Introduction to Logic

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Contents

狭义的逻辑学一般不研究什么?

- 特定领域的规律、机制:市场逻辑、强盗逻辑、道德律令
- 辩论技巧:预设结论,通过修辞、共情等手段说服对方
- 批判性思维:非形式谬误
- 侦探小说的诡计
- 语用推理

逻辑学研究什么?

- 逻辑是求真的 — 研究关于真的普遍性的规律
- 有效的推理形式(保真:前提真保证结论真)
- 任何结构上都真的命题集合

Contents

逻辑与哲学、数学、计算机

- Logic vs (Analytic) Philosophy.
  sense & reference / extension & intension / use & mention / truth & provability / mutual vs distributed vs common knowledge / knowledge update / belief revision / preference change / information flow / action & strategy / multi-agent interaction / counterfactual / causation / possible world / cross-world identity / essentialism / induction / ontological commitment / concept analysis / laws of thought / extent & limit / paradoxes ...
  Leibniz, Peirce, Frege, Russell, Wittgenstein, Ramsey, Carnap, Quine, Putnam, Kripke, Chomsky, Cantor, Hilbert, Gödel, Tarski, Turing ...
- Logic vs Mathematics.
  Logicism / Formalism / Intuitionism / Constructivism / Finitism / Structuralism / Homotopy Type Theory
- Logic vs Computer Science.

$$\frac{\text{Logic}}{\text{Computer Science}} \approx \frac{\text{Calculus}}{\text{Physics}}$$

逻辑学的几个主要分支

Mathematical Logic

- **First Order Logic**
- Set Theory
- Model Theory
- Proof Theory
- Recursion Theory
- (Homotopy) Type Theory
- Category / Topos Theory / Categorical Logic

Computational Logic

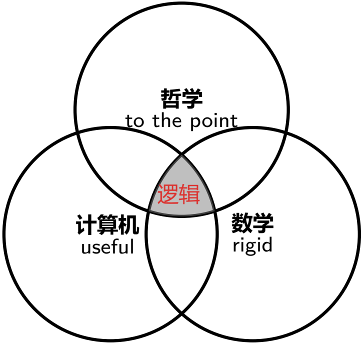
- Automata Theory
- Computational Complexity
- Finite Model Theory
- Model Checking
- Lambda Calculus
- Theorem Proving
- Description Logic
- Fixpoint Logic
- Dynamic Logic
- Linear Logic
- Temporal Logic
- Process Algebra
- Hoare Logic
- Inductive Logic
- Fuzzy Logic
- Non-monotonic Logic
- Computability Logic
- Default Logic
- Markov Logic Networks
- Situation/Event Calculus

Philosophical Logic

- Intuitionistic Logic
- Modal Logic
- Algebraic Logic
- Epistemic Logic
- Doxastic Logic
- Preference Logic
- Provability Logic
- Spatial Logic
- Justification Logic
- Hybrid Logic
- Substructural Logic
- Free Logic
- Counterfactual Logic
- Relevance Logic
- Quantum Logic
- Paraconsistent Logic
- Intensional Logic
- Partial Logic
- Diagrammatic Logic
- Deontic Logic

跨学科视角下的逻辑

- 逻辑学
  1. 就主题而言主要是哲学
  2. 就方法而言主要是数学
  3. 就应用而言主要是计算机科学
- 逻辑学家希望成为
  1. 哲学家的哲学家
  2. 数学家的数学家
  3. 计算机科学家的计算机科学家
- 却往往成为了
  1. 哲学家眼中的数学家
  2. 数学家眼中的计算机科学家
  3. 计算机科学家眼中的哲学家



Why Study Logic? — The glory of the human spirit!

- 逻辑是心灵的免疫系统.
- 世界不仅比我们想象的更古怪, 甚至比我们能够想象的更古怪.
- 逻辑可以扩展我们的抽象想象力, 重塑我们的心灵.
- 几行逻辑推理可能改变我们看待世界的方式.
- 从逻辑的观点看 (世界、语言与世界的关系、哲学……)
- 通往柏拉图理念世界的桥梁, 为了人类心智的荣耀!
- 逻辑自身就很有趣 — 理性的音乐.

- 鉴赏是视听艺术. 视读乐符, 演奏乐章.
- 创作是言说艺术. 言说什么? 如何言说?

世界逻辑日: 每年的 1 月 14 日

因为担心失衡跌倒,
我们的思想紧紧抓住逻辑
这个扶手.

— 纪德

时光飞逝, 勿浪费时间;
方法会教你赢得每一天;
年轻的朋友, 听我一句劝,
大学逻辑是起点!

— 歌德《浮士德》

"工欲善其事, 必先利其器."

— 孔子《论语》

"君子性非异也, 善假于物也."

— 荀子《劝学》

逻辑是笨人的学问

- 蠢人会把对方的智商拉低到自己的水平, 然后用丰富的经验打败他.
- 笨人会把对方的问题翻译为自己的逻辑语言, 然后用机械的逻辑工具暴力破解它 ☺
  - 学逻辑唯有 "笨办法", 要不得小聪明取不得巧

杀不死我的, 使我更强大!

— 尼采

Contents

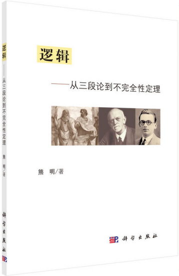
中文参考书

参考书 **熊明: 《逻辑 — 从三段论到不完全性定理》**

参考书 郝兆宽、杨睿之、杨跃: 《数理逻辑: 证明及其限度》

漫画书 Doxiadis, Papadimitriou: 《Logicomix 疯狂的罗素》

科普书 侯士达: 《哥德尔、艾舍尔、巴赫 — 集异璧之大成》



侯世达定律: 做事所花费的时间总是比你预期的要长, 即使你的预期中考虑了**侯世达定律**.

- Dangerous Knowledge 危险的知识
- The Imitation Game 模仿游戏

- libgen
- sci-hub

课程目标 & 大纲 & 成绩

课程目标:

- 论证的形式化表达 ♡
- 论证有效性的判定 ♡
- 数学哲学 🚲
- 形式认识论 🚲
- 逻辑在哲学、数学、计算机科学、人工智能、语言学、认知科学、物理学、信息论、博弈论、社会科学等领域的应用 🚲

大纲:

1. 命题逻辑 ♡
2. 谓词逻辑 ♡
3. 模态逻辑 🚲

平时成绩 40%、**期末成绩** 60%

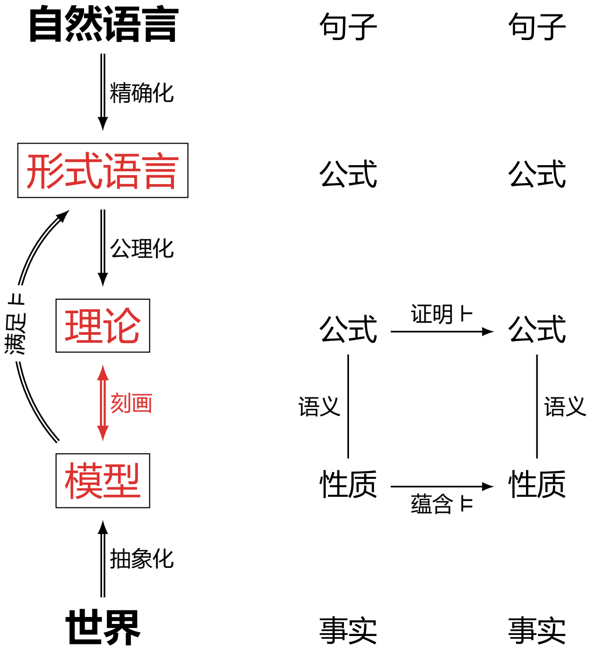
- 问答、讨论
- 作业 ⚡
- 练习 ⚡
- 考试 ⚡
- Paper
- Techniques e.g. LaTeX, Coq, Lean, Prover9, Vampire ...

课件记号的意思:

⚡ 考试相关

♡ 需要掌握

🚲 可略过


自然语言
精确化
形式语言
公理化
理论
刻画
模型
抽象化
世界
满足 ⊨
句子
句子
公式
公式
公式
证明 ⊢
公式
语义
语义
性质
蕴含 ⊨
性质
事实
事实

一个超级简单的 Toy Logic

- **语法:** 符号集 $\mathrm{Var} = \{X, Y, \ldots\}$, 公式 $X \to Y$
- **语义:** 结构 $\mathcal{M} := (M, [\![\,]\!])$, 其中 $[\![X]\!] \subset M$

$$\mathcal{M} \vDash X \to Y \quad \text{当且仅当} \quad [\![X]\!] \subset [\![Y]\!]$$

- 逻辑蕴涵: $\Gamma \vDash \varphi$ 当且仅当对任意结构 $\mathcal{M}$: 若 $\mathcal{M} \vDash \Gamma$ 则 $\mathcal{M} \vDash \varphi$
- **形式系统:**

$$\frac{X \to Y \quad Y \to Z}{X \to Z} \qquad \frac{}{X \to X}$$

- 证明: $\Gamma \vdash \varphi$ 当且仅当存在一棵以 φ 为 "根" 节点的有穷 "树", 其每一节点或是公理或属于前提 Γ, 或通过推理规则由前面的节点生成
- 例子: 怎么证明 $A \to B, B \to C, C \to D \vdash A \to D$

$$\cfrac{\cfrac{A \to B \quad B \to C}{A \to C} \qquad C \to D}{A \to D}$$

- **元定理:** 能推出的都有效? 有效的都能推出? $\Gamma \vdash \varphi \iff \Gamma \vDash \varphi$
- 思考一下: 怎么证明推不出? $A \to B \nvdash B \to A$

$$
\begin{tikzcd}
\text{形式语言} \arrow[d, dash, "\text{语义}" description] & A \to B, B \to C \arrow[rr, "\text{证明}"] \arrow[d, dash, "\text{语义}"'] & & A \to C \arrow[d, dash, "\text{语义}"] \\
\text{模型} & \llbracket A \rrbracket \subset \llbracket B \rrbracket, \llbracket B \rrbracket \subset \llbracket C \rrbracket \arrow[rr, "\text{蕴含}"] & & \llbracket A \rrbracket \subset \llbracket C \rrbracket
\end{tikzcd}
$$

Theorem (Toy Logic 的可靠性定理 🚲)

$\Gamma \vdash \varphi \implies \Gamma \vDash \varphi$

Proof.

对证明树 $\Gamma \vdash \varphi$ 线性排列, 对其证明长度应用数学归纳法.

- 长度为 1: $\varphi \in \Gamma$ 或 φ 是公理 $X \to X$ 的特例. 显然 $\Gamma \vDash \varphi$.
- 假设证明长度不超过 n 时都有 $\Gamma \vdash \varphi \implies \Gamma \vDash \varphi$, 下证长度为 $n+1$ 时也成立.
  第 $n+1$ 步或是公理或属于前提或由推理规则得到.
  若由推理规则 $\dfrac{X \to Y \quad Y \to Z}{X \to Z}$ 得到, 由归纳假设, $\Gamma \vDash X \to Y$ 且 $\Gamma \vDash Y \to Z$.
  即 $[\![X]\!] \subset [\![Y]\!]$ 且 $[\![Y]\!] \subset [\![Z]\!]$.
  所以 $[\![X]\!] \subset [\![Z]\!]$, 即 $\Gamma \vDash X \to Z$.

□

Remark: 公理有效, 推理规则保真

Theorem (Toy Logic 的完备性定理)

$\Gamma \vDash \varphi \implies \Gamma \vdash \varphi$

Proof.

- 不妨证其逆否命题: 假设 $\Gamma \nvdash \varphi$, 往证 $\Gamma \nvDash \varphi$,
  即, 找一个结构 $\mathcal{M}$, 使得 $\mathcal{M} \vDash \Gamma$ 但 $\mathcal{M} \nvDash \varphi$.
- 令 $\mathcal{M}^\Gamma := (M, [\![\,]\!])$, 其中

$$M := \mathrm{Var}, \quad [\![X]\!] := \{Y : \Gamma \vdash Y \to X\}$$

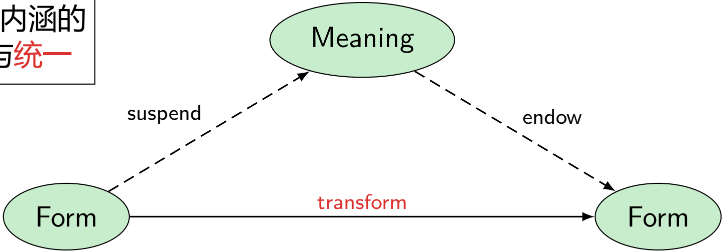
1. 检查 $\mathcal{M}^\Gamma \vDash \Gamma$.
   任给 $A \to B \in \Gamma$, 根据 $[\![A]\!], [\![B]\!]$ 的定义和 $\dfrac{Y \to A \quad A \to B}{Y \to B}$, 可得
   $Y \in [\![A]\!] \implies Y \in [\![B]\!]$, 即 $\mathcal{M}^\Gamma \vDash A \to B$.
2. 检查 $\mathcal{M}^\Gamma \nvDash \varphi$. (往证: $\mathcal{M}^\Gamma \vDash \varphi \implies \Gamma \vdash \varphi$, 从而与假设 $\Gamma \nvdash \varphi$ 矛盾)
   假设 $\varphi = A \to B$, 若 $\mathcal{M}^\Gamma \vDash \varphi$, 则 $[\![A]\!] \subset [\![B]\!]$.
   即, 对任意 Y, $Y \in [\![A]\!] \implies Y \in [\![B]\!]$.
   因为$\overline{A \to A}$, 所以 $A \in [\![A]\!]$. 从而 $A \in [\![B]\!]$. 因此 $\Gamma \vdash \varphi$.

逻辑求真

“真”为逻辑指引方向，正如“美”为美学、“善”为伦理指引方向.

— 弗雷格

形式与内涵的
分离与统一



数理逻辑是先于一切科学的科学，包含着位于一切科学底层的观念和原理.

— 哥德尔

一个概念是“逻辑”的，当且仅当，它对“论域”到自身的所有可能的一一变换都保持不变.

— 塔斯基

Homework ↯

1. 自选自然语言的句子, 将其翻译为逻辑语言.
   — 所选的句子既要有「趣味性」又要有一点儿「难度」, 可以是名人名言、名篇名句、古典诗词、歇后语、网络流行语等等, 自己作诗也欢迎 ☺
   — 判断所选的句子是否构成一个有效或无效的命题或论证, 给出证明或反模型.
2. 自选习题、自学笔记……

Google / Wikipedia / Stanford Encyclopedia / Internet Encyclopedia / StackExchange

Contents

Contents

莱布尼茨 Gottfried Wilhelm Leibniz 1646-1716

Don't argue. Let us Calculate!

- 最后的 "通才", 创立了单子论, 发展了微积分, 改进了二进制系统, 发明了能进行加减乘除四则运算的计算器.
- 被罗素、欧拉、哥德尔、维纳、曼德布洛特、鲁滨逊、柴汀等人认为是 数理逻辑[a]、拓扑学、博弈论、控制论、分形几何、非标准分析、算法信息论、计算主义哲学的先驱.
- 通用文字、理性演算

[a]Wolfgang Lenzen: Leibniz's Logic.

布尔 George Boole 1815-1864

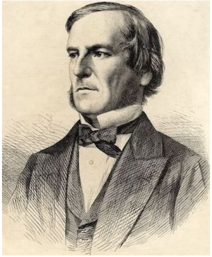


Figure: 自学成才的逻辑学家、数学家、哲学家、诗人

- 《思维规律》*1854.*
- 布尔代数
  命题逻辑
- 逻辑还原为代数

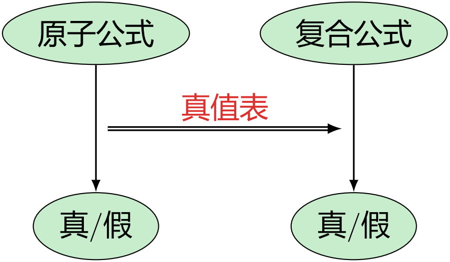
$$\begin{tikzcd}
\text{命题} \arrow[r, "\text{逻辑}", "\text{推导}"'] \arrow[d, "\text{翻译}"'] & \text{命题} \\
\text{方程} \arrow[r, "\text{计算}", "\text{代数}"'] & \text{方程} \arrow[u, "\text{翻译}"']
\end{tikzcd}$$

Contents

命题逻辑

- 语言
  原子公式、连接词
- 语法

- 语义



- 形式系统

语言

$$\mathscr{L}^0 := \{\neg, \wedge, \vee, \to, \leftrightarrow, (,)\} \cup \mathrm{Var}$$

其中 $\mathrm{Var} := \{p_1, p_2, p_3, \dots\}$.

Definition (公式 Well-Formed Formula Wff — Top Down)

1. 原子公式 $p \in \mathrm{Var}$ 是公式.
2. 如果 A 是公式, 则 $(\neg A)$ 也是公式.
3. 如果 A, B 是公式, 则 $(A \wedge B), (A \vee B), (A \to B), (A \leftrightarrow B)$ 也是公式.
4. 除此之外, 别无其他公式.

$$A := p \mid (\neg A) \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid (A \leftrightarrow A)$$

- $\bot := (A \wedge (\neg A))$
- $\top := (\neg \bot)$

Example

1. You are beautiful.
2. You are not beautiful.
3. You are beautiful inside and out.
4. You win, or you learn.
5. If wishes are horses, then beggars will ride.
6. You are beautiful if and only if you are smart.
7. If you want to get good, get your hands dirty and start solving problems.
8. 侠之大者, 为国为民.
9. 只要功夫深, 铁杵磨成针. / 砍头不要紧, 只要主义真.
10. 只有你请她才来. / 除非你请, 否则她不会来.
11. 不积跬步, 无以至千里. /没有共产党, 就没有新中国.
12. 不是你死, 就是我亡.
13. 鱼与熊掌不可兼得.
14. 欲寄君衣君不还, 不寄君衣君又寒. 寄与不寄间, 妾身千万难.

括号的作用

他手里不是有 K 或有黑桃的话就有 A.

$$\neg(K \vee S) \to A$$

$$((\neg K) \vee S) \to A$$

Definition (公式构造算子)

$$\begin{aligned} f_\neg(A) &:= (\neg A) \\ f_\wedge(A,B) &:= (A \wedge B) \\ f_\vee(A,B) &:= (A \vee B) \\ f_\to(A,B) &:= (A \to B) \\ f_\leftrightarrow(A,B) &:= (A \leftrightarrow B) \end{aligned}$$

$$\boxed{\begin{aligned} f_\neg(A) &:= \neg A \\ f_\wedge(A,B) &:= \wedge AB \\ f_\vee(A,B) &:= \vee AB \\ f_\to(A,B) &:= \to AB \\ f_\leftrightarrow(A,B) &:= \leftrightarrow AB \end{aligned}}$$

省略括号的约定 ♡

1. 公式最外层的括号可以省略

$$\frac{A \to B}{(A \to B)}$$

2. $\neg, \land, \lor, \to, \leftrightarrow$ 的组合强度依次减弱

$$\frac{\neg A \lor B \to C \land D}{((\neg A) \lor B) \to (C \land D)}$$

Remark: 类似先乘除后加减 $1 + 2 * 3 = ?$

3. 同一连接词相邻出现时, 右边的组合力更强

$$\frac{A \to B \to C \to D}{A \to (B \to (C \to D))}$$

生成 — 自下而上 vs 自上而下

Problem (生成 Generation)

给定集合 U 和初始的 $V \subset U$, 怎么通过 U 上的函数类 $\mathcal{F}$ 生成 U 的某个子集?

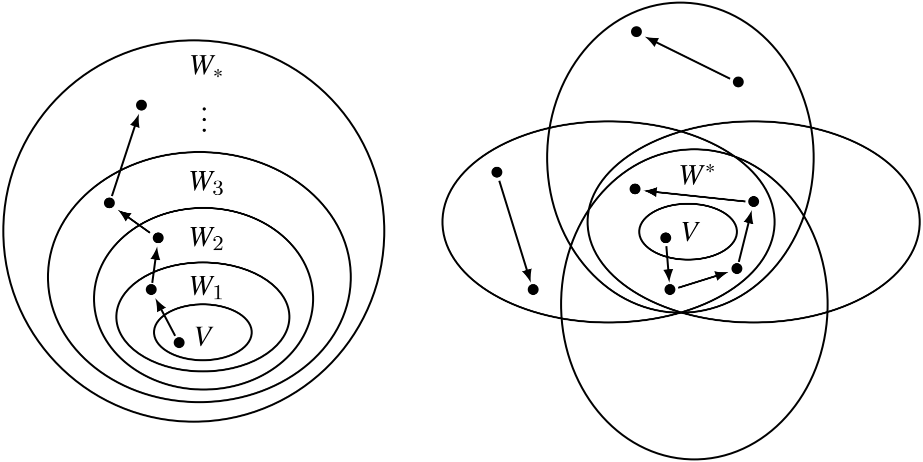
Definition (自下而上)

$$W_0 := V$$
$$W_{n+1} := W_n \cup \bigcup_{f \in \mathcal{F}} \{f(\mathbf{x}) : \mathbf{x} \in W_n\} \qquad \deg(\mathbf{x}) := \mu n\,[\mathbf{x} \in W_n]$$
$$W_* := \bigcup_{n \in \mathbb{N}} W_n$$

Definition (自上而下)

- 集合 S 对函数 f 封闭, 当且仅当, 对任意 $\mathbf{x}$, 若 $\mathbf{x} \in S$ 则 $f(\mathbf{x}) \in S$.
- 集合 S 是归纳集, 当且仅当, $V \subset S$ 且对任意 $f \in \mathcal{F}$: S 对 f 封闭.
- $W^* := \bigcap\{S : S\text{是归纳集}\}$

生成 — 自下而上 vs 自上而下 🚲



Example ($10 元钱可以喝多少瓶啤酒?)

- $2 元钱可以买 1 瓶啤酒.
- 4 个瓶盖可以换 1 瓶啤酒.
- 2 个空瓶也可以换 1 瓶啤酒.

$\frac{a_1}{1-\frac{3}{4}}$

公式上的归纳法

Theorem (公式上的归纳法)

令 P 是一个关于公式的性质. 假设

1. 所有原子公式都有性质 P.
2. 对任意公式 A, B, 若 A, B 有性质 P, 则 $\neg A, A \wedge B, A \vee B, A \to B, A \leftrightarrow B$ 都有性质 P.

那么, 所有公式都有性质 P.

Proof.

$$\mathrm{Wff}_* = \mathrm{Wff}^* \subset P$$

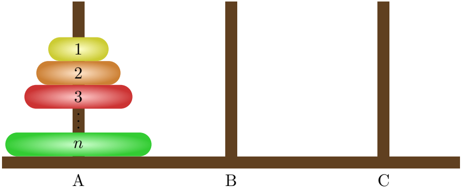
□

$$P(0) \wedge \forall n(P(n) \to P(n+1)) \to \forall n P(n)$$

$$P(n) := P(\mathrm{Wff}_n)$$

$$\dfrac{P(0) \qquad \begin{array}{c}[P(n)] \\ \vdots \\ P(n+1)\end{array}}{\forall n P(n)}$$

归纳 vs 递归



$P(n) :=$ "n rings needs $2^n - 1$ moves."

Example (怎么让送奶工天天留奶？)

1. 如果某天留奶, 那么第二天也留奶
2. 今天留奶

$\boxed{\text{今天留奶, 并且, 明天再读一遍这个字条}}$

子公式

Definition (子公式 — 归纳定义)

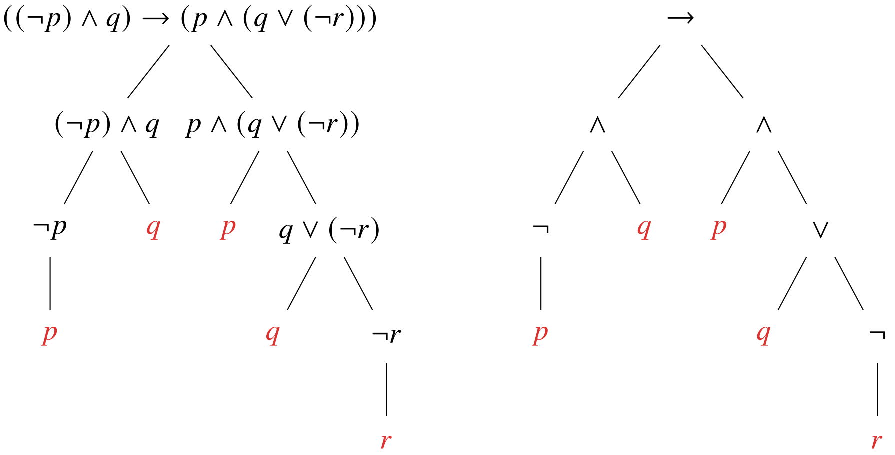
公式 A 的子公式集 $\mathrm{Sub}(A)$ 是满足以下条件的最小集合 Γ:

1. $A \in \Gamma$
2. $\neg B \in \Gamma \implies B \in \Gamma$
3. $B \to C \in \Gamma \implies B, C \in \Gamma$

Definition (子公式 — 递归定义)

$$\mathrm{Sub}(A) := \begin{cases} \{A\} & \text{if } A = p \\ \{A\} \cup \mathrm{Sub}(B) & \text{if } A = \neg B \\ \{A\} \cup \mathrm{Sub}(B) \cup \mathrm{Sub}(C) & \text{if } A = B \to C \end{cases}$$

唯一可读性 = 唯一语法树



- 主连接词：树根
- 子公式：子树

翻译技巧 ♡

- **Negation 否定:** not / it is false that ...
  - Jay and Kay are married, but not to each other. $J \wedge K \wedge \neg M$
- **Conjunction 合取:** and / moreover / furthermore / but / yet / although / though / even though / however / whereas
- **Disjunction 析取:** or / either or
- **Implies 蕴含/只要就:** "if then" / provided that / in case / on the condition that
- **Neither Nor 既不也不:** negation of "either or"
- **Only If 仅当/只有才:** "in order that ...it is necessary that ..."
- **Unless 除非:** if not / if and only if not
  1. I will not graduate unless I pass logic. ¬PassLogic→¬Graduate
  2. The store is open unless/except it is sunday. Open↔¬Sunday
- **Even If 即使:**
  - I'm going to the party even if it rains. $(\neg R \to P) \wedge (R \to P) \equiv P$
  - The Allies would have won even if the U.S. had not entered the war.[1] $W \wedge (\neg E \to W) \equiv W$
  - The Axis powers would have won if the U.S. had not entered the war. $\neg W \wedge (\neg E \to W)$

[1] 虚拟语气的表达需要借助 Counterfactual Logic.

Example ☹

1. The programmer's wife tells him: "Run to the store and pick up a loaf of bread. If they have eggs, get a dozen."
2. The programmer comes home with 12 loaves of bread.
3. "Why did you buy 12 loaves of bread!?", his wife screamed.
4. "Because they had eggs!"

