \to X\}_{i \in I}$ is a *covering* if it induces an effective epimorphism $\amalg U_i \to X$. Equivalently, $\{f_i : U_i \to X\}$ is a covering if, for every subobject $X_0 \subseteq X$ such that each f_i factors through X_0, we must have $X_0 = X$ (as subobjects of X).

Warning 2. In Lecture 8, we defined a Grothendieck topology on any coherent category $\mathcal{C}$ using a very similar notion of covering. However, Definition 1 is different because we do not require that every covering family admits a finite subcover.

Theorem 3. *Let $\mathcal{X}$ be a category. The following conditions are equivalent:*

(1) *The category $\mathcal{X}$ is a topos: that is, it can be realized as a category of sheaves $\mathrm{Shv}(\mathcal{C})$, where $\mathcal{C}$ is a small category with finite limits which is equipped with a Grothendieck topology.*

(2) *There exists a small category $\mathcal{C}$ and a fully faithful embedding $\mathcal{X} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ which admits a left adjoint $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X}$ for which L preserves finite limits.*

(3) *The category $\mathcal{X}$ satisfies Giraud's axioms:*

   ($G1$) *The category $\mathcal{X}$ admits finite limits.*

   ($G2$) *Every equivalence relation in $\mathcal{X}$ is effective.*

   ($G3$) *The category $\mathcal{X}$ admits small coproducts, and coproducts in $\mathcal{X}$ are disjoint.*

   ($G4$) *The collection of effective epimorphisms in $\mathcal{X}$ is closed under pullback.*

   ($G5$) *The formation of coproducts commutes with pullback: that is, for every morphism $f : X \to Y$ in $\mathcal{X}$, the pullback functor $f^* : \mathcal{X}_{/Y} \to \mathcal{X}_{/X}$ preserves coproducts.*

   ($G6$) *There exists a* set *of objects $\mathcal{U}$ of $\mathcal{X}$ which generate $\mathcal{X}$ in the following sense: for every object $X \in \mathcal{X}$, there exists a covering $\{U_i \to X\}$, where each U_i belongs to $\mathcal{U}$.*

Remark 4. In the statement of Theorem 3, we do not assume that the category $\mathcal{X}$ is small (in practice, this will only happen in trivial cases). However, we always implicitly assume that it is *locally small*: that is, the collection of morphisms $\mathrm{Hom}_{\mathcal{X}}(X, Y)$ forms a set, for every pair of objects $X, Y \in \mathcal{X}$. Note that axiom ($G6$) would be vacuous if the category $\mathcal{X}$ were small (because we could take $\mathcal{U}$ to consist of *all* the objects of $\mathcal{X}$).

Remark 5. In Lecture 8, we defined the notion of Grothendieck topology only under the assumption that $\mathcal{C}$ admits finite limits. If we had given the definition more generally, we could add the following equivalent characterization to Theorem 3:

(1.5) The category $\mathcal{X}$ is equivalent to $\mathrm{Shv}(\mathcal{C})$, where $\mathcal{C}$ is a small category with a Grothendieck topology.

The implication (1) $\Rightarrow$ (1.5) would then be vacuous, and the implication (1.5) $\Rightarrow$ (2) follows from the sheafification construction of Lecture 9.

In the previous lecture, we showed that (1) implies (2) and that (2) implies axioms $(G1)$ through $(G5)$. Let us show that it also implies $(G6)$. Assume for simplicity that $\mathcal{X}$ is given as a full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$, where $\mathcal{C}$ is a small category, and that the inclusion functor admits a left exact left adjoint $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X}$. For each object $X \in \mathcal{C}$, let $h_X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ denote the functor represented by X, given by the formula $h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$. We claim that the collection $\{Lh_X\}_{X \in \mathcal{C}}$ is a set of generators for $\mathcal{X}$. To prove this, we note that every presheaf $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ admits an effective epimorphism

$$\amalg_{X \in \mathcal{C}, \eta \in \mathscr{F}(X)} h_X \to \mathscr{F} .$$

If $\mathscr{F}$ belongs to $\mathcal{X}$, then the induced maps $\{Lh_X \to \mathscr{F}\}_{X \in \mathcal{C}, \eta \in \mathscr{F}(X)}$ are a covering of $\mathscr{F}$ in the category $\mathcal{X}$.

Our goal for the rest of this lecture is to prove the implication (3) $\Rightarrow$ (1) of Theorem 3. Let $\mathcal{X}$ be a category satisfying $(G1)$ through $(G6)$. Using $(G6)$, we can choose a small full subcategory $\mathcal{C} \subseteq \mathcal{X}$ whose objects generate $\mathcal{X}$, in the sense of $(G6)$. Enlarging $\mathcal{C}$ if necessary, we can assume that $\mathcal{C}$ is closed under finite limits (meaning that every finite diagram in $\mathcal{C}$ admits a limit in $\mathcal{X}$ which also belongs to $\mathcal{C}$). We will prove the following:

Theorem 6. *Let $\mathcal{X}$ be a category satisfying axioms $(G1)$ through $(G5)$ and let $\mathcal{C} \subseteq \mathcal{X}$ be a full subcategory of $\mathcal{X}$ which is closed under finite limits and generates $\mathcal{X}$ (in the sense of $(G6)$). Let us say that a family of morphisms $\{U_i \to X\}$ in $\mathcal{C}$ is a covering if it is a covering in $\mathcal{X}$ (in the sense of Definition 1). Then:*

- (a) *The collection of coverings families determines a Grothendieck topology on $\mathcal{C}$.*
- (b) *For every object $Y \in \mathcal{X}$, let $h_Y : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ denote the functor represented by Y* on the subcategory *$\mathcal{C}$, given by the formula $h_Y(X) = \mathrm{Hom}_{\mathcal{X}}(X, Y)$. Then h_Y is a sheaf with respect to the Grothendieck topology of (a).*
- (c) *The construction $Y \mapsto h_Y$ induces an equivalence of categories $\mathcal{X} \to \mathrm{Shv}(\mathcal{C})$.*

Proof. We first prove (a). Suppose first that we are given a collection of morphisms $\{U_i \to X\}$ in $\mathcal{C}$ and a morphism $Y \to X$ in $\mathcal{C}$; we wish to show that the collection of projection maps $\{U_i \times_X Y \to Y\}$ is also a covering. In other words, we are given that the map $\amalg U_i \to X$ is an effective epimorphism in $\mathcal{X}$, and we wish to show that the induced map $\amalg(U_i \times_X Y) \to Y$ is also an effective epimorphism in $\mathcal{X}$. This is clear, since axioms $(G4)$ and $(G5)$ guarantee that pullback along the map $Y \to X$ preserves the formation of coproducts and the property of being an effective epimorphism.

Now suppose that we are given a covering $\{U_i \to X\}$ in $\mathcal{C}$ and, for each index i, a covering $\{V_{i,j} \to U_i\}$ in $\mathcal{C}$. In this case, we would like to show that the composite maps $\{V_{ij} \to X\}$ are also a covering. We proceed as in proof of the analogous statement for a coherent category. Suppose we are given a subobject $X_0 \subseteq X$ (not necessarily in $\mathcal{C}$) such that each of the composite maps $V_{i,j} \to X$ factors through X_0. Then, for each i, the maps $V_{i,j} \to U_i$ factor through $X_0 \times_X U_i$. Since the $V_{i,j}$ cover U_i, we have $X_0 \times_X U_i = U_i$: that is, each of the maps $U_i \to X$ factors through X_0. Since the maps $\{U_i \to X\}$ are a cover of X, we deduce that $X_0 = X$.

To complete the proof that the collection of coverings determines a Grothendieck topology, we should verify that if a collection of maps $\{f_i : U_i \to X\}$ is a covering whenever some f_i admits a section. This is clear (in this case, the morphism f_i itself is an effective epimorphism in $\mathcal{C}$).

We now prove (b). Fix an object $Y \in \mathcal{X}$; we wish to show that the functor $h_Y : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is a sheaf. Choose any covering $\{U_i \to X\}$ of an object $Y \in \mathcal{C}$; we wish to show that the diagram of sets

$$h_Y(X) \longrightarrow \prod h_Y(U_i) \rightrightarrows \prod h_Y(U_i \times_X U_j)$$

is an equalizer. Unwinding the definitions, we can rewrite this diagram as

$$\mathrm{Hom}_{\mathcal{X}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{X}}(\amalg_i U_i, Y) \rightrightarrows \mathrm{Hom}_{\mathcal{X}}(\amalg_{i,j} U_i \times_X U_j, Y)$$

Our assumption that $\{U_i \to X\}$ is a covering guarantees that the map $\amalg_i U_i \to X$ is an effective epimorphism. We are therefore reduced to proving that the natural map

$$\amalg_{i,j}(U_i \times_X U_j) \to (\amalg_i U_i) \times_X (\amalg_j U_j)$$

is an isomorphism, which follows from axiom $(G5)$.

To verify (c), we will need the following:

Lemma 7. *Let $\{U_i \to X\}_{i \in I}$ be a covering in $\mathcal{X}$ (not necessarily between objects of $\mathcal{C}$). Then the induced map $\amalg h_{U_i} \to h_X$ is an effective epimorphism in* $\mathrm{Shv}(\mathcal{C})$

Proof. Suppose we are given a section $s \in h_X(C)$, for some $C \in \mathcal{C}$, given by a morphism $C \to X$ in the category $\mathcal{X}$. Since $\{U_i \to X\}$ is a covering, the induced map $\amalg U_i \to X$ is an effective epimorphism in the category $\mathcal{X}$. Using $(G4)$ and $(G5)$, we deduce that the collection of maps $\{U_i \times_X C \to C\}$ is also a covering in $\mathcal{X}$. Since the objects of $\mathcal{C}$ generate $\mathcal{X}$, each $U_i \times_X C$ admits a covering $\{V_{i,j} \to U_i \times_X C\}$, where $V_{i,j} \in \mathcal{C}$. Then the collection of composite maps $\{V_{i,j} \to C\}$ are a covering in the category $\mathcal{C}$. By construction, for each pair (i, j), the image $s_{i,j} \in h_X(V_{i,j})$ of s belongs to the image of the map $h_{U_i}(V_{i,j}) \to h_X(V_{i,j})$. Allowing C and s to vary, we conclude that the maps $\{h_{U_i} \to h_X\}$ are a covering of h_X in the topos $\mathrm{Shv}(\mathcal{C})$. $\square$

Our next goal is to show that the functor $h : \mathcal{X} \to \mathrm{Shv}(\mathcal{C})$ is fully faithful: that is, for every pair of objects $X, Y \in \mathcal{C}$, the natural map

$$\theta_X : \mathrm{Hom}_{\mathcal{X}}(X, Y) \to \mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(h_X, h_Y)$$

is bijective. Let us regard Y as fixed. We will say that X is *good* if θ_X is bijective. The argument now proceeds in several steps:

(i) Every object $X \in \mathcal{C}$ is good: this follows from Yoneda's lemma.

(ii) Suppose that $X \in \mathcal{X}$ admits a covering $\{U_i \to X\}_{i \in I}$. If each U_i and each fiber product $U_i \times_X U_j$ is good, then X is good. To prove this, we observe that we have effective epimorphisms

$$\amalg_{i \in I} U_i \to X \qquad \amalg_{i \in I} h_{U_i} \to h_X$$

in the categories $\mathcal{X}$ and $\mathrm{Shv}(\mathcal{C})$, respectively (Lemma 7). Using the fact that both categories satisfy $(G5)$ (and the observation that h preserves finite limits), we obtain coequalizer diagrams

$$\amalg_{i,j \in I} U_i \times_X U_j \rightrightarrows \amalg_{i \in I} U_i \longrightarrow X$$

$$\amalg_{i,j \in I} h_{U_i \times_X U_j} \rightrightarrows \amalg_{i \in I} h_{U_i} \longrightarrow h_X.$$

It follows that the map θ_X fits into a commutative diagram of sets

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{X}}(X, Y) \arrow[r] \arrow[d, "\theta_X"] & \prod_i \mathrm{Hom}_{\mathcal{X}}(U_i, Y) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{i,j} \mathrm{Hom}_{\mathcal{X}}(U_i \times_X U_j, Y) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(h_X, h_Y) \arrow[r] & \prod_i \mathrm{Hom}_{\mathcal{X}}(h_{U_i}, h_Y) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} \mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(h_{U_i \times_X U_j}, h_Y)
\end{tikzcd}$$

where the rows are equalizer diagrams. Since the vertical maps in the middle and on the right are isomorphisms, it follows that θ_X is also an isomorphism.

(*iii*) Let X be an object of $\mathcal{C}$ and let $U \subseteq X$ be a subobject. Then U is good. To prove this, choose a covering $\{U_i \to U\}$, where each U_i belongs to $\mathcal{C}$. Since $\mathcal{C}$ is closed under finite limits, each of the fiber products $U_i \times_U U_j \simeq U_i \times_X U_j$ belongs to $\mathcal{C}$. It follows from (*i*) that the objects U_i and $U_i \times_X U_j$ are good, so that U is good by (*ii*).

(*iv*) Let X be an arbitrary object of $\mathcal{C}$. Choose a covering $\{U_i \to X\}$, where each U_i belongs to $\mathcal{C}$. Then each fiber product $U_i \times_X U_j$ is a subobject of $U_i \times U_j \in \mathcal{C}$, and is therefore good by (*iii*). Using (*ii*), we deduce that X is good.

This completes the proof that the functor $X \mapsto h_X$ is fully faithful. We wish to show that it is essentially surjective. Note that we have not yet used the full strength of our axioms: specifically, we have not used the fact that coproducts in $\mathcal{X}$ are disjoint, or that equivalence relations in $\mathcal{X}$ are effective.

Applying Lemma 7 in the case where $I = \emptyset$, we deduce that h carries the initial object of $\mathcal{X}$ to an initial object of $\mathrm{Shv}(\mathcal{C})$. We claim that h preserves coproducts. Fix a collection of objects $\{X_i\} \in \mathcal{X}$ with coproduct X; we wish to show that the canonical map

$$\theta : \amalg h_{X_i} \to h_X$$

is an isomorphism in $\mathcal{C}$. By virtue of Lemma 7, it is an effective epimorphism. It will therefore suffice to show that it is also a monomorphism: that is, that the diagonal map

$$\amalg h_{X_i} \to (\amalg h_{X_i} \times_{h_X} \amalg h_{X_j})$$

is an isomorphism. Using the fact that $\mathrm{Shv}(\mathcal{C})$ satisfies $(G5)$ and that h is right exact, we can rewrite the codomain of this map as $\amalg_{i,j} h_{X_i \times_X X_j}$. We are therefore reduced to showing that diagonal maps $h_{X_i} \to h_{X_i \times_X X_i}$ are isomorphisms and that $h_{X_i \times_X X_j}$ is an initial object of $\mathrm{Shv}(\mathcal{C})$ for $i \neq j$. This follows from our assumption that coproducts in $\mathcal{X}$ are disjoint (axiom $(G3)$).

We now show that the functor h is essentially surjective. Choose an object $\mathscr{F} \in \mathrm{Shv}(\mathcal{C})$; we wish to show that $\mathscr{F}$ belongs to the essential image of h. We first treat the special case where $\mathscr{F} \subseteq h_X$ for some object $X \in \mathcal{X}$. Choose an effective epimorphism $\amalg h_{C_i} \to \mathscr{F}$, where each C_i belongs to $\mathcal{C}$. Setting $U = \amalg_i C_i$, we obtain an effective epimorphism $h_U \to \mathscr{F}$ for some $U \in \mathcal{X}$. In this case, the composite map $h_U \to \mathscr{F} \hookrightarrow h_X$ arises from some morphism $u : U \to X$ in $\mathcal{X}$. Since $\mathcal{X}$ is a pretopos, the morphism u factors as a composition $U \xrightarrow{u'} Y \xrightarrow{u''} X$, where u' is an effective epimorphism and u'' is a monomorphism. Note that the induced map $h_U \xrightarrow{u'} h_Y$ is an effective epimorphism in $\mathrm{Shv}(\mathcal{C})$ (by virtue of Lemma 7), and the map $h_Y \xrightarrow{u''} X$ is a monomorphism in $\mathrm{Shv}(\mathcal{C})$ (since the functor h is left exact). From the uniqueness of images in the pretopos $\mathrm{Shv}(\mathcal{C})$, we conclude that $\mathscr{F}$ is isomorphic to h_Y.

We now treat the general case. Let $\mathscr{F}$ be any sheaf on $\mathcal{C}$. As before, we can choose an effective epimorphism $h_U \to \mathscr{F}$, for some $U \in \mathcal{X}$. In this case, the fiber product $h_U \times_{\mathscr{F}} h_U$ is a sheaf on $\mathcal{C}$ which can be viewed as a subobject of $h_U \times h_U = h_{U \times U}$. It follows from the analysis above that we can choose an isomorphism $h_U \times_{\mathscr{F}} h_U \simeq h_R$ for some object $R \in \mathcal{X}$. Note that for any object $Y \in \mathcal{X}$, we have a canonical bijection

$$\mathrm{Hom}_{\mathcal{X}}(Y, R) \simeq \mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(h_Y, h_U \times_{\mathscr{F}} h_U).$$

From this description, we see that R can be viewed as an equivalence relation on U in the pretopos $\mathcal{X}$. It follows from $(G2)$ that this equivalence relation is effective: that is, there exists an effective epimorphism $U \to X$ in $\mathcal{X}$ with $R = U \times_X U$ (as subobjects of $U \times U$). Applying Lemma 7 to the covering $\{U \to X\}$, we obtain an isomorphism

$$h_X \simeq \mathrm{Coeq}(h_R \rightrightarrows h_U) \simeq \mathrm{Coeq}(h_U \times_{\mathscr{F}} h_U \rightrightarrows h_U) \simeq \mathscr{F}\,.$$

$\square$

Lecture 11: Coherent Topoi

February 22, 2018

Recall that if X is an object of a topos $\mathcal{X}$, we say that a collection of morphisms $\{f_i : U_i \to X\}_{i \in I}$ is a *covering* if the induced map $\amalg U_i \to X$ is an effective epimorphism.

Definition 1. Let $\mathcal{X}$ be a topos. We say that an object $X \in \mathcal{X}$ is *quasi-compact* if every covering of X has a finite subcovering. In other words, for every covering $\{U_i \to X\}_{i \in I}$, we can choose a finite subset $I_0 \subseteq I$ such that $\{U_i \to X\}_{i \in I_0}$ is also a covering.

Remark 2. Let X be an object of a topos $\mathcal{X}$. Then a collection of morphisms $\{f_i : U_i \to X\}$ is a covering if and only if the collection of subobjects $\{\mathrm{Im}(f_i) \subseteq X\}$ is a covering. Consequently, X is quasi-compact if and only if every covering of X by subobjects $\{U_i \subseteq X\}_{i \in I}$ admits a finite subcover.

Proposition 3. *Let $\mathcal{X}$ be a topos and let $f : X \to Y$ be an effective epimorphism in $\mathcal{X}$. If X is quasi-compact, then so is Y.*

Proof. Let $\{U_i \to Y\}_{i \in I}$ be a covering of Y. Then $\{U_i \times_Y X \to X\}_{i \in I}$ is a covering of X. Since X is quasi-compact, this cover admits a finite subcover $\{U_i \times_Y X \to X\}_{i \in I_0}$. Since f is an effective epimorphism, it follows that the collection of morphisms $\{U_i \times_Y X \to Y\}_{i \in I_0}$ is a covering of Y. In particular, the maps $\{U_i \to Y\}_{i \in I_0}$ give a covering of Y. $\square$

Exercise 4. Let $\mathcal{X}$ be a topos and let $\{X_i\}_{i \in I}$ be a collection of objects indexed by a finite set I, having a coproduct $X = \amalg_{i \in I} X_i$. Show that X is quasi-compact if and only if each X_i is quasi-compact. In particular, the initial object of $\mathcal{X}$ is always quasi-compact.

Definition 5. Let $\mathcal{X}$ be a topos. We will say that an object $X \in \mathcal{X}$ is *quasi-separated* if, for every pair of morphisms $U \to X \leftarrow V$, where U and V are quasi-compact, the fiber product $U \times_X V$ is also quasi-compact.

Beware that the requirement of Definition 5 is sometimes satisfied for uninteresting reasons. For example, if $\mathcal{X} = \mathrm{Shv}(\mathbb{R}^n)$ is the category of sheaves on the Euclidean space $\mathbb{R}^n$, then the only quasi-compact object of $\mathcal{X}$ is the initial object. In this case, every object of $\mathcal{X}$ is quasi-separated. For Definition 5 to be meaningful, we need to ensure that there exists a good supply of quasi-compact objects.

Definition 6. Let $\mathcal{X}$ be a topos. We will say that $\mathcal{X}$ is *coherent* if there exists a collection of objects $\mathcal{U}$ satisfying the following conditions:

- The collection $\mathcal{U}$ generates $\mathcal{X}$: that is, every object $X \in \mathcal{X}$ admits a covering $\{U_i \to X\}$, where each U_i belongs to $\mathcal{U}$.
- The collection $\mathcal{U}$ is closed under finite products. In particular, it contains a final object of $\mathcal{X}$.
- Every object of $\mathcal{U}$ is quasi-compact and quasi-separated.

Remark 7. Let $\mathcal{X}$ be a coherent topos. Then the final object of $\mathcal{X}$ is quasi-separated. It follows that the collection of quasi-compact objects of $\mathcal{X}$ is closed under finite products.

We now describe a large class of examples of coherent topoi.

Definition 8. Let $\mathcal{C}$ be a category which admits finite limits. We say that a Grothendieck topology on $\mathcal{C}$ is *finitary* if, for every covering $\{U_i \to X\}_{i \in I}$ in $\mathcal{C}$, there exists a finite subset $I_0 \subseteq I$ such that $\{U_i \to X\}_{i \in I_0}$ is also a covering.

Proposition 9. *Let $\mathcal{C}$ be a small category which admits finite limits which is equipped with a finitary Grothendieck topology. Then the topos* $\mathrm{Shv}(\mathcal{C})$ *is coherent.*

Proof. Let $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Shv}(\mathcal{C})$ be the sheafification functor, and let $\mathcal{U}$ be the collection of all objects of the form Lh_C for $C \in \mathcal{C}$. We claim that $\mathcal{U}$ satisfies the requirements of Definition 6. We saw in the previous lecture that $\mathcal{U}$ generates $\mathrm{Shv}(\mathcal{C})$. Moreover, since the functor L preserves finite limits, the construction $C \mapsto Lh_C$ preserves finite limits. It follows that $\mathcal{U}$ is closed under finite products. It will therefore suffice to show that for each object $C \in \mathcal{C}$, the sheaf Lh_C is quasi-compact and quasi-separated. We first verify quasi-compactness. Choose a covering $\{\mathscr{F}_i \to Lh_C\}_{i \in I}$. Note that the identity map $\mathrm{id}_C : C \to C$ determines a section $s \in (Lh_C)(C)$. It follows that there exists a covering $\{C_j \to C\}_{j \in J}$ in the category $\mathcal{C}$ such that, for each $j \in J$, the image $s_j \in (Lh_C)(C_j)$ of s can be lifted to an element $\widetilde{s}_j \in \mathscr{F}_{i_j}(C_j)$ for some $i_j \in I$. Since the topology on $\mathcal{C}$ is finitary, we may assume without loss of generality that J is finite. Setting $I_0 = \{i_j\}_{j \in J} \subseteq I$, we deduce that $\{\mathscr{F}_i \to Lh_C\}_{i \in I_0}$ is a finite subcover of $\{\mathscr{F}_i \to Lh_C\}_{i \in I}$.

We now argue that each Lh_C is quasi-separated. Choose quasi-compact objects $\mathscr{F}, \mathscr{G} \in \mathrm{Shv}(\mathcal{C})$ equipped with maps $\mathscr{F} \to Lh_C \leftarrow \mathscr{G}$; we wish to show that the fiber product $\mathscr{F} \times_{Lh_C} \mathscr{G}$ is quasi-compact. Note that $\mathscr{F}$ admits a covering $\{\mathscr{F}_i \to \mathscr{F}\}_{i \in I}$, where each $\mathscr{F}_i$ belongs to $\mathcal{U}$. Since $\mathscr{F}$ is quasi-compact, we may assume that I is finite. Then $\{\mathscr{F}_i \times_{Lh_C} \mathscr{G}\}_{i \in I}$ is a finite covering of $\mathscr{F} \times_{Lh_C} \mathscr{G}$. It will therefore suffice to show that each $\mathscr{F}_i \times_{Lh_C} \mathscr{G}$ is quasi-compact. Replacing $\mathscr{F}$ by $\mathscr{F}_i$, we are reduced to the case where $\mathscr{F}$ has the form Lh_D for some object $D \in \mathcal{C}$. In this case, the map $\mathscr{F} \to Lh_C$ can be identified with an element of $(Lh_C)(D)$. Passing to a covering of D (which we can also replace by a finite subcover), we may assume that this element lies in the image of the map $h_C(D) \to (Lh_C)(D)$. In other words, we may assume that the map $\mathscr{F} \to Lh_C$ arises from applying the functor $Lh_\bullet$ to a morphism $D \to C$ in the category $\mathcal{C}$. Similarly, we may assume that the map $\mathscr{G} \to Lh_C$ arises by applying $Lh_\bullet$ to a morphism $E \to C$ in $\mathcal{C}$. In this case, the fiber product $\mathscr{F} \times_{Lh_C} \mathscr{G}$ can be identified with $Lh_{D \times_C E}$, which is quasi-compact by the first part of the argument. $\square$

Our next goal is to show that every coherent topos $\mathcal{X}$ arises from the construction of Proposition 9 in a canonical (though not unique) fashion.

Definition 10. Let $\mathcal{X}$ be a coherent topos. We will say that an object $X \in \mathcal{X}$ is *coherent* if it is quasi-compact and quasi-separated.

It will be convenient to employ another characterization of the class of quasi-separated objects.

Lemma 11. *Let $\mathcal{X}$ be a coherent topos. Then an object $X \in \mathcal{C}$ is quasi-separated if and only if it satisfies the following condition:*

($*$) *For every quasi-compact object $U \in \mathcal{X}$ and every pair of morphisms $f, g : U \to X$, the equalizer* $\mathrm{Eq}(U \rightrightarrows X)$ *is quasi-compact.*

Proof. Suppose first that X is quasi-separated, and that we are given a pair of morphisms $f, g : U \to X$; we wish to show that the equalizer $\mathrm{Eq}(U \rightrightarrows X)$ is quasi-compact. Let $\mathcal{U}$ be as in Definition 6, and choose a covering $\{U_i \to U\}_{i \in I}$ where each U_i belongs to $\mathcal{U}$. Since U is quasi-compact, we can assume that this covering is finite. Then f and g induce maps $f_i, g_i : U_i \to X$, having an equalizer $\mathrm{Eq}(U_i \rightrightarrows X) \simeq \mathrm{Eq}(U \rightrightarrows X) \times_U U_i$. It follows that $\mathrm{Eq}(U \rightrightarrows X)$ admits a finite covering by the objects $\mathrm{Eq}(U_i \rightrightarrows X)$. It will therefore suffice to show that each $\mathrm{Eq}(U_i \rightrightarrows X)$ is quasi-compact. In other words, we may replace U by U_i and thereby reduce to the case where $U \in \mathcal{U}$.

Unwinding the definitions, we see that the equalizer $\mathrm{Eq}(U \rightrightarrows X)$ can be identified with the fiber product

$$(U \times_X U) \times_{U \times U} U.$$

Note that the product $U \times U$ belongs to $\mathcal{U}$, and is therefore quasi-separated. Since U and and $U \times_X U$ are quasi-compact (the second by virtue of our assumption that X is quasi-separated), it follows that $\mathrm{Eq}(U \rightrightarrows X)$ is also quasi-compact.

We now prove the converse. Assume that condition $(*)$ is satisfied; we wish to prove that X is quasi-separated. Choose quasi-compact objects $U, V \in \mathcal{X}$ equipped with maps $U \to X \leftarrow V$; we wish to show that the fiber product $U \times_X V$ is quasi-compact. Unwinding the definitions, we can identify $U \times_X V$ with the equalizer of a diagram $(U \times V) \rightrightarrows X$. The desired result now follows from $(*)$, since $U \times V$ is a compact object of $\mathcal{X}$ (Remark 7). $\square$

Lemma 12. *Let $\mathcal{X}$ be a coherent topos. Then the collection of quasi-separated objects of $\mathcal{X}$ is closed under finite products.*

Proof. Let X and Y be quasi-separated objects of $\mathcal{X}$. Suppose we are given a quasi-compact object $U \in \mathcal{X}$ and a pair of maps $f, g : U \to X \times Y$. Since X satisfies condition $(*)$ of Lemma 11, we deduce that the equalizer $\mathrm{Eq}(U \rightrightarrows X)$ is a quasi-compact subobject of U. Note that the equalizer $\mathrm{Eq}(U \rightrightarrows X \times Y)$ can be identified with the equalizer of a diagram

$$\mathrm{Eq}(U \rightrightarrows X) \rightrightarrows Y,$$

and is therefore also quasi-compact (since Y satisfies the condition $(*)$. $\square$

Lemma 13. *Let $\mathcal{X}$ be a coherent topos. Then the collection of coherent objects of $\mathcal{X}$ is closed under finite limits.*

Proof. It follows immediately from the definition that the final object of $\mathcal{X}$ is coherent. It will therefore suffice to show that the collection of coherent objects is closed under fiber products. Suppose we are given maps $X \to Y \leftarrow Z$, where X, Y, and Z are coherent. Then X and Z are quasi-compact and Y is quasi-separated, so the fiber product $X \times_Y Z$ is quasi-compact. Moreover, $X \times_Y Z$ is a subobject of the product $X \times Z$, which is quasi-separated by Lemma 12. It follows that $X \times_Y Z$ is also quasi-separated. $\square$

Lemma 14. *Let $\mathcal{X}$ be any topos and let $X \in \mathcal{X}$ be an object. Then* $\mathrm{Sub}(X)$ *is a set.*

Remark 15. Since a topos $\mathcal{X}$ is usually a large category, Lemma 14 is not automatic. The collection of isomorphism classes of objects of $\mathcal{X}$ will almost always be a proper class. However, Lemma 14 asserts that the collection of isomorphism classes of *subobjects* of a fixed object X is bounded in size.

Proof of Lemma 14. Let $\mathcal{U}$ be a set of generators for $\mathcal{X}$. Let $\mathrm{Sub}_0(X)$ denote the collection of all subobjects of X of the form $\mathrm{Im}(f)$, where $f : U \to X$ is a morphism with $U \in \mathcal{U}$. Since $\mathcal{U}$ is a set and $\mathrm{Hom}_{\mathcal{X}}(U, X)$ is a set for each $U \in \mathcal{U}$, the collection $\mathrm{Sub}_0(X)$ is a set. Note that every subobject $X_0 \subseteq X$ admits a covering $\{U_i \to X_0\}$, where each U_i belongs to $\mathcal{U}$. It follows that, as an element of $\mathrm{Sub}(X)$, we can identify X_0 with the least upper bound of $\{Y \in \mathrm{Sub}_0(X) : Y \subseteq X_0\}$. In particular, the construction

$$(X_0 \in \mathrm{Sub}(X)) \mapsto (\{Y \in \mathrm{Sub}_0(X) : Y \subseteq X_0\} \subseteq \mathrm{Sub}_0(X))$$

determines a monomorphism from $\mathrm{Sub}(X)$ to the power set of $\mathrm{Sub}_0(X)$, so that $\mathrm{Sub}(X)$ is also a set. $\square$

Lemma 16. *Let $\mathcal{X}$ be a topos and let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the full subcategory of $\mathcal{X}$ spanned by the quasi-compact objects. Then $\mathcal{X}_0$ is essentially small (that is, it is equivalent to a small category).*

Proof. Let $\mathcal{U}$ be a set of generators for $\mathcal{X}$. Enlarging $\mathcal{U}$ if necessary, we may assume that $\mathcal{U}$ is closed under finite coproducts. For each object $X \in \mathcal{X}$, we can choose a covering $\{U_i \to X\}$, where each U_i belongs to U. If X is quasi-compact, this covering admits a finite subcover. We can therefore choose a single object $U \in \mathcal{U}$ and an effective epimorphism $U \to X$. It follows that X can be identified with the coequalizer of a diagram $R \rightrightarrows U$, where $R = U \times_X U \subseteq U \times U$. Note that for each $U \in \mathcal{U}$, there is a bounded number of possibilities for what the equivalence relation R can be (since $\mathrm{Sub}(U \times U)$ is a set, by virtue of Lemma 14). $\square$

Theorem 17. *Let $\mathcal{X}$ be a coherent topos and let $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ be the full subcategory spanned by the coherent objects. Then:*

(a) *The category $\mathcal{X}_{\mathrm{coh}}$ admits a finitary Grothendieck topology, where a collection of morphisms $\{U_i \to X\}$ in $\mathcal{X}_{\mathrm{coh}}$ is a covering if and only if it is a covering in X.*

(b) *For each $X \in \mathcal{X}$, let $h_X : \mathcal{X}_{\mathrm{coh}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ denote the functor represented by X (given by $h_X(Y) = \mathrm{Hom}_{\mathcal{X}}(Y, X)$). Then the construction $X \mapsto h_X$ induces an equivalence of categories $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{X}_{\mathrm{coh}})$.*

Proof. Since $\mathcal{X}_{\mathrm{coh}}$ is an essentially small subcategory of $\mathcal{X}$ (Lemma 16) which is closed under finite limits (Lemma 13) and generates $\mathcal{X}$ (Definition 6), assertions (a) and (b) follow from Giraud's theorem (see Lecture 10). Note that the topology on $\mathcal{X}_{\mathrm{coh}}$ is finitary because every object of $\mathcal{X}_{\mathrm{coh}}$ is quasi-compact. $\square$

Lecture 12: Geometric Morphisms

February 21, 2018

Definition 1. Let $\mathcal{X}$ and $\mathcal{Y}$ be topoi. A *geometric morphism* from $\mathcal{X}$ to $\mathcal{Y}$ is a functor $f^* : \mathcal{Y} \to \mathcal{X}$ which preserves finite limits, effective epimorphisms, and (small) coproducts. We let $\mathrm{Fun}^*(\mathcal{Y}, \mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{Y}, \mathcal{X})$ spanned by the geometric morphisms from $\mathcal{X}$ to $\mathcal{Y}$.

Remark 2. The terminology of Definition 1 is motivated by topology. If $f : X \to Y$ is a continuous map of topological spaces, then pullback of sheaves determines a functor $f^* : \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$ satisfying the requirements of Definition 1. Under the mild assumption that Y is *sober*, this construction establishes a bijection between the set of continuous maps from X to Y and the set of isomorphism classes of geometric morphisms from the topos $\mathrm{Shv}(X)$ to the topos $\mathrm{Shv}(Y)$. We will return to this point later.

Definition 1 admits several reformulations:

Proposition 3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topoi and let $f^* : \mathcal{Y} \to \mathcal{X}$ be a functor which preserves finite limits. The following conditions are equivalent:*

(1) *The functor f^* is a geometric morphism from $\mathcal{X}$ to $\mathcal{Y}$: that is, it preserves effective epimorphisms and coproducts.*

(2) *The functor f^* preserves small colimits.*

(3) *The functor f^* admits a right adjoint.*

Sketch. The equivalence (2) $\Leftrightarrow$ (3) follows from the adjoint functor theorem and the implication (2) $\Rightarrow$ (1) is clear. We complete the proof by showing that (1) $\Rightarrow$ (2). Assume that f^* is a geometric morphism; we wish to show that f^* preserves small colimits. By general nonsense, all small colimits can be constructed from coproducts and coequalizers. It will therefore suffice to show that for every pair of morphisms $u, v : U \to Y$ in $\mathcal{Y}$, the canonical map $\mathrm{Coeq}(f^*U \rightrightarrows f^*Y) \to f^* \mathrm{Coeq}(U \rightrightarrows Y)$ is an isomorphism in $\mathcal{X}$.

Note that the canonical map $Y \to \mathrm{Coeq}(U \rightrightarrows Y)$ cannot factor through any proper subobject of Y, and is therefore an effective epimorphism. We therefore have $\mathrm{Coeq}(U \rightrightarrows Y) \simeq \mathrm{Coeq}(R \rightrightarrows Y)$, where $R \subseteq Y \times Y$ is the equivalence relation $Y \times_{\mathrm{Coeq}(U \rightrightarrows Y)} Y$. In this case, we can characterize R as the equivalence relation *generated by* U: that is, it is the smallest equivalence relation on Y which contains the image of the map $(u, v) : U \to Y \times Y$. Similarly, we can identify $\mathrm{Coeq}(f^*U \rightrightarrows f^*Y)$ with the quotient of f^*Y by the equivalence relation $R' \subseteq f^*(Y) \times f^*(Y)$ generated by f^*U. It will therefore suffice to show that $R' = f^*R$ (as subobjects of $f^*Y \times f^*Y$.

We now describe passage from the object U to the equivalence relation R. Note that R depends only on the image of the map $(u, v) : U \to Y \times Y$. We may therefore replace U by $\mathrm{Im}(u, v)$ and thereby reduce to the case where U is a subobject of $Y \times Y$. Let U^{op} denote the image of U under the automorphism of $Y \times Y$ given by swapping the factors. Replacing U by the join $U \vee U^{\mathrm{op}}$, we can reduce to the case where U is a *symmetric* relation on Y. For each $n \geq 0$, let V_n denote the n-fold fiber product

$$U_f \times_{Yg} U_f \times_{Yg} U_f \times_{Xg} \cdots_f \times_{Yg} U,$$

(so that $V_0 = Y$) and let U_n denote the image of V_n in the product $Y \times Y$ (under the maps from the outermost factors of U). The equivalence relation R can then be described as the join $\bigvee_{n \geq 0} U_n$ (Exercise: Check this.)

Note that the equivalence relation R was constructed from U by a combination of taking images of morphisms, fiber products, and joins of subobjects. By assumption, the functor $f^* : \mathcal{Y} \to \mathcal{X}$ is compatible with each of these operations. It follows that f^*R agrees with the equivalence relation R' generated by f^*U, as desired. $\square$

Remark 4. Let $\mathcal{X}$ and $\mathcal{Y}$ be topoi and let $f^* : \mathcal{Y} \to \mathcal{X}$ be a geometric morphism from $\mathcal{X}$ to $\mathcal{Y}$. Proposition 3 implies that f^* admits a right adjoint, which we will denote by $f_* : \mathcal{X} \to \mathcal{Y}$. In this situation, the functors f^* and f_* are equivalent pieces of data: either can be recovered (up to canonical isomorphism) from the other. We will sometimes abuse terminology by saying that f_* is a *geometric morphism* from $\mathcal{X}$ to $\mathcal{Y}$.

Our next goal is to understand how to compute the category $\mathrm{Fun}^*(\mathcal{Y}, \mathcal{X})$ of geometric morphisms in the case where $\mathcal{Y} = \mathrm{Shv}(\mathcal{C})$ is described as a category of sheaves.

Theorem 5. *Let $\mathcal{C}$ be a small category which admits finite limits which is equipped with a Grothendieck topology, and let $j : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ be the composition of the Yoneda embedding $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ with the sheafification functor $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{S}$. Then, for any topos $\mathcal{X}$, composition with j induces a fully faithful embedding*

$$\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X}),$$

whose essential image consists of those functors $f : \mathcal{C} \to \mathcal{X}$ which satisfy the following pair of conditions:

(*a*) *The functor f is left exact: that is, it preserves finite limits.*

(*b*) *For every covering $\{C_i \to C\}$ in the category $\mathcal{C}$, the maps $\{f(C_i) \to f(C)\}$ form a covering with respect to the canonical topology on $\mathcal{X}$.*

We begin by proving Theorem 5 in a special case, where we can ignore the topology on $\mathcal{C}$.

Exercise 6. Let $\mathcal{C}$ be a small category (which admits finite limits). Show that $\mathcal{C}$ admits a Grothendieck topology where the covering families $\{f_i : C_i \to C\}$ are those collections of maps where some f_i admits a section. We will refer to this Grothendieck topology as the *trivial topology* on $\mathcal{C}$.

Exercise 7. Let $\mathcal{C}$ be a small category which admits finite limits. Show that every presheaf on $\mathcal{C}$ is a sheaf with respect to the trivial topology. That is, if we equip $\mathcal{C}$ with the trivial topology, then we have $\mathrm{Shv}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$.

We first prove Theorem 5 in the case where the topology on $\mathcal{C}$ is trivial. We will use the following general fact about presheaf categories:

Proposition 8. *Let $\mathcal{C}$ be a small category and let $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ be the Yoneda embedding. Then h exhibits the $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ as the category freely generated by $\mathcal{C}$ under small colimits. In other words, if $\mathcal{X}$ is any category which admits small colimits and $\mathrm{LFun}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}), \mathcal{X})$ is the category of functors from $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ to $\mathcal{X}$ which preserve small colimits, then composition with h induces an equivalence of categories*

$$\mathrm{LFun}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}), \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X}).$$

Remark 9. The inverse functor $\theta : \mathrm{Fun}(\mathcal{C}, \mathcal{X}) \to \mathrm{LFun}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}), \mathcal{X})$ can be described explicitly as follows. Let $f : \mathcal{C} \to \mathcal{X}$ be any functor. Then every object $X \in \mathcal{X}$ determines a presheaf $\mathscr{F}_X$ on $\mathcal{C}$, given by the formula $\mathscr{F}_X(C) = \mathrm{Hom}_{\mathcal{X}}(f(C), X)$. Then $\theta(f) : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X}$ can be descrbed as the left adjoint of the functor

$$\mathcal{X} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \qquad X \mapsto \mathscr{F}_X .$$

Proof of Theorem 5 for a trivial topology. Let $\mathcal{C}$ be a category which admits finite limits and let $\mathcal{X}$ be a topos. By virtue of Proposition 8, composition with h induces an equivalence

$$\{\text{Functors } F : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X} \text{ preserving colimits}\} \to \{\text{All functors } f : \mathcal{C} \to \mathcal{X}\}$$

We wish to show that this restricts to an equivalence of categories

$$\mathrm{Fun}^*(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}), \mathcal{X}) \to \{\text{Left exact functors } f : \mathcal{C} \to \mathcal{X}\}$$

(note that condition (b) is of Theorem 5 is automatic when the Grothendieck topology on $\mathcal{C}$ is trivial). In other words, we wish to show that if $F : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X}$ is a functor which preserves small colimits, then F preserves finite limits if and only if the composition $F \circ h : \mathcal{C} \to \mathcal{X}$ preserves finite limits. The "only if" direction is clear (and does not require the assumption that $\mathcal{X}$ is a topos), since the Yoneda embedding $C \mapsto h_C$ preserves finite limits.

To prove the converse, assume that $f = F \circ h$ preserves finite limits. Let $\mathcal{D} \subseteq \mathcal{X}$ be a small full subcategory which contains a set of generators for $\mathcal{X}$, contains the image of f, and is closed under finite limits. Then the canonical topology on $\mathcal{X}$ determines a Grothendieck topology on $\mathcal{D}$, and we proved in Lecture 10 that $\mathcal{X}$ can be identified with the category of sheaves $\mathrm{Shv}(\mathcal{D})$. Let $h' : \mathcal{D} \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ be the Yoneda embedding of $\mathcal{D}$ and let $L : \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X}$ be the sheafification functor. Using Proposition 8, we see that the functor $h' \circ f : \mathcal{C} \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ admits an essentially unique extension to a functor $F' : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ which preserves small colimits. Moreover, the functor $L \circ F' : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X}$ preserves small colimits, and $L \circ F' \circ h \simeq L \circ h' \circ f \simeq f$. Using Proposition 8 again, we can identify $L \circ F'$ with F. We wish to show that F preserves finite limits. Since the sheafification functor L preserves finite limits, we are reduced to proving that F' preserves finite limits. Concretely, the functor F' is given by left Kan extension along the functor $f^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. In particular, if $\mathscr{F}$ is a presheaf on $\mathcal{C}$, then $F'(\mathscr{F})$ is the presheaf on $\mathcal{D}$ given by the formula

$$F'(\mathscr{F})(D) = \varinjlim_{D \to f(C)} \mathscr{F}(C).$$

We wish to show that, for each object $D \in \mathcal{D}$, the construction $\mathscr{F} \mapsto F'(\mathscr{F})(D)$ preserves finite limits. This follows from the observation that the colimit $\varinjlim_{D \to f(C)} \mathscr{F}(C)$ is indexed by a *filtered* diagram (in fact, the category $\mathcal{C}^{\mathrm{op}} \times_{\mathcal{D}^{\mathrm{op}}} (\mathcal{D}_{D/})^{\mathrm{op}}$ admits finite colimits, since $\mathcal{C}$ has finite limits which are preserved by the functor f). $\square$

Proof of Theorem 5 in general. Now suppose that $\mathcal{C}$ is a category which admits finite limits which is equipped with an arbitrary Grothendieck topology. Let $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Shv}(\mathcal{C})$ be the sheafification functor and let $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ be the Yoneda embedding. For any topos $\mathcal{X}$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X}) \arrow[r, "\circ j"] \arrow[d, "\circ L"] & \mathrm{Fun}(\mathcal{C}, \mathcal{X}) \arrow[d, "\mathrm{id}"] \\
\mathrm{Fun}^*(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}), \mathcal{X}) \arrow[r, "\circ h"] & \mathrm{Fun}(\mathcal{C}, \mathcal{X})
\end{tikzcd}$$

The first part of the proof shows that the lower horizontal map is a fully faithful embedding, whose essential image consists of those functors $f : \mathcal{C} \to \mathcal{X}$ which preserve finite limits. Moreover, the left vertical map is also fully faithful; its essential image consists of those geometric morphisms $F^* : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X}$ having the property that the right adjoint $F_* : \mathcal{X} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ factors through the full subcategory $\mathrm{Shv}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$. Using the description of F_* given in Remark 9, we see that the upper vertical map induces an equivalence from $\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X})$ to the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{X})$ spanned by those functors $f : \mathcal{C} \to \mathcal{S}\mathrm{et}$ which satisfy the following pair of conditions:

- (a) The functor f preserves finite limits.
- (b') For each object $X \in \mathcal{X}$, the functor $\mathscr{F}_X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ given by $\mathscr{F}_X(C) = \mathrm{Hom}_{\mathcal{X}}(f(C), X)$ is a sheaf.

Unwinding the definitions, we can rephrase (b') as follows:

(b'') For each object $X \in \mathcal{X}$ and each covering $\{C_i \to C\}$ in $\mathcal{C}$, the diagram

$$\mathrm{Hom}_{\mathcal{X}}(f(C), X) \to \prod_i \mathrm{Hom}_{\mathcal{X}}(f(C_i), X) \rightrightarrows \prod_{i,j} \mathrm{Hom}_{\mathcal{X}}(f(C_i \times_C C_j), X)$$

is an equalizer.

Allowing X to vary, we can rephrase this as:

(b''') For each covering $\{C_i \to C\}$ in $\mathcal{C}$, the diagram

$$\coprod_{i,j} f(C_i \times_C C_j) \rightrightarrows \coprod_i f(C_i) \to f(C)$$

is a coequalizer in $\mathcal{X}$.

Assuming (a), we can rewrite the diagram of (b''') as

$$\coprod_{i,j} f(C_i) \times_{f(C)} f(C_j) \rightrightarrows \coprod_i f(C_i) \to f(C),$$

so that (b''') is equivalent to condition (b) of Theorem 5. $\square$

Lecture 13: Elimination of Imaginaries

February 23, 2018

Let us now return to the discussion of coherent topoi from Lecture 11. Recall that, if $\mathcal{X}$ is a coherent topos, then we can identify $\mathcal{X}$ with $\mathrm{Shv}(\mathcal{X}_{\mathrm{coh}})$, where $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ is the full subcategory of coherent objects. We now note a few closure properties of this subcategory.

Lemma 1. *Let $\mathcal{X}$ be a coherent topos. Then the collection of coherent objects is closed under finite coproducts.*

Proof. Let $\{X_i\}_{i \in I}$ be a collection of coherent objects of $\mathcal{X}$ indexed by a finite set I, having coproduct $X = \coprod_{i \in I} X_i$. Then X is quasi-compact; we claim that it is also quasi-separated. Choose quasi-compact objects $U, V \in \mathcal{X}$ with maps $U \to X \leftarrow V$. For each $i \in I$, set $U_i = U \times_X X_i$ and $V_i = V \times_X X_i$. Then $U \times_X V$ can be identified with the coproduct $\coprod_{i \in I} U_i \times_{X_i} V_i$. Since U_i and V_i are quasi-compact and X_i is quasi-separated, the fiber product $U_i \times_{X_i} V_i$ is also quasi-compact. It follows that $U \times_X V$ is quasi-compact. $\square$

Lemma 2. *Let $\mathcal{X}$ be a coherent topos. Suppose that we are given an effective epimorphism $f : U \to X$ in $\mathcal{X}$. If U is coherent and the equivalence relation $U \times_X U$ is quasi-compact, then X is coherent.*

Proof. Since X is a quotient of U, it is quasi-compact. We will show that it is quasi-separated by verifying condition $(*)$ of Lecture 11. Suppose we are given a quasi-compact object Y and a pair of maps $g, g' : Y \to X$; we wish to show that the equalizer $\mathrm{Eq}(Y \rightrightarrows X)$ is quasi-compact. Choose an effective epimorphism $Y' \to (U \times U) \times_{X \times X} Y$, where Y' is quasi-compact. Then we have an effective epimorphism

$$\mathrm{Eq}(Y' \rightrightarrows X) \simeq Y' \times_Y \mathrm{Eq}(Y \rightrightarrows X) \to \mathrm{Eq}(Y \rightrightarrows X).$$

It will therefore suffice to show that $\mathrm{Eq}(Y' \rightrightarrows X)$ is quasi-compact. We may therefore replace Y by Y' and thereby reduce to the case where $g = f \circ \overline{g}$ and $g' = f \circ \overline{g}'$ for some pair of maps $\overline{g}, \overline{g}' : Y \to U$. In this case, we have a canonical isomorphism $\mathrm{Eq}(Y \rightrightarrows X) \simeq (Y \times_{U \times U} (U \times_X U))$. Since Y and $U \times_X U$ are quasi-compact and $U \times U$ is quasi-separated, it follows that $\mathrm{Eq}(Y \rightrightarrows X)$ is quasi-compact, as desired. $\square$

Proposition 3. *Let $\mathcal{X}$ be a coherent topos and let $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ be the full subcategory spanned by the coherent objects. Then $\mathcal{X}_{\mathrm{coh}}$ is an (essentially small) pretopos.*

Proof. We proved in Lecture 9 that $\mathcal{X}$ is a pretopos. In particular, it admits finite limits, finite coproducts, and every equivalence relation $R \subseteq U \times U$ can be obtained as the fiber product $U \times_X U$, for some effective epimorphism $U \to X$. We proved in Lecture 11 that the subcategory $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ is closed under the formation of finite limits, and Lemmas 1 and 2 guarantee that it is also closed finite coproducts, and quotients by equivalence relations. From this it is easy to see that $\mathcal{X}_{\mathrm{coh}}$ is also a pretopos (check this as an exercise), and we saw in Lecture 11 that it is essentially small. $\square$

Let $\mathcal{C}$ be a coherent category. Recall that $\mathcal{C}$ can be equipped with a Grothendieck topology, where a collection of morphisms $\{u_i : U_i \to X\}_{i \in I}$ is a covering if there exists a finite subset $I_0 \subseteq I$ such that $X = \bigvee_{i \in I_0} \mathrm{Im}(u_i)$. In Lecture 8, we saw that this Grothendieck topology is subcanonical: that is, the Yoneda embedding determines a functor $h : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$. Moreover, it is also finitary, so that $\mathrm{Shv}(\mathcal{C})$ is a coherent topos and the functor h takes values in the subcategory $\mathrm{Shv}(\mathcal{C})_{\mathrm{coh}} \subseteq \mathrm{Shv}(\mathcal{C})$ of coherent objects.

Exercise 4. Let $\mathcal{C}$ and $\mathcal{D}$ be coherent categories and let $f : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite limits. Show that the following conditions are equivalent:

(1) The functor f is a morphism of coherent categories: that is, it preserves effective epimorphisms and (finite) joins of subobjects.

(2) The functor f carries each covering $\{U_i \to X\}$ in $\mathcal{C}$ to a covering $\{f(U_i) \to f(X)\}_{i \in I}$ in $\mathcal{D}$.

Proposition 5. *Let $\mathcal{C}$ be a coherent category and let $h : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ be the Yoneda embedding. Then:*

(1) *Let $\mathscr{G}$ be a coherent object of $\mathrm{Shv}(\mathcal{C})$ and let $\mathscr{F} \subseteq \mathscr{G}$ be a coherent subobject. If $\mathscr{G}$ belongs to the essential image of h, then so does $\mathscr{F}$.*

(2) *If $\mathcal{C}$ is a pretopos, then the Yoneda embedding h induces an equivalence of categories $\mathcal{C} \to \mathrm{Shv}(\mathcal{C})_{\mathrm{coh}}$.*

Proof. We first prove (1). Assume that $\mathscr{G} = h_X$ for some object $X \in \mathcal{C}$. Let $\mathscr{F} \subseteq \mathscr{G}$ be a coherent subobject, and choose a covering $\{h_{U_i} \to \mathscr{F}\}_{i \in I}$ in $\mathrm{Shv}(\mathcal{C})$. Since $\mathscr{F}$ is quasi-compact, we can assume that I is finite. Note that each of the maps $h_{U_i} \to \mathscr{F}$ can be identified with a map from h_{U_i} to h_X, and therefore (by Yoneda's lemma) arises from a map $u_i : U_i \to X$ in the category $\mathcal{C}$. Since the category $\mathcal{C}$ is coherent, we can form the join $X_0 = \bigvee_{i \in I} \mathrm{Im}(u_i)$. Since the functor h preserves images and joins of subobjects (Exercise 4), it follows that $\mathscr{F} \simeq h_{X_0}$ belongs to the essential image of h.

We now prove (2). The Yoneda embedding $h : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ is a morphism of coherent categories (Exercise 4), and $\mathrm{Shv}(\mathcal{C})$ is a pretopos. If $\mathcal{C}$ is also a pretopos, then h preserves finite coproducts. Let $\mathscr{F} \in \mathrm{Shv}(\mathcal{C})$ be a coherent object, and choose a covering $\{h_{X_i} \to \mathscr{F}\}_{i \in I}$. Since $\mathscr{F}$ is quasi-compact, we can assume that I is finite. Setting $X = \coprod_i X_i$ (and noting that h preserves coproducts), we obtain an effective epimorphism $h_X \to \mathscr{F}$. Note that $h_X \times_{\mathscr{F}} h_X$ can be identified with a subobject of $h_{X \times X}$. Using (1), we can write $h_X \times_{\mathscr{F}} h_X = h_R$ for some subobject $R \subseteq X \times X$. Then R is an equivalence relation on X, and our assumption that $\mathcal{C}$ is a pretopos guarantees that we have $R = X \times_Y X$ for some effective epimorphism $X \to Y$. It then follows that $\mathscr{F} \simeq h_Y$ belongs to the essential image of h. $\square$

It follows from Proposition 5 that the datum of a coherent topos $\mathcal{X}$ is equivalent to the datum of an essentially small pretopos $\mathcal{C}$: from an essentially small pretopos $\mathcal{C}$ we can construct a coherent topos $\mathrm{Shv}(\mathcal{C})$, and from a coherent topos $\mathcal{X}$ we can extract an essentially small pretopos $\mathcal{X}_{\mathrm{coh}}$; these processes are mutually inverse to one another. Beware, however, that the 2-category of pretopoi (with maps given by morphisms of coherent categories) is not quite equivalent to the 2-category of coherent topoi (with maps given by geometric morphisms): see Corollary 7 below.

Proposition 6. *Let $\mathcal{C}$ be a small coherent category and let $\mathcal{X}$ be a topos. Then composition with the Yoneda embedding $h : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ induces a fully faithful embedding*

$$\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X}),$$

whose essential image consists of those functors $f : \mathcal{C} \to \mathcal{X}$ which are morphisms of coherent categories.

Proof. By virtue of the main result of Lecture 12, it will suffice to show that a functor $f : \mathcal{C} \to \mathcal{X}$ is a morphism of coherent categories if and only if it preserves finite limits and carries coverings in $\mathcal{C}$ to coverings in $\mathcal{X}$. This is a special case of Exercise 4.

$\square$

Corollary 7. *Let $\mathcal{C}$ be a small coherent category and let $\mathcal{D}$ be a small pretopos. Then the category $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D})$ of morphisms of coherent categories from $\mathcal{C}$ to $\mathcal{D}$ can be identified with the full subcategory of $\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathrm{Shv}(\mathcal{D}))$ spanned by those geometric morphisms $f^* : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ which carry coherent objects to coherent objects.*

Proof. Set $\mathrm{Shv}(\mathcal{D})$. Let us abuse notation by identifying $\mathcal{D}$ with the full subcategory $\mathrm{Shv}(\mathcal{D})_{\mathrm{coh}}$ of coherent objects of $\mathrm{Shv}(\mathcal{D})$ (Proposition 5). Then Proposition 6 supplies an equivalence $\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathrm{Shv}(\mathcal{D})) \simeq \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathrm{Shv}(\mathcal{D}))$. It will therefore suffice to show that a morphism of coherent categories $f : \mathcal{C} \to \mathrm{Shv}(\mathcal{D})$ sends each object of $\mathcal{C}$ into $\mathrm{Shv}(\mathcal{D})_{\mathrm{coh}}$ if and only if the induced geometric morphism $F : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ carries each coherent object of $\mathrm{Shv}(\mathcal{C})$ into $\mathrm{Shv}(\mathcal{D})_{\mathrm{coh}}$. The "if" direction is obvious; we leave the converse as an exercise. $\square$

Construction 8. Let $\mathcal{C}$ be a small coherent category. We let $\mathcal{C}_{\mathrm{eq}}$ denote the full subcategory of $\mathrm{Shv}(\mathcal{C})$ spanned by the coherent objects. Note that the Yoneda embedding $h : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ determines a morphism of coherent categories $h : \mathcal{C} \to \mathcal{C}_{\mathrm{eq}}$.

Proposition 9. *Let $\mathcal{C}$ be a small coherent category. Then the functor $h : \mathcal{C} \to \mathcal{C}_{\mathrm{eq}}$ exhibits $\mathcal{C}_{\mathrm{eq}}$ as a pretopos completion of $\mathcal{C}$, in the sense of Lecture 7.*

Proof. Let $\mathcal{D}$ be a pretopos; we wish to show that composition with h induces an equivalence $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}_{\mathrm{eq}}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D})$. Writing $\mathcal{D}$ as a filtered union of small pretopoi, we can reduce to the case where $\mathcal{D}$ is essentially small. Using Proposition 5, we can reduce to the case where $\mathcal{D} = \mathcal{Y}_{\mathrm{coh}}$, where $\mathcal{Y}$ is a coherent topos.

Set $\mathcal{X} = \mathrm{Shv}(\mathcal{C})$. Then $\mathcal{X}$ is a coherent topos, and can therefore be identified with the category of sheaves $\mathrm{Shv}(\mathcal{X}_{\mathrm{coh}}) = \mathrm{Shv}(\mathcal{C}_{\mathrm{eq}})$. Let $\mathrm{Fun}'(\mathcal{X}, \mathcal{Y})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}, \mathcal{Y})$ spanned by those functors which preserve small colimits, finite limits, and coherent objects. We have restriction functors

$$\mathrm{Fun}'(\mathcal{X}, \mathcal{Y}) \to \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}_{\mathrm{eq}}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D}).$$

It follows from Corollary 7 that the left map and the composite map are equivalences of categories, so the right map is an equivalence of categories as well. $\square$

We close this lecture with a (hopefully) illuminating example.

Definition 10. Let $\mathcal{C}$ be a category which admits finite limits. A *group object* of $\mathcal{C}$ is an object $G \in \mathcal{C}$ equipped with a map $m : G \times G \to G$ with the following property: for each object $C \in \mathcal{C}$, the induced multiplication

$$\mathrm{Hom}_{\mathcal{C}}(C, G) \times \mathrm{Hom}_{\mathcal{C}}(C, G) \simeq \mathrm{Hom}_{\mathcal{C}}(C, G \times G) \xrightarrow{m\circ} \mathrm{Hom}_{\mathcal{C}}(C, G)$$

endows $\mathrm{Hom}_{\mathcal{C}}(C, G)$ with the structure of a group. The collection of group objects of $\mathcal{C}$ forms a category, which we will denote by $\mathrm{Group}(\mathcal{C})$.

Example 11. In the case $\mathcal{C} = \mathcal{S}\mathrm{et}$, we will denote $\mathrm{Group}(\mathcal{C})$ simply by Group; this is the usual category of groups.

Remark 12. Let $\mathcal{C}$ be a category which admits finite limits and let G be a group object of $\mathcal{C}$. For every group $\Gamma \in \mathrm{Group}$, the construction

$$(C \in \mathcal{C}) \mapsto \mathrm{Hom}_{\mathrm{Group}}(\Gamma, \mathrm{Hom}_{\mathcal{C}}(C, G))$$

determines a functor from $\mathcal{C}^{\mathrm{op}}$ to the category of sets, which we will denote by G^{Γ}.

Note that if Γ is given by generators $\{x_i\}_{i \in I}$ and relations $\{u_j = v_j\}_{j \in J}$, then the presheaf G^{Γ} can be realized as an equalizer

$$G^{\Gamma} \to \prod_{i \in I} h_G \rightrightarrows \prod_{j \in J} h_G.$$

In particular, if Γ is finitely generated, then the presheaf G^{Γ} is representable by an object of $\mathcal{C}$; we will abuse notation by identifying this object with G^{Γ}.

Exercise 13. Let $\mathcal{C}$ be a category which admits finite limits and let $\mathrm{Group}_{\mathrm{fp}}$ denote the full subcategory of Group spanned by the finitely presented groups. Show that the construction

$$(G \in \mathrm{Group}(\mathcal{C})) \mapsto \{G^{\Gamma}\}_{\Gamma \in \mathrm{Group}_{\mathrm{fp}}}$$

induces a fully faithful embedding

$$\mathrm{Group}(\mathcal{C}) \to \mathrm{Fun}(\mathrm{Group}_{\mathrm{fp}}^{\mathrm{op}}, \mathcal{C}).$$

Moreover, the essential image of this embedding consists of those functors $\mathrm{Group}_{\mathrm{fp}}^{\mathrm{op}} \to \mathcal{C}$ which preserve finite limits.

Hint: to go the reverse direction, suppose we are given a functor $F : \mathrm{Group}_{\mathrm{fp}}^{\mathrm{op}} \to \mathcal{C}$ which preserves finite limits. Let $\langle x \rangle \simeq \mathbf{Z}$ be the free group on one generator and let $\langle x_0, x_1 \rangle$ be the free group on 2 generators. Set $G = F(\langle x \rangle)$, and let

$$m : G \times G \simeq F(\langle x_0 \rangle) \times F(\langle x_1 \rangle) \simeq F(\langle x_0, x_1 \rangle) \to F(\langle x \rangle) \simeq G$$

be the map obtained by applying F to the group homomorphism

$$\langle x \rangle \to \langle x_0, x_1 \rangle \qquad x \mapsto x_0 x_1.$$

Show that m exhibits G as a group object of $\mathcal{C}$, and that the construction $F \mapsto G$ is inverse to the construction $(G \in \mathrm{Group}(\mathcal{C})) \mapsto \{G^{\Gamma}\}_{\Gamma \in \mathrm{Group}_{\mathrm{fp}}}$.

Definition 14. Let $\mathcal{X} = \mathrm{Fun}(\mathrm{Group}_{\mathrm{fp}}, \mathcal{S}\mathrm{et})$ denote the category of presheaves on $\mathrm{Group}_{\mathrm{fp}}^{\mathrm{op}}$. We will refer to $\mathcal{X}$ as the *classifying topos of groups*.

For any topos $\mathcal{Y}$, let $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ be the category of geometric morphisms from $\mathcal{Y}$ to $\mathcal{X}$. The main result of Lecture 12 shows that composition with the Yoneda embedding $h : \mathrm{Group}_{\mathrm{fp}}^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(\mathrm{Group}_{\mathrm{fp}}, \mathcal{S}\mathrm{et}) = \mathcal{X}$ induces an equivalence of categories

$$\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathrm{Group}_{\mathrm{fp}}^{\mathrm{op}}, \mathcal{Y}) \simeq \mathrm{Group}(\mathcal{Y}).$$

In particular, the category of geometric morphisms from $\mathcal{S}\mathrm{et}$ to $\mathcal{X}$ can be identified with the category $\mathrm{Group} = \mathrm{Group}(\mathcal{S}\mathrm{et})$ of groups.

Note that the topos $\mathcal{X}$ is coherent. Applying Proposition 6 to the pretopos $\mathcal{X}_{\mathrm{coh}}$, we obtain equivalences

$$\mathrm{Mod}(\mathcal{X}_{\mathrm{coh}}) = \mathrm{Fun}^{\mathrm{coh}}(\mathcal{X}_{\mathrm{coh}}, \mathcal{S}\mathrm{et}) \simeq \mathrm{Fun}^*(\mathcal{X}, \mathcal{S}\mathrm{et}) \simeq \mathrm{Group}(\mathcal{S}\mathrm{et}) = \mathrm{Group}\,.$$

That is, $\mathcal{X}_{\mathrm{coh}}$ is a pretopos whose models are groups.

Remark 15. In the above discussion, we can replace groups by other mathematical structures of a formally similar nature: abelian groups, monoids, rings, commutative rings, Lie algebras, $\cdots$.

Lecture 14X: Pro-Objects

March 19, 2018

Let $\mathcal{C}$ and $\mathcal{D}$ be categories which admit finite limits. We let $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which are *left exact*: that is, which preserve finite limits.

Definition 1. Let $\mathcal{C}$ be an essentially small category which admits finite limits. We let $\mathrm{Pro}(\mathcal{C})$ denote the category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})^{\mathrm{op}}$. We will refer to the objects of $\mathrm{Pro}(\mathcal{C})$ as *pro-objects* of $\mathcal{C}$, and to $\mathrm{Pro}(\mathcal{C})$ as *the category of pro-objects of* $\mathcal{C}$.

Remark 2. Let $\mathcal{C}$ be a category which admits finite limits. For each object $C \in \mathcal{C}$, the functor $D \mapsto \mathrm{Hom}_{\mathcal{C}}(C, D)$ preserves finite limits, and can therefore be regarded as an object of $\mathrm{Pro}(\mathcal{C})$. The Yoneda embedding $C \mapsto \mathrm{Hom}_{\mathcal{C}}(C, \bullet)$ induces a fully faithful functor $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$. In what follows, we will generally abuse notation by identifying $\mathcal{C}$ with its essential image in $\mathrm{Pro}(\mathcal{C})$.

Remark 3. In the category of sets, the formation of finite limits commutes with filtered colimits. It follows that the full subcategory $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})$ is closed under filtered colimits in $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$. In particular, the category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})$ admits filtered colimits, so that $\mathrm{Pro}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})^{\mathrm{op}}$ admits filtered limits.

Example 4. Let $\mathcal{C}$ be a small category which admits finite limits. Suppose we are given a diagram $\{C_\alpha\}_{\alpha \in \mathcal{A}^{\mathrm{op}}}$ indexed by (the opposite of) a filtered category $\mathcal{A}$. Then we can also regard $\{C_\alpha\}$ as a diagram in the category $\mathrm{Pro}(\mathcal{C})$ (via the Yoneda embedding), where we can take the inverse limit. We will abuse notation by denoting this inverse limit also by $\{C_\alpha\}$. When viewed as a functor from $\mathcal{C}$ to the category of sets, it is given by the construction $D \mapsto \varinjlim_\alpha \mathrm{Hom}_{\mathcal{C}}(C_\alpha, D)$.

Remark 5. Let $\mathcal{C}$ be an essentially small category which admits finite limits, and let $F : \mathcal{C} \to \mathcal{S}\mathrm{et}$ be a functor. Then F has a canonical presentation

$$\varinjlim_{(C,\eta) \in \mathcal{A}} \mathrm{Hom}_{\mathcal{C}}(C, \bullet)$$

has a colimit of corepresentable functors, indexed by the category $\mathcal{A}$ whose objects are pairs (C, η) where $C \in \mathcal{C}$ and $\eta \in F(C)$, where

$$\mathrm{Hom}_{\mathcal{A}}((C, \eta), (C', \eta')) = \{f \in \mathrm{Hom}_{\mathcal{C}}(C', C) : F(f)(\eta') = \eta\}.$$

If the functor F preserves finite limits, then the category $\mathcal{A}$ is filtered. It follows that every object of $\mathrm{Pro}(\mathcal{C})$ has a (canonical) presentation as a filtered limit of objects of $\mathcal{C}$.

Remark 6. Let $\mathcal{C}$ be an essentially small category which admits finite limits. From the above discussion, we see that the category $\mathrm{Pro}(\mathcal{C})$ can be described more informally as follows:

- The objects of $\mathrm{Pro}(\mathcal{C})$ are diagrams $\{C_\alpha\}$ in $\mathcal{C}$, indexed by (the opposite of) a small filtered category.
- Given two such diagrams $\{C_\alpha\}$ and $\{D_\beta\}$, we have

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\{C_\alpha\}, \{D_\beta\}) = \varprojlim_\beta \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\{C_\alpha\}, D_\beta) = \varprojlim_\beta \varinjlim_\alpha \mathrm{Hom}_{\mathcal{C}}(C_\alpha, D_\beta).$$

Remark 7. The category $\mathrm{Pro}(\mathcal{C})$ can be characterized by a universal property. Let $\mathcal{D}$ be any category which admits small filtered limits, and let $\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ be the full subcategory of $\mathrm{Fun}(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ spanned by those functors which preserve small filtered limits. Then composition with the inclusion $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ induces an equivalence of categories $\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. In other words, every functor $f : \mathcal{C} \to \mathcal{D}$ admits an essentially unique extension to a functor $F : \mathrm{Pro}(\mathcal{C}) \to \mathcal{D}$ which preserves small filtered limits.

Let $\mathcal{C}$ be an essentially small category which admits finite limits and let $\mathcal{I}$ be any small category. Since $\mathrm{Pro}(\mathcal{C})$ admits small filtered limit, the functor category $\mathrm{Fun}(\mathcal{I}, \mathrm{Pro}(\mathcal{C}))$ also admits small filtered limits (which are computed pointwise). Consequently, the inclusion functor $\mathrm{Fun}(\mathcal{I}, \mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{I}, \mathrm{Pro}(\mathcal{C}))$ admits an essentially unique extension to a functor

$$\mathrm{Pro}(\mathrm{Fun}(\mathcal{I}, \mathcal{C})) \to \mathrm{Fun}(\mathcal{I}, \mathrm{Pro}(\mathcal{C}))$$

which preserves small filtered limits. We will use the following standard result:

Proposition 8. *Let $\mathcal{C}$ be an essentially small category which admits finite limits and let I be a finite poset. Then the map*

$$\mathrm{Pro}(\mathrm{Fun}(I, \mathcal{C})) \to \mathrm{Fun}(I, \mathrm{Pro}(\mathcal{C}))$$

is an equivalence of categories. In particular, every diagram $I \to \mathrm{Pro}(\mathcal{C})$ can be written as a filtered limit of diagrams $I \to \mathcal{C}$.

Example 9. Applying Proposition 8 in the case $I = \{0 < 1\}$, we see that every morphism $f : C \to D$ in $\mathrm{Pro}(\mathcal{C})$ can be obtained as the limit of a filtered diagram of morphisms $\{f_\alpha : C_\alpha \to D_\alpha\}$ between objects of $\mathcal{C}$.

Corollary 10. *Let $\mathcal{C}$ be an essentially small category which admits finite limits. Then the category* $\mathrm{Pro}(\mathcal{C})$ *admits finite limits. Moreover, the inclusion $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ preserves finite limits.*

Proof. Let $\{C_i\}_{i \in I}$ be a finite diagram in $\mathcal{C}$ having a limit $C \in \mathcal{C}$, and let $\{D_\alpha\}$ be a filtered diagram in $\mathcal{C}$ which we identify with an object of $\mathrm{Pro}(\mathcal{C})$. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\{D_\alpha\}, C) &\simeq \varinjlim_{\alpha} \mathrm{Hom}_{\mathcal{C}}(D_\alpha, C) \\
&\simeq \varinjlim_{\alpha} \varprojlim_{i} \mathrm{Hom}_{\mathcal{C}}(D_\alpha, C_i) \\
&\simeq \varprojlim_{i} \varinjlim_{\alpha} \mathrm{Hom}_{\mathcal{C}}(D_\alpha, C_i) \\
&\simeq \varprojlim_{i} \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\{D_\alpha\}, C).
\end{aligned}$$

where we have invoked the fact that filtered colimits commute with finite limits in the category of sets. This proves that the inclusion $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ preserves finite limits. In particular, $\mathrm{Pro}(\mathcal{C})$ has an initial object. To complete the proof, it will suffice to show that every diagram $C \to D \leftarrow E$ in $\mathrm{Pro}(\mathcal{C})$ admits a fiber product. Using Proposition 8, we can realize our diagram as a filtered limit of diagrams $\{C_\alpha \to D_\alpha \leftarrow E_\alpha\}$ in $\mathcal{C}$. Then the filtered diagram $\{C_\alpha \times_{D_\alpha} E_\alpha\}$ represents a fiber product $C \times_D E$ in the category $\mathrm{Pro}(\mathcal{C})$. $\square$

We will be particularly interested in studying $\mathrm{Pro}(\mathcal{C})$ in the case where $\mathcal{C}$ is a pretopos.

Proposition 11. *Let $\mathcal{C}$ be a category which admits finite limits. Assume that every morphism $f : X \to Z$ in $\mathcal{C}$ factors as a composition $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g is an effective epimorphism and h is a monomorphism. Then every morphism in* $\mathrm{Pro}(\mathcal{C})$ *factors as a composition $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g is an effective epimorphism and h is a monomorphism.*

Proof. Let $f : X \to Z$ be a morphism in $\mathrm{Pro}(\mathcal{C})$, which we can realize as a filtered limit of morphisms $\{f_\alpha : X_\alpha \to Z_\alpha\}$ in $\mathcal{C}$. Factor each f_α as a composition $X_\alpha \xrightarrow{g_\alpha} Y_\alpha \xrightarrow{h_\alpha} Z_\alpha$, where g_α is an effective epimorphism and h_α is a monomorphism. This factorization is functorial, so we can regard $Y = \{Y_\alpha\}$ as a pro-object of $\mathcal{C}$ equipped with morphisms $g : X \to Y$ and $h : Y \to Z$ with $f = h \circ g$. Note that $Y \times_Z Y \simeq \{Y_\alpha \times_{Z_\alpha} Y_\alpha\} \simeq \{Y_\alpha\} = Y$, so that h is a monomorphism in $\mathrm{Pro}(\mathcal{C})$. We will complete the proof by showing that g is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$. For this, we wish to show that for each object $C \in \mathrm{Pro}(\mathcal{C})$, the diagram

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(Y, C) \to \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, C) \rightrightarrows \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X \times_Y X, C)$$

is an equalizer. Writing C as a filtered limit of objects of $\mathcal{C}$, we can assume that $C \in \mathcal{C}$. In this case, the diagram above is given by a filtered colimit of diagrams

$$\mathrm{Hom}_{\mathcal{C}}(Y_\alpha, C) \to \mathrm{Hom}_{\mathcal{C}}(X_\alpha, C) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(X_\alpha \times_{Y_\alpha} X_\alpha, C).$$

We conclude by observing that each of these diagrams is an equalizer (since g_α is an effective epimorphism in $\mathcal{C}$), and the collection of equalizer diagrams in $\mathcal{S}\mathrm{et}$ is closed under filtered colimits. $\square$

Remark 12. The proof of Proposition 11 shows that a morphism $f : X \to Y$ in $\mathrm{Pro}(\mathcal{C})$ is a monomorphism (effective epimorphism) if and only if it can be realized as a filtered limit of morphisms $\{f_\alpha : X_\alpha \to Y_\alpha\}$ which are monomorphisms (effective epimorphisms) in $\mathcal{C}$.

Remark 13. In the situation of Proposition 11, suppose that the formation of images in $\mathcal{C}$ is compatible with pullback (or equivalently, the collection of effective epimorphisms is stable under pullback). Then the category $\mathrm{Pro}(\mathcal{C})$ has the same property: any diagram $X \xrightarrow{f} Y \xleftarrow{g} Z$ can be realized as a filtered limit of diagrams $\{X_\alpha \xrightarrow{f_\alpha} Y_\alpha \xleftarrow{g_\alpha} Z_\alpha\}$, in which case we have

$$\begin{aligned} \mathrm{Im}(X \times_Y Z \to Z) &\simeq \{\mathrm{Im}(X_\alpha \times_{Y_\alpha} Z_\alpha \to Z_\alpha) \\ &\simeq \{\mathrm{Im}(X_\alpha \to Y_\alpha) \times_{Y_\alpha} Z_\alpha\} \\ &\simeq \mathrm{Im}(X \to Y) \times_Y Z. \end{aligned}$$

Let $\mathcal{C}$ be an essentially small category which admits finite limits. Then $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})$ is closed under limits in $\mathcal{S}\mathrm{et}$, and therefore admits small limits. It follows that the category $\mathrm{Pro}(\mathcal{C})$ admits small colimits. Moreover, the inclusion functor $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ preserves all colimits which exist in $\mathcal{C}$ (this is immediate from the definitions).

Proposition 14. *Let $\mathcal{C}$ be an essentially small category which admits finite limits and finite coproducts. Then the category* $\mathrm{Pro}(\mathcal{C})$ *admits finite coproducts, given by the formula*

$$\{C_\alpha\} \coprod \{D_\beta\} = \{C_\alpha \coprod D_\beta\}.$$

Proof. It suffices to observe that for any object $E \in \mathcal{C}$, we have

$$\varinjlim_{\alpha,\beta} \mathrm{Hom}_{\mathcal{C}}(C_\alpha \coprod D_\beta, E) \simeq (\varinjlim_{\alpha} \mathrm{Hom}_{\mathcal{C}}(C_\alpha, E)) \times (\varinjlim_{\beta} \mathrm{Hom}_{\mathcal{C}}(D_\beta, E)).$$

$\square$

Given objects $C, D \in \mathrm{Pro}(\mathcal{C})$, we can use Proposition 8 to write $C = \{C_\alpha\}$ and $D = \{D_\alpha\}$ as limits of diagrams indexed by the same category. In this case, the coproduct $C \amalg D$ is given by $\{C_\alpha \amalg D_\alpha\}$.

Remark 15. In the situation of Proposition 14, suppose that the formation of coproducts in $\mathcal{C}$ is preserved by pullback. Then the same is true in $\mathrm{Pro}(\mathcal{C})$. Given morphisms $f : C \to X$, $g : D \to X$, and $h : Y \to X$ in $\mathrm{Pro}(\mathcal{C})$, we can apply Proposition 8 to realize f, g, and h as filtered limits of maps $f_\alpha : C_\alpha \to X_\alpha$, $g_\alpha : D_\alpha \to X_\alpha$, and $h_\alpha : Y_\alpha \to X_\alpha$ (indexed by the same category), so that both $(C \amalg D) \times_X Y$ and $(C \times_X Y) \amalg (D \times_X Y)$ are represented by the diagram

$$\{(C_\alpha \amalg D_\alpha) \times_{X_\alpha} Y_\alpha\} \simeq \{(C_\alpha \times_{X_\alpha} Y_\alpha) \amalg (D_\alpha \times_{X_\alpha} Y_\alpha)\}.$$

Remark 16. In the situation of Proposition 14, suppose that coproducts in $\mathcal{C}$ are disjoint. Then, for every pair of objects $C = \{C_\alpha\}$ and $D = \{D_\alpha\}$ in $\mathcal{C}$, we deduce that

$$C \times_{C \amalg D} D \simeq \{C_\alpha \amalg_{C_\alpha \amalg D_\alpha} D_\alpha\} = \{\emptyset\}$$

is an initial object of $\mathrm{Pro}(\mathcal{C})$: that is, coproducts are disjoint in $\mathrm{Pro}(\mathcal{C})$.

Combining the above results, we obtain the following:

Proposition 17. *Let $\mathcal{C}$ be an essentially small coherent category with disjoint coproducts (for example, a pretopos). Then* $\mathrm{Pro}(\mathcal{C})$ *is also a coherent category with disjoint coproducts.*

Warning 18. It is not true that if $\mathcal{C}$ is a pretopos, then $\mathrm{Pro}(\mathcal{C})$ is also a pretopos. For example, let $\mathcal{C}$ be the category of finite sets. Then the category $\mathrm{Pro}(\mathcal{C})$ of profinite sets can be identified with the category of Stone spaces: that is, the category whose objects are totally disconnected compact Hausdorff spaces, and whose morphisms are continuous maps. Let $C \in \mathrm{Pro}(\mathcal{C})$ be the Cantor set, which we identify with the collection of infinite sequences $(n_1, n_2, n_3, \ldots)$ where $n_i \in \{0, 1\}$. The construction

$$(n_1, n_2, n_3, \ldots) \mapsto \sum \frac{n_i}{2^i}$$

defines a continuous surjection $C \to [0, 1]$, and the fiber product $R = C \times_{[0,1]} C$ can be regarded as an equivalence relation on C in the category of Stone spaces. However, this equivalence relation is *not* effective: given any Stone space X, a continuous map $C \to X$ which equalizes the two projection maps $R \rightrightarrows C$ must factor through a continuous map $[0, 1] \to X$. Such a map is automatically constant (since X is totally disconnected), so that $C \times_X C = C \times C$ is larger than R.

Lecture 15X: Pro-Étale Sheaves

March 20, 2018

Throughout this lecture, we let $\mathcal{C}$ denote an essentially small coherent category with disjoint coproducts (for example, a small pretopos). In the previous lecture, we proved that $\mathrm{Pro}(\mathcal{C})$ is also a coherent category with disjioont coproducts. In particular, we can endow $\mathrm{Pro}(\mathcal{C})$ with a finitary Grothendieck topology, where a finite collection of morphisms $\{U_i \to X\}$ is a covering if the induced map $\coprod U_i \to X$ is an effective epimorphism. We let $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ denote the category of sheaves with respect to this topology.

Warning 1. The category $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ is *not* a topos (note that $\mathrm{Pro}(\mathcal{C})$ is not small).

Example 2. Let X be a quasi-compact and quasi-separated scheme, and let $\mathrm{Sch}_X^{\mathrm{et}}$ denote the category of quasi-compact, quasi-separated schemes U equipped with an étale map $U \to X$. Then $\mathrm{Sch}_X^{\mathrm{et}}$ is an essentially small coherent category, and $\mathrm{Shv}(\mathrm{Pro}(\mathrm{Sch}_X^{\mathrm{et}}))$ can be identified with the category of *pro-étale sheaves on X* introduced by Bhatt-Scholze.

Similarly, Scholze's category of pro-étale sheaves on a (quasi-compact, quasi-separated) perfectoid space X can be realized as $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$, where $\mathcal{C}$ is the category of (quasi-compact, quasi-separated) perfectoid spaces which are étale over X.

Our first goal is to understand the relationship of $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ with the topos $\mathrm{Shv}(\mathcal{C})$.

Proposition 3. *Let $\mathcal{C}$ be as above and let $\mathscr{F} : \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor. Then:*

(1) *If $\mathscr{F}$ is a sheaf on the category $\mathrm{Pro}(\mathcal{C})$, then the restriction $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf on $\mathcal{C}$.*

(2) *If $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf on $\mathcal{C}$ and the functor $\mathscr{F}$ commutes with filtered colimits, then $\mathscr{F}$ is a sheaf on $\mathrm{Pro}(\mathcal{C})$.*

Proof. We will prove (2) and leave (1) as an exercise for the reader. Assume that $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf and that $\mathscr{F}$ commutes with filtered colimits; we wish to show that $\mathscr{F}$ is a sheaf. For this, we must prove the following:

(a) The functor $\mathscr{F}$ carries finite coproducts in $\mathrm{Pro}(\mathcal{C})$ to products of sets.

(b) For each effective epimorphism $U \to X$ in $\mathrm{Pro}(\mathcal{C})$, the diagram

$$\mathscr{F}(X) \to \mathscr{F}(U) \rightrightarrows \mathscr{F}(U \times_X U)$$

is an equalizer.

We begin with (a). Suppose we are given a finite collection of objects $C_1, \ldots, C_n \in \mathrm{Pro}(\mathcal{C})$, each of which is the limit of a pro-system $\{C_{i,\alpha}\}$ in $\mathcal{C}$; without loss of generality, we may assume that each of these pro-systems is indexed by the same category. Then the coproduct $C_1 \amalg \cdots \amalg C_n$ is given by the limit of the pro-system $\{C_{1,\alpha} \amalg \cdots \amalg C_{n,\alpha}\}$. Since $\mathscr{F}$ carries filtered limits in $\mathrm{Pro}(\mathcal{C})$ to filtered colimits of sets, we are reduced to showing that the canonical map

$$\varinjlim_{\alpha} \mathscr{F}(C_{1,\alpha} \amalg \cdots \amalg C_{n,\alpha}) \to \prod_{1 \leq i \leq n} \varinjlim_{\alpha} \mathscr{F}(C_{i,\alpha})$$

is an isomorphism, which follows from the fact that filtered colimits of sets commute with products and our assumption that $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf.

We now prove (b). Let $f : U \to X$ be an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$. Then we can write f as the limit of a diagram $\{f_\alpha : U_\alpha \to X_\alpha$ of effective epimorphisms in $\mathcal{C}$. Using our assumption that $\mathscr{F}$ is compatible with filtered limits in $\mathrm{Pro}(\mathcal{C})$, we are reduced to showing that the diagram

$$\varinjlim_\alpha \mathscr{F}(X_\alpha) \to \varinjlim_\alpha \mathscr{F}(U_\alpha) \rightrightarrows \varinjlim_\alpha \mathscr{F}(U_\alpha \times_{X_\alpha} U_\alpha).$$

This follows from our assumption that $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf, since the collection of equalizer diagrams in $\mathcal{S}\mathrm{et}$ is closed under filtered colimits. $\square$

The universal property of $\mathrm{Pro}(\mathcal{C})$ implies that any presheaf $\mathscr{F}_0 \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ admits an essentially unique extension to a presheaf $\mathscr{F} \in \mathrm{Fun}(\mathrm{Pro}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ which preserves filtered colimits. It follows from Proposition 3 that $\mathscr{F}_0$ is a sheaf if and only if $\mathscr{F}$ is a sheaf. This proves the following:

Proposition 4. *Let* $\mathrm{Shv}_c(\mathrm{Pro}(\mathcal{C}))$ *denote the full subcategory of* $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ *consisting of those sheaves* $\mathscr{F} : \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ *which preserve filtered colimits. Then the restriction functor* $\mathscr{F} \mapsto \mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ *induces an equivalence of categories* $\mathrm{Shv}_c(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathcal{C})$.

Proposition 4 is the starting point of a strategy for understanding the topos $\mathrm{Shv}(\mathcal{C})$: its objects can also be understood as sheaves on the larger coherent category $\mathrm{Pro}(\mathcal{C})$, satisfying a certain continuity condition. This is convenient because $\mathrm{Pro}(\mathcal{C})$ contains many useful objects that do not belong to $\mathcal{C}$:

Definition 5. Recall that a *model* of $\mathcal{C}$ is a morphism of coherent categories $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$: that is, a functor which satisfies the following axioms:

(1) The functor M commutes with finite limits.

(2) The functor M carries effective epimorphisms in $\mathcal{C}$ to surjections of sets.

(3) The functor M preserves finite coproducts.

Let $\mathrm{Mod}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ spanned by the models of $\mathcal{C}$. By definition, $\mathrm{Pro}(\mathcal{C})$ is the opposite of the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ spanned by those functors which satisfy condition (1). We can therefore identify $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$ with a full subcategory of $\mathrm{Pro}(\mathcal{C})$. Note that objects of $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$ very rarely belong to $\mathcal{C}$ itself (regarded as a full subcategory of $\mathrm{Pro}(\mathcal{C})$ via the Yoneda embedding.

We will say that an object $M \in \mathrm{Pro}(\mathcal{C})$ is *weakly projective* if it satisfies conditions (1) and (2). We let $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Pro}(\mathcal{C})$ spanned by the weakly projective objects.

Example 6. Any model of $\mathcal{C}$ is weakly projective when viewed as an object of $\mathrm{Pro}(\mathcal{C})$. That is, we have inclusions

$$\mathrm{Mod}(\mathcal{C}) \subseteq \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}} \subseteq \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}).$$

Example 7. Suppose that $\mathcal{C}$ is the category of finite sets. Then every effective epimorphism in $\mathcal{C}$ admits a section, so condition (2) of Definition 5 is automatic: that is, we have $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) = \mathrm{Pro}(\mathcal{C})$.

Remark 8. By definition, an object $X \in \mathrm{Pro}(\mathcal{C})$ is weakly projective if and only if, for every effective epimorphism $C \to D$ in $\mathcal{C}$, the map $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, C) \to \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, D)$ is surjective: that is, every map from X to D factors through C. It follows that $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ is closed under (possibly infinite) coproducts in $\mathrm{Pro}(\mathcal{C})$.

Beware that the map $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, C) \to \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, D)$ is generally *not* surjective if we assume only that $C \to D$ is an effective epimorphism in $\mathcal{C}$ (this is the motivation for the using the modifier "weakly" to describe the condition of Definition 4).

Remark 9. The full subcategory $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$ is closed under filtered inverse limits (since the collection of surjections in $\mathcal{S}\mathrm{et}$ is closed under filtered direct limits).

The following result allows us to "resolve" any object of $\mathrm{Pro}(\mathcal{C})$ by weakly projective objects:

Proposition 10. *For every object $X \in \mathrm{Pro}(\mathcal{C})$, there exists an effective epimorphism $\rho_X : \lambda(X) \to X$ in $\mathrm{Pro}(\mathcal{C})$ where $\lambda(X)$ is weakly projective. Moreover, we can arrange that $\lambda(X)$ is a functor of X, that ρ_X is a natural transformation of functors, and that the functor λ commutes with filtered limits.*

Proof. We use the small object argument of Quillen. Let $\{C_i \to D_i\}_{i \in I}$ be a set of representatives for all isomorphism classes of effective epimorphisms in $\mathcal{C}$. For each object $X \in \mathrm{Pro}(\mathcal{C})$, set

$$C(X) = \prod_{i \in I} \prod_{\eta \in \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, D_i)} C_i \qquad D(X) = \prod_{i \in I} \prod_{\eta \in \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, D_i)} D_i,$$

where both products are formed in the category $\mathrm{Pro}(\mathcal{C})$. We have a tautological map $X \to D(X)$; we define $\lambda_1(X) = C(X) \times_{D(X)} X$. Note that that there is a projection map $\lambda_1(X) \to X$ in $\mathrm{Pro}(\mathcal{C})$, which is easily seen to be an effective epimorphism. For $n > 1$, we define $\lambda_n(X)$ by the formula $\lambda_n(X) = \lambda_1(\lambda_{n-1}(X))$, so that we have an inverse system

$$\cdots \to \lambda_3(X) \to \lambda_2(X) \to \lambda_1(X) \to X.$$

Set $\lambda(X) = \varprojlim \lambda_n(X)$. Note that each map $f : \lambda(X) \to D_i$ factors through $f_n : \lambda_n(X) \to D_i$ for some $n \gg 0$. By construction, the composite map $\lambda_{n+1}(X) \to \lambda_n(X) \xrightarrow{f_n} D_i$ factors through C_i, so that $f : \lambda(X) \to D_i$ factors through C_i. It follows that $\lambda(X)$ is weakly projective. By inspection, the construction of $\lambda(X)$ (and the projection map $\lambda(X) \to X$) is functorial in X and commutes with filtered limits. $\square$

We will say that a collection of morphisms $\{U_i \to X\}_{i \in I}$ in $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ is a *covering* if it is a covering in $\mathrm{Pro}(\mathcal{C})$: that is, if there is a finite subset $I_0 \subseteq I$ such that $\coprod_{i \in I_0} U_i \to X$ is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$ (note that in this case, $\coprod_{i \in I_0} U_i$ is also weakly projective). This determines a Grothendieck topology on the category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$.

Warning 11. In Lecture 8, we defined the notion of a *Grothendieck topology* on a category $\mathcal{E}$ under the assumption that $\mathcal{E}$ admits finite limits. In general, the category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ need not admit finite limits. In such cases, we must replace condition $(T1)$ appearing in Lecture 8 with the following:

$(T1')$ For every covering $\{U_i \to X\}$ in $\mathcal{E}$ and every morphism $Y \to X$ in $\mathcal{E}$, there exists a covering $\{V_j \to Y\}$ for which each of the maps $V_j \to Y \to X$ factors through some U_i.

We also need to revise the notion of sheaf. A functor $\mathscr{F} : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is said to be a sheaf if, for every covering $\{U_i \to X\}$ in $\mathcal{E}$, the canonical map

$$\mathscr{F}(X) \to \varprojlim \mathscr{F}(U)$$

is a bijection, where the limit is taken over the sieve on X generated by the objects U_i (see Definition 13 of Lecture 9).

Example 12. Let $\mathcal{C}$ be the category of finite sets. Then $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) = \mathrm{Pro}(\mathcal{C})$ can be identified with the category of Stone spaces. The preceding topology can be described as follows: a finite collection of maps of Stone spaces $\{Y_i \to X\}$ is a covering if and only if the induced map $\coprod Y_i \to X$ is surjective.

Proposition 13. *The construction $\mathscr{F} \mapsto \mathscr{F}|_{\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}}}$ induces an equivalence of categories $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}))$. Moreover, a sheaf $\mathscr{F} : \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ commutes with filtered colimits if and only if $\mathscr{F}|_{\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}}}$ commutes with filtered colimits.*

Proof. Let $\mathscr{F} \in \mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$. For each object $X \in \mathrm{Pro}(\mathcal{C})$, let $\lambda(X)$ be defined as in Proposition 11, and set $\mu(X) = \lambda(\lambda(X) \times_X \lambda(X))$. We then have an equalizer diagram

$$\mathscr{F}(X) \to \mathscr{F}(\lambda(X)) \rightrightarrows \mathscr{F}(\mu(X)),$$

so that we can functorially recover $\mathscr{F}(X)$ from the values of $\mathscr{F}$ on weakly projective objects. This gives an explicit left inverse to the restriction functor

$$\mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})) \qquad \mathscr{F} \mapsto \mathscr{F}\,|_{\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}}};$$

we leave it to the reader to verify that it is a right inverse as well.

It is clear that if $\mathscr{F}$ commutes with filtered colimits, then so does the restriction $\mathscr{F}\,|_{\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}}}$. The converse follows from the formula

$$\mathscr{F}(X) = \mathrm{Eq}(\mathscr{F}(\lambda(X)) \rightrightarrows \mathscr{F}(\mu(X))),$$

since the constructions $X \mapsto \lambda(X)$ and $X \mapsto \mu(X)$ both preserve filtered inverse limits (as functors from $\mathrm{Pro}(\mathcal{C})$ to itself). □

Corollary 14. *Let* $\mathrm{Shv}_c(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}))$ *be the full subcategory of* $\mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}))$ *spanned by those sheaves* $\mathscr{F} : \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ *which preserve filtered colimits. Then there is a canonical equvialence of categories* $\mathrm{Shv}(\mathcal{C}) \simeq \mathrm{Shv}_c(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}))$.

Proof. Combine Propositions 14 and 4. □

Lecture 16X: X-Models

March 22, 2018

As we have seen, topos theory provides a simultaneous generalization of point-set topology and first-order logic. We now take advantage of this.

Definition 1. Let $\mathcal{C}$ be an essentially small pretopos, which we regard as fixed throughout this lecture. Let X be a topological space. An *X-model of* $\mathcal{C}$ is a geometric morphism of topoi from $\mathrm{Shv}(X)$ to $\mathrm{Shv}(\mathcal{C})$.

Let us unwind Definition 1. As we saw in Lecture 13, giving a geometric morphism from $\mathrm{Shv}(X)$ to $\mathrm{Shv}(\mathcal{C})$ is equivalent to giving a functor $\mathcal{O}_X : \mathcal{C} \to \mathrm{Shv}(X)$ which preserves finite limits, effective epimorphisms, and finite coproducts. We will denote the value of $\mathcal{O}_X$ on an object $C \in \mathcal{C}$ by $\mathcal{O}_X^C$. Here $\mathcal{O}_X^C$ denotes a sheaf of sets on X which we can evaluate on an open subset $U \subseteq X$ to obtain a set $\mathcal{O}_X^C(U)$. We can also take the *stalk* of $\mathcal{O}_X^C$ at a point $x \in X$, to obtain a set $\mathcal{O}_{X,x}^C = \varinjlim_{x \in U} \mathcal{O}_X^C(U)$. We can therefore reformulate Definition 1 as follows:

Definition 2. Let X be a topological space. An *X-model of* $\mathcal{C}$ is a functor

$$\mathcal{O}_X : \mathcal{C} \times \mathcal{U}(X)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$$

with the following properties:

(a) For each $C \in \mathcal{C}$, the functor $U \mapsto \mathcal{O}_X^C(U)$ is a sheaf of sets on X. Given an element $s \in \mathcal{O}_X^C(U)$ and an open subset $V \subseteq U$, we let $s|_V$ denote the image of s in $\mathcal{O}_X^C(V)$.

(b) For each open set $U \subseteq X$, the functor $C \mapsto \mathcal{O}_X^C(U)$ preserves finite limits.

(c) Given an effective epimorphism $C' \to C$ in the pretopos $\mathcal{C}$, the induced map $\mathcal{O}_X^{C'} \to \mathcal{O}_X^C$ is an effective epimorphism of sheaves. That is, given a section $s \in \mathcal{O}_X^C(U)$ for some open set $U \subseteq X$, we can choose an open covering $\{U_\alpha\}$ of U such that each $s|_{U_\alpha}$ can be lifted to an element of $\mathcal{O}_X^{C'}(U_\alpha)$.

(d) For every finite collection of objects $\{C_i\}_{i \in I}$ in $\mathcal{C}$ having coproduct C, we can regard $\mathcal{O}_X^C$ as the coproduct of $\mathcal{O}_X^{C_i}$ in the category $\mathrm{Shv}(X)$. In other words, giving an element $s \in \mathcal{O}_X^C(U)$ is equivalent to giving a decomposition $U = \amalg_{i \in I} U_i$ and a collection of elements $s_i \in \mathcal{O}_X^{C_i}(U_i)$.

Remark 3. In the situation of Definition 2, conditions (b), (c) and (d) can also be formulated stalkwise as follows:

(b') For every point $x \in X$, the functor $C \mapsto \mathcal{O}_{X,x}^C$ preserves finite limits.

(c') For every point $x \in X$, the functor $C \mapsto \mathcal{O}_{X,x}^C$ carries effective epimorphisms in $\mathcal{C}$ to surjections of sets.

(d') For every point $x \in X$, the functor $C \mapsto \mathcal{O}_{X,x}^C$ preserves finite coproducts.

Together, these conditions assert that for each point $x \in X$, the functor $C \mapsto \mathcal{O}_{X,x}^C$ is a model of $\mathcal{C}$. We will denote this model by $\mathcal{O}_{X,x}$.

Roughly speaking, we can think of an X-model $\mathcal{O}_X$ of $\mathcal{C}$ as a family of models $\{\mathcal{O}_{X,x}\}_{x \in X}$ of $\mathcal{C}$, which in some sense depend *continuously* on the point $x \in X$.

The following observation will be useful for verifying condition (d) of Definition 2:

Lemma 4. *Let $\mathcal{C}$ and $\mathcal{D}$ be pretopoi and let $f : \mathcal{C} \to \mathcal{D}$ be a left exact functor. The following conditions are equivalent:*

(1) *The functor f preserves finite coproducts.*

(2) *The functor f preserves initial objects, and the canonical map $f(\mathbf{1}) \amalg f(\mathbf{1}) \to f(\mathbf{1} \amalg \mathbf{1})$ is an isomorphism.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Conversely, suppose that (2) is satisfied. To show that f preserves finite coproducts. Since f preserves empty coproducts, it will suffice to show that for every pair of objects $C, D \in \mathcal{C}$, the canonical map $f(C) \amalg f(D) \to f(C \amalg D)$ is an equivalence. Note that we have a canonical map $f(C \amalg D) \to f(\mathbf{1} \amalg \mathbf{1})$. Using condition (2), we are reduced to showing that the maps

$$(f(C) \amalg f(D)) \times_{f(\mathbf{1} \amalg \mathbf{1})} f(\mathbf{1}) \to (f(C \amalg D)) \times_{f(\mathbf{1} \amalg \mathbf{1})} f(\mathbf{1})$$

are isomorphisms (for each of the summand inclusions $\mathbf{1} \to \mathbf{1} \amalg \mathbf{1}$.). By symmetry, it suffices to treat the case of the inclusion of the first summand. Using the assumption that coproducts are pullback-stable in $\mathcal{D}$ and that f preserves finite limits, we can rewrite this map as

$$f(C \times_{\mathbf{1} \amalg \mathbf{1}} \mathbf{1}) \amalg f(D \times_{\mathbf{1} \amalg \mathbf{1}} \mathbf{1}) \to f((C \amalg D) \times_{\mathbf{1} \amalg \mathbf{1}} \mathbf{1}).$$

Using the disjointness of coproducts in $\mathcal{C}$, we can further rewrite this as

$$f(C) \amalg f(\emptyset) \to f(C),$$

which is an isomorphism since f preserves initial objects. $\square$

Remark 5 (Functoriality)**.** Let $f : X \to Y$ be a continuous map of topological spaces and let $\mathcal{O}_Y$ be a Y-model of $\mathcal{C}$. Then we can construct an X-model $f^* \mathcal{O}_Y$ of $\mathcal{C}$, given concretely by the formula $(f^* \mathcal{O}_Y)^C = f^* \mathcal{O}_Y^C$.

Note that, if we think of $\mathcal{O}_Y$ as encoding the data of a geometric morphism of topoi $\mathrm{Shv}(Y) \to \mathrm{Shv}(\mathcal{C})$, then $f^* \mathcal{O}_Y$ simply encodes the composite geometric morphism $\mathrm{Shv}(X) \xrightarrow{f} \mathrm{Shv}(Y) \to \mathrm{Shv}(\mathcal{C})$. Note also that the stalk of $(f^* \mathcal{O}_Y)$ at a point $x \in X$ is given by $\mathcal{O}_{Y,f(x)}$

Definition 6. We define a category $\mathrm{Top}_{\mathcal{C}}$ as follows:

- An object of $\mathrm{Top}_{\mathcal{C}}$ consists of a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is an X-model of $\mathcal{C}$.
- A morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ consists of a continuous map of topological spaces $f : X \to Y$ together with a natural transformation of functors $f^* \mathcal{O}_Y \to \mathcal{O}_X$.

Example 7. Let $\mathcal{C} = \mathcal{S}\mathrm{et}_{\mathrm{fin}}$ be the category of finite sets. Then, for every topological space X, there is an essentially unique X-model of $\mathcal{C}$, given by the formula $\mathcal{O}_X^S = \underline{S}$ (here $\underline{S}$ denotes the constant sheaf associated to the finite set S). The construction $(X, \mathcal{O}_X) \mapsto X$ induces an equivalence from the category $\mathrm{Top}_{\mathcal{C}}$ of Definition 6 to the category Top of topological spaces.

Example 8. Let $\mathcal{C}$ be the category of coherent objects of the classifying topos of commutative rings, given by $\mathrm{Fun}(\{\text{Finitely presented commutative rings}\}, \mathcal{S}\mathrm{et})$. Then the datum of an X-model of $\mathcal{C}$ is equivalent to the datum of a sheaf of commutative rings on $\mathcal{C}$, and $\mathrm{Top}_{\mathcal{C}}$ is equivalent to the category of ringed spaces.

Proposition 9. *Let X be a topological space and let $\mathcal{O}_X$ be an X-model of $\mathcal{C}$. We let $\Gamma(X; \mathcal{O}_X) : \mathcal{C} \to \mathcal{S}\mathrm{et}$ denote the functor given by the construction*

$$(C \in \mathcal{C}) \mapsto (\mathcal{O}_X^C(X) \in \mathcal{S}\mathrm{et}).$$

Then:

(1) *The functor $\Gamma(X; \mathcal{O}_X)$ preserves finite limits, and can therefore be regarded as a pro-object of $\mathcal{C}$.*

(2) *If X is a Stone space, then the functor $\Gamma(X; \mathcal{O}_X)$ also preserves effective epimorphisms, and is therefore* weakly projective *as a pro-object of $\mathcal{C}$.*

Proof. Assertion (1) is immediate from part (b) of Definition 2. To prove (2), we note that if $C' \to C$ is an effective epimorphism in $\mathcal{C}$, then part (c) of Definition 2 guarantees that the induced map $\mathcal{O}_X^{C'} \to \mathcal{O}_X^C$ is an effective epimorphism of *sheaves*. In particular, for every global section $s \in \mathcal{O}_X^C(X)$, we can choose an open covering $\{U_\alpha\}$ of X such that each $s|_{U_\alpha}$ lifts to an element $\widetilde{s}_\alpha \in \mathcal{O}_X^{C'}(U_\alpha)$. If X is a Stone space, then the open covering $\{U_\alpha\}$ can be refined to an open covering by *disjoint* open subsets of X, in which case we can amalgamate the sections $\widetilde{s}_\alpha$ to a single element $\widetilde{s} \in \mathcal{O}_X^{C'}(X)$ lying over s. □

Note that if $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism in the category $\mathrm{Top}_{\mathcal{C}}$, then we have canonical maps

$$\Gamma(Y; \mathcal{O}_Y)(C) = \mathcal{O}_Y^C(Y) \to (f^* \mathcal{O}_Y^C)(X) \to \mathcal{O}_X^C(X) = \Gamma(X; \mathcal{O}_X)(C),$$

depending functorially on C. This determines a natural transformation of functors from $\Gamma(Y; \mathcal{O}_Y)$ to $\Gamma(X; \mathcal{O}_X)$, or equivalently a morphism

$$\Gamma(X; \mathcal{O}_X) \to \Gamma(Y; \mathcal{O}_Y)$$

in the category $\mathrm{Pro}(\mathcal{C})$. In other words, we can regard the construction $(X, \mathcal{O}_X) \mapsto \Gamma(X; \mathcal{O}_X)$ as a functor

$$\Gamma : \mathrm{Top}_{\mathcal{C}} \to \mathrm{Pro}(\mathcal{C}).$$

Notation 10. We let $\mathrm{Stone}_{\mathcal{C}}$ denote the full subcategory of $\mathrm{Top}_{\mathcal{C}}$ spanned by those pairs $(X, \mathcal{O}_X)$ where X is a Stone space. It follows from Proposition 9 that the global sections functor restricts to a functor

$$\Gamma : \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}).$$

Theorem 11. *The functor $\Gamma : \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ is an equivalence of categories.*

Sketch. We sketch an explicit construction of an inverse to the functor Γ. First, recall that there is an essentially unique pretopos morphism $\iota : \mathcal{S}\mathrm{et}_{\mathrm{fin}} \to \mathcal{C}$, given by $\iota(S) = \coprod_{s \in S} \mathbf{1}$. Precomposition with ι determines a forgetful functor

$$\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}) = \mathrm{Pro}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}) = \mathrm{Stone},$$

where Stone is the category of Stone spaces. Explicitly, if M is an object of $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$, viewed as a functor $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ which preserves finite limits and effective epimorphisms, then this forgetful functor carries M to a Stone space X characterized by the existence of natural bijections $\mathrm{Hom}_{\mathrm{Top}}(X, S) \simeq M(\iota(S))$. For every integer $n \geq 0$, let $\mathbf{n}$ denote the coproduct $\mathbf{1} \amalg \cdots \amalg \mathbf{1}$ of n copies of the final object of $\mathcal{C}$. We can then restate our characterization of X as follows: for each n, we can identify $M(\mathbf{n})$ with the set of all n-tuples $(U_1, \ldots, U_n)$ of disjoint clopen subsets $U_1, \ldots, U_n \subseteq X$ satisfying $X = U_1 \amalg \cdots \amalg U_n$.