- wife.

$$\text{1Bread} \wedge (\text{Egg} \rightarrow \text{12Egg})$$

- programmer.

$$(\neg\text{Egg} \rightarrow \text{1Bread}) \wedge (\text{Egg} \rightarrow \text{12Bread})$$

练习: 翻译 — Now it's your turn ⚡

1. 我不擅表达.
2. 如果你表述不清楚, 那么你就没搞明白.
3. 只有你能把它形式化, 才算你真正理解了.
4. 你和我, 都没道理.
5. 一门科学只有在成功地运用数学时, 才算发展成熟了. — 马克思
6. 小艾非蠢即坏.
7. 如果你只有在受到威胁时才努力工作, 那么你是不会成功的.
8. 只要你不笨, 你就会通过考试, 除非你偷懒.
9. 如果, 只有开除小白, 小艾才会认真工作, 那么, 为了顺利完成工作, 就要开除小白.
10. 如果小艾和小白都没有档期, 为了电影顺利开拍, 必须启用新人.
11. 如果你做了承诺又不遵守, 那么我会生气的, 除非你有好的借口.
12. 我要打篮球, 除非天下雨; 那样的话, 我就打乒乓球或游泳.
13. 只要天晴且不冷, 我就打篮球; 否则, 我就打乒乓球或游泳.
14. 考试拿 100 分是优秀的必要但不充分条件.

Digression — 为什么要学一门 “形式语言”?

逻辑**仅仅**是一门语言吗?

- 语言是人与其他动物最大的区别.
- 阿拉伯数字 vs 汉语数字: 仅仅是语言的区别吗?
- 你甚至可以将 DNA 视为一种语言: 一种程序语言.

怎么以**简洁的**方式**准确地**表达我们的思想?

什么是理性的**范围与限度**?

“少于十八个汉字不可**定义**的最小自然数.”

— *Berry* 悖论

Contents

真值赋值 Truth Assignment & 真值表 Truth Table ♡

- 真值赋值是一个从原子公式到真、假的函数

$$\nu : \mathrm{Var} \longrightarrow \{0,1\}$$

- 真值赋值 ν 可以在满足下列条件的情况下递归地扩展到所有公式上 $\nu : \mathrm{Wff} \longrightarrow \{0,1\}$.

p	$\neg p$
0	1
1	0

p	q	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

$$\frac{\nu(A \to B) = 1}{\nu(A) \leq \nu(B)}$$

$$\begin{tikzcd} 0 \arrow[loop above] \arrow[r, "\leq"] & 1 \arrow[loop above] \end{tikzcd}$$

$$B^A$$

$$A \to B$$

充分 $\to$ 必要

Example:

- 如果 $0 = 1$, 那么罗素是上帝.
- 雪是白的当且仅当 $1 + 1 = 2$.

Example

令 $v(p)=1, v(q)=0, v(r)=1$, 求 $v(p \vee r \to \neg(p \to q))$.

p	q	r	$p \vee r$	$p \to q$	$\neg(p \to q)$	$p \vee r \to \neg(p \to q)$
1	0	1	1	0	1	1

Remark: 在不引起歧义的情况下, 我们经常省略 v 不写.

$$
\begin{aligned}
p \vee r \to \neg(p \to q) &= 1 \vee 1 \to \neg(1 \to 0) \\
&= 1 \to \neg 0 \\
&= 1 \to 1 \\
&= 1
\end{aligned}
$$

“实质蕴含” — 反直觉吗?

翻哪几张卡才能验证 “如果一面是偶数, 另一面肯定是红色”?

未满 18 岁禁止饮酒!

p	q	$p \to q$	如果 x 是有理数, 那么 x^2 是有理数
0	0	1	$x = \pi$
0	1	1	$x = \sqrt{2}$
1	0	0	
1	1	1	$x = \frac{1}{2}$

“实质蕴含”

Problem (辩护律师靠谱吗？)

某人因涉嫌参与盗窃而受审.

检察官　如果被告有罪, 那么他有搭档伙同作案.

辩护律师　这不是实话!

Example:

记者　如果你有一百万, 你愿意捐给国家吗?
农民　我愿意.
记者　如果你有一头牛, 你愿意捐给国家吗?
农民　我不愿意.
记者　为什么?
农民　我真的有一头牛!

Remark

- “如果我不来上课, 那么宇宙会爆炸.” — “反事实蕴含”
- 数学语境中的“蕴含”都是“实质蕴含”.

Find the Error ☺

怎么解方程 $y + 2 = y$?

$$
\begin{aligned}
y + 2 = y &\implies (y+2)^2 = y^2 \\
&\implies y^2 + 4y + 4 = y^2 \\
&\implies 4y + 4 = 0 \\
&\implies y = -1
\end{aligned}
$$

真值赋值、真值表[2]

如果雪是白的, 那么, 窦娥是杀人犯蕴含雪是白的.

	p	q	$q \to p$	$p \to q \to p$
v_1	0	0	1	1
v_2	0	1	0	1
v_3	1	0	1	1
v_4	1	1	1	1

n 个原子命题有 2^n 种真值赋值.

<table>
<tr><td>永真
重言
$A \to A$</td><td>偶真/偶假
(综合命题?)
$A \to B$</td><td>永假
矛盾
$A \leftrightarrow \neg A$</td></tr>
<tr><td>有效</td><td colspan="2">无效</td></tr>
<tr><td colspan="2">可满足</td><td>不可满足</td></tr>
</table>

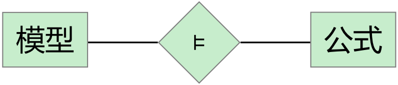
[2] Truth Table Generator

可满足、逻辑蕴含、有效式、逻辑等价 ♡

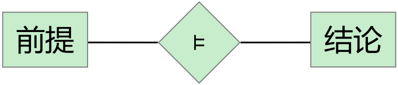
- 可满足: 公式 A 是可满足的, 当且仅当, 存在赋值 ν 使得 $\nu(A) = 1$. (我们通常把 $\nu(A) = 1$ 记为 $\nu \vDash A$)
- 逻辑蕴含: $\Gamma \vDash A$ 当且仅当, 对任意 ν: 若 $\nu \vDash \Gamma$ 则 $\nu \vDash A$. (其中, $\nu \vDash \Gamma$ 指: 对任意 $A \in \Gamma$ 都有 $\nu \vDash A$)
- 有效式/重言式: 公式 A 是有效的 $\vDash A$ 当且仅当 $\emptyset \vDash A$, 换句话说, 对任意赋值 ν 都有 $\nu \vDash A$.
- 逻辑等价: $A \equiv B$ 当且仅当, $A \vDash B$ 且 $B \vDash A$.

逻辑蕴含	有效	可满足	逻辑等价
entailment	validity	satisfiability	logical equivalence
$A \vDash B$	$\vDash A \to B$	$A \wedge \neg B$ 不可满足	$A \to B \equiv \top$
$\top \vDash A$	$\vDash A$	$\neg A$ 不可满足	$A \equiv \top$
$A \nvDash \bot$	$\nvDash \neg A$	A 可满足	$\neg A \not\equiv \top$
$A \vDash B$ 且 $B \vDash A$	$\vDash A \leftrightarrow B$	$A \leftrightarrow \neg B$ 不可满足	$A \equiv B$

满足 $\nu \vDash A$



逻辑蕴含 $\Gamma \vDash A$



直观理解	形式定义
可能世界/模型	真值赋值
连接词的语义	真值表
在所有可能世界上都真的命题	重言式/有效式
有效 (“保真” 的) 论证	逻辑蕴含

Remark

- 二值原则: 每个命题有且只有两个真值中的一个: 0 或 1
- 组合原则: 复合命题的真值由组成它的命题的真值以及这些命题的组合方式唯一确定.

$$\frac{\nu(A) = \mu(A)}{\nu(\neg A) = \mu(\neg A)} \qquad \frac{\nu(A) = \mu(A) \quad \nu(B) = \mu(B)}{\nu(A \to B) = \mu(A \to B)}$$

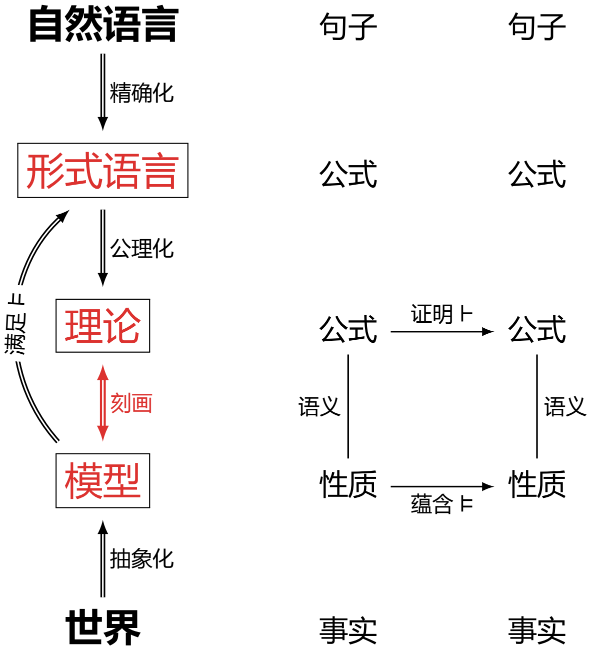
Example

证明 $p \wedge (\neg q \vee \neg r) \to (p \to \neg q)$ 既不是矛盾式也不是有效式.

p	q	r	$p \wedge (\neg q \vee \neg r) \to (p \to \neg q)$
0	0	0	1
1	1	0	0

$$
\begin{aligned}
p \wedge (\neg q \vee \neg r) \to (p \to \neg q) &= 0 \wedge (\neg 0 \vee \neg 0) \to (0 \to \neg 0) \\
&= 0 \wedge (1 \vee 1) \to (0 \to 1) \\
&= 0 \wedge 1 \to 1 \\
&= 0 \to 1 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
p \wedge (\neg q \vee \neg r) \to (p \to \neg q) &= 1 \wedge (\neg 1 \vee \neg 0) \to (1 \to \neg 1) \\
&= 1 \wedge (0 \vee 1) \to (1 \to 0) \\
&= 1 \wedge 1 \to 0 \\
&= 1 \to 0 \\
&= 0
\end{aligned}
$$


自然语言
精确化
形式语言
公理化
满足 ⊨
理论
刻画
模型
抽象化
世界
句子
句子
公式
公式
公式
证明 ⊢
公式
语义
语义
性质
蕴含 ⊨
性质
事实
事实

题外话

维特根斯坦 — 逻辑原子主义哲学

- 世界是事实的总和. 语言是命题的总和.
- 事实是事态的存在. 原子事态是对象的结合. 原子事态彼此独立.
- 命题是事实的图像.
  — 原子命题对应原子事态. 复合命题对应复杂事态.
- 复合命题是原子命题的真值函数.
- 图像与其描绘的实在拥有共同的逻辑结构.
  — 唱片、音乐思想、乐谱、声波, 彼此处在图示的内在关系中, 这也是语言和世界的关系.
- **理解一个命题, 意味着知道其为真的情形.**
- 真命题的总和是世界的图像.

Remark: 在翻译自然语言时, 我们根据需要来设定原子命题, 尽量保证原子命题彼此独立.

每个命题对应
一类可能世界

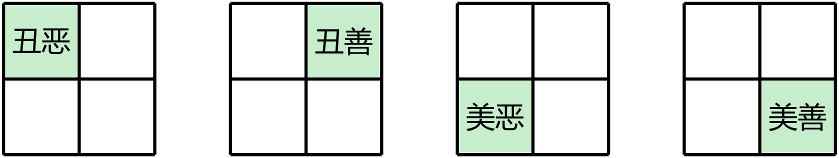
$v : \mathrm{Var} \to \{0,1\}$

$[\![A]\!] = \{v : v \vDash A\}$

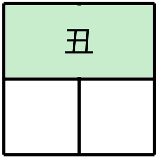
- p: 人美
- q: 心善

		$q=0$	$q=1$
p	0	v_1	v_2
	1	v_3	v_4

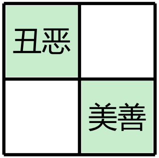
	p	q
v_1	0	0
v_2	0	1
v_3	1	0
v_4	1	1



$$\frac{(\neg p \wedge \neg q) \vee (\neg p \wedge q)}{\neg p}$$



$p \leftrightarrow q$

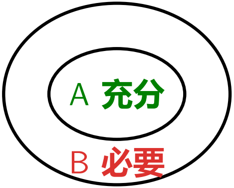


真值赋值 $v : \mathrm{Var} \to \{0, 1\}$ 可以在满足下列条件的情况下递归地扩展到所有公式上:

1. $[\![p]\!] := \{v : v(p) = 1\}$
2. $[\![\top]\!] := \{0, 1\}^{\mathrm{Var}}$
3. $[\![\bot]\!] := \emptyset$
4. $[\![\neg A]\!] := \overline{[\![A]\!]}$
5. $[\![A \wedge B]\!] := [\![A]\!] \cap [\![B]\!]$
6. $[\![A \vee B]\!] := [\![A]\!] \cup [\![B]\!]$
7. $[\![A \to B]\!] := \overline{[\![A]\!]} \cup [\![B]\!]$
8. $[\![A \leftrightarrow B]\!] := \overline{[\![A]\!] \Delta [\![B]\!]}$

1. $[\![p]\!] := v(p)$
2. $[\![\top]\!] := 1$
3. $[\![\bot]\!] := 0$
4. $[\![\neg A]\!] := 1 - [\![A]\!]$
5. $[\![A \wedge B]\!] := \min\left\{[\![A]\!], [\![B]\!]\right\}$
6. $[\![A \vee B]\!] := \max\left\{[\![A]\!], [\![B]\!]\right\}$
7. $[\![A \to B]\!] := \max\left\{1 - [\![A]\!], [\![B]\!]\right\}$
8. $[\![A \leftrightarrow B]\!] := 1 - \left|[\![A]\!] - [\![B]\!]\right|$

$$\begin{array}{c} \models A \to B \\ \hline\hline A \models B \\ \hline\hline [\![A]\!] \subset [\![B]\!] \end{array}$$



$$\begin{array}{c} \models A \to B \\ \hline\hline A \models B \\ \hline\hline [\![A]\!] \leq [\![B]\!] \end{array}$$

$$\begin{tikzcd} 0 \arrow[loop above] \arrow[r, "\leq"] & 1 \arrow[loop above] \end{tikzcd}$$

你的论证有效 (“保真”) 吗? ♡

问: 一个从前提集 $\{A_1, \ldots, A_n\}$ 到结论 B 的论证

$$\frac{A_1, \ldots, A_n}{B}$$

怎么才算有效?

答: 逻辑蕴含即有效

$$A_1, \ldots, A_n \vDash B$$

1. 如果前提 $A_1, \ldots, A_n$ 为真, 那么结论 B 必为真
2. 如果结论 B 为假, 那么至少有一个前提 A_i 为假

$$\frac{\text{如果猪会飞, 那么这瓜保熟} \quad \text{猪会飞}}{\text{这瓜保熟}}$$

论证保真

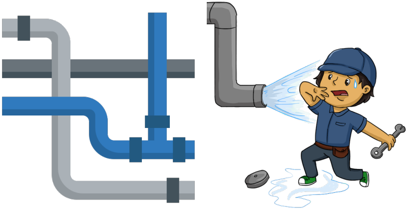
你的论证有效 ("保真") 吗? — Example

$$\frac{\text{整数 } x \text{ 或是奇数或是偶数} \quad \text{如果 } x \text{ 是奇数, 那么 } x+x \text{ 是偶数} \quad \text{如果 } x \text{ 是偶数, 那么 } x+x \text{ 是偶数}}{x+x \text{ 是偶数}}$$

$$\frac{A \vee B \quad A \to C \quad B \to C}{C}$$

$$A \vee B, A \to C, B \to C \vDash C$$

兴, 百姓苦; 亡, 百姓苦.



逻辑学家是干啥的? — 水管工!

组装密封水管, 保证干净的水能到达我们想要的地方.

有效论证 — Example ☹

- 克莱因是教授, 因此, 克莱因或者是教授或者是连环杀手.

$$\frac{A}{A \vee B}$$

- 琳达, 31 岁, 单身, 一位直率又聪明的女士, 大学主修哲学. 在学生时代, 她就关心种族歧视和社会公正问题, 还参加了反核示威游行.
  1. 琳达是银行出纳.
  2. 琳达是银行出纳, 还是女权主义者.

$$\frac{A \wedge B}{A}$$

有效性判定 — 真值表

$$\vDash (p \to q \to r) \to (p \to q) \to p \to r$$

p	q	r	$q \to r$	$p \to q \to r$	$p \to q$	$p \to r$	$(p \to q) \to p \to r$	$(p \to q \to r) \to (p \to q) \to p \to r$
0	0	0						1
0	0	1						1
0	1	0						1
0	1	1						1
1	0	0						1
1	0	1						1
1	1	0						1
1	1	1						1

$$p \vee q, p \to r, q \to r \vDash r$$

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \wedge (p \to r) \wedge (q \to r) \to r$
0	0	0		1	1	1
0	0	1		1	1	1
0	1	0	1	1		1
0	1	**1**	1	1	1	1
1	0	0	1		1	1
1	0	**1**	1	1	1	1
1	1	0	1			1
1	1	1	1	1	1	1

$$(p \to q \to r) \to (p \to q) \to p \to r$$

p	$\to$	q	$\to$	r	$\to$	p	$\to$	q	$\to$	p	$\to$	r
					$\underline{\mathbf{0}}$							
	1				$\mathbf{0}$				$\underline{\mathbf{0}}$			
	1				$\mathbf{0}$		1		$\mathbf{0}$		$\underline{\mathbf{0}}$	
1	1			0	$\mathbf{0}$	1	$\underline{\mathbf{1}}$		$\mathbf{0}$	1	$\mathbf{0}$	0
1	$\underline{\mathbf{1}}$	1		0	$\mathbf{0}$	1	$\mathbf{1}$	1	$\mathbf{0}$	1	$\mathbf{0}$	0
1	$\mathbf{1}$	1	$\underline{\mathbf{1}}$	0	$\mathbf{0}$	1	$\mathbf{1}$	1	$\mathbf{0}$	1	$\mathbf{0}$	0
			×									

有效式的判定 — 合取范式 CNF

$$\begin{aligned}
A \leftrightarrow B &\equiv (\neg A \vee B) \wedge (A \vee \neg B) \\
A \rightarrow B &\equiv \neg A \vee B \\
\neg(A \vee B) &\equiv \neg A \wedge \neg B \\
\neg(A \wedge B) &\equiv \neg A \vee \neg B \\
\neg\neg A &\equiv A \\
(A \wedge B) \vee C &\equiv (A \vee C) \wedge (B \vee C) \\
C \vee (A \wedge B) &\equiv (C \vee A) \wedge (C \vee B)
\end{aligned}$$

Example:

$$\begin{aligned}
p \vee q \rightarrow q \vee r &\equiv \neg(p \vee q) \vee (q \vee r) && \text{Elim} \rightarrow \\
&\equiv (\neg p \wedge \neg q) \vee (q \vee r) && \text{Push } \neg \text{ in} \\
&\equiv (\neg p \vee q \vee r) \wedge (\neg q \vee q \vee r) && \text{Push } \vee \text{ in} \\
&\equiv \neg p \vee q \vee r && \text{Simplify}
\end{aligned}$$

令 $p = 1, q = 0, r = 0$, 则 $p \vee q \rightarrow q \vee r = 0$, 故无效.

练习: 有效式的判定 — Now it's your turn ↯

1. $\neg\neg A \to A$ 双重否定消去
2. $A \to \neg\neg A$ 双重否定引入
3. $A \vee \neg A$ 排中律
4. $\neg(A \wedge \neg A)$ 无矛盾律
5. $A \wedge \neg A \to B$ 爆炸律
6. $(A \to B) \wedge (\neg A \to B) \to B$ 二难推理
7. $(A \to B) \wedge (A \to \neg B) \to \neg A$ 归谬法
8. $(\neg A \to B) \wedge (\neg A \to \neg B) \to A$ 反证法
9. $(\neg A \to \bot) \to A$
10. $(\neg A \to A) \to A$
11. $((A \to B) \to A) \to A$ Peirce 律

Problem (阿基米德有支点吗？)

大力士 听说 "如果你有一个支点, 就能翘起地球", 这是真的吗?

阿基米德 如果这是真的, 那么我有一个支点.

有效论证的判定 — Example: $\dfrac{A \vee B \quad A \to C \quad B \to C}{C}$

Proof.1.

画真值表. $\square$

Proof.2.

- 往证: 对任意 v, 若 $v \vDash A \vee B$ 且 $v \vDash A \to C$ 且 $v \vDash B \to C$, 则 $v \vDash C$.
- 假设 $v \vDash A \vee B$, 则 $v \vDash A$ 或 $v \vDash B$.
- 若 $v \vDash A$, 则由 $v \vDash A \to C$ 可得 $v \vDash C$.
- 若 $v \vDash B$, 则由 $v \vDash B \to C$ 可得 $v \vDash C$.

$\square$

Proof.3.

$$
\begin{aligned}
(A \vee B) \wedge (A \to C) \wedge (B \to C) \to C &\equiv \neg[(A \vee B) \wedge (\neg A \vee C) \wedge (\neg B \vee C)] \vee C \\
&\equiv \neg(A \vee B) \vee \neg(\neg A \vee C) \vee \neg(\neg B \vee C) \vee C \\
&\equiv (\neg A \wedge \neg B) \vee (A \wedge \neg C) \vee (B \wedge \neg C) \vee C \\
&\equiv \dots \\
&\equiv \top
\end{aligned}
$$

练习: 有效论证的判定 — Now it's your turn ⚡

$$\frac{A \vee B}{\neg A \to B} \qquad \frac{A \wedge B}{\neg(A \to \neg B)} \qquad \frac{A \leftrightarrow B}{(A \to B) \wedge (B \to A)} \qquad \frac{A \vee B}{(A \to B) \to B}$$

$$\frac{A \vee (B \vee C)}{(A \vee B) \vee C} \qquad \frac{A \wedge (B \wedge C)}{(A \wedge B) \wedge C}$$ 结合律

$$\frac{A \vee B}{B \vee A} \qquad \frac{A \wedge B}{B \wedge A}$$ 交换律

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B} \qquad \frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$ 德摩根律

$$\frac{A \vee (A \wedge B)}{A} \qquad \frac{A \wedge (A \vee B)}{A}$$ 吸收律

$$\frac{A \wedge (B \vee C)}{(A \wedge B) \vee (A \wedge C)} \qquad \frac{A \vee (B \wedge C)}{(A \vee B) \wedge (A \vee C)}$$ 分配律

$$\frac{A \vee B \qquad \neg B \vee C}{A \vee C} \qquad \frac{\Gamma, A \vDash B}{\Gamma \vDash A \to B}$$

无效论证

$$\frac{A \to B \quad \neg A}{\neg B} \qquad \frac{A \to B \quad B}{A} \qquad \frac{A \vee B \quad A}{\neg B} \qquad \frac{\neg(A \wedge B) \quad \neg A}{B}$$

我思故我在 我不思	葡萄酸故我不吃 我不吃	此论证或是有效的或是无效的 此论证是有效的
故我不在	故葡萄酸	此论证不是无效的

- 家里有一头猪和一头驴, 你说我是杀猪呢, 还是杀驴呢?
- 杀驴.
- 猪也是这么想的.

妈妈: "你只要把西蓝花吃了就可以去吃冰激凌了."
儿子: "我把猪蹄吃了可以去吃冰激凌吗?"
妈妈: "你只有把西蓝花吃了才能去吃冰激凌!"

> *By all means marry; if you get a good wife, you'll be happy. If you get a bad one, you'll become a philosopher.* — Socrates

$$\frac{\begin{aligned}\text{Marry} &\leftrightarrow \text{GoodWife} \vee \text{BadWife}\\ \text{GoodWife} &\to \text{HappyLife}\\ \text{BadWife} &\to \text{Philosopher}\\ \text{HappyLife} &\to \text{PerfectEnding}\\ \text{Philosopher} &\to \text{PerfectEnding}\\ &\text{PerfectEnding}\end{aligned}}{\text{Marry}}$$

令 $\text{Marry} = \text{GoodWife} = \text{BadWife} = 0$ 且 $\text{HappyLife} = \text{Philosopher} = \text{PerfectEnding} = 1$.

$$\text{则} \quad \begin{aligned}\text{Marry} \leftrightarrow \text{GoodWife} \vee \text{BadWife} &= 1\\ \text{GoodWife} \to \text{HappyLife} &= 1\\ \text{BadWife} \to \text{Philosopher} &= 1\\ \text{HappyLife} \to \text{PerfectEnding} &= 1\\ \text{Philosopher} \to \text{PerfectEnding} &= 1\\ \text{PerfectEnding} &= 1\end{aligned} \quad \text{但} \quad \text{Marry} = 0$$

梁实秋 vs 鲁迅

说我是资本家的走狗，是哪一个资本家，还是所有的资本家？我还不知道我的主子是谁．

— 梁实秋《资本家的走狗》

凡走狗，虽或为一个资本家所豢养，其实是属于所有的资本家的，所以它遇见所有的阔人都驯良，遇见所有的穷人都狂吠．
不知道谁是它的主子，正是它遇见所有阔人都驯良的原因，也就是属于所有的资本家的证据．
即使无人豢养，饿的精瘦，变成野狗了，但还是遇见所有的阔人都驯良，遇见所有的穷人都狂吠的，不过这时它就愈不明白谁是主子了．

— 鲁迅《丧家的资本家的乏走狗》

$$\begin{array}{c} \text{有主子豢养} \to \text{是走狗} \\ \text{无主子豢养} \\ \hline \text{不是走狗} \end{array} \quad \times$$

两小儿辩日

《列子 · 汤问》

孔子东游, 见两小儿辩斗, 问其故.

- 一儿曰: “我以日始出时去人近, 而日中时远也.”
- 一儿曰: “我以日初出远, 而日中时近也.”
- 一儿曰: “日初出大如车盖, 及日中则如盘盂, 此不为远者小而近者大乎?”
- 一儿曰: “日初出沧沧凉凉, 及其日中如探汤, 此不为近者热而远者凉乎?”
- 孔子不能决也.
- 两小儿笑曰: “孰为汝多知乎?”

$$\frac{\text{日出} \to \text{大}, \text{日中} \to \text{小} \quad \text{大者} \to \text{近}, \text{小者} \to \text{远}}{\text{日出} \to \text{近}, \text{日中} \to \text{远}}$$

$$\frac{\text{日出} \to \text{凉}, \text{日中} \to \text{热} \quad \text{凉者} \to \text{远}, \text{热者} \to \text{近}}{\text{日出} \to \text{远}, \text{日中} \to \text{近}}$$

Example

明·浮白斋主人《雅谑》

叶衡罢相归，一日病，问诸客曰："我且死，但未知死后佳否?" 一士曰："甚佳". 叶惊问曰："何以知之?" 答曰：**"使死而不佳，死者皆逃回矣．一死不返，以是知其佳也."**

$$\frac{\neg\text{佳} \rightarrow \text{回} \qquad \neg\text{回}}{\text{佳}}$$

好货不贱，贱货不好．
$$\frac{\text{好} \rightarrow \neg\text{贱}}{\text{贱} \rightarrow \neg\text{好}}$$

痞子蔡《第一次的亲密接触》

1. 如果把整个太平洋的水倒出，也浇不灭我对你爱情的火焰．整个太平洋的水倒得出吗? 不行．所以，我不爱你．
2. 如果把整个浴缸的水倒出，也浇不灭我对你爱情的火焰．整个浴缸的水倒得出吗? 可以．所以，是的，我爱你．

二难推理

- 如果你工作, 就能挣钱; 如果你赋闲在家, 就能悠然自在. 你或者工作或者赋闲, 总之, 你或者能挣钱或者能悠然自在.
- 如果你工作, 就不能悠然自在; 如果你赋闲在家, 就不能挣钱. 你或者工作或者赋闲, 总之, 你或者不能悠然自在或者不能挣钱.

$$\frac{\begin{array}{c} A \rightarrow C \\ B \rightarrow D \\ A \vee B \end{array}}{C \vee D} \qquad \frac{\begin{array}{c} A \rightarrow \neg D \\ B \rightarrow \neg C \\ A \vee B \end{array}}{\neg D \vee \neg C}$$

- 老婆婆有俩儿子, 老大卖阳伞, 老二卖雨伞, 晴天雨伞不好卖, 雨天阳伞不好卖……
- 被困失火的高楼, 走楼梯会被烧死, 跳窗会摔死……

蒋介石

反腐, 亡党; 不反, 亡国.