Let M and X be as above, and let $\mathcal{U}_0(X)$ denote the Boolean algebra of clopen subsets of X. For each object $C \in \mathcal{C}$ and each $U \in \mathcal{U}_0(X)$,

$$\mathcal{O}_X^C(U) = M(\mathbf{1} \amalg C) \times_{M(\mathbf{2})} \{(X - U, U)\}.$$

Note that if $V \subseteq U$ is a smaller clopen subset, then we have a pullback square

$$\begin{tikzcd} \mathbf{1} \amalg C \amalg C \arrow[r] \arrow[d] & \mathbf{1} \amalg C \arrow[d] \\ \mathbf{3} \arrow[r] & \mathbf{2} \end{tikzcd}$$

in $\mathcal{C}$, so that $\mathcal{O}_X^C(U)$ can also be identified with the set $M(\mathbf{1} \amalg C \amalg C) \times_{M(\mathbf{3})} \{(X - U, U - V, V)\}$. We therefore have a canonical map

$$\mathcal{O}_X^C(U) = M(\mathbf{1} \amalg C \amalg C) \times_{M(\mathbf{3})} \{(X - U, U - V, V)\} \to M(\mathbf{1} \amalg \mathbf{1} \amalg C) \times_{M(\mathbf{3})} \{(X - U, U - V, V)\} \simeq \mathcal{O}_X^C(V).$$

For fixed $C \in \mathcal{C}$, these maps endow $\mathcal{O}_X^C$ with the structure of a *presheaf of sets* on the poset $\mathcal{U}_0(X)$ (Exercise: check this.)

In fact, we claim that this presheaf is actually a sheaf. Since every open covering of a Stone space can be refined to a finite covering by disjoint clopen sets, it will suffice to prove the following:

- For $V \subseteq U$ as above, the restriction maps $\mathcal{O}_X^C(U) \to \mathcal{O}_X^C(V)$ and $\mathcal{O}_X^C(U) \to \mathcal{O}_X^C(U - V)$ induce a bijection $\mathcal{O}_X^C(U) \to \mathcal{O}_X^C(V) \times \mathcal{O}_X^C(U - V)$. This follows from the fact that the diagram of sets

$$\begin{tikzcd} M(\mathbf{1} \amalg C \amalg C) \arrow[r] \arrow[d] & M(\mathbf{1} \amalg \mathbf{1} \amalg C) \arrow[d] \\ M(\mathbf{1} \amalg C \amalg \mathbf{1}) \arrow[r] & M(\mathbf{1} \amalg \mathbf{1} \amalg \mathbf{1}) \end{tikzcd}$$

  is a pullback square (since M is left exact).

- When $U = \emptyset$, the set $\mathcal{O}_X^C(U)$ is a singleton. We leave this as an exercise.

It follows that that for each object $C \in \mathcal{C}$, the construction $(U \in \mathcal{U}_0(X)) \mapsto \mathcal{O}_X^C(U)$ admits an essentially unique extension to a sheaf of sets on X, which we will also denote by $\mathcal{O}_X^C$. We claim that the functor

$$\mathcal{O}_X : \mathcal{C} \to \mathrm{Shv}(X) \qquad C \mapsto \mathcal{O}_X^C$$

is an X-model of $\mathcal{C}$, in the sense of Definition 2. The verification of (a) was sketched above, and condition (b) follows from our assumption that M preserves finite limits. Since M preserves effective epimorphisms, an effective epimorphism $C' \to C$ induces a surjection $\mathcal{O}_X^{C'}(U) \to \mathcal{O}_X^C(U)$ for every *clopen* subset $U \subseteq X$, and therefore an effective epimorphism of sheaves $\mathcal{O}_X^{C'} \to \mathcal{O}_X^C$; this proves (c). To verify condition (d), we note that for every finite set S and each clopen subset $U \subseteq X$, we have

$$\begin{aligned} \mathcal{O}_X^{\iota S}(U) &= M(\mathbf{1} \amalg \iota(S)) \times_{M(\mathbf{1} \amalg \mathbf{1})} \{(X - U, U)\} \\ &= \{\text{Clopen decompositions } X = \textstyle\coprod_{s \in S \cup \{0\}} X_s \text{ with } X_0 = X - U \ \} \\ &= \{\text{Clopen decompositions } U = \textstyle\coprod_{s \in S} U_s \ \}. \end{aligned}$$

so that $\mathcal{O}_X^{\iota S}$ can be identified with the constant sheaf with value S. Condition (d) now follows from Lemma 4.

Summarizing the above discussion, from a functor $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ which preserves finite limits and effective epimorphisms, we can construct an object $(X, \mathcal{O}_X)$ in $\mathrm{Stone}_{\mathcal{C}}$. Note that for $C \in \mathcal{C}$, we have canonical isomorphisms

$$\begin{aligned} \Gamma(X; \mathcal{O}_X)(C) &= \mathcal{O}_X^C(X) \\ &= M(\mathbf{1} \amalg C) \times_{M(\mathbf{1} \amalg \mathbf{1})} \{(\emptyset, X)\} \\ &\simeq M(C). \end{aligned}$$

since the diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathbf{1} \amalg C \arrow[d] \\ \mathbf{1} \arrow[r] & \mathbf{1} \amalg \mathbf{1} \end{tikzcd}$$

is a pullback and $M(\mathbf{1})$ is a singleton. In other words, the construction $M \mapsto (X, \mathcal{O}_X)$ is right inverse to the functor $\Gamma : \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ (up to isomorphism).

Consider now the composition in the other direction. Let Y be a Stone space and $\mathcal{O}_Y$ a Y-model of $\mathcal{C}$, and suppose that we apply the above construction to the functor $M = \Gamma(Y, \mathcal{O}_Y)$. For every finite set S, we have $M(\iota(S)) = \mathcal{O}_Y^{\iota(S)}(Y) = \underline{S}(Y)$, where $\underline{S}$ is the constant sheaf on Y with the value S. The value of this sheaf on Y can be identified with the set of continuous maps $Y \to S$, functorially in S. It follows that the Stone space X constructed above is canonically homeomorphic to Y. Let us identify X with Y. If C is an object of $\mathcal{C}$ and U is a clopen subset of X, we have canonical bijections

$$\begin{aligned} \mathcal{O}_X^C(U) &= M(\mathbf{1} \amalg C) \times_{M(\mathbf{1} \amalg \mathbf{1})} \{(X - U, U)\} \\ &\simeq \mathcal{O}_Y^{\mathbf{1} \amalg C}(Y) \times_{\mathcal{O}_Y^{\mathbf{1} \amalg \mathbf{1}}(Y)} \{(X - U, U)\} \\ &\simeq (\underline{1} \amalg \mathcal{O}_Y^C)(Y) \times_{(\underline{1} \amalg \underline{1})(Y)} \{(X - U, U)\}. \end{aligned}$$

where $\underline{1}$ denotes the final object of $\mathrm{Shv}(Y)$ and coproducts are formed in the category of sheaves; we conclude by observing that this fiber product is canonically isomorphic to the set $\mathcal{O}_Y^C(U)$. $\square$

Lecture 17X: Deligne's Theorem; Continuity of Sheaves

March 25, 2018

Throughout this lecture, we fix an essentially small pretopos $\mathcal{C}$. In the previous lecture, we showed that the construction

$$(X, \mathcal{O}_X) \mapsto \Gamma(X; \mathcal{O}_X)$$

induces an equivalence of categories $\mathrm{Stone}_{\mathcal{C}} \simeq \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$. Let us note the following immediate consequence:

Theorem 1 (Deligne's Completeness Theorem)**.** *Let $U \subsetneq \mathbf{1}$ be a subobject of the final object of $\mathcal{C}$, which is not equal to $\mathbf{1}$. Then there exists a model M of $\mathcal{C}$ such that $M(U) = \emptyset$.*

Proof. In Lecture 15X, we showed that there exists an effective epimorphism $q : P \to \mathbf{1}$ in $\mathrm{Pro}(\mathcal{C})$, where P is weakly projective. Using Lecture 16X, we can assume that $P = \Gamma(X; \mathcal{O}_X)$, where X is a Stone space and $\mathcal{O}_X$ is an X-model of $\mathcal{C}$. Then $\mathcal{O}_X^U$ can be identified with a subobject of the final object of $\mathrm{Shv}(X)$: that is, with an open subset $V \subseteq X$. Since q is an effective epimorphism, it cannot factor through U. It follows that we must have $V \neq X$. We can therefore choose a point $x \in X$ which does not belong to V. In this case, the stalk $\mathcal{O}_{X,x}^U$ is empty, so that $M = \mathcal{O}_{X,x}$ is a model of $\mathcal{C}$ satisfying $M(U) = \emptyset$. $\square$

Recall that the topos $\mathrm{Shv}(\mathcal{C})$ can be identified with the full subcategory of $\mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}))$ spanned by those sheaves

$$\mathscr{F} : \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$$

which carry filtered limits in the category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ to filtered colimits in $\mathcal{S}\mathrm{et}$. To parse this condition, we need to understand filtered limits in the category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \simeq \mathrm{Stone}_{\mathcal{C}}$. These can be described in two different ways:

- Filtered limits in $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ correspond to filtered colimits in the opposite category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}}$, which we can identify with the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ spanned by those functors which preserve finite limits and effective epimorphisms. The collection of such functors is closed under filtered colimits, so that filtered colimits in $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}}$ are computed "pointwise."

- If we identify $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ with the category $\mathrm{Stone}_{\mathcal{C}}$, then filtered limits can be understood "geometrically." Suppose we are given a diagram $\{(X_\alpha, \mathcal{O}_{X_\alpha})\}$ in the category $\mathrm{Stone}_{\mathcal{C}}$, indexed by (the opposite of) a filtered category. The limit of this diagram is given by $(X, \mathcal{O}_X)$, where $X = \varprojlim_\alpha X_\alpha$ is the limit of the diagram $\{X_\alpha\}$ in the category of topological spaces, and $\mathcal{O}_X^C \simeq \varinjlim_\alpha \pi_\alpha^* \mathcal{O}_{X_\alpha}^C$ for each object $C \in \mathcal{C}$ (here $\pi_\alpha : X \to X_\alpha$ denotes the projection map). To check this, the main ingredient we need is the observation that $\mathcal{O}_X$ is again an X-model of $\mathcal{C}$: that is, for every point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a model of $\mathcal{C}$. This is clear: as a functor from $\mathcal{C}$ to the category of sets, the stalk $\mathcal{O}_{X,x}$ can be identified with the filtered colimit of the diagram $\{\mathcal{O}_{X_\alpha, x_\alpha}\}$ (where $x_\alpha = \pi_\alpha(x) \in X_\alpha$); this is a model of $\mathcal{C}$, since the subcategory $\mathrm{Mod}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ is closed under filtered colimits.

Remark 2. From the equivalence of these two descriptions of filtered limits, we can extract the following consequence: for any filtered diagram $\{(X_\alpha, \mathcal{O}_{X_\alpha})\}$ in the category $\mathrm{Stone}_{\mathcal{C}}$ having limit $(X, \mathcal{O}_X)$ and any object $C \in \mathcal{C}$, the canonical map

$$\varinjlim \mathcal{O}_{X_\alpha}^C(X_\alpha) \to \mathcal{O}_X^C(X)$$

is bijective.

Example 3. Let $\mathcal{C}$ be the category of coherent objects of the classifying topos $\mathrm{Fun}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{S}\mathrm{et})$, and let $E \in \mathcal{C}$ be the inclusion functor $\mathcal{S}\mathrm{et}_{\mathrm{fin}} \hookrightarrow \mathcal{S}\mathrm{et}$. Then the construction $(X, \mathcal{O}_X) \mapsto (X, \mathcal{O}\, X^E)$ induces an equivalence from $\mathrm{Stone}_{\mathcal{C}}$ with the category of pairs $(X, \mathscr{G})$, where X is a Stone space and $\mathscr{G}$ is a sheaf of sets on X. In this case, Remark 2 asserts that for any filtered diagram $\{(X_\alpha, \mathscr{G}_\alpha)\}$ in this category, the canonical map

$$\varinjlim_{\alpha} \mathscr{G}_\alpha(X_\alpha) \to (\varinjlim_{\alpha} \pi_\alpha^* \mathscr{G}_\alpha)(X)$$

is bijective, where $X = \varprojlim X_\alpha$ and $\pi_\alpha : X \to X_\alpha$ denote the projection maps. From this, Remark 2 is immediate.

Exercise 4. Show that the map

$$\varinjlim_{\alpha} \mathscr{G}_\alpha(X_\alpha) \to (\varinjlim_{\alpha} \pi_\alpha^* \mathscr{G}_\alpha)(X)$$

is bijective more generally under the assumption that each X_α is a compact Hausdorff space (in which case X has the same properties).

Example 5. Let $\mathrm{Mod}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ be the category of models of $\mathcal{C}$. Then the opposite category $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$ can be identified with the full subcategory of $\mathrm{Stone}_{\mathcal{C}}$ spanned by those pairs $(X, \mathcal{O}_X)$ where the Stone space X is a single point. As a subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$, the category $\mathrm{Mod}(\mathcal{C})$ is closed under filtered colimits. In particular, the category $\mathrm{Mod}(\mathcal{C})$ admits filtered colimits, so the opposite category $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$ admits filtered limits. These are preserved by the inclusion $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}} \hookrightarrow \mathrm{Stone}_{\mathcal{C}}$.

Example 6. Let X be a Stone space and let $\mathcal{O}_X$ be an X-model of $\mathcal{C}$. For every point $x \in X$, we have a filtered diagram $\{(U, \mathcal{O}_X|_U)\}_{x \in U}$ in the category $\mathrm{Stone}_{\mathcal{C}}$, where U ranges over all *clopen* neighborhoods of x. This diagram admits an inverse limit, given by $(\{x\}, \mathcal{O}_{X,x})$, which belong to the essential image of the inclusion $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}} \hookrightarrow \mathrm{Stone}_{\mathcal{C}}$.

It turns out that the filtered diagrams appearing in Examples 5 and 6 are the only ones that we need to consider.

Theorem 7. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor which carries finite coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to products in $\mathcal{S}\mathrm{et}$ (this condition is automatically satisfied if $\mathscr{F}$ corresponds to a sheaf on the category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$). Then $\mathscr{F}$ carries filtered limits in $\mathrm{Stone}_{\mathcal{C}}$ to filtered colimits in $\mathcal{S}\mathrm{et}$ if and only if it satisfies the following pair of conditions:*

(*a*) *The composite functor $\mathrm{Mod}(\mathcal{C}) \hookrightarrow \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathcal{S}\mathrm{et}$ preserves filtered colimits.*

(*b*) *For every object $(X, \mathcal{O}_X)$ in $\mathrm{Stone}_{\mathcal{C}}$ and every point $x \in X$, the canonical map*

$$\varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X|_U) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$$

is a bijection. Here the colimit is taken over all clopen neighborhoods of x in X.

Proof. The necessity of conditions (*a*) and (*b*) follows from Examples 5 and 6. Let us therefore assume that (*a*) and (*b*) are satisfied, and consider an arbitrary filtered diagram $\{(X_\alpha, \mathcal{O}_{X_\alpha})\}$ in the category $\mathrm{Stone}_{\mathcal{C}}$. Set $X = \varprojlim X_\alpha$ and, for each index α, let $\pi_\alpha : X \to X_\alpha$ denote the projection map and let $\mathcal{O}_X$ be the X-model of $\mathcal{C}$ given by the formula $\mathcal{O}_X^C = \varinjlim \pi_\alpha^* \mathcal{O}_{X_\alpha}^C$ for $C \in \mathcal{C}$.

Let $\mathcal{U}_0(X)$ denote the collection of clopen subsets of X and define $\mathcal{U}_0(X_\alpha)$ similarly. We define functors

$$\mathscr{G} : \mathcal{U}_0(X)^{\mathrm{op}} \to \mathcal{S}\mathrm{et} \qquad \mathscr{G}_\alpha : \mathcal{U}_0(X_\alpha)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$$

by the formulae

$$\mathscr{G}(U) = \mathscr{F}(U, \mathcal{O}_X|_U) \qquad \mathscr{G}_\alpha(U) = \mathscr{F}(U, \mathcal{O}_{X_\alpha}|_U).$$

Using our assumption that $\mathscr{F}$ carries finite coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to finite products in the category of sets, we see that the presheaf $\mathscr{G}$ carries finite disjoint unions in $\mathcal{U}_0(X)$ to products of sets. It follows that $\mathscr{G}$ admits an essentially unique extension to a sheaf of sets on X, which we will also denote by $\mathscr{G}$. Similarly, each $\mathscr{G}_\alpha$ can be extended to a sheaf of sets on X_α. Moreover, the map $(X, \mathcal{O}_X) \to (X_\alpha, \mathcal{O}_{X_\alpha})$ in $\mathrm{Stone}_{\mathcal{C}}$ induces a map of sheaves $f_\alpha : \pi_\alpha^* \mathscr{G}_\alpha \to \mathscr{G}$. These maps are compatible as α varies, and induce a map

$$f_\infty : \varinjlim_\alpha \pi_\alpha^* \mathscr{G}_\alpha \to \mathscr{G}$$

in the category $\mathrm{Shv}(X)$ of $\mathcal{S}\mathrm{et}$-valued sheaves on X.

We wish to show that the canonical map

$$\varinjlim_\alpha \mathscr{F}(X_\alpha, \mathcal{O}_{X_\alpha}) \to \mathscr{F}(X, \mathcal{O}_X)$$

is a bijection. Unwinding the definitions, we can write this map as a composition

$$\varinjlim_\alpha \mathscr{G}_\alpha(X_\alpha) \to (\varinjlim_\alpha \pi_\alpha^* \mathscr{G}_\alpha)(X) \xrightarrow{f_\infty} \mathscr{G}(X).$$

Here the first map is bijective by virtue of Example 3. To complete the proof, it will suffice to show that f_∞ is an isomorphism of sheaves on X. For this, it will suffice to show that for each point $x \in X$, the induced map of stalks

$$\varinjlim_\alpha (\pi_\alpha^* \mathscr{G}_\alpha)_x = \varinjlim_\alpha \mathscr{G}_{\alpha, x_\alpha} \to \mathscr{G}_x$$

is a bijection; here x_α denotes the point $\pi_\alpha(x) \in X_\alpha$. Using assumption (b), we can rewrite this map as

$$\varinjlim_\alpha \mathscr{F}(\{x_\alpha\}, \mathcal{O}_{X_\alpha, x_\alpha}) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x}).$$

This map is bijective by virtue of (a) (and the construction of $\mathcal{O}_X$). $\square$

Combining Theorem 7 with the results of Lecture 15X, we obtain the following:

Corollary 8. *There is a fully faithful embedding* $\mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$*, whose essential image consists of those functors* $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ *which are sheaves (with respect to the topology given by finite coverings in* $\mathrm{Pro}(\mathcal{C})$*) and which satisfy conditions* (a) *and* (b) *of Theorem 7.*

Lecture 18X-Coverings in $\mathrm{Stone}_{\mathcal{C}}$

April 6, 2018

Let $\mathcal{C}$ be an essentially small pretopos, which we regard as fixed throughout this lecture. In Lecture 16X, we saw that the global sections functor induces an equivalence of categories

$$\mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$$

$$(X, \mathcal{O}_X) \mapsto \Gamma(X; \mathcal{O}_X).$$

The category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ is equipped with a Grothendieck topology, where a collection of morphisms $\{P_i \to Q\}_{i \in I}$ is a covering if there exists some finite subset $I_0 \subseteq I$ such that the induced map

$$\coprod_{i \in I_0} P_i \to Q$$

is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$. Transporting this Grothendieck topology along the equivalence $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \simeq \mathrm{Stone}_{\mathcal{C}}$, we obtain a Grothendieck topology on the category $\mathrm{Stone}_{\mathcal{C}}$. Our goal in this lecture is to describe this topology more explicitly. Note that a collection of maps

$$\{(X_i, \mathcal{O}_{X_i}), (Y, \mathcal{O}_Y)\}_{i \in I}$$

in $\mathrm{Stone}_{\mathcal{C}}$ is a covering if and only if, for some finite subset $I_0 \subseteq I$, the single map

$$(\amalg_{i \in I_0} X_i, \mathcal{O}_{\amalg_{i \in I_0} X_i}) \to (Y, \mathcal{O}_Y)$$

is a covering (where $\mathcal{O}_{\amalg_{i \in I_0} X_i}$ denotes the sheaf whose restriction to each X_i is given by $\mathcal{O}_{X_i}$). It will therefore be enough to characterize the singleton coverings. First, we need to introduce a bit of (nonstandard) terminology.

Proposition 1. *Let $f : M \to N$ be a morphism of models of $\mathcal{C}$. The following conditions are equivalent:*

(1) *For every object $C \in \mathcal{C}$, the map $M(C) \to N(C)$ is injective.*

(2) *For every object $C \in \mathcal{C}$ and every subobject $C_0 \subseteq C$, the diagram*

$$\begin{tikzcd} M(C_0) \arrow[r] \arrow[d] & N(C_0) \arrow[d] \\ M(C) \arrow[r] & N(C) \end{tikzcd}$$

is a pullback square (in the category of sets).

Proof. Suppose that condition (2) is satisfied. Then, for each object $C \in \mathcal{C}$, the diagram

$$\begin{tikzcd} M(C) \arrow[r] \arrow[d] & N(C) \arrow[d] \\ M(C \times C) \arrow[r] & N(C \times C) \end{tikzcd}$$

is a pullback square, which is equivalent to the injectivity of the map $M(C) \to N(C)$.

We now prove the converse. Let C be an object of $\mathcal{C}$ and let $C_0 \subseteq C$ be a subobject. Let R be the equivalence relation on $(C \amalg \mathbf{1})$ given by the join of C (embedded diagonally in $C \times C$), $C_0 \times C_0$, $C_0 \times \mathbf{1}$, $\mathbf{1} \times C_0$, and $\mathbf{1} \times \mathbf{1}$. Since $\mathcal{C}$ is a pretopos, the equivalence relation R is effective and we can form the quotient $D = (C \amalg \mathbf{1})/R$. The object D can be described more simply as the pushout $C \amalg_{C_0} \mathbf{1}$, and this pushout is preserved by any morphism of pretopoi. In particular, for each model M of $\mathcal{C}$, we can identify $M(D)$ with the pushout $M(C) \amalg_{M(C_0)} \{*\}$ obtained by from $M(C)$ by collapsing the subset $M(C_0) \subseteq M(C)$ to a point.

Suppose that $f : M \to N$ is a map of models, so that f induces a map $f_D : M(D) \to N(D)$. Let us abuse notation by writing $*$ for the distinguished point in both $M(D)$ and $N(D)$. Then we have a canonical bijection

$$f_D^{-1}\{*\} \simeq \{*\} \cup \{x \in M(C) \setminus M(C_0) : f_C(x) \in N(C_0)\}.$$

Consequently, if f_D is injective, then the set $\{x \in M(C) \setminus M(C_0) : f_C(x) \in N(C_0)\}$ is empty, which is equivalent to the statement that the diagram

$$\begin{array}{ccc} M(C_0) & \longrightarrow & N(C_0) \\ \downarrow & & \downarrow \\ M(C) & \longrightarrow & N(C) \end{array}$$

is a pullback. □

Definition 2. Let M and N be models of $\mathcal{C}$. We will say that a morphism $f : M \to N$ is *elementary* if it satisfies the equivalent conditions of Proposition 1.

Remark 3. The terminology of Definition 2 is intended to evoke the notion of an *elementary embedding* in the setting of model theory: we can think of the subobject $C_0 \subseteq C$ as a "proposition" that can hold of elements of $M(C)$, where M is a model of $\mathcal{C}$; elementary morphisms are those which preserve the truth of such propositions.

Example 4. Let $\mathcal{C}$ be a Boolean pretopos. Then *every* morphism $f : M \to N$ in $\mathrm{Mod}(\mathcal{C})$ is elementary. Given $C_0 \subseteq C$ in $\mathcal{C}$, we tautologically have an inclusion

$$M(C_0) \subseteq N(C_0) \times_{N(C)} M(C);$$

the reverse inclusion follows by applying the same argument to a complement of C_0 in C.

Warning 5. Let $\mathcal{C}$ be a pretopos and let $\mathcal{C}'$ be its Booleanization. Then every model M of $\mathcal{C}$ can be promoted (in an essentially unique way) to a model of $\mathcal{C}'$. Let us abuse notation by denoting this model also by M. For every pair of models $M, N \in \mathrm{Mod}(\mathcal{C})$, we have a canonical injection

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}')}(M, N) \to \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M, N).$$

If $f : M \to N$ is a morphism in $\mathrm{Mod}(\mathcal{C})$ which belongs to the image of this map, then f is necessarily elementary. However, the converse need not be true (so the terminology of Definition 2 is perhaps misleading).

The relevance of Definition 2 for us is the following:

Theorem 6. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism in $\mathrm{Stone}_{\mathcal{C}}$. The following conditions are equivalent:*

(1) *The induced map $\Gamma(X; \mathcal{O}_X) \to \Gamma(Y; \mathcal{O}_Y)$ is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$.*

(2) *For each point $y \in Y$, there exists a point $x \in X$ such that $f(x) = y$ and the induced map of models $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is elementary.*

Corollary 7. *Suppose that the pretopos $\mathcal{C}$ is Boolean. Then a map $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ induces an effective epimorphism in* $\mathrm{Pro}(\mathcal{C})$ *if and only if the underlying map of topological spaces $X \to Y$ is surjective.*

Proof. Combine Theorem 6 with Example 4. □

Proof of Theorem 6. We proceed in several steps. Let $f : P \to Q$ be an arbitrary morphism in $\mathrm{Pro}(\mathcal{C})$. Consider the following assertion:

(i) The map $f : P \to Q$ is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$.

We claim that (i) is equivalent to the following:

(ii) For every monomorphism $U \hookrightarrow V$ in $\mathrm{Pro}(\mathcal{C})$ and every commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"] & U \arrow[d] \\ Q \arrow[r] \arrow[ur, dashed] & V, \end{tikzcd}$$

there exists a dotted arrow as indicated.

The implication (i) $\Rightarrow$ (ii) is clear. Conversely, if (ii) is satisfied for the inclusion $\mathrm{Im}(u) \hookrightarrow Q$, then assertion ($i$) follows. We saw in Lecture 14X that every monomorphism in $\mathrm{Pro}(\mathcal{C})$ can be realized as a filtered inverse limit of monomorphisms in $\mathcal{C}$. Consequently, (ii) is equivalent to the following *a priori* weaker condition:

(iii) For every object $C \in \mathcal{C}$ and every subobject $C_0 \subseteq C$, and every commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & C_0 \arrow[d] \\ Q \arrow[r] \arrow[ur, dashed] & C, \end{tikzcd}$$

there exists a dotted arrow as indicated.

Let us now suppose that $P = \Gamma(X; \mathcal{O}_X)$ and $Q = \Gamma(Y; \mathcal{O}_Y)$ for some objects $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y) \in \mathrm{Stone}_{\mathcal{C}}$. Unwinding the definitions, we can rephrase (iii) as follows:

(iv) Let $C \in \mathcal{C}$ be an object, let $s_Y \in \Gamma(Y; \mathcal{O}_Y)(C) = \mathcal{O}_Y^C(Y)$, and let $s_X \in \mathcal{O}_X^C(X)$ be the image of s_Y. Suppose that C_0 is a subobject of C and that s_X can be lifted to a global section of the subsheaf $\mathcal{O}_X^{C_0} \subseteq \mathcal{O}_X^C$. Then s_Y can be lifted to a global section of the subsheaf $\mathcal{O}_Y^{C_0} \subseteq \mathcal{O}_Y^C$.

We can restate (v) in contrapositive form:

(v) Let $C \in \mathcal{C}$ be an object and let $C_0 \subseteq C$ be a subobject. Suppose we are given a global section $s_Y \in \mathcal{O}_Y^C(Y)$ having image $s_X \in \mathcal{O}_X^C(X)$. If there exists a point $y \in Y$ such that the stalk $s_{Y,y}$ does not belong to $\mathcal{O}_{Y,y}^{C_0}$, then there exists a point $x \in X$ such that $s_{X,x}$ does not belong to $\mathcal{O}_{X,x}^{C_0}$.

Note that assertion (v) follows immediately from (2) (it suffices to choose $x \in X$ for which $f(x) = y$ and the induced map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an elementary morphism in $\mathrm{Mod}(\mathcal{C})$). We will complete the proof by showing that assertion (v) implies (2).

Assume that (iv) is satisfied, and fix a point $y \in Y$. We wish to show that there exists a point $x \in X$ such that $f(x) = y$ and the induced map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is elementary. Suppose otherwise. Then, for each point $x \in f^{-1}(y)$, the induced map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is not elementary. It follows that we can choose an object $C(x) \in \mathcal{C}$, a subobject $C_0(x) \subseteq C(x)$, and an element of $\eta_x \in \mathcal{O}_{Y,y}^{C(x)}$ which does not belong to $\mathcal{O}_{Y,y}^{C_0(x)}$, but the image of η_x in $\mathcal{O}_{X,x}^{C(x)}$ belongs to $\mathcal{O}_{X,x}^{C_0(x)}$. Let U_x be an open neighborhood of x in $f^{-1}\{y\}$ for which the

image of η_x in $\mathcal{O}_{X,x'}^{C(x)}$ belongs to $\mathcal{O}_{Y,y}^{C_0(x)}$, for each $x' \in U_x$. Since the fiber $f^{-1}\{y\}$ is compact, we can choose finitely many points $x_1, \ldots, x_n \in f^{-1}\{y\}$ for which the open sets $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ cover the fiber $f^{-1}\{y\}$. Set $C = C(x_1) \times \cdots \times C(x_n)$, and let $C_0 \subseteq C$ be the union of the subobjects $C_0(x_i) \times \prod_{j \neq i} C(x_j)$. Then we can identify $\{\eta_{x_i}\}_{1 \leq i \leq n}$ with a point $\eta \in \mathcal{O}_{Y,y}^{C}$. By construction, η does not belong to $\mathcal{O}_{Y,y}^{C_0}$, but the image of η in $\mathcal{O}_{X,x}^{C}$ belongs to $\mathcal{O}_{X,x}^{C_0}$ for each $x \in f^{-1}\{y\}$.

Choose a lift of η to a point $s_V \in \mathcal{O}_Y^C(V)$, for some open neighborhood V of Y. Let $s_{f^{-1}(V)}$ denote the image of V in $\mathcal{O}_X^C(f^{-1}(V))$. Then there is a largest open subset $W \subseteq f^{-1}(V)$ for which the restriction $s_{f^{-1}(V)}|_W$ is a section of the subsheaf $\mathcal{O}_X^{C_0} \subseteq \mathcal{O}_X^C$. By construction, the open set W contains $f^{-1}\{y\}$. Since f is a proper map, we can choose a smaller open set $V' \subseteq V$ such that $y \in V'$ and $f^{-1}(V') \subseteq W$. Replacing V by V', we can assume that $s_{f^{-1}(V)}$ belongs to $\mathcal{O}_X^{C_0}(f^{-1}(V))$.

Shrinking V further if necessary, we can arrange that V is both open and closed. In this case, we can extend s_V to a global section s_Y of the sheaf $\mathcal{O}_Y^{C \amalg \mathbf{1}} \simeq \mathcal{O}_Y^C \amalg \mathbf{1}$ (which is equal to s_V on the open set V, and carries the complement of V to the second summand of $\mathcal{O}_Y^{C \amalg \mathbf{1}}$). Replacing C by the coproduct $C \amalg \mathbf{1}$ and C_0 by the coproduct $C_0 \amalg \mathbf{1}$, we can assume that $V = Y$: that is, that s_V is a global section of $\mathcal{O}_Y^C$. It then follows from (v) (or (iv)) that s_V is also a global section of the subsheaf $\mathcal{O}_Y^{C_0} \subseteq \mathcal{O}_Y^C$, contradicting our choice of η. $\square$

Corollary 8 (Amalgamation)**.** *Let $f : M \to N$ be an arbitrary morphism in* $\mathrm{Mod}(\mathcal{C})$*, and let $g : M \to M'$ be an elementary morphism in* $\mathrm{Mod}(\mathcal{C})$*. Then there exists a commutative diagram*

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"] & M' \arrow[d] \\ N \arrow[r, "g'"] & N' \end{tikzcd}$$

in $\mathrm{Mod}(\mathcal{C})$*, where g' is also elementary.*

Proof. Let us regard g as a morphism $(*, M') \to (*, M)$ in the category $\mathrm{Stone}_{\mathcal{C}}$. By virtue of Theorem 6, this is a covering (that is, it induces an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$). We can therefore choose a commutative diagram

$$\begin{tikzcd} (*, M) & (*, M') \arrow[l] \\ (*, N) \arrow[u] & (X, \mathcal{O}_X) \arrow[l] \arrow[u] \end{tikzcd}$$

in $\mathrm{Stone}_{\mathcal{C}}$, where the bottom horizontal map is also a covering. Using Theorem 6 again, we can choose a point $x \in X$ for which the induced map $N \to \mathcal{O}_{X,x}$ is elementary. We then obtain a diagram in the category $\mathrm{Mod}(\mathcal{C})$

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"] & M' \arrow[d] \\ N \arrow[r, "g'"] & \mathcal{O}_{X,x} \end{tikzcd}$$

with the desired properties. $\square$

Lecture 19X-Sheaves on $\mathrm{Stone}_{\mathcal{C}}$

March 30, 2018

Let $\mathcal{C}$ be an essentially small pretopos, which we regard as fixed throughout this lecture. In Lecture 15X, we constructed a fully faithful embedding

$$\mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Stone}_{\mathcal{C}}) \subseteq \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et}).$$

The essential image of this embedding consists of those functors $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ which commute with filtered colimits and are sheaves with respect to the Grothendieck topology of Lecture 18X. Any such functor must satisfy the following condition:

(a) The functor $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ preserves finite products: that is, it carries finite coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to finite products in the category of sets.

In Lecture 17X, we proved that if $\mathscr{F}$ is a functor satisfying (a), then it commutes with filtered colimits if and only if it satisfies the following additional conditions:

(b) For every object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ and every point $x \in X$, the canonical map

$$\varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X|_U) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$$

is bijective; here the colimit is taken over all clopen neighborhoods $U \subseteq X$ of the point x.

(c) The composite functor

$$\mathrm{Mod}(\mathcal{C}) \hookrightarrow \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathcal{S}\mathrm{et}$$

commutes with filtered colimits.

Our goal in this section is to characterize those functors $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ which are sheaves. It is easy to see that if $\mathscr{F}$ is a sheaf, then it must satisfy condition (a) above. We will prove the following partial converse:

Theorem 1. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor satisfying conditions (a) and (b). Then $\mathscr{F}$ is a sheaf if and only if it satisfies the following further condition:*

(d) *For every elementary morphism $f : M \to N$ in $\mathrm{Mod}(\mathcal{C})$, we have an equalizer diagram*

$$\mathscr{F}(M) \to \mathscr{F}(N) \rightrightarrows \prod \mathscr{F}(P)$$

where the product is taken over all commutative diagrams

$$M \xrightarrow{f} N \rightrightarrows P$$

in $\mathrm{Mod}(\mathcal{C})$.

Here we identify $\mathrm{Mod}(\mathcal{C})$ *with the full subcategory of* $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}$ *spanned by those pairs* $(X, \mathcal{O}_X)$*, where* X *is a singleton.*

Warning 2. In the formulation of condition (d), the product $\prod \mathscr{F}(P)$ is an ill-defined object, because it is indexed by a proper class. However, the equalizer of the diagram $\mathscr{F}(N) \rightrightarrows \prod \mathscr{F}(P)$ is still well-defined as a subset of $\mathscr{F}(N)$.

Remark 3. Condition (d) is equivalent to the requirement that the restriction $\mathscr{F}|_{\mathrm{Mod}(\mathcal{C})}$ is a sheaf on the category $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$, where we consider a collection of morphisms $\{M \to N_i\}_{i \in I}$ in $\mathrm{Mod}(\mathcal{C})$ to be a covering in $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$ if at least one of the maps $M \to N_i$ is elementary (it follows from the amalgamation property of the previous lecture that this notion of covering defines a Grothendieck topology on $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$).

Corollary 4. *The topos* $\mathrm{Shv}(\mathcal{C})$ *can be identified with the full subcategory of* $\mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ *spanned by those functors* $\mathscr{F}$ *which satisfy conditions* (a)*,* (b)*,* (c) *and* (d) *above.*

Before giving the proof of Theorem 1, it will be convenient to revisit a construction from Lecture 17X, which gives a convenient reformulation of conditions (a) and (b).

Notation 5. Let $(X, \mathcal{O}_X)$ be an object of $\mathrm{Stone}_{\mathcal{C}}$, and let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor. Let $\mathcal{U}_0(X)$ denote the collection of all clopen subsets of X. We define a functor

$$\mathscr{F}(\mathcal{O}_X) : \mathcal{U}_0(X)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$$

by the formula

$$\mathscr{F}(\mathcal{O}_X)(U) = \mathscr{F}(U, \mathcal{O}_X|_U).$$

Note that the functor $\mathscr{F}$ satisfies condition (a) above if and only if, for every object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$, the functor $\mathscr{F}(\mathcal{O}_X) : \mathcal{U}_0(X)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ carries disjoint unions in $\mathcal{U}_0(X)$ to products in $\mathcal{S}\mathrm{et}$. In this case, $\mathscr{F}(\mathcal{O}_X)$ extends uniquely to a sheaf of sets on X, which we will also denote by $\mathscr{F}(\mathcal{O}_X)$.

By construction, the stalk of $\mathscr{F}(\mathcal{O}_X)$ at a point $x \in X$ is given by the direct limit $\varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X|_U)$. We therefore have a canonical map $\mathscr{F}(\mathcal{O}_X)_x \to \mathscr{F}(\mathcal{O}_{X,x})$ (here we abuse notation by identifying the model $\mathcal{O}_{X,x}$ with the object $(\{x\}, \mathcal{O}_{X,x}) \in \mathrm{Stone}_{\mathcal{C}}$). Condition (b) can then be restated as follows:

(b') For each $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ and each point $x \in X$, the map $\mathscr{F}(\mathcal{O}_X)_x \to \mathscr{F}(\mathcal{O}_{X,x})$ is a bijection.

Remark 6. Let $(Y, \mathcal{O}_Y)$ be an object of $\mathrm{Stone}_{\mathcal{C}}$ and suppose we are given a map of Stone spaces $f : X \to Y$, so that $f^* \mathcal{O}_Y$ is an X-model of $\mathcal{C}$. If $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is a functor satisfying condition (a), then Notation 5 determines set-valued sheaves

$$\mathscr{F}(\mathcal{O}_Y) \in \mathrm{Shv}(Y) \qquad \mathscr{F}(f^* \mathcal{O}_Y) \in \mathrm{Shv}(X)$$

together with a comparison map $f^* \mathscr{F}(\mathcal{O}_Y) \to \mathscr{F}(f^* \mathcal{O}_Y)$ in $\mathrm{Shv}(X)$. If $\mathscr{F}$ satisfies condition (b), then this comparison map is an isomorphism of sheaves on X (this can be checked on stalks, where it follows from (b')).

Remark 7. Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor satisfying (a) and (b). Then, for each $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$, the canonical map

$$\mathscr{F}(X, \mathcal{O}_X) \to \prod_{x \in X} \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$$

is injective. This follows from the fact that a section s of the sheaf $\mathscr{F}(\mathcal{O}_X) \in \mathrm{Shv}(X)$ is determined by its stalks $\{s_x\}_{x \in X}$.

We can now prove the "easy" direction of Theorem 1. Let

$$\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$$

be a sheaf (so that is satisfies condition (a)), and suppose that $\mathscr{F}$ also satisfies condition (b). We wish to show that it satisfies condition (d). Suppose we are given an elementary morphism $M \to N$ in $\mathrm{Mod}(\mathcal{C})$. Then the induced map

$$(*, N) \to (*, M)$$

is a covering in $\mathrm{Stone}_{\mathcal{C}}$ (see Lecture 18X). It follows that the canonical map $\mathscr{F}(M) \to \mathscr{F}(N)$ is injective, and that its image consists of those elements $s \in \mathscr{F}(N)$ which satisfy the following condition:

$(*)$ For every object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ equipped with a pair of maps $(X, \mathcal{O}_X) \rightrightarrows (*, N)$ which are coequalized by $(*, N) \to (*, M)$, the element s belongs to the equalizer $\mathrm{Eq}(\mathscr{F}(N) \rightrightarrows \mathscr{F}(X, \mathcal{O}_X))$.

To verify (d), we must show that it suffices to check the criterion of $(*)$ in the case where X is a single point. This follows from the injectivity of the map $\mathscr{F}(X, \mathcal{O}_X) \to \prod_{x \in X} \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$ (Remark 7).

We now tackle the hard direction. Assume that $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ satisfies conditions (a), (b), and (d); we wish to show that $\mathscr{F}$ is a sheaf. Choose a covering $\{(X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)\}_{i \in I}$ in the category $\mathrm{Stone}_{\mathcal{C}}$. For every pair $i, j \in I$, we can identify $(X_i, \mathcal{O}_{X_i})$, $(X_j, \mathcal{O}_{X_j})$, and $(X, \mathcal{O}_X)$ with weakly projective pro-objects $\Gamma(X_i; \mathcal{O}_{X_i})$, $\Gamma(X_j; \mathcal{O}_{X_j})$, and $\Gamma(X; \mathcal{O}_X)$. We can then form the fiber product

$$\Gamma(X_i; \mathcal{O}_{X_i}) \times_{\Gamma(X; \mathcal{O}_X)} \Gamma(X_j; \mathcal{O}_{X_j})$$

in $\mathrm{Pro}(\mathcal{C})$. This fiber product might not be weakly projective. However, it we can choose a *covering* by a weakly projective pro-object, which we can then write in the form $\Gamma(X_{ij}, \mathcal{O}_{X_{ij}})$ for some $(X_{ij}, \mathcal{O}_{X_{ij}}) \in \mathrm{Stone}_{\mathcal{C}})$. In order to show that $\mathscr{F}$ is a sheaf, we must verify that the diagram

$$\mathscr{F}(X, \mathcal{O}_X) \to \prod_i \mathscr{F}(X_i, \mathcal{O}_{X_i}) \rightrightarrows \prod_{i,j} \mathscr{F}(X_{ij}, \mathcal{O}_{X_{ij}})$$

is an equalizer diagram in the category of sets. Since every covering admits a finite subcover, it suffices to check this in the case where the set I is finite. In this case, we can form the coproducts

$$(Y, \mathcal{O}_Y) = \coprod_{i \in I} (X_i, \mathcal{O}_{X_i}) \qquad (Z, \mathcal{O}_Z) = \coprod_{i,j \in I} (X_{ij}, \mathcal{O}_{X_{ij}}).$$

Using condition (a), we are reduced to showing that the diagram

$$\mathscr{F}(X, \mathcal{O}_X) \to \mathscr{F}(Y, \mathcal{O}_Y) \rightrightarrows \mathscr{F}(Z, \mathcal{O}_Z)$$

is an equalizer diagram of sets.

Let $\mathscr{G}$ denote the direct image of $\mathscr{F}(\mathcal{O}_Y) \in \mathrm{Shv}(Y)$ along the projection map $Y \to X$, and let $\mathscr{H}$ denote the direct image of $\mathscr{F}(\mathcal{O}_Z)$ along the projection $Z \to X$. We then have a commutative diagram

$$\mathscr{F}(\mathcal{O}_X) \to \mathscr{G} \rightrightarrows \mathscr{H}$$

in $\mathrm{Shv}(X)$, and we wish to show that it becomes an equalizer diagram in $\mathcal{S}\mathrm{et}$ after taking global sections. To prove this, it will suffice to show that the above diagram is an equalizer in $\mathrm{Shv}(X)$. This can be checked stalkwise: that is, we are reduced to showing that the map

$$\mathscr{F}(\mathcal{O}_X)_x \to \mathscr{G}_x \rightrightarrows \mathscr{H}_x$$

is an equalizer diagram of sets, for each point $x \in X$. Let $Y_x \subseteq Y$ and $Z_x \subseteq Z$ denote the inverse images of x. Using (b) (in the form of Remark 7), we are reduced to showing that the diagram

$$\mathscr{F}(\{x\}, \mathcal{O}_{X,x}) \xrightarrow{\psi} \mathscr{F}(Y_x, \mathcal{O}_Y |_{Y_x}) \rightrightarrows \mathscr{F}(Z_x, \mathcal{O}_Z |_{Z_x})$$

is an equalizer diagram of sets.

Since the map $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a covering, Lecture 18X shows that we an choose a point $y \in Y_x$ for which the map of stalks $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is elementary. Using condition (d), we deduce that the composite map

$$\mathscr{F}(\{x\}, \mathcal{O}_{X,x}) \xrightarrow{\psi} \mathscr{F}(Y_x, \mathcal{O}_Y |_{Y_x}) \to \mathscr{F}(\{y\}, \mathcal{O}_{Y,y})$$

is injective, so that ψ is injective. We will complete the proof by showing that every element s of the equalizer

$$\mathrm{Eq}(\mathscr{F}(Y_x, \mathcal{O}_Y |_{Y_x}) \rightrightarrows \mathscr{F}(Z_x, \mathcal{O}_Z |_{Z_x}))$$

belongs the image of ψ. Let $s_y \in \mathscr{F}(\{y\}, \mathcal{O}_{Y,y})$ denote the stalk of s at the point y. We first claim that s_y belongs to the image of the composite map

$$\mathscr{F}(\{x\}, \mathcal{O}_{X,x}) \xrightarrow{\psi} \mathscr{F}(Y_x, \mathcal{O}_Y |_{Y_x}) \to \mathscr{F}(\{y\}, \mathcal{O}_{Y,y}).$$

By virtue of (d), it will suffice to prove the following:

$(*')$ Given a model $P \in \mathrm{Mod}(\mathcal{C})$ and a commutative diagram of models

$$\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y} \rightrightarrows P,$$

the stalk s_y belongs to the equalizer $\mathrm{Eq}(\mathscr{F}(\mathcal{O}_{Y,y}) \rightrightarrows \mathscr{F}(P))$.

Let us identify P with an object of $\mathrm{Pro}(\mathcal{C})$, and form a pullback diagram

$$\begin{array}{ccc} \widetilde{P} & \longrightarrow & \Gamma(Z; \mathcal{O}_Z) \\ \downarrow & & \downarrow \\ P & \longrightarrow & \Gamma(Y; \mathcal{O}_Y) \times_{\Gamma(X; \mathcal{O}_X)} \Gamma(Y; \mathcal{O}_Y). \end{array}$$

in $\mathrm{Pro}(\mathcal{C})$. Here the right vertical map is an effective epimorphism, so the left vertical map is an effective epimorphism as well. The object $\widetilde{P}$ might not be weakly projective. However, we can choose an effective epimorphism $Q \to \widetilde{P}$, where Q is weakly projective. We can then write $Q = \Gamma(W, \mathcal{O}_W)$, for some object $(W, \mathcal{O}_W)$ in $\mathrm{Stone}_{\mathcal{C}}$. By construction, the map $(W, \mathcal{O}_W) \to (*, P)$ is a covering in $\mathrm{Stone}_{\mathcal{C}}$. Using Lecture 18X, we see that there exists a point $w \in W$ for which the map of models $P \to \mathcal{O}_{W,w}$ is elementary. Using condition (d), we see that the map $\mathscr{F}(P) \to \mathscr{F}(\mathcal{O}_{W,w})$ is injective. Consequently, to verify $(*')$, we are free to replace P by $\mathcal{O}_{W,w}$. Let $z \in Z$ denote the image of w under the map $(W, \mathcal{O}_W) \to (Z, \mathcal{O}_Z)$ in $\mathrm{Stone}_{\mathcal{C}}$, so that the diagram of $(*')$ refines to a diagram

$$\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y} \rightrightarrows \mathcal{O}_{Z,z} \to P.$$

We are therefore reduced to showing that s_y belongs to the equalizer $\mathrm{Eq}(\mathscr{F}(\mathcal{O}_{Y,y}) \rightrightarrows \mathscr{F}(\mathcal{O}_{Z,z}))$, which follows from our assumption that $s \in \mathrm{Eq}(\mathscr{F}(Y_x, \mathcal{O}_Y |_{Y_x}) \rightrightarrows \mathscr{F}(Z_x, \mathcal{O}_Z |_{Z_x}))$.

The above argument shows that we can write s_y as the image of an element $\overline{s} \in \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$. We will complete the proof by showing that $\psi(\overline{s}) = s$. For this, it will suffice to show that $\psi(\overline{s})$ and s have the same image in the stalk $\mathscr{F}(\{y'\}, \mathcal{O}_{Y,y'})$ for each point y' in the fiber Y_x. Using the amalgamation property of the previous lecture, we see that there exists a commutative diagram of models σ :

$$\begin{array}{ccc} \mathcal{O}_{X,x} & \longrightarrow & \mathcal{O}_{Y,y} \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y,y'} & \longrightarrow & N \end{array}$$

where the bottom horizontal map is elementary. As above, we can form a pullback diagram

$$\begin{tikzcd} \widetilde{N} \arrow[r] \arrow[d] & \Gamma(Z; \mathcal{O}_Z) \arrow[d] \\ N \arrow[r] & \Gamma(Y; \mathcal{O}_Y) \times_{\Gamma(X; \mathcal{O}_X)} \Gamma(Y; \mathcal{O}_Y) \end{tikzcd}$$

in $\mathrm{Pro}(\mathcal{C})$ and choose an effective epimorphism $\Gamma(V, \mathcal{O}_V) \to \widetilde{N}$ for some $(V, \mathcal{O}_V) \in \mathrm{Stone}_{\mathcal{C}}$. The map $(V, \mathcal{O}_V) \to (*, N)$ is a covering, so there exists some point $v \in V$ for which the map of models $N \to \mathcal{O}_{V,v}$ is elementary. Our assumption that s belongs to the equalizer $\mathrm{Eq}(\mathscr{F}(Y_x, \mathcal{O}_Y|_{Y_x}) \rightrightarrows \mathscr{F}(Z_x, \mathcal{O}_Z|_{Z_x}))$ then implies that the stalks $s_y = \psi(\overline{s})_y$ and $s_{y'}$ have the same image in $\mathscr{F}(\{v\}, \mathcal{O}_{V,v})$. It follows that $\psi(\overline{s})_{y'}$ and $s_{y'}$ have the same image in $\mathscr{F}(\{v\}, \mathcal{O}_{V,v})$. Since the composite map $\mathcal{O}_{Y,y'} \to N \to \mathcal{O}_{V,v}$ is elementary, assumption (d) guarantees the injectivity of $\mathscr{F}(\{y\}, \mathcal{O}_{Y,y'}) \to \mathscr{F}(\{v\}, \mathcal{O}_{V,v})$, so that we must have $\psi(\overline{s})_{y'} = s_{y'}$, as desired.

Lecture 20X-Ultraproducts

March 31, 2018

In this lecture, we review the theory of ultrafilters and ultraproducts.

Definition 1. Let I be a set. An *ultrafilter on* I is a collection $\mathcal{U}$ of subsets of I satisfying the following conditions:

(a) The set $\mathcal{U}$ is closed under finite intersections. That is, the set I belongs to $\mathcal{U}$, and for every $J, J' \in \mathcal{U}$, the intersection $J \cap J'$ also belongs to $\mathcal{U}$.

(b) The set $\mathcal{U}$ is closed upwards: that is, if $J \subseteq J'$ and J is contained in $\mathcal{U}$, then J' is also contained in $\mathcal{U}$.

(c) For every subset $J \subseteq I$, exactly one of the sets J and $I - J$ belongs to $\mathcal{U}$.

Exercise 2. In Definition 1, show that (b) can be deduced from (a) and (c).

Remark 3. Let I be a set. Then the datum of an ultrafilter $\mathcal{U}$ on I is equivalent to the datum of a finitely additive measure

$$\mu : \{\text{Subsets of } I\} \to \{0, 1\};$$

the equivalence is implemented by taking $\mathcal{U} = \{J \subseteq I : \mu(J) = 1\}$.

Example 4 (Principal Ultrafilters)**.** Let I be a set containing an element i, and let $\mathcal{U}_i$ be the collection of all subsets of I which contain i. Then $\mathcal{U}_i$ is an ultrafilter on I. We refer to $\mathcal{U}_i$ as the *principal ultrafilter associated to* i.

Exercise 5. Let $\mathcal{U}$ be a collection of subsets of a set I. We say that $\mathcal{U}$ is a *filter on* I if it satisfies conditions (a) and (b) of Definition 1. Show that if $\mathcal{U}$ is a filter on I such that $\emptyset \notin \mathcal{U}$, then $\mathcal{U}$ can be enlarged to an ultrafilter on I.

Construction 6 (Ultraproducts)**.** Let $\{M_i\}_{i \in I}$ be a collection of sets indexed by a set I, and let $\mathcal{U}$ be an ultrafilter on I. We let $(\prod_{i \in I} \mathcal{M}_i)/\mathcal{U}$ denote the direct limit

$$\varinjlim_{J \in \mathcal{U}} \prod_{i \in J} M_i.$$

We will refer to $(\prod_{i \in I} \mathcal{M}_i)/\mathcal{U}$ as the *ultraproduct of the sets* M_i *with respect to the ultrafilter* $\mathcal{U}$.

Exercise 7. In the situation of Construction 6, suppose that each of the sets M_i is nonempty. Show that the ultraproduct $(\prod_{i \in I} M_i)/\mathcal{U}$ can be identified with the quotient of $\prod_{i \in I} M_i$ by an equivalence relation $\sim$, where $\{x_i\}_{i \in I} \simeq \{y_i\}_{i \in I}$ if $\{i \in I : x_i = y_i\}$ belongs to the ultrafilter $\mathcal{U}$ (in this case, we say that the sequences $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$ agree *almost everywhere* with respect to $\mathcal{U}$).

Beware that this is not necessarily true if some M_j is empty. In this case, the product $\prod_{i \in I} M_i$ is also empty. However, the ultraproduct $(\prod_{i \in I} M_i)/\mathcal{U}$ will be nonempty if the set $\{i \in I : M_i \neq \emptyset\}$ belongs to the ultrafilter $\mathcal{U}$.

Example 8. In the situation of Construction 6, suppose that $\mathcal{U} = \mathcal{U}_j$ is the principal ultrafilter associated to an element $j \in I$. Then the ultraproduct $(\prod_{i \in I} M_i)/\mathcal{U}$ can be identified with M_j.

Ultraproducts appear in mathematical logic because they behave well with respect to the truth of first-order formulas.

Theorem 9 (Łos's Ultraproduct Theorem, Pretopos Version)**.** *Let $\mathcal{C}$ be a pretopos, let $\{M_i\}_{i \in I}$ be a collection of models of $\mathcal{C}$ indexed by a set I, and let $\mathcal{U}$ be an ultrafilter on I. Then the construction*

$$(C \in \mathcal{C}) \mapsto (\prod_{i \in I} M_i(C))/\,\mathcal{U}$$

is also a model of $\mathcal{C}$

Corollary 10 (Łos's Ultraproduct Theorem, Classical Version)**.** *Let T be a first-order theory in a language $\{P_j\}_{j \in J}$. Let $\{M_i\}_{i \in I}$ be a collection of models of T, and assume for simplicity that each M_i is nonempty. Suppose we are given an ultrafilter $\mathcal{U}$ on the set I, and set $M = (\prod_{i \in I} M_i)/\,\mathcal{U}$. Regard M as a structure for the language L by declaring*

$$(M \vDash P_j(\{\vec{c}_i\}_{i \in I})) \Leftrightarrow \{i \in I : M_i \vDash P_j(\vec{c}_i)\} \in \mathcal{U}\,.$$

Then M is also a model of T. Moreover, for any formula $\varphi(\vec{x})$ in the language L, we have

$$(M \vDash \varphi(\{\vec{c}_i\}_{i \in I})) \Leftrightarrow \{i \in I : M_i \vDash \varphi(\vec{c}_i)\} \in \mathcal{U}\,.$$

Proof. Apply Theorem 9 to the syntactic category $\mathrm{Syn}(T)$. (Note that the desired conclusion can be restated as $M[\varphi] \simeq (\prod_{i \in I} M_i[\varphi])/\,\mathcal{U}$.) □

It is not difficult to give a direct proof of Theorem 9 (or Corollary 10): the essential point is that the formation of ultraproducts commutes with the formation of finite limits, finite coproducts, and images. However, we will give a different explanation of Theorem 9, which connects up with the material of the last few lectures.

For the remainder of this lecture, let $\mathcal{C}$ be a small pretopos. Recall that the category $\mathrm{Pro}(\mathcal{C})$ has small limits and colimits.

Proposition 11. (1) *The subcategory $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$ of weakly projective pro-objects of $\mathcal{C}$ has (possibly infinite) coproducts, which are preserved by the inclusion $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \hookrightarrow \mathrm{Pro}(\mathcal{C})$.*

(2) *For every object $C \in \mathcal{C}$, the construction $M \mapsto M(C)$ determines a functor $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ which preserves (possibly infinite) products: that is, it carries coproducts in $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ to products of sets.*

(3) *The category $\mathrm{Stone}_{\mathcal{C}}$ has (possibly infinite) coproducts. Moreover, for each object $C \in \mathcal{C}$, the functor $(X, \mathcal{O}_X) \mapsto \mathcal{O}_X^C(X)$ carries coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to products of sets.*

Proof. Recall that $\mathrm{Pro}(\mathcal{C})$ can be defined as the opposite of the category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})$ of left exact functors from $\mathcal{C}$ to $\mathcal{S}\mathrm{et}$. Since the class of left exact functors is closed under inverse limits, it follows that colimits in $\mathrm{Pro}(\mathcal{C})$ are computed pointwise. In particular, given a collection of pro-objects $\{M_i\}_{i \in I}$, the coproduct $M = \amalg_{i \in I} M_i$ in the category $\mathrm{Pro}(\mathcal{C})$ is given by the formula $M(C) = \prod_{i \in I} M_i(C)$. From this description, it is clear that that if each M_i is weakly projective, then so is M (note that a product of surjections in the category of sets is again a surjection). This proves (1) and (2), and assertion (3) is just a restatement. □

Example 12 (Ultrafilters)**.** Let $\mathcal{C} = \mathcal{S}\mathrm{et}_{\mathrm{fin}}$ be the category of finite sets, so that $\mathrm{Stone}_{\mathcal{C}} \simeq \mathrm{Stone}$ is the category of Stone spaces. Proposition 11 implies that the category Stone admits coproducts. Beware that the inclusion $\mathrm{Stone} \hookrightarrow \mathrm{Top}$ does *not* preserve coproducts: a coproduct of Stone spaces is Hausdorff and totally disconnected, but usually not compact.

For example, let I be a set, and consider the coproduct $\amalg_{i \in I}\{i\}$, formed in the category Stone. We denote this coproduct by βI and refer to it as the *Stone-Čech compactification of I*. It is characterized by the following universal property: there is a map $\rho : I \to \beta I$ such that composition with ρ induces a bijection

$$\mathrm{Hom}_{\mathrm{Top}}(\beta I, X) \to \prod_{i \in I} X$$

for any Stone space X (or, more generally, any compact Hausdorff space X). In particular, taking X to be a two-point space, we obtain a bijection

$$\{\text{Clopen subsets of } \beta I\} \simeq \{\text{Arbitrary subsets of } I\}.$$

In other words, we can describe βI as the spectrum of the Boolean algebra $P(I)$ of subsets of I. It follows that βI can be identified with the set of Boolean algebra homomorphisms $\mu : P(I) \to \{0, 1\}$: that is, with the collection of all ultrafilters on I (see Remark 3). The topology on βI is generated by open (and closed) sets of the form

$$U_J := \{\mathcal{U} \in \beta I : J \in \mathcal{U}\},$$

where J ranges over all subsets of I (in fact, the construction $J \mapsto U_J$ implements the isomorphism of $P(I)$ with the Boolean algebra of clopen subsets of βI).

Remark 13. In the situation of Example 12, the canonical map $\rho : I \to \beta I$ carries each element $i \in I$ to the principal ultrafilter $\mathcal{U}_i$ of Example 4.

Example 14 (Ultraproducts)**.** Let us now return to the situation where $\mathcal{C}$ is an arbitrary small pretopos. Suppose we are given a collection of models $\{M_i \in \mathrm{Mod}(\mathcal{C})\}_{i \in I}$. We can then regard each pair $(\{i\}, M_i)$ as an object of $\mathrm{Stone}_{\mathcal{C}}$, and form the coproduct

$$(X, \mathcal{O}_X) = \amalg_{i \in I}(\{i\}, M_i)$$

in $\mathrm{Stone}_{\mathcal{C}}$.

Note that the forgetful functor $\mathrm{Stone}_{\mathcal{C}} \to \mathrm{Stone}$ preserves coproducts: it is given by the composition

$$\mathrm{Stone}_{\mathcal{C}} \simeq \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \hookrightarrow \mathrm{Pro}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{C})^{\mathrm{op}} = \mathrm{Pro}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}) = \mathrm{Stone}$$

induced by the morphism of pretopoi $\mathcal{S}\mathrm{et}_{\mathrm{fin}} \to \mathcal{C}$. It follows that we can identify the Stone space X with the Stone-Čech compactification βI. In particular, the construction

$$(J \subseteq I) \mapsto U_J = \{\mathcal{U} \in \beta I : J \in \mathcal{U}\}$$

induces a bijection from the collection $P(I)$ of subsets of I to the collection of clopen subsets of X. Unwinding the definitions, we see that $\mathcal{O}_X$ is given by the formula

$$\mathcal{O}_X^C(U_J) = \prod_{i \in J} M_i(C).$$

In particular, given a point $x \in X$ corresponding to an ultrafilter $\mathcal{U}$ on the set I, we have

$$\begin{aligned} \mathcal{O}_{X,x}^C &= \varinjlim_{x \in U_J} \mathcal{O}_X^C(U_J) \\ &= \varinjlim_{J \in \mathcal{U}} \prod_{i \in J} M_i(C) \\ &= (\prod_{i \in I} M_i(C))/\mathcal{U}. \end{aligned}$$

Proof of Theorem 9. Let $\mathcal{C}$ be a pretopos, let $\{M_i\}_{i \in I}$ be a collection of models of $\mathcal{C}$ indexed by a set I, and let $\mathcal{U}$ be an ultrafilter on I. Forming the coproduct $(X, \mathcal{O}_X) = \amalg_{i \in I}(\{i\}, M_i)$ in $\mathrm{Stone}_{\mathcal{C}}$, we observe that $\mathcal{U}$ can be identified with a point $x \in X \simeq \beta I$, and that the stalk $\mathcal{O}_{X,x}$ is a model of $\mathcal{C}$ given by the formula $C \mapsto (\prod_{i \in I} M_i(C))/\mathcal{U}$. □

We can summarize the situation informally as follows: given a collection of models $\{M_i\}_{i \in I}$ of a pretopos $\mathcal{C}$, we can construct a larger family of models parametrized by the Stone-Čech compactification βI, which assigns to each ultrafilter $\mathcal{U} \in \beta I$ the corresponding ultraproduct $(\prod_{i \in I} M_i)/\mathcal{U}$.

Definition 15. We will say that an object $M \in \mathrm{Pro}(\mathcal{C})$ is *free* if it can be written as a coproduct $\coprod_{i \in I} M_i$ in $\mathrm{Pro}(\mathcal{C})$, where each M_i is a model of $\mathcal{C}$. Note that in this case, M is automatically weakly projective.

We say that an object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ is *free* if it corresponds to a free object of $\mathrm{Pro}(\mathcal{C})$ under the equivalence $\mathrm{Stone}_{\mathcal{C}} \simeq \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$: that is, if it can be written as a coproduct

$$\coprod_{i \in I} (\{i\}, M_i)$$

in the category $\mathrm{Stone}_{\mathcal{C}}$.

Proposition 16. (1) *For every object $Z \in \mathrm{Pro}(\mathcal{C})$, there exists an effective epimorphism $M \to Z$, where M is free.*

(2) *For every object $(X, \mathcal{O}_X)$, there exists a covering $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ in $\mathrm{Stone}_{\mathcal{C}}$, where $(Y, \mathcal{O}_Y)$ is free.*

Proof. To prove (1) we may assume without loss of generality that Z is weakly projective. In this case, (1) and (2) are equivalent. Let us therefore consider (2). Fix an object $(X, \mathcal{O}_X)$ in $\mathrm{Stone}_{\mathcal{C}}$, and form the coproduct

$$(Y, \mathcal{O}_Y) = \coprod_{x \in X} (\{x\}, \mathcal{O}_{X,x}).$$

We claim that the tautological map $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a covering. Using the criterion of Lecture 18X, we are reduced to showing that for each point $x \in X$, we can choose a point $y \in Y$ lying over x for which the induced map of models $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is an isomorphism. Identifying Y with the set βX of ultrafilters on X, it suffices to choose y to correspond to the principal ultrafilter $\mathcal{U}_x$; in this case, the canonical map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is an isomorphism (Example 8). $\square$

Lecture 21X-Characterization of $\mathcal{C}$

April 4, 2018

Let $\mathcal{C}$ be an essentially small pretopos, which we regard as fixed throughout this lecture. We have fully faithful embeddings

$$\mathcal{C} \hookrightarrow \mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Stone}_{\mathcal{C}}) \subseteq \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et}).$$

Moreover, in Lectures 17X and 19X we established the following:

Proposition 1. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor. Then $\mathscr{F}$ belongs to the essential image of the embedding $\mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ if and only if it satisfies the following conditions:*

- (a) *The functor $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ preserves finite products: that is, it carries finite coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to finite products in the category of sets.*
- (b) *For every object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ and every point $x \in X$, the canonical map*
$$\varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X|_U) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$$
is bijective; here the colimit is taken over all clopen neighborhoods $U \subseteq X$ of the point x.
- (c) *The composite functor*
$$\mathrm{Mod}(\mathcal{C}) \hookrightarrow \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathcal{S}\mathrm{et}$$
commutes with filtered colimits.
- (d) *For every elementary morphism $f : M \to N$ in $\mathrm{Mod}(\mathcal{C})$, we have an equalizer diagram*
$$\mathscr{F}(M) \to \mathscr{F}(N) \rightrightarrows \prod \mathscr{F}(P)$$
where the product is taken over all commutative diagrams
$$M \xrightarrow{f} N \rightrightarrows P$$
in $\mathrm{Mod}(\mathcal{C})$.

Our goal in this lecture is to explain what additional conditions need to be satisfied for the functor $\mathscr{F}$ to belong to the essential image of the embedding $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$. This embedding is easy to describe: to an object $C \in \mathcal{C}$, it associates the functor

$$\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et} \qquad (X, \mathcal{O}_X) \mapsto \mathcal{O}_X^C(X),$$

which corresponds under the equivalence $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \simeq \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ to the evaluation functor $F \mapsto F(C)$. In the last lecture, we noted that the category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ admits small coproducts, which are computed as (pointwise) products in the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$. It follows that if $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is given by evaluation at an object $C \in \mathcal{C}$, then it satisfies the following stronger version of condition (a):

(a^+) The functor $\mathscr{F}$ carries (possibly infinite) coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to products in the category of sets.

We will show that, conversely, a functor $\mathscr{F}$ satisfying (a^+) together with conditions (b), (c), and (d) of Proposition 1 belongs to the essential image of $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$. Moreover, it suffices to check (a^+) in a restricted class of examples.

Theorem 2. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor which satisfies the conditions of Proposition 1, so that $\mathscr{F}$ is isomorphic to the image of some object $\mathscr{F}_0 \in \mathrm{Shv}(\mathcal{C})$. The following conditions are equivalent:*

(1) *The sheaf $\mathscr{F}_0 \in \mathrm{Shv}(\mathcal{C})$ is representable by an object $C \in \mathcal{C}$.*

(2) *The functor $\mathscr{F}$ satisfies condition (a^+) above.*

(3) *The functor $\mathscr{F}$ satisfies the following weaker version of (a^+):*

   (a') *For every collection of models $\{M_i \in \mathrm{Mod}(\mathcal{C})\}_{i \in I}$, the canonical map*

$$\mathscr{F}(\coprod_{i \in I}(\{i\}, M_i)) \to \prod_{i \in I} \mathscr{F}(\{i\}, M_i)$$

   is a bijection.

The implication (1) $\Rightarrow$ (2) was noted above, and the implication (2) $\Rightarrow$ (3) is immediate. We will complete the proof by showing that (3) $\Rightarrow$ (1). For this, we will need a variant of Deligne's completeness theorem.

Notation 3. Recall that every model $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ admits an essentially unique extension to a functor $\mathrm{Shv}(\mathcal{C}) \to \mathcal{S}\mathrm{et}$ which preserves small colimits and finite limits (that is, to a *point* of the topos $\mathrm{Shv}(\mathcal{C})$). In what follows, we will denote this extension by $\widehat{M} : \mathrm{Shv}(\mathcal{C}) \to \mathcal{S}\mathrm{et}$.

Lemma 4. *Let $u : \mathscr{G} \to \mathscr{F}$ be a morphism in the topos $\mathrm{Shv}(\mathcal{C})$. If u is not an effective epimorphism, then there exists a model M of $\mathcal{C}$ for which the map $\widehat{M}(\mathscr{G}) \to \widehat{M}(\mathscr{F})$ is not surjective.*

Proof. Since $\mathscr{F}$ admits a covering by representable functors, our assumption that u is not an effective epimorphism guarantees that we can choose an object $C \in \mathcal{C}$ and a morphism $h_C \to \mathscr{F}$ for which the projection map

$$h_C \times_{\mathscr{F}} \mathscr{G} \to h_C$$

is not an effective epimorphism. For any model $M \in \mathrm{Mod}(\mathcal{C})$, we have a pullback diagram of sets

$$\begin{array}{ccc} \widehat{M}(h_C \times_{\mathscr{F}} \mathscr{G}) & \longrightarrow & M(C) \\ \downarrow & & \downarrow \\ \widehat{M}(\mathscr{G}) & \longrightarrow & \widehat{M}(\mathscr{F}). \end{array}$$

Consequently, if the upper horizontal map is not surjective, then the lower horizontal map is also not surjective. We may therefore replace $\mathscr{F}$ by h_C (and $\mathscr{G}$ by the fiber product $h_C \times_{\mathscr{F}} \mathscr{G}$) and thereby reduce to the case where $\mathscr{F}$ is representable by an object $C \in \mathcal{C}$.

Choose an effective epimorphism $u : P \to C$ in $\mathrm{Pro}(\mathcal{C})$, where P is weakly projective. Under the equivalence $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \simeq \mathrm{Stone}_{\mathcal{C}}$, we can identify P with an object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$. Moreover, the map u determines a global section s of $\mathcal{O}_X^C(X)$. For each point $x \in X$, let us regard $\mathcal{O}_{X,x}$ as a model of $\mathcal{C}$, so that s determines an element $s_x \in \mathcal{O}_{X,x}^C$. Assume, for a contradiction, that each of the maps

$$\widehat{\mathcal{O}}_{X,x}(\mathscr{G}) \to \widehat{\mathcal{O}}_{X,x}(h_C) = \mathcal{O}_{X,x}^C$$

is surjective. Then each s_x can be lifted to an element $\widetilde{s}_x \in \widehat{\mathcal{O}}_{X,x}(\mathscr{G})$. Choose a covering $\{h_{C_i} \to \mathscr{G}\}_{i \in I}$ in the topos $\mathrm{Shv}(\mathcal{C})$. Then, for each point $x \in X$, we can choose an index $i(x) \in I$ such that $\widetilde{s}_x$ lifts to a

point $t_x \in \widehat{\mathcal{O}}_{X,x}(h_{C_{i(x)}}) \simeq \mathcal{O}_{X,x}^{C_{i(x)}}$. Choose an open set $U(x) \subseteq X$ containing x such that $\overline{s}_x$ can be lifted to $t \in \mathcal{O}_X^{C_{i(x)}}(U(x))$. Shrinking $U(x)$ if necessary, we may assume that the image of t in $\mathcal{O}_X^C(U(x))$ agrees with the restriction $s|_{U(x)}$.

Note that the open sets $\{U(x)\}_{x \in X}$ cover the topological space X. Since X is compact, we can choose a finite collection of points $x_1, x_2, \ldots, x_n \in X$ for which the open sets $U(x_1), \ldots, U(x_n)$ cover X. By construction, each restriction $s|_{U(x_j)}$ can be lifted to a section of $\mathcal{O}_X^{C_{i(x_j)}}$ over the open set $U(x_j)$. It follows that s is a global section of the subsheaf $\mathcal{O}_X^{C_0} \subseteq \mathcal{O}_X^C$, where $C_0 = \mathrm{Im}(\coprod C(x_j) \to C)$. Our assumption that u is an effective epimorphism then shows that we must have $C_0 = C$, contradicting our assumption that the map $\mathscr{G} \to h_C$ is not an effective epimorphism. $\square$

Proof of Theorem 2. Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor which satisfies the conditions of Proposition 1, so that $\mathscr{F}$ arises from a sheaf $\mathscr{F}_0 \in \mathrm{Shv}(\mathcal{C})$. Assume further that $\mathscr{F}$ satisfies condition (a'). We wish to prove that $\mathscr{F}_0$ belongs to the essential image of the Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Shv}(\mathcal{C})$. Since $\mathcal{C}$ is a pretopos, the sheaf $\mathscr{F}_0 \in \mathrm{Shv}(\mathcal{C})$ is representable by an object of $\mathcal{C}$ if and only if it is quasi-compact and quasi-separated.

We first show that $\mathscr{F}_0$ is quasi-compact. Choose a collection $\{u_i : h_{C_i} \to \mathscr{F}_0\}_{i \in I}$ of representatives for all maps from representable sheaves to $\mathscr{F}_0$. Since $\mathscr{F}_0$ is not quasi-compact, none of these maps is an effective epimorphism. For each index $i \in I$, we can use Lemma 4 to choose a model M_i and a point $\eta_i \in \mathscr{F}(\{i\}, M_i)$ which does not belong to the image of the map $M_i(C_i) \to \widehat{M_i}(\mathscr{F}_0) = \mathscr{F}(\{i\}, M_i)$. Set $(X, \mathcal{O}_X) = \amalg_{i \in I}(\{i\}, M_i)$, where the coproduct is formed in the category $\mathrm{Stone}_{\mathcal{C}}$. Using condition (a'), we see that the system $\{\eta_i\}_{i \in I}$ can be lifted (uniquely) to a point $\eta \in \mathscr{F}(X, \mathcal{O}_X)$ under the bijection $\mathscr{F}(X, \mathcal{O}_X) \to \prod_{i \in I} \mathscr{F}(\{i\}, M_i)$.

For each point $x \in X$, let η_x denote the image of η in $\mathscr{F}(\{x\}, \mathcal{O}_{X,x}) \simeq \widehat{\mathcal{O}}_{X,x}(\mathscr{F}_0)$. Then there exists some $i(x) \in I$ such that η_x can be lifted to an element $\widetilde{\eta}_x \in \widehat{\mathcal{O}}_{X,x}(h_{C_{i(x)}}) \simeq \mathcal{O}_{X,x}^{C_{i(x)}}$. Choose a clopen open set $U(x)$ containing x and lift of $\widetilde{\eta}_x$ to some $s_x \in \mathcal{O}_X^{C_{i(x)}}(U(x))$. Let $\overline{s}_x$ denote the image of s_x in $\mathscr{F}(U(x), \mathcal{O}_X|_{U(x)})$. By construction, $\overline{s}_x$ and η have the same image in $\mathscr{F}(\{x\}, \mathcal{O}_{X,x})$. It follows from (b) that we can assume, after shrinking $U(x)$ if necessary, that $\overline{s}_x = \eta|_{U(x)}$.

Since X is compact, we can choose finitely many points $x_1, \ldots, x_n$ for which the open sets $U(x_1), \ldots, U(x_n)$ cover X. Then the map

$$(u_{i(x_1)}, \ldots, u_{i(x_n)}) : (h_{C_{i(x_1)}} \amalg \cdots \amalg h_{C_{i(x_n)}} \to \mathscr{F}_0$$

can be identified with $u_j : h_{C_j} \to \mathscr{F}_0$ for some $j \in I$. Let y denote the image of j in $X = \beta I$ (corresponding to the principal ultrafilter associated to j). Then we have $y \in U(x)$ for some $x \in \{x_1, \ldots, x_n\}$. By construction, it follows that $\eta|_{U(x)}$ can be lifted to the point $s_x \in \mathcal{O}_X^{C_{i(x)}}(U(x))$, so that the stalk η_y belongs to the image of the map

$$\mathcal{O}_{X,y}^{C_{i(x)}} \simeq \widehat{\mathcal{O}}_{X,y}(h_{C_{i(x)}}) \to \widehat{\mathcal{O}}_{X,y}(\mathscr{F}_0)$$

determined by the map $u_{i(x)} : h_{C_{i(x)}} \to \mathscr{F}_0$. However, the map $u_{i(x)}$ factors through $u_j : h_{C_j} \to \mathscr{F}_0$, so that η_y also belongs to the image of the map

$$\mathcal{O}_{X,y}^{C_j} \simeq M_j(C_j) \to \widehat{\mathcal{O}}_{X,y}(\mathscr{F}_0),$$

contradicting our choice of M_j. This completes the proof that $\mathscr{F}_0$ is quasi-compact.

We now complete the proof by showing that $\mathscr{F}_0$ is quasi-separated. Choose a pair of quasi-compact objects $\mathscr{G}_0, \mathscr{H}_0 \in \mathrm{Shv}(\mathcal{C})_{/\mathscr{F}_0}$; we wish to show that the fiber product $\mathscr{G}_0 \times_{\mathscr{F}_0} \mathscr{H}_0$ is quasi-compact. Covering $\mathscr{G}$ and $\mathscr{H}$ by representable sheaves, we may assume that $\mathscr{G}_0$ and $\mathscr{H}_0$ are representable by objects of $\mathcal{C}$. Let $\mathscr{G}$ and $\mathscr{H}$ denote the images of $\mathscr{G}_0$ and $\mathscr{H}_0$ in the category $\mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$. Then $\mathscr{G}$ and $\mathscr{H}$ satisfy condition (a') (even condition (a^+)), so that $\mathscr{G} \times_{\mathscr{F}} \mathscr{H}$ also satisfies condition (a'). The preceding argument then shows that $\mathscr{G}_0 \times_{\mathscr{F}_0} \mathscr{H}_0$ is quasi-compact, as desired. $\square$

Lecture 22X-Embedding into an Ultrapower

April 6, 2018

Let $\mathcal{C}$ be a small pretopos, which we regard as fixed through this lecture. We let $\mathrm{Pro}^{\mathrm{fr}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Pro}(\mathcal{C})$ spanned by the *free* objects (that is, those which can be written as coproducts of models), and we define $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \subseteq \mathrm{Stone}_{\mathcal{C}}$ similarly, so that we have an equivalence $\mathrm{Pro}^{\mathrm{fr}}(\mathcal{C}) \simeq \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$. We have fully faithful embeddings

$$\mathcal{C} \hookrightarrow \mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Pro}^{\mathrm{fr}}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}) \subseteq \mathrm{Fun}(\mathrm{Stone}^{\mathrm{fr},\mathrm{op}}_{\mathcal{C}}, \mathcal{S}\mathrm{et}).$$

We have seen that every functor $\mathscr{F} : \mathrm{Stone}^{\mathrm{fr},\mathrm{op}}_{\mathcal{C}} \to \mathcal{S}\mathrm{et}$ which belongs to the essential image of the composite embedding must satisfy the following pair of conditions:

(a') For every collection of models $\{M_i\}_{i \in I}$ of $\mathcal{C}$, the canonical map

$$\mathscr{F}(\coprod_{i \in I}(\{i\}, M_i)) \to \prod_{i \in I} \mathscr{F}(\{i\}, M_i)$$

is a bijection.

(b') For every object $(X, \mathcal{O}_X) \in \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$ and every point $x \in X$, the canonical map $\varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X |_U) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$ is a bijection. Here U ranges over all clopen neighborhoods of x in X.

Remark 1. Suppose that $\mathscr{F} : \mathrm{Stone}^{\mathrm{fr},\mathrm{op}}_{\mathcal{C}} \to \mathcal{S}\mathrm{et}$ satisfies conditions (a') and (b'). Let $\{M_i\}_{i \in I}$ be a collection of models of M, and set $(X, \mathcal{O}_X) = \coprod_{i \in I}(\{i\}, M_i)$ in $\mathrm{Stone}_{\mathcal{C}}$. Suppose that x is a point of X, corresponding to an ultrafilter $\mathcal{U}$ on the set I. Then every clopen neighborhood of x in X has the form $U_J = \{\mathcal{U}' \in X : J \in \mathcal{U}'\}$, where J belongs to the ultrafilter $\mathcal{U}$. We then compute

$$\begin{aligned}
\mathscr{F}((\prod_{i \in I} M_i)/\mathcal{U}) &\simeq \mathscr{F}(\{x\}, \mathcal{O}_{X,x}) \\
&\simeq \varinjlim_{x \in U_J} \mathscr{F}(U_J, \mathcal{O}_X |_{U_J}) \\
&\simeq \varinjlim_{J \in \mathcal{U}} \mathscr{F}(\coprod_{i \in J}(\{i\}, M_i)) \\
&\simeq \varinjlim_{J \in \mathcal{U}} \prod_{i \in J} \mathscr{F}(\{i\}, M_i) \\
&\simeq (\prod_{i \in I} \mathscr{F}(M_i))/\mathcal{U}.
\end{aligned}$$

In other words, the functor $\mathscr{F} |_{\mathrm{Mod}(\mathcal{C})}$ "commutes with ultraproducts". We will return to this observation later.

Remark 2. Since every object of $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$ can be written as a coproduct of models, condition (a') is equivalent to the following *a priori* stronger condition:

(a'') The functor $\mathscr{F}$ carries coproducts in $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ to products in $\mathcal{S}\mathrm{et}$.

Over the next few lectures, we will prove the converse: any functor $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr,op}} \to \mathcal{S}\mathrm{et}$ satisfying (a') and (b') arises from an object of $\mathcal{C}$. In this lecture, we will carry out the first step by proving the following:

Theorem 3. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr,op}} \to \mathcal{S}\mathrm{et}$ be a functor satisfying (a') and (b'). Then $\mathscr{F}$ also satisfies the following:*

(d) *For every elementary morphism $f : M \to N$ in $\mathrm{Mod}(\mathcal{C})$, we have an equalizer diagram*

$$\mathscr{F}(M) \to \mathscr{F}(N) \rightrightarrows \prod \mathscr{F}(P)$$

where the product is taken over all commutative diagrams $M \xrightarrow{f} N \rightrightarrows P$.

Corollary 4. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr,op}} \to \mathcal{S}\mathrm{et}$ be a functor satisfying (a') and (b'). Then $\mathscr{F}$ is a sheaf (with respect to the topology on $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ generated by the finite coverings in $\mathrm{Stone}_{\mathcal{C}}$).*

Proof. This follows by exactly the same argument we used in Lecture 19X, replacing $\mathrm{Stone}_{\mathcal{C}}$ by the subcategory $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$. □

We begin with some preliminaries. Suppose we are given a collection of models $\{M_i\}_{i \in I}$ of $\mathcal{C}$, indexed by a set I. In Lecture 20X, we defined the ultraproduct $(\prod_{i \in I} M_i)/\mathcal{U}$ associated to an ultrafilter $\mathcal{U}$ on I: it is the functor from $\mathcal{C}$ to $\mathcal{S}\mathrm{et}$ given by the construction $C \mapsto (\prod_{i \in I} M_i(C))/\mathcal{U}$. In the special case where each M_i is equal to some fixed model M, we denote this ultraproduct by $M^I/\mathcal{U}$ and refer to it as the *ultrapower of M with respect to the ultrafilter $\mathcal{U}$*. Note that the diagonal embedding $M(C) \mapsto M(C)^I$ induces a map

$$M(C) \to M(C)^I \to \varinjlim_{J \in \mathcal{U}} M(C)^J = (M^I/\mathcal{U})(C).$$

This map depends functorially on C, and can therefore be regarded as a morphism of models $\delta_M : M \to M^I/\mathcal{U}$. We will deduce Theorem 3 from the following:

Theorem 5. *Let $f : M \to N$ be an elementary map between models of $\mathcal{C}$. Then there exists a set I, an ultrafilter $\mathcal{U}$ on I, and a map of models $g : N \to M^I/\mathcal{U}$ for which the composite map*

$$M \xrightarrow{f} N \xrightarrow{g} M^I/\mathcal{U}$$

coincides with the diagonal embedding δ_M.

Exercise 6. Let M be a model of $\mathcal{C}$, I a set, and $\mathcal{U}$ an ultrafilter on I. Show that the map of models $\delta_M : M \to M^I/\mathcal{U}$ is elementary.

Proof of Theorem 3 from Theorem 5. Suppose that $f : M \to N$ is an elementary map between models of $\mathcal{C}$, and that we are given a point $\eta \in \mathscr{F}(N)$ which belongs to the equalizer

$$\mathrm{Eq}(\mathscr{F}(N) \rightrightarrows \prod_{M \to N \rightrightarrows P} \mathscr{F}(P)).$$

We wish to show that η can be lifted uniquely to an element of $\mathscr{F}(M)$.

Choose $g : N \to M^I/\mathcal{U}$ as in Theorem 5, and let $\overline{\eta}$ denote the image of η in $\mathscr{F}(M^I/\mathcal{U})$. Since $\mathscr{F}$ commutes with ultraproducts (Remark 1), we can identify $\overline{\eta}$ with an element of $\mathscr{F}(M)^I/\mathcal{U}$. Choose a representative $\{\eta_i \in \mathscr{F}(M)\}_{i \in I}$ for $\overline{\eta}$. We have a commutative diagram of models

$$\begin{tikzcd} M \arrow[r, "\delta_M"] \arrow[d, "\delta_M"] & M^I/\mathcal{U} \arrow[d, "(\delta_M)^I/\mathcal{U}"] \\ M^I/\mathcal{U} \arrow[r, "\delta_{M^I/\mathcal{U}}"] & (M^I/\mathcal{U})^I/\mathcal{U}. \end{tikzcd}$$

Consequently, our hypothesis on η guarantees that the maps

$$\delta_M^I/\mathcal{U}, \delta_{M^I/\mathcal{U}} : M^I/\mathcal{U} \to (M^I/\mathcal{U})^I/\mathcal{U}$$

carry $\overline{\eta}$ to the same element of

$$\mathscr{F}((M^I/\mathcal{U})^I/\mathcal{U}) \simeq (\mathscr{F}(M)^I/\mathcal{U})^I/\mathcal{U}\,.$$

Unwinding the definitions, this tells us that the set $\{i \in I : \{j \in I : \eta_i = \eta_j\} \in \mathcal{U}\}$ belongs to $\mathcal{U}$. In particular, it is nonempty: that is, there exists some $i \in I$ such that $\eta_i = \eta_j$ for almost all $j \in I$ (with respect to the ultrafilter $\mathcal{U}$). We will complete the proof by showing that η is the image of $\eta_i \in \mathscr{F}(M)$. To prove this, we use the commutativity of the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle \delta_M} & & \downarrow{\scriptstyle \delta_N} \\ M^I/\mathcal{U} & \xrightarrow{f^I/\mathcal{U}} & N^I/\mathcal{U} \end{array}$$

to observe that the maps $(f^I/\mathcal{U}) \circ g, \delta_N : N \to N^I/\mathcal{U}$ agree on M, and therefore carry η to the same element of $N^I/\mathcal{U}$. It follows that the set $\{j \in J : \mathscr{F}(f)(\eta_j) = \eta\}$ belongs to $\mathcal{U}$, and therefore has nonempty intersection with the set $\{j \in J : \eta_j = \eta_i\}$. $\square$

Proof of Theorem 5. To avoid confusion, let us use the notation T_M to denote the image of a model $M \in \mathrm{Mod}(\mathcal{C})$ under the inclusion $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}} \hookrightarrow \mathrm{Pro}(\mathcal{C})$. The elementary map $f : M \to N$ can then be identified with a map of pro-objects $T_N \to T_M$, which we will denote by T_f. Our assumption that f is elementary guarantees that T_f is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$, and can therefore be realized as the limit of an inverse system $\{u_\alpha : C_\alpha \to D_\alpha\}$, where each u_α is an effective epimorphism in $\mathcal{C}$. Without loss of generality, we may assume that this inverse limit is indexed by the opposite of partially ordered set I which is *directed* (so that every finite subset of I has an upper bound in I).

For each $\alpha \in I$, set $P_\alpha = C_\alpha \times_{D_\alpha} T_M$ (where the fiber product is formed in $\mathrm{Pro}(\mathcal{C})$. Since M is a model, each of the maps $T_M \to D_\alpha$ factors through u_α. A choice of factorization determines a section $s_\alpha : T_M \to P_\alpha$ of the projection map $P_\alpha \to T_M$. To avoid confusion, let us write F_α for the image of P_α in the opposite category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et}) \simeq \mathrm{Pro}(\mathcal{C})^{\mathrm{op}}$, so that each s_α can be viewed as a natural transformation of functors $F_\alpha \to M$. Note that we have $T_N \simeq \varprojlim_\alpha P_\alpha$ in $\mathrm{Pro}(\mathcal{C})$, so that $N \simeq \varinjlim_\alpha F_\alpha$ in $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})$.

Let $\mathcal{U}_0$ be the collection of all subsets $J \subseteq I$ for which there exists some $\alpha \in I$ such that $\{\beta \in I : \beta \geq \alpha\} \subseteq J$. Our assumption that I is directed guarantees that $\mathcal{U}_0$ is a (nontrivial) filter on I. We can therefore choose an ultrafilter $\mathcal{U}$ which contains $\mathcal{U}_0$. For each object $C \in \mathcal{C}$ and each $\alpha \in I$, we have a canonical map $F_\alpha(C) \to \prod_{\beta \geq \alpha} F_\beta(C)$ given by the transition maps in the direct system $\{F_\beta(C)\}_{\beta \in I}$, which induces a map from $F_\alpha(C)$ to the ultraproduct $(\prod_{\beta \in I} F_\beta(C))/\mathcal{U}$. This construction depends functorially on α, and therefore yields a map

$$N(C) = \varinjlim_{\alpha \in I} F_\alpha(C) \to \varinjlim_{\alpha \in I} (\prod_{\beta \geq \alpha} F_\beta(C)) \to \varinjlim_{J \in \mathcal{U}} \prod_{\beta \in J} F_\beta(C) = (\prod_{\beta \in I} F_\beta(C))/\mathcal{U}\,.$$

Composing with the map

$$(\prod_{\beta \in I} F_\beta(C))/\mathcal{U} \xrightarrow{\{s_\beta\}} (\prod_{\beta \in I} M(C))/\mathcal{U} = (M^I/\mathcal{U})(C),$$

we obtain a map $N(C) \to (M^I/\mathcal{U})(C)$ which depends functorially on C, and can therefore be regarded as a map of models $N \to M^I/\mathcal{U}$. Our assumption that each s_α is a section of the projection map guarantees the the composition $M \to N \to M^I/\mathcal{U}$ agrees with the diagonal embedding. $\square$

Lecture 23X-Compatibility with Filtered Colimits

April 6, 2018

Let $\mathcal{C}$ be a small pretopos, which we regard as fixed through this lecture. We have fully faithful embeddings

$$\mathcal{C} \hookrightarrow \mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Stone}_{\mathcal{C}}) \simeq \mathrm{Shv}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}) \subseteq \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr,op}}, \mathcal{S}\mathrm{et}).$$

Our goal in this lecture is to prove the following:

Theorem 1. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr,op}} \to \mathcal{S}\mathrm{et}$ be a functor. The $\mathscr{F}$ belongs to the essential image of $\mathcal{C}$ if and only if it satisfies the following pair of conditions:*

(a') *For every collection of models $\{M_i\}_{i \in I}$ of $\mathcal{C}$, the canonical map*

$$\mathscr{F}(\coprod_{i \in I}(\{i\}, M_i)) \to \prod_{i \in I} \mathscr{F}(\{i\}, M_i)$$

is a bijection.

(b') *For every object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ and every point $x \in X$, the canonical map $\varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X|_U) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$ is a bijection. Here U ranges over all clopen neighborhoods of x in X.*

We have already seen that conditions (a') and (b') are necessary. Moreover, we proved in Lecture 22X that conditions (a') and (b') imply that $\mathscr{F}$ is a sheaf on the category $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$, and therefore admits an essentially unique extension to a sheaf on the entire category $\mathrm{Stone}_{\mathcal{C}}$. Consequently, to show that $\mathscr{F}$ belongs to the essential image of $\mathrm{Shv}(\mathcal{C})$, it will suffice to show that this extension commutes with filtered colimits (Lecture 15X). In this case, we have seen that condition (a') guarantees that $\mathscr{F}$ also belongs to the essential image of $\mathcal{C}$ (Lecture 21X). We may therefore reformulate Theorem 1 as follows:

Theorem 2. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a sheaf and suppose that $\mathscr{F}|_{\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr,op}}}$ satisfies conditions (a') and (b') of Theorem 1. Then $\mathscr{F}$ preserves filtered colimits (that is, it carries filtered limits in $\mathrm{Stone}_{\mathcal{C}}$ to filtered colimits in $\mathcal{S}\mathrm{et}$). Using the criterion of Lecture 17X, we can state this more concretely as follows:*

(b) *For every object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ and every point $x \in X$, the canonical map*

$$\varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X|_U) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$$

is bijective; here the colimit is taken over all clopen neighborhoods $U \subseteq X$ of the point x.

(c) *The composite functor*

$$\mathrm{Mod}(\mathcal{C}) \hookrightarrow \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathcal{S}\mathrm{et}$$

commutes with filtered colimits.

Proof. We begin by proving (c). Fix a diagram of models $\{M_\alpha\}_{\alpha \in I}$ indexed by a directed partially ordered set I. Set $M = \varinjlim_{\alpha \in I} M_\alpha$; we wish to show that the canonical map

$$\rho : \varinjlim_{\alpha} \mathscr{F}(M_\alpha) \to \mathscr{F}(M)$$

is bijective.

As in Lecture 22X, we can choose an ultrafilter $\mathcal{U}$ on the set I such that, for each $\beta \in I$, the subset $I_{\geq \beta} = \{\alpha \in I : \alpha \geq \beta\}$ is contained in $\mathcal{U}$. For each object $C \in \mathcal{C}$ and each index $\beta \in I$, the transition maps in the diagram $\{M_\alpha\}_{\alpha \in I}$ determine a canonical map

$$M_\beta(C) \to \prod_{\alpha \in I_{\geq \beta}} M_\alpha(C) \to \varinjlim_{J \in \mathcal{U}} \prod_{\alpha \in J} M_\alpha(C) = ((\prod_{\alpha \in I} M_\alpha)/\mathcal{U})(C).$$

Passing to the colimit over β, we obtain a map $M(C) \to ((\prod_{\alpha \in I} M_\alpha)/\mathcal{U})(C)$ depending functorially on C, which we can identify with a map of models $f : M \to (\prod_{\alpha \in I} M_\alpha)/\mathcal{U}$

Note that the canonical maps $M_\alpha \to M$ induce a map g from $(\prod_{\alpha \in I} M_\alpha)/\mathcal{U}$ to the ultrapower $M^I/\mathcal{U}$. By construction, the composition $M \xrightarrow{f} (\prod_{\alpha \in I} M_\alpha)/\mathcal{U} \xrightarrow{g} M^I/\mathcal{U}$ agrees with the diagonal map $\delta_M : M \to M^I/\mathcal{U}$.

We now prove the surjectivity of ρ. Suppose we are given an element $x \in \mathscr{F}(M)$. Recall that conditions (a') and (b') imply that $\mathscr{F}$ commutes with the formation of ultraproducts (see Lecture 22X). We may therefore identify $\mathscr{F}(f)(x) \in \mathscr{F}((\prod_{\alpha \in I} M_\alpha)/\mathcal{U})$ with an element of $(\prod_{\alpha \in I} \mathscr{F}(M_\alpha))/\mathcal{U}$, which we can represent by a tuple of elements $\{x_\alpha \in \mathscr{F}(M_\alpha)\}_{\alpha \in J}$ for some subset $J \subseteq I$ which belong to the ultrafilter $\mathcal{U}$. Each x_α has some image y_α in $\mathscr{F}(M)$, and we can identify $\{y_\alpha\}_{\alpha \in J}$ with the image of x under the composite map

$$\mathscr{F}(M) \to \mathscr{F}((\prod_{\alpha \in I} M_\alpha)/\mathcal{U}) \to \mathscr{F}(M^I/\mathcal{U}).$$

Since this composite map agrees with the diagonal δ_M, the equality $y_\alpha = x$ must hold almost everywhere: that is, we can choose some $J' \subseteq J$ belonging to $\mathcal{U}$ such that $y_\alpha = x$ for $\alpha \in J'$. Then x belongs to the image of the map $\mathscr{F}(M_\alpha) \to \mathscr{F}(M)$ for each $\alpha \in J'$, and in particular belongs to the image of ρ.

We now show that ρ is injective. Fix an index $\beta \in I$ and a pair of elements $x, y \in \mathscr{F}(M_\beta)$ having the same image in $\mathscr{F}(M)$; we wish to show that x and y have the same image in $\mathscr{F}(M_\alpha)$ for some $\alpha \geq \beta$. Replacing I by the set $I_{\geq \beta}$, we can assume without loss of generality that β is a least element of I. Note that we have a commutative diagram of models

$$\begin{array}{ccc} M_\beta & \xrightarrow{\delta} & M_\beta^I/\mathcal{U} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & (\prod_{\alpha \in I} M_\alpha)/\mathcal{U}. \end{array}$$

It follows that x and y have the same image under the composite map

$$\mathscr{F}(M_\beta) \to \mathscr{F}(M_\beta^I/\mathcal{U}) \to \mathscr{F}((\prod_{\alpha \in I} M_\alpha)/\mathcal{U}) \simeq (\prod_{\alpha \in I} \mathscr{F}(M_\alpha))/\mathcal{U}.$$

In other words, x and y have the same image in $\mathscr{F}(M_\alpha)$ for almost every $\alpha \in I$; this completes the proof of (c).

We now prove (b). Fix an object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ and a point $x \in X$; we wish to show that the canonical map

$$\phi : \varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X|_U) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$$

is bijective. We first show that ϕ is injective. Suppose we are given a clopen subset $U \subseteq X$ containing x and a pair of elements $u, v \in \mathscr{F}(U, \mathcal{O}_X|_U)$ with the same image in $\mathscr{F}(\{x\}, \mathcal{O}_{X,x})$. We wish to show that u and v have the same image $\mathscr{F}(U', \mathcal{O}_X|_{U'})$ for some clopen subset $U' \subseteq U$ containing x. Choose a covering $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$, where $(Y, \mathcal{O}_Y)$ is free. Then u and v have the same image in $\mathscr{F}(\{y\}, \mathcal{O}_{Y,y})$ for each point $y \in Y_x = f^{-1}\{x\}$. Using condition (b'), we see that u and v have the same image in $\mathscr{F}(V_y, \mathcal{O}_Y|_{V_y})$ for some clopen neighborhood V_y of y. Covering the fiber Y_x by finitely many sets of the form V_y, we conclude that there is a clopen set $V \subseteq f^{-1}(U)$ containing Y_x such that u and v have the same image in $\mathscr{F}(V, \mathcal{O}_Y|_V)$. Since the map of topological spaces $f : Y \to X$ is closed, we can assume without loss of generality that $V = f^{-1}(U_0)$ for some clopen set $U_0 \subseteq U$ containing x. Note that the map $(V, \mathcal{O}_Y|_V) \to (U_0, \mathcal{O}_X|_{U_0})$ is a covering in $\mathrm{Stone}_{\mathcal{C}}$, so that the map of sets $\mathscr{F}(U_0, \mathcal{O}_X|_{U_0}) \to \mathscr{F}(V, \mathcal{O}_Y|_V)$ is injective. It follows that u and v have the same image in $\mathscr{F}(U_0, \mathcal{O}_X|_{U_0})$, as desired.

We now show that ϕ is surjective. Fix an element $s \in \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$. For each point $y \in Y_x$, let s_y denote the image of s in $\mathscr{F}(\{y\}, \mathcal{O}_{Y,y})$. Using assumption (b'), we can lift s_y to an element $\widetilde{s}_y \in \mathscr{F}(V_y, \mathcal{O}_Y|_{V_y})$ for some clopen set $V_y \subseteq Y$ containing y. Applying the first part of the proof to the object $(Y_x, \mathcal{O}_Y|_{Y_x})$, we conclude that there is clopen set $T_y \subseteq Y_x$ containing y such that $\widetilde{s}_y$ and s have the same image in $\mathscr{F}(T_y, \mathcal{O}_Y|_{T_y})$. Shrinking V_y if necessary, we may assume that $T_y = V_y \cap Y_x$. Since Y_x is compact, it is contained in the union $V = V_{y_1} \cup \cdots \cup V_{y_n}$ for finitely many elements $y_1, \ldots, y_n \in Y_x$. By passing to a disjoint refinement of the covering of V by the sets V_{y_i}, we can amalgamate the sections $\{\widetilde{s}_y\}$ to a single element $\widetilde{s} \in \mathscr{F}(V, \mathcal{O}_Y|_V)$ such that s and $\widetilde{s}$ have the same image in $\mathscr{F}(Y_x, \mathcal{O}_Y|_{Y_x})$.

Let us abuse notation by identifying $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ with objects of $\mathrm{Pro}(\mathcal{C})$, and form the fiber product $(Y, \mathcal{O}_Y) \times_{(X, \mathcal{O}_X)} (Y, \mathcal{O}_Y)$ in $\mathrm{Pro}(\mathcal{C})$. Choose an effective epimorphism

$$(Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y) \times_{(X, \mathcal{O}_X)} (Y, \mathcal{O}_Y),$$

where $(Z, \mathcal{O}_Z) \in \mathrm{Stone}_{\mathcal{C}}$. We then have a pair of projection maps $\pi, \pi' : Z \to Y$. Let t and t' denote the images of $\widetilde{s}$ in $\mathscr{F}(\pi^{-1}(V), \mathcal{O}_Z|_{\pi^{-1}(V)})$ and $\mathscr{F}(\pi'^{-1}(V), \mathcal{O}_Z|_{\pi'^{-1}(V)})$, respectively. Note that t and t' have the same image in $\mathscr{F}(\{z\}, \mathcal{O}_{Z,z})$ for each $z \in Z \times_X \{x\}$. Using the first part of the proof, we see that there exists a clopen set $W \subseteq \pi^{-1}(V) \cap \pi'^{-1}(V)$ containing $Z \times_X \{x\}$ such that t and t' have the same image in $\mathscr{F}(W, \mathcal{O}_Z|_W)$. Since the projection maps $Z \to X \leftarrow Y$ are closed, we can choose a clopen subset $U \subseteq X$ containing x such that $U \times_X Y \subseteq V$ and $U \times_X Z \subseteq W$. Replacing V and W by the inverse images of U, we can invoke our assumption that $\mathscr{F}$ is a sheaf to deduce that the diagram of sets

$$\mathscr{F}(U, \mathcal{O}_X|_U) \to \mathscr{F}(V, \mathcal{O}_Y|_V) \rightrightarrows \mathscr{F}(Z, \mathcal{O}_Z|_W)$$

is an equalizer, so that $\widetilde{s} \in \mathscr{F}(V, \mathcal{O}_Y|_V)$ lifts uniquely to an element $s' \in \mathscr{F}(U, \mathcal{O}_X|_U)$. Using the commutativity of the diagram

$$\begin{tikzcd}
\mathscr{F}(U, \mathcal{O}_X|_U) \arrow[r] \arrow[d] & \mathscr{F}(\{x\}, \mathcal{O}_{X,x}) \arrow[d] \\
\mathscr{F}(V, \mathcal{O}_Y|_V) \arrow[r] & \mathscr{F}(Y_x, \mathcal{O}_Y|_{Y_x})
\end{tikzcd}$$

(and the fact that the right vertical map is injective), we deduce that the upper horizontal map carries s' to s, so that s belongs to the image of ϕ. $\square$

Lecture 24X-Ultracategory Fibrations

April 11, 2018

Let $\mathcal{C}$ be a small pretopos, which we regard as fixed throughout this lecture. Recall that an object of $\mathrm{Stone}_{\mathcal{C}}$ is *free* if it can be written as a coproduct $\coprod_{i \in I}(\{i\}, M_i)$, where $\{M_i\}_{i \in I}$ is a collection of models of $\mathcal{C}$ indexed by a set I. Let $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ denote the full subcategory of $\mathrm{Stone}_{\mathcal{C}}$ spanned by the free objects. By definition, the category $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ contains the category of models $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$ as a full subcategory, and every object of $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ can be written as a coproduct of objects of $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$. Our goal in this lecture is to address the following:

Question 1. What is the structure of the category $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$? To what extent can it be reconstructed from the category $\mathrm{Mod}(\mathcal{C})$ of models of $\mathcal{C}$?