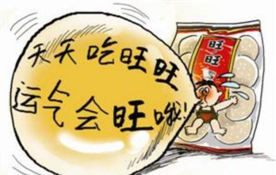
¬ 上帝万能

上帝能否创造一块自己举不起来的石头?

诉讼悖论

- 曾有师生签订合同: 上学期间不收费, 学生毕业打赢第一场官司后交学费.
- 可学生毕业后并未从事律师职业, 于是老师威胁起诉学生.
- 老师说: 如果我赢了, 根据法庭判决, 你必须交学费; 如果你赢了, 根据合同, 你也必须交学费. 要么我赢要么你赢, 你都必须交学费.
- 学生说: 如果我赢了, 根据法庭判决, 我不用交学费; 如果你赢了, 根据合同, 我不用交学费. 要么我赢要么你赢, 我都不用交学费.

$$\frac{\begin{array}{c} W \to P \\ \neg W \to P \\ W \vee \neg W \end{array}}{P} \qquad \frac{\begin{array}{c} W \wedge J \to P \\ \neg W \wedge C \to P \\ W \vee \neg W \end{array}}{P}\ ? \qquad \frac{\begin{array}{c} \neg W \wedge J \to \neg P \\ W \wedge C \to \neg P \\ W \vee \neg W \end{array}}{\neg P}\ ? \qquad \frac{\begin{array}{c} W \wedge J \to P \\ \neg W \wedge C \to P \\ (W \wedge J) \vee (\neg W \wedge C) \end{array}}{P}$$



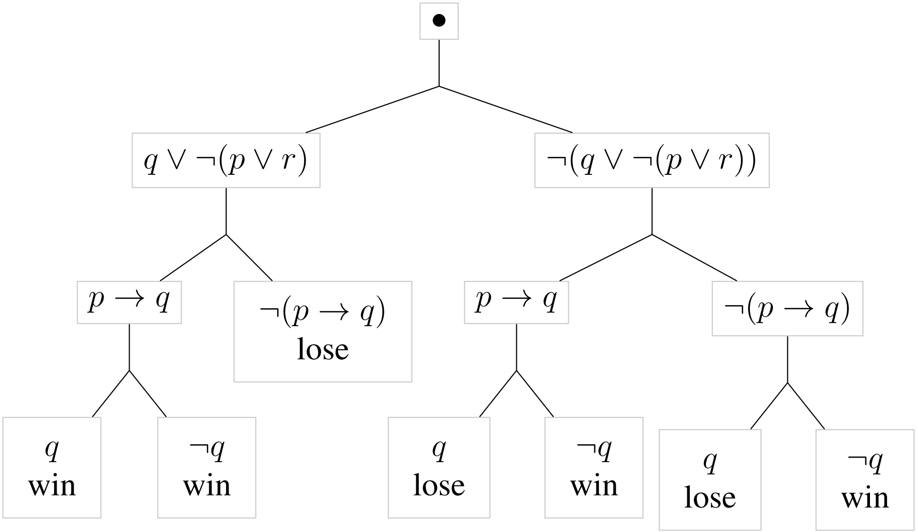
中世纪的逻辑考试

- 老师通过一种 n 轮的游戏来测试学生的逻辑水平.
- 每一轮中老师会给出一个命题 A_i.
- 学生必须选择 “接受” 或 “拒绝” 该命题.
- 如果接受该命题, 则将 A_i 放入已有命题的集合, 否则将 $\neg A_i$ 放入.
- 如果放入后的命题集合矛盾了, 则学生失败. 如果 n 轮后得到命题集合仍然没有矛盾, 学生通过测试.

Remark: 如果学生看到了老师的试题库 $\{A_1, A_2, \ldots, A_n\}$, 怎么提前备考?

比如：假设老师的出题顺序是

1. $q \vee \neg(p \vee r)$
2. $p \to q$
3. q



反解真值表

Problem (“君子” 与 “小人”)

一个岛上有“君子”、“小人”两类人. “君子”只说真话, “小人”只说假话.

1. 什么话“君子”可以说, 但“小人”不可以说?
2. 什么话“小人”可以说, 但“君子”不可以说?
3. 什么话“君子”和“小人”都可以说?
4. 什么话“君子”和“小人”都不可以说?

- p: 我是君子

身份	p	可说 x 吗?	x
小人	0	0	$v_1(x) = 1$
君子	1	0	$v_2(x) = 0$

$$x = \neg p$$

反解真值表

Problem (招驸马)

一个岛上有“君子”、“小人”两类人．“君子”只说真话，“小人”只说假话．他们有的富有的穷．国王现在欲招一位“穷君子”做驸马．“穷君子”可以说一句什么话证明自己的身份？

- p：我是穷人
- q：我是君子

身份	p	q	可说 x 吗?	x
富小人	0	0	0	$v_1(x) = 1$
富君子	0	1	0	$v_2(x) = 0$
穷小人	1	0	0	$v_3(x) = 1$
穷君子	1	1	1	$v_4(x) = 1$

$p \backslash q$	0	1
0	1	0
1	1	1

$$x = \neg(\neg p \wedge q) \equiv p \vee \neg q \equiv q \to p \equiv \cdots$$

Remark: 析取/合取范式 $x = \bigvee_{k:v_k(x)=1} \bigwedge_{i=1}^{n} p_i^{v_k(p_i)} \equiv \bigwedge_{k:v_k(x)=0} \bigvee_{i=1}^{n} p_i^{1-v_k(p_i)}$

where $p^1 := p$, $p^0 := \neg p$.

可能世界集 vs 语言

ab \ cd	00	01	11	10
00	0	0	1	1
01	1	1	1	1
11	1	1	1	1
10	1	1	0	0

$$(\neg a \wedge c) \vee (a \wedge \neg c) \vee b$$

ab \ cd	00	01	11	10
00	0	0	1	1
01	1	1	1	1
11	1	1	1	1
10	1	1	0	0

$$\neg((\neg a \wedge \neg b \wedge \neg c) \vee (a \wedge \neg b \wedge c))$$

卡诺图: 在环面上尽量画大圈 (只含 2^n 个相邻项), 圈的个数尽量少.

Problem (鳄鱼困境)

I will return your child iff you can correctly predict what I will do next.

$$x = ? \implies x \leftrightarrow r \vDash r$$

r	x	$x \leftrightarrow r$	$(x \leftrightarrow r) \to r$
0	$v_1(x) = ?$	?	1
1	$v_2(x) = ?$	?	1

r	x	$x \leftrightarrow r$	$(x \leftrightarrow r) \to r$
0	$v_1(x) = ?$	0	1
1	$v_2(x) = ?$	0/1	1

r	x	$x \leftrightarrow r$	$(x \leftrightarrow r) \to r$
0	$v_1(x) = 1$	0	1
1	$v_2(x) = 0/1$	0/1	1

$$x = \begin{cases} \neg r & \text{if } v_2(x) = 0 \\ r \vee \neg r & \text{if } v_2(x) = 1 \end{cases}$$

Remark: 若要求 $x \leftrightarrow r$ 可满足, 则需舍弃 $x = \neg r$.

Problem (怎么大奖小奖全都拿？)

- 说真话得一个大奖或一个小奖.
- 说假话不得奖.

- b: 我会得大奖.
- s: 我会得小奖.

$b \backslash s$	0	1
0	1	0
1	0	0/1

$$x = ? \implies x \leftrightarrow b \lor s \vDash b \land s$$

b	s	x	$b \lor s$	$x \leftrightarrow b \lor s$	$b \land s$	$(x \leftrightarrow b \lor s) \to b \land s$
0	0	$v_1(x) = 1$	0	0	0	1
0	1	$v_2(x) = 0$	1	0	0	1
1	0	$v_3(x) = 0$	1	0	0	1
1	1	$v_4(x) = 0/1$	1	0/1	1	1

$$x = \begin{cases} \neg b \land \neg s & \text{if } v_4(x) = 0 \\ b \leftrightarrow s & \text{if } v_4(x) = 1 \end{cases}$$

Remark: 若要求 $x \leftrightarrow b \lor s$ 可满足, 则需舍弃 $x = \neg b \land \neg s$.

代入

$$p_i[C/p] \coloneqq \begin{cases} C & \text{if } p_i = p \\ p_i & \text{otherwise} \end{cases}$$

$$(\neg A)[C/p] \coloneqq \neg A[C/p]$$

$$(A \to B)[C/p] \coloneqq A[C/p] \to B[C/p]$$

Theorem (Equivalent Substitution)

$$\frac{\Gamma \vDash B \leftrightarrow C}{\Gamma \vDash A[B/p] \leftrightarrow A[C/p]}$$

$$\vDash A \implies \vDash A[B/p]$$

Example: $\vDash p \vee \neg p \implies \vDash (p \wedge \neg p) \vee \neg(p \wedge \neg p)$

- 何谓 “形式逻辑”? 对任意代入 $[C/p]$:

$$\frac{\Gamma \vDash A}{\Gamma[C/p] \vDash A[C/p]}$$

- 给定一个命题形式, 对相同的变元处处以相同的公式代入, 得到的公式是这个命题形式的特例.

$$p \to p \vee \neg q$$

$$A \to A \vee \neg A$$

$$(A \wedge B) \to (A \wedge B) \vee \neg(A \to B)$$

- 一个公式 A 的**命题形式**是一个命题逻辑公式 B, 使得 A 就是对 B 中相同的命题变元 p_i 处处以相同的公式 C_i 代入的结果.

$$A = B[C_1/p_1, \ldots, C_n/p_n]$$

夏目漱石《梦十夜》

- “他怎能那样行云流水，凿刀所到之处，自然地雕琢出内心所想的眉毛、鼻子的样子？”
- “不难啊，那不是凿刻出眉毛、鼻子，而是眉毛、鼻子本来就埋藏在木头中，他只是用锤子、凿子将其挖出来而已.”

- 雕刻家凿掉无关的木头，使得雕像显现.
- 数学家抽象掉无关的细节，使得模式显现.

Problem (我父亲年龄多大?)

我父亲的年龄是我的年龄的两倍, 十年前, 他的年龄是我的年龄的三倍.

Problem (这个袋子里有几个苹果?)

这个袋子里的苹果数量是那个袋子的两倍, 如果从两个袋子里各取出十个, 那么, 这个袋子里剩下的苹果数量是那个袋子的三倍.

$$
\begin{aligned}
x &= 2y \\
x - 10 &= 3(y - 10)
\end{aligned}
$$

$$
\begin{aligned}
x - 2y &= 0 \\
x - 3y &= -20
\end{aligned}
\qquad
\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -3 & -20 \end{array}\right]
$$

$$
\begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -20 \end{bmatrix}
\qquad
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -20 \end{bmatrix} = \begin{bmatrix} 40 \\ 20 \end{bmatrix}
$$

Theorem (等价替换定理)

假设 B 是 A 的子公式, $A[C/\!\!/B]$ 是用 C 替换 B 在 A 中的一些出现后得到的结果. 则

$$\frac{\Gamma \vDash B \leftrightarrow C}{\Gamma \vDash A \leftrightarrow A[C/\!\!/B]}$$

Proof.

归纳证明. □

Example ☹

1. A logician's wife is having a baby.
2. The doctor immediately hands the newborn to the dad.
3. His wife asks impatiently: "So, is it a boy or a girl"?
4. The logician replies: "yes".

- wife.

$$B?$$

- logician.
  1. $A = B \vee G \qquad A?$
  2. $G \leftrightarrow \neg B$
  3. $A[\neg B /\!\!/ G] = B \vee \neg B \qquad \checkmark$

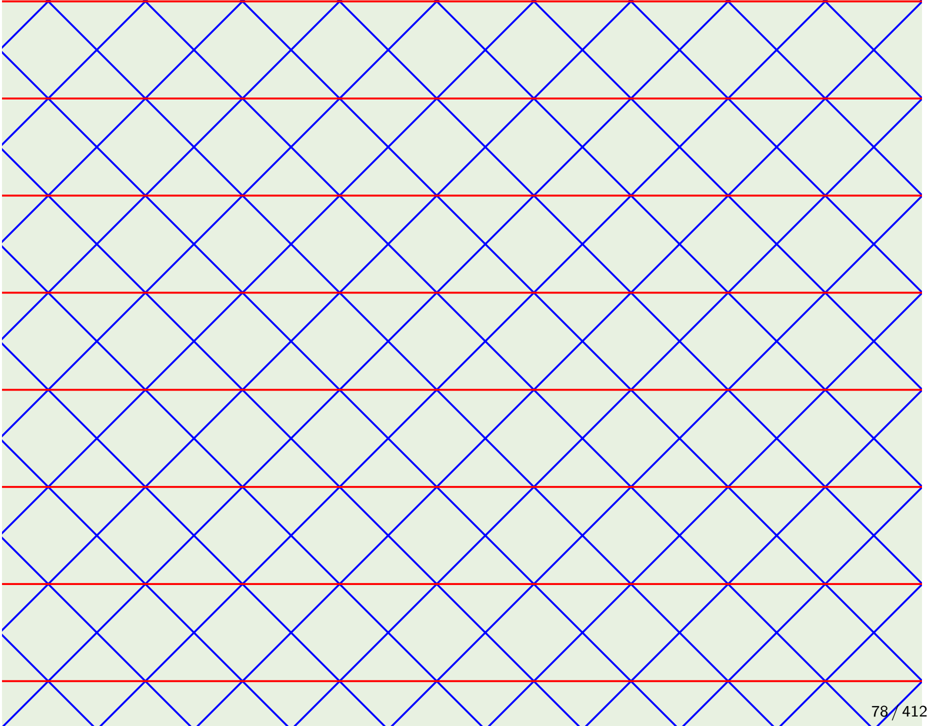
Contents

Why Study Formal System?

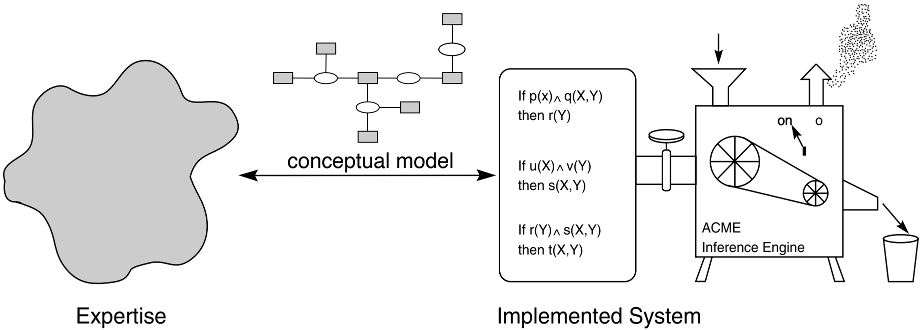
Why truth tables are not sufficient?

- Exponential size
  - How many times would you have to fold a piece of paper($0.1mm$) onto itself to reach the Moon?
  - Common Ancestors of All Humans
    (1) Someone alive $1000BC$ is an ancestor of everyone alive today;
    (2) Everyone alive $2000BC$ is either an ancestor of nobody alive today or of everyone alive today;
    (3) Most of the people you are descended from are no more genetically related to you than strangers are.
    (4) Even if everyone alive today had exactly the same set of ancestors from $2000BC$, the distribution of one's ancestors from that population could be very different.
- Inapplicability beyond Boolean connectives.

42



Formal Systems



- Tree Method
- Natural Deduction
- Sequent Calculus
- Hilbert System
- Resolution
- ...

命题逻辑的树形方法 ♡

$$\begin{array}{c} \neg\neg A \\ | \\ A \end{array} \qquad \begin{array}{ccc} & A \to B & \\ \neg A & & B \end{array} \qquad \begin{array}{c} \neg(A \to B) \\ | \\ A \\ \neg B \end{array}$$

$$\begin{array}{c} A \wedge B \\ | \\ A \\ B \end{array} \qquad \begin{array}{ccc} & \neg(A \wedge B) & \\ \neg A & & \neg B \end{array} \qquad \begin{array}{ccc} & A \vee B & \\ A & & B \end{array} \qquad \begin{array}{c} \neg(A \vee B) \\ | \\ \neg A \\ \neg B \end{array}$$

$$\begin{array}{ccc} & A \leftrightarrow B & \\ A & & \neg A \\ B & & \neg B \end{array} \qquad \begin{array}{ccc} & \neg(A \leftrightarrow B) & \\ A & & \neg A \\ \neg B & & B \end{array}$$

✓

证明 = 闭树

$$\boxed{\dfrac{A \to B \quad \neg B \vee C}{\neg A \vee C}}$$

$A \to B$ ✓

$\neg B \vee C$ ✓

$\neg(\neg A \vee C)$ ✓

$\neg\neg A$

$\neg C$

$\neg A$ $\qquad$ B

$\times$

$\neg B$ $\qquad$ C

$\times$ $\qquad$ $\times$

树形方法指南[3] ♡

- **文字** 一个原子公式或原子公式的否定.
- **闭枝** 一枝上出现了某个公式及其否定.
- **闭树** 所有枝都是闭枝的树.

1. 以前提和结论的否定作为根节点画树.
2. 检查每一个开路径, 如果在其上发现了矛盾, 则闭掉这个路径 $\times$.
3. 如果一个非文字公式在所有开路径上都拆过了, 则用 $\checkmark$ 标记.
4. 如果在所有开路径上都没有没标记过的非文字公式, 则停止画树!
5. 否则的话, 在开路径上选一个没标记过的非文字公式继续拆.
6. Goto 2.

[3] Tree Proof Generator: https://www.umsu.de/trees/

什么是“证明”? ♡

Definition (证明)

$A_1, \ldots, A_n \vdash B$ 当且仅当, 存在一棵从 $\{A_1, \ldots, A_n, \neg B\}$ 开始的闭树.

小技巧

1. 少生枝节: 既能生一枝又能生两枝时, 优先生一枝.
2. 能闭掉的枝尽早闭掉.

Theorem (可靠性定理 & 完备性定理)

$$\frac{A_1, \ldots, A_n \vdash B}{A_1, \ldots, A_n \vDash B}$$

$\vdash$ *captures* $\vDash$

No more, no less

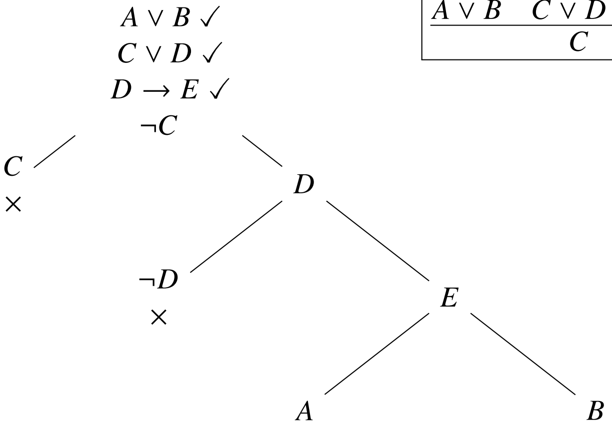
可靠 $\vdash \implies \vDash$ 不多: 所有证明出来的论证都是有效的

完备 $\vDash \implies \vdash$ 不少: 所有有效的论证都能够证明出来

Remark: 如果一个命题逻辑的论证不是有效的, 那么, 至少有一枝闭不掉. 通过闭不掉的开枝可以构造使得论证无效的反模型.

若无效, “开枝” 给出 “反模型”

$$\frac{A \vee B \quad C \vee D \quad D \rightarrow E}{C} \; ?$$

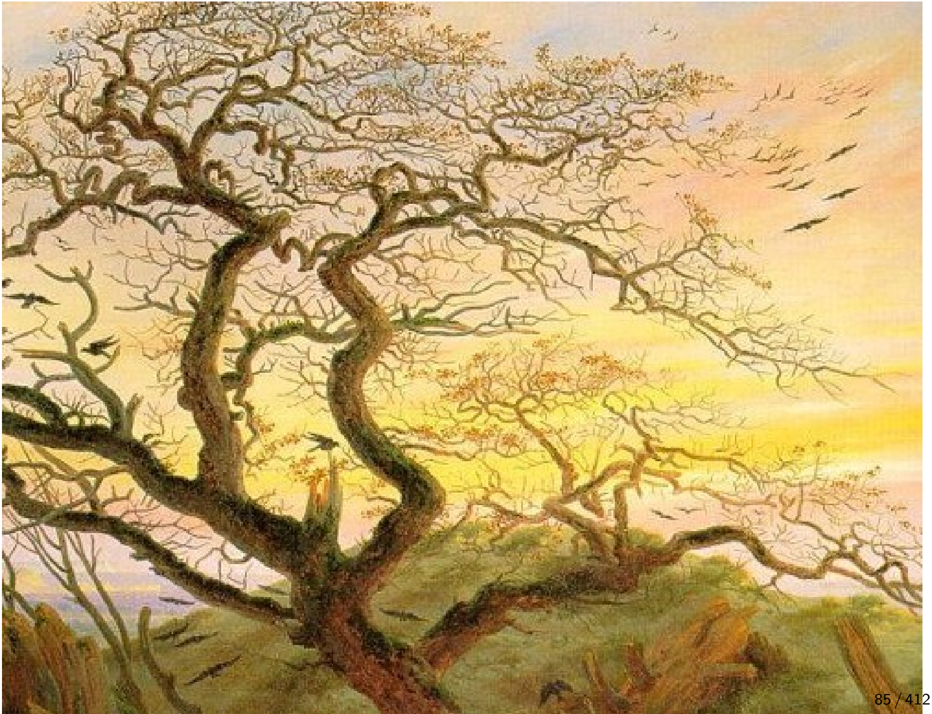


令 $A = 1, \quad E = 1, \quad D = 1 \quad C = 0 \quad B = 1$ (or 0)

$\quad B = 1, \quad E = 1, \quad D = 1 \quad C = 0 \quad A = 1$ (or 0)

则 $A \vee B = 1, \quad C \vee D = 1, \quad D \rightarrow E = 1$

但 $C = 0$



这个岛上有金子吗?

一个岛上有 “君子”、“小人” 两类人. “君子” 只说真话, “小人” 只说假话. 你是一个淘金客, 来岛上遇到了一个土著.

- 你: “这个岛上有金子吗?”
- 土著: “如果我是君子, 那么这个岛上有金子.”

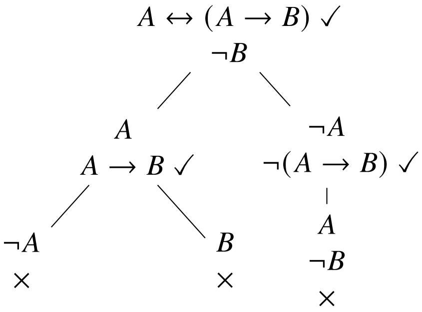
淘金客

这个岛上有金子吗?

一个岛上有 “君子”、“小人” 两类人. “君子” 只说真话, “小人” 只说假话. 你是一个淘金客, 来岛上遇到了一个土著.

- 你: “这个岛上有金子吗?”
- 土著: “如果我是君子, 那么这个岛上有金子.”

$$\boxed{\dfrac{A \leftrightarrow (A \to B)}{B}}$$



Curry's Paradox ☹

如果这句话是真的, 那么上帝存在.

$$\frac{A \leftrightarrow (A \to B)}{B}$$

Proof.

1. $A \leftrightarrow (A \to B)$
2. $A \to A \to B$
3. $(A \to A) \to A \to B$
4. $A \to B$
5. A
6. B

□

1. 甲: 如果我没说错, 那么上帝存在.
2. 乙: **如果你没说错,** 那么上帝存在.
3. 甲: 你承认我没说错了?
4. 乙: 当然.
5. 甲: 可见我没说错. 你已经承认: **如果我没说错, 那么上帝存在.** 所以, 上帝存在.

这句话是假的, 并且, 上帝不存在.

$$\frac{A \leftrightarrow (\neg A \wedge \neg B)}{B}$$

Curry's Paradox — How to Flirt with a Beauty ☺

Smullyan 实力撩妹 ♡◎♡

1. “我说一句话, 如果它是真的, 可以给我签个名吗?”
2. “没问题呀.”
3. “如果它是假的, 就不要给我签名了.”
4. “好的.”
5. 然后 Smullyan 说了一句话. 美女发现必须给他一个吻!

$$x =? \implies a \leftrightarrow x \vDash k$$

Hi 美女, 问你个问题呗

如果我问你“你能做我女朋友吗”, 那么你的答案和**这个问题**的答案是一样的吗?

$$\frac{Q \leftrightarrow (G \leftrightarrow Q)}{G}$$

Don't just read it; fight it!

练习: 树形证明 — Now it's your turn ⚡

$$\begin{array}{c} \\ \hline (A \to B) \lor (B \to C) \end{array} \qquad \begin{array}{c} A \to C \quad B \to D \\ \hline A \land B \to C \land D \end{array} \qquad \begin{array}{c} A \to C \quad B \to D \\ \hline \neg C \lor \neg D \to \neg A \lor \neg B \end{array}$$

$$\begin{array}{c} A \to B \quad \neg A \to B \\ \hline B \end{array} \qquad \begin{array}{c} A \to C \quad B \to D \\ \hline A \lor B \to C \lor D \end{array} \qquad \begin{array}{c} (A \to C) \land (B \to C) \\ \hline\hline A \lor B \to C \end{array}$$

$$\begin{array}{c} (A \to B) \to C \\ \hline B \to C \end{array} \qquad \begin{array}{c} (A \to C) \lor (B \to C) \\ \hline\hline A \land B \to C \end{array} \qquad \begin{array}{c} C \to A \quad C \to B \\ \hline C \to A \land B \end{array}$$

$$\begin{array}{c} A \to \neg B \to B \\ \hline\hline A \to B \end{array} \qquad \begin{array}{c} \neg(A \leftrightarrow B) \\ \hline\hline \neg A \leftrightarrow B \end{array} \qquad \begin{array}{c} A \leftrightarrow B \\ \hline\hline A \lor B \to A \land B \end{array}$$

$$\begin{array}{c} A \land (B \lor C) \\ \hline\hline (A \land B) \lor (A \land C) \end{array} \qquad \begin{array}{c} A \lor (\neg A \land B) \\ \hline\hline A \lor B \end{array} \qquad \begin{array}{c} A \to B \to C \\ \hline\hline A \land B \to C \end{array}$$

练习: 有效性判定 — Now it's your turn ↯

以下推理是否有效? 若无效, 请构造反模型.

$$\frac{B \to C}{(A \to B) \to C}\ ? \qquad \frac{A \vee (B \wedge C)}{(A \vee B) \wedge C}\ ? \qquad \frac{(A \leftrightarrow B) \to C}{A \leftrightarrow (B \to C)}\ ?$$

$$\frac{(A \to C) \to C}{(A \to B) \to C}\ ? \qquad \frac{A \to B \wedge C \quad \neg(A \vee B \to C)}{A}\ ?$$

$$\frac{(A \to C) \vee (B \to C)}{A \vee B \to C}\ ? \qquad \frac{A \to B}{(A \to C) \to (B \to C)}\ ?$$

1. 如果你不愿意做我女朋友, 那么 "如果我表白, 你答应" 是不可能的. 我不表白. 所以, 你愿意.
2. If you concentrate only if you are threatened, then you will not pass unless you are threatened — provided that concentrating is a necessary condition for passing.