Let us begin by treating the case where $\mathcal{C} = \mathcal{S}\mathrm{et}_{\mathrm{fin}}$ is the category of finite sets. Then $\mathrm{Stone}_{\mathcal{C}}$ can be identified with the category Stone of Stone spaces, and $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ can be identified with the category $\mathrm{Stone}^{\mathrm{fr}}$ of *free* Stone spaces: that is, Stone spaces which have the form βI, for some set I. This category can be described explicitly as follows:

- The objects of $\mathrm{Stone}^{\mathrm{fr}}$ are Stone spaces of the form βI, where I is a set.
- Morphisms in $\mathrm{Stone}^{\mathrm{fr}}$ are given by continuous maps of Stone spaces $\beta I \to \beta J$. Using the universal property of βI, we see that a morphism from βI to βJ is just given by a map of sets $I \to \beta J$, or equivalently a collection $\{\mathcal{U}_i\}_{i \in I}$ of ultrafilters on J which parametrized by the set I.

Remark 2. The construction $I \mapsto \beta I$ determines a faithful and essentially surjective functor from the category $\mathcal{S}\mathrm{et}$ to the category $\mathrm{Stone}^{\mathrm{fr}}$. For every pair of sets I and J, we can identify $\mathrm{Hom}_{\mathcal{S}\mathrm{et}}(I, J)$ with the subset of $\mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}}(\beta I, \beta J) \simeq \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(I, \beta J)$ consisting of those maps which carry I into J. In other words, it corresponds to the subset of $\mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}}(\beta I, \beta J)$ corresponding to those collections of ultrafilters $\{\mathcal{U}_i\}_{i \in I}$ where each $\mathcal{U}_i$ is principal.

Exercise 3. Let I, J, and K be sets, and suppose we are given morphisms

$$\beta I \xrightarrow{f} \beta J \xrightarrow{g} \beta K,$$

so that f determines a collection $\{\mathcal{U}_i\}_{i \in I}$ of ultrafilters on J, and g determiens a collection $\{\mathcal{V}_j\}_{j \in J}$ of ultrafilters on K. Show that the composition $g \circ f$ corresponds to the collection of ultrafilters $\{\mathcal{W}_i\}_{i \in I}$ on K, where

$$(K_0 \in \mathcal{W}_i) \Leftrightarrow (\{j \in J : K_0 \in \mathcal{V}_j\} \in \mathcal{U}_i).$$

Let's now return to the case of a general pretopos $\mathcal{C}$. Note that we have a forgetful functor

$$\pi : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}} \to \mathrm{Stone}^{\mathrm{fr}} \qquad \pi(X, \mathcal{O}_X) = X.$$

We now articulate a special feature of the functor π.

Definition 4. Let $\pi : \mathcal{E} \to \mathcal{B}$ be a functor between categories. For each object $B \in \mathcal{B}$, we let $\mathcal{E}_B$ denote the fiber product $\mathcal{E} \times_{\mathcal{B}} \{B\}$.

Let $\overline{f} : E' \to E$ be a morphism in $\mathcal{E}$ having image $f : B' \to B$ in the category $\mathcal{B}$. We will say that $\overline{f}$ is *locally π-Cartesian* if the following condition is satisfied: for every object $E'' \in \mathcal{E}_{B'}$ with composition with $\overline{f}$ induces a bijection

$$\mathrm{Hom}_{\mathcal{E}_{B'}}(E'', E') \to \mathrm{Hom}_{\mathcal{E}}(E'', E) \times_{\mathrm{Hom}_{\mathcal{B}}(B', B)} \{f\}.$$

We say that π is a *local Grothendieck fibration* if, for every object $E \in \mathcal{E}$ and every morphism $f : B' \to \pi(E)$ in $\mathcal{B}$, there exists a locally π-Cartesian morphism $\overline{f} : E' \to E$ with $\pi(f) = f$. In this case, it follows from Yoneda's lemma that the object E' is well-defined up to canonical isomorphism as an object of the category $\mathcal{E}_{B'}$. Moreover, the construction $E \mapsto E'$ determines a functor $f^* : \mathcal{E}_B \to \mathcal{E}_{B'}$.

Remark 5. Let $\pi : \mathcal{E} \to \mathcal{B}$ be a local Grothendieck fibration of categories, and suppose we are given morphisms

$$B'' \xrightarrow{g} B' \xrightarrow{f} B$$

in the category $\mathcal{B}$. For each object $E \in \mathcal{E}_B$, we can lift f to a locally π-Cartesian morphism $\overline{f} : f^*E \to E$, and we can lift g to a locally π-Cartesian morphism $\overline{g} : g^*f^*E \to f^*E$. The composition $(\overline{f} \circ \overline{g}) : g^*f^*E \to E$ induces a comparison map $\rho_E : g^*f^*E \to (f \circ g)^*E$. This construction depends functorially on E, and therefore determines a natural transformation of functors $\rho : g^* \circ f^* \to (f \circ g)^*$.

The following conditions on π are equivalent:

- The collection of locally π-Cartesian morphisms of $\mathcal{E}$ is closed under composition.
- For every pair of morphisms $B'' \xrightarrow{g} B' \xrightarrow{f} B$ in $\mathcal{B}$, the comparison map $g^* \circ f^* \to (f \circ g)^*$ is an isomorphism.

If these conditions are satisfied, we say that π is a *Grothendieck fibration.*

Proposition 6. *The forgetful functor $\pi : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \to \mathrm{Stone}^{\mathrm{fr}}$ is a local Grothendieck fibration.*

Proof. Suppose we are given an object $(X, \mathcal{O}_X) \in \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$ and a morphism $f : Y \to X$ in $\mathrm{Stone}^{\mathrm{fr}}$; we wish to lift f to a locally π-Cartesian morphism

$$\overline{f} : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

in $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$. Write $Y = \beta I$ for some set I, so that f determines a map from I to X (which we will also denote by f). We take $(Y, \mathcal{O}_Y)$ to be the coproduct $\coprod_{i \in I}(\{i\}, \mathcal{O}_{X,f(i)})$. The canonical maps $(\{i\}, \mathcal{O}_{X,f(i)}) \to (X, \mathcal{O}_X)$ then amalgamate to a map $\overline{f} : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$. We claim that $\overline{f}$ is locally π-Cartesian. To prove this, we must show that for any other object $(Y, \mathcal{O}'_Y) \in \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$, the canonical map

$$\theta : \mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}}((Y, \mathcal{O}'_Y), (Y, \mathcal{O}_Y)) \times_{\mathrm{Hom}_{\mathrm{Stone}}(Y,Y)} \{\mathrm{id}_Y\} \to \mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}}((Y, \mathcal{O}'_Y), (X, \mathcal{O}_X)) \times_{\mathrm{Hom}_{\mathrm{Stone}}(Y,X)} \{f\}$$

is bijective. Since $(Y, \mathcal{O}'_Y)$ is free, we can write it as a coproduct $\coprod_{i \in I}(\{i\}, M_i)$ for some collection of models $\{M_i\}_{i \in I}$. We can then factor θ as a product of maps

$$\theta_i : \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(\mathcal{O}_{Y,i}, M_i) \to \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(\mathcal{O}_{X,f(i)}, M_i).$$

Each of these maps is bijective, because $\overline{f}$ induces an isomorphism of models $\mathcal{O}_{X,f(i)} \simeq \mathcal{O}_{Y,i}$. $\square$

For each object $X \in \mathrm{Stone}^{\mathrm{fr}}$, let $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},X}$ denote the fiber product $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \times_{\mathrm{Stone}^{\mathrm{fr}}} \{X\}$. It follows from Proposition 6 that every continuous map $f : X \to Y$ in $\mathrm{Stone}^{\mathrm{fr}}$ induces a pullback functor $f^* : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},Y} \to \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},X}$.

Proposition 7. *Let I be a set and set $X = \beta I$. For each $i \in I$, let $f_i : \{i\} \hookrightarrow X$ be the inclusion map. Then the pullback functors f_i^* induce an equivalence of categories*

$$\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},X} \to \prod_{i \in I} \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\{i\}} \simeq \prod_{i \in I} \mathrm{Mod}(\mathcal{C})^{\mathrm{op}}.$$

Proof. Concretely, this functor is given by

$$((X, \mathcal{O}_X) \in \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},X}) \mapsto \{\mathcal{O}_{X,i}\}_{i \in I}.$$

An inverse functor is given by $\{M_i\}_{i \in I} \mapsto \coprod_{i \in I}(\{i\}, M_i)$. □

Remark 8. In what follows, it will be convenient to use Proposition 7 to identify each of the categories $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\beta I}$ with $(\mathrm{Mod}(\mathcal{C})^{\mathrm{op}})^I$. Suppose that $f : \beta I \to \beta J$ is a continuous map of Stone spaces, corresponding to a family of ultrafilters $\{\mathcal{U}_i\}_{i \in I}$ on the set J. Then the associated pullback functor

$$f^* : (\mathrm{Mod}(\mathcal{C})^{\mathrm{op}})^J \simeq \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\beta J} \to \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\beta I} \simeq (\mathrm{Mod}(\mathcal{C})^{\mathrm{op}})^I$$

is given by the construction

$$\{M_j\}_{j \in J} \mapsto \{(\prod_{j \in J} M_j)/\,\mathcal{U}_i\}_{i \in I}.$$

In other words, the local Grothendieck fibration π encodes the operation of "forming ultraproducts in $\mathrm{Mod}(\mathcal{C})$".

Warning 9. The local Grothendieck fibration $\pi : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \to \mathrm{Stone}^{\mathrm{fr}}$ is usually not a Grothendieck fibration. To see this, consider a pair of composable morphisms $\beta I \xrightarrow{g} \beta J \xrightarrow{f} \beta K$ in the category $\mathrm{Stone}^{\mathrm{fr}}$, corresponding to a family of ultrafilters $\{\mathcal{U}_i\}_{i \in I}$ on the set J and a family of ultrafilters $\{\mathcal{V}_j\}_{j \in J}$ on the set K. The composition $f \circ g$ then corresponds to a family of ultrafilters $\{\mathcal{W}_i\}_{i \in I}$ on the set K, as described in Exercise 3. We then have a natural transformation $g^* \circ f^* \to (f \circ g)^*$ of functors from $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\beta K} \simeq (\mathrm{Mod}(\mathcal{C})^{\mathrm{op}})^K$ to $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\beta I} \simeq (\mathrm{Mod}(\mathcal{C})^{\mathrm{op}})^I$. To a collection of models $\{M_k\}_{k \in K}$, this natural transformation associates a collection of maps

$$\rho_i : (\prod_{k \in K} M_k)/\,\mathcal{W}_i \to (\prod_{j \in J}((\prod_{k \in K} M_k)/\,\mathcal{V}_j))/\,\mathcal{U}_i\,.$$

of models of $\mathcal{C}$. These maps are usually not isomorphisms. For example, suppose we are given an object $C \in \mathcal{C}$ for which each $M_k(C)$ is nonempty. In this case, we obtain a map of sets

$$\rho_i(C) : (\prod_{k \in K} M_k(C))/\,\mathcal{W}_i \to (\prod_{j \in J}((\prod_{k \in K} M_k(C))/\,\mathcal{V}_j))/\,\mathcal{U}_i\,.$$

whose domain can be identified with a quotient of the product $\prod_{k \in K} M_k(C)$, and whose codomain can be identified with a quotient of the product $\prod_{k \in K} M_k(C)^J$. This map is injective but usually not surjective.

Example 10. In the situation of Warning 9, suppose that we take I and K to be one-element sets, so that $f : \beta J \to \beta K$ is uniquely determined and $g : \beta I \to \beta J$ is given by specifying an ultrafilter $\mathcal{U}$ on the set J. Let M be a model of $\mathcal{C}$, which we can identify with an object of $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\beta K}$. In this case, we can identify $(f \circ g)^* M = \mathrm{id}^*\, M$ with M, and $(g^* \circ f^*)(M)$ with the ultrapower $M^J/\,\mathcal{U}$. Under these identifications, the natural transformation $g^* \circ f^* \to (f \circ g)^*$ induces the diagonal embedding

$$\delta_M : M \hookrightarrow M^J/\,\mathcal{U}\,.$$

We now describe a situation where the issue raised in Warning 9 does not arise.

Proposition 11. *Let $g : \beta I \to \beta J$ and $f : \beta J \to \beta K$ be maps in* $\mathrm{Stone}^{\mathrm{fr}}$*, and suppose that g carries I into J (that is, g arises from a map of sets $I \to J$). Then the natural transformation $g^* \circ f^* \to (f \circ g)^*$ is an equivalence of functors from* $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\beta K}$ *to* $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}\,\beta I}$*.*

Proof. Let us identify g and f with collections of ultrafilters $\{\mathcal{U}_i\}_{i \in I}$ on the set J and $\{\mathcal{V}_j\}_{j \in J}$ on the set K, respectively. Our assumption is that each $\mathcal{U}_i$ is the principal ultrafilter associated to some element $g(i) \in J$. In this case, the composite map $f \circ g : \beta I \to \beta K$ corresponds to the family of ultrafilters $\{\mathcal{V}_{g(i)}\}_{i \in I}$ on K. The desired result now follows from the observation that for any collection of sets $\{S_k\}_{k \in K}$, the canonical map

$$(\prod_{k \in K} S_k)/\,\mathcal{V}_{g(i)} \to (\prod_{j \in J}((\prod_{k \in K} S_k)/\,\mathcal{V}_j))/\,\mathcal{U}_i$$

is bijective (this is immediate from our assumption that $\mathcal{U}_i$ is principal). □

We now propose the following preliminary answer to Question 1:

Definition 12. An *ultracategory fibration* is a category $\mathcal{E}$ together with a functor $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ with the following properties:

(1) The functor π is a local Grothendieck fibration.

(2) Let I be a set and let $f_i : \{i\} \hookrightarrow \beta I$ denote the inclusion map for each $i \in I$. Then the construction

$$(M \in \mathcal{E}_{\beta I}) \mapsto \{f_i^* M \in \mathcal{E}_{\{i\}}\}_{i \in I}$$

induces an equivalence of categories

$$\mathcal{E}_{\beta I} \to \prod_{i \in I} \mathcal{E}_{\{i\}}\,.$$

(3) Let $g : \beta I \to \beta J$ and $f : \beta J \to \beta K$ be maps in $\mathrm{Stone}^{\mathrm{fr}}$, and suppose that g carries I into J. Then the natural transformation $g^* \circ f^* \to (f \circ g)^*$ is an equivalence of functors from $\mathcal{M}_{\beta K}$ to $\mathcal{E}_{\beta I}$.

Example 13. Let $\mathcal{C}$ be a small pretopos. Then the forgetful functor $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \to \mathrm{Stone}^{\mathrm{fr}}$ is an ultracategory fibration.

Lecture 25X-Ultracategories

April 13, 2018

We begin by recalling the following definition from Lecture 24X:

Definition 1. An *ultracategory fibration* is a category $\mathcal{E}$ together with a functor $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ with the following properties:

(1) The functor π is a local Grothendieck fibration.

(2) Let I be a set and let $f_i : \{i\} \hookrightarrow \beta I$ denote the inclusion map for each $i \in I$. Then the construction

$$(M \in \mathcal{E}_{\beta I}) \mapsto \{f_i^* M \in \mathcal{E}_{\{i\}}\}_{i \in I}$$

induces an equivalence of categories

$$\mathcal{E}_{\beta I} \to \prod_{i \in I} \mathcal{E}_{\{i\}} .$$

(3) Let $g : \beta I \to \beta J$ and $f : \beta J \to \beta K$ be maps in $\mathrm{Stone}^{\mathrm{fr}}$, and suppose that g carries I into J. Then the natural transformation $g^* \circ f^* \to (f \circ g)^*$ is an equivalence of functors from $\mathcal{M}_{\beta K}$ to $\mathcal{E}_{\beta I}$.

In this lecture, we will describe the structure of an arbitrary ultracategory fibration $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$.

Notation 2. Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ be an ultracategory fibration. We let $\mathcal{M}$ denote the fiber product $(\mathcal{E} \times_{\mathrm{Stone}^{\mathrm{fr}}} \{*\})^{\mathrm{op}}$. We will refer to $\mathcal{M}$ as the *underlying category* of the ultracategory fibration π.

Example 3. Let $\mathcal{C}$ be a small pretopos. Then the underlying category of the ultracategory fibration $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \to \mathrm{Stone}^{\mathrm{fr}}$ is the category $\mathrm{Mod}(\mathcal{C})$ of models of $\mathcal{C}$.

Let's now return to the general case. Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ be an ultracategory fibration with underlying category $\mathcal{M}$. For every set I, assumption (2) of Definition 1 supplies an equivalence of categories

$$\gamma_I : \mathcal{E}^{\mathrm{op}}_{\beta I} \xrightarrow{\sim} (\mathcal{M}^I).$$

Let $\mathcal{U}$ be an ultrafilter on I, which we can identify with a point of βI. Then $\mathcal{U}$ determines a map of spaces $* \to \beta I$, which gives rise to a pullback functor $\psi_{\mathcal{U}} : \mathcal{E}^{\mathrm{op}}_{\beta I} \to \mathcal{E}^{\mathrm{op}}_{\{\mathcal{U}\}} \simeq \mathcal{M}$. We let $P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M}$ denote the functor given by the composition $\psi_{\mathcal{U}} \circ \gamma_I^{-1}$.

Example 4. Let $\mathcal{C}$ be a small pretopos and let $\pi : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \to \mathrm{Stone}^{\mathrm{fr}}$ be the forgetful functor. Then, for any set I and any ultrafilter $\mathcal{U}$ on I, the functor $P^{\mathcal{U}} : \mathrm{Mod}(\mathcal{C})^I \to \mathrm{Mod}(\mathcal{C})$ is given by

$$P^{\mathcal{U}}(\{M_i\}_{i \in I}) = (\prod_{i \in I} M_i)/\mathcal{U} .$$

We again return to the general case. Suppose we are given a continuous map $f : \beta I \to \beta J$, given by a collection of ultrafilters $\{\mathcal{U}_i\}_{i \in I}$ on the set J. Suppose we are given a pair of objects $E \in \mathcal{E}_{\beta I}$ and $E' \in \mathcal{E}_{\beta J}$, having images

$$\gamma_I(E) = \{M_i \in \mathcal{M}\}_{i \in I} \qquad \gamma_J(E') = \{N_j \in \mathcal{M}\}_{j \in J}.$$

Let's try to describe the set

$$\mathrm{Hom}_{\mathcal{E}}(E, E') \times_{\mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}}(\beta I, \beta J)} \{f\}.$$

We then have canonical bijections

$$\begin{aligned} \mathrm{Hom}_{\mathcal{E}}(E, E') \times_{\mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}}(\beta I, \beta J)} \{f\} &\simeq \mathrm{Hom}_{\mathcal{E}_{\beta I}}(E, f^* E') \\ &\simeq \mathrm{Hom}_{\mathcal{M}^I}(\gamma_I(f^* E'), \gamma_I(E)) \\ &\simeq \prod_{i \in I} \mathrm{Hom}_{\mathcal{M}}(P^{\mathcal{U}_i}\{N_j\}_{j \in J}, M_i); \end{aligned}$$

Here we are using condition (3) of Definition 1 to identify $\gamma_I(f^* E')$ with the tuple $\{P^{\mathcal{U}_i} \gamma_J(E')\}_{i \in I}$.

By virtue of this calculation, we can attempt to reconstruct the category $\mathcal{E}$ (up to equivalence) from the data of the category $\mathcal{M}$ and the functors $P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M}$. Let's attempt to define a category $\overline{\mathcal{E}}$ as follows:

- The objects of $\overline{\mathcal{E}}$ are pairs $(I, \{M_i\}_{i \in I})$, where I is a set and $\{M_i\}_{i \in I}$ is a family of objects of $\mathcal{M}$ indexed by I.
- Given a pair of objects $(I, \{M_i\}_{i \in I})$ and $(J, \{N_j\})_{j \in J}$, we set

  $$\mathrm{Hom}_{\overline{\mathcal{E}}}((I, \{M_i\}_{i \in I}), (J, \{N_j\})_{j \in J}) = \coprod_{f : \beta I \to \beta J} \prod_{i \in I} \mathrm{Hom}_{\mathcal{M}}(P^{\mathcal{U}_i}\{N_j\}_{j \in J}, M_i)$$

  where the coproduct is taken over all continuous maps $f : \beta I \to \beta J$, which we identify with families of ultrafilters $\{\mathcal{U}_i\}_{i \in I}$ on the set J.

By virtue of the above discussion, choosing an inverse γ_I^{-1} to each of the functors γ_I gives a construction F

$$(I, \{M_i\}) \mapsto \gamma_I^{-1}\{M_i\}$$

which carries objects of $\overline{\mathcal{E}}$ to objects of $\mathcal{E}$, and we have canonical bijections

$$\mathrm{Hom}_{\overline{\mathcal{E}}}(\overline{E}, \overline{E}') = \mathrm{Hom}_{\mathcal{E}}(F(\overline{E}), F(\overline{E}'))$$

for every pair of objects $\overline{E}, \overline{E}' \in \overline{\mathcal{E}}$. It follows that there is a unique composition law on $\overline{\mathcal{E}}$ for which F is a functor (and therefore an equivalence of categories). We now give an explicit description of this structure in terms of the functors $P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M}$.

Remark 5 (Identity Morphisms)**.** Let I be a set containing an element i, and let $\mathcal{U}_i$ denote the principal ultrafilter determined by the element i. By construction, the diagram of categories

$$\begin{tikzcd} \mathcal{E}_{\beta I}^{\mathrm{op}} \arrow[rr, "\psi_{\mathcal{U}_i}"] \arrow[dr, "\gamma_I"'] & & \mathcal{M} \\ & \mathcal{M}^I \arrow[ur, "\mathrm{ev}_i"'] & \end{tikzcd}$$

commutes up to canoncial isomorphism, where ev_i is the functor given by evaluation on the ith coordinate. It follows that there is a canonical isomorphism

$$\epsilon_{I,i} : P^{\mathcal{U}_i} \simeq \mathrm{ev}_i$$

of functors from $\mathcal{M}^I$ to $\mathcal{M}$.

For any collection of objects $\{M_i\}_{i \in I}$, the identity morphism from $(I, \{M_i\}_{i \in I})$ to itself in $\overline{\mathcal{E}}$ is encoded by the family of maps

$$\{\epsilon_{I,i}(\{M_j\}_{j \in I} : P^{\mathcal{U}_i}\{M_j\}_{j \in I} \simeq M_i\}_{i \in I}.$$

Remark 6 (Composition)**.** Let $f : \beta J \to \beta K$ be a morphism in $\mathrm{Stone}_{\mathcal{C}}$, given by a collection $\{\mathcal{V}_j\}_{j \in J}$ of ultrafilters on K. Suppose we are given an ultrafilter $\mathcal{U}$ on J, which we can identify with a map $g : * \to \beta J$. Let $\mathcal{U} \wr \mathcal{V}_\bullet$ denote the ultrafilter on K given by the composition $f \circ g$, so that

$$(K_0 \in \mathcal{U} \wr \mathcal{V}_\bullet) \Leftrightarrow (\{j \in J : K_0 \in \mathcal{V}_j\} \in \mathcal{U})$$

Then we have a natural transformation of functors $g^* \circ f^* \to (f \circ g)^*$ from $\mathcal{E}_{\beta K}$ to $\mathcal{E}_*$. Passing to opposite categories and composing with the equivalence γ_K, we obtain a map

$$P^{\mathcal{U} \wr \mathcal{V}_\bullet}\{M_k\}_{k \in K} \to P^{\mathcal{U}}\{P^{\mathcal{V}_j}\{M_k\}_{k \in K}\}_{j \in J}$$

depending functorially on $\{M_k\}_{k \in K} \in \mathcal{M}^K$; we will write this as a natural transformation

$$\mu_{\mathcal{U}, \mathcal{V}_\bullet} : P^{\mathcal{U} \wr \mathcal{V}_\bullet} \to P^{\mathcal{U}} \circ \{P^{\mathcal{V}_j}\}_{j \in J}.$$

Using the natural transformations $\mu_{\mathcal{U}, \mathcal{V}_\bullet}$, we can describe the composition of morphisms in the category $\overline{\mathcal{E}}$. Let $f : \beta J \to \beta K$ be as above, and suppose we are given another map $g : \beta I \to \beta J$, given by a collection of ultrafilters $\{\mathcal{U}_i\}_{i \in I}$ on the set J. For each $i \in I$, let $\mathcal{U}_i \wr \mathcal{V}_\bullet \in \beta K$ be the image of $\mathcal{U}_i$ under the map f. Suppose that we lift f and g to morphisms

$$\overline{g} : (I, \{M_i\}_{i \in I}) \to (J, \{M'_j\}_{j \in J})$$

$$\overline{f} : (J, \{M'_j\}_{j \in J}) \to (K, \{M''_k\}_{k \in K})$$

in the category $\overline{\mathcal{E}}$. Then $\overline{f}$ is given by specifying a collection of maps

$$\{\overline{f}_j : P^{\mathcal{V}_j}\{M''_k\}_{k \in K} \to M'_j\}_{j \in J},$$

in the category $\mathcal{M}$, and $\overline{g}$ is given by specifying a family of maps

$$\{\overline{g}_i : P^{\mathcal{U}_i}\{M'_j\}_{j \in J} \to M_i\}_{i \in I}$$

in the category $\mathcal{M}$. Unwinding the definitions, we see that the composition $\overline{f} \circ \overline{g}$ in $\mathcal{E}$ is encoded by the family of composite maps

$$P^{\mathcal{U}_i \wr \mathcal{V}_\bullet}\{M''_k\}_{k \in K} \xrightarrow{\mu_{\mathcal{U}_i, \mathcal{V}_\bullet}} P^{\mathcal{U}_i}\{P^{\mathcal{V}_j}\{M''_k\}_{k \in K}\}_{j \in J} \xrightarrow{\overline{f}_j} P^{\mathcal{U}_i}\{M'_j\}_{j \in J} \xrightarrow{\overline{g}_i} M_i.$$

It follows from the above discussion that all of the data needed to construct the category $\overline{\mathcal{E}}$ is encoded by the functors $P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M}$, the natural transformations $\epsilon_{I,i}$ (which encode identity morphisms in $\overline{\mathcal{E}}$), and the natural transformations $\mu_{\mathcal{U}, \mathcal{V}_\bullet}$. This motivates the following definition, which is a variant of a notion introduced by Makkai:

Definition 7 (Ultracategories)**.** An *ultracategory* consists of the following data:

(1) A category $\mathcal{M}$.

(2) For every set I and every ultrafilter $\mathcal{U}$ on I, a functor $P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M}$.

(3) For every set I and every element $i \in I$, an isomorphism of functors $\epsilon_{I,i} : P^{\mathcal{U}_i} \simeq \mathrm{ev}_i$, where $\mathcal{U}_i$ denotes the principal ultrafilter associated to i (and $\mathrm{ev}_i : \mathcal{M}^I \to \mathcal{M}$ is given by projection onto the ith factor).

(4) For every pair of sets I and J, every ultrafilter $\mathcal{U}$ on I, and every family $\{\mathcal{V}_i\}_{i \in I}$ of ultrafilters on J, a natural transformation

$$\mu_{\mathcal{U}, \mathcal{V}_\bullet} : P^{\mathcal{U} \wr \mathcal{V}_\bullet} \to P^{\mathcal{U}} \circ \{P^{\mathcal{V}_i}\}_{i \in I}$$

of functors from $\mathcal{M}^J$ to $\mathcal{M}$.

These maps are required to satisfy the following axioms:

(A) In the situation of (4), suppose that $\mathcal{U}$ is the principal ultrafilter associated to some element $i_0 \in I$, so that $\mathcal{U} \wr \mathcal{V}_\bullet = \mathcal{V}_{i_0}$. Then, for any collection of objects $\{M_j\}_{j \in J}$, we have a commutative diagram

$$\begin{tikzcd} P^{\mathcal{U} \wr \mathcal{V}_\bullet}\{M_j\}_{j \in J} \arrow[rr, "\mu_{\mathcal{U}, \mathcal{V}_\bullet}"] \arrow[dr, "="'] & & P^{\mathcal{U}}\{P^{\mathcal{V}_j}\{M_j\}_{j \in J}\}_{i \in I} \arrow[dl, "\epsilon_{I,i_0}"] \\ & P^{\mathcal{V}_{i_0}}\{M_j\}_{j \in J} & \end{tikzcd}$$

(B) In the situation of (4), suppose that $I = J$ and that each $\mathcal{V}_i$ is the principal ultrafilter associated to i, so that $\mathcal{U} \wr \mathcal{V}_\bullet = \mathcal{U}$. Then, for any collection of objects $\{M_j\}_{j \in J}$, we have a commutative diagram

$$\begin{tikzcd} P^{\mathcal{U} \wr \mathcal{V}_\bullet}\{M_j\}_{j \in J} \arrow[rr, "\mu_{\mathcal{U}, \mathcal{V}_\bullet}"] \arrow[dr, "="'] & & P^{\mathcal{U}}\{P^{\mathcal{V}_j}\{M_j\}_{j \in J}\}_{i \in I} \arrow[dl, "\prod \epsilon_{J,\bullet}"] \\ & P^{\mathcal{U}}\{M_i\}_{i \in I}. & \end{tikzcd}$$

(C) Suppose we are given a diagram $* \xrightarrow{f} \beta I \xrightarrow{g} \beta J \xrightarrow{h} \beta K$ in $\mathrm{Stone}^{\mathrm{fr}}$, corresponding to an ultrafilter $\mathcal{U}$ on I, a collection of ultrafilters $\{\mathcal{V}_i\}_{i \in I}$ on J and a collection of ultrafilters $\{\mathcal{W}_j\}_{j \in J}$ on K. Then, for every collection $\{M_k\}_{k \in K}$, we have a commutative diagram

$$\begin{tikzcd} P^{\mathcal{U} \wr \mathcal{V}_\bullet \wr \mathcal{W}_\bullet}\{M_k\}_{k \in K} \arrow[r, "\mu_{\mathcal{U}, \mathcal{V}_\bullet \wr \mathcal{W}_\bullet}"] \arrow[d, "\mu_{\mathcal{U} \wr \mathcal{V}_\bullet, \mathcal{W}_\bullet}"] & P^{\mathcal{U}}\{P^{\mathcal{V}_i \wr \mathcal{W}_\bullet}\{M_k\}_{k \in K}\}_{i \in I} \arrow[d, "\mu_{\mathcal{V}_i, \mathcal{W}_\bullet}"] \\ P^{\mathcal{U} \wr \mathcal{V}_\bullet}\{P^{\mathcal{W}_j}\{M_k\}_{k \in K}\}_{j \in J} \arrow[r, "\mu_{\mathcal{U}, \mathcal{V}_\bullet}"] & P^{\mathcal{U}}\{P^{\mathcal{V}_i}\{P^{\mathcal{W}_j}\{M_k\}_{k \in K}\}_{j \in J}\}_{i \in I}. \end{tikzcd}$$

Remark 8. Roughly speaking, we can think of an ultracategory as a category $\mathcal{M}$ equipped with a notion of "how to take ultraproducts of objects in $\mathcal{M}$": that is, given a collection of objects $\{M_i\}_{i \in I}$ and an ultrafilter $\mathcal{U}$ on the index category I, we can form a new object $P^{\mathcal{U}}\{M_i\}_{i \in I} \in \mathcal{M}$ which we think of as the ultraproduct of the objects M_i with respect to the ultrafilter $\mathcal{U}$. Datum (3) asserts that an ultraproduct indexed by a *principal* ultrafilter returns one of the objects that we started with, and datum (4) encodes natural comparison maps from ultraproducts to iterated ultraproducts (such as a "diagonal" map from any object M to the ultrapower $P^{\mathcal{U}}\{M\}_{i \in I}$).

Given an ultracategory $\mathcal{M}$, we can define a category $\overline{\mathcal{E}}$ with objects and morphisms defined above, where the identity morphisms are determined by the natural isomorphisms $\epsilon_{I,i}$ and the composition is determined by the morphisms $\mu_{\mathcal{U},\{\mathcal{V}_i\}}$. Conditions ($A$), ($B$), and ($C$) are exactly what is needed to guarantee that the resulting composition law is unital (on both sides) and associative. Moreover, the construction

$$(I, \{M_i\}_{i \in I}) \mapsto \beta I$$

determines a forgetful functor $\pi : \overline{\mathcal{E}} \to \mathrm{Stone}^{\mathrm{fr}}$, which is *essentially* a local Grothendieck fibration. More precisely, it defines a local Grothendieck fibration from $\overline{\mathcal{E}}$ to the full subcategory of $\mathrm{Stone}^{\mathrm{fr}}$ spanned by spaces which are identical to (rather than merely homeomorphic to) βI, for some set I. Moreover, the functor $\overline{\pi}$ satisfies the obvious analogues of conditions (2) and (3) of Definition 1. We can summarize the situation informally as follows:

Proposition 9. *The constructions of this lecture establish an equivalence between the following data:*

- *Ultracategory fibrations $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ (in the sense of Definition 1).*
- *Ultracategories $\mathcal{M}$ (in the sense of Definition 7).*

Lecture 26X-Ultrasets

April 14, 2018

Our goal in this lecture is to unwind the definitions of the preceding lectures in a particularly simple case.

Definition 1 (Ultrasets-Version 1)**.** An *ultraset* is an ultracategory fibration $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ with the property that each fiber $\pi^{-1}\{\beta I\}$ is a category with only identity morphisms: that is, it is a set, regarded as a category.

Note that if $\pi : \mathcal{E} \to \mathcal{B}$ is any local Grothendieck fibration whose fibers are sets, then π is automatically a Grothendieck fibration (since every natural transformation between functors $\mathcal{E}_{B'} \to \mathcal{E}_B$ is automatically invertible). It is therefore a *fibration in sets*, which is classified by a functor $\mathcal{B}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, given by $B \mapsto \mathcal{E}_B$. In the case where $\mathcal{B} = \mathrm{Stone}^{\mathrm{fr}}$, such a fibration $\pi : \mathcal{E} \to \mathcal{B}$ is an ultracategory fibration if and only if the associated functor $\mathcal{B}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ preserves (possibly infinite) products. We can therefore reformulate Definition 1 as follows:

Definition 2 (Ultrasets-Version 2)**.** An *ultraset* is a functor $F : \mathrm{Stone}^{\mathrm{fr,op}} \to \mathcal{S}\mathrm{et}$ which carries coproducts in $\mathrm{Stone}^{\mathrm{fr}}$ to products in $\mathcal{S}\mathrm{et}$ (so that we have canonical isomorphisms $F(\beta I) = F(*)^I$).

We can obtain another equivalent formulation of Definition 2 using the notion of ultracategory introduced in Lecture 25X. Giving an ultraset (in the sense of either Definition 1 or 2) is equivalent to giving an ultracategory for which the underlying category $\mathcal{M}$ is a set. In this case, the definition of an ultracategory simplifies substantially:

Definition 3 (Ultrasets-Version 3)**.** An *ultraset* consists of the following data:

(1) A set M.

(2) For every set I and every ultrafilter $\mathcal{U}$ on I, a map of sets $P^{\mathcal{U}} : M^I \to M$.

This data is required to satisfy the following conditions:

(3) If $\mathcal{U}$ is the principal ultrafilter associated to an element $i \in I$, then $P^{\mathcal{U}} : M^I \to M$ is given by projection onto the ith factor.

(4) Given sets I and J, an ultrafilter $\mathcal{U}$ on I, and a collection $\{\mathcal{V}_i\}_{i \in I}$ of ultrafilters on J, we have an equality
$$P^{\mathcal{U} \wr \mathcal{V}_\bullet} = P^{\mathcal{U}} \circ \{P^{\mathcal{V}_i}\}_{i \in I}$$
of functions from M^J into M.

Suppose that M is an ultraset in the sense of Definition 3. Taking $I = M$ and $\mathrm{id}_M \in M^M$ to be the identity function, we see that the construction $\mathcal{U} \mapsto P^{\mathcal{U}}(\mathrm{id}_M)$ determines a function $r : \beta M \to M$. The structure of M as an ultraset is completely encoded by this function:

Lemma 4. *Let M be an ultraset (in the sense of Definition 3). Then, for any set I, any ultrafilter $\mathcal{U}$ on I, and any $f \in M^I$ (which we view as a function from I to M), we have*

$$P^{\mathcal{U}}(f) = u(f_* \mathcal{U}).$$

Here $f_* \mathcal{U}$ denotes the ultrafilter on M given by

$$(M_0 \in f_* \mathcal{U}) \Leftrightarrow (f^{-1} M_0 \in \mathcal{U}).$$

Proof. Set $J = M$. For each $i \in I$, let $\mathcal{V}_i$ be the principal ultrafilter on J associated to the element $f(i) \in M$. Axiom (3) then gives $P^{\mathcal{V}_i}(\mathrm{id}_M) = \mathrm{id}_M(f_i) = f(i)$. Note that the composite ultrafilter $\mathcal{U} \wr \mathcal{V}_\bullet$ can be identified with $f_* \mathcal{U}$. Applying axiom (4), we obtain

$$u(f_* \mathcal{U}) = P^{\mathcal{U} \wr \mathcal{V}_\bullet}(\mathrm{id}_M) = P^{\mathcal{U}}\{P^{\mathcal{V}_i}\{\mathrm{id}_M\}\}_{i \in I} = P^{\mathcal{U}}(f).$$

□

Proposition 5. *Let M be an ultraset (in the sense of Definition 3). Then the map $r : \beta M \to M$ defined above has the following properties:*

(*a*) *The restriction of r to M is the identity map id_M (where we identify M with a subset of βM).*

(*b*) *Let $\rho : \beta(\beta M) \to \beta M$ be the unique continuous map whose restriction to βM is the identity (where we identify βM with an open subset of $\beta(\beta M)$). Then we have a commutative diagram*

$$\begin{tikzcd} \beta(\beta M) \arrow[r, "\rho"] \arrow[d, "\beta(u)"] & \beta M \arrow[d, "u"] \\ \beta M \arrow[r, "u"] & M. \end{tikzcd}$$

Proof. Assertion (*a*) is an immediate consequence of axiom (3) of Definition 3. To prove (*b*) let us apply axiom (4) in the case where $I = \beta M$, $J = M$, the construction $i \mapsto \mathcal{V}_i$ is the identity (that is, it associates to each element of βM the corresponding ultrafilter on M). Let $\mathcal{U}$ be an ultrafilter on βM, which we can identify with a point of $\beta(\beta M)$. Then axiom (4) gives an equality

$$P^{\mathcal{U} \wr \mathcal{V}_\bullet}(\mathrm{id}_M) = P^{\mathcal{U}}\{P^{\mathcal{V}}(\mathrm{id}_M)\}_{\mathcal{V} \in \beta M} = P^{\mathcal{U}}(u)$$

Unwinding the definitions, we see that $\mathcal{U} \wr \mathcal{V}_\bullet$ coincides with $\rho(\mathcal{U})$ as an ultrafilter on M, so the left hand side is given by $r(\rho(\mathcal{U}))$. On the other hand, Lemma 4 allows us to rewrite the right hand side as $r(r_* \mathcal{U})$. □

Proposition **??** has a converse:

Proposition 6. *Let M be a set equipped with a map $r : \beta M \to M$ satisfying conditions (a) and (b) of Proposition 5. Then M inherits the structure of an ultraset, given by*

$$P^{\mathcal{U}}(f) = u(f_* \mathcal{U})$$

for $f \in M^I$ and $\mathcal{U} \in \beta I$.

Proof. We must show that $P^{\mathcal{U}}$ satisfies conditions (3) and (4) of Definition 3. Suppose first that $\mathcal{U}$ is a principal ultrafilter on a set I and that $f \in M^I$. Then $f_* \mathcal{U}$ is the principal ultrafilter on $f(i) \in M$, so that $\mathcal{P}^{\mathcal{U}}(f) = u(f_*(\mathcal{U})) = u(f(i)) = f(i)$ by virtue of (*a*).

Now suppose we are given sets I and J and a collection of ultrafilters $\{\mathcal{V}_i\}_{i \in I}$ on the set J, so that the construction $i \mapsto \mathcal{V}_i$ determines a map of sets $g : I \to \beta J$. We wish to show that we have an equality

$$P^{\mathcal{U} \wr \mathcal{V}_\bullet}(f) = P^{\mathcal{U}}\{P^{\mathcal{V}_i}(f)\}_{i \in I}.$$

for each ultrafilter $\mathcal{U}$ on the set I. Unwinding the definitions, we see that the left hand side is given (as a function of $\mathcal{U}$) by the composition

$$\beta I \xrightarrow{\beta g} \beta\beta J \xrightarrow{\beta\beta f} \beta\beta M \xrightarrow{\rho} \beta M \xrightarrow{u} M$$

while the right hand side is given by

$$\beta I \xrightarrow{\beta g} \beta\beta J \xrightarrow{\beta f_*} \beta\beta M \xrightarrow{\beta(u)} \beta M \xrightarrow{u} M;$$

these maps coincide by virtue of property (b). □

We leave it to the reader to check that for a fixed set M, these constructions establish mutually inverse bijections

$$\{\text{Functions } P^{\mathcal{U}} : M^I \to M \text{ satisfying Definition 3}\} \simeq \{\text{Functions } r : \beta M \to M \text{ satisfying } (a) \text{ and } (b)\}.$$

This leads to another formulation:

Definition 7 (Ultrasets-Version 4)**.** An *ultraset* is a set M equipped with a map $r : \beta M \to M$ satisfying conditions (a) and (b) of Proposition 5.

Let us give another interpretation of this definition. Let $\mathrm{Top}_{\mathrm{ch}}$ denote the category of compact Hausdorff topological spaces and continuous maps. This category is equipped with a forgetful functor $G : \mathrm{Top}_{\mathrm{ch}} \to \mathcal{S}\mathrm{et}$. The forgetful functor G has a left adjoint F, which associates to each set I its Stone-Čech compactification βI, regarded as a topological space. The composition $\beta = G \circ F$ is the functor which associates to each set I the Stone-Čech compactification βI, regarded merely as a set. This realization endows β with the structure of a *monad*: that is, as an associative algebra in the category $\mathrm{Fun}(\mathcal{S}\mathrm{et}, \mathcal{S}\mathrm{et})$ of endofunctors of $\mathcal{S}\mathrm{et}$. Explicitly, this algebra structure is encoded by a unit map

$$u : \mathrm{id}_{\mathcal{S}\mathrm{et}} \to \beta \qquad \rho : \beta \circ \beta \to \beta.$$

Unwinding the definitions, we see that u associates to each set M the canonical inclusion $M \hookrightarrow \beta M$, while ρ associates to each set M the map $\beta\beta M \to \beta M$ appearing in Proposition 5. We can therefore restate Definition 7 in the language of monads:

Definition 8 (Ultrasets-Version 5)**.** An *ultraset* is an algebra over the monad $\beta : \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$.

By general nonsense, the forgetful functor $G : \mathrm{Top}_{\mathrm{ch}} \to \mathcal{S}\mathrm{et}$ factors through the category of algebras for the monad β. In other words, every compact Hausdorff space X can be regarded as an ultraset, taking $r : \beta X \to X$ to be the unique map which agrees with the identity on X and is continuous (where βX is regarded as the Stone-Čech compactifiction of X as a discrete set, and the right hand side uses the compact Hausdorff topology on X).

Theorem 9. *The adjunction* $\mathcal{S}\mathrm{et} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Top}_{\mathrm{ch}}$ *is monadic. In other words, it induces an equivalence of categories*

$$\{\textit{Compact Hausdorff spaces}\} \xrightarrow{\sim} \{\textit{Ultrasets}\}.$$

By virtue of Theorem 9, we obtain yet another formulation of the notion of an ultraset:

Definition 10 (Ultrasets-Version 6)**.** An *ultraset* is a compact Hausdorff space.

Remark 11. We can make the equivalence of Definitions 10 and 2 explicit, without passing through the theory of ultracategories. Note that every topological space X represents a functor

$$F_X : \mathrm{Stone}^{\mathrm{fr,op}} \to \mathcal{S}\mathrm{et},$$

given by $F_X(\beta I) = \mathrm{Hom}_{\mathrm{Top}}(\beta I, X)$. If X is compact Hausdorff, then the functor F_X carries coproducts in $\mathrm{Stone}^{\mathrm{fr}}$ to products of sets, and is therefore an ultraset in the sense of Definition 2.

Proof Sketch of Theorem 9. We will explicitly describe an inverse to the functor

$$\{\text{Compact Hausdorff spaces}\} \to \{\text{Ultrasets}\}.$$

Let X be an ultraset: that is, a set equipped with a map $r : \beta X \to X$ satisfying the requirements of Proposition 5. We wish to show that X can be equipped with the structure of a compact Hausdorff space for which the map r is continuous (in which case r will be the unique continuous extension of the identity map $\mathrm{id}_X : X \to X$, by the universal property of βX). Note that r determines an equivalence relation $R \subseteq (\beta X) \times_X (\beta X)$, given by

$$((x, y) \in R) \Leftrightarrow (r(x) = r(y)).$$

Since r is surjective, we can identify X with the quotient $(\beta X)/R$. Let us endow X with the quotient topology, so that X is automatically compact and r is automatically continuous. The only nontrivial point is to verify that this topology is Hausdorff: that is, that the equivalence relation R is closed.

Consider the projection maps $\pi, \pi' : R \to \beta X$, and set $Y = \beta R$. Note that π and π' extend uniquely to continuous maps $\overline{\pi}, \overline{\pi}' : \beta R \to \beta X$. The pair

$$(\overline{\pi}, \overline{\pi}') : \beta R \to \beta X \times \beta X$$

is a continuous map of compact Hausdorff spaces, and therefore has closed image. Clearly, this image contains R. We will complete the proof by showing that the image is exactly R: that is, that the diagram

$$\begin{tikzcd} \beta R \arrow[r, "\overline{\pi}"] \arrow[d, "\overline{\pi'}"] & \beta X \arrow[d, "r"] \\ \beta X \arrow[r, "r"] & X \end{tikzcd}$$

commutes. Note that the maps $r \circ \overline{\pi}$ and $r \circ \overline{\pi}'$ are maps from βR to X in the category of algebras for the monad β. Consequently, to show that they coincide, it will suffice to show that they coincide when restricted to R (since βR is the free β-algebra generated by R). We are therefore reduced to proving the commutativity of the diagram

$$\begin{tikzcd} R \arrow[r, "\pi"] \arrow[d, "\pi'"] & \beta X \arrow[d, "r"] \\ \beta X \arrow[r, "r"] & X, \end{tikzcd}$$

which is immediate from the definition of R. $\square$

Lecture 27X-Ultrafunctors

April 17, 2018

We now study functors between ultracategories.

Definition 1. Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ and $\pi' : \mathcal{E}' \to \mathrm{Stone}^{\mathrm{fr}}$ be ultracategory fibrations. A *morphism of ultracategory fibrations* from $\mathcal{E}$ to $\mathcal{E}'$ is a functor $F : \mathcal{E} \to \mathcal{E}'$ with the following properties:

(1) The diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "F"] \arrow[dr, "\pi"'] & & \mathcal{E}' \arrow[dl, "\pi'"] \\ & \mathrm{Stone}^{\mathrm{fr}} & \end{tikzcd}$$

commutes. In particular, for every $X \in \mathrm{Stone}^{\mathrm{fr}}$, F induces a functor $F_X : \mathcal{E}_X \to \mathcal{E}'_X$.

(2) The functor F carries locally π-Cartesian morphisms of $\mathcal{E}$ to locally π'-Cartesian morphisms of $\mathcal{E}'$.

We let $\mathrm{Mor}(\mathcal{E}, \mathcal{E}')$ denote the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathcal{E}') \times_{\mathrm{Fun}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}})} \{\pi\}$ consisting of morphisms of ultracategory fibrations from $\mathcal{E}$ to $\mathcal{E}'$.

We will be primarily interested in the following example:

Proposition 2. *Let $\mathcal{C}$ and $\mathcal{C}'$ be small pretopoi, and let $f : \mathcal{C} \to \mathcal{C}'$ be a pretopos morphism (that is, a functor which preserves finite limits, finite coproducts, and effective epimorphisms). If X is a topological space and $\mathcal{O}_X : \mathcal{C}' \to \mathrm{Shv}(X)$ is an X-model of $\mathcal{C}'$, then $\mathcal{O}_X \circ f : \mathcal{C} \to \mathrm{Shv}(X)$ is an X-model of $\mathcal{C}$. It follows that the construction $(X, \mathcal{O}_X) \to (X, \mathcal{O}_X \circ f)$ determines a functor $\widetilde{F} : \mathrm{Top}_{\mathcal{C}'} \to \mathrm{Top}_{\mathcal{C}}$. Then:*

(1) *The functor $\widetilde{F}$ carries $\mathrm{Stone}_{\mathcal{C}'}$ into $\mathrm{Stone}_{\mathcal{C}}$ and $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}'}$ into $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$.*

(2) *The induced map $F : \mathrm{Stone}_{\mathcal{C}'} \to \mathrm{Stone}_{\mathcal{C}}$ is a morphism of ultracategory fibrations.*

Proof. It is clear that $\widetilde{F}$ carries $\mathrm{Stone}_{\mathcal{C}'}$ into $\mathrm{Stone}_{\mathcal{C}}$. Let us identify $\mathrm{Stone}^{\mathrm{op}}_{\mathcal{C}}$ and $\mathrm{Stone}^{\mathrm{op}}_{\mathcal{C}'}$ with the full subcategories of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ and $\mathrm{Fun}(\mathcal{C}', \mathcal{S}\mathrm{et})$ spanned by those functors which preserve finite limits and effective epimorphisms. Under these identifications, the functor $\widetilde{F}|_{\mathrm{Stone}_{\mathcal{C}'}}$ is given by precomposition with f. It follows that $\widetilde{F}|_{\mathrm{Stone}_{\mathcal{C}'}} : \mathrm{Stone}_{\mathcal{C}'} \to \mathrm{Stone}_{\mathcal{C}}$ commutes with coproducts (since these correspond to products in $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ and $\mathrm{Fun}(\mathcal{C}', \mathcal{S}\mathrm{et})$). Since $\widetilde{F}$ carries $\mathrm{Mod}(\mathcal{C}')^{\mathrm{op}} \subseteq \mathrm{Stone}_{\mathcal{C}'}$ into $\mathrm{Mod}(\mathcal{C}) \subseteq \mathrm{Stone}_{\mathcal{C}}$, it restricts to a functor $F : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}'} \to \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$. By construction, this functor fits into a commutative diagram

$$\begin{tikzcd} \mathrm{Stone}_{\mathcal{C}'} \arrow[rr, "F"] \arrow[dr, "\pi'"'] & & \mathrm{Stone}_{\mathcal{C}} \arrow[dl, "\pi"] \\ & \mathrm{Stone}^{\mathrm{fr}} . & \end{tikzcd}$$

We claim that F carries locally π-Cartesian morphisms in $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}'}$ to locally π-Cartesian morphisms in $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$. Note that a morphism $g : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in the category $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}'}$ is locally π'-Cartesian if and only if, for every *isolated* point $x \in X$, the induced map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism in $\mathrm{Mod}(\mathcal{C}')$. It then follows that $\mathcal{O}_{Y,f(x)} \circ f \to \mathcal{O}_{X,x} \circ f$ is an isomorphism of models of $\mathcal{C}$, so that $F(g)$ is locally π-Cartesian. $\square$

We can now give a precise statement of Makkai's theorem:

Theorem 3 (Strong Conceptual Completeness)**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be small pretopoi, and let* $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}')$ *denote the category of pretopos morphisms from $\mathcal{C}$ to $\mathcal{C}'$. Then the preceding construction induces an equivalence of categories*

$$\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}') \to \mathrm{Mor}(\mathrm{Stone}_{\mathcal{C}'}, \mathrm{Stone}_{\mathcal{C}})^{\mathrm{op}}.$$

We will give the proof of Theorem 3 over the next two lectures. First, let us try to describe more concretely what it is saying. Let's return to the case of a general pair of ultracategory fibrations

$$\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}} \qquad \pi' : \mathcal{E}' \to \mathrm{Stone}^{\mathrm{fr}}.$$

Suppose we are given a continuous map $f : \beta I \to \beta J$ in $\mathrm{Stone}^{\mathrm{fr}}$. Then pullback along f induces functors

$$\mathcal{E}_{\beta J} \to \mathcal{E}_{\beta I} \qquad \mathcal{E}'_{\beta J} \to \mathcal{E}'_{\beta I},$$

both of which we will denote by f^*. Condition (2) guarantees that the diagram

$$\begin{tikzcd} \mathcal{E}_{\beta J} \arrow[r, "f^*"] \arrow[d, "F_{\beta J}"] & \mathcal{E}_{\beta I} \arrow[d, "F_{\beta I}"] \\ \mathcal{E}'_{\beta J} \arrow[r, "f^*"] & \mathcal{E}'_{\beta I}. \end{tikzcd}$$

Let $\mathcal{M} = \mathcal{E}_*^{\mathrm{op}}$ and $\mathcal{M}' = \mathcal{E}_*'^{\mathrm{op}}$ denote the underlying categories of $\mathcal{E}$ and $\mathcal{E}'$, respectively. Then F induces a functor $F_0 : \mathcal{M} \to \mathcal{M}'$. For every set I, we have equivalences

$$\mathcal{E}_{\beta I}^{\mathrm{op}} \simeq \mathcal{M}^I \qquad \mathcal{E}_{\beta I}'^{\mathrm{op}} \simeq \mathcal{M}'^I$$

which fit into a commutative diagram

$$\begin{tikzcd} \mathcal{E}_{\beta I}^{\mathrm{op}} \arrow[r, "F_{\beta I}"] \arrow[d, "\sim"] & \mathcal{E}_{\beta I}'^{\mathrm{op}} \arrow[d, "\sim"] \\ \mathcal{M}^I \arrow[r, "F_0^I"] & \mathcal{M}'^I. \end{tikzcd}$$

Consequently, a morphism of ultracategory fibrations $F : \mathcal{E} \to \mathcal{E}'$ is largely determined by the underlying functor $F_0 : \mathcal{M} \to \mathcal{M}'$. However, the functor F_0 is not arbitrary: applying the preceding paragraph in the case where $I = *$ is a single point and J is an arbitrary set, (so that $f : \beta I \to \beta J$ determines an ultrafilter $\mathcal{U}$ on J), we deduce that F_0 "commutes with ultraproducts indexed by $\mathcal{U}$", in the sense that we have a commutative diagram

$$\begin{tikzcd} \mathcal{M}^J \arrow[r, "F_0^J"] \arrow[d, "P^{\mathcal{U}}"] & \mathcal{M}'^J \arrow[d, "P'^{\mathcal{U}}"] \\ \mathcal{M} \arrow[r, "F_0"] & \mathcal{M}', \end{tikzcd}$$

where $P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M}$ and $P'^{\mathcal{U}} : \mathcal{M}'^I \to \mathcal{M}'$ are the maps given by f^*. This motivates the following:

Definition 4. Let $\mathcal{M}$ and $\mathcal{M}'$ be ultracategories, equipped with functors

$$P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M} \qquad P'^{\mathcal{U}} : \mathcal{M}'^I \to \mathcal{M}'$$

where $\mathcal{U}$ is an ultrafilter on a set I, together with isomorphisms

$$\epsilon_{I,i} : P^{\mathcal{U}} \simeq \mathrm{ev}_i \qquad \epsilon'_{I,i} : P'^{\mathcal{U}} \simeq \mathrm{ev}_i$$

when $\mathcal{U}$ is the principal ultrafilter associated to an element $i \in I$, and "diagonal" maps

$$\mu_{\mathcal{U},\mathcal{V}_\bullet} : P^{\mathcal{U} \wr \mathcal{V}_\bullet} \to P^{\mathcal{U}} \circ \{P^{\mathcal{V}_i}\}_{i \in I} \qquad \mu'_{\mathcal{U},\mathcal{V}_\bullet} : P'^{\mathcal{U} \wr \mathcal{V}_\bullet} \to P'^{\mathcal{U}} \circ \{P'^{\mathcal{V}_i}\}_{i \in I}.$$

when $\mathcal{U}$ is an ultrafilter on a set I and $\{\mathcal{V}_i\}_{i \in I}$ is an I-indexed collection of ultrafilters on a set J.

An *ultrafunctor from* $\mathcal{M}$ *to* $\mathcal{M}'$ consists of the following data:

- A functor $F_0 : \mathcal{M} \to \mathcal{M}'$.
- For every set I and every ultrafilter $\mathcal{U}$ on a set I, an isomorphism $\gamma_{\mathcal{U}} : P'^{\mathcal{U}} \circ F_0^I \simeq F_0 \circ P^{\mathcal{U}}$ of functors from $\mathcal{M}^I$ to $\mathcal{M}'$.

These isomorphisms are required to satisfy the following conditions:

- If $\mathcal{U}$ is the principal ultrafilter associated to an element $i \in I$, then the diagram

$$\begin{tikzcd} P'^{\mathcal{U}} \circ F_0^I \arrow[r, "\gamma_{\mathcal{U}}"] \arrow[d, "\epsilon'_{I,i}"] & F_0 \circ P^{\mathcal{U}} \arrow[d, "\epsilon_{I,i}"] \\ \mathrm{ev}_i \circ F_0^I \arrow[r, "="] & F_0 \circ \mathrm{ev}_i \end{tikzcd}$$

  commutes (in the category of functors from $\mathcal{M}^I$ to $\mathcal{M}'$).

- If $\mathcal{U}$ is an ultrafilter on a set I and $\{\mathcal{V}_i\}_{i \in I}$ is an I-indexed collection of ultrafilters on a set J, then the diagram

$$\begin{tikzcd} P'^{\mathcal{U} \wr \mathcal{V}_\bullet} \circ F_0^J \arrow[rr, "\gamma_{\mathcal{U} \wr \mathcal{V}_\bullet}"] \arrow[d, "\mu'_{\mathcal{U},\mathcal{V}_\bullet}"] & & F_0 \circ P^{\mathcal{U} \wr \mathcal{V}_\bullet} \arrow[d, "\mu_{\mathcal{U},\mathcal{V}_\bullet}"] \\ P'^{\mathcal{U}} \circ \{P'^{\mathcal{V}_i}\}_{i \in I} \circ F_0^J \arrow[r, "\gamma_{\mathcal{V}_\bullet}"] & P'^{\mathcal{U}} \circ F_0^I \circ \{P^{\mathcal{V}_i}\}_{i \in I} \arrow[r, "\gamma_{\mathcal{U}}"] & F_0 \circ P^{\mathcal{U}} \circ \{P^{\mathcal{V}_i}\}_{i \in I} \end{tikzcd}$$

  commutes (in the category of functors from $\mathcal{M}^J$ to $\mathcal{M}'$).

Given ultrafunctors $(F_0, \{\gamma_{\mathcal{U}}\})$ and $(F'_0, \{\gamma'_{\mathcal{U}}\})$ from $\mathcal{M}$ to $\mathcal{M}'$, we will say that a natural transformation $\rho : F_0 \to F'_0$ is a *morphism of ultrafunctors* if, for every ultrafilter $\mathcal{U}$ on a set I, the diagram of natural transformations

$$\begin{tikzcd} P'^{\mathcal{U}} \circ F_0^I \arrow[r, "\rho^I"] \arrow[d, "\gamma_{\mathcal{U}}"] & P'^{\mathcal{U}} \circ F_0'^I \arrow[d, "\gamma'_{\mathcal{U}}"] \\ F_0 \circ P^{\mathcal{U}} \arrow[r, "\rho"] & F'_0 \circ P^{\mathcal{U}}. \end{tikzcd}$$

commutes. We let $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{M}')$ denote the category whose objects are ultrafunctors from $\mathcal{M}$ to $\mathcal{M}'$ and whose morphisms are morphisms are ultrafunctors.

Warning 5. Makkai introduced a notion of ultrafunctor between ultracategories which is *a priori* more restrictive than our Definition 4: that is, Makkai requires a larger collection of diagrams to commute. However, the difference turns out to be irrelevant in the primary case of interest to us (where $\mathcal{M}$ is the category of models of a small pretopos $\mathcal{C}$), by virtue of Theorem 8 below.

Example 6. Let X and X' be compact Hausdorff spaces. We saw in the previous lecture that we can think of X and X' as *ultrasets*: that is, as ultracategories with only trivial morphisms. In this situation, Definition 4 simplifies considerably: an ultrafunctor from X to X' is simply a map of sets $F_0 : X \to X'$ with the following property: for every map of sets $f : I \to X$ and every ultrafilter $\mathcal{U}$ on I, we have $F_0(P^{\mathcal{U}}(f)) = P'^{\mathcal{U}}(F_0 \circ f)$

in X'. By general nonsense, it suffices to check this equality in the case where $I = X$ and f is the identity. It follows that F_0 is an ultrafunctor if and only if it is a morphism of β-algebras: that is, the diagram

$$\begin{tikzcd} \beta X \arrow[r, "\beta(F_0)"] \arrow[d] & \beta X' \arrow[d] \\ X \arrow[r, "F_0"] & X' \end{tikzcd}$$

commutes, where the vertical maps are the continuous extensions of id_X and $\mathrm{id}_{X'}$, respectively. This is equivalent to the requirement that F_0 is continuous.

Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ and $\pi' : \mathcal{E}' \to \mathrm{Stone}^{\mathrm{fr}}$ be ultracategory fibrations, with underlying ultracategories $\mathcal{M}$ and $\mathcal{M}'$, respectively. If $F : \mathcal{E} \to \mathcal{E}'$ is a morphism of ultracategory fibrations (in the sense of Definition 1), then it is not difficult to see that the underlying functor $F_0 : \mathcal{M} \to \mathcal{M}'$ has the structure of an ultrafunctor (with isomorphisms $\gamma_{\mathcal{U}}$ defined as in the discussion preceding Definition 4). Passage from F to F_0 determines a functor

$$\mathrm{Mor}(\mathcal{E}, \mathcal{E}') \to \mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{M}')^{\mathrm{op}}.$$

Moreover, just as an ultracategory fibration can be recovered (up to equivalence) from its underlying ultracategory, a morphism of ultracategory fibrations can be recovered (up to isomorphism) from its underlying ultrafunctor. More precisely, we can apply the analysis of Lecture 25X to obtain the following:

Proposition 7. *Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ and $\pi' : \mathcal{E}' \to \mathrm{Stone}^{\mathrm{fr}}$ be ultracategory fibrations with underlying ultracategories $\mathcal{M} = \mathcal{E}_*^{\mathrm{op}}$ and $\mathcal{M}' = \mathcal{E}_*'^{\mathrm{op}}$. Then the preceding construction induces an equivalence of categories*

$$\mathrm{Mor}(\mathcal{E}, \mathcal{E}')^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{M}').$$

Combining this result with Theorem 3, we obtain the following reformulation of Theorem 3:

Theorem 8 (Strong Conceptual Completeness)**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be small pretopoi and let $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}')$ denote the category of pretopos morphisms from $\mathcal{C}$ to $\mathcal{C}'$. Then there is a canonical equivalence of categories*

$$\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}') \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}'), \mathrm{Mod}(\mathcal{C})).$$

Lecture 28X-Ultrafunctors to $\mathcal{S}\mathrm{et}$

April 18, 2018

Let $\mathcal{S}\mathrm{et}$ denote the category of sets. Then $\mathcal{S}\mathrm{et}$ can be regarded as an ultracategory: for every ultrafilter $\mathcal{U}$ on a set I, we define $P^{\mathcal{U}} : \mathcal{S}\mathrm{et}^I \to \mathcal{S}\mathrm{et}$ by the formula

$$P^{\mathcal{U}}\{S_i\}_{i \in I} = (\prod_{i \in I} S_i)/\,\mathcal{U}\,.$$

Our goal in this lecture is to describe the category $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et})$ of $\mathcal{S}\mathrm{et}$-valued ultrafunctors on $\mathcal{M}$, where $\mathcal{M}$ is an arbitrary ultracategory. For this, it will be more convenient to work at the level of ultracategory fibrations.

Construction 1. We define a category $\mathrm{Stone}_{\mathcal{S}\mathrm{et}}$ as follows:

- The objects of $\mathrm{Stone}_{\mathcal{S}\mathrm{et}}$ are pairs $(X, \mathcal{O}_X)$, where X is a Stone space and $\mathcal{O}_X$ is a sheaf of sets on X.
- A morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ in $\mathrm{Stone}_{\mathcal{S}\mathrm{et}}$ consists of a continuous map $f : X \to Y$ together with a map $f^* \mathcal{O}_Y \to \mathcal{O}_X$ of $\mathcal{S}\mathrm{et}$-valued sheaves on X.

We say that an object of $\mathrm{Stone}_{\mathcal{S}\mathrm{et}}$ is *free* if it can be written as a coproduct $\coprod_{i \in I}(\{i\}, S_i)$, for some family of sets $\{S_i\}_{i \in I}$ indexed by a set I. Note that this coproduct can be written as $(X, \mathcal{O}_X)$, where $X = \beta I$ is the Stone-Čech compactification of I and the stalk of $\mathcal{O}_X$ at an ultrafilter $\mathcal{U} \in \beta I$ is givenby

$$\mathcal{O}_{X,\mathcal{U}} = (\prod_{i \in I} S_i)/\,\mathcal{U}\,.$$

We let $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$ denote the full subcategory of $\mathrm{Stone}_{\mathcal{S}\mathrm{et}}$ spanned by the free objects.

Remark 2. Less explicitly, we can describe $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$ as the category $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$, where $\mathcal{C}$ is the pretopos of coherent objects of the classifying topos $\mathcal{E}_{\mathcal{S}\mathrm{et}} = \mathrm{Fun}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{S}\mathrm{et})$ (so that $\mathrm{Mod}(\mathcal{C})$ equivalent to the category of sets).

Note that we have an evident forgetful functor $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}} \to \mathrm{Stone}^{\mathrm{fr}}$, which is an ultracategory fibration. Consequently, if $\mathcal{M}$ is an ultracategory with associated ultracategory fibration $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$, then the category $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et})$ of ultrafunctors from $\mathcal{M}$ to $\mathcal{S}\mathrm{et}$ can be identified with the opposite of the category $\mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})$ of morphisms of ultracategory fibrations from $\mathcal{E}$ to $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$. Before we can describe the classification of such functors, we need a few preliminary remarks.

Lemma 3. *Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ be an ultracategory fibration. Suppose we are given an object $E \in \mathcal{E}$ with $\pi(E) = \beta I$ for some set I. For each element $i \in I$, choose a locally π-Cartesian morphism $f_i : E_i \to E$ lying over the inclusion map $\iota_i : \{i\} \hookrightarrow \beta I$ in $\mathrm{Stone}^{\mathrm{fr}}$. Then the morphisms f_i exhibit E as the coproduct of the objects E_i in $\mathcal{E}$.*

Proof. Let E' be any other object of $\mathcal{E}$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{E}}(E, E') & \xrightarrow{\circ f_\bullet} & \prod_{i \in I} \mathrm{Hom}_{\mathcal{E}}(E_i, E') \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}}(\beta I, \pi(E')) & \longrightarrow & \prod_{i \in I} \mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}}(\{i\}, \pi(E')) \end{array}$$

and we wish to show that the upper horizontal map is bijective. Since the bottom horizontal map is bijective, it will suffice to show that the diagram is a pullback square. Equivalently, we wish to show that for every continuous map $g : \beta I \to \pi(E')$, the induced map

$$\mathrm{Hom}_{\mathcal{E}_{\beta I}}(E, g^* E') \to \prod_{i \in I} \mathrm{Hom}_{\mathcal{E}_{\{i\}}}(\iota_i^* E, (g \circ \iota_i)^* E')$$

is a bijection. This follows from our assumptions that the functor $(\prod \iota(i)^*) : \mathcal{E}_{\beta I} \to \prod_{i \in I} \mathcal{E}_{\{i\}}$ is an equivalence of categories and that the comparison maps $\iota_i^* \circ g^* \to (g \circ \iota_i)^*$ are isomorphisms. □

Corollary 4. *Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ be an ultracategory fibration. Then the category $\mathcal{E}$ admits coproducts.*

Proof. Let $\mathcal{E}_* = \pi^{-1}\{*\}$. By virtue of Lemma 3, every object of $\mathcal{E}$ can be written as a coproduct of objects of $\mathcal{E}_*$. It will therefore suffice to show that every collection of objects $\{E_i \in \mathcal{E}_*\}_{i \in I}$ admits a coproduct. This follows from Lemma 3, since the functor $(\prod \iota(i)^*) : \mathcal{E}_{\beta I} \to \prod_{i \in I} \mathcal{E}_{\{i\}}$ is essentially surjective. □

Remark 5. It follows from the proof of Corollary 4 that if $\mathcal{D}$ is any category which admits coproducts, then a functor $F : \mathcal{E} \to \mathcal{D}$ preserves coproducts if and only if it preserves coproducts of families $\{E_i\}_{i \in I}$ where each E_i belongs to $\mathcal{E}_*$. In particular, the projection map $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ preserves coproducts.

Corollary 6. *Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ and $\pi' : \mathcal{E}' \to \mathrm{Stone}^{\mathrm{fr}}$ be ultracategory fibrations. Then any morphism of ultracategory fibrations $F : \mathcal{E} \to \mathcal{E}'$ preserves coproducts.*

Proof. By virtue of Remark 5, it will suffice to show that F preserves the coproduct of any family of objects $\{E_i\}_{i \in I}$ belonging to $\mathcal{E}_*$. Choose a collection of maps $f_i : E_i \to E$ in $\mathcal{E}$ which exhibit E as a coproduct of $\{E_i\}_{i \in I}$. It follows from Lemma 3 (and the uniqueness of coproducts) that each f_i is locally π-Cartesian. It follows that each $F(f_i)$ is locally π'-Cartesian, so that the morphisms $F(f_i)$ exhibit $F(E)$ as a coproduct of the family $\{F(E_i)\}_{i \in I}$ (by virtue of Lemma 3, applied to the ultracategory fibration π'). □

Notation 7. For every object $(X, \mathcal{O}_X) \in \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$, we let $\Gamma(X; \mathcal{O}_X)$ denote the set $\mathcal{O}_X(X)$. Then the construction $(X, \mathcal{O}_X) \mapsto \Gamma(X; \mathcal{O}_X)$ determines a functor $\Gamma : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}} \to \mathcal{S}\mathrm{et}^{\mathrm{op}}$. This functor is representable: we have canonical bijections

$$\Gamma(X; \mathcal{O}_X) \simeq \mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}}((X, \mathcal{O}_X), (*, \mathbf{1})),$$

where $*$ denotes the one-point space and $\mathbf{1} \in \mathrm{Shv}(*)$ denotes the final object (corresponding to the one-element set). It follows that the functor Γ carries coproducts in $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$ to products in $\mathcal{S}\mathrm{et}$.

We can now state our main result.

Theorem 8. *Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ be an ultracategory fibration. Then composition with the functor $\Gamma : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}} \to \mathcal{S}\mathrm{et}^{\mathrm{op}}$ induces a fully faithful embedding*

$$\mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}) \to \mathrm{Fun}(\mathcal{E}, \mathcal{S}\mathrm{et}^{\mathrm{op}}),$$

whose essential image consists of those functors $T : \mathcal{E} \to \mathcal{S}\mathrm{et}^{\mathrm{op}}$ which satisfy the following pair of conditions:

(*a*) *The functor T carries coproducts in $\mathcal{E}$ to products of sets.*

(*b*) *For each object $E \in \mathcal{E}$ having image $X = \pi(E)$ and each point $x \in X$, the canonical map*

$$\varinjlim_{x \in U} T(E_U) \to T(E_{\{x\}})$$

is a bijection. Here the direct limit is taken over all clopen neighborhoods U of x, $E_U \in \mathcal{E}_U$ denotes the pullback of $E \in \mathcal{E}_X$ under the inclusion $U \hookrightarrow X$, and $E_{\{x\}} \in \mathcal{E}_{\{x\}}$ is defined similarly.

Proof. If $F : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$ is a morphism of ultracategory fibrations, then F preserves coproducts (Corollary 6), so that $\Gamma \circ F$ carries coproducts in $\mathcal{E}$ to products of sets (Notation 7). Moreover, if E and X are as in (*b*), then we can write $F(E) = (X, \mathcal{O}_X)$ for some sheaf of sets $\mathcal{O}_X$ on X. Our assumption that F is a morphism of ultractegory fibrations (and therefore preserves locally Cartesian morphisms) supplies a canonical isomorphisms $F(E_U) \simeq (U, \mathcal{O}_X|_U)$ for $U \subseteq X$ clopen and $F(E_{\{x\}}) \simeq (\{x\}, \mathcal{O}_{X,x})$, so that

$$(\Gamma \circ F)(E_{\{x\}} \simeq \mathcal{O}_{X,x} \simeq \varinjlim_{x \in U} \mathcal{O}_X(U) = \varinjlim_{x \in U} (\Gamma \circ F)(E_U).$$

Consequently, every functor belonging to the essential image of composition with Γ satisfies conditions (*a*) and (*b*).

Conversely, suppose we are given a functor $T : \mathcal{E} \to \mathcal{S}\mathrm{et}^{\mathrm{op}}$ satisfying (*a*) and (*b*). We define a functor $F_T : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$ by the formula $F_T(E) = (X, \mathcal{O}_X)$, where $X = \pi(E)$ and $\mathcal{O}_X(U) = T(E_U)$ for $U \subseteq X$ clopen (condition (*a*) guarantees that the construction $U \mapsto T(E_U)$ carries coproducts of disjoint clopen subsets of X to products of sets, so this formula determines a sheaf $\mathcal{O}_X$ on X which is unique up to canonical isomorphism). For each clopen set $U \subseteq X$, we can identify E_U with the coproduct of the objects $E_{\{x\}}$ where x ranges over the isolated points of U (Lemma 3), so that condition (*a*) gives

$$\mathcal{O}_X(U) \simeq \prod_{x \in U, x \text{ isolated}} \mathcal{O}_{X,x}\,.$$

It follows that $(X, \mathcal{O}_X)$, is given by the coproduct of $(\{x\}, \mathcal{O}_{X,x})$ as x ranges over the isolated points of X, so that $(X, \mathcal{O}_X)$ belongs to $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$.

By construction, F_T is a functor from $\mathcal{E}$ to $\mathrm{Stone}_{\mathcal{S}\mathrm{et}}$ which fits into a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "F_T"] \arrow[dr, "\pi"'] & & \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}} \arrow[dl, "\pi'"] \\ & \mathrm{Stone}^{\mathrm{fr}}\,. & \end{tikzcd}$$

We claim that F_T is a morphism of ultracategory fibrations. In other words, we wish to show that for each locally π-Cartesian morphism $g : E \to E'$ in $\mathcal{E}$, the image $F_T(g)$ is a locally π'-Cartesian morphism in $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$. Write $F_T(E) = (X, \mathcal{O}_X)$ and $F_T(E') = (X', \mathcal{O}_{X'})$, so that $\pi(g)$ is a continuous map from X to X'. We then have a canonical map $\pi(g)^* \mathcal{O}_{X'} \to \mathcal{O}_X$, and we wish to show that this map of sheaves is an isomorphism after taking the stalk at each *isolated* point $x \in X$. Equivalently, we must show that the canonical map

$$\varinjlim_{\pi(g)(x) \in U} T(E'_U) \to T(E_{\{x\}})$$

is bijective, where U ranges over the collection of clopen subsets of X' containing $\pi(g)(x)$. This follows immediately from (*b*).

We leave it to the reader to verify that the constructions $F \mapsto \Gamma \circ F$ and $T \mapsto F_T$ are mutually inverse (up to canonical isomorphism). $\square$

If $\mathcal{C}$ is a small pretopos, then each object $C \in \mathcal{C}$ determines a morphism of ultracategory fibrations $F^C : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \to \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$, given by $(X, \mathcal{O}_X) \mapsto (X, \mathcal{O}_X^C)$. The construction $C \mapsto F^C$ then determines a functor $\mathcal{C} \to \mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})^{\mathrm{op}}$. In Lecture 23X, we showed that the composition

$$\mathcal{C} \to \mathrm{Mor}(\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})^{\mathrm{op}} \xrightarrow{\Gamma \circ} \mathrm{Fun}(\mathrm{Stone}^{\mathrm{fr},\mathrm{op}}_{\mathcal{C}}, \mathcal{S}\mathrm{et})$$

$$C \mapsto ((X, \mathcal{O}_X) \mapsto \Gamma(X; \mathcal{O}_X^C))$$

is a fully faithful embedding, whose essential image is spanned by those functors $\mathrm{Stone}^{\mathrm{fr},\mathrm{op}}_{\mathcal{C}} \to \mathcal{S}\mathrm{et}$ satisfying conditions (a) and (b) of Theorem 8. Combining this with Theorem 8, we obtain the following:

Corollary 9. *Let $\mathcal{C}$ be a small pretopos. Then the construction $C \mapsto F^C$ induces an equivalence of categories*

$$\mathcal{C} \to \mathrm{Mor}(\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})^{\mathrm{op}}.$$

Or, stated in terms of ultrafunctors:

Corollary 10 (Makkai)**.** *Let $\mathcal{C}$ be a small pretopos. Then there is an equivalence of categories*

$$\mathcal{C} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathcal{S}\mathrm{et}),$$

which carries an object C to the ultrafunctor $\mathrm{Mod}(\mathcal{C}) \to \mathcal{S}\mathrm{et}$ given by evaluation at C.

Lecture 29X-Makkai Duality

April 22, 2018

To every small pretopos $\mathcal{C}$, we have associated an ultracategory fibration $\pi : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \to \mathrm{Stone}^{\mathrm{fr}}$, with underlying ultracategory $\mathrm{Mod}(\mathcal{C})$. Here the assumption that $\mathcal{C}$ is small is not really essential: for any pretopos $\mathcal{C}$, we can consider the category $\mathrm{Mod}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ of pretopos morphisms from $\mathcal{C}$ to $\mathcal{S}\mathrm{et}$, and equip it with the structure of an ultracategory. In this lecture, we will study a right adjoint to this construction.

We begin with some preliminary remarks.

Proposition 1. *Let I be a set and let $X = \beta I$. Then the direct image functor*

$$\iota_* : \mathcal{S}\mathrm{et}^I \simeq \mathrm{Shv}(I) \to \mathrm{Shv}(X)$$

is a morphism of pretopoi: that is, it preserves finite limits, finite coproducts, and effective epimorphisms.

Proof. It is clear that ι_* preserves finite limits. The axiom of choice implies that every epimorphism in $\mathcal{S}\mathrm{et}^I$ admits a section, so that ι_* preserves effective epimorphisms. Consequently, to show that it is a morphism of pretopoi, it will suffice to show that for every finite set J, the functor ι_* carries the constant sheaf $\underline{J}_I$ on the discrete space I to the constant sheaf $\underline{J}_X$ on X (see Lemma 4 of Lecture 16X). This follows from the observation that every map $I \to J$ extends uniquely to a locally constant map $\beta I \to J$ (since J is compact), and that an analogous statement holds for each clopen subset of X. $\square$

Corollary 2. *Let I be a set and let $\mathcal{U}$ be an ultrafilter on I. Then the ultraproduct functor*

$$P^{\mathcal{U}} : \mathcal{S}\mathrm{et}^I \to \mathcal{S}\mathrm{et} \qquad \{M_i\}_{i \in I} \mapsto (\prod_{i \in I} M_i)/\mathcal{U}$$

is a morphism of pretopoi.

Proof. The functor $P^{\mathcal{U}}$ is given by the composition

$$\mathcal{S}\mathrm{et}^I \xrightarrow{\iota_*} \mathrm{Shv}(\beta I) \xrightarrow{\mathcal{U}^*} \mathcal{S}\mathrm{et}$$

where $\mathcal{U}^*$ denotes the functor given by taking the stalk at the point of βI corresponding to the ultrafilter $\mathcal{U}$. $\square$

Remark 3. Corollary 2 is a version of the Łos ultraproduct theorem: it implies that for any pretopos $\mathcal{C}$, composition with $P^{\mathcal{U}}$ induces a functor

$$\mathrm{Mod}(\mathcal{C})^I = \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{S}\mathrm{et}^I) \xrightarrow{P^{\mathcal{U}} \circ} \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{S}\mathrm{et}) = \mathrm{Mod}(\mathcal{C}),$$

where $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D})$ denotes the category of pretopos morphisms from $\mathcal{C}$ to $\mathcal{D}$.

Corollary 4. *Let $\pi' : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}} \to \mathrm{Stone}^{\mathrm{fr}}$ denote the forgetful functor. Then:*

- *For each $X \in \mathrm{Stone}^{\mathrm{fr}}$, the category $(\pi'^{-1}\{X\})^{\mathrm{op}}$ is a pretopos.*

- *For each morphism $f : Y \to X$ in $\mathrm{Stone}^{\mathrm{fr}}$, the pullback functor $f^* : (\pi'^{-1}\{X\})^{\mathrm{op}} \to (\pi'^{-1}\{Y\})^{\mathrm{op}}$ is a morphism of pretopoi.*

Proof. If $X = \beta I$, then $(\pi'^{-1}\{X\})^{\mathrm{op}}$ is equivalent to $\mathcal{S}\mathrm{et}^I$. Moreover, if $Y = \beta J$ and $f : Y \to X$ is a map given by a family of ultrafilters $\{\mathcal{U}_j\}_{j \in J}$ on the set I, then the pullback functor f^* can be identified with the construction

$$(\{M_i\}_{i \in I} \in \mathcal{S}\mathrm{et}^I) \mapsto (\{\prod_{i \in I} M_i)/\,\mathcal{U}_j\}_{j \in J} \in \mathcal{S}\mathrm{et}^J),$$

and is therefore a morphism of pretopoi (Corollary 2). □

Corollary 5. *Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ be an ultracategory fibration and let $\mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})$ denote the category of ultracategory fibration morphisms from $\mathcal{E}$ to $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$. Then the category $\mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})^{\mathrm{op}}$ is a pretopos (not necessarily small).*

Proof. It follows from Corollary 4 that we can compute finite limits, finite coproducts, and quotients by equivalence relations in the category $\mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})^{\mathrm{op}}$ "fiberwise." □

Or, stated in terms of ultracategories:

Corollary 6. *Let $\mathcal{M}$ be an ultracategory. Then the category $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et})$ is a pretopos.*

Remark 7. In the situation of Corollary 6, the forgetful functor $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{M}, \mathcal{S}\mathrm{et})$ is a morphism of pretopoi. Equivalently, for each $M \in \mathcal{M}$, the evaluation functor $F \mapsto F(M)$ determines a morphism of pretopoi $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et}) \to \mathcal{S}\mathrm{et}$. In order words, finite limits, finite coproducts, and quotients by equivalence relations in $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et})$ can be computed "pointwise".

Let $\mathcal{C}$ be a (small) pretopos. Then the category $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$ can be identified with the full subcategory of

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}) \times_{\mathrm{Fun}(\mathcal{C}, \mathrm{Stone}^{\mathrm{fr}})} \mathrm{Stone}^{\mathrm{fr}}$$

spanned by those pairs $(F : \mathcal{C} \to \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}, X)$ with the property that F determines a morphism of pretopoi $\mathcal{C} \to \mathrm{Shv}(X)$. Consequently, for any ultracategory fibration $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$, we have a canonical equivalence

$$\mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}})^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})^{\mathrm{op}}).$$

Or, stated in terms of ultracategories:

Corollary 8. *Let $\mathcal{M}$ be an ultracategory and let $\mathcal{C}$ be a pretopos. Then there is a canonical equivalence of categories*

$$\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathrm{Mod}(\mathcal{C})) \simeq \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et})).$$

We can regard the construction $\mathcal{C} \mapsto \mathrm{Mod}(\mathcal{C})$ as a contravariant functor from the 2-category of pretopoi (where morphisms are functors that preserve finite limits, finite coproducts, and effective epimorphisms) to the 2-category of ultracategories (where morphisms are ultrafunctor). We can interpret Corollary 8 as asserting that that this construction admits a right adjoint

$$\{\mathrm{Ultracategories}\}^{\mathrm{op}} \to \{\mathrm{Pretopoi}\} \qquad \mathcal{M} \mapsto \mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et}).$$

In particular, for every pretopos $\mathcal{C}$, we have a unit map

$$u : \mathcal{C} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathcal{S}\mathrm{et}),$$

which carries each object $C in \mathcal{C}$ to the ultrafunctor given by evaluation at C. We saw in the previous lecture that this functor is an equivalence when $\mathcal{C}$ is small.

Corollary 9 (Strong Conceptual Completeness)**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be pretopoi. If $\mathcal{C}'$ is small, then the canonical map*

$$\theta : \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}') \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}'), \mathrm{Mod}(\mathcal{C}))$$

is an equivalence of categories.

Proof. Using the adjunction of Corollary 8, we can identify θ with the map

$$\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}') \to \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}'), \mathcal{S}\mathrm{et}))$$

given by composition with the unit map $u : \mathcal{C}' \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}'), \mathcal{S}\mathrm{et})$. □

Remark 10. Makkai's strong conceptual completeness theorem can be interpeted as a kind of duality between the theory of pretopoi (having to do with the syntax of first-order logic) and the theory of ultracategories (which is a way of capturing the semantics of first-order logic). Concretely, this duality is implemented by the category $\mathcal{S}\mathrm{et}$ of sets, which simultaneously has the structure of a pretopos and an ultracategory (moreover, these structures are "compatible" by virtue of Corollary 2). It follows that for every pretopos $\mathcal{C}$, the category $\mathrm{Mod}(\mathcal{C})$ of pretopos morphisms from $\mathcal{C}$ to $\mathcal{S}\mathrm{et}$ has the structure of an ultracategory, and that for every ultracategory $\mathcal{M}$ the category $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et})$ of ultrafunctors from $\mathcal{M}$ to $\mathcal{S}\mathrm{et}$ has the structure of a pretopos.

Corollary 11. *The construction $\mathcal{C} \mapsto \mathrm{Mod}(\mathcal{C})$ determines a fully faithful embedding of 2-categories*

$$\{\textit{Small pretopoi}\}^{\mathrm{op}} \hookrightarrow \{\textit{Ultracategories}\}.$$

It follows immediatey from the definitions that an ultrafunctor $F : \mathcal{M} \to \mathcal{M}'$ is invertible (in the 2-category of ultracategories) if and only if it is an equivalence (in particular if F is an equivalence of categories, then a homotopy inverse F^{-1} to F can also be regarded as an ultrafunctor). We therefore obtain the following:

Corollary 12 (Makkai-Reyes Conceptual Completeness Theorem)**.** *Let $f : \mathcal{C} \to \mathcal{C}'$ be a morphism of small pretopoi. If the induced map $\mathrm{Mod}(\mathcal{C}') \to \mathrm{Mod}(\mathcal{C})$ is an equivalence of categories, then f is an equivalence of categories.*

Remark 13. Note that the analogue Corollary **??** would *not* hold if we were to allow $\mathcal{C}$ and $\mathcal{C}'$ to be arbitrary coherent categories: the canonical map from a coherent category $\mathcal{C}$ to its pretopos completion $\mathcal{C}_{\mathrm{eq}}$ induces an equivalence of categories of models $\mathrm{Mod}(\mathcal{C}_{\mathrm{eq}}) \to \mathrm{Mod}(\mathcal{C})$, but is generally not an equivalence. Consequently, we can regard Corollary **??** as evidence that pretopoi are a good class of objects to work with. For example, if T is a first-order theory, then the syntactic category $\mathrm{Syn}(T)$ is the "maximal" enlargement of the weak syntactic category $\mathrm{Syn}_0(T)$ with the same semantics.

Corollary 11 can be considered as a categorified version of Stone duality, which establishes a fully faithful embedding

$$\{\text{Boolean algebras}\}^{\mathrm{op}} \hookrightarrow \{\text{Compact Hausdorff spaces}\}$$

(whose essential image is the category Stone of Stone spaces). In fact, it is even a generalization of Stone duality: there is a commutative diagram

$$\begin{tikzcd}
\{\text{Boolean algebras}\}^{\mathrm{op}} \arrow[r] \arrow[d] & \{\text{Compact Hausdorff Spaces}\} \arrow[d] \\
\{\text{Small pretopoi}\}^{\mathrm{op}} \arrow[r, "\mathrm{Mod}"] & \{\text{Ultracategories}\}
\end{tikzcd}$$

where the left vertical map is given by pretopos completion and the right vertical map associates to each compact Hausdorff space the associated ultraset (see Lecture 26X).

Warning 14. The fully faithful embedding of Corollary 11 is not essentially surjective. For example, if X is a compact Hausdorff space which is not a Stone space, then X (regarded as an ultracategory with only identity morphisms) does not belong to the essential image.

Lecture 30X-Higher Categorical Logic

April 24, 2018

In this final lecture, we sketch (without proof) how some of the ideas of this course can be extended to the setting of higher category theory. We assume that the reader is familiar with the language of ∞-categories (following the terminology of Higher Topos Theory).

Let $\mathcal{C}$ be an ∞-category which admits finite limits. To each morphism $f : X \to Y$ in $\mathcal{C}$, we can associate a simplicial object $X_\bullet$ of $\mathcal{C}$, where X_n is the $(n+1)$-fold fiber power of X over Y. We refer to $X_\bullet$ as the *Čech nerve* of f. It is an example of a *groupoid object* of $\mathcal{C}$. We will say that a morphism $f : X \to Y$ is an *effective epimorphism* if it exhibits Y as a geometric realization of the Čech nerve of f.

Definition 1. Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is an ∞-*pretopos* if it satisfies the following axioms:

$(A1)$ The ∞-category $\mathcal{C}$ admits finite limits.

$(A2)$ Every groupoid object $X_\bullet$ is *effective*: that is, it arises as the Čech nerve of an effective epimorphism $X_0 \to Y$ in $\mathcal{C}$ (in particular, $X_\bullet$ admits a geometric realization $Y \simeq |X_\bullet|$).

$(A3)$ The ∞-category $\mathcal{C}$ admits finite coproducts, and coproducts are disjoint (that is, the maps $X \hookrightarrow X \amalg Y \hookleftarrow Y$ are monomorphisms, and the fiber product $X \times_{X \amalg Y} Y$ is initial).

$(A4)$ The collection of effective epimorphisms in $\mathcal{C}$ is closed under pullbacks.

$(A5)$ The formation of finite coproducts in $\mathcal{C}$ is preserved by pullback.

Let $n \geq -1$ be an integer. We say that an object X of an ∞-category $\mathcal{C}$ is n-*truncated* if the mapping spaces $\mathrm{Map}_{\mathcal{C}}(Y, X)$ are n-truncated for each object $Y \in \mathcal{C}$: that is, if the homotopy groups $\pi_* \mathrm{Map}_{\mathcal{C}}(Y, X)$ vanish for $* > n$. We say that an object of $\mathcal{C}$ is *discrete* if it is 0-truncated.

Proposition 2. *Let $\mathcal{C}$ be an ∞-pretopos. For every integer n, let $\mathcal{C}_{\leq n}$ be the full subcategory of $\mathcal{C}$ spanned by the n-truncated objects. Then:*

- *The inclusion functor $\mathcal{C}_{\leq n} \hookrightarrow \mathcal{C}$ admits a left adjoint, which we will denote by $X \mapsto \tau_{\leq n} X$.*
- *The category $\mathcal{C}_{\leq 0}$ of discrete objects of $\mathcal{C}$ is a pretopos.*

Definition 3. Let $\mathcal{C}$ and $\mathcal{C}'$ be ∞-pretopoi. A *morphism of ∞-pretopoi* from $\mathcal{C}$ to $\mathcal{C}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ which preserves finite limits, finite coproducts, and effective epimorphisms. We let $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}')$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ spanned by those morphisms of ∞-pretopoi.

If $\mathcal{C}$ is an ∞-pretopos, we define a *model* of $\mathcal{C}$ to be a morphism of ∞-pretopoi $\mathcal{C} \to \mathcal{S}$, where $\mathcal{S}$ denotes the ∞-category of spaces. The collection of models of $\mathcal{C}$ forms an ∞-category which we will denote by $\mathrm{Mod}(\mathcal{C})$.

Remark 4. It follows from Proposition 2 that the construction $\mathcal{C} \mapsto \mathcal{C}_{\leq 0}$ determines a forgetful functor

$$\{\infty\text{-pretopoi}\} \to \{\text{Pretopoi}\}.$$

One can show that this functor has a left adjoint, which carries an ordinary pretopos $\mathcal{C}$ to an ∞-pretopos that we will denote by $\mathcal{C}^+$. Roughly speaking, one can think of objects of $\mathcal{C}^+$ as "stacky" objects of $\mathcal{C}$.

Example 5. Let $\mathcal{S}$ be the ∞-category of spaces. Then $\mathcal{S}$ is an ∞-pretopos, whose underlying ordinary pretopos $\mathcal{S}_{\leq 0}$ can be identified with the category $\mathcal{S}\mathrm{et}$ of sets. The ∞-pretopos $\mathcal{S}\mathrm{et}^+$ can be identified with the union $\bigcup_{n \geq 0} \mathcal{S}_{\leq n}$, regarded as a full subcategory of $\mathcal{S}$.

Example 6. Let $\mathcal{S}\mathrm{et}_{\mathrm{fin}}$ be the pretopos of finite sets. Then $\mathcal{S}\mathrm{et}^+_{\mathrm{fin}}$ is the ∞-pretopos of *π-finite spaces*: that is, the full subcategory of $\mathcal{S}$ spanned by those spaces X for which the homotopy groups $\pi_n(X, x)$ are finite for each $n \geq 0$ (and each base point $x \in X$), and which vanish for $n \gg 0$. This is the initial object in the setting of ∞-pretopoi.

Remark 7. Let $\mathcal{C}$ be a pretopos and let $\mathcal{C}^+$ be the associated ∞-pretopos. Then we have a canonical equivalence

$$\mathrm{Mod}(\mathcal{C}^+) = \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}^+, \mathcal{S}) \simeq \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{S}_{\leq 0}) \simeq \mathrm{Mod}(\mathcal{C}).$$

In other words, the ∞-category of models of $\mathcal{C}^+$ can be identified with the ordinary category of models of $\mathcal{C}$. Consequently, the pretopos $\mathcal{C}$ and the ∞-pretopos $\mathcal{C}^+$ can be regarded as incarnations of the same object.

Remark 8. One can think of the relationship between ∞-pretopoi and ordinary pretopoi as analogous to the relationship between ordinary pretopoi and distributive lattices. Every ordinary pretopos $\mathcal{C}$ determines a distributive lattice, given by the poset $\mathrm{Sub}(\mathbf{1}_{\mathcal{C}})$. This construction has a left adjoint, given by the pretopos completion. The left adjoint is fully faithful (in both settings), but not essentially surjective: passing from an ∞-pretopos to an ordinary pretopos and passing from an ordinary pretopos to a distributive lattice typically loses a lot of information.

Definition 9. Let $\mathcal{C}$ be an ∞-pretopos. We will say that $\mathcal{C}$ is *bounded* if every object X of $\mathcal{C}$ is n-truncated, for some $n \gg 0$ (which might depend on X).

If $\mathcal{C}$ is an ∞-pretopos, then the full subcategory $\mathcal{C}_{<\infty} = \bigcup_{n \geq 0} \mathcal{C}_{\leq n}$ is a bounded ∞-pretopos. Moreover, the ∞-pretopoi $\mathcal{C}$ and $\mathcal{C}_{<\infty}$ have the same models (every model $M : \mathcal{C}_{<\infty} \to \mathcal{S}$ extends uniquely to a model of $\mathcal{C}$, given by the construction

$$(C \in \mathcal{C}) \mapsto (\varprojlim M(\tau_{\leq n} C) \in \mathcal{S}).$$

We will henceforth restrict our attention to the study of bounded ∞-pretopoi.

Remark 10. Remark 7 admits a converse. Let $\mathcal{C}$ be a small ∞-pretopos. Then $\mathcal{C}$ is of the form $\mathcal{C}^+_{\leq 0}$ if and only if it is bounded and the ∞-category $\mathrm{Mod}(\mathcal{C})$ is (equivalent to) an ordinary category.

Example 11. Let CRing denote the category of commutative rings, and let $\mathcal{X}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CRing}, \mathcal{S}\mathrm{et})$ spanned by those functors which preserve filtered colimits. Then $\mathcal{X}$ is a coherent topos. Let $\mathcal{X}^{\mathrm{coh}}$ denote the pretopos of coherent objects of $\mathcal{X}$. Then the category $\mathrm{Mod}(\mathcal{X}^{\mathrm{coh}})$ can be identified with the category CRing of commutative rings. Here we can replace the category CRing by an arbitrary compactly generated category.

This construction has an ∞-categorical analogue. Let $\mathcal{E}$ denote the ∞-category of connective E_∞-ring spectra (or any other compactly generated ∞-category), and let $\mathcal{X} \subseteq \mathrm{Fun}(\mathcal{E}, \mathcal{S})$ be the full subcategory spanned by those functors which preserve filtered colimits. Then $\mathcal{X}$ is an example of an *∞-topos*, and it has a full subcategory $\mathcal{X}^{\mathrm{coh}}_{<\infty} \subseteq \mathcal{X}$ of truncated coherent objects. This subcategory is a bounded ∞-pretopos, and the ∞-category $\mathrm{Mod}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$ can be identified with $\mathcal{E}$.

The study of ∞-pretopoi can be regarded as a generalization of classical first order logic, suitable for studying the model theory of objects which behave in a homotopy-theoretic way (such as structured ring spectra).

Let $\mathcal{C}$ be a bounded ∞-pretopos. We let $\mathrm{Stone}_{\mathcal{C}}$ denote the ∞-category whose objects are pairs $(X, \mathcal{O}_X)$, where X is a Stone space and $\mathcal{O}_X : \mathcal{C} \to \mathrm{Shv}_{\mathcal{S}}(X)$ is a morphism of ∞-pretopoi (here $\mathrm{Shv}_{\mathcal{S}}(X)$ denotes the ∞-category of $\mathcal{S}$-valued sheaves on X). We can regard the ∞-category $\mathrm{Mod}(\mathcal{C})$ of models of $\mathcal{C}$ as a full subcategory of $\mathrm{Stone}^{\mathrm{op}}_{\mathcal{C}}$, spanned by those pairs $(X, \mathcal{O}_X)$ where X consists of a single point. As in the

1-categorical case, one can show that the category $\mathrm{Stone}_{\mathcal{C}}$ admits coproducts, so that every collection of models $\{M_i\}_{i \in I}$ admits a coproduct

$$\coprod_{i \in I} (\{i\}, M_i)$$

in the ∞-category $\mathrm{Stone}_{\mathcal{C}}$. This is a pair $(X, \mathcal{O}_X)$, where X can be identified with the Stone-Čech compactification βI, and the stalk of $\mathcal{O}_X$ at a point x corresponding to an ultrafilter $\mathcal{U}$ is given by the formula

$$\mathcal{O}_{X,x}^C = \varinjlim_{J \in \mathcal{U}} \prod_{i \in J} M_i(C)$$

(which we can think of as the ultraproduct of the models M_i indexed by $\mathcal{U}$). We let $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ denote the full subcategory of $\mathrm{Stone}_{\mathcal{C}}$ spanned by the objects of this form, so that we have a forgetful functor $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}} \to \mathrm{Stone}^{\mathrm{fr}}$. The following definition encodes the essential properties of this fibration:

Definition 12. An *∞-ultracategory* is a locally Cartesian fibration of ∞-categories $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ with the following properties:

(1) Let $\mathcal{M} = \mathcal{E}_*^{\mathrm{op}}$. Then, for every set I, the pullback functors associated to the inclusion maps $* \simeq \{i\} \hookrightarrow \beta I$ induce an equivalence of ∞-categories $\mathcal{E}_{\beta I}^{\mathrm{op}} \simeq \prod_{i \in I} \mathcal{M}$.

(2) If $f : E \to E'$ and $g : E' \to E''$ are locally π-Cartesian morphisms of $\mathcal{E}$, and $\pi(f) : \pi(E) \to \pi(E')$ carries isolated points of $\pi(E)$ to isolated points of $\pi(E')$, then $g \circ f$ is also locally π-Cartesian.

In this case, we will refer to $\mathcal{M}$ as the *underlying ∞-category* of the ultracategory $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ (and will abuse terminology by simply referring to $\mathcal{M}$ as an ultracategory).

In the case where $\mathcal{M}$ is an ordinary category, Definition 12 specializes to the notion of an ultracategory (or, more precisely, the notion of an ultracategory fibration) introduced earlier in this course. In this case, it was possible to rewrite Definition 12 explicitly in terms of structure on $\mathcal{M}$. This is already somewhat cumbersome in the 1-categorical case, and is much more so in the ∞-categorical setting: roughly speaking, the structure on $\mathcal{M}$ is given by a collection of "ultraproduct functors" $P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M}$, together with some natural transformations between iterated ultraproducts which satisfy an infinite hierarchy of coherence conditions. These coherence conditions are encoded neatly in the structure of the locally Cartesian fibration $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$.

Example 13. Let $\mathcal{C}$ be an ∞-pretopos. Then the forgetful functor $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}} \to \mathrm{Stone}^{\mathrm{fr}}$ is an ∞-ultracategory, with underlying ∞-category $\mathrm{Mod}(\mathcal{C})$.

Definition 14. Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ and $\pi' : \mathcal{E}' \to \mathrm{Stone}^{\mathrm{fr}}$ be ∞-ultracategories, with underlying ∞-categories $\mathcal{M}$ and $\mathcal{M}'$, respectively. We let $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{M}')$ denote the full subcategory of

$$\mathrm{Fun}(\mathcal{E}, \mathcal{E}')^{\mathrm{op}} \times_{\mathrm{Fun}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}})^{\mathrm{op}}} \{\pi\}$$

spanned by those functors $F : \mathcal{E} \to \mathcal{E}'$ satisfying $\pi = \pi' \circ F$, for which F carries locally π-Cartesian morphisms to locally π'-Cartesian morphisms. We will refer to $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{M}')$ as the *∞-category of ultrafunctors from $\mathcal{M}$ to $\mathcal{M}'$*.

Makkai duality then admits the following ∞-categorical generalization:

Theorem 15. *Let $\mathcal{C}$ and $\mathcal{C}'$ be small bounded ∞-pretopoi. Then the canonical map*

$$\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}') \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}'), \mathrm{Mod}(\mathcal{C}))$$

is an equivalence of ∞-categories. (In fact, it is enough to assume that $\mathcal{C}$ is bounded and $\mathcal{C}'$ is small.)

In other words, a small bounded ∞-pretopos $\mathcal{C}$ can be recovered from its ∞-category $\mathrm{Mod}(\mathcal{C})$ of models, regarded as an ∞-ultracategory. This can be proven using essentially the same strategy that we used to deduce the 1-categorical version of Makkai duality.

Remark 16. It is not so hard to classify ∞-ultracategories for which the underlying ∞-category $\mathcal{M}$ is a Kan complex. The ∞-ultracategories for which $\mathcal{M}$ is an n-truncated Kan complex can be identified with the n-truncated objects of $\mathrm{Top}_{\mathrm{ch}}^{+}$, where $\mathrm{Top}_{\mathrm{ch}}$ is the pretopos of compact Hausdorff spaces. (In other words, they are "stacky" versions of compact Hausdorff spaces: when $n = 0$, this recovers the description of ultrasets given in Lecture 26X.

Lecture 14: Locales and Topoi

March 5, 2018

Recall that, if X is an object of a coherent category $\mathcal{C}$, then the poset $\mathrm{Sub}(X)$ is a distributive lattice. If $\mathcal{C}$ is a topos, we can say more.

Definition 1. A *locale* is a poset $\mathcal{U}$ with the following properties:

(a) Every subset $S \subseteq \mathcal{U}$ has a least upper bound $\bigvee S$.

It follows from (a) that every subset $S \subseteq \mathcal{U}$ also has a greatest lower bound $\bigwedge S$, given by the least upper bound of the set $\{U \in \mathcal{U} : (\forall V \in S) U \leq V\}$ of all lower bounds for S. In particular, every pair of elements $U, V \in \mathcal{U}$ have a meet $U \wedge V$.

(b) For each element $V \in \mathcal{U}$ and every set of elements $\{U_\alpha\}$, we have a distributive law

$$(\bigvee_\alpha U_\alpha) \wedge V = \bigvee_\alpha (U_\alpha \wedge V).$$

Remark 2. Every locale is a distributive lattice.

Exercise 3. Let $\mathcal{U}$ be a poset satisfying condition (a) of Definition 1. Show that $\mathcal{U}$ is a locale if and only if it is a *Heyting algebra*: that is, if and only if for every pair of elements $U, V \in \mathcal{U}$, there is an element $(U \Rightarrow V) \in \mathcal{U}$ such that $W \leq (U \Rightarrow V)$ if and only if $U \wedge W \leq V$.

Example 4. Let B be a complete Boolean algebra (that is, a Boolean algebra satisfying condition (a) of Definition 1). Then B is a locale.

Example 5. Let X be a topological space and let $\mathcal{U}(X)$ be the collection of open subsets of X, partially ordered with respect to inclusion. Then $\mathcal{U}(X)$ is a locale. Moreover, the join $\bigvee U_\alpha$ of a collection of elements $U_\alpha \in \mathcal{U}(X)$ coincides with the set-theoretic union $\bigcup U_\alpha$, and the meet of a pair $U, V \in \mathcal{U}(X)$ is given by the set-theoretic intersection $U \cap V$.