自然推演 Natural Deduction

$$\frac{A \quad B}{A \wedge B}\wedge^{+} \qquad \frac{A \wedge B}{A}\wedge^{-} \qquad \frac{A \wedge B}{B}\wedge^{-}$$

$$\frac{A}{A \vee B}\vee^{+} \qquad \frac{B}{A \vee B}\vee^{+} \qquad \frac{A \vee B \quad \begin{matrix}[A]^n \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]^n \\ \vdots \\ C\end{matrix}}{C}\vee^{-n}$$

$$\frac{\begin{matrix}[A]^n \\ \vdots \\ B\end{matrix}}{A \longrightarrow B}\longrightarrow^{+n} \qquad \frac{A \longrightarrow B \quad A}{B}\longrightarrow^{-}$$

$$\frac{\begin{matrix}[A]^n \\ \vdots \\ \bot\end{matrix}}{\neg A}\neg^{+n} \qquad \frac{\begin{matrix}[\neg A]^n \\ \vdots \\ \bot\end{matrix}}{A}\neg^{-n} \qquad \frac{\neg A \quad A}{\bot}\bot^{+} \qquad \frac{\bot}{A}\bot^{-}$$

试给出关于命题连接词 $\leftrightarrow$ 的引入和消去规则.

Examples

$$\boxed{\dfrac{A \lor B \quad \neg B}{A}}$$

Proof.

$$\dfrac{A \lor B \qquad \begin{matrix}[A]^1 \\ \vdots \\ A\end{matrix} \qquad \dfrac{\dfrac{[B]^1 \quad \neg B}{\bot}\,\bot^+}{A}\,\bot^-}{A}\,\lor^{-1}$$

□

$$\boxed{\dfrac{A \longrightarrow B \quad \neg B}{\neg A}}$$

Proof.

$$\dfrac{\dfrac{\dfrac{A \longrightarrow B \quad [A]^1}{B}\,\longrightarrow^- \qquad \neg B}{\bot}\,\bot^+}{\neg A}\,\neg^{+1}$$

Examples

$$\boxed{\frac{\neg\neg A}{A}}$$

Proof.

$$\dfrac{\dfrac{\neg\neg A \quad [\neg A]^1}{\bot}\,\bot^+}{A}\,\neg^{-1}$$

□

$$\boxed{\frac{A}{\neg\neg A}}$$

Proof.

$$\dfrac{\dfrac{A \quad [\neg A]^1}{\bot}\,\bot^+}{\neg\neg A}\,\neg^{+1}$$

□

$$\boxed{\frac{\neg A \longrightarrow A}{A}}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\neg A \longrightarrow A \quad [\neg A]^1}{A}\,\longrightarrow^- \qquad [\neg A]^1}{\bot}\,\bot^+}{A}\,\neg^{-1}$$

□

Examples

$$\boxed{\dfrac{A \lor B \to C}{(A \to C) \land (B \to C)}}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{[A]^1}{A \lor B}\,\lor^+ \qquad A \lor B \to C}{C}\,\to^-}{A \to C}\,\to^{+1} \qquad \dfrac{\dfrac{\dfrac{[B]^2}{A \lor B}\,\lor^+ \qquad A \lor B \to C}{C}\,\to^-}{B \to C}\,\to^{+2}}{(A \to C) \land (B \to C)}\,\land^+$$

□

$$\boxed{\dfrac{(A \to C) \land (B \to C)}{A \lor B \to C}}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{(A \to C) \land (B \to C)}{A \to C}\,\land^- \qquad [A]^1}{C}\,\to^- \qquad \dfrac{\dfrac{(A \to C) \land (B \to C)}{B \to C}\,\land^- \qquad [B]^1}{C}\,\to^- \qquad [A \lor B]^2}{C}\,\lor^{-1}}{A \lor B \to C}\,\to^{+2}$$

Examples

$$\boxed{\dfrac{\neg A \to \bot}{A}}$$

Proof.

$$\dfrac{\dfrac{\neg A \to \bot \qquad [\neg A]^1}{\bot}\ {\to^-}}{A}\ \neg^{-1}$$

□

$$\boxed{\dfrac{A \wedge \neg A}{B}}$$

Proof.

$$\dfrac{\dfrac{\dfrac{A \wedge \neg A}{A}}{A \vee B} \qquad \dfrac{A \wedge \neg A}{\neg A}}{B}$$

□

$$\boxed{\dfrac{}{A \vee \neg A}}$$

Proof.

$$\dfrac{\dfrac{[\neg(A \vee \neg A)]^2 \qquad \dfrac{\dfrac{\dfrac{[\neg(A \vee \neg A)]^2 \qquad \dfrac{[A]^1}{A \vee \neg A}\ \vee^+}{\bot}\ \bot^+}{\neg A}\ \neg^{+1}}{A \vee \neg A}\ \vee^+}{\bot}\ \bot^+}{A \vee \neg A}\ \neg^{-2}$$

□

$$\boxed{\dfrac{\neg A \vee \neg B}{\neg(A \wedge B)}}$$

$$\dfrac{\dfrac{\dfrac{\neg A \vee \neg B \quad \dfrac{[A \wedge B]^1}{A}}{\neg B} \quad \dfrac{[A \wedge B]^1}{B}}{\bot}}{\neg(A \wedge B)}\ \neg^{+1}$$

$$\boxed{\dfrac{\neg(A \wedge B)}{\neg A \vee \neg B}}$$

$$\dfrac{\dfrac{\neg(A \wedge B) \quad \dfrac{\dfrac{\dfrac{[\neg(\neg A \vee \neg B)]^3 \quad \dfrac{[\neg A]^1}{\neg A \vee \neg B}}{\bot}}{A}\ \neg^{-1} \quad \dfrac{\dfrac{[\neg(\neg A \vee \neg B)]^3 \quad \dfrac{[\neg B]^2}{\neg A \vee \neg B}}{\bot}}{B}\ \neg^{-2}}{A \wedge B}}{\bot}}{\neg A \vee \neg B}\ \neg^{-3}$$

Example — 福尔摩斯《血字的研究》

- 这起谋杀的目的不是抢劫 $\neg R$, 因为死者身上的东西没有少 $\neg S$.
- 不是抢劫, 那么是政治暗杀 P 呢? 还是情杀 Q 呢?
- 我倾向后者 Q.
- 因为在政治暗杀中 P, 凶手一经得手势必立即逃离现场 L.
- 而在这起谋杀案中, 凶手没有立即离开现场 $\neg L$, 因为在屋子里到处留下了足迹 F.

$$\boxed{\neg S, \neg S \to \neg R, \neg R \to P \vee Q, P \to L, F \to \neg L, F \vdash Q}$$

$$\dfrac{\dfrac{\dfrac{\neg S \quad \neg S \to \neg R}{\neg R} \qquad \neg R \to P \vee Q}{P \vee Q} \qquad \dfrac{P \to L \qquad \dfrac{F \to \neg L \quad F}{\neg L}}{\neg P}}{Q}$$

Example — 谁是盗贼?

谁偷了《白玉美人》?

1. 《白玉美人》是白展堂或楚留香偷的.
2. 如果是白展堂偷的, 则偷窃时间不会在午夜前.
3. 如果楚留香的证词正确, 则午夜时烛光未灭.
4. 如果楚留香的证词不正确, 则偷窃发生在午夜前.
5. 午夜时没有烛光.

1. $B \vee C$
2. $B \to \neg M$
3. $T \to L$
4. $\neg T \to M$
5. $\neg L$

$$\dfrac{B \vee C \qquad \dfrac{B \to \neg M \qquad \dfrac{\neg T \to M \qquad \dfrac{T \to L \qquad \neg L}{\neg T}}{M}}{\neg B}}{C}$$

Example — 自然推演

如果在有限长的线段 L 上有无穷多个点的话, 那么, 如果这些点都有长度, 则 L 将无限长; 如果这些点都没有长度, 则 L 将没有长度. 而一个有限长的线段不可能无限长, 也不可能没有长度. 因此, 在有限长的线段上不可能有无穷多个点.

$$\begin{array}{c} D \to (L \to I) \wedge (\neg L \to N) \\ \neg I \\ \neg N \\ \hline \neg D \end{array}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[D]^1 \quad D \to (L \to I) \wedge (\neg L \to N)}{(L \to I) \wedge (\neg L \to N)}}{L \to I} \qquad \dfrac{\dfrac{\dfrac{[D]^1 \quad D \to (L \to I) \wedge (\neg L \to N)}{(L \to I) \wedge (\neg L \to N)}}{\neg L \to N} \qquad \neg N}{L}}{I} \qquad \neg I}{\dfrac{\bot}{\neg D}\ \neg^{+1}}$$

有效论证 & 证明方法

1. **Direct Proof(直接证明):**

$$\frac{\begin{array}{l} A \to B \\ A \end{array}}{B}$$

   Example: 两个有理数之和是有理数.

2. **Backward Reasoning(反向推理):** to prove B, find A and $A \to B$.
   Example: 如果 x 和 y 是非负实数, 那么

$$\frac{x+y}{2} \geq \sqrt{xy}$$

3. **Proof by Contraposition(逆否证明):**

$$\frac{A \to B}{\neg B \to \neg A}$$

   Example: 如果 $3n+2$ 是奇数, 那么 n 也是奇数.

4. **Proof by Cases(分情况证明):**

$$\frac{A \vee B \to C}{\overline{(A \to C) \wedge (B \to C)}}$$

Example: 如果 n 是整数, 那么 $n(n+1)$ 是偶数.

5. **Proof by Elimination(排除法):**

$$\frac{A \to B \vee C}{\overline{A \wedge \neg B \to C}}$$

Example: 对于复数 $a+bi$ 和 $c+di$, 如果 $(a+bi)(c+di)=1$, 那么 $a \neq 0$ 或 $b \neq 0$.

6. **Proof by Contradiction(反证法):**

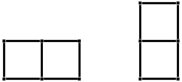
$$\frac{\begin{array}{c}\neg A \rightarrow B \\ \neg A \rightarrow \neg B\end{array}}{A}$$

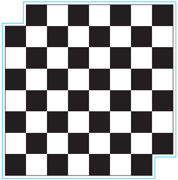
Example: 欧几里得定理 — 素数有无穷多.

7. **Reductio Ad Absurdum (归谬法):**

$$\frac{\begin{array}{c}A \rightarrow B \\ A \rightarrow \neg B\end{array}}{\neg A}$$

Example: 没有最大的自然数.
Example: $\sqrt{2}$ 是无理数.
Example: 理发师悖论.
Example: 下面缺角的棋盘不能被多米诺骨牌平铺.





Hilbert Formal System = Axiom + Inference Rule

公理模式

1. $A \to B \to A$
2. $(A \to B \to C) \to (A \to B) \to A \to C$
3. $(\neg A \to \neg B) \to (\neg A \to B) \to A$

推理规则

$$\frac{A \quad A \to B}{B}\ \text{MP}$$

什么是“证明”?

Definition (证明 $\Gamma \vdash A$)

$\Gamma \vdash A$ 当且仅当, 存在以 A 结尾的有穷公式序列 $(C_1, \ldots, C_n)$, 其中 $C_n = A$, 使得序列中的每个公式 $C_k, k \leq n$:

1. 或者是公理;
2. 或者在 Γ 里;
3. 或者由前面的公式通过推理规则得到.

当 $\Gamma = \emptyset$ 时, A 是定理 $\vdash A$.

A Joke ☺

数学家的房子起火了. 他的妻子用水将火扑灭了. 然后发生了煤气泄漏. 数学家点燃了它.

Example

Theorem

$$\vdash A \to A$$

Proof.

1. $A \to (A \to A) \to A$ — A1
2. $(A \to (A \to A) \to A) \to (A \to A \to A) \to A \to A$ — A2
3. $(A \to A \to A) \to A \to A$ — 1,2 MP
4. $A \to A \to A$ — A1
5. $A \to A$ — 3,4 MP

$\square$

Remark☺ Logic is like love; a simple idea, but it can get complicated.

- 这 TM 也用证?
- 这 TM 也能证?

演绎定理 "$\vdash$" vs "$\to$"

Theorem (Deduction Theorem)

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

Proof.

Prove by induction on the length of the deduction sequence $(C_1, \ldots, C_n)$ of B from $\Gamma \cup \{A\}$.

Base step $n = 1$:

case1. B is an axiom. (use Axiom1.)

case2. $B \in \Gamma$.

case3. $B = A$.

Inductive step $n > 1$:

case1. B is either an axiom, or $B \in \Gamma$, or $B = A$.

case2. $C_i = C_j \to B$

$\Gamma, A \vdash C_j \implies \Gamma \vdash A \to C_j$

$\Gamma, A \vdash C_j \to B \implies \Gamma \vdash A \to C_j \to B$

$\Gamma \vdash A \to B$

Contents

什么是“理论”?

- $\mathrm{Mod}(A) := \{\nu : \nu \vDash A\}$
- $\mathrm{Mod}(\Gamma) := \bigcap\limits_{A\in\Gamma} \mathrm{Mod}(A)$
- $\mathrm{Th}(\nu) := \{A : \nu \vDash A\}$
- $\mathrm{Th}(\mathcal{K}) := \bigcap\limits_{\nu\in\mathcal{K}} \mathrm{Th}(\nu)$
- $\mathrm{Cn}(\Gamma) := \{A : \Gamma \vDash A\}$

$$A \vDash B \iff \mathrm{Mod}(A) \subset \mathrm{Mod}(B)$$

What is “theory”?

- 句子集 Γ 是一个理论, 当且仅当, $\Gamma = \mathrm{Cn}(\Gamma)$.
- 理论 Γ 是完备的, 当且仅当, 对任意句子 A: 或者 $A \in \Gamma$ 或者 $\neg A \in \Gamma$.
- 理论 Γ 是可公理化的, 当且仅当, 有一个可判定的句子集 Σ 使得 $\Gamma = \mathrm{Cn}(\Sigma)$.

一致性 & 可满足性

定义:

- Γ 是一致的, 当且仅当, $\Gamma \nvdash \bot$.
- Γ 是极大的, 当且仅当, 对任意公式 A: 或者 $A \in \Gamma$ 或者 $\neg A \in \Gamma$.
- Γ 是极大一致的, 当且仅当, 它既是极大的又是一致的.
- Γ 是可满足的, 当且仅当, $\mathrm{Mod}(\Gamma) \neq \emptyset$.
- Γ 是有穷可满足的, 当且仅当, 其任意有穷子集都是可满足的.

性质:

- 如果 Γ 是一致的, 且 $\Gamma \vdash A$, 那么 $\Gamma \cup \{A\}$ 是一致的.
- $\Gamma \cup \{\neg A\}$ 不一致, 当且仅当, $\Gamma \vdash A$.
- 如果 Γ 是极大一致的, 那么, $A \notin \Gamma$ 蕴含着 $\Gamma \cup \{A\}$ 不一致.
- 集合 $\mathrm{Th}(\nu) = \{A : \nu \vDash A\}$ 是极大一致的.

命题逻辑的可靠性、完备性定理

Theorem (可靠性定理)

$$\Gamma \vdash A \implies \Gamma \vDash A$$

Corollary

可满足的公式集是一致的.

Theorem (完备性定理)

$$\Gamma \vDash A \implies \Gamma \vdash A$$

Corollary

一致的公式集是可满足的.

$$\begin{array}{ccc}
\Gamma \vDash A & \Longleftrightarrow & \Gamma \vdash A \\
\Updownarrow & & \Updownarrow \\
\Gamma \cup \{\neg A\} & & \Gamma \cup \{\neg A\} \\
\text{不可满足} & \Longleftrightarrow & \text{不一致}
\end{array}$$

$\vdash$ captures $\vDash$
No more, no less

Theorem (紧致性定理)

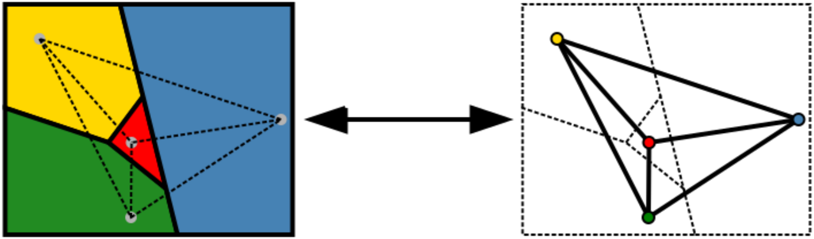
一个公式集是可满足的, 当且仅当, 它是有穷可满足的.

Corollary

如果 $\Gamma \vDash A$, 则存在有穷子集 $\Gamma_0 \subset \Gamma$ 使得 $\Gamma_0 \vDash A$.

Remark: 一个公式集是一致的, 当且仅当, 它的任何有穷子集是一致的.
Remark: 在一个允许无穷析取的语言里紧致性失效.

$$\left\{ \bigvee_{i=1}^{\infty} p_i, \neg p_1, \neg p_2, \ldots \right\}$$



$\boxed{\text{An infinite graph } (V, E) \text{ is } n\text{-colorable iff every finite subgraph of } (V, E) \text{ is } n\text{-colorable.}}$

Proof.

Take $\{p_v^i : v \in V, 1 \le i \le n\}$ as the set of atoms.

$\Gamma := \{p_v^1 \vee \cdots \vee p_v^n : v \in V\} \cup \left\{\neg\left(p_v^i \wedge p_v^j\right) : v \in V, 1 \le i < j \le n\right\} \cup$
$\left\{\neg\left(p_v^i \wedge p_w^i\right) : (v, w) \in E, 1 \le i \le n\right\}$ □

波斯特 Emil Post 1897-1954

- Truth table
- Completeness of propositional logic
- Post machine
- Post canonical system
- Post correspondence problem
- Post problem

soundness

Provability
($syntactic \vdash$)
"there exists a proof..."

Validity
($semantic \vDash$)
"for all assignments..."

completeness

可判定性

$$\overset{?}{\vDash} A \xrightarrow{\text{input}} \boxed{\text{Machine}} \xrightarrow{\text{output}} Y/N$$

- 存在机械程序, 可以通过有限步骤判定任一符号串是否为公式.
- 存在机械程序, 可以通过有限步骤判定任一公式是否为公理.
- 存在机械程序, 可以通过有限步骤判定任一公式串是否为一个证明.

Theorem (Decidability — Post1921)

存在机械程序, 给定任意有穷公式集 $\Gamma \cup \{A\}$, 可以判定是否 $\Gamma \vDash A$.

Theorem

如果公式集 Γ 是可判定的, 那么 $\mathrm{Cn}(\Gamma)$ 是递归可枚举的.

模型检测、可满足性检测、有效性检测[4]

- 给定赋值 ν 和命题 A, 是否 $\nu \vDash A$? —P
- 给定命题 A, 是否存在赋值 ν 使得 $\nu \vDash A$? —NP
- 给定命题 A, 是否 $\vDash A$? —co-NP

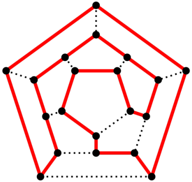


Figure: Hamiltonian Circle(NPC)

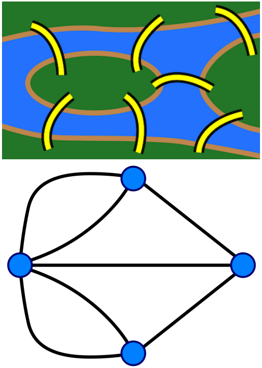


Figure: Eulerian Circle(P)

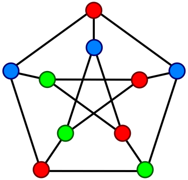
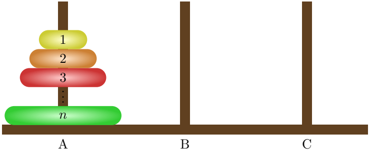


Figure: Graph Coloring(NPC)



[4] Aaronson: Why philosophers should care about computational complexity.

Contents

$\bot$	$\top$	$\vee$	$\wedge$	$\neg$
0	1	$+$	$\cdot$	$\overline{}$

$$(B, 0, 1, +, \cdot, \overline{})$$

- $x + (y + z) = (x + y) + z$
  $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- $x + y = y + x \quad x \cdot y = y \cdot x$
- $x + (x \cdot y) = x \quad x \cdot (x + y) = x$
- $x + (y \cdot z) = (x + y) \cdot (x + z)$
  $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
- $\overline{\overline{x}} = x$
- $\overline{x + y} = \overline{x} \cdot \overline{y} \quad \overline{x \cdot y} = \overline{x} + \overline{y}$
- $x + \overline{x} = 1 \quad x \cdot \overline{x} = 0 \quad 0 \neq 1$
- $x + 0 = x \quad x \cdot 0 = 0$
  $x + 1 = 1 \quad x \cdot 1 = x$

$$x \leq y := x\overline{y} = 0$$

1. 解方程: $ax = b$
   $$ax = b \iff \overline{a}b + a\overline{b}x + b\overline{x} = 0$$
   $$\overline{a}b = 0$$
   $$a\overline{b}x = 0$$
   $$b\overline{x} = 0$$
   x 的解集为:
   $$b \leq x \leq \overline{a} + b$$
   且满足有解的条件: $\overline{a}b = 0$.
2. 解方程: $ax + b\overline{x} + c = 0$
   $$ax = 0$$
   $$b\overline{x} = 0$$
   $$c = 0$$
   x 的解集为:
   $$b \leq x \leq \overline{a}$$
   且满足有解的条件: $c = 0$ & $b \leq \overline{a}$.

更多布尔运算和性质

- $x - y := x\overline{y}$
- $\overline{x} = 1 - x$
- $x(y - z) = xy - xz$
- $x \leq y \iff \overline{x} + y = 1 \iff x - y = 0 \iff xy = x \iff x + y = y$
- $x \oplus y := x\overline{y} + \overline{x}y$
- $x = y \iff x \oplus y = 0$
- $\overline{x} = 1 \oplus x$
- $x + x = x$
- $xx = x$ **Remark:** $x = x^2$ 意味着无矛盾律.

$$x = x^2 \implies x - x^2 = 0 \implies x(1 - x) = 0$$

- $x + y = 1 \;\; \& \;\; xy = 0 \implies y = \overline{x}$
- $x + \overline{x}y = x + y$
- $xy + \overline{x}z + yzw = xy + \overline{x}z = (x + z)(\overline{x} + y)$
- $(x + y)(\overline{x} + z)(y + z + w) = (x + y)(\overline{x} + z) = xz + \overline{x}y$

布尔代数 $(B, 0, 1, +, \cdot, ^-)$ 的例子

- $\left(\mathrm{P}(X), \emptyset, X, \cup, \cap, ^-\right)$
- $\left(\{k : k \mid n\}, 1, n, \mathrm{lcm}, \gcd, \frac{n}{-}\right)$ where n is square-free.

$$\begin{tikzcd}
 & 30 & \\
6 \arrow[ur] & 10 \arrow[u] & 15 \arrow[ul] \\
2 \arrow[u] \arrow[ur] & 3 \arrow[ul] \arrow[ur] & 5 \arrow[ul] \arrow[u] \\
 & 1 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

$n = 20$ 则不成, $\overline{2} = \frac{20}{2} = 10$, $2 \cdot \overline{2} = \gcd(2, 10) = 2 \neq 1$.

- $\mathbf{2} := \left(\{0, 1\}, 0, 1, \max, \min, 1 -\right)$
- $\mathrm{Lin} := \left(\{[A] : A \in \mathrm{Wff}\}, 0, 1, +, \cdot, ^-\right)$ where $[A] := \{B : \vdash A \leftrightarrow B\}$

$$\begin{aligned}
0 &:= [\bot] \\
1 &:= [\top] \\
[A] + [B] &:= [A \vee B] \\
[A] \cdot [B] &:= [A \wedge B] \\
\overline{[A]} &:= [\neg A]
\end{aligned}$$

$$\begin{tikzcd}
\mathrm{Wff} \arrow[d, "A\mapsto[A]"'] \arrow[dr, "v"] & \\
\mathrm{Lin} \arrow[r, "h"'] & \mathbf{2}
\end{tikzcd}$$

命题逻辑的布尔代数语义

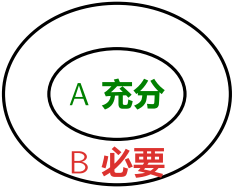
给定布尔代数 $\mathcal{M} := (M, 0, 1, +, \cdot, ^{-}, \leq)$, $\mathcal{M}$ 上的赋值 $\nu : \mathrm{Var} \to M$ 可以在满足下列条件的情况下递归地扩展到所有公式上:

1. $[\![p]\!] := \nu(p)$
2. $[\![\top]\!] := 1$
3. $[\![\bot]\!] := 0$
4. $[\![\neg A]\!] := \overline{[\![A]\!]}$
5. $[\![A \wedge B]\!] := [\![A]\!] \cdot [\![B]\!]$
6. $[\![A \vee B]\!] := [\![A]\!] + [\![B]\!]$

当 $[\![A]\!] = 1$ 时, 我们记 $\mathcal{M}, \nu \vDash A$. 若对任意 ν 都成立, 则记 $\mathcal{M} \vDash A$.