Beware that the meet of an *infinite* set of elements $U_\alpha \in \mathcal{U}(X)$ usually does not coincide with the intersection $\bigcap U_\alpha$, because the intersection $\bigcap U_\alpha$ need not be open; instead, $\bigwedge U_\alpha$ is given by the interior of $\bigcap U_\alpha$. In particular, we generally have

$$(\bigwedge_\alpha U_\alpha) \vee V \neq \bigwedge_\alpha (U_\alpha \vee V).$$

Proposition 6. *Let $\mathcal{X}$ be a topos and let X be an object of $\mathcal{X}$. Then the poset* $\mathrm{Sub}(X)$ *is a locale.*

Proof. Every collection of objects $\{U_i \subseteq X\}_{i \in I}$ has a join, given by the image of the map $\coprod_{i \in I} U_i \to X$. For

$V \subseteq X$, we compute

$$\begin{aligned}
(\bigvee U_i) \wedge V &= (\bigvee U_i) \times_X V \\
&= \mathrm{Im}(\coprod_{i \in I} U_i \to X) \times_X V \\
&= \mathrm{Im}((\coprod_{i \in I} U_i) \times_X V \to V) \\
&= \mathrm{Im}(\coprod_{i \in I} (U_i \times_X V) \to V) \\
&= \bigvee_{i \in I} U_i \wedge V.
\end{aligned}$$

□

Definition 7. Let $\mathcal{X}$ be a topos and let $\mathbf{1}$ be the final object of $\mathcal{X}$. Then $\mathrm{Sub}(\mathbf{1})$ is a locale. We will refer to $\mathrm{Sub}(\mathbf{1})$ as the *underlying locale of* $\mathcal{X}$.

In the situation of Definition 7, the poset $\mathrm{Sub}(\mathbf{1})$ can be regarded as a full subcategory of $\mathcal{X}$.

Definition 8. Let $\mathcal{X}$ be a topos. We say that $\mathcal{X}$ is *localic* if it is generated by $\mathrm{Sub}(\mathbf{1})$: that is, if every object $X \in \mathcal{X}$ admits a covering $\{U_i \to X\}$, where each U_i is a subobject of $\mathbf{1}$.

Example 9. Let $\mathcal{C}$ be a category which admits finite limits, equipped with a Grothendieck topology. Suppose that $\mathcal{C}$ is a poset (that is, every object of $\mathcal{C}$ can be identified with a subobject of the final object). Then the topos $\mathrm{Shv}(\mathcal{C})$ is localic: it is generated by objects of the form Lh_C, each of which is a subobject of the final object of $\mathrm{Shv}(\mathcal{C})$.

Example 10. Let X be a topological space. Then the topos $\mathrm{Shv}(X)$ is localic (this is a special case of Example 9).

We now prove a converse to Example 9.

Exercise 11. Let $\mathcal{U}$ be a locale. Show that $\mathcal{U}$ admits a Grothendieck topology, where a collection of maps $\{U_i \to X\}$ is a covering if $X = \bigvee U_i$.

Proposition 12. *Let $\mathcal{X}$ be a localic topos, and regard the underlying locale $\mathcal{U} = \mathrm{Sub}(\mathbf{1})$ as equipped with the Grothendieck topology of Exercise 11. Then we have a canonical equivalence $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{U})$.*

Proof. We can regard $\mathcal{U}$ as an essentially small full subcategory of $\mathcal{X}$ which is closed under finite limits. If $\mathcal{X}$ is localic, then $\mathcal{U}$ generates $\mathcal{X}$, so the desired result follows as in the proof of Giraud's theorem. □

We now proceed in the reverse direction.

Proposition 13. *Let $\mathcal{U}$ be a locale. Then the Yoneda embedding $h : \mathcal{U} \to \mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ induces an equivalence from $\mathcal{U}$ to the poset of subobjects of $\mathbf{1}$ in $\mathrm{Shv}(\mathcal{U})$.*

Proof. We first show that, for each $U \in \mathcal{U}$, the presheaf h_U is a sheaf. Suppose we are given a covering $\{V_i \to V\}_{i \in I}$ in $\mathcal{U}$; we wish to show that the canonical map

$$h_U(V) \to \prod_i h_U(V_i) \rightrightarrows \prod_{i,j} h_U(V_i \wedge V_j)$$

is an equalizer diagram. Equivalently, we wish to show that $V \leq U$ if and only if each $V_i \leq U$, which follows from the identity $V = \bigvee_{i \in I} V_i$.

It is clear that each h_U is a subobject of the final object of $\mathrm{Shv}(\mathcal{U})$ (note that $h_U(V)$ is a singleton for $V \leq U$, and empty otherwise). Conversely, let $\mathscr{F} \in \mathrm{Shv}(\mathcal{U})$ be a subobject of the final object, so that $\mathscr{F}(V)$ has at most one element for each $V \in \mathcal{U}$. Set $U = \bigvee_{\mathscr{F}(V) \neq \emptyset} V$. Then we have a covering $\{V \to U\}_{\mathscr{F}(V) \neq \emptyset}$. Invoking the assumption that $\mathscr{F}$ is a sheaf, we conclude that $\mathscr{F}(U) \neq \emptyset$. We therefore have $\mathscr{F}(V) = \begin{cases} * & \text{if } V \leq U \\ \emptyset & \text{otherwise.} \end{cases}$, so that $\mathscr{F} \simeq h_U$. $\square$

We can summarize Propositions 12 and 13 more informally by saying that we have an equivalence

$$\{ \text{ Localic topoi } \} \simeq \{ \text{ Locales } \}.$$

To every localic topos $\mathcal{X}$, we can associate the locale $\mathrm{Sub}(\mathbf{1})$ of subobjects of the final object; to any locale $\mathcal{U}$, we can associate a topos $\mathrm{Shv}(\mathcal{U})$, and these constructions are mutually inverse (up to equivalence). In fact, we can be a bit more precise.

Definition 14. Let $\mathcal{U}$ and $\mathcal{V}$ be locales. A *morphism of locales* from $\mathcal{V}$ to $\mathcal{U}$ is an order-preserving map $f^* : \mathcal{U} \to \mathcal{V}$ such that f^* preserves finite meets and arbitrary joins (equivalently, it preserves finite limits and small colimits, if we view $\mathcal{U}$ and $\mathcal{V}$ as categories). We let $\mathrm{Fun}^*(\mathcal{U}, \mathcal{V})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{U}, \mathcal{V})$ spanned by the morphisms of locales from $\mathcal{V}$ to $\mathcal{U}$ (note that $\mathrm{Fun}^*(\mathcal{U}, \mathcal{V})$ is a poset).

Proposition 15. *Let $\mathcal{U}$ be a locale and let $\mathcal{X}$ be a topos with underlying locale* $\mathrm{Sub}(\mathbf{1})$. *Then composition with the Yoneda embedding $h : \mathcal{U} \to \mathrm{Shv}(\mathcal{U})$ induces an equivalence of categories*

$$\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{U}), \mathcal{X}) \to \mathrm{Fun}^*(\mathcal{U}, \mathrm{Sub}(\mathbf{1})).$$

In other words, the category of geometric morphisms from $\mathcal{X}$ to $\mathrm{Shv}(\mathcal{U})$ *is equivalent to the poset of morphisms of locales from* $\mathrm{Sub}(\mathbf{1})$ *to $\mathcal{U}$.*

Proof. We proved in Lecture 12 that composition with h induces an equivalence of categories $\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{U}), \mathcal{X}) \to \mathrm{Fun}'(\mathcal{U}, \mathcal{X})$, where $\mathrm{Fun}'(\mathcal{U}, \mathcal{X})$ is the full subcategory of $\mathrm{Fun}(\mathcal{U}, \mathcal{X})$ spanned by those functors $f : \mathcal{U} \to \mathcal{X}$ which preserve finite limits and coverings. Since every object of $\mathcal{U}$ is a subobject of the final object, any functor $f : \mathcal{U} \to \mathcal{X}$ which preserves finite limits automatically carries each element of $\mathcal{U}$ to a subobject of the final object $\mathbf{1} \in \mathcal{X}$, and can therefore be identified with a map of posets $g : \mathcal{U} \to \mathrm{Sub}(\mathbf{1})$. In this case, the assumption that f preserves finite limits translates into the assumption that g preserves finite meets, and the assumption that f preserves coverings translates into the assumption that g preserves infinite joins. $\square$

We can summarize the situation as follows: there are adjoint functors (of 2-categories)

$$\{\mathrm{Topoi}\} \underset{\mathcal{U} \mapsto \mathrm{Shv}(\mathcal{U})}{\overset{\mathcal{X} \mapsto \mathrm{Sub}(\mathbf{1})}{\rightleftarrows}} \{\mathrm{Locales}\}.$$

where the construction $\mathcal{U} \mapsto \mathrm{Shv}(\mathcal{U})$ is fully faithful by virtue of Proposition 13; its essential image is the 2-category of localic topoi. It follows that for every topos $\mathcal{X}$, there is a universal example of a localic topos which admits a geometric morphism from $\mathcal{X}$, given by $\mathrm{Shv}(\mathrm{Sub}(\mathbf{1}))$. We refer to this topos as the *localic reflection* of $\mathcal{X}$.

Example 16. Let X be a topological space equipped with an action of a (discrete) group G. Then the category $\mathrm{Shv}_G(X)$ of G-equivariant sheaves on X is a topos. The subobjects of the final object of $\mathrm{Shv}(X)$ can be identified with open subsets of X. It follows that subobjects of the final object of $\mathrm{Shv}_G(X)$ can be identified with G-equivariant open subsets of X, or equivalently with open subsets of the quotient X/G (where we endow X/G with the quotient topology). It follows that there is a canonical map $\mathrm{Shv}_G(X) \to \mathrm{Shv}(X/G)$ which exhibits $\mathrm{Shv}(X/G)$ as the localic reflection of $\mathrm{Shv}_G(X)$.

Lecture 15: Spaces and Locales

February 28, 2018

In the previous lecture, we introduced the notion of a *locale*. Every topological space X determines a locale $\mathcal{U}(X)$, given by the poset of open subsets of X. Note that if $f : X \to Y$ is a continuous map of topological spaces, then the formation of inverse images $f^{-1} : \mathcal{U}(Y) \to \mathcal{U}(X)$ preserves arbitrary unions (which are joins in $\mathcal{U}(X)$ and $\mathcal{U}(Y)$) and finite intersections (which are meets in $\mathcal{U}(X)$ and $\mathcal{U}(Y)$), and can therefore be regarded as a morphism of locales from $\mathcal{U}(X)$ to $\mathcal{U}(Y)$. Consequently, the construction

$$X \mapsto \mathcal{U}(X)$$

can be regarded as a functor from the category of topological spaces to the category of locales. In this lecture, we will show that this functor is not too far from being an equivalence of categories.

We now construct a functor in the opposite direction.

Definition 1. Let $\{0 < 1\}$ be the poset with two elements 0 and 1, which we can identify with the locale of open subsets of the one-point space $*$.

Let $\mathcal{U}$ be any locale. A *point* of $\mathcal{U}$ is a morphism of locales $\{0 < 1\} \to \mathcal{U}$, which is given by a map of posets $x^* : \mathcal{U} \to \{0 < 1\}$ which preserves finite meets and arbitrary joins. Note that such a map is determined by the subset $\mathcal{U}_x = \{U \in \mathcal{U} : x^*(U) = 1\} \subseteq \mathcal{U}$, which is required to satisfy the following conditions:

(a) If $U \leq V$ and U belongs to $\mathcal{U}_x$, then V also belongs to $\mathcal{U}_0$. (This guarantees that the map x^* is order-preserving.)

(b) If a join $\bigvee U_i$ belongs to $\mathcal{U}_x$, then some U_i belongs to $\mathcal{U}_x$ (this guarantees that the map x^* preserves joins).

(c) The subset $\mathcal{U}_x$ is closed under finite meets (this guarantees that the map x^* preserves meets).

Note that conditions (a) and (b) are equivalent to the requirement that there exists an element $U(x) \in \mathcal{U}$ such that $\mathcal{U}_x = \{U \in \mathcal{U} : U \not\leq U_x\}$; we can define $U(x)$ as the join $\bigvee_{U \notin \mathcal{U}_x} U$. In this case, we can restate (c) as follows:

(c') The element $U(x) \in \mathcal{U}$ is *prime*: that is, it is not equal to the largest element $\mathbf{1} \in \mathcal{U}$ and, whenever $U(x) = V \wedge W$, we have either $U(x) = V$ or $U(x) = W$.

Remark 2. In the situation of Definition 1, it is useful to think of $\mathcal{U}_x$ as "the collection of $U \in \mathcal{U}$ which contain x" and $U(x)$ as "the largest element of $\mathcal{U}$ which does not contain x."

Definition 3. Let $\mathcal{U}$ be a locale. We let $\mathrm{Pt}(\mathcal{U})$ denote the collection of all points of $\mathcal{U}$, which we can identify either with the collection of subsets $\mathcal{U}_x \subseteq \mathcal{U}$ satisfying (a), (b), and (c), or with the collection of elements $U(x) \in \mathcal{U}$ satisfying (c').

For each $U \in \mathcal{U}$, let $\widetilde{U} \subseteq \mathrm{Pt}(\mathcal{U})$ be the collection of all points $x \in \mathrm{Pt}(\mathcal{U})$ such that $U \in \mathcal{U}_x$. Note that the construction $U \mapsto \widetilde{U}$ carries joins in $\mathcal{U}$ to unions of subsets of $\mathrm{Pt}(\mathcal{U})$ (by virtue of (b)) and finite meets in $\mathcal{U}$ to finite intersections of subsets of $\mathrm{Pt}(\mathcal{U})$ (by virtue of (c)). It follows that the collection of subsets of $\mathrm{Pt}(\mathcal{U})$ having the form $\widetilde{U}$ determines a topology on $\mathrm{Pt}(\mathcal{U})$. We will henceforth regard $\mathrm{Pt}(\mathcal{U})$ as equipped with this topology.

Now suppose that we are given a topological space X and a locale $\mathcal{U}$. Note that a map $f : X \to \mathrm{Pt}(\mathcal{U})$ is continuous if and only if, for each $U \in \mathcal{U}$, the set $f^{-1}(\widetilde{U}) = \{x \in X : U \in \mathcal{U}_{f(x)}\}$ is open in X. In this case, we the construction $U \mapsto f^{-1}\widetilde{U}$ determines a morphism of locales from $\mathcal{U}(X)$ to $\mathcal{U}$. Conversely, if $g^* : \mathcal{U} \to \mathcal{U}(X)$ is morphism of locales from $\mathcal{U}(X)$ to $\mathcal{U}$, then the construction

$$(x \in X) \mapsto \{U \in \mathcal{U} : x \in g^* U\}$$

determines a continuous map from X to $\mathrm{Pt}(\mathcal{U})$. These constructions are mutually inverse:

Proposition 4. *For any topological space X and any locale $\mathcal{U}$, we have a canonical bijection*

$$\mathrm{Hom}_{\mathrm{Top}}(X, \mathrm{Pt}(\mathcal{U})) \simeq \mathrm{Hom}_{Locales}(\mathcal{U}(X), \mathcal{U}) = \mathrm{Fun}^*(\mathcal{U}, \mathcal{U}(X)).$$

Consequently, we have a pair of adjoint functors

$$\text{Topological Spaces} \underset{\mathcal{U} \mapsto \mathrm{Pt}(\mathcal{U})}{\overset{X \mapsto \mathcal{U}(X)}{\rightleftarrows}} \text{Locales} .$$

Let X be a topological space. Then we can identify elements of $\mathrm{Pt}(\mathcal{U}(X))$ with open subsets $U \subseteq X$ which are *prime*, in the following sense:

($*$) $U \neq X$. Moreover, if U is an intersection $V_0 \cap V_1$ of open subsets $V_0, V_1 \subseteq X$, then either $U = V_0$ or $U = V_1$.

We can rephrase this as a condition on the *complement* $K = X - U$:

($*'$) The set K is nonempty. Moreover, if $K = K_0 \cup K_1$ written as a union of closed subsets of X, then either $K = K_0$ or $K = K_1$.

We say that a closed subset $K \subseteq X$ is *irreducible* if it satisfies condition ($*'$). We have proven:

Proposition 5. *Let X be a topological space. Then there is a canonical bijection*

$$\{ \text{ Irreducible closed subsets } K \subseteq X \ \} \simeq \{ \text{ Points of } \mathcal{U}(X) \ \}$$

This bijection carries an irreducible closed subset $K \subseteq X$ to the point x of $\mathcal{U}(X)$ given by $\mathcal{U}(X)_x = \{U \subseteq X : U \cap K \neq \emptyset\}$.

For any topological space X, we have a unit map $X \to \mathrm{Pt}(\mathcal{U}(X))$, which we can think of as a map of sets

$$X \to \{\text{Irreducible closed subsets } K \subseteq X\}.$$

Unwinding the definitions, we see that this map carries a point $x \in X$ to the subset $\overline{\{x\}} \subseteq X$.

Definition 6. Let X be a topological space. We say that X is *sober* if the map

$$X \to \{\text{Irreducible closed subsets } K \subseteq X\}.$$

is bijective. In other words, X is sober if every irreducible closed subset K of X has a unique "generic point" $x \in K$, characterized by the requirement that $K = \overline{\{x\}}$.

Proposition 7. *Let X be a topological space. Then X is sober if and only if the canonical map $X \to \mathrm{Pt}(\mathcal{U}(X))$ is a homeomorphism.*

Proof. The "only if" direction is obvious (since any homeomorphism is bijective). To prove the converse, we observe that every open set $U \subseteq X$ is the inverse image of an open subset of $\mathrm{Pt}(\mathcal{U}(X))$: namely, the open subset $\widetilde{U}$. $\square$

We now give some examples of sober topological spaces.

Example 8. Let X be a Hausdorff space. Then a subset $K \subseteq X$ is closed and irreducible if and only if $K = \{x\}$ for some $x \in X$. In particular, every irreducible closed subset of X has a unique generic point, so that X is sober.

Example 9. Let X be the underlying topological space of a scheme. Then X is sober.

Proposition 10. *Let $\mathcal{U}$ be a locale. Then the topological space $\mathrm{Pt}(\mathcal{U})$ is sober.*

Proof. We wish to show that the unit map

$$u_{\mathrm{Pt}(\mathcal{U})} : \mathrm{Pt}(\mathcal{U}) \to \mathrm{Pt}(\mathcal{U}(\mathrm{Pt}(\mathcal{U})))$$

is bijective. Let $v_{\mathcal{U}} : \mathcal{U}(\mathrm{Pt}(\mathcal{U})) \to \mathcal{U}$ be the counit map (regarded as a map of locales), so that the composition

$$\mathrm{Pt}(\mathcal{U}) \xrightarrow{u_{\mathrm{Pt}(\mathcal{U})}} \mathrm{Pt}(\mathcal{U}(\mathrm{Pt}(\mathcal{U}))) \xrightarrow{\mathrm{Pt}(v_{\mathcal{U}})} \mathrm{Pt}(\mathcal{U})$$

is the identity. It follows immediately that $u_{\mathrm{Pt}(\mathcal{U})}$ is injective. To show surjectivity, it will suffice to show that the map of topological spaces $\mathrm{Pt}(v_{\mathcal{U}}) : \mathrm{Pt}(\mathcal{U}(\mathrm{Pt}(\mathcal{U}))) \to \mathrm{Pt}(\mathcal{U})$ is injective. In other words, it will suffice to show that a point x of the locale $\mathcal{U}(\mathrm{Pt}(\mathcal{U}))$ is determined by the set $\{U \in \mathcal{U} : x \in \widetilde{U}\}$. This is clear, because the map of posets

$$v_{\mathcal{U}}^* : \mathcal{U} \to \mathcal{U}(\mathrm{Pt}(\mathcal{U})) \qquad U \mapsto \widetilde{U}$$

is surjective. is surjective (by definition, every open subset of $\mathrm{Pt}(\mathcal{U})$ has the form $\widetilde{U}$ for some $U \in \mathcal{U}$). $\square$

Corollary 11. *Let X be a topological space. Then X is sober if and only it is homeomorphic to $\mathrm{Pt}(\mathcal{U})$, for some locale $\mathcal{U}$.*

Definition 12. Let $\mathcal{U}$ be a locale. We say that $\mathcal{U}$ is *spatial* if the counit $\mathcal{U}(\mathrm{Pt}(\mathcal{U})) \to \mathcal{U}$ is an isomorphism of locales. In other words, $\mathcal{U}$ is spatial if the construction

$$(U \in \mathcal{U}) \mapsto (\widetilde{U} \subseteq \mathrm{Pt}(\mathcal{U}))$$

determines a bijection from $\mathcal{U}$ to the collection of open subsets of $\mathrm{Pt}(\mathcal{U})$.

Remark 13. Note that the construction

$$(U \in \mathcal{U}) \mapsto (\widetilde{U} \subseteq \mathrm{Pt}(\mathcal{U}))$$

is automatically surjective (by the definition of the topology on $\mathrm{Pt}(\mathcal{U})$). Consequently, $\mathcal{U}$ is spatial if and only if it is injective. In other words, $\mathcal{U}$ is spatial if it has *enough points*, in the sense that for any pair of distinct elements $U \neq V$ in $\mathcal{U}$, we can find a point x of $\mathcal{U}$ which distinguishes U from V (in the sense that $U \in \mathcal{U}_x$ and $V \notin \mathcal{U}_x$, or vice versa).

Proposition 14. *Let $\mathcal{U}$ be a locale. Then $\mathcal{U}$ is spatial if and only if it is isomorphic to $\mathcal{U}(X)$, for some topological space X.*

Proof. If $\mathcal{U}$ is spatial, then it is isomorphic to $\mathcal{U}(X)$ for $X = \mathrm{Pt}(\mathcal{U})$. Conversely, if X is any topological space, then the locale $\mathcal{U}(X)$ is spatial, since every pair of distinct open sets $U, V \subseteq X$ can be distinguished by a point of X (and therefore also by its image in $\mathrm{Pt}(\mathcal{U}(X))$). $\square$

It follows that the adjunction

$$\text{Topological Spaces} \underset{\mathcal{U} \mapsto \mathrm{Pt}(\mathcal{U})}{\overset{X \mapsto \mathcal{U}(X)}{\rightleftarrows}} \text{Locales} .$$

restricts to an equivalence of categories

$$\{ \text{ Sober topological spaces } \} \simeq \{ \text{ Spatial locales } \}.$$

Since every locale of the form $\mathcal{U}(X)$ is spatial and every topological space of the form $\mathrm{Pt}(\mathcal{U})$ is sober, we also have the following:

Corollary 15. *The inclusion functor*

$$\{ \textit{ Spatial Locales } \} \hookrightarrow \{ \textit{ Locales } \}$$

admits a right adjoint, given by the construction $\mathcal{U} \mapsto \mathcal{U}(\mathrm{Pt}(\mathcal{U}))$. *Concretely, this construction carries a locale* $\mathcal{U}$ *to the quotient* $\mathcal{U} / \sim$, *where* $\sim$ *denotes the equivalence relation*

$$(U \sim V) \Leftrightarrow (\widetilde{U} = \widetilde{V})$$

Corollary 16. *The inclusion functor*

$$\{ \textit{ Sober topological spaces } \} \hookrightarrow \{ \textit{ Topological Spaces } \}$$

admits a left adjoint, given by the construction $X \mapsto \mathrm{Pt}(\mathcal{U}(X))$. *Concretely, this left adjoint assigns to a topological space* X *the collection of all irreducible closed subsets of* X, *endowed with an appropriate topology.*

Not every topological space is sober. For example, if X is a topological space with the trivial topology (defined by $\mathcal{U}(X) = \{\emptyset, X\}$), then X is sober if and only if it has cardinality ≤ 1. However, this can be viewed as a pathology of the notion of topological space: we are generally not very interested in topological spaces with the trivial topology. More generally, replacing a topological space X by its "soberification" $\mathrm{Pt}(\mathcal{U}(X))$ is in practice an inoffensive procedure.

Example 17. Let X be an algebraic variety over $\mathbf{C}$, which we identify with its set of $\mathbf{C}$-valued points. Then we can endow X with the Zariski topology. This space is usually not sober (unless X is 0-dimensional): its irreducible closed subsets are the subvarieties of X. The associated sober space $\mathrm{Pt}(\mathcal{U}(X))$ is the underlying topological space of X as a scheme (where every subvariety has a generic point).

By contrast, replacing a locale $\mathcal{U}$ by the associated spatial locale $\mathcal{U}(\mathrm{Pt}(\mathcal{U}))$ is often very destructive. In this class, we will meet some interesting and useful examples of locales which have no points at all.

Example 18 (Deligne)**.** Let $\mathcal{U}$ denote the collection of equivalence classes of measurable subsets of the interval $[0, 1]$, where two measurable subsets X and Y are considered to be equivalent if $X - Y$ and $Y - X$ have measure zero. Then $\mathcal{U}$ is locale (in fact, it is a complete Boolean algebra) for which the associated topological space $\mathrm{Pt}(\mathcal{U})$ is empty.

Lecture 16: Enumerations

March 2, 2018

The contents of the course up to this point can be roughly summarized in the following diagram:

$$\begin{tikzcd}
\{\text{(Typed) First-Order Theories}\} \arrow[d, "\sim"', "T \mapsto \mathrm{Syn}(T)"] & & \{\text{(Sober) Topological Spaces}\} \arrow[d, "\sim"', "X \mapsto \mathcal{U}(X)"] \\
\{\text{(Small) Boolean Pretopoi}\} \arrow[d, hook] & & \{\text{Spatial Locales}\} \arrow[d, hook] \\
\{\text{(Small) Pretopoi}\} \arrow[d, "\sim"', "\mathcal{C} \mapsto \mathrm{Shv}(\mathcal{C})"] & & \{\text{Locales}\} \arrow[d, "\sim"', "\mathcal{X} \mapsto \mathrm{Shv}(\mathcal{X})"] \\
\{\text{Coherent Topoi}\} \arrow[rr, hook] & & \{\text{Topoi}\} & & \{\text{Localic Topoi}\} \arrow[ll, hook']
\end{tikzcd}$$

Our ultimate goal in this course is to try to use the right side of this diagram to study the left side. To every first-order theory T, we can associate a *classifying topos* $\mathrm{Shv}(\mathrm{Syn}(T)) \simeq \mathrm{Shv}(\mathrm{Syn}_0(T))$. Our strategy will be to analyze T by comparing its classifying topos with localic topoi.

Question 1. Given an arbitrary topos $\mathcal{X}$, how well can $\mathcal{X}$ be approximated by a localic topos?

Our more immediate goal in this course is to prove the following result of Joyal and Tierney, which gives a precise answer to Question 1:

Theorem 2 (Joyal-Tierney)**.** *Let $\mathcal{X}$ be a topos. Then there exists a localic topos $\mathcal{U}$ and a geometric morphism $\mathcal{U} \to \mathcal{X}$ which is an effective epimorphism in the 2-categorical sense: that is, $\mathcal{X}$ can be identified with the colimit of the diagram*

$$\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \Rrightarrow \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \rightrightarrows \mathcal{U} ,$$

formed in the 2-category of topoi and geometric morphisms. Moreover, every topos appearing in this diagram is localic.

We will give a more precise formulation (and proof) of Theorem 2 slowly over the course of the next several lectures. Let us begin by outlining our strategy. Let $\mathcal{X}$ be a topos. If $\mathcal{X}$ is localic, then there is nothing to do (we can simply take $\mathcal{U} = \mathcal{X}$). Otherwise, there exists some object $X \in \mathcal{X}$ which cannot be covered by subobjects of the final object of $\mathcal{X}$. Let us proceed immediately to the "hardest" case, where we know nothing at all about the object X.

Definition 3. Let $\mathcal{S}\mathrm{et}_{\mathrm{fin}}$ denote the category of finite sets, and let $\mathcal{X}_{\mathcal{S}\mathrm{et}} = \mathrm{Fun}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{S}\mathrm{et})$. We will refer to $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ as the *classifying topos of sets.*

Remark 4. We can also describe $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ as the category of functors from $\mathcal{S}\mathrm{et}$ to $\mathcal{S}\mathrm{et}$ which preserve filtered colimits.

The topos $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ contains a distinguished object X_0, given by the inclusion functor $\mathcal{S}\mathrm{et}_{\mathrm{fin}} \hookrightarrow \mathcal{S}\mathrm{et}$ (under the alternate description of Remark 4, this corresponds to the identity functor $\mathrm{id} : \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$).

Proposition 5. *For any topos $\mathcal{X}$, evaluation on the object $X_0 \in \mathcal{X}_{\mathcal{S}\mathrm{et}}$ induces an equivalence of categories*

$$\mathrm{Fun}^*(\mathcal{X}_{\mathcal{S}\mathrm{et}}, \mathcal{X}) \to \mathcal{X}\,.$$

In other words, we can identify geometric morphisms from $\mathcal{X}$ to $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ with objects of the topos $\mathcal{X}$.

Remark 6. We can paraphrase Proposition 5 as follows: the classifying topos $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ is freely generated, as a topos, by the object X_0.

Proof of Proposition 5. Note that $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ can be described as the topos of presheaves on the category $\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}$. Using Lecture 12, we deduce that composition with the Yoneda embedding induces an equivalence of categories

$$\mathrm{Fun}^*(\mathcal{X}_{\mathcal{S}\mathrm{et}}, \mathcal{X}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}, \mathcal{X}),$$

where $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}, \mathcal{X})$ denotes the category of functors $F : \mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}} \to \mathcal{X}$ which preserve finite limits.

For every finite set S, we can write S as a coproduct $\coprod_{s \in S} \{s\}$ in the category $\mathcal{S}\mathrm{et}_{\mathrm{fin}}$, or equivalently as a product of singletons in the category $\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}$. It follows that for any functor F as above, we have canonical isomorphisms $F(S) = \prod_{s \in S} F(\{s\}) \simeq F(*)^S$, so that F is determined by its value on the one-point set $*$. Conversely, for any $X \in \mathcal{X}$, the construction $S \mapsto X^S$ determines a functor $\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}} \to \mathcal{X}$ preserving finite limits. It follows that evaluation at $*$ induces an equivalence of categories

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}, \mathcal{X}) \to \mathcal{X}\,.$$

Note that the composite equivalence

$$\mathrm{Fun}^*(\mathcal{X}_{\mathcal{S}\mathrm{et}}, \mathcal{X}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}, \mathcal{X}) \simeq \mathcal{X}$$

is given by evaluation on the representable presheaf $h_* \in \mathcal{X}_{\mathcal{S}\mathrm{et}} = \mathrm{Fun}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{S}\mathrm{et})$, which is the object X_0. $\square$

The topos $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ is not localic. In some sense, it is as far from localic as possible: note that if there were to exist a covering $\{U_i \to X_0\}$ of the object $X_0 \in \mathcal{X}_{\mathcal{S}\mathrm{et}}$ by subobjects of the final object in $\mathcal{X}_{\mathcal{S}\mathrm{et}}$, then we would be able to find such a covering for *any* object X of *any* topos $\mathcal{X}$, since Proposition 5 supplies a geometric morphism $f : \mathcal{X} \to \mathcal{X}_{\mathcal{S}\mathrm{et}}$ satisfying $X \simeq f^* X_0$.

Remark 7. Let $\mathcal{X}$ be a topos containing an object X, so that we can write $X = \pi^* X_0$ for an essentially unique geometric morphism π from $\mathcal{X} \to \mathcal{X}_{\mathcal{S}\mathrm{et}}$. Then the right adjoint π_* of π^* is given by the formula

$$\begin{aligned}
\pi_*(Y)(S) &\simeq \mathrm{Hom}_{\mathcal{X}_{\mathcal{S}\mathrm{et}}}(h_S, \pi_*(Y)) \\
&\simeq \mathrm{Hom}_{\mathcal{X}_{\mathcal{S}\mathrm{et}}}(\prod_{s \in S} X_0, \pi_*(Y)) \\
&\simeq \mathrm{Hom}_{\mathcal{X}}(\pi^* \prod_{s \in S} X_0, Y) \\
&\simeq \mathrm{Hom}_{\mathcal{X}}(\prod_{s \in S} \pi^* X_0, Y) \\
&\simeq \mathrm{Hom}_{\mathcal{X}}(X^S, Y)
\end{aligned}$$

for $Y \in \mathcal{X}$ and $S \in \mathcal{S}\mathrm{et}_{\mathrm{fin}}$.

It will be convenient to introduce a slight variant of Definition 3.

Definition 8. Let us regard the category $\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}$ as equipped with a Grothendieck topology, where a collection of maps of finite sets $\{S \to T_i\}_{i \in I}$ is a covering if $S \to T_i$ is injective for some i. We will denote the topos $\mathrm{Shv}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}) \subseteq \mathcal{X}_{\mathcal{S}\mathrm{et}}$ by $\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}$, and refer to it as the *classifying topos for nonempty sets.*

Exercise 9. Show that a functor $F : \mathcal{S}\mathrm{et}_{\mathrm{fin}} \to \mathcal{S}\mathrm{et}$ is a sheaf for the topology of Definition 8 if and only if the diagram of sets

$$F(\emptyset) \to F(*) \rightrightarrows F(* \amalg *)$$

is an equalizer. In particular, the object $X_0 \in \mathcal{X}_{\mathcal{S}\mathrm{et}}$ belongs to the subcategory $\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}$.

Exercise 10. Let $\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\neq \emptyset}$ denote the category of nonempty finite sets. Show that composition with the inclusion functor $\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\neq \emptyset}$ induces an equivalence of categories

$$\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}} \subseteq \mathrm{Fun}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\neq \emptyset}, \mathcal{S}\mathrm{et}).$$

We have the following analogue of Proposition 5:

Proposition 11. *For any topos $\mathcal{X}$, evaluation on the object $X_0 \in \mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}$ induces a fully faithful embedding*

$$\mathrm{Fun}^*(\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}, \mathcal{X}) \to \mathcal{X}$$

whose essential image consists of those objects $X \in \mathcal{X}$ for which the map $X \to \mathbf{1}$ is an effective epimorphism in $\mathcal{X}$.

Proof. Using Lecture 12, we deduce that composition with the Yoneda embedding induces a fully faithful embedding

$$\mathrm{Fun}^*(\mathcal{X}_{\mathcal{S}\mathrm{et}}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}, \mathcal{X})$$

whose essential image is spanned by those functors $F : \mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}} \to \mathcal{X}$ which preserve finite limits and coverings. The first condition guarantees that we can write $F(S) = X^S$ for some object $X \in \mathcal{X}$ (as in the proof of Proposition 5). In this case, preservation of coverings translates to the condition that for any injective map $S \to T$ of finite set, the induced map $X^T \to X^S$ is an effective epimorphism of sets. If $S \neq \emptyset$, then this is automatic (since the map $X^T \to X^S$ admits a section). Consequently, preservation of coverings translates to the condition that the projection map $X^T \to \mathbf{1}$ is an effective epimorphism in $\mathcal{X}$. We conclude by observing that if this condition is satisfied when T is a singleton, then it is satisfied for all T (since the collection of effective epimorphisms in $\mathcal{X}$ is closed under products). $\square$

Construction 12. Let $\mathrm{Equiv}(\mathbf{Z})$ denote the set of all equivalence relations on the set $\mathbf{Z}$ of integers. We will regard $\mathrm{Equiv}(\mathbf{Z})$ as a topological space, where the collection of subsets

$$U_{i,j} = \{E \in \mathrm{Equiv}(\mathbf{Z}) : iEj\}$$

forms a sub-basis of open sets. (In other words, we equip $\mathrm{Equiv}(\mathbf{Z})$ with the coarsest topology for which each $U_{i,j}$ is an open set).

Let $\mathscr{F}$ denote the sheaf of sets on $\mathrm{Equiv}(\mathbf{Z})$ whose stalk at a point $E \in \mathrm{Equiv}(\mathbf{Z})$ is given by the quotient $\mathbf{Z}/E$. More formally, we can describe $\mathscr{F}$ as the quotient $\underline{\mathbf{Z}}/\mathscr{R}$, where $\underline{\mathbf{Z}}$ denotes the constant sheaf with value $\mathbf{Z}$, and $\mathscr{R} \subseteq \underline{\mathbf{Z}} \times \underline{\mathbf{Z}}$ is the equivalence relation given by $\amalg_{i,j \in \mathbf{Z}} : h_{U_{i,j}}$.

Note that the canonical map $\mathscr{F} \to \mathbf{1}$ is an effective epimorphism in the topos $\mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z}))$ (by construction, it comes with many sections). Applying Proposition 11, we see that there is an essentially unique geometric morphism

$$\varphi : \mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z})) \to \mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}$$

satisfying $\varphi^* X_0 = \mathscr{F}$. This gives a localic "approximation" to the topos $\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}$.

Construction 13. Let $\mathcal{X}$ be a topos and let $X \in \mathcal{X}$ be an object for which $X \to \mathbf{1}$ is an effective epimorphism, so we can write $X = \varphi^* X_0$ for an essentially unique geometric morphism $\psi : \mathcal{X} \to \mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}$. We let $\mathrm{Enum}(X)$ denote the fiber product $\mathcal{X} \times_{\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}} \mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z}))$, formed in the 2-category of topoi. We will refer to $\mathrm{Enum}(X)$ as the *topos of enumerations of X*.

Remark 14. In the situation of Construction 13, we have a commutative diagram of topoi

$$\begin{tikzcd} \mathrm{Enum}(X) \arrow[r, "\pi"] \arrow[d, "\psi'"] & \mathcal{X} \arrow[d, "\psi"] \\ \mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z})) \arrow[r, "\varphi"] & \mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}} \end{tikzcd} ,$$

hence canonical isomorphisms

$$\pi^* X \simeq \pi^* \psi^* X_0 \simeq \psi'^* \varphi^* X_0 \simeq \psi'^* \mathscr{F} \simeq \psi'^*(\underline{\mathbf{Z}})/\psi'^* \mathscr{R} .$$

It follows that, after pulling back along the geometric morphism π', the object X can be equipped with an *enumeration*: that is, a cover by countably many copies of the final object.

The proof of Theorem 2 will proceed via Construction 13. Given a topos $\mathcal{X}$, we can choose a set of generators $\{X_i\}_{i \in I}$. Enlarging this set if necessary (say by adding the final object $\mathbf{1}$), we can assume that the coproduct $X = \coprod_{i \in I} X_i$ has the property that the projection $X \to \mathbf{1}$ is an effective epimorphism. In this case, we will show (following Joyal and Tierney) that the projection map

$$\pi : \mathrm{Enum}(X) \to \mathcal{X}$$

satisfies the requirements of Theorem 2.

Warning 15. Recall that a *point* of a topos $\mathcal{X}$ is a geometric morphism to $\mathcal{X}$ from the topos $\mathcal{S}\mathrm{et}$ of sets. Then:

- Points of the topos $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ can be identified with sets (Proposition 5), and points of the topos $\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}$ can be identified with nonempty sets (Proposition 11).
- One can show that the space $\mathrm{Equiv}(\mathbf{Z})$ is sober, so that points of the topos $\mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z}))$ can be identified with points of $\mathrm{Equiv}(\mathbf{Z})$: that is, with equivalence relations E on $\mathbf{Z}$. Put differently, we can think of a point of $\mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z}))$ as a set S equipped with a surjection $\mathbf{Z} \to S$.

Suppose now that we are in the situation of Construction 13 and that we are given a point $x : \mathcal{S}\mathrm{et} \to \mathcal{X}$. Then lifting x to a point of $\mathrm{Enum}(X) = \mathcal{X} \times_{\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}} \mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z}))$ is equivalent to choosing a surjection $\mathbf{Z} \to x^*(X)$.

It follows that Construction 13 can easily produce examples of topoi that have no points at all. For example, we could take $\mathcal{X} = \mathcal{S}\mathrm{et}$ to be the topos of sets, and $X \in \mathcal{X}$ to be any uncountable set. In this case, the enumeration topos $\mathrm{Enum}(X)$ will not have any points (because there are no surjections from $\mathbf{Z}$ to X). Nevertheless, we will see that the projection map $\pi : \mathrm{Enum}(X) \to \mathcal{X}$ still satisfies the requirements of Theorem 2. (In particular, $\mathrm{Enum}(X)$ is a localic topos whose associated locale is *non-spatial*.)

Lecture 17: Open Morphisms

March 10, 2018

Let $f : X \to Y$ be a continuous map of topological spaces. We say that f is *open* if, for every open subset $U \subseteq X$, the image $f(U)$ is an open subset of Y. In this case, the construction $U \mapsto f(U)$ determines a map of posets

$$f_! : \mathcal{U}(X) \to \mathcal{U}(Y).$$

We observe that $f_!$ can be characterized as the *left adjoint* of the inverse image map

$$f^* : \mathcal{U}(Y) \to \mathcal{U}(X) \qquad V \mapsto f^{-1}(V).$$

In other words, for open subsets $U \subseteq X$ and $V \subseteq Y$, we have $(f(U) \subseteq V) \Leftrightarrow (U \subseteq f^{-1}(V))$. This motivates the following:

Definition 1. Let $\mathcal{U}$ and $\mathcal{V}$ be locales, and let $f : \mathcal{U} \to \mathcal{V}$ be a morphism of locales, which we identify with a map of posets $f^* : \mathcal{V} \to \mathcal{U}$ which preserves finite meets and arbitrary joins. We say that f is *open* if it satisfies the following conditions:

(1) The map f^* admits a left adjoint $f_! : \mathcal{U} \to \mathcal{V}$. That is, for each $U \in \mathcal{U}$, there exists an object $f_!(U) \in \mathcal{V}$ such that $(f_!(U) \leq V) \Leftrightarrow (U \leq f^*(V))$, for each $V \in \mathcal{V}$.

(2) The map $f_!$ satisfies the projection formula $f_!(U \wedge f^*V) = f_!(U) \wedge V$ for $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Example 2. Let $f : X \to Y$ be an open map of topological spaces. Then the induced map of locales $\mathcal{U}(X) \to \mathcal{U}(Y)$ is open. The projection formula of Definition 1 follows from the identity $f(U \cap f^{-1}(V)) = f(U) \cap V$ for $U \subseteq X$ and $V \subseteq Y$.

Warning 3. The converse of Example 2 need not hold, even if X and Y are assumed to be sober.

Proposition 4. *Let $f : X \to Y$ be a map of topological spaces, and suppose that every point of Y is closed (this condition is satisfied, for example, if Y is Hausdorff). If the map of posets $f^* : \mathcal{U}(Y) \to \mathcal{U}(X)$ admits a left adjoint $f_!$, then f is open. In particular, if the map of locales $\mathcal{U}(X) \to \mathcal{U}(Y)$ is open, then f is open.*

Proof. Let $U \subseteq X$ be an open set; we wish to show that $f(U) \subseteq Y$ is open. We will complete the proof by showing that $f(U) = f_!(U)$. By definition, $f_!(U)$ is the smallest open subset of Y which contains $f(U)$. Conversely, suppose that $y \in Y$ is a point which does not belong to $f(U)$. Since y is closed, the complement $Y - \{y\}$ is an open subset of Y which contains $f(U)$, so we have $f_!(U) \subseteq Y - \{y\}$. It follows that y does not belong to $f_!(U)$. $\square$

Example 5. Let Y be the Sierpinski space, consisting of a closed point x and an open point η. Let $X = \{x\}$ be the subset consisting only of the closed point. Then:

- The inclusion map $X \hookrightarrow Y$ is not open (as a map of topological spaces).
- The map of posets $f^{-1} : \mathcal{U}(Y) \to \mathcal{U}(X)$ has a left adjoint $f_! : \mathcal{U}(X) \to \mathcal{U}(Y)$.

- The map of locales $\mathcal{U}(X) \to \mathcal{U}(Y)$ is not open, because the map $f_!$ does not satisfy the projection formula: we have
$$\{\eta\} = f_!(X) \cap \{\eta\} \neq f_!(X \cap f^{-1}\{\eta\}) = f_!(\emptyset) = \emptyset.$$

Definition 6. Let $f : \mathcal{U} \to \mathcal{V}$ be a morphism of locales. We will say that f is an *open surjection* if it is open and $f_!(\mathbf{1}_{\mathcal{U}}) = \mathbf{1}_{\mathcal{V}}$, where $\mathbf{1}_{\mathcal{U}}$ and $\mathbf{1}_{\mathcal{V}}$ denote the largest elements of $\mathcal{U}$ and $\mathcal{V}$, respectively.

Remark 7. Let $f : \mathcal{U} \to \mathcal{V}$ be an open surjection. Then, for each $V \in \mathcal{V}$, we have
$$V = \mathbf{1}_{\mathcal{V}} \wedge V = f_!(\mathbf{1}_{\mathcal{U}}) \wedge V = f_!(\mathbf{1}_{\mathcal{U}} \wedge f^* V) = f_! f^* V.$$
It follows that an open morphism f is an open surjection if and only if the counit map $f_! f^* \to \mathrm{id}_{\mathcal{V}}$ is an isomorphism.

Example 8. Let $f : X \to Y$ be a map of topological spaces. If f is an open surjection, then the induced map of locales $\mathcal{U}(X) \to \mathcal{U}(Y)$ is an open surjection. The converse holds if every point of Y is closed.

Remark 9. Let $f : \mathcal{U} \to \mathcal{V}$ and $g : \mathcal{V} \to \mathcal{W}$ be morphisms of locales. If f and g are open (respectively open surjections), then the composition $g \circ f$ is also open (respectively an open surjection). The map $(g \circ f)_!$ is given by $g_! \circ f_!$, and the projection formula follows from the calculation
$$(g \circ f)_!(U) \wedge W = g_!(f_!(U)) \wedge W = g_!(f_!(U) \wedge g^* W) = g_!(f_!(U \wedge f^* g^* W)) = (g \circ f)_!(U \wedge (g \circ f)^* W).$$

Example 10. Let $\mathcal{X}$ be a topos and let $f : X \to Y$ be a morphism between objects of $\mathcal{X}$. Then the construction
$$(V \subseteq Y) \mapsto (V \times_Y X \subseteq X)$$
determines a map of locales $\mathrm{Sub}(X) \to \mathrm{Sub}(Y)$. This map is automatically open: it has a left adjoint given by the construction
$$(U \subseteq X) \mapsto (\mathrm{Im}(U \to Y) \subseteq Y),$$
and the projection formula follows from the calculation
$$\begin{aligned} \mathrm{Im}(U \to Y) \times_Y V &= \mathrm{Im}(U \times_Y V \to V) \\ &= \mathrm{Im}(U \times_X (X \times_Y V) \to Y). \end{aligned}$$

Remark 11. In the situation of Example 10, the map of locales $\mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ is an open surjection if and only if f is an effective epimorphism.

Definition 12. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. We say that f is *open* if it satisfies the following conditions: Then, for every object $Y \in \mathcal{Y}$, the pullback functor f^* determines a map from $\mathrm{Sub}(Y)$ to $\mathrm{Sub}(f^* Y)$ which preserves arbitrary joins and finite meets; that is, it determines a map of locales $f_Y : \mathrm{Sub}(f^*(Y)) \to \mathrm{Sub}(Y)$.

We say that f is *open* if it satisfies the following:

(1) For each $Y \in \mathcal{Y}$, the map of posets $f_Y^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(f^*))$ admits a left adjoint $f_{Y!}$.

(2) For every morphism $Y' \to Y$ in $\mathcal{Y}$ and each subobject $U \subseteq Y$, we have $Y' \times_Y (f_{Y!} U) = f_{Y'!}(f^* Y' \times_{f^* Y} U)$ (as subobjects of Y').

Note that (1) and (2) imply that each $f_Y : \mathrm{Sub}(f^*(Y)) \to \mathrm{Sub}(Y)$ is an open morphism of locales (the projection formula follows by applying (2) in the case where Y' is a subobject of Y). We say that f is an *open surjection* if each f_Y is an open surjection of locales.

Remark 13. Let $f : \mathcal{X} \to \mathcal{Y}$ be an open geometric morphism (respectively open surjection) of topoi. Then the induced map of locales $\mathrm{Sub}(\mathbf{1}_{\mathcal{X}}) \to \mathrm{Sub}(\mathbf{1}_{\mathcal{Y}})$ is open (respectively an open surjection).

In the situation of Definition 12, we do not need to consider *every* object $Y \in \mathcal{Y}$; it suffices to check on a set of generators.

Proposition 14. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Suppose that there exists a collection of generators $\mathcal{Y}_0$ satisfying the following:*

(1′) *For each $Y \in \mathcal{Y}_0$, the map of posets $f_Y^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(f^*))$ admits a left adjoint $f_{Y!}$.*

(2′) *For every morphism $Y' \to Y$ between objects of $\mathcal{Y}_0$ and each subobject $U \subseteq Y$, we have $Y' \times_Y (f_{Y!}U) = f_{Y'!}(f^*Y' \times_{f^*Y} U)$ (as subobjects of Y').*

Then f is open. Moreover, if the map of locales $f_Y : \mathrm{Sub}(f^*Y) \to \mathrm{Sub}(Y)$ is an open surjection for each $Y \in \mathcal{Y}_0$, then f is an open surjection.

Proof. Let Y be an arbitrary object of $\mathcal{Y}$, and choose a covering $\{Y_i \to Y\}_{i \in I}$ where each Y_i belongs to $\mathcal{Y}_0$. Then $\{f^*Y_i \to f^*Y\}_{i \in I}$ is a covering in the topos X. Then then ahve

$$\begin{aligned}
(U \subseteq f^*V) &\Leftrightarrow (\forall i \in I)(U \times_{f^*Y} f^*Y_i \subseteq f^*V \times_{f^*Y} f^*Y_i) \\
&\Leftrightarrow (\forall i \in I)(U \times_{f^*Y} f^*Y_i \subseteq f^*(V \times_Y Y_i) \\
&\Leftrightarrow (\forall i \in I) f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \subseteq V \times_Y Y_i \\
&\Leftrightarrow (\forall i \in I)\, \mathrm{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \to Y) \subseteq V \\
&\Leftrightarrow \bigvee_{i \in I} \mathrm{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \to Y) \subseteq V.
\end{aligned}$$

It follows that the map of posets f_Y^* has a left adjoint $f_{Y!}$, given by the formula

$$f_{Y!}(U) = \bigvee_{i \in I} \mathrm{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \to Y).$$

That is, f satisfies condition (1) of Definition 12.

We now verify condition (2). Let $\{Y_i \to Y\}$ be as above, and suppose we are given a morphism $Y' \to Y$ in $\mathcal{Y}$. For each $i \in I$, choose a covering $\{Y'_{i,j} \to Y_i \times_Y Y'\}$ where each $Y'_{i,j}$ belongs to $\mathcal{Y}_0$, so that the $Y'_{i,j}$ comprise a covering of Y'. For $U \in \mathrm{Sub}(f^*Y)$, we compute

$$\begin{aligned}
f_{Y'!}(U \times_{f^*Y} f^*Y') &= \bigvee_{i,j} \mathrm{Im}(f_{Y'_{i,j}!}(U \times_{f^*Y} f^*Y' \times_{f^*Y'} f^*Y'_{i,j}) \to Y') \\
&= \bigvee_{i,j} \mathrm{Im}(f_{Y'_{i,j}!}(U \times_{f^*Y} f^*Y'_{i,j}) \to Y') \\
&= \bigvee_{i,j} \mathrm{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \times_{Y_i} Y'_{i,j} \to Y') \\
&= \bigvee_{i} \mathrm{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \times_{Y_i} (Y_i \times_Y Y') \to Y') \\
&= \bigvee_{i} \mathrm{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \times_Y Y' \to Y') \\
&= \bigvee_{i} (\mathrm{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \to Y) \times_Y Y') \\
&= (\bigvee_{i} \mathrm{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \to Y)) \times_Y Y' \\
&= (f_{Y!}U) \times_Y Y'.
\end{aligned}$$

Here the first equality follows from the construction of $f_{Y'!}$, the third from assumption (2′), the fourth from the fact that $\{Y'_{i,j} \to Y_i \times_Y Y'\}$ is a covering for each $i \in I$, the sixth from the fact that the formation of

images commutes with pullback, the seventh from the fact that joins commute with pullback, and the eighth from the construction of $f_{Y!}$.