- 通常的真值赋值即取 $\mathcal{M} := \mathbf{2} := \Big(\{0, 1\}, 0, 1, \max, \min, 1-, \leq\Big)$
- 若取 $\mathcal{M} := (\mathcal{P}(X), \emptyset, X, \cup, \cap, ^{-}, \subset)$, 则

$$\begin{array}{c} \vDash A \to B \\ \hline\hline A \vDash B \\ \hline\hline [\![A]\!] \leq [\![B]\!] \\ \hline\hline [\![A]\!] \subset [\![B]\!] \end{array}$$



命题逻辑 vs 布尔代数

- For a Boolean algebra $(B, 0, 1, +, \cdot, ^{-}, \leq)$, build language $\mathscr{L}$ by having a propositional constant P_x for each $x \in B$.
- Construct propositional theory T_B in $\mathscr{L}$ by adding for $x, y \in B$ an axiom

$$P_x \to P_y \quad \text{if } x \leq y$$

  and axioms

$$P_x \wedge P_y \leftrightarrow P_{x \cdot y}$$
$$P_x \vee P_y \leftrightarrow P_{x+y}$$
$$\neg P_x \leftrightarrow P_{\overline{x}}$$

- Then Boolean algebra B is isomorphic to the Lindenbaum algebra $\mathrm{Lin}_{\mathrm{T}_B}$ of its theory T_B.

$$B \cong \mathrm{Lin}_{\mathrm{T}_B}$$

- We call two propositional theories equivalent if their Lindenbaum algebras are isomorphic. Then for a propositional theory T,

$$\mathrm{T} \equiv \mathrm{T}_{\mathrm{Lin}_{\mathrm{T}}}$$

命题逻辑 vs 布尔代数

$$\frac{x \vDash y}{\overline{\overline{x \leq y}}}$$

$$\begin{array}{c} a \vee b, a \to c, b \to d \vDash c \vee d \\ \hline\hline (a+b)(\overline{a}+c)(\overline{b}+d) \leq c+d \\ \hline\hline (a+b)(\overline{a}+c)(\overline{b}+d)\overline{c}\overline{d} = 0 \end{array}$$

$$\begin{aligned} & (a+b)(\overline{a}+c)(\overline{b}+d)\overline{c}\overline{d} \\ = {} & (a+b)(\overline{a}+c)\overline{c}(\overline{b}+d)\overline{d} \\ = {} & (a+b)\,\overline{a}\,\overline{c}\,\overline{b}\,\overline{d} \\ = {} & b\,\overline{a}\,\overline{c}\,\overline{b}\,\overline{d} \\ = {} & 0 \end{aligned}$$

布尔方程的通解

Theorem (布尔方程的通解)

Let $f : B \to B$ be a Boolean function for which $f(0) \cdot f(1) = 0$. Then

$$f(x) = 0$$

$$\Updownarrow$$

$$f(0) \le x \le \overline{f(1)}$$

$$\Updownarrow$$

$$x = f(0) + \theta \cdot \overline{f(1)} \quad \textit{for } \theta \in B$$

Example: $x = ? \implies \models (a \leftrightarrow x) \to k$

$$f(x) = \overline{\overline{(a \cdot x + \overline{a} \cdot \overline{x})} + k} = 0$$

$$\overline{a} \cdot \overline{k} \le x \le \overline{a} + k$$

$$x = \overline{a} \cdot \overline{k} + \theta \cdot (\overline{a} + k)$$

Problem (天堂之门)

1. 你面前有左右两护卫镇守左右两门.
2. 一人只说真话, 一人只说假话.
3. 一门通天堂, 一门通地狱.
4. 你只能向其中一人提一个 “是/否” 的问题.
5. 怎么问出去天堂的门?

- 令 t 表示: 第一个人说真话.
- 设想你问第一个人问题 x, 第一个人对问题 x 的回答:

$$a := tx + \bar{t}\,\bar{x}$$

- 设想你问第二个人问题 Q: 如果我问另一个人问题 “x”, 他会说 “是” 吗?
- 第二个人对问题 Q 的回答:

$$t\,\bar{a} + \bar{t}a = t\,\overline{tx + \bar{t}\,\bar{x}} + \bar{t}\,(tx + \bar{t}\,\bar{x}) = \bar{x}$$

- 令 x 为 “左门通天堂”, 然后问 Q, 根据其回答, 选择相反的门.

Problem (天堂之门)

1. 你面前有左右两护卫镇守左右两门.
2. 一人只说真话, 一人只说假话.
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4. 你只能向其中一人提一个 “是/否” 的问题.
5. 怎么问出去天堂的门?

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{h} \\
 & & 0 & 1 \\
\hline
\multirow{2}{*}{t} & 0 & 1 & 0 \\
\cline{3-4}
 & 1 & 0 & 1 \\
\hline
\end{array}
$$

- t: 你说真话
- h: 左门通天堂

$$x = ? \implies \vDash (t \to (x \leftrightarrow h)) \land (\neg t \to (x \leftrightarrow \neg h))$$

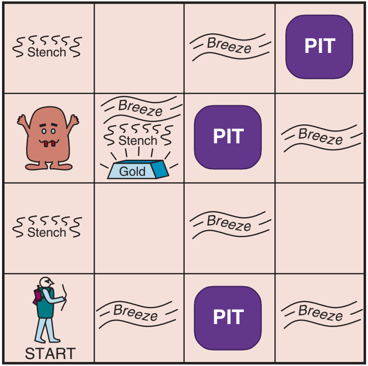
$$f(x) = t(x\bar{h} + \bar{x}h) + \bar{t}(xh + \bar{x}\bar{h}) = 0 \iff x = th + \bar{t}\bar{h}$$

t	h	x	$x \leftrightarrow h$	$x \leftrightarrow \neg h$	$t \to (x \leftrightarrow h)$	$\neg t \to (x \leftrightarrow \neg h)$	A	守卫
0	0	$v_1(x) = 1$	0/1	1	1	1	1	N
0	1	$v_2(x) = 0$	0/1	1	1	1	1	Y
1	0	$v_3(x) = 0$	1	0/1	1	1	1	N
1	1	$v_4(x) = 1$	1	0/1	1	1	1	Y

$$x = t \leftrightarrow h$$

Contents

Wumpus 游戏



- squares adjacent to wumpus are smelly
- squares adjacent to pit are breezy
- glitter iff gold is in the same square
- shooting kills wumpus if you are facing it
- shooting uses up the only arrow
- grabbing picks up gold if in same square
- releasing drops the gold in same square

KB = wumpus-world rules + observations

Example: $B_{21} \leftrightarrow P_{11} \lor P_{22} \lor P_{31}$

Automated Theorem Prover: Prover9 is an automated theorem prover for first-order and equational logic, and Mace4 searches for finite models and counter-examples.[5] Vampire is more powerful.

[5] Adrian Groza: Modelling Puzzles in First Order Logic.

数独游戏

	8	6				2	9	
4			1		5			8
7				9				4
1								9
	5						1	
		8				3		
			5		9			
				2				

$p(i,j,n)$:= the cell in row i and column j contains the number n

- Every row/column contains every number.

$$\bigwedge_{i=1}^{9}\bigwedge_{n=1}^{9}\bigvee_{j=1}^{9} p(i,j,n) \qquad \bigwedge_{j=1}^{9}\bigwedge_{n=1}^{9}\bigvee_{i=1}^{9} p(i,j,n)$$

- Every 3×3 block contains every number.

$$\bigwedge_{r=0}^{2}\bigwedge_{s=0}^{2}\bigwedge_{n=1}^{9}\bigvee_{i=1}^{3}\bigvee_{j=1}^{3} p(3r+i,3s+j,n)$$

- No cell contains more than one number.
  for all $1 \le i,j,n,n' \le 9$ and $n \neq n'$:

$$p(i,j,n) \to \neg p(i,j,n')$$

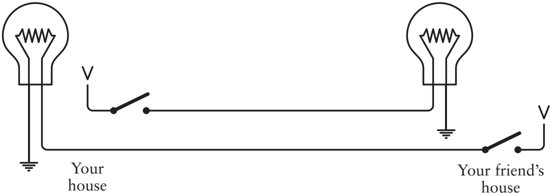
- 一个数独可以看作一个理论.
- 如果有格子填不了任何数字, 则 “公理” 不一致.
- 如果有格子有两种或两种以上的方式可以填数字, 则 “公理” 不完备.

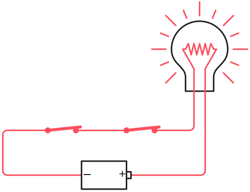
化学反应

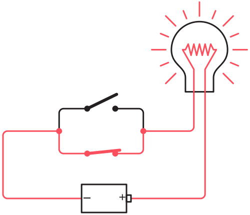
$$\frac{\begin{array}{c} C \wedge O_2 \longrightarrow CO_2 \\ H_2O \wedge CO_2 \longrightarrow H_2CO_3 \\ HCl \wedge NaOH \longrightarrow NaCl \wedge H_2O \end{array}}{HCl \wedge NaOH \wedge C \wedge O_2 \longrightarrow NaCl \wedge H_2CO_3}$$

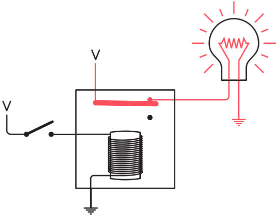
$$\frac{\dfrac{HCl \quad NaOH}{NaCl} \qquad \dfrac{\dfrac{HCl \quad NaOH}{H_2O} \qquad \dfrac{C \quad O_2}{CO_2}}{H_2CO_3}}{NaCl \wedge H_2CO_3}$$

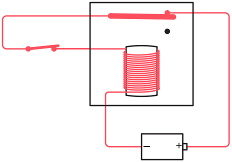
电源、电线、开关、灯泡能用来做什么？



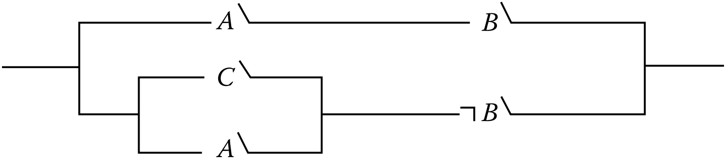




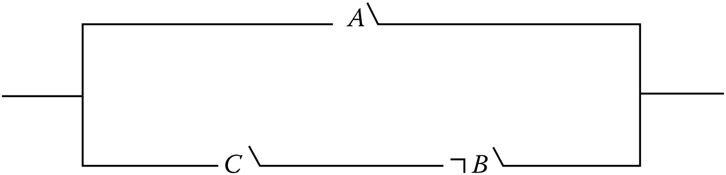




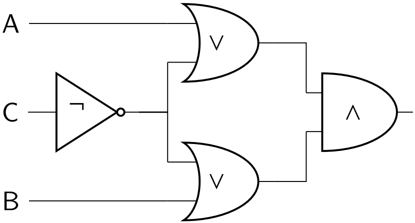
用逻辑做电路优化

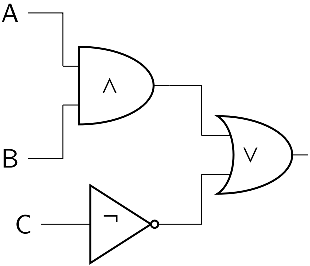


$$\frac{(A \wedge B) \vee ((C \vee A) \wedge \neg B)}{A \vee (C \wedge \neg B)}$$

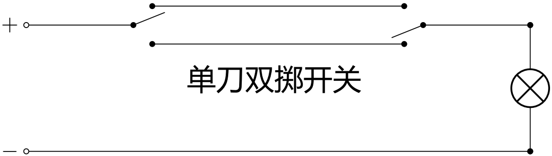


用逻辑做电路优化



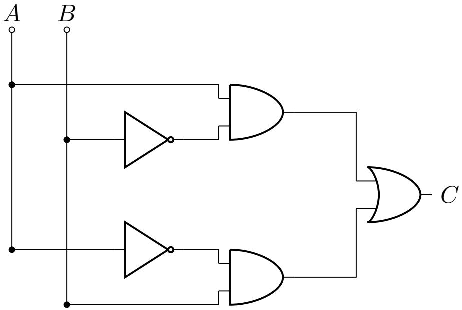


$$\frac{(A \lor \neg C) \land (B \lor \neg C)}{(A \land B) \lor \neg C}$$



- 怎么用两个“单刀单掷开关”独立地控制一盏灯?

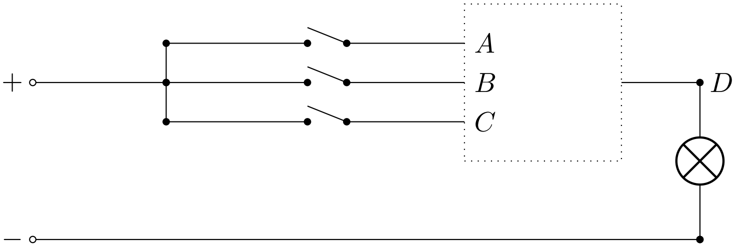
$$(A \wedge \neg B) \vee (\neg A \wedge B)$$



用逻辑做电路设计

- 怎么用三个“单刀单掷开关”独立地控制一盏灯？

$$(A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C) \vee (A \wedge B \wedge C)$$



Problem

设计一个配备三把钥匙的保险箱，要求至少同时使用其中的两把钥匙才能开启.

用逻辑做算术计算

- 二进制自然数:
  $0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, \ldots$
- 二进制加法

$$\begin{array}{cccc} & 1 & 0 & 1 \\ & \bullet & 1_{\bullet} & 1 \\ \hline 1 & 0 & 0 & 0 \end{array}$$

- 对于某一位上的加法, 我们给定的输入是被加数的值 a, 加数的值 b, 以及上一位的进位 c.
  1. 这一位需要进位当且仅当 a, b, c 中至少有两个 1

  $$(a \wedge b) \vee (b \wedge c) \vee (a \wedge c)$$

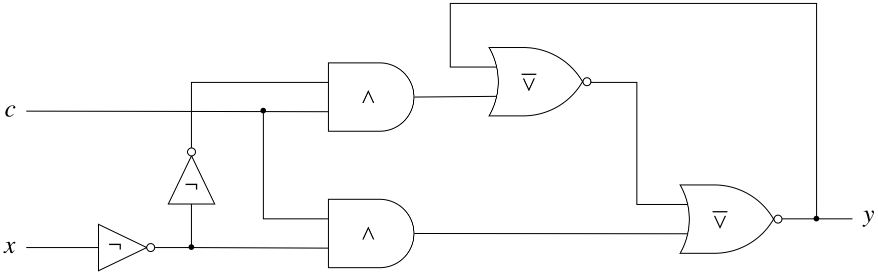
  2. 这一位实际输出为 1 当且仅当 a, b, c 中恰好有一个 1 或三个 1

  $$a \oplus b \oplus c$$

Remark: 乘法呢?

- 整数因式分解问题可以归约到 SAT 问题.
- 整数因式分解是 RSA 密码的基础. 若 P = NP, 则 RSA 将失效.

用逻辑做记忆存储



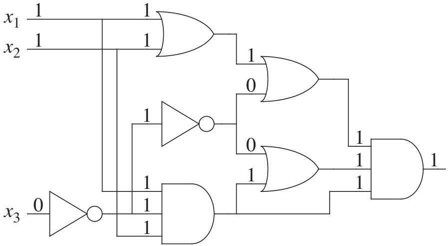
$$y \equiv (c \wedge \neg x) \mathbin{\overline{\vee}} \left[(c \wedge x) \mathbin{\overline{\vee}} y\right]$$

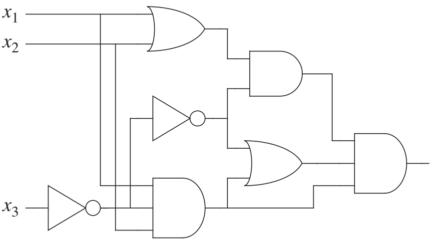
$$y = \overline{c\bar{x} + \overline{cx + y}}$$

$$y = \begin{cases} x & \text{if } c = 1 \\ y & \text{if } c = 0 \end{cases}$$

Remark: 锁存器. 当 $c = 1$ 时, 数据 x 会被 y 记住, 当 $c = 0$ 时, y 不会被修改.

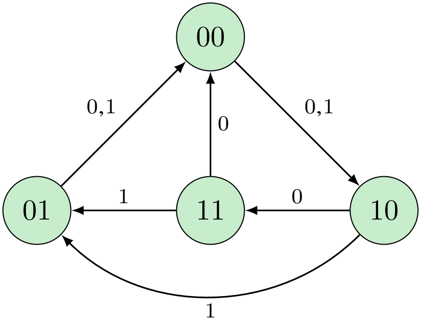
电路可满足性问题 Circuit-Satisfiability Problem (CSAT)





给定一个布尔电路，是否存在一组输入使得输出为 1?

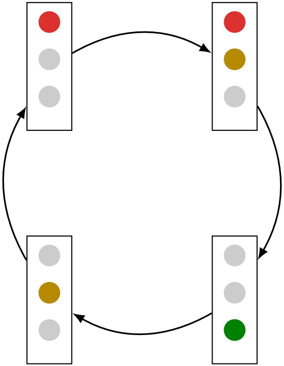
- CSAT 是 NP 完全的.
- CSAT 可以归于到 SAT; SAT 是 NP 完全的.
- 3SAT 是 NP 完全的. (3SAT 是每个子句只含有 3 个文字的合取范式的满足性问题)



y_1	y_2	x	y_1'	y_2'
0	0	0	1	0
0	0	1	1	0
0	1	0	0	0
0	1	1	0	0
1	0	0	1	1
1	0	1	0	1
1	1	0	0	0
1	1	1	0	1

$$y_1' = \overline{y}_1\overline{y}_2 + \overline{y}_2\overline{x}$$
$$y_2' = y_1\overline{y}_2 + y_1 x$$

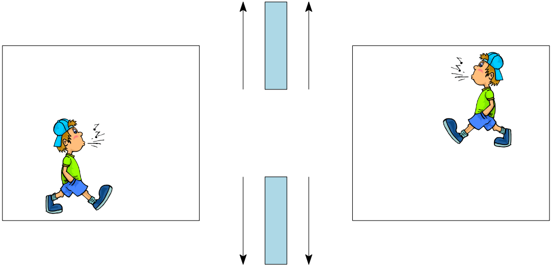
红绿灯



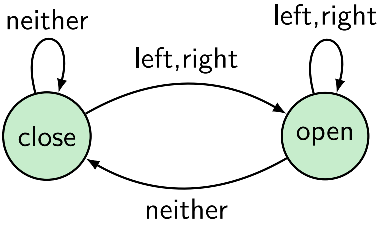
r	a	g	r'	a'	g'
1	0	0	1	1	0
1	1	0	0	0	1
0	0	1	0	1	0
0	1	0	1	0	0

$$r' = r \oplus a$$
$$a' = \overline{a}$$
$$g' = ra$$

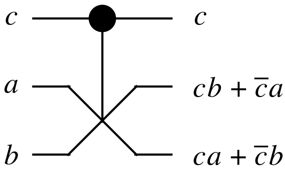
自动门

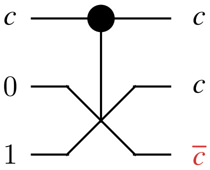


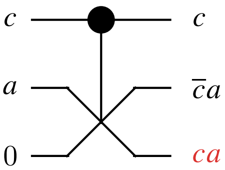
- States = {close, open}
- Input = {left, right, neither}



可逆计算 — Fredkin Gate: CSWAP







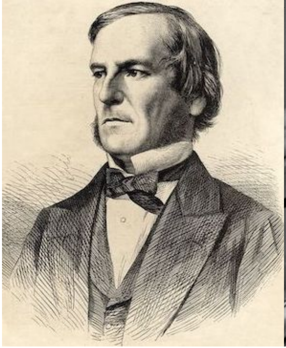
控制位 c 保持不变; 当且仅当 $c = 1$ 时, a 和 b 交换.

$$f : (c, a, b) \mapsto (c, cb + \bar{c}a, ca + \bar{c}b)$$

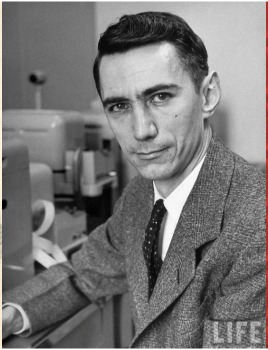
c	a	b	c	x	y
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	1	0
1	1	0	1	0	1
1	1	1	1	1	1

- Fredkin 门可模拟任何经典逻辑门.
- 在可逆计算中, 没有信息被擦除, 原则上没有能量耗散, 没有熵增.
- Landauer 原理: 擦除 1 比特信息会向环境中耗散至少 $kT \ln 2$ 的热量.

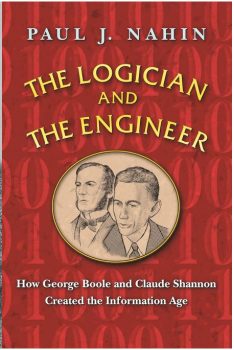
- 真值表 (表示唯一)
- 布尔表达式 (方便变换)
- 电路图 (方便应用)



(a) 布尔

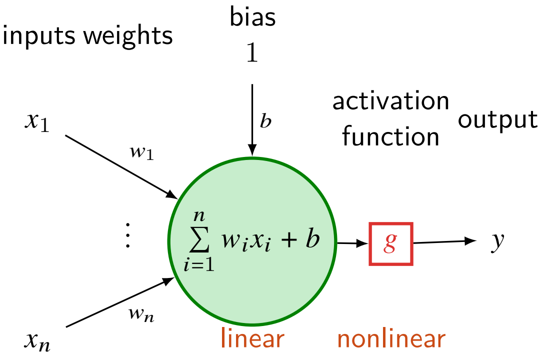


(b) 香农

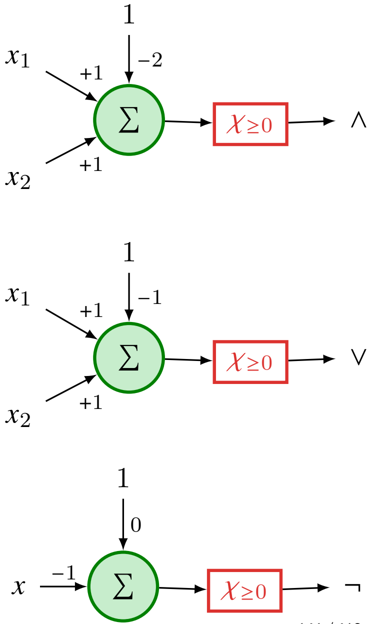


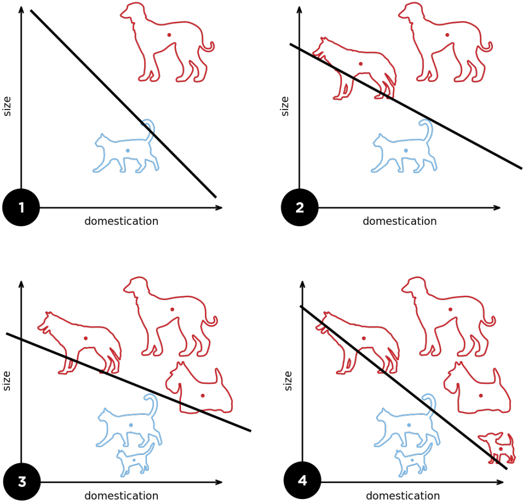
(c) 布尔 & 香农

McCulloch-Pitts 人工神经网络 (神经的逻辑演算)



$$y = g\left(\sum_{i=1}^{n} w_i x_i + b\right)$$





1-layer NN

$$y = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} w_i x_i + b \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

线性不可分问题

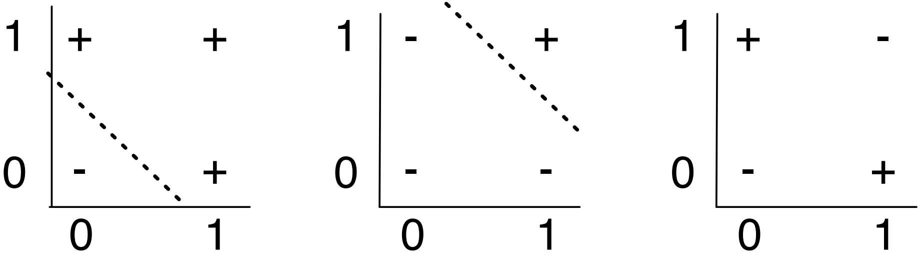


Figure: $\vee, \wedge, \oplus$

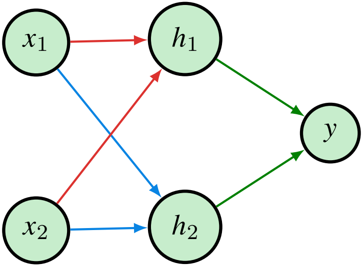
x_1	x_2	$x_1 \oplus x_2$
0	0	0
0	1	1
1	0	1
1	1	0

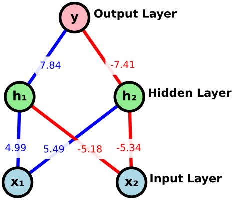
$$
\begin{aligned}
w_1 0 + w_2 0 + b &< 0 & b &< 0 \\
w_1 0 + w_2 1 + b &\geq 0 & w_2 + b &\geq 0 \\
w_1 1 + w_2 0 + b &\geq 0 & w_1 + b &\geq 0 \\
w_1 1 + w_2 1 + b &< 0 & w_1 + w_2 + b &< 0
\end{aligned}
$$

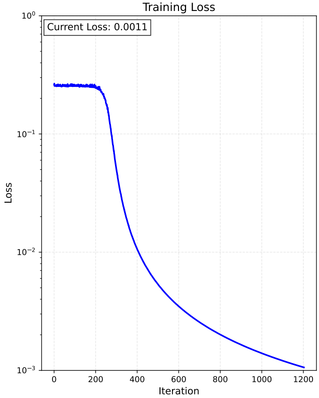
单层感知机无法解决线性不可分问题 (比如异或问题).

异或问题

$$\underbrace{x_1 \oplus x_2}_{y} \equiv (\underbrace{\neg x_1 \wedge x_2}_{h_1}) \vee (\underbrace{x_1 \wedge \neg x_2}_{h_2})$$







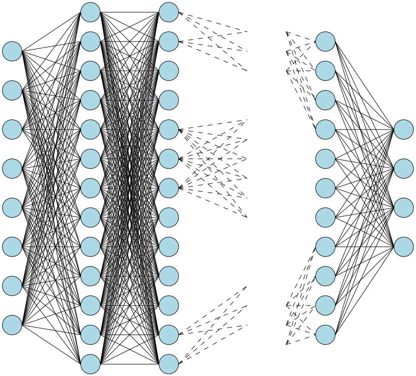
《三体》— 人列计算机

- 秦始皇: 朕当然需要预测太阳的运行, 但你们让我集结三千万大军, 至少要首先向朕演示一下这种计算如何进行吧?
- 冯诺依曼: 陛下, 请给我三个士兵, 我将为您演示. ... 我们组建一千万个这样的门部件, 再将这些部件组合成一个系统, 这个系统就能进行我们所需要的运算, 解出那些预测太阳运行的微分方程[a].

[a] 即 $\frac{\mathrm{d}^2\mathbf{r}_i}{\mathrm{d}t^2} = -\sum_{j\neq i} Gm_j \frac{\mathbf{r}_i-\mathbf{r}_j}{|\mathbf{r}_i-\mathbf{r}_j|^3} \qquad i = 1, 2, 3$

用连续信号模拟离散信号会怎样?

深度神经网络



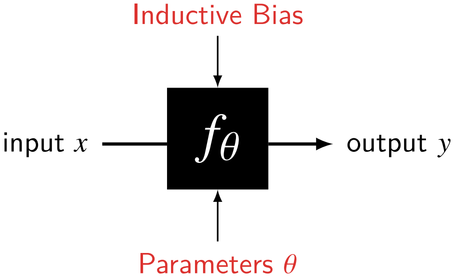
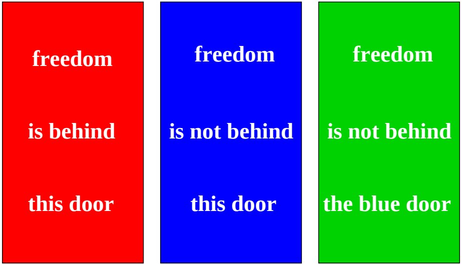


Figure: Walter Pitts & Jeff Hinton

自由之门

Problem (哪扇门通往自由?)

1. 三扇门中只有一扇是自由之门, 另外两扇是死亡之门.
2. 门上的话至少有一句是真的.
3. 门上的话至少有一句是假的.