We conclude by observing that if f_{Y_i} is an open surjection, then we have

$$f_{Y!}(f^*(Y)) = \bigvee_{i \in I} \mathrm{Im}(f_{Y_i!}(f^*Y \times_{f^*Y} f^*Y_i) \to Y) = \bigvee_{i \in I} \mathrm{Im}(Y_i \to Y) = Y,$$

so that f_Y is also an open surjection. □

Corollary 15. *Let $\mathcal{X}$ be a topos and let $\mathcal{U}$ be a locale. Suppose we are given a geometric morphism $f : \mathcal{X} \to \mathrm{Shv}(\mathcal{U})$, which we can identify with a morphism of locales $g : \mathrm{Sub}(\mathbf{1}_{\mathcal{X}}) \to \mathcal{U}$. Then f is an open morphism of topoi (in the sense of Definition 12) if and only if g is an open morphism of locales (in the sense of Definition 1). Similarly, f is an open surjection if and only if g is an open surjection.*

Proof. The "only if" direction follows from Remark 13. Conversely suppose that g is an open morphism (open surjection) of locales; we wish to show that f is open (an open surjection). Let $h : \mathcal{U} \to \mathrm{Shv}(\mathcal{U})$ be the Yoneda embedding, so that $\mathcal{U}$ is generated by objects of the form h_U. For each $U \in \mathcal{U}$, we can identify $\mathrm{Sub}(h_U)$ with the subset $\{V \in \mathcal{U} : V \leq U\}$, and $\mathrm{Sub}(f^*h_U)$ with the subset $\{W \in \mathrm{Sub}(\mathbf{1}_{\mathcal{X}}) : W \subseteq g^*U\}$. Consequently, we can identify $f^*_{h_U}$ with the map

$$\{V \in \mathcal{U} : V \leq U\} \to \{W \in \mathrm{Sub}(\mathbf{1}_{\mathcal{X}}) : W \subseteq g^*U\}$$

given by restriction of g^*. Since g is open, this map admits a left adjoint $f_{h_U!}$, which is given by the restriction of $g_! : \mathcal{U} \to \mathrm{Sub}(\mathbf{1}_{\mathcal{X}})$. Consequently, f satisfies condition $(1')$ of Proposition 14. Condition $(2')$ follows from the projection formula for $g_!$. It follows that f is open. If g is an open surjection, then we have $f_{h_U!}(f^*_{h_U}U) = g_!g^*U = U$ by virtue of Remark 7. □

Corollary 16. *Let $f : \mathcal{U} \to \mathcal{V}$ be a morphism of locales. Then f is open (respectively an open surjection) if and only if the induced map $\mathrm{Shv}(\mathcal{U}) \to \mathrm{Shv}(\mathcal{V})$ is an open geometric morphism of topoi.*

Corollary 17. *Let $\mathcal{X}$ be any topos and let $\mathrm{Sub}(\mathbf{1}_{\mathcal{X}})$ be its underlying locale. Then the unit map $\mathcal{X} \to \mathrm{Shv}(\mathrm{Sub}(\mathbf{1}_{\mathcal{X}}))$ is an open surjection of topoi.*

Lecture 18: Localic Morphisms

March 10, 2018

In Lecture 17, we introduced the notion of an open morphism (and of an open surjection) between topoi. Our first goal in this lecture is to provide a nontrivial example:

Proposition 1. *The geometric morphism* $\varphi : \mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z})) \to \mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}$ *constructed in Lecture 16 is an open surjection.*

Proof. Let us identify the classifying topos $\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}$ of nonempty sets with the category of functors $\mathrm{Fun}(\mathcal{S}\mathrm{et}^{\neq \emptyset}_{\mathrm{fin}}, \mathcal{S}\mathrm{et})$. For every nonempty finite set S, let $h_S : \mathcal{S}\mathrm{et}^{\neq \emptyset}_{\mathrm{fin}} \to \mathcal{S}\mathrm{et}$ denote the functor corepresented by S, given by the formula $h_S(T) = \mathrm{Hom}(S, T) = T^S$. Equivalently, we can identify h_S with X_0^S, where $X_0 \in \mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}$ is the "universal nonempty object" appearing in Lecture 16 (given by the inclusion map $\mathcal{S}\mathrm{et}^{\neq \emptyset}_{\mathrm{fin}} \hookrightarrow \mathcal{S}\mathrm{et}$).

The objects h_S generate the topos $\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}$. We first show that for each S, the associated morphism of locales

$$\mathrm{Sub}(\varphi^* h_S) \to \mathrm{Sub}(h_S)$$

is an open surjection. Note that we can identify $\varphi^* h_S$ with $\mathscr{F}^S$, where $\mathscr{F}$ is the sheaf on $\mathrm{Equiv}(\mathbf{Z})$ whose stalk at a point $E \in \mathrm{Equiv}(\mathbf{Z})$ is given $\mathscr{F}_E = \mathbf{Z}/E$.

Let's begin by analyzing the poset $\mathrm{Sub}(h_S)$. By definition, we can identify the elements of h_S with *subfunctors* of h_S. Such a subobject is specified by giving a property P of maps between finite sets $S \to T$, having the property that for every map $f : S \to T$ with the property P and any map $g : T \to T'$, the composite map $(g \circ f) : S \to T'$ also has the property P. Note that each f factors canonically as a composition

$$S \to S/E_f \hookrightarrow T,$$

where E_f is the equivalence relation given by $(sE_f s') \Leftrightarrow (f(s) = f(s'))$. It follows that if the quotient map $S \to S/E_f$ has the property P, then the map f also has the property P. Conversely, if f has the property P, then the quotient map $S \to S/E_f$ must also have the property P, since there exists a retraction of T back onto the quotient S/E_f (here we use the fact that S is not empty).

Let $\mathrm{Equiv}(S)$ denote the collection of all equivalence relations on S. From the preceding discussion, we obtain an injective map

$$\{\text{Subobjects of } h_S\} \hookrightarrow \{\text{Subsets of } \mathrm{Equiv}(S)\}$$

which carries a subobject $F \subseteq h_S$ to the collection of all equivalence relations $E \in \mathrm{Equiv}(S)$ for which the quotient map $S \to S/E$ belongs to $F(S/E) \subseteq h_S(S/E) = \mathrm{Hom}(S, S/E)$. This map is not surjective: its image consists of the collection of all subsets $U \subseteq \mathrm{Equiv}(S)$ having the property that if any refinement of an equivalence relation E belongs to U, then E also belongs to U. Put another way, the image consists of the collection of all *open* subsets of $\mathrm{Equiv}(S)$, where we equip $\mathrm{Equiv}(S)$ with the topology generated by sub-basic open sets $U_{s,s'} = \{E \in \mathrm{Equiv}(S) : sEs'\}$.

We can identify the sheaf $\mathscr{F}^S \in \mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z}))$ with a topological space $\widetilde{\mathrm{Equiv}(\mathbf{Z})}_S$ equipped with a local homeomorphism $\widetilde{\mathrm{Equiv}(\mathbf{Z})}_S \to \mathrm{Equiv}(\mathbf{Z})$. The points of $\widetilde{\mathrm{Equiv}(\mathbf{Z})}_S$ are given by pairs (E, ρ), where E is an equivalence relation on $\mathbf{Z}$ and $\rho : S \to \mathbf{Z}/E$ is a map of sets. Under this identification, subobjects of

$\mathscr{F}^S$ correspond to open subsets of $\widetilde{\mathrm{Equiv}(\mathbf{Z})}$, and the map of locales $\mathrm{Sub}(\varphi^* h_S) \to \mathrm{Sub}(h_S)$ arises from a continuous map of topological spaces

$$\pi_S : \widetilde{\mathrm{Equiv}(\mathbf{Z})}_S \to \mathrm{Equiv}(S)$$

which carries a pair $(E, \rho : S \to \mathbf{Z}/E)$ to the equivalence relation

$$E_\rho := \{(s, s') \in S^2 : \rho(s) = \rho(s')\} \in \mathrm{Equiv}(S).$$

To complete the proof, it will suffice to show that π is an open surjection of topological spaces. It is clearly a surjection: every quotient of S can be embedded into a suitable quotient of $\mathbf{Z}$ (here we invoke the fact that $\mathbf{Z}$ is infinite). To show that it is open, let $U \subseteq \widetilde{\mathrm{Equiv}(S)}_S$ be an open set containing a point (E, ρ). We wish to show that $\pi_S(U)$ contains an open neighborhood of E_ρ. In other words, we wish to show that if E_ρ is a refinement of some equivalence relation $\simeq$ on S, then we can find some other $(E', \rho') \in U$ such that $E'_{\rho'} = \simeq$. For this, we simply take E' to be the equivalence relation on $\mathbf{Z}$ corresponding to the quotient map $\mathbf{Z} \to (\mathbf{Z}/E) \amalg_{\mathrm{Im}(\rho)} (S/\simeq)$, and ρ' to be the composite map $S \xrightarrow{\rho} \mathbf{Z}/E \to \mathbf{Z}/E'$.

To complete the proof, it will suffice to verify clause $(2')$ appearing in Proposition 14 of the previous lecture (for the set of generators $\{h_S\}_{S \in \mathcal{S}\mathrm{et}^{\emptyset}_{\mathrm{fin}}}$), it will suffice to show that for every map of nonempty finite sets $\alpha : S \to T$, the diagram of topological spaces

$$\begin{array}{ccc} \widetilde{\mathrm{Equiv}(\mathbf{Z})}_T & \xrightarrow{\phi} & \widetilde{\mathrm{Equiv}(\mathbf{Z})}_S \\ \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi_S} \\ \mathrm{Equiv}(T) & \xrightarrow{\psi} & \mathrm{Equiv}(S) \end{array}$$

has the property that $\psi^{-1}(\pi_S(U)) = \pi_T(\phi^{-1} U)$ for each open set $U \subseteq \widetilde{\mathrm{Equiv}(\mathbf{Z})}_S$ as above. Suppose we are given a point (E, ρ) of U and that E_ρ is the restriction an equivalence relation E_T on the set T. We wish to show that, after replacing (E, ρ) by another point of U, we can arrange that ρ extends to a map $\widetilde{\rho} : T \to \mathbf{Z}/E$ satisfying $E_T = E_{\widetilde{\rho}}$. Equivalently, we wish to show that, possibly after changing (E, ρ), the monomorphism $S/E_\rho \hookrightarrow \mathbf{Z}/E$ can be extended to a monomorphism $T/E_T \hookrightarrow \mathbf{Z}/E$. This is always possible when the quotient $\mathbf{Z}/E$ is infinite, which can always be arranged by refining the equivalence relation E. $\square$

Let's now return to the picture of Lecture 16. Let $\mathcal{X}$ be a topos containing a set of generators $\{X_i\}_{i \in I}$. Set $X = \coprod_{i \in I}$, assume that $X \to \mathbf{1}$ is an effective epimorphism (which can always be arranged by including $\mathbf{1}$ as a generator), and form a pullback diagram

$$\begin{array}{ccc} \mathrm{Enum}(X) & \xrightarrow{\pi} & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z})) & \longrightarrow & \mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}} \end{array}$$

(we'll show later that such a thing exists). We would like to argue that $\mathrm{Enum}(X)$ is localic. To prove this, it will be convenient to introduce a relative version of the condition of being localic.

Definition 2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. We will say that f is *localic* (or that $\mathcal{X}$ is *localic relative to* $\mathcal{Y}$) if there exist generators $\{X_i\}$ for $\mathcal{X}$, each of which appears as a subobject of $f^* Y_i$ for some $Y_i \in \mathcal{Y}$.

Example 3. Let $\mathcal{X}$ be a topos with a set of generators $\{X_i\}_{i \in I}$, and set $X = \coprod_{i \in I} X_i$. Then X is classified by a geometric morphism $\rho : \mathcal{X} \to \mathcal{X}_{\mathcal{S}\mathrm{et}}$, characterized by the requirement that $X \simeq \rho^* X_0$ where $X_0 \in \mathcal{X}_{\mathcal{S}\mathrm{et}}$ is the "universal" object. The geometric morphism ρ is localic: by construction, $\mathcal{X}$ is generated by subobjects of $\rho^* X_0$. (If $X \to \mathbf{1}$ is an effective epimorphism, then the classifying map $\mathcal{X} \to \mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}$ is likewise localic).

Proposition 4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Then:*

(1) *If $\mathcal{X}$ is localic, f is localic.*

(2) *If $\mathcal{Y}$ is localic and f is localic, then $\mathcal{X}$ is localic.*

Proof. If $\mathcal{X}$ is localic, then it is generated by subobjects of $\mathbf{1}_{\mathcal{X}} \simeq f^* \mathbf{1}_{\mathcal{Y}}$, so that f is localic. This proves (1). To prove (2), assume that $\mathcal{Y}$ is localic and that f is localic. Since f is localic, every object $X \in \mathcal{X}$ admits a covering $\{U_i \to X\}$, where each U_i appears as a subobject of $f^* Y_i$ for some $Y_i \in \mathcal{Y}$. Since $\mathcal{Y}$ is localic, each Y_i admits a covering $\{V_{i,j} \to Y_i\}$, where each $V_{i,j}$ is a subobject of $\mathbf{1}_{\mathcal{Y}}$. Then X admits a covering $\{(f^* V_{i,j}) \times_{f^* Y_i} U_i \to X\}$, where each $(f^* V_{i,j}) \times_{f^* Y_i} U_i$ is a subobject of $f^* \mathbf{1}_{\mathcal{Y}} \simeq \mathbf{1}_{\mathcal{X}}$. $\square$

Corollary 5. *Let $\mathcal{X}$ be a topos, so that there is an essentially unique geometric morphism $f : \mathcal{X} \to \mathcal{S}\mathrm{et}$. Then $\mathcal{X}$ is localic if and only if f is localic.*

We now consider a relative version of Proposition 4:

Proposition 6. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be geometric morphisms of topoi. Then:*

(1) *If $g \circ f$ is localic, then f is localic.*

(2) *If f and g are localic, then $g \circ f$ is localic.*

Proof. Assertion (1) follows from the observation that every subobject of $(g \circ f)^*(Z)$ is a subobject of an object of the form $f^* Y$, by taking $Y = g^* Z$. To prove (2), assume that f and g are localic. For each object $X \in \mathcal{X}$, we can find a cover $\{U_i \to X\}$, where each U_i is a subobject of some $f^* Y_i$. Then each Y_i admits a cover $\{V_{i,j} \to Y_i\}$, where each $V_{i,j}$ is a subobject of some $g^* Z_{i,j} \in \mathcal{Z}$. It follows that X admits a covering $\{f^*(V_{i,j}) \times_{f^* Y_i} U_i \to X\}$ where each $f^*(V_{i,j}) \times_{f^* Y_i} U_i$ can be regarded as a subobject of $(g \circ f)^* Z_{i,j}$. $\square$

In Lecture 14, we showed that the datum of a localic topos $\mathcal{X}$ is determined by the datum of its underlying localic $\mathrm{Sub}(\mathbf{1}_{\mathcal{X}})$. Our next goal will be to establish a relative version of this observation, where we encode the datum of a localic geometric morphism $f : \mathcal{X} \to \mathcal{Y}$ in terms of a single "partially ordered" object of $\mathcal{Y}$. First, we need the following general fact:

Proposition 7. *Let $\mathcal{X}$ be a topos, and regard the construction $X \mapsto \mathrm{Sub}(X)$ as a contravariant functor from $\mathcal{X}$ to the category of sets (carrying each morphism $f : X \to Y$ to the inverse image map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$). Then the functor $X \mapsto \mathrm{Sub}(X)$ is representable. In other words, there exists an object $\Omega_{\mathcal{X}}$ and bijections $\mathrm{Sub}(X) \simeq \mathrm{Hom}_{\mathcal{X}}(X, \Omega_{\mathcal{X}})$ depending functorially on X.*

Remark 8. The object $\Omega_{\mathcal{X}}$ appearing in the statement of Proposition 7 is called a *subobject classifier* of $\mathcal{X}$.

Proof of Proposition 7. The proof of Giraud's theorem shows that the Yoneda embedding $h : \mathcal{X} \to \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ induces an equivalence of $\mathcal{X}$ with the category of sheaves on itself, where we equip $\mathcal{X}$ with the topology given by the coverings. It will therefore suffice to show that the construction $X \mapsto \mathrm{Shv}(X)$ is a sheaf. In other words, we must show that for every covering $\{U_i \to X\}_{i \in I}$, the diagram of sets

$$\mathrm{Sub}(X) \to \prod_{i \in I} \mathrm{Sub}(U_i) \rightrightarrows \prod_{i,j \in I} \mathrm{Sub}(U_i \times_X U_j)$$

is an equalizer. Note that a subobject $V \subseteq X$ be recovered as the join $\bigvee_{i \in I} \mathrm{Im}(U_i \times_X V \to X)$, so the map $\mathrm{Sub}(X) \to \prod_{i \in I} \mathrm{Sub}(U_i)$ is injective. Conversely, suppose we are given an element of $\prod_{i \in I} \mathrm{Sub}(U_i)$, given by a collection of subobjects $V_i \subseteq U_i$. Set $V = \bigvee_{i \in I} \mathrm{Im}(V_i \to X)$. Then, for each $j \in I$, we have

$$\begin{aligned} V \times_X U_j &= (\bigvee_{i \in I} \mathrm{Im}(V_i \to X)) \times_X U_j \\ &= \bigvee_{i \in I} (\mathrm{Im}(V_i \to X) \times_X U_j) \\ &= \bigvee_{i \in I} (\mathrm{Im}(V_i \times_X U_j \to U_j). \end{aligned}$$

If $\{V_i\}_{i \in I}$ belongs to the equalizer, then we can rewrite this as

$$\begin{aligned}
\bigvee_{i \in I} \mathrm{Im}(U_i \times_X V_j \to U_j) &= \bigvee_{i \in I} \mathrm{Im}(U_i \times_X V_j \to V_j) \\
&= \bigvee_{i \in I} (\mathrm{Im}(U_i \to X) \times_X V_j) \\
&= (\bigvee_{i \in I} \mathrm{Im}(U_i \to X)) \times_X V_j \\
&= X \times_X V_j \\
&= V_j.
\end{aligned}$$

□

Corollary 9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Then the functor*

$$\mathcal{Y} \to \mathcal{S}\mathrm{et} \qquad Y \mapsto \mathrm{Sub}(f^* Y)$$

is representable by an object $\Omega_{\mathcal{X} / \mathcal{Y}} \in \mathcal{Y}$.

Proof. Take $\Omega_{\mathcal{X} / \mathcal{Y}} = f_* \Omega_{\mathcal{X}}$, where $\Omega_{\mathcal{X}}$ is a subobject classifier of $\mathcal{X}$ and f_* is right adjoint to f^*. □

Example 10. In the situation of Corollary 9, suppose that $\mathcal{Y} = \mathcal{S}\mathrm{et}$ is the topos of sets. Then we have

$$\begin{aligned}
\Omega_{\mathcal{X} / \mathcal{Y}} &= \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(\mathbf{1}_{\mathcal{S}\mathrm{et}}, \Omega_{\mathcal{X} / \mathcal{Y}}) \\
&\simeq \mathrm{Sub}(f^* \mathbf{1}_{\mathcal{S}\mathrm{et}}) \\
&\simeq \mathrm{Sub}(\mathbf{1}_{\mathcal{X}}).
\end{aligned}$$

In other words, we can identify $\Omega_{\mathcal{X} / \mathcal{Y}}$ with the underlying locale of the topos $\mathcal{X}$.

We will see in the next lecture that a localic morphism $f : \mathcal{X} \to \mathcal{Y}$ can be recovered from the object $\Omega_{\mathcal{X} / \mathcal{Y}} \in \mathcal{Y}$ (together with a suitable partial ordering of it).

Lecture 19: Reconstruction of Localic Morphisms

March 9, 2018

Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. In the previous lecture, we saw that there is an object $\Omega_{\mathcal{X}/\mathcal{Y}}$ of $\mathcal{Y}$ with the following universal property: for each object $Y \in \mathcal{Y}$, we have a canonical bijection

$$\mathrm{Hom}_{\mathcal{Y}}(Y, \Omega_{\mathcal{X}/\mathcal{Y}}) \simeq \mathrm{Sub}(f^* Y).$$

Our goal in this lecture is to explain that, if the morphism f is localic, then we can recover $\mathcal{X}$ from the topos $\mathcal{Y}$ and the object $\Omega_{\mathcal{X}/\mathcal{Y}}$. For this, we need to endow $\Omega_{\mathcal{X}/\mathcal{Y}}$ with a little bit of additional structure.

Exercise 1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Show that the functor

$$(Y \in \mathcal{Y}) \mapsto \{U, V \in \mathrm{Sub}(f^* Y) : U \subseteq V\}$$

is representable by an object $\Omega^{\subseteq}_{\mathcal{X}/\mathcal{Y}}$ of $\mathcal{Y}$. Note that $\Omega^{\subseteq}_{\mathcal{X}/\mathcal{Y}}$ can then be viewed as a subobject of $\Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}}$.

We now consider several (essentially equivalent) ways of looking at the object $\Omega_{\mathcal{X}/\mathcal{Y}}$:

(a) We can think of $\Omega_{\mathcal{X}/\mathcal{Y}}$ as a *partially ordered object of* $\mathcal{Y}$ (with partial order given by $\Omega^{\subseteq}_{\mathcal{X}/\mathcal{Y}} \subseteq \Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}}$). Concretely, this means that the intersection

$$\Omega^{\subseteq}_{\mathcal{X}/\mathcal{Y}} \cap (\Omega^{\subseteq}_{\mathcal{X}/\mathcal{Y}})^{\mathrm{op}}$$

coincides with the image of the diagonal map $\Omega_{\mathcal{X}/\mathcal{Y}} \to \Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}}$ (here $(\Omega^{\subseteq}_{\mathcal{X}/\mathcal{Y}})^{\mathrm{op}}$ denote the image of $\Omega^{\subseteq}_{\mathcal{X}/\mathcal{Y}}$ under the automorphism of $\Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}}$ given by swapping the two factors), and that we have

$$\pi_1^* \Omega^{\subseteq}_{\mathcal{X}/\mathcal{Y}} \cap \pi_3^* \Omega^{\subseteq}_{\mathcal{X}/\mathcal{Y}} \subseteq \pi_2^* \Omega^{\subseteq}_{\mathcal{X}/\mathcal{Y}}$$

as subobjects of $\Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}}$, where

$$\pi_i : \Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}} \to \Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}}$$

denotes the projection map which omits the ith factor.

The first demand encodes the requirement that the relation $\Omega^{\subseteq}_{\mathcal{X}/\mathcal{Y}}$ is reflexive and antisymmetric, and the second encodes transitivity.

(b) Rather than viewing $\Omega_{\mathcal{X}/\mathcal{Y}}$ as an object of $\mathcal{Y}$, we can identify it with the functor that it represents, given by

$$Y \mapsto \mathrm{Hom}_{\mathcal{Y}}(Y, \Omega_{\mathcal{X}/\mathcal{Y}}) \simeq \mathrm{Sub}(f^* Y).$$

The axioms of (a) translate to the requirement that, for each $Y \in \mathcal{Y}$, the image of the inclusion map

$$\mathrm{Hom}_{\mathcal{Y}}(Y, \Omega^{\subseteq}_{\mathcal{X}/\mathcal{Y}}) \hookrightarrow \mathrm{Hom}_{\mathcal{Y}}(Y, \Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}}) \simeq \mathrm{Sub}(f^* Y) \times \mathrm{Sub}(f^* Y)$$

determines a partial ordering of $\mathrm{Sub}(f^* Y)$. Of course, this is why those axioms are satisfied in the first place (by construction, this is just the partial order on $\mathrm{Sub}(f^* Y)$ given by inclusions of subobjects).

In other word, we can think of $\Omega_{\mathcal{X}/\mathcal{Y}}$ as encoding a functor

$$\mathcal{Y}^{\mathrm{op}} \to \{\text{Partially Ordered Sets}\}.$$

Such a functor is representable by a partially ordered object of $\mathcal{Y}$ (in the sense of (a)) if and only if it is a sheaf with respect to the canonical topology on $\mathcal{Y}$.

(b') In the situation of (b), we can be more specific. Each the partially ordered sets $\mathrm{Sub}(f^*Y)$ is a locale, and each $Y \to Y'$ in $\mathcal{Y}$ induces a map of posets $\mathrm{Sub}(f^*Y') \to \mathrm{Sub}(f^*Y)$ which preserves finite meets and arbitrary joins; that is, it can be regarded as a locale morphism from $\mathrm{Sub}(f^*Y)$ to $\mathrm{Sub}(f^*Y')$. We can therefore identify $\Omega_{\mathcal{X}/\mathcal{Y}}$ with a functor

$$\mathcal{Y} \to \{\text{Locales}\}.$$

We can do even better: in Lecture 17, we saw that each of the locale morphisms $\mathrm{Sub}(f^*Y) \to \mathrm{Sub}(f^*Y')$ is open. We therefore obtain a functor

$$\mathcal{Y} \to \{\text{Locales, Open Morphisms of Locales}\}.$$

(c) Given any category $\mathcal{C}$ and a functor $P : \mathcal{C}^{\mathrm{op}} \to \{\text{Partially Ordered Sets}\}$, we can form a new category $\widetilde{\mathcal{C}}$ described as follows:

- The objects of $\widetilde{\mathcal{C}}$ are pairs (C, U), where $C \in \mathcal{C}$ and $U \in P(C)$.
- A morphism from (C, U) to (C', U') in $\widetilde{\mathcal{C}}$ is a morphism $g : C \to C'$ with the property that $U \leq P(f)(U')$ (in the poset $P(C)$).

Note that the category $\widetilde{\mathcal{C}}$ is equipped with a forgetful functor $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$, given on objects by $\pi(C, U) = C$. Moreover, we can recover the original functor P from $\widetilde{\mathcal{C}}$ and the functor π. For each $C \in \mathcal{C}$, we can identify the poset $P(C)$ with the fiber $\pi^{-1}\{C\}$. If $g : C \to C'$ is a morphism in $\mathcal{C}$ and U' is an element of $P(C')$, identified with an object $\widetilde{C}' \in \widetilde{\mathcal{C}}$ satisfying $\pi(\widetilde{C}') = C'$, then we can identify $P(f)(U') \in P(C)$ with the largest element of the poset $\{\widetilde{C} \in \pi^{-1}\{C\} : (\exists \widetilde{g} : \widetilde{C} \to \widetilde{C}')[\pi(\widetilde{g}) = g]\}$. In fact, this construction determines an equivalence

$$\{\text{Functors } P : \mathcal{C}^{\mathrm{op}} \to \{\text{Posets}\}\} \simeq \{\text{Functors } \pi : \widetilde{\mathcal{C}} \to \mathcal{C} \text{ which are fibered in posets}\}.$$

In the case of interest, we take $\mathcal{C} = \mathcal{Y}$ and $P : \mathcal{Y}^{\mathrm{op}} \to \{\text{Posets}\}$ to be the functor $Y \mapsto \mathrm{Sub}(f^* \mathcal{Y})$. In this case, we will denote the category $\widetilde{\mathcal{C}}$ by $\mathrm{Loc}(f)$; it can be described concretely as follows:

- The objects of $\mathrm{Loc}(f)$ are pairs (Y, U) where Y is an object of $\mathcal{Y}$ and $U \subseteq f^*Y$ is a subobject in $\mathcal{X}$. As a mnemonic aide, we will denote such an object by $(U \subseteq f^*Y)$.
- A morphism from $(U \subseteq f^*Y)$ to $(U' \subseteq f^*Y')$ is a morphism $g : Y \to Y'$ in the topos $\mathcal{Y}$ satisfying $U \subseteq U' \times_{f^*Y'} f^*Y$ (as subobjects of f^*Y).

In what follows, it will be convenient to adopt perspective (c).

Remark 2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Then the category $\mathrm{Loc}(f)$ has a forgetful functor to $\mathcal{X}$, given by the construction $(U \subseteq f^*Y) \mapsto U$.

Remark 3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Then the category $\mathrm{Loc}(f)$ admits finite limits. For example, given a pair of morphisms

$$(U_0 \subseteq f^*Y_0) \to (U_{01} \subseteq f^*Y_{01}) \leftarrow (U_1 \subseteq f^*Y_1),$$

the fiber product is given by $(U_0 \times_{U_{01}} U_1 \subseteq f^*(Y_0 \times_{Y_{01}} Y_1))$.

Exercise 4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Let us say that a collection of morphisms $\{(U_i \subseteq f^* Y_i) \to (U \subseteq f^* Y)\}_{i \in I}$ in the category $\mathrm{Loc}(f)$ is a *covering* if the diagram $\{U_i \to U\}$ is a covering in $\mathcal{X}$. Show that this determines a Grothendieck topology on the category $\mathrm{Sub}(f)$. Note that we do *not* require that the objects Y_i cover Y.

Our next goal is to prove the following:

Theorem 5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a localic geometric morphism. Then there is a canonical equivalence $\mathcal{X} \simeq \mathrm{Shv}(\mathrm{Loc}(f))$. In particular, the topos $\mathcal{X}$ can be recovered from the fibration $\mathrm{Loc}(f) \to \mathcal{Y}$ (or, equivalently, from $\mathcal{Y}$ together with the partially ordered object $\Omega_{\mathcal{X}/\mathcal{Y}}$).*

The proof of Theorem 5 will require some preliminaries.

Notation 6. Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the full subcategory spanned by those objects $X \in \mathcal{X}$ for which there exists a monomorphism $X \hookrightarrow f^* Y$, for some Y in $\mathcal{Y}$. Note that the forgetful functor

$$\mathrm{Loc}(f) \to \mathcal{X} \qquad (U \subseteq f^* Y) \mapsto U$$

factors through $\mathcal{X}_0$ (in fact, $\mathcal{X}_0$ is defined as the essential image of this forgetful functor).

Remark 7. The subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ is closed under finite limits. For example, if we are given a fiber product $X_0 \times_{X_{01}} X_1$ where X_0, X_{01}, and X_1 belong to $\mathcal{X}_0$, then we can choose monomorphisms $u : X_0 \hookrightarrow f^* Y_0$ and $v : X_1 \hookrightarrow f^* Y_1$ for some $Y_0, Y_1 \in \mathcal{Y}$. In this case, u and v induce a monomorphism $X_0 \times_{X_{01}} X_1 \hookrightarrow f^*(Y_0 \times Y_1)$.

Lemma 8. *Let $\mathcal{Z}$ be a topos and let $\mathcal{Z}_0 \subseteq \mathcal{Z}$ be a full subcategory satisfying the following conditions:*

(1) *The full subcategory $\mathcal{Z}_0 \subseteq \mathcal{Z}$ contains a set of generators for $\mathcal{Z}$.*

(2) *The full subcategory $\mathcal{Z}_0 \subseteq \mathcal{Z}$ is closed under finite limits.*

Then the Yoneda embedding induces an equivalence of categories $\mathcal{Z} \simeq \mathrm{Shv}(\mathcal{Z}_0)$, where we equip $\mathcal{Z}_0$ with the Grothendieck topology given by the covering families in $\mathcal{Z}$.

Proof. When $\mathcal{Z}_0$ is small, we proved this in Lecture 10. The general case follows by writing $\mathcal{Z}_0$ as a union of small subcategories satisfying (1) and (2). □

Corollary 9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a localic geometric morphism of topoi, and let $\mathcal{X}_0 \subseteq \mathcal{X}$ be as in Notation* **??**. *Then the Yoneda embedding induces an equivalence $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{X}_0)$, where $\mathcal{X}_0$ is equipped with the Grothendieck topology given by the covering families in $\mathcal{X}$.*

In the situation of Theorem 5, we have a forgetful functor

$$\pi : \mathrm{Loc}(f) \to \mathcal{X}_0 \qquad \pi(U \subseteq f^* Y) = U.$$

This functor preserves finite limits (Remark 3) and coverings, so that composition with π induces a functor $\mathrm{Shv}(\mathcal{X}_0) \to \mathrm{Shv}(\mathrm{Loc}(f))$. We will complete the proof by establishing the following (which does not require the assumption that f is localic):

Proposition 10. *Composition with π induces an equivalence of categories $\mathrm{Shv}(\mathcal{X}_0) \to \mathrm{Shv}(\mathrm{Loc}(f))$.*

Sketch. Let $\mathscr{F} : \mathrm{Loc}(f)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a sheaf. Essentially, we need to show that for an object $(U \subseteq f^* Y) \in \mathrm{Loc}(f)$, the set $\mathscr{F}(U \subseteq f^* Y)$ depends only on $U \in \mathcal{X}_0$, and not on the particular realization of U as a subobject of $f^* Y$. To this end, let us regard the object $U \in \mathcal{X}_0$ as fixed, and suppose we are given two different monomorphisms $U \hookrightarrow f^* Y$ and $U \hookrightarrow f^* Y'$. The product then defines a monomorphism $U \hookrightarrow f^*(Y \times Y')$. We therefore have a diagram of sets

$$\mathscr{F}(U \subseteq f^* Y) \to \mathscr{F}(U \subseteq f^*(Y \times Y')) \leftarrow \mathscr{F}(U \subseteq f^* Y').$$

We claim that these maps are bijective. This is a special case the following:

($*$) Let U be a subobject of f^*Y, and suppose we are given a morphism $g : Y \to Z$ in $\mathcal{Y}$ such that the composite map $U \hookrightarrow f^*Y \to f^*Z$ is still a monomorphism. Then the induced map $\mathscr{F}(U \subseteq f^*Z) \to \mathscr{F}(U \subseteq f^*Y)$ is bijective.

In the situation of ($*$), the map $(U \subseteq f^*Y) \to (U \subseteq f^*Z)$ is a covering (for the Grothendieck topology of Exercise 4). We therefore have an equalizer diagram of sets

$$\mathscr{F}(U \subseteq f^*Z) \to \mathscr{F}(U \subseteq f^*Y) \rightrightarrows \mathscr{F}(U \subseteq f^*(Y \times_Z Y)).$$

Consequently, to prove ($*$), it will suffice to show that two different projection maps $Y \times_Z Y \to Y$ induce the same map from $\mathscr{F}(U \subseteq f^*Y)$ to $\mathscr{F}(U \subseteq f^*(Y \times_Z Y))$. It is clear that these maps agree after composition with the map

$$\mathscr{F}(U \subseteq f^*(Y \times_Z Y)) \to \mathscr{F}(U \subseteq f^*Y)$$

given by composition with the diagonal map $\delta : Y \to Y \times_Z Y$. Consequently, to prove assertion ($*$) for the map $g : Y \to Z$, it will suffice to prove ($*$) for the map $\delta : Y \to Y \times_Z Y$. In particular, assertion ($*$) is true whenever g is a monomorphism (since in this case δ is an isomorphism). However, the map δ is always a monomorphism, and therefore satisfies ($*$); it follows that ($*$) is true in general.

Using ($*$) (and the discussion which precedes it), we see that we can identify $\mathscr{F}(U \subseteq f^*Y)$ with $\mathscr{G}(U)$, for some set $\mathscr{G}(U)$ which is independent of the embedding $U \hookrightarrow f^*Y$. We leave it to the reader to verify that the construction $U \mapsto \mathscr{G}(U)$ is functorial (hint: realize $\mathscr{G}$ as the left Kan extension along the projection $\mathrm{Loc}(f)^{\mathrm{op}} \to \mathcal{X}_0^{\mathrm{op}}$). We claim that $\mathscr{G}$ is a sheaf on $\mathcal{X}_0$. To prove this, suppose we are given a covering $\{U_i \to U\}$ in $\mathcal{X}_0$. Choose monomorphisms

$$U \hookrightarrow f^*Y \qquad U_i \hookrightarrow f^*Y_i.$$

Then the diagram

$$\{(U_i \subseteq f^*(Y_i \times Y)) \to (U \subseteq f^*Y)\}$$

is a covering in $\mathrm{Loc}(f)$. Our assumption that $\mathscr{F}$ is a sheaf then supplies an equalizer diagram

$$\mathscr{F}(U \subseteq f^*Y) \to \prod_i \mathscr{F}(U_i \subseteq f^*(Y_i \times Y)) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \times_U U_j \subseteq f^*(Y_i \times Y_j \times Y)),$$

which we can rewrite as

$$\mathscr{G}(U) \to \prod_{i \in I} \mathscr{G}(U_i) \rightrightarrows \prod_{i,j} \mathscr{G}(U_i \times_U U_j).$$

It is now easy to verify that the construction $\mathscr{F} \mapsto \mathscr{G}$ determines an inverse to the functor $\mathrm{Shv}(\mathcal{X}_0) \to \mathrm{Shv}(\mathrm{Loc}(f))$ given by composition with π. $\square$

Lecture 20: $\mathcal{X}$-Locales

March 12, 2018

Let $f : \mathcal{Y} \to \mathcal{X}$ be a geometric morphism of topoi. In the previous lecture, we proved that if f is localic, then it can be recovered from any of the following three interchangeable pieces of data:

(a) The object $\Omega_{\mathcal{Y}/\mathcal{X}} \in \mathcal{X}$, together with its partial order $\Omega^{\subseteq}_{\mathcal{Y}/\mathcal{X}} \subseteq \Omega_{\mathcal{Y}/\mathcal{X}} \times \Omega_{\mathcal{Y}/\mathcal{X}}$.

(b) The functor $\mathcal{X}^{\mathrm{op}} \to \{\mathrm{Posets}\}$ given by $X \mapsto \mathrm{Sub}(f^* X)$.

(c) The category $\mathrm{Loc}(f)$ obtained by applying the Grothendieck construction to the functor $X \mapsto \mathrm{Sub}(f^* X)$.

In this lecture, it will be convenient to adopt perspective (b). Our goal is to address the following:

Question 1. Let $\mathcal{L} : \mathcal{X}^{\mathrm{op}} \to \{\mathrm{Posets}\}$ be a functor. Under what conditions does there exist a geometric morphism $f : \mathcal{Y} \to \mathcal{X}$ and a natural isomorphism $\mathcal{L} \simeq \mathrm{Sub}(f^*(\bullet))$?

We begin by recording some necessary conditions.

Proposition 2. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a geometric morphism of topoi and define $\mathcal{L} : \mathcal{X}^{\mathrm{op}} \to \{\mathrm{Posets}\}$ by the formula $\mathcal{L}(X) = \mathrm{Sub}(f^*(X))$. Then:*

(1) *The functor $\mathcal{L} : \mathcal{X}^{\mathrm{op}} \to \{\mathrm{Posets}\}$ is a sheaf (with respect to the canonical topology on $\mathcal{X}$).*

(2) *For each $X \in \mathcal{X}$, the poset $\mathcal{L}(X)$ is a locale.*

(3) *For each morphism $g : X' \to X$ in $\mathcal{X}$, let $g^* = \mathcal{L}(g)$ denote the associated map of posets from $\mathcal{L}(X)$ to $\mathcal{L}(X')$. Then g^* determines an open morphism of locales $\mathcal{L}(X') \to \mathcal{L}(X)$. In particular, it has a left adjoint $g_! : \mathcal{L}(X') \to \mathcal{L}(X)$.*

(4) *For every pullback diagram*

$$\begin{tikzcd} U' \arrow[r, "g'"] \arrow[d, "h'"] & U \arrow[d, "h"] \\ X' \arrow[r, "g"] & X, \end{tikzcd}$$

we have an equality $h'_! \circ g'^* = g^* \circ h_!$ of maps from $\mathcal{L}(U)$ to $\mathcal{L}(X')$.

Proof. Assertions (1), (2), and (3) were noted in the previous lecture. To prove (4), let $V \in \mathcal{L}(U) = \mathrm{Sub}(f^* U)$. Note that we have a pullback diagram

$$\begin{tikzcd} f^* U' \arrow[r] \arrow[d] & f^* U \arrow[d] \\ f^* X' \arrow[r] & f^* X. \end{tikzcd}$$

Then $h_! V$ is the image of the composite map $V \subseteq f^* U \to f^* X$, and $h'_! g'^*(V)$ is the image of the map

$$V \times_{f^* U} f^* U' \simeq V \times_{f^* X} f^* X' \to f^* X'.$$

The equality $(g^* \circ h_!(V) = (h'_! \circ g'^*)(V)$ now follows from the fact that the formation of images in the topos $\mathcal{Y}$ is compatible with pullback. $\square$

Definition 3. Let $\mathcal{X}$ be a topos. An $\mathcal{X}$-*locale* is a functor $\mathcal{L} : \mathcal{X}^{\mathrm{op}} \to \{\mathrm{Posets}\}$ which satisfies conditions (1) through (4) of Proposition 2.

Remark 4. We will refer to condition (4) of Proposition 2 as the *Beck-Chevalley condition* for the functor $\mathcal{L}$. Note that for *any* commutative diagram σ :

$$\begin{tikzcd} U' \arrow[r, "g'"] \arrow[d, "h'"] & U \arrow[d, "h"] \\ X' \arrow[r, "g"] & X, \end{tikzcd}$$

in the topos $\mathcal{X}$, we automatically have an inequality $h'_! \circ g'^* \leq g^* \circ h_!$. The Beck-Chevalley condition asserts that the reverse inequality holds when σ is a pullback square.

Remark 5. Conditions (1), (2), and (3) of Proposition 2 can be summarized by saying that the functor $\mathcal{L} : \mathcal{X}^{\mathrm{op}} \to \{\mathrm{Posets}\}$ can be regarded as a sheaf on $\mathcal{X}$ taking values in the opposite of the category of locales and open morphisms.

We can restate Proposition 2 as follows: for every geometric morphism $f : \mathcal{Y} \to \mathcal{X}$, the construction $X \mapsto \mathrm{Sub}(f^* X)$ determines an $\mathcal{X}$-locale. We now prove the converse:

Proposition 6. *Let $\mathcal{X}$ be a topos and let $\mathcal{L} : \mathcal{X}^{\mathrm{op}} \to \{Posets\}$ be an $\mathcal{X}$-locale. Then there exists a localic geometric morphism $f : \mathcal{Y} \to \mathcal{X}$ and a natural isomorphism $\mathcal{L}(X) \simeq \mathrm{Sub}(f^* X)$.*

Remark 7. In Lecture 19, we saw that a localic geometric morphism $f : \mathcal{Y} \to \mathcal{X}$ can be recovered from the associated $\mathcal{X}$-locale. Combining this with Proposition 6, we obtain a dictionary

$$\{\text{Topoi localic over } \mathcal{X}\} \simeq \{\mathcal{X}\text{-locales}\}.$$

We will later refine this picture to an equivalence of categories.

Proof of Proposition 6. Lecture 19 tells us what we need to do. Let $\widetilde{\mathcal{X}}$ denote the category obtained by applying the Grothendieck construction to the functor $\mathcal{L}$. More concretely, it can be described as follows:

- The objects of $\widetilde{\mathcal{X}}$ are pairs (X, U), where $X \in \mathcal{X}$ and $U \in \mathcal{L}(X)$.
- A morphism from (X', U') to (X, U) is a morphism $g : X' \to X$ in the topos $\mathcal{X}$ satisfying $U' \leq g^*(U)$ (or equivalently $g_!(U') \leq U$).

Let us say that a collection of maps $\{g_i : (X_i, U_i) \to (X, U)\}_{i \in I}$ in $\widetilde{\mathcal{X}}$ is a *covering* if $U = \bigvee g_{i!} U_i$ in $\mathcal{L}(X)$. We claim that this determines a Grothendieck topology on $\widetilde{\mathcal{X}}$. We check that the collection of coverings is stable under pullback (the remaining axioms for a Grothendieck topology are easy and left as an exercise).

Suppose we are given a covering $\{g_i : (X_i, U_i) \to (X, U)$ and an arbitrary morphism $h : (Y, V) \to (X, U)$. For each $i \in I$, we can form the fiber product $(X_i, U_i) \times_{(X,U)} (Y, V)$ in the category $\widetilde{X}$; it is given by $(X_i \times_X Y, \pi_i^* U_i \wedge \pi_i'^* V)$, where $\pi_i : U_i \times_X Y \to U_i$ and $\pi'_i : U_i \times_X Y \to Y$ are the two projection maps. We

claim that the family of maps $\{\pi'_i : (X_i \times_X Y, \pi_i^* U_i \wedge \pi_i'^* V) \to (Y, V)\}_{i \in I}$ is also a covering. For this, we compute

$$\begin{aligned}
\bigvee_{i \in I} \pi'_{i!}(\pi_i^* U_i \wedge \pi_i'^* V) &= \bigvee_{i \in I} (\pi'_{i!}(\pi_i^* U_i) \wedge V) \\
&= (\bigvee_{i \in I} \pi'_{i!}(\pi_i^* U_i)) \wedge V \\
&= (\bigvee_{i \in I} h^* g_{i!} U_i) \wedge V \\
&= h^*(\bigvee_{i \in I} g_{i!} U_i) \wedge V \\
&= h^*(U) \wedge V \\
&= V.
\end{aligned}$$

Here the first equality follows from the projection formula for the open morphism of locales $\mathcal{L}(U_i \times_X Y) \to \mathcal{L}(Y)$, the second equality from the distributive law in the locale $\mathcal{L}(Y)$, the third from the Beck-Chevalley property for $\mathcal{L}$, the fourth from the fact that the map h^* preserves joins, the fifth from the assumption that $\{(X_i, U_i) \to (X, U)\}_{i \in I}$ is a covering, and the sixth from the inequality $V \subseteq h^* U$.

We now encounter a bit of a technical annoyance: the category $\widetilde{\mathcal{X}}$ is not small, so it is not *a priori* clear that the category $\mathrm{Shv}(\widetilde{\mathcal{X}})$ is a topos. To address this, let us choose a small full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ which generates $\mathcal{X}$ and is closed under finite limits, and set $\widetilde{\mathcal{X}}_0 = \widetilde{\mathcal{X}} \times_{\mathcal{X}} \mathcal{X}_0$. Then $\widetilde{\mathcal{X}}_0$ inherits a Grothendieck topology (with the notion of covering defined as above), so that $\mathrm{Shv}(\widetilde{\mathcal{X}}_0)$ is a topos. Moreover, the inclusion functor $\mathrm{Shv}(\widetilde{\mathcal{X}}_0) \hookrightarrow \mathrm{Fun}(\widetilde{\mathcal{X}}_0^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ admits a left adjoint (given by sheafification), which we will denote by L. (One can show that the restriction map $\mathrm{Shv}(\widetilde{\mathcal{X}}) \to \mathrm{Shv}(\widetilde{\mathcal{X}}_0)$ is an equivalence of categories, but we will not need this). Let $\widetilde{h} : \widetilde{\mathcal{X}}_0 \to \mathrm{Shv}(\widetilde{\mathcal{X}}_0)$ denote the sheafified Yoneda embedding $\widetilde{h}(X, U) = L h_{(X,U)}$ (beware that the topology on $\widetilde{\mathcal{X}}_0$ is not subcanonical, so the sheafification is necessary). Since L and the Yoneda embedding preserve finite limits, the functor $\widetilde{h}$ also preserves finite limits.

For each object $X \in \mathcal{X}$, let $\mathbf{1}_X$ denote the largest element of $\mathcal{L}(X)$. Then the construction $X \mapsto (X, \mathbf{1}_X)$ determines a functor $\mathcal{X} \to \widetilde{\mathcal{X}}$ which preserves finite limits. We claim that it also preserves coverings: that is, for every covering $\{g_i : X_i \to X\}$ in $\mathcal{X}$, the family $\{(X_i, \mathbf{1}_{X_i}) \to (X, \mathbf{1}_X)\}$ is a covering in $\mathcal{X}$. To prove this, we must show that if $U \in \mathcal{L}(X)$ has the property that $\mathbf{1}_{X_i} \leq g_i^* U$ for each $i \in I$, then $U = \mathbf{1}_X$. This follows from our assumption that $\mathcal{L}$ is a sheaf (since $g_i^* U = \mathbf{1}_{X_i}$ for each i). It follows that there is an essentially unique geometric morphism of topoi $f' : \mathrm{Shv}(\widetilde{\mathcal{X}}_0) \to \mathrm{Shv}(\mathcal{X}_0)$ satisfying $f'^* h_X \simeq \widetilde{h}(X, U)$ for each $X \in \mathcal{X}_0$. Composing with the equivalence $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{X}_0)$ given by the Yoneda embedding, we obtain a geometric morphism $f : \mathrm{Shv}(\widetilde{\mathcal{X}}_0) \to \mathcal{X}$ with the property that $f^* X \simeq \widetilde{h}(X, \mathbf{1}_X)$ for each $X \in \mathcal{X}_0$. We will complete the proof by showing that this functor has the desired properties.

We first note that the topos $\mathrm{Shv}(\widetilde{\mathcal{X}}_0)$ is generated by objects of the form $\widetilde{h}(X, U)$, where $X \in \mathcal{X}_0$ and $U \in \mathcal{L}(X)$. For every such object, we have a monomorphism $(X, U) \to (X, \mathbf{1}_X)$ in the category $\widetilde{\mathcal{X}}_0$, hence also a homomorphism $\widetilde{h}(X, U) \to \widetilde{h}(X, \mathbf{1}_X) \simeq f^* X$ in the category $\mathrm{Shv}(\widetilde{\mathcal{X}}_0)$. It follows that the geometric morphism f is localic.

To complete the proof, it will suffice to construct a natural isomorphism $\mathcal{L} \simeq \mathrm{Sub}(f^*(\bullet))$ of functors $\mathcal{X}^{\mathrm{op}} \to \{\mathrm{Posets}\}$. For each $X \in \mathcal{X}_0$, we have a map

$$\rho_X : \mathcal{L}(X) \to \mathrm{Sub}(f^* X) \simeq \mathrm{Sub}(\widetilde{h}(X, \mathbf{1}_X)),$$

given by $\rho_X(U) = \widetilde{h}(X, U)$. Note that ρ_X depends functorially on X. We will show that each ρ_X is bijective, so that $\{\rho_X\}_{X \in \mathcal{X}_0}$ is a natural isomorphism of functors $\mathcal{L}|_{\mathcal{X}_0^{\mathrm{op}}} \simeq \mathrm{Sub}(f^*(\bullet))|_{\mathcal{X}_0^{\mathrm{op}}}$. Since both sides are sheaves with respect to the canonical topology on $\mathcal{X}$, it follows that this natural isomorphism extends uniquely to an isomorphism of functors $\mathcal{L} \simeq \mathrm{Sub}(f^*(\bullet))$.

Let us henceforth regard $X \in \mathcal{X}_0$ as fixed. We will construct an inverse to the map ρ_X. Let $\mathscr{F}$ be a subsheaf of $f^*X = Lh_{(X,\mathbf{1}_X)}$. For each $U \in \mathcal{L}(X)$, let $\iota_U : (X,U) \to (X,\mathbf{1}_X)$ denote the morphism in $\widetilde{\mathcal{X}}_0$ given by the identity map from X to itself, which we can regard as an element of $h_{(X,\mathbf{1}_X)}(X,U)$. Let $\widetilde{\iota}_U$ denote the image of ι_U as a section of the associated sheaf $Lh_{(X,\mathbf{1}_X)} = f^*X$. Consider the set

$$\mathcal{U} = \{U \in \mathcal{L}(X) : \widetilde{\iota}_U \in \mathscr{F}(X,U) \subseteq (f^*X)(X,U)\}.$$

Note that $\mathcal{U}$ is closed under joins (this follows by applying the sheaf condition to $\mathscr{F}$, since the family $\{(X,U_i) \to (X, \bigvee U_i)\}$ is a covering for any family of elements $\{U_i \in \mathcal{L}(X)\}_{i \in i}$). It follows that $\mathcal{U}$ contains a largest element U. The construction $\mathscr{F} \mapsto U$ determines a map of posets $\psi : \mathrm{Sub}(f^*X) \to \mathcal{L}(X)$.

We first claim that the composition $\mathcal{L}(X) \xrightarrow{\rho_X} \mathrm{Sub}(f^*X) \xrightarrow{\psi} \mathcal{L}(X)$ is the identity. Fix an object $U \in \mathcal{L}(X)$. Then ι_U belongs to the sub-presheaf $h_{(X,U)} \subseteq h_{(X,\mathbf{1})}$ (when evaluated at (X,U)), so that $\widetilde{\iota}_U$ belongs to the subsheaf $\widetilde{h}(X,U) \subseteq \widetilde{h}(X,\mathbf{1}_X)$ (when evaluated at (X,U)). We wish to show that U is the largest element of $\mathcal{L}(X)$ with this property. Let V be any element of $\mathcal{L}(X)$, and assume that $\widetilde{\iota}_V$ belongs to the subsheaf $\widetilde{h}(X,U) \subseteq \widetilde{h}(X,\mathbf{1}_X)$ (when evaluated at (X,V)). Then we can choose a covering $\{g_i : (X_i,V_i) \to (X,V)\}$ in $\mathcal{X}_0$ with the property that each of the composite maps $(X_i,V_i) \to (X,V) \subseteq (X,\mathbf{1}_X)$ belongs to $h_{(X,U)}(X_i,V_i)$. It follows that we have $g_{i!}V_i \leq U$ for each i, so that $V = \bigvee g_{i!}V_i \leq U$. This completes the proof that $\psi \circ \rho_X$ is the identity.

We now prove that the composition $\rho_X \circ \psi : \mathrm{Sub}(f^*X) \to \mathrm{Sub}(f^*X)$ is the identity. Let $\mathscr{F}$ be a subsheaf of f^*X and form a pullback diagram of presheaves

$$\begin{tikzcd} \mathscr{F}_0 \arrow[r] \arrow[d] & \mathscr{F} \arrow[d] \\ h_{(X,\mathbf{1}_X)} \arrow[r] & f^*X. \end{tikzcd}$$

Set $U = \psi(\mathscr{F})$. By construction, U is the largest element of $\mathcal{U}(X)$ for which $\mathscr{F}_0$ contains $h_{(X,U)}$ (as subfunctors of $h_{(X,\mathbf{1}_X)}$. We will complete the proof by showing that $\mathscr{F}_0 = h_{(X,U)}$; it then follows by the left exactness of sheafification that $\mathscr{F} = L\,\mathscr{F}_0 = Lh_{(X,U)} = \rho_X(U)$. Fix an object $(Y,V) \in \mathcal{X}_0$ and a point η of $\mathscr{F}_0(Y,V)$, which we can identify with a map $g : Y \to X$. The map $(Y,V) \to (X,g_!V)$ is a covering in $\mathcal{X}_0$. Since membership in the subsheaf $\mathscr{F} \subseteq f^*X$ can be tested locally, we conclude that $\widetilde{\iota}_{g_!V}$ belongs to $\mathscr{F}(X,g_!V)$, so that $g_!V \subseteq U$. It follows that η belongs to the sub-pre-sheaf $h_{(X,U)} \subseteq \mathscr{F}_0$, as desired. $\square$