1. $(r \wedge \neg b \wedge \neg g) \vee (\neg r \wedge b \wedge \neg g) \vee (\neg r \wedge \neg b \wedge g)$
2. $r \vee \neg b \vee \neg b$
3. $\neg r \vee \neg\neg b \vee \neg\neg b$

Solution

r	b	g	1	2	3
0	0	0	0	1	1
0	0	1	1	1	1
0	1	0	1	0	1
0	1	1	0	0	1
1	0	0	1	1	0
1	0	1	0	1	0
1	1	0	0	1	1
1	1	1	0	1	1

r	b	g	
0	0	0	1×
0	0	1	
0	1	0	2×
0	1	1	1×
1	0	0	3×
1	0	1	1×
1	1	0	1×
1	1	1	1×

$$(r \land \neg b \land \neg g) \lor (\neg r \land b \land \neg g) \lor (\neg r \land \neg b \land g), r \lor \neg b, \neg r \lor b \vdash g$$

$$\dfrac{\dfrac{\dfrac{\neg r \lor b}{\neg r \lor b \lor g}}{\neg(r \land \neg b \land \neg g)} \qquad \dfrac{\dfrac{r \lor \neg b}{r \lor \neg b \lor g}}{\neg(\neg r \land b \land \neg g)} \qquad (r \land \neg b \land \neg g) \lor (\neg r \land b \land \neg g) \lor (\neg r \land \neg b \land g)}{\neg r \land \neg b \land g}$$

$$(r\overline{b}\,\overline{g} + \overline{r}b\overline{g} + \overline{r}\,\overline{b}g)(r + \overline{b})(\overline{r} + b) = \overline{r}\,\overline{b}g$$

Problem (宝藏在哪里?)

你面前有三扇门, 只有一扇门后是宝藏. 门上各有一句话, 只有一扇门上的是真话.

① 宝藏不在这儿.

② 宝藏不在这儿.

③ 宝藏在②号门.

Solution

- ① $\neg t_1$; ② $\neg t_2$; ③ t_2

1. 只有一扇门上的是真话.
   $(\neg t_1 \wedge \neg\neg t_2 \wedge \neg t_2) \vee (\neg\neg t_1 \wedge \neg t_2 \wedge \neg t_2) \vee (\neg\neg t_1 \wedge \neg\neg t_2 \wedge t_2)$
2. 只有一扇门后是宝藏.
   $(t_1 \wedge \neg t_2 \wedge \neg t_3) \vee (\neg t_1 \wedge t_2 \wedge \neg t_3) \vee (\neg t_1 \wedge \neg t_2 \wedge t_3)$

$$\bar{t}_1\bar{\bar{t}}_2\bar{t}_2 + \bar{\bar{t}}_1\bar{t}_2 + \bar{\bar{t}}_1\bar{\bar{t}}_2 t_2 = 1 \implies t_1 = 1$$
$$t_1\bar{t}_2\bar{t}_3 + \bar{t}_1 t_2\bar{t}_3 + \bar{t}_1\bar{t}_2 t_3 = 1 \implies t_2 = t_3 = 0$$

哪条大路通罗马?

Problem (哪条大路通罗马?)

你面前有左、中、右三条大路. 三条路分别被三个**说谎的**守卫守护着.

1. 左路护卫: “左路通罗马. 如果右路通罗马的话, 中路也通罗马”.
2. 中路护卫: “左路、右路都不通罗马.”
3. 右路护卫: “左路通罗马, 中路不通.”

Solution

1. $\neg(x \wedge (z \to y))$
2. $\neg(\neg x \wedge \neg z)$
3. $\neg(x \wedge \neg y)$

$$\overline{x(\overline{z} + y)}\ \overline{\overline{x}\,\overline{z}}\ \overline{x\overline{y}} = 1$$

$$x(\overline{z} + y) + \overline{x}\,\overline{z} + x\overline{y} = 0$$

$$\overline{z} + x = 0$$

$$z = 1, x = 0$$

竞赛排名

Problem (每人的名次是多少?)

A, B, C, D 四人参加了一场竞赛.

- A 猜测: C 第 1, B 第 2.
- B 猜测: C 第 2, D 第 3.
- C 猜测: D 第 4, A 第 2.

结果每人都只猜对了一半, 求每人的名次.

Solution

令 X_i 表示 X 是第 i 名.
显然, 当 $i \neq j$ 时, $X_i X_j = 0$. 当 $X \neq Y$ 时, $X_i Y_i = 0$.

$$(C_1 \overline{B}_2 + \overline{C}_1 B_2)(C_2 \overline{D}_3 + \overline{C}_2 D_3)(D_4 \overline{A}_2 + \overline{D}_4 A_2) = 1$$

$$\Downarrow$$

$$C_1 \overline{C}_2 A_2 D_3 \overline{D}_4 \overline{B}_2 = 1$$

Problem (只从一个盒子里取一个水果观察，能否把标签重贴正确?)

- 三个盒子．一个只装苹果，一个只装橙子，一个两样都装．
- 标签都是错的．

Apple	Both	Orange

令 Px 表示标签为 P 的盒子装的是 x．

1. $Ao \vee Ab$
2. $Ba \vee Bo$
3. $Oa \vee Ob$
4. $Px \to \neg Py$ 当 $x \neq y$ 时．其中 $x \in \{a, b, o\}$．
5. $Px \to \neg Qx$ 当 $P \neq Q$ 时．其中 $P, Q \in \{A, B, O\}$．

$$\dfrac{\dfrac{\dfrac{\dfrac{\color{red}{Bo} \quad Bo \to \neg Ao}{\neg Ao} \qquad Ao \vee Ab}{Ab} \qquad Ab \to \neg Ob}{\neg Ob} \qquad Oa \vee Ob}{Oa}$$

A	B	O	
a	b	o	×
a	o	b	×
b	a	o	×
b	o	a	✓
o	a	b	✓
o	b	a	×

自指问题

Problem (下面哪一项是这个问题的正确答案?)

1. 下面所有项.
2. 下面没有一项.
3. 上面所有项.
4. 上面某一项.
5. 上面没有一项.
6. 上面没有一项.

1. $p_1 = p_2p_3p_4p_5p_6$
2. $p_2 = \overline{p}_3\overline{p}_4\overline{p}_5\overline{p}_6$
3. $p_3 = p_1p_2$
4. $p_4 = p_1\overline{p}_2\overline{p}_3 + \overline{p}_1p_2\overline{p}_3 + \overline{p}_1\overline{p}_2p_3$
5. $p_5 = \overline{p}_1\overline{p}_2\overline{p}_3\overline{p}_4$
6. $p_6 = \overline{p}_1\overline{p}_2\overline{p}_3\overline{p}_4\overline{p}_5$

$$\begin{array}{l}
\underline{2 \quad 3} \\
\underline{p_3 = 0 \quad 1} \\
\quad \underline{p_1 = 0 \qquad 4} \\
\qquad \underline{p_4 = p_2 \qquad 2} \\
\qquad \underline{p_2 = p_4 = 0 \qquad 5} \\
\qquad\qquad \underline{p_5 = 1 \qquad\qquad 6} \\
\qquad\qquad\qquad p_6 = 0
\end{array}$$

Problem (谁是君子? 谁是小人?)

一个岛上有 “君子”、“小人” 两类人. “君子” 只说真话, “小人” 只说假话. 你来岛上遇到了小艾、小白、小菜三个土著.

1. 小艾: “如果小菜说谎, 我或小白说的就是真话”.
2. 小白: “如果小艾或小菜说真话, 那么, 我们三人中有且只有一人说真话是不可能的”.
3. 小菜: “小艾或小白说谎当且仅当小艾或我说真话”.

1. $a = c + a + b$
2. $b = \overline{a + c} + a\overline{b}\,\overline{c} + \overline{a}b\overline{c} + \overline{a}\,\overline{b}c$
3. $c = (\overline{a} + \overline{b})(a + c) + ab\,\overline{a + c}$

$$c = (\overline{a} + \overline{b})(a + c) + ab\overline{a}\,\overline{c} = (\overline{a} + \overline{b})(a + c) + 0 = (\overline{a} + \overline{b})a = a\overline{b}$$

$$b = \overline{a + c} + \overline{c\overline{c} + \overline{c + a + b}\ b\overline{c} + \overline{c + a + b}\ \overline{b}c} = \overline{a + c} + \overline{0 + 0 + 0} = 1$$

$$a = c + a + 1 = 1$$

$$c = a\overline{b} = 0$$

练习 — Now it's your turn ↯

Problem (我在做啥?)

1. 如果我不在打网球, 那就在看网球.
2. 如果我不在看网球, 那就在读网球杂志.
3. 但我不能同时做两件或两件以上的事.

Problem (你想下厨为全家准备晚餐. 做点儿什么呢?)

1. 香菇青菜、红烧鱼、千页豆腐三个菜里你希望至少做出两个来.
2. 爸爸红烧鱼和千页豆腐两样中必须吃一样, 但不同时吃.
3. 妈妈除非同时吃香菇青菜, 否则不吃千页豆腐, 而且, 只要吃红烧鱼就不吃香菇青菜.

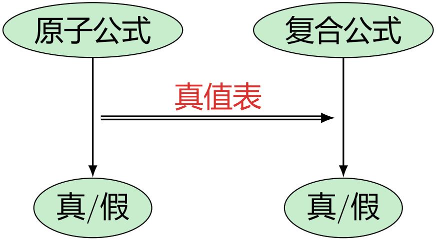
Problem (A, B, C, D 四个嫌疑人, 谁有罪?)

1. 如果 A 有罪, 那么 B 也有罪.
2. 如果 B 有罪, 那么或者 A 无罪或者 C 有罪.
3. 如果 D 有罪, 那么 A 也有罪.
4. 如果 D 无罪, 那么 A 有罪而 C 无罪.

总结

- 语法

- 语义



- 形式系统

- 表达力 / 简洁性
- 可满足性 / 有效性
- 可靠性 / 完备性 / 紧致性
- 可判定性 / 计算复杂性

Contents

Contents

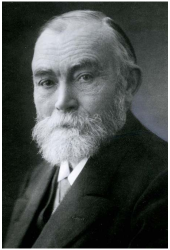
弗雷格 Gottlob Frege 1848-1925

皮尔士 Charles Peirce 1839-1914

- 《概念文字: 一种模仿算术语言构造的纯思维的形式语言》*1879.*
- $\boxed{\text{逻辑主义}}$ 数学可还原为逻辑.[a]
- 谓词逻辑之父
  (关系 & 量词)
  (Every boy loves some girl.)

$$\frac{\text{subject}}{\text{predicate}} \approx \frac{\text{argument}}{\text{function}}$$

- 语言哲学
  The evening star is the morning star.[b]



[a] Frege: The Foundations of Arithmetic. 1884.
[b] Frege: On Sense and Reference. 1892.

Why Study Predicate Logic?

- 命题逻辑预设世界由事实构成
- 谓词逻辑预设世界包含
  1. 个体: 人、狗、书、自然数、实数、城市、国家 ...
  2. 关系: 红的、圆的、大于、爱上、父子、朋友、老师 ...
  3. 函数: 平方、加法、母亲、老婆、最好的朋友、导师 ...
- 谓词逻辑的表达力更强

$$\frac{\mathrm{Father}(\mathrm{Father}(\mathrm{alice})) = \mathrm{Father}(\mathrm{Mother}(\mathrm{bob}))}{\mathrm{Cousin}(\mathrm{alice}, \mathrm{bob})}$$

语言	本体论承诺	认识论承诺
Propositional Logic	facts	true/false/unknown
Predicate Logic	facts, objects, relations	true/false/unknown
Temporal Logic	facts, objects, relations, times	true/false/unknown
Probability Theory	facts	degree of belief $[0, 1]$
Fuzzy Logic	facts with degree of truth $[0, 1]$	known interval value

Example ☹

What will a logician choose: philosophy or money?

Philosophy! Because Nothing is better than Money, and Philosophy is better than Nothing.

$$
\begin{aligned}
p > 0 > m &\implies p > m \\
\neg\exists x(x > m) &\implies 0 \not> m
\end{aligned}
$$

A cat has nine lives ☹

No cat has eight lives. A cat has one more life than no cat.

爱丽丝镜中奇遇 — Lewis Carroll

- "I see nobody on the road." said Alice.
- "I only wish I had such eyes," the King remarked in a fretful tone. "To be able to see Nobody! And at that distance too!"

Contents

什么是“量词”?

- All students work hard.
- Some students are asleep.
- At least six students are awake.
- Exactly five students pass the exam.
- Eight out of ten students are good at logic.
- Nobody knows logic better than Donald Trump.
- The present King is smart.
- Most/Many/Few students love logic.
- There are finitely/infinitely/countably/uncountably many elements such that...
- For all but finitely many elements...
- There are more enemies than friends.
- **Some** girls admire only one another.

语言

$$\mathscr{L}^1 := \{\neg, \wedge, \vee, \to, \leftrightarrow, \forall, \exists, =, (,)\} \cup \mathrm{Var} \cup \overbrace{\mathrm{Cst} \cup \mathrm{Fun} \cup \mathrm{Pred}}^{\text{signature}}$$

其中

$$\mathrm{Var} := \{x_i : i \in \mathbb{N}\}$$

$$\mathrm{Cst} := \{c_i : i \in \mathbb{N}\}$$

$$\mathrm{Fun} := \bigcup_{n\in\mathbb{N}} \mathrm{Fun}^n \quad \mathrm{Fun}^n := \left\{f_1^n, f_2^n, f_3^n, \dots\right\}$$

$$\mathrm{Pred} := \bigcup_{n\in\mathbb{N}} \mathrm{Pred}^n \quad \mathrm{Pred}^n := \left\{P_1^n, P_2^n, P_3^n, \dots\right\}$$

- c 是常元符号
- f^n 是 n 元函数符号
- P^n 是 n 元谓词符号

项 & 公式 ♡

Definition (项 Term)

$$t := x \mid c \mid f(t, \ldots, t)$$

where $x \in \mathrm{Var}$, $c \in \mathrm{Cst}$ and $f \in \mathrm{Fun}$.

- Term is freely generated from Var by Fun.

Definition (公式 Well-Formed Formula Wff)

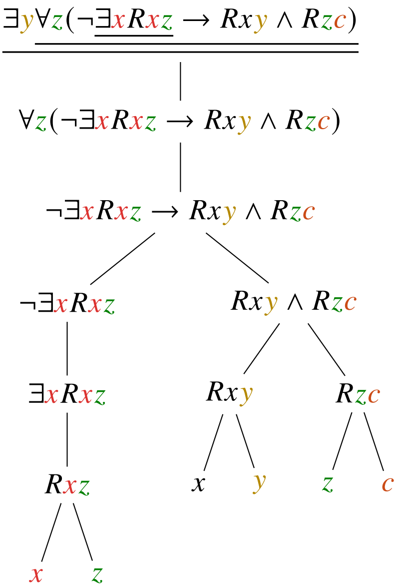
$$A := \overbrace{t = t \mid P(t, \ldots, t)}^{\text{atomic formula}} \mid \neg A \mid A \wedge A \mid A \vee A \mid A \to A \mid A \leftrightarrow A \mid \forall x A \mid \exists x A$$

where $t \in \mathrm{Term}$ and $P \in \mathrm{Pred}$.

- Wff is freely generated from atomic formulas by connectives and quantifier operators.

语法 ♡

- $A \wedge B := \neg(A \to \neg B)$
- $A \vee B := \neg A \to B$
- $A \leftrightarrow B := (A \to B) \wedge (B \to A)$
- $\exists x A := \neg \forall x \neg A$
- $\bot := A \wedge \neg A$
- $\top := \neg \bot$
- 项、子项、公式、子公式的定义
- 项和公式上的归纳法
- 项和公式的唯一可读性
- 省略括号的约定:
  1. 公式最外层的括号可以省略
  2. $\neg, \forall, \exists, \wedge, \vee, \to, \leftrightarrow$ 的组合强度依次减弱
  3. 同一连接词相邻出现时，右边的组合力更强



- 变元 x 出现在形如 $\forall xA$ (或 $\exists xA$) 的子公式中是**约束的**(bound).
- 变元 x 的出现不是约束的即为**自由的**(free).
- 不含变元的项为**闭项**(closed/ground term).
- 不含自由变元的公式为**闭公式/句子**(closed formula/sentence).

记号:

- $\mathrm{Var}(t)$: 项 t 中变元的集合
- $\mathrm{Bv}(A)$: 公式 A 中约束变元的集合
- $\mathrm{Fv}(A)$: 公式 A 中自由变元的集合

Remark

- **命题**是具有真值的表达式，是“真值承担者”，或为真或为假，但不能既真又假.
  1. 简奥斯汀是《傲慢与偏见》的作者. ✓
  2. 爱因斯坦是《傲慢与偏见》的作者. ×
  3. 谁是《傲慢与偏见》的作者? ?
  4. 因为这个论证有效，所以这个论证无效. ?
- 闭公式是命题.
  1. 某人是《傲慢与偏见》的作者. ✓
  2. 所有人都是《傲慢与偏见》的作者. ×
  3. 某人是某书的作者. ✓
  4. 某人是所有书的作者. ×
- 包含自由变元的公式是**命题函数**，给定自由变元的取值后变为命题.
  1. x 是 y 的作者. ?
  2. x 是《傲慢与偏见》的作者. ?
  3. 简奥斯汀是 y 的作者. ?
  4. 简奥斯汀是《傲慢与偏见》的作者. ✓

A	所有学生都漂亮.	$\forall x(Sx \to Px)$
E	没有学生漂亮.	$\forall x(Sx \to \neg Px)$
I	有些学生漂亮.	$\exists x(Sx \wedge Px)$
O	有些学生不漂亮.	$\exists x(Sx \wedge \neg Px)$

1. 有些学生爱所有漂亮的学生.

$$\exists x(Sx \wedge \forall y(Sy \wedge Py \to Lxy))$$

2. 愚蠢的人自信满满, 而聪明的人却充满怀疑. —罗素

$$\forall x(Sx \to Cx) \wedge \forall x(Ix \to Dx)$$

3. 敌人的敌人是朋友.

$$\forall x\forall y\forall z(Exy \wedge Eyz \to Fxz)$$

4. 爷爷是爸爸的爸爸.

$$\forall x\forall y\big(Gxy \leftrightarrow \exists z(Fxz \wedge Fzy)\big) \qquad \forall x\big(g(x) = f(f(x))\big)$$

5. 有爹就有娘.

$$\forall x(\exists yFyx \to \exists yMyx)$$

$\forall x$

Every boy loves some girl

$\rightarrow$

If x is a boy then x loves some girl

Bx

x is a boy

$\exists y$

x loves some girl

$\wedge$

y is a girl whom x loves

Gy

y is a girl

Lxy

x loves y

- Every boy loves some girl. $\forall x(Bx \rightarrow \exists y(Gy \wedge Lxy))$
- There is a girl whom every boy loves. $\exists x(Gx \wedge \forall y(By \rightarrow Lyx))$

$\neg$

No girl who loves a boy is not loved by some boy

$\exists x$

Some girl who loves a boy is not loved by some boy

$\wedge$

x is a girl who loves a boy is not loved by some boy

$\wedge$

x is a girl who loves a boy

Gx

x is a girl

$\exists y$

x loves a boy

$\wedge$

y is a boy who is loved by x

By

y is a boy

Lxy

x loves y

$\neg$

x is not loved by some boy

$\exists z$

x is loved by some boy

$\wedge$

z is a boy who loves x

Bz

z is a boy

Lzx

z loves x

$$\neg\exists x(Gx \wedge \exists y(By \wedge Lxy) \wedge \neg\exists z(Bz \wedge Lzx))$$

翻译 — 学说 “逻辑语”

1. 谁与你的敌人为友就是与我为敌.

$$\forall x \forall y \big(Eyu \wedge Fxy \to Exm\big)$$

2. 既没有朋友又没有敌人是寂寞的.

$$\forall x (\neg \exists y Fxy \wedge \neg \exists z Exz \to Lx)$$

3. 朋友之间要么都抽烟要么都不抽烟.

$$\forall x \forall y \big(Fxy \to (Sx \leftrightarrow Sy)\big)$$

4. 最可怕的敌人是最亲密的朋友.

$$\forall x \forall y \big(Exy \wedge \forall z (Exz \to Tyz) \to Fxy \wedge \forall z (Fxz \to Cyz)\big)$$

5. 名, 可名, 非常名.

$$\forall x \Big(\exists y \mathsf{Name}(x, y) \wedge \exists z \mathsf{Name}(z, x) \to \neg \mathsf{Unchanging}(x)\Big) ?$$

翻译 — 学说 “逻辑语”

1. If bad things happen to good people, then God is either not omnipotent or not benevolent.

$$\exists x \exists y(\mathsf{Bad}(x) \wedge \mathsf{Good}(y) \wedge \mathsf{Happen}(x, y)) \to \neg\mathsf{Omnip}(g) \vee \neg\mathsf{Benev}(g)$$

2. Every book that Alice lends to Bob she steals from Chris.

$$\forall x\big(Bx \wedge Lxab \to Sxac\big)$$

3. For every professor, there is a student who likes every book the professor recommends to the student.

$$\forall x\Big(Px \to \exists y\big(Sy \wedge \forall z(Bz \wedge Rzxy \to Lyz)\big)\Big)$$

4. When a mathematical or philosophical author writes with a misty profundity, he is talking nonsense. — *Alfred Whitehead*

$$\forall x\Big(\big(\mathsf{Math}(x) \vee \mathsf{Pilo}(x)\big) \wedge \mathsf{Write}(x) \to \mathsf{TalkNonsense}(x)\Big)$$

5. No dolphin sings unless it jumps.

$$\forall x\Big(\mathsf{Dolphin}(x) \to \neg\mathsf{Jump}(x) \to \neg\mathsf{Sing}(x)\Big)$$

翻译 — 学说 “逻辑语”

1. If all dancers have knee injuries, then *they* will be frustrated.

$$\forall x\Big(\mathsf{Dancer}(x) \to \mathsf{Knee}(x)\Big) \to \forall x\Big(\mathsf{Dancer}(x) \to \mathsf{Frustrated}(x)\Big)$$

2. If all dancers have knee injuries, then *some of them* will be frustrated.

$$\forall x\Big(\mathsf{Dancer}(x) \to \mathsf{Knee}(x)\Big) \to \exists x\Big(\mathsf{Dancer}(x) \wedge \mathsf{Frustrated}(x)\Big)$$

3. If a student takes a course and the course covers a concept, then the student knows that concept.

$$\forall x\forall y\forall z\Big(\mathsf{Student}(x) \wedge \mathsf{Course}(y) \wedge \mathsf{Take}(x,y) \wedge \mathsf{Concept}(z) \wedge \mathsf{Cover}(y,z) \to \mathsf{Know}(x,z)\Big)$$

4. Alice is the first to have completed the test.

$$Cat \wedge \forall x(Cxt \wedge x \neq a \to Baxt)$$

Alice is the oldest daughter of the King.

5. Alice is the second to have completed the test.

$$Cat \wedge \exists x\Big(Cxt \wedge x \neq a \wedge Bxat \wedge \forall y(Cyt \wedge y \neq a \wedge y \neq x \to Bayt)\Big)$$

Example — 究竟有几个神?

有神论 至少有一个神

$$\exists x Gx$$

无神论 没有神

$$\neg \exists x Gx$$

泛神论 万物皆神

$$\forall x Gx$$

不可知论 至多有一个神

$$\forall x (Gx \to \forall y (Gy \to y = x))$$

一神论 有且仅有一个神

$$\exists x (Gx \wedge \forall y (Gy \to y = x))$$

二元神论 有且仅有两个神

$$\exists x (Gx \wedge \exists y (Gy \wedge x \neq y) \wedge \forall z (Gz \to z = x \vee z = y))$$

至多/至少/恰好

1. 至多有 1 个元素.

$$\forall x \forall y (x = y)$$

2. 至少有 2 个元素.

$$\exists x \exists y (x \neq y)$$

3. 有且仅有 2 个元素.

$$\exists x \exists y \big(x \neq y \wedge \forall z (z = x \vee z = y)\big)$$

4. 有且仅有 n 个元素.

$$\exists x_1 \ldots \exists x_n \left(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \wedge \forall y \left(\bigvee_{i=1}^{n} y = x_i \right) \right)$$

5. 有且仅有 1 个元素有 P 性质. (缩写为: $\exists! x Px$)

$$\exists x Px \wedge \forall y \forall z (Py \wedge Pz \rightarrow y = z)$$

$$\exists x \big(Px \wedge \forall y (Py \rightarrow y = x)\big)$$

$$\exists x \forall y (Py \leftrightarrow y = x)$$

- "You are sad," the Knight said in an anxious tone: "let me sing you a song to comfort you."
- "Is it very long?" Alice asked.
- "It's long," said the Knight, "but it's very, very beautiful. Everybody that hears me sing it — either it brings the tears into their eyes, or else —"

$$\exists x \forall y ((\mathsf{Song}(x) \leftrightarrow y = x) \wedge \mathsf{Long}(x) \wedge \mathsf{Beautiful}(x))$$

- "Or else what?" said Alice.
- "Or else it doesn't, you know. The name of the song is called 'Haddocks' Eyes."'

$$\exists! x (\mathsf{Song}(x) \wedge \forall y (\mathsf{Hear}(y, x) \to \mathsf{Tear}(x, y) \vee \neg \mathsf{Tear}(x, y)))$$

$$\mathsf{callname}(\mathsf{name}(\mathsf{the\text{-}song})) = \mathsf{Haddocks'\ Eyes}$$

- “Oh, that’s the name of the song, is it?” said Alice.

$$\text{name(the-song)} = \text{Haddocks' Eyes}$$

- “No, you don’t understand,” the Knight said. “That’s what the name is called. The name really is ‘The Aged Aged Man.”’

$$\text{name(the-song)} = \text{The Aged Aged Man}$$

- “Then I ought to have said ‘That’s what the song is called’?” Alice corrected herself.

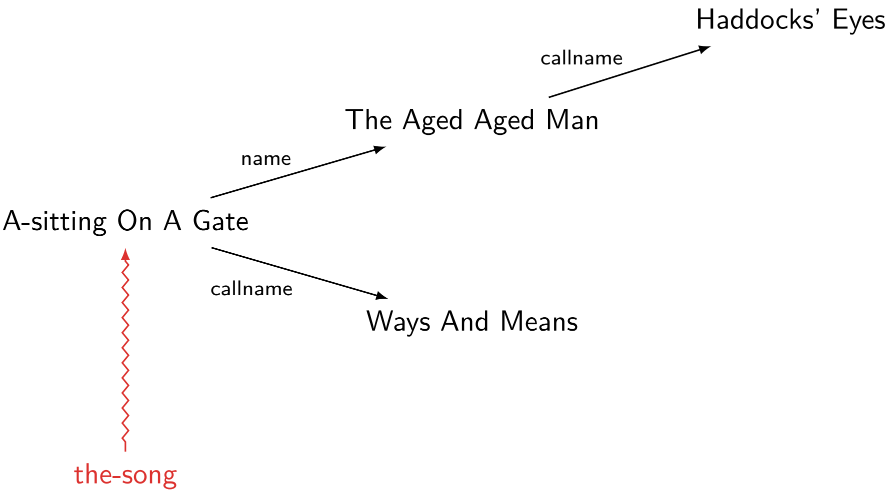
$$\text{callname(the-song)} = \text{The Aged Aged Man}$$

- “No, you oughtn’t: that’s quite another thing! The song is called ‘Ways And Means’: but that’s only what it’s called, you know!”

$$\text{callname(the-song)} = \text{Ways And Means}$$

- “Well, what is the song, then?” said Alice.
- “I was coming to that.” the Knight said. “The song really is ‘A-sitting On A Gate’.”

$$\text{the-song} = \text{A-sitting On A Gate}$$


Haddocks' Eyes
callname
The Aged Aged Man
name
A-sitting On A Gate
callname
Ways And Means
the-song

1. No dogs must be brought to this Park except on a lead.

$$\forall d \forall t (Bdt \rightarrow Ldt)$$

Objection: this is consistent with $\exists d \exists t (Pdt \wedge \neg Ldt)$.

2. Dogs are not allowed in this Park without leads.

$$\forall d \forall t (Pdt \rightarrow Ldt)$$

Objection? The order should be addressed to the owners, not to the dogs.

3. Owners of dogs are not allowed in this Park unless they keep them on leads.

$$\forall p \forall d \forall t (Opd \wedge Ppt \rightarrow Lpdt)$$

Objection: doesn't allow $\exists t \exists p \exists d (Opd \wedge Ppt \wedge \neg Pdt \wedge \neg Lpdt)$.

4. Nobody without his dog on a lead is allowed in this Park.

$$\forall p \forall t (Ppt \rightarrow \exists d (Opd \wedge Lpdt))$$

Objection: doesn't allow $\exists p \exists t (Ppt \wedge \neg \exists d Opd)$.

5. Dogs must be led in this Park.

$$\forall d \forall t (Ldt \wedge Pdt) \qquad \forall d \forall t (Ldt \rightarrow Pdt)$$

6. All dogs in this Park must be kept on the lead.

$$\forall d \forall t (Pdt \rightarrow Ldt)$$

$$\forall d \forall t \forall p (Pdt \wedge Opd \rightarrow Lpdt)$$

翻译 — 学说 “逻辑语”

1. He who learns but does not think, is lost. He who thinks but does not learn is in danger. — *Confucius*

$$\forall x(\mathsf{Learn}(x) \land \neg\mathsf{Think}(x) \to \mathsf{Lost}(x)) \land$$
$$\forall x(\mathsf{Think}(x) \land \neg\mathsf{Learn}(x) \to \mathsf{InDanger}(x))$$

2. He who can, does. He who cannot, teaches. — *Bernard Shaw*

$$\forall xy(\mathsf{Can}(x,y) \to \mathsf{Do}(x,y)) \land \forall xy(\neg\mathsf{Can}(x,y) \to \mathsf{Teach}(x,y))$$

3. Don't interrupt me, while I'm interrupting. — *Churchill*

$$\exists x \mathsf{Interrupt}(i,x) \to \forall y \neg\mathsf{Interrupt}(y,i)$$

4. There are no shortcuts to any place worth going. — *Beverly Sills*

$$\forall x(\mathsf{Place}(x) \land \mathsf{WorthGo}(x) \to \neg\exists y \mathsf{Shortcut}(y,x))$$

5. The old believe everything; the middle-aged suspect everything; the young know everything. — *Oscar Wilde*

$$\forall x(\mathsf{Old}(x) \to \forall y \mathsf{Believe}(x,y)) \land$$
$$\forall x(\mathsf{Middle}(x) \to \forall y \mathsf{Suspect}(x,y)) \land \forall x(\mathsf{Young}(x) \to \forall y \mathsf{Know}(x,y))$$

6. No married man is ever attractive except to his wife. — *Oscar Wilde*

$$\forall x(\mathsf{Man}(x) \land \mathsf{Married}(x) \to \forall y(\mathsf{Attractive}(x,y) \to y = \mathsf{wife}(x)))$$

7. Behind every great fortune there is a crime. — *Balzac*

$$\forall x(\text{GreatFortune}(x) \to \exists y(\text{Crime}(y) \wedge \text{Behind}(y,x)))$$

8. Always two there are: a Master and an Apprentice. — *Yoda*

$$\exists xy(x \neq y \wedge \forall z(z = x \vee z = y) \wedge \text{Master}(x) \wedge \text{Apprentice}(y))$$

9. There are two ways to live your life. One is as though nothing is a miracle. The other is as though everything is a miracle. — *Einstein*

$$\exists xy(x \neq y \wedge \forall z(\text{WayLive}(z) \leftrightarrow z = x \vee z = y) \wedge$$
$$(\text{WayLive}(x) \to \neg\exists z\text{Miracle}(z)) \wedge (\text{WayLive}(y) \to \forall z\text{Miracle}(z)))$$

10. The weakest link in a chain is the strongest because it can break it. — *Stanislaw J. Lec*

$$\forall xy((\text{Link}(x) \wedge \text{Chain}(y) \wedge \text{In}(x,y) \wedge \text{Weakest}(x) \to \text{Break}(x,y))$$
$$\to \text{Strongest}(x))$$

11. A man with only a hammer sees every problem as a nail. Our age's greatest hammer is the algorithm. — *Poundstone*

$$\forall x(\text{Man}(x) \wedge \exists! y(\text{Hammer}(y) \wedge \text{With}(x,y)) \to \forall z(\text{Problem}(z) \to$$
$$\exists n(\text{Nail}(n) \wedge \text{Seeas}(x,z,n)))) \quad \wedge \quad \exists x(\text{Hammer}(x) \wedge \text{Age}(x,\text{we}) \wedge$$
$$\forall y(\text{Hammer}(y) \wedge \text{Age}(y,\text{we}) \to \text{Greater}(x,y)) \wedge x = \text{algorithm}))$$

1. $\text{Think}(i) \rightarrow \exists x(x = i)$ *Descartes*
2. $\exists x(x = i) \vee \neg\exists x(x = i)$ *Shakespeare*
3. $\forall x\big(\text{Month}(x) \rightarrow \text{Crueler}(\text{april}, x)\big)$ *Eliot*
4. $\forall x(\neg\text{Weep}(x) \rightarrow \neg\text{See}(x))$ *Hugo*
5. $\forall x\big(\text{Time}(x) \rightarrow \text{Better}(t, x)\big) \wedge \forall x\big(\text{Time}(x) \rightarrow \text{Better}(x, t)\big)$ *Dickens*
6. $\exists x\Big(\text{Child}(x) \wedge \neg\text{Growup}(x) \wedge \forall y\big(\text{Child}(y) \wedge y \neq x \rightarrow \text{Growup}(y)\big)\Big)$ *Barrie*
7. $\forall x \forall y\big(Fx \wedge Fy \rightarrow (Hx \wedge Hy \rightarrow Axy) \wedge (\neg Hx \wedge \neg Hy \rightarrow \neg Axy)\big)$ *Tolstoi*
8. $\forall x(Px \rightarrow \exists y(Ty \wedge Fuxy)) \wedge \exists x(Px \wedge \forall y(Ty \rightarrow Fuxy)) \wedge \neg\forall x(Px \rightarrow \forall y(Ty \rightarrow Fuxy))$ *Lincoln*
9. $\forall x\big(\text{Problem}(x) \wedge \text{Philo}(x) \wedge \text{Serious}(x) \leftrightarrow x = \text{suicide}\big)$ *Camus*
10. $\forall x\big(\text{Feather}(x) \wedge \text{Perch}(x, \text{soul}) \leftrightarrow x = \text{hope}\big)$ *Dickinson*
11. $\forall x\Big(\text{Enter}(x) \rightarrow \forall y\big(\text{Hope}(y) \rightarrow \text{Abandon}(x, y)\big)\Big)$ *Dante*
12. $\exists x \forall y\big(\text{For}(y, x) \wedge \text{For}(x, y)\big)?$ $\forall x \forall y\big(\text{For}(y, x) \leftrightarrow \text{For}(x, y)\big)?$ *Dumas*
13. $\forall x\big(\text{Fear}(\text{we}, x) \leftrightarrow x = \text{fear}\big)?$ *Roosevelt*
14. $\forall x \forall y(Ax \wedge Ay \rightarrow Exy) \wedge \exists x \exists y\big(Ax \wedge Ay \wedge [\![Exx]\!] > [\![Eyy]\!]\big)?$ *Orwell*

1. I think, therefore I am. *Descartes*
2. To be or not to be. *Shakespeare*
3. April is the cruellest month. *Eliot*
4. Those who do not weep, do not see. *Hugo*
5. It was the best of times, it was the worst of times. *Dickens*
6. All Children, except one, grow up. *Barrie*
7. All happy families are alike; each unhappy family is unhappy in it's own way. *Tolstoi*
8. You can fool all the people some of the time, and some of the people all the time, but you can't fool all the people all the time. *Lincoln*
9. There is but one truly serious philosophical problem and that is suicide. *Camus*
10. Hope is the thing with feathers that perches in the soul. *Dickinson*
11. All hope abandon, all you who enter here. *Dante*
12. One for all, all for one. *Dumas*
13. The only thing we have to fear is fear itself. *Roosevelt*
14. All animals are equal, but some animals are more equal than others. *Orwell*

翻译 — “逻辑语言” 到 “自然语言”

1. $\mathsf{Love}(\mathsf{alice}, \mathsf{bob}) \wedge \exists x(\mathsf{Girl}(x) \wedge x \neq \mathsf{alice} \wedge \mathsf{Love}(\mathsf{bob}, x))$
2. $\forall x\big(\mathsf{Buy}(\mathsf{alice}, x) \to \mathsf{Buy}(\mathsf{benny}, x)\big)$
3. $\forall x\big(\exists y \mathsf{Buy}(y, x) \vee \exists z \mathsf{Sell}(z, x) \to \neg\mathsf{Human}(x)\big)$
4. $\forall x\Big(\big(\mathsf{Petted}(x) \to \mathsf{Jump}(x)\big) \to \mathsf{Dog}(x) \vee \mathsf{Dolphin}(x)\Big)$
5. $\forall x\Big(\mathsf{Girl}(x) \wedge \mathsf{Love}(\mathsf{bob}, x) \to \forall y\big(\mathsf{PhilosophyBook}(y) \to \mathsf{Read}(x, y)\big)\Big)$

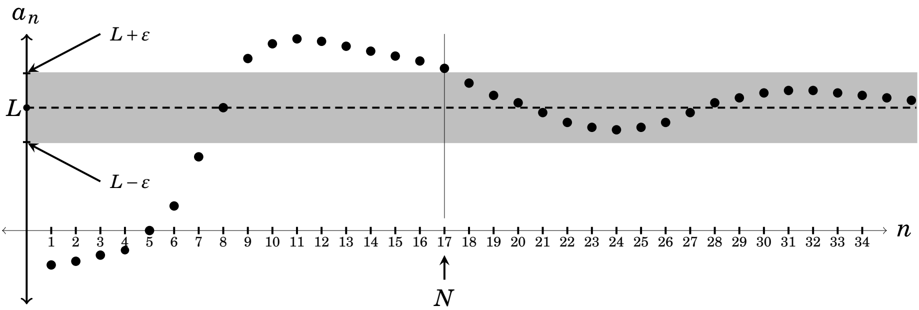
- One driver dies in car accident every minute.
- Every minute one driver dies in car accident.
- $\exists x\Big(\mathsf{Driver}(x) \wedge \forall y\big(\mathsf{Minite}(y) \to \mathsf{Die}(x, y)\big)\Big)$
- $\forall x\Big(\mathsf{Minute}(x) \to \exists y\big(\mathsf{Driver}(y) \wedge \mathsf{Die}(y, x)\big)\Big)$

翻译 — “逻辑语言” 到 “自然语言”

1. $\exists x\Big(Gx \wedge \forall y\big(By \wedge \forall z(Gz \wedge z \neq x \to \neg Lzy) \to Lxy\big)\Big) \to \forall x\Big(Bx \to \exists y\big(Gy \wedge Lyx\big)\Big)$
2. $\forall x \forall y\Big(\big(Gx \wedge \forall y(By \to \neg Lxy)\big) \wedge \big(Gy \wedge \exists x(Bx \wedge Lyx)\big) \to \neg Lxy\Big)$

1. $\exists x\Big(Gx \wedge \forall y\big(By \wedge \forall z(Gz \wedge z \neq x \to \neg Lzy) \to Lxy\big)\Big) \to \forall x\Big(Bx \to \exists y\big(Gy \wedge Lyx\big)\Big)$
2. $\forall x \forall y\Big(\big(Gx \wedge \forall y(By \to \neg Lxy)\big) \wedge \big(Gy \wedge \exists x(Bx \wedge Lyx)\big) \to \neg Lxy\Big)$

安得圣母爱渣男，大庇天下雄性有红颜!



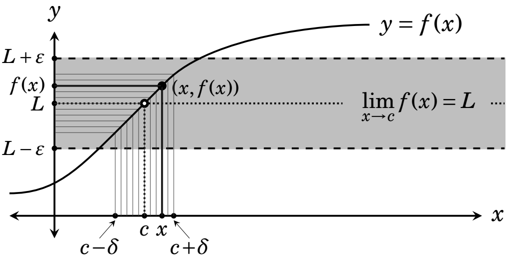
Definition (Convergence)

A sequence $\{a_n\}$ converges to a number $L \in \mathbb{R}$ provided that for any real number $\varepsilon > 0$ there is an $N \in \mathbb{N}$ for which $n > N$ implies $|a_n - L| < \varepsilon|$.

$$\lim_{n\to\infty} a_n = L \iff \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \big(|a_n - L| < \varepsilon\big)$$

$$\lim_{n\to\infty} a_n = \infty \iff \forall L \in \mathbb{R} \exists N \in \mathbb{N} \forall n \geq N \big(a_n > L\big)$$

翻译 — "自然语言" 到 "逻辑语言"



- Informal definition of a limit: $f(x)$ is arbitrarily close to L provided that x is sufficiently close to c.
- Precise definition: for any real number $\varepsilon > 0$, there is a real number $\delta > 0$ for which $|f(x) - L| < \varepsilon$ provided that $0 < |x - c| < \delta$.
- Formal defnition:

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \big(0 < |x - c| < \delta \rightarrow |f(x) - L| < \varepsilon\big)$$

- Divergence

$$\lim_{x \to c} f(x) \uparrow \iff \forall L \in \mathbb{R} \exists \varepsilon > 0 \forall \delta > 0 \exists x \big(0 < |x - c| < \delta \wedge |f(x) - L| \geq \varepsilon\big)$$

翻译 — “自然语言” 到 “逻辑语言”

- Continuity

$$\forall x \in I \forall \varepsilon > 0 \exists \delta > 0 \forall y \in I \Big(|x - y| < \delta \to |f(x) - f(y)| < \varepsilon \Big)$$

$$\forall (x_n)_{n\in\mathbb{N}} \subset I : \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$$

- Uniform Continuity

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in I \forall y \in I \Big(|x - y| < \delta \to |f(x) - f(y)| < \varepsilon \Big)$$

- Any two distinct points determine a unique line.

$$\forall x \forall y \Big(\mathsf{Point}(x) \wedge \mathsf{Point}(y) \wedge x \neq y \to \exists! z \mathsf{Line}(z) \wedge \mathsf{On}(x, z) \wedge \mathsf{On}(y, z) \Big)$$

- There are infinitely many Prime numbers.

$$\forall x \exists y \big(y > x \wedge \mathsf{Prime}(y) \big)$$

- There are only finitely many Even Prime numbers.

$$\exists x \forall y \big(\mathsf{Even}(y) \wedge \mathsf{Prime}(y) \to y < x \big)$$

练习: 翻译 — Now it's your turn ↯

1. Everyone hears only what he understands. — Goethe
2. If you can't solve a problem, then there is an easier problem that you can't solve. — Polya
3. Men *and* women are welcome to apply.
4. Alice can't do *every* job right.
5. Alice can't do *any* job right.
6. 除了熟香蕉都不可以吃.
7. 只有苏格拉底和柏拉图是人.
8. 除了苏格拉底和柏拉图都是人.
9. 每个男孩爱着至少两个女孩.
10. 闪光的不都是金子.
11. 同一个城市的人的身份证号的第一位数字相同.
12. 有女朋友的男孩都幸福.
13. 有女朋友的男孩都宠她.
14. 所有的偶数都可以被 2 整除, 但只有一些可以被 4 整除.

练习: 翻译 — Now it's your turn ⚡

15. 喜欢所有女生的男生没有女生喜欢.
16. 每个班都有一个同学被所有同班同学喜欢.
17. 如果狗是动物, 那么狗的头就是动物的头.
18. 苏格拉底的老婆的脸只有她亲妈才不嫌弃.
19. 只有段正淳才会喜欢两个或两个以上的女孩.
20. 黄药师鄙视所有不自我鄙视的人.
21. 乔峰只爱也爱他的那个女孩.
22. 爱着王语嫣的那个人不是王语嫣所爱的那个人.
23. 最高的男孩爱着最矮的女孩.
24. If a clown enters the room, then it will be displeased if no person is surprised.
25. Everyone alive $2000BC$ is either an ancestor of nobody alive today or of everyone alive today.
26. No man loves children unless he has his own.
27. Alice and Bob have the same maternal grandmother. Mother(x, y)
28. Someone *other than the girl* who loves Bob is stupid.

代入 vs 可代入[6]

Definition: 代入 $[t/x]$ 用项 t 代入变元 x 的所有自由出现.

Definition (可代入)

项 t 对变元 x 在公式 A 中可代入, 当且仅当, 用 t 替换 x 在 A 中的所有自由出现后, t 中的变元不会被 A 中的量词所约束.

Remark: 我们写 $A[t/x]$ 时通常默认 t 对 x 在 A 中可代入, 有时会简写为 $A(t)$.

Counter-example: $A = \exists y(y \neq x) \quad t = y \quad A[t/x]? \quad \forall x A \vdash A(t)$

[6] A term t is substitutable for x in

- any atomic formula A.
- $\neg A$ if it is substitutable for x in A.
- $A \to B$ if it is substitutable for x in A and B.
- $\forall y A$ if either
  1. $x \notin \mathrm{Fv}(A)$ or
  2. $y \notin \mathrm{Var}(t)$ and t is substitutable for x in A.

Contents

塔斯基 Alfred Tarski 1901-1983

什么是 "真"?

「雪是白的」是真的, 当且仅当, 雪是白的.

"I am not true."[7]

模型论

Undefinability of truth Theorem

Arithetical truth can't be defined in arithmetic.

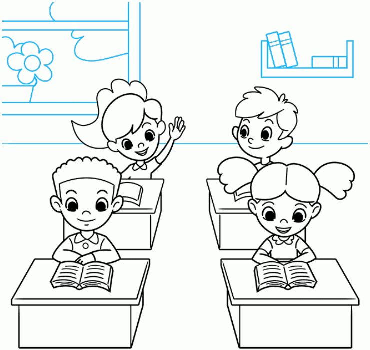
The theory of real closed fields / elementary geometry is complete and decidable.

Banach-Tarski Paradox

[7] Tarski: On the Concept of Truth in Formalized Languages. 1933.
Tarski: The Semantic Conception of Truth and the Foundations of Semantics. 1944.

结构！结构！什么是“结构”？

结构



语言

- 班里有四名同学
- 两名男同学
- 两名女同学
- 有个女同学叫小艾
- 小艾在举手
- 小白在看小艾
- 小艾的大姐在小白前面
- 某男同学在小艾前面
- 小艾前面是一头粉色大象

$$(\mathbb{Z}^+, 0, 1, +, \cdot, <) \models \exists xyz(x^2 + y^2 = z^2)$$

$$(\mathbb{Z}^+, 0, 1, +, \cdot, <) \not\models \exists xyz(x^3 + y^3 = z^3)$$

结构 ♡

Definition (结构)

结构 $\mathcal{M} := (M, [\![\,]\!])$ 由非空集合 M 及其上的映射 $[\![\,]\!]$ 构成, 其中 $[\![\,]\!]$ 把

- 常元符号 c 映射为元素 $[\![c]\!] \in M$
- n 元函数符号 f 映射为函数 $[\![f]\!] : M^n \to M$
- n 元谓词符号 P 映射为关系 $[\![P]\!] \subset M^n$

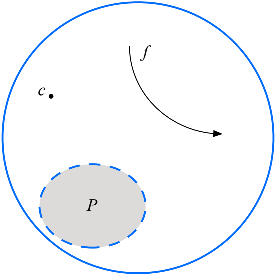
Remark: 为了方便, 我们通常把结构 $\mathcal{M}$ 简记为 $(M, c^{\mathcal{M}}, f^{\mathcal{M}}, P^{\mathcal{M}})$.

语言 $\mathscr{L}$ | **结构 $\mathcal{M}$**

常元符号 c $\longrightarrow$ 元素 $c^{\mathcal{M}} \in M$

函数符号 f $\longrightarrow$ 函数 $f^{\mathcal{M}} : M^n \to M$

谓词符号 P $\longrightarrow$ 关系 $P^{\mathcal{M}} \subset M^n$

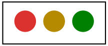


荀子: 制 “名” 以指 “实”

$$[\![\text{名}]\!] = \text{实}$$

$$\text{名}^{\mathcal{M}} = \text{实}$$

红灯停; 绿灯行; 黄灯表示警示的功能.

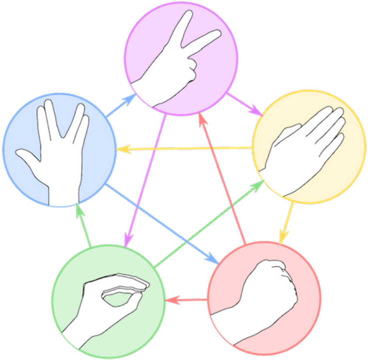


$$[\![\,\textcolor{red}{\bullet}\,]\!] = \text{Stop}$$

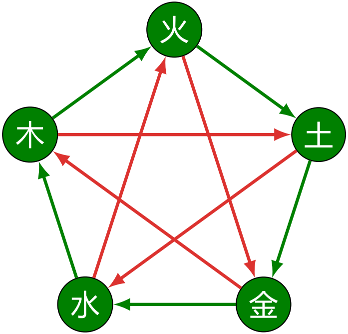
$$[\![\,\textcolor{olive}{\bullet}\,]\!] = \text{Caution}$$

$$[\![\,\textcolor{green}{\bullet}\,]\!] = \text{Go}$$

Example — 结构



$\cong$



$$M = \{金, 木, 水, 火, 土\}$$

$$c^{\mathcal{M}}_{金} = 金, \quad c^{\mathcal{M}}_{木} = 木, \quad c^{\mathcal{M}}_{水} = 水, \quad c^{\mathcal{M}}_{火} = 火, \quad c^{\mathcal{M}}_{土} = 土$$

$$R^{\mathcal{M}}_{生} = \{(金, 水), (水, 木), (木, 火), (火, 土), (土, 金)\}$$

$$R^{\mathcal{M}}_{克} = \{(金, 木), (木, 土), (土, 水), (水, 火), (火, 金)\}$$

Question: 你能看出来这两幅图是同构的吗?

项的解释: “名” 副其 “实” ♡

Definition (变元赋值)

结构 $\mathcal{M}$ 上的变元赋值是一个函数 $\nu : \mathrm{Var} \to M$.
变元赋值 ν 可以递归地扩展到所有项上 $\overline{\nu} : \mathrm{Term} \to M$:

- $\overline{\nu}(x) \coloneqq \nu(x)$
- $\overline{\nu}(c) \coloneqq c^{\mathcal{M}}$
- $\overline{\nu}(f(t_1, \ldots, t_n)) \coloneqq f^{\mathcal{M}}(\overline{\nu}(t_1), \ldots, \overline{\nu}(t_n))$

$$\begin{tikzcd} \mathrm{Term} \arrow[r, "\overline{\nu}"] \arrow[d, "f"'] & M \arrow[d, "f^{\mathcal{M}}"] \\ \mathrm{Term} \arrow[r, "\overline{\nu}"'] & M \end{tikzcd}$$

Remark: 在不引起歧义的情况下, 我们把 $\overline{\nu}$ 写作 ν.

公式的解释: “满足关系” ♡

Definition (满足关系 $\mathcal{M}, \nu \vDash A$)

- $\mathcal{M}, \nu \vDash t_1 = t_2$ iff $\nu(t_1) = \nu(t_2)$
- $\mathcal{M}, \nu \vDash P(t_1, \ldots, t_n)$ iff $(\nu(t_1), \ldots, \nu(t_n)) \in P^{\mathcal{M}}$
- $\mathcal{M}, \nu \vDash \neg A$ iff $\mathcal{M}, \nu \nvDash A$
- $\mathcal{M}, \nu \vDash A \wedge B$ iff $\mathcal{M}, \nu \vDash A$ and $\mathcal{M}, \nu \vDash B$
- $\mathcal{M}, \nu \vDash A \vee B$ iff $\mathcal{M}, \nu \vDash A$ or $\mathcal{M}, \nu \vDash B$
- $\mathcal{M}, \nu \vDash A \to B$ iff $\mathcal{M}, \nu \nvDash A$ or $\mathcal{M}, \nu \vDash B$
- $\mathcal{M}, \nu \vDash A \leftrightarrow B$ iff $\mathcal{M}, \nu \vDash A \iff \mathcal{M}, \nu \vDash B$
- $\mathcal{M}, \nu \vDash \forall x A$ iff for every $a \in M : \mathcal{M}, \nu(a/x) \vDash A$

$$\text{where} \quad \nu(a/x)(y) := \begin{cases} a & \text{if } y = x \\ \nu(y) & \text{otherwise} \end{cases}$$

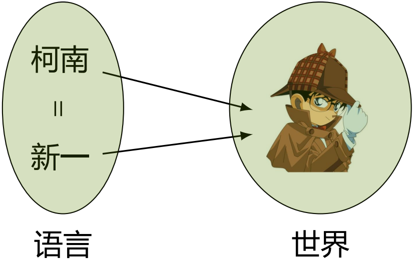
- $\mathcal{M}, \nu \vDash \exists x A$ iff there is an $a \in M : \mathcal{M}, \nu(a/x) \vDash A$

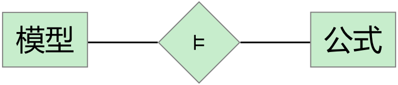
Remark: $\nu(a/x)$ 就是改变 ν 对 x 的赋值为 a, 其它不变.

$\mathcal{M}, \nu \vDash \forall x A$ iff for all $\nu' \sim_x \nu : \mathcal{M}, \nu' \vDash A$. where $\nu' \sim_x \nu$ iff for all $y \neq x : \nu'(y) = \nu(y)$

公式的解释: 从“满足关系”到“真” ♡

- $\mathcal{M}, v \vDash t_1 = t_2$ iff $v(t_1) = v(t_2)$





- 满足关系 $\mathcal{M}, v \vDash A$
  Remark: 先把符号串 A 里的常元符号、函数符号、谓词符号按照结构 $\mathcal{M}$ 的规定来解释, 把量词的论域限定在集合 M 上, 把自由变元 x 解释成它的赋值 $v(x)$, 从而把公式 A 翻译成一个元语言中关于结构 $\mathcal{M}$ 的命题, 而利用结构 $\mathcal{M}$ 的知识, 就可以知道命题 A 是否成立.
- $\mathcal{M} \vDash A$ 当且仅当, 对任意变元赋值 v 都有: $\mathcal{M}, v \vDash A$. (真)
  Remark: 此时, 我们说, A 在 $\mathcal{M}$ 上为真, 或 $\mathcal{M}$ 是 A 的模型.

Notation

当项 t 和公式 A 中的自由变元都在 $\{x_1, \ldots, x_n\}$ 里时, 如果变元赋值 $v(x_i) = a_i$ for $1 \le i \le n$, 那么[8], 我们有时会把

$$v(t) \quad \text{写成} \quad t^{\mathcal{M}}[a_1, \ldots, a_n]$$

$$\mathcal{M}, v \vDash A \quad \text{写成} \quad \mathcal{M} \vDash A[a_1, \ldots, a_n]$$

Remark: When $\mathrm{Fv}(A) = \{x\}$,

$$\mathcal{M} \vDash \forall x A \iff \text{for every } a \in M : \mathcal{M} \vDash A[a]$$

$$\mathcal{M} \vDash \exists x A \iff \text{for some } a \in M : \mathcal{M} \vDash A[a]$$

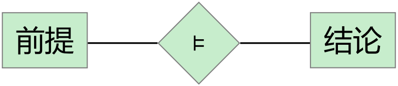
[8]这一记号的合理性由 Coincidence Lemma 保证.

Remark: 借助 Substitution Lemma, $\mathcal{M} \vDash A[a_1, \ldots, a_n]$ 相当于: 把论域 M 里的个体 a_i 命名为 a_i, 然后把名字 a_i 代入到 $A(x_1, \ldots, x_n)$ 中的自由变元 x_i, 得到的句子 $A(a_1, \ldots, a_n)$ 在结构 $\mathcal{M}$ 上是真的.

Remark: 无名, 天地之始; 有名, 万物之母. — 老子 ☺

可满足、逻辑蕴含、有效式 ♡

- $\mathcal{M}, \nu \vDash \Gamma$ 当且仅当, 对任意 $A \in \Gamma$ 都有 $\mathcal{M}, \nu \vDash A$.
- Γ 是**可满足的**, 当且仅当, 存在 $\mathcal{M}, \nu$ 使得 $\mathcal{M}, \nu \vDash \Gamma$. (可满足)
- $\mathcal{M} \vDash \Gamma$ 当且仅当, 对任意 $A \in \Gamma$ 都有 $\mathcal{M} \vDash A$.
- $\Gamma \vDash A$ 当且仅当, 对任意 $\mathcal{M}, \nu$: 若 $\mathcal{M}, \nu \vDash \Gamma$ 则 $\mathcal{M}, \nu \vDash A$. (逻辑蕴含)[9]

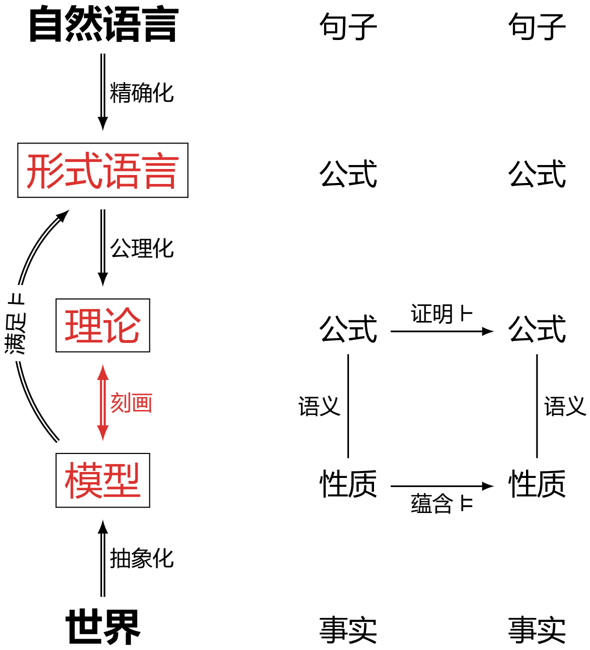


- A 是**有效的** $\vDash A$, 当且仅当, $\emptyset \vDash A$. (有效式)

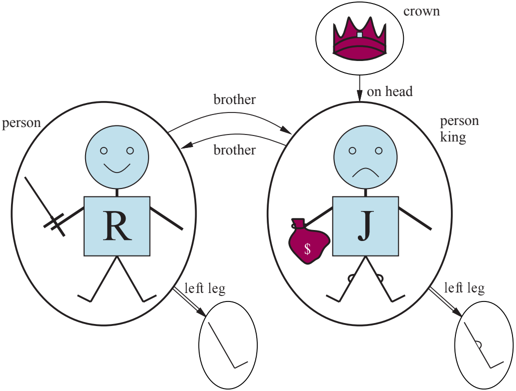
  说是者为非, 或非者为是, 是假的; 说是者为是, 或非者为非, 是真的. — 亚里士多德

[9] If we define $\Gamma \vDash^* A$ iff for all $\mathcal{M}$: $\mathcal{M} \vDash \Gamma \implies \mathcal{M} \vDash A$, then we have $Px \vDash^* \forall x Px$.

- 逻辑蕴含 $\vDash$ 是 “保赋值” 的, 而 $\vDash^*$ 是 “保真” 或 “保模型” 的, 我们还可以定义 “保有效”: $\vDash \Gamma \implies \vDash A$.
- 显然, “保赋值” 强于 “保真” 强于 “保有效”.


自然语言
精确化
形式语言
公理化
理论
满足 ⊨
刻画
模型
抽象化
世界
句子
句子
公式
公式
公式
证明 ⊢
公式
语义
语义
性质
蕴含 ⊨
性质
事实
事实

Example — 模型

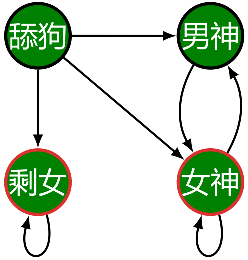


1. $\text{Brother}(r, j)$
2. $\neg\text{King}(r) \to \text{King}(j)$
3. $\text{King}(j) \wedge \exists x \big(\text{Crown}(x) \wedge \text{OnHead}(x, j)\big)$
4. $\exists x (\text{Person}(x) \wedge \exists y (\text{Crown}(y) \wedge \text{OnHead}(y, x)))$
5. $\neg\text{Brother}\big(\text{leftLeg}(r), j\big)$
6. $\forall x \big(\text{King}(x) \to \text{Person}(x)\big)$

Example — 模型

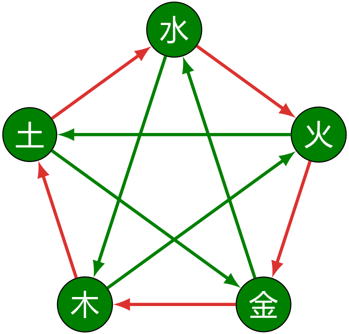
$$
\begin{aligned}
M &= \{\text{舔狗}, \text{男神}, \text{女神}, \text{剩女}\}\\
c^{\mathcal{M}} &= \text{女神}\\
B^{\mathcal{M}} &= \{\text{舔狗}, \text{男神}\}\\
G^{\mathcal{M}} &= \{\text{剩女}, \text{女神}\}\\
L^{\mathcal{M}} &= \{(\text{舔狗}, \text{男神}), (\text{舔狗}, \text{女神}), (\text{女神}, \text{女神}) \cdots\cdots\}
\end{aligned}
$$

$(M, c^{\mathcal{M}}, B^{\mathcal{M}}, G^{\mathcal{M}}, L^{\mathcal{M}})$

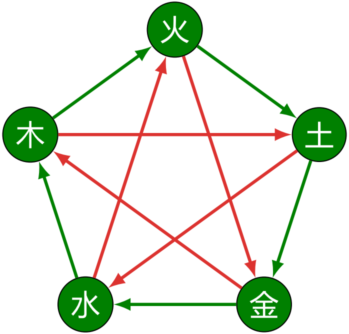


1. $\mathcal{M} \overset{?}{\vDash} Bc$
2. $\mathcal{M} \overset{?}{\vDash} Bc \vee Lcc$
3. $\mathcal{M} \overset{?}{\vDash} \exists x \neg Lxc$
4. $\mathcal{M} \overset{?}{\vDash} \forall x(Bx \vee Lxx)$
5. $\mathcal{M} \overset{?}{\vDash} \forall x \exists y Lxy$
6. $\mathcal{M} \overset{?}{\vDash} \exists x \forall y(y = x \vee Lxy)$
7. $\mathcal{M} \overset{?}{\vDash} \exists x(Bx \wedge \forall y(Gy \to \neg Lyx))$

Example — 模型



$\cong$

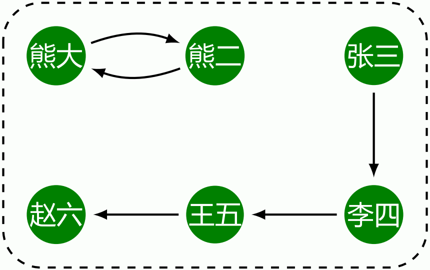


1. $\forall x \exists y R_{生}(x, y)$
2. $\forall x \exists y R_{克}(x, y)$
3. $\forall x \exists y R_{生}(y, x)$
4. $\forall x \exists y R_{克}(y, x)$
5. $\forall x \exists y_1 y_2 y_3 y_4 \left[R_{生}(x, y_1) \wedge R_{生}(y_1, y_2) \wedge R_{生}(y_2, y_3) \wedge R_{生}(y_3, y_4) \wedge R_{生}(y_4, x) \right]$
6. $\forall x \exists y_1 y_2 y_3 y_4 \left[R_{克}(x, y_1) \wedge R_{克}(y_1, y_2) \wedge R_{克}(y_2, y_3) \wedge R_{克}(y_3, y_4) \wedge R_{克}(y_4, x) \right]$

Transitive Relation

$$\forall xyz(Rxy \wedge Ryz \rightarrow Rxz)$$

在下图上为真吗? 如果不为真, 添加尽量少的箭头使之为真.

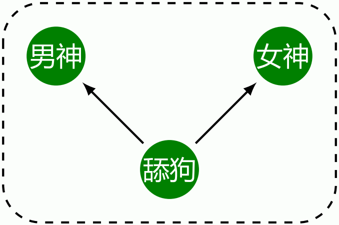


如果是删箭头呢?

Euclidean Relation

$$\forall xyz(Rxy \wedge Rxz \to Ryz)$$

在下图上为真吗? 如果不为真, 添加尽量少的箭头使之为真.



Example — 模型

$\exists x \forall y Rxy$

R	1	2	3	4
1	✓	✓	✓	✓
2				
3				
4				

$\forall y \exists x Rxy$

R	1	2	3	4
1	✓	✓		
2			✓	
3				✓
4				

$\exists y \forall x Rxy$

R	1	2	3	4
1				✓
2				✓
3				✓
4				✓

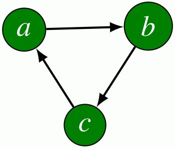
$\forall x \exists y Rxy$

R	1	2	3	4
1		✓		
2			✓	
3				✓
4				✓

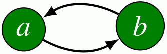
若 $A \nvDash B$, 则存在反模型$\mathcal{M}$ 使得 $\mathcal{M} \vDash A$ 但 $\mathcal{M} \nvDash B$

构造反模型

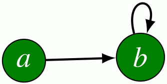
$$\frac{\forall x \exists y Lxy}{\exists x \forall y Lxy} \times$$



$$\frac{\forall x \exists y Lxy}{\exists y \forall x Lxy} \times$$



$$\frac{\exists y \forall x Lxy}{\forall y \exists x Lxy} \times$$



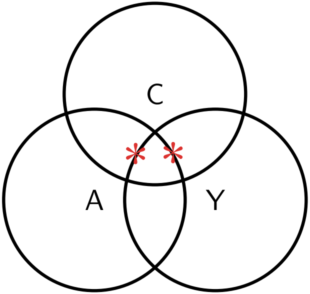
$$\cdots \to -2 \to -1 \to 0 \to 1 \to 2 \to 3 \to 4 \to 5 \to \cdots$$

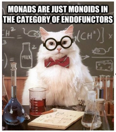
$$(\mathbb{Z}, <)$$

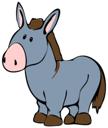
有些动物是猫
有些猫是黄色的

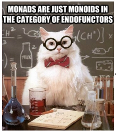
有些动物是黄色的

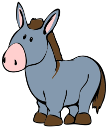
$$\frac{\exists x(Ax \land Cx) \quad \exists x(Cx \land Yx)}{\exists x(Ax \land Yx)}$$

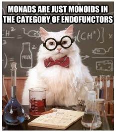


$M = \Big\{$ , 

,  $\Big\}$

$A^M = \Big\{$ 

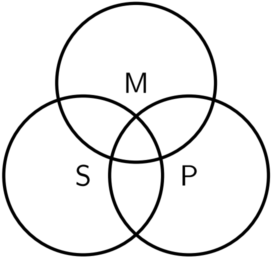
,  $\Big\}$

$C^M = \Big\{$ , 

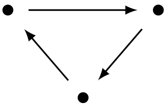
$\Big\}$

$Y^M = \Big\{$  $\Big\}$

- 只含有一元谓词的公式集对应欧拉图



- 只含有一个二元谓词的公式集对应有向图



Example — 反模型

$$\frac{\text{Everybody loves somebody} \qquad \text{Everybody loves all persons who are loved by his loved ones}}{\text{There is at least a pair of persons who love each other}}$$

$$\frac{\text{万物皆有因} \qquad \text{原因的原因也是原因}}{\text{某物是自身的原因}}$$

$$\frac{\forall x \exists y Rxy \qquad \forall xyz(Rxy \wedge Ryz \to Rxz)}{\exists xy(Rxy \wedge Ryx)} \times \qquad\qquad \frac{\forall x \exists y Ryx \qquad \forall xyz(Rxy \wedge Ryz \to Rxz)}{\exists x Rxx} \times$$

$$\cdots \to -2 \to -1 \to 0 \to 1 \to 2 \to 3 \to 4 \to 5 \to \cdots$$

$$(\mathbb{Z}, <)$$

Remark: 存在有穷反模型吗?

Problem (写一个只能被无穷结构满足的公式)

$$\forall x \exists y Rxy \ \wedge \ \forall xyz(Rxy \wedge Ryz \to Rxz) \ \wedge \ \forall x \neg Rxx$$

- 假设有满足它的有穷结构 $\mathcal{M}$, 不妨假设 $|M| = n$.
- 由 $\mathcal{M} \vDash \forall x \exists y Rxy$, 存在序列 $(a_1, a_2, \ldots, a_{n+1})$ 使得对任意 $1 \leq i \leq n+1$ 都有 $R^{\mathcal{M}} a_i a_{i+1}$.
- 又由 $\mathcal{M} \vDash \forall xyz(Rxy \wedge Ryz \to Rxz)$, 对任意 $1 \leq i < j \leq n+1$ 都有 $R^{\mathcal{M}} a_i a_j$.
- 因为 $|M| = n$, 所以存在 $1 \leq i < j \leq n+1$ 使得 $a_i = a_j$, 这意味着 $R^{\mathcal{M}} a_i a_i$. 这与 $\mathcal{M} \vDash \forall x \neg Rxx$ 矛盾.

Examples — 模型 & 反模型

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

1. $\forall x \exists y : x + y = 0$

$$(\mathbb{N}, 0, 1, +, \cdot, <) \nvDash \forall x \exists y : x + y = 0$$
$$(\mathbb{Z}, 0, 1, +, \cdot, <) \vDash \forall x \exists y : x + y = 0$$

2. $\forall x \neq 0 \exists y : x \cdot y = 1$

$$(\mathbb{Z}, 0, 1, +, \cdot, <) \nvDash \forall x \neq 0 \exists y : x \cdot y = 1$$
$$(\mathbb{Q}, 0, 1, +, \cdot, <) \vDash \forall x \neq 0 \exists y : x \cdot y = 1$$

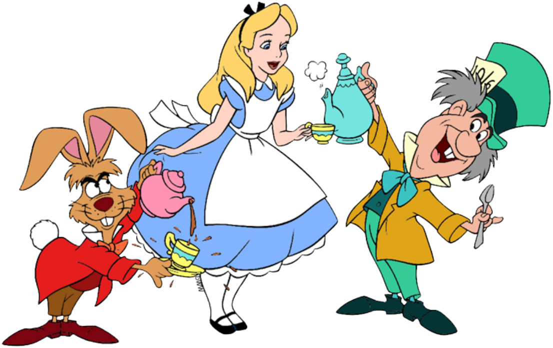
3. $\forall x \geq 0 \exists y : x = y \cdot y$

$$(\mathbb{Q}, 0, 1, +, \cdot, <) \nvDash \forall x \geq 0 \exists y : x = y^2$$
$$(\mathbb{R}, 0, 1, +, \cdot, <) \vDash \forall x \geq 0 \exists y : x = y^2$$

4. $\forall x \exists y : x = y \cdot y$

$$(\mathbb{R}, 0, 1, +, \cdot, <) \nvDash \forall x \exists y : x = y^2$$
$$(\mathbb{C}, 0, 1, +, \cdot, <) \vDash \forall x \exists y : x = y^2$$

数学的本质在于它的自由. — 康托尔



$$\frac{\forall x(A \to B)}{\forall x(B \to A)} \times$$

- 三月兔: 那你怎么想就怎么说.
- 爱丽丝: 我说的就是我想的 — 这是一回事.
- 疯帽匠: 不是一回事, 那样的话你可以说 — "凡是我吃的东西我都能看见" 和 "凡是我能看见的东西我都吃" 是一回事.
- 三月兔: 那样的话你可以说 — "凡是我拥有的东西我都喜欢" 和 "凡是我喜欢的东西我都拥有" 是一回事.

易犯的“错误” ♡

$$\forall x(Bx \to Sx)$$

$$\exists x(Bx \land Sx)$$

- $\forall x(Bx \land Sx) \equiv \forall xBx \land \forall xSx$
  万物皆男孩, 并且, 万物皆聪明.
- $\exists x(Bx \to Sx) \equiv \exists x\neg Bx \lor \exists xSx$
  只要有不是男孩的东西, 这句话就为真.

下面这句话为真吗?

All the elephants in this room are purple.

$$\forall x(\mathsf{Elephant}(x) \land \mathsf{In}(x, \mathsf{room}) \to \mathsf{Purple}(x))$$

$$\forall x(\mathsf{Elephant}(x) \land \mathsf{In}(x, \mathsf{room}) \land \mathsf{Purple}(x))$$

Coincidence Lemma

- 赋值 $\nu_1, \nu_2 : \mathrm{Var} \to M$ 在项 t 上一致, 记为 $\nu_1 \equiv \nu_2 \pmod{t}$, 当且仅当, 对任意 $x \in \mathrm{Var}(t)$: $\nu_1(x) = \nu_2(x)$.
- 赋值 $\nu_1, \nu_2 : \mathrm{Var} \to M$ 在公式 A 上一致, 记为 $\nu_1 \equiv \nu_2 \pmod{A}$, 当且仅当, 对任意 $x \in \mathrm{Fv}(A)$: $\nu_1(x) = \nu_2(x)$.

Lemma (Coincidence Lemma)

- *If* $\nu_1 \equiv \nu_2 \pmod{t}$, *then*

$$\nu_1(t) = \nu_2(t)$$

- *If* $\nu_1 \equiv \nu_2 \pmod{A}$, *then*

$$\mathcal{M}, \nu_1 \vDash A \iff \mathcal{M}, \nu_2 \vDash A$$

Remark:

- If A is a closed formula, then either $\mathcal{M} \vDash A$ or $\mathcal{M} \vDash \neg A$.
- $\mathcal{M} \vDash A \implies \mathcal{M} \vDash \forall x A$
- $\mathcal{M}, \nu \vDash \forall x A$ iff for all $\nu' \equiv \nu \pmod{\forall x A} : \mathcal{M}, \nu' \vDash A$.
- $\mathcal{M}, \nu \vDash \exists x A$ iff for some $\nu' \equiv \nu \pmod{\exists x A} : \mathcal{M}, \nu' \vDash A$.

Substitution Lemma

Lemma (Substitution Lemma)

- $\nu\,(s[t/x]) = \nu(\nu(t)/x)(s)$
- *If the term t is substitutable for the variable x in the formula A, then*

$$\mathcal{M}, \nu \vDash A[t/x] \iff \mathcal{M}, \nu(\nu(t)/x) \vDash A$$

$$\begin{tikzcd}
s \arrow[rr, "{[t/x]}"] \arrow[dr, "\nu(\nu(t)/x)"'] & & s[t/x] \arrow[dl, "\nu"] \\
& M &
\end{tikzcd}$$

Remark:

$$\mathscr{L}_M := \mathscr{L} \cup \{c_a : a \in M\} \quad \text{and} \quad c_a^{\mathcal{M}_M} = a \quad \text{and} \quad \mathcal{M}_M := (\mathcal{M}, a)_{a \in M}$$

$$\mathcal{M}_M, \nu \vDash A[c_a/x] \iff \mathcal{M}, \nu(a/x) \vDash A$$

How to Check Validity? — Example

$$\boxed{\forall x A \to A[t/x]}$$

$\mathcal{M}, v \vDash \forall x A \implies$ for all $a \in M : \mathcal{M}, v(a/x) \vDash A \implies \mathcal{M}, v(v(t)/x) \vDash A$
According to Substitution Lemma, $\mathcal{M}, v \vDash A[t/x]$.

$$\boxed{\forall x(B \to A) \to (\exists x B \to A) \quad \text{where } x \notin \mathrm{Fv}(A)}$$

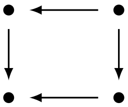
Assume $\mathcal{M}, v \vDash \exists x B$ and $\mathcal{M}, v \nvDash A$. Then there exists $a \in M$ s.t. $\mathcal{M}, v(a/x) \vDash B$. According to Coincidence Lemma and $x \notin \mathrm{Fv}(A)$, we have $\mathcal{M}, v(a/x) \nvDash A$. Therefore $\mathcal{M}, v(a/x) \nvDash B \to A$. This contradicts $\mathcal{M}, v \vDash \forall x(B \to A)$.

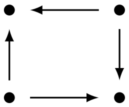
$$\boxed{(\exists x B \to A) \to \forall x(B \to A) \quad \text{where } x \notin \mathrm{Fv}(A)}$$

$\mathcal{M}, v \vDash \exists x B \to A \implies \mathcal{M}, v \nvDash \exists x B$ or $\mathcal{M}, v \vDash A$.
If $\mathcal{M}, v \nvDash \exists x B$, then for all $a \in M$, $\mathcal{M}, v(a/x) \nvDash B$. It follows that $\mathcal{M}, v(a/x) \vDash B \to A$. Therefore $\mathcal{M}, v \vDash \forall x(B \to A)$.
If $\mathcal{M}, v \vDash A$, then according to Coincidence Lemma and $x \notin \mathrm{Fv}(A)$, for all $a \in M$, $\mathcal{M}, v(a/x) \vDash A$. It follows that $\mathcal{M}, v(a/x) \vDash B \to A$. Therefore $\mathcal{M}, v \vDash \forall x(B \to A)$.

1. 公式 $\exists x \forall y Rxy$ 在此图上为真吗? 如果不为真, 添加尽量少的箭头使之为真.

2. 写一个公式, 使其在左图上为真, 右图上为假.

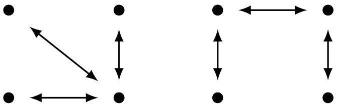




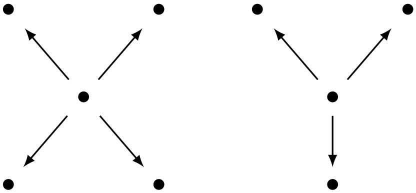
3. 画两幅图, 使公式在一幅图上为真, 另一幅图上为假.
   - 3.1 $\forall xy(Rxy \to Ryy)$
   - 3.2 $\forall x \exists y Rxy$
   - 3.3 $\forall xyz(Rxy \wedge Rxz \to Ryz \vee Rzy)$
   - 3.4 $\forall x(\exists y Ryx \to \forall z Rzx)$
   - 3.5 $\forall xyz(Rxy \wedge Rxz \to \exists w(Ryw \wedge Rzw))$

练习: 模型 & 反模型 — Now it's your turn ⚡

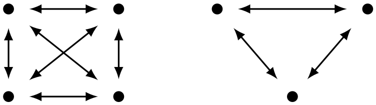
1. 写一个公式, 使其在左图上为真, 右图上为假.



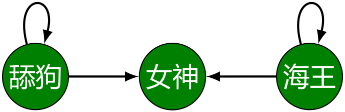
2. 写一个公式, 使其在左图上为真, 右图上为假.



3. 写一个公式, 使其在左图上为真, 右图上为假.

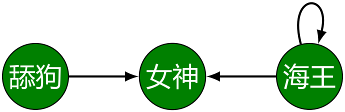


思考一下



假设 $\mathscr{L} = \{L\}$.

1. "女神" 可定义吗? $x =$ 女神 $\iff \mathcal{M} \vDash \forall y(y \neq x \rightarrow Lyx)$
2. $\{$舔狗, 海王$\}$ 可定义吗?
   $x \in \{$舔狗, 海王$\} \iff \mathcal{M} \vDash \neg\exists y(y \neq x \wedge Lyx)$
3. "舔狗" 可定义吗? 对称 $\implies$ 不可区分 $\implies$ 不可定义



现在的 $\mathcal{M}$ 有什么特点? 可定义 $\implies$ 可区分 $\implies$ 不对称

1. 是否存在非平凡的自同构?
2. 是否任意两个个体都可区分? $\mathcal{M} \vDash A[a]$ & $\mathcal{M} \nvDash A[b]$ $(\mathbb{R}^{\geq 0}, <)$
3. 是否每个个体都可定义? $x = a \iff \mathcal{M} \vDash A(x)$ $(\mathbb{R}, 0, 1, +, \cdot, <)$

- 能否找一个句子, 使其在一个结构上为真、另一个结构上为假?

思考一下

Problem

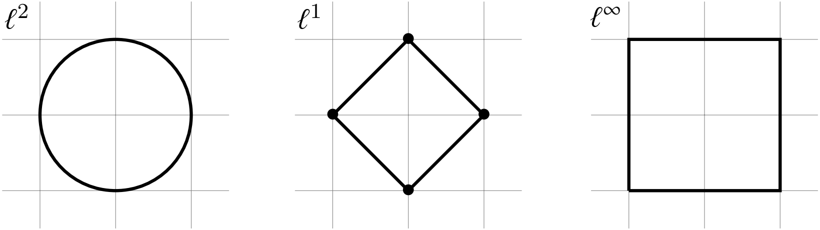
- 一个图是完全的，当且仅当，图中任意两个不同节点之间至少有一个箭头.
- 一个节点是社牛，当且仅当，它可以通过至多两个箭头指到图中任何一个节点.
- 问题：怎么用公式表达“完全图”和“社牛节点”？

$$\begin{aligned} \mathsf{Complete}(G) &\iff G \vDash A \\ \mathsf{Social}(x) &\iff G \vDash A(x) \end{aligned}$$

- 证明：任何有穷的、自反的完全图都有一个社牛.
- 无穷图呢？ $(\mathbb{N}, \geq)$

$$\cdots \longrightarrow n+1 \longrightarrow n \longrightarrow n-1 \longrightarrow \cdots$$

什么是圆?



- 什么是圆? — 到定点 $\mathbf{c}$ 的距离等于定长 r 的点的集合.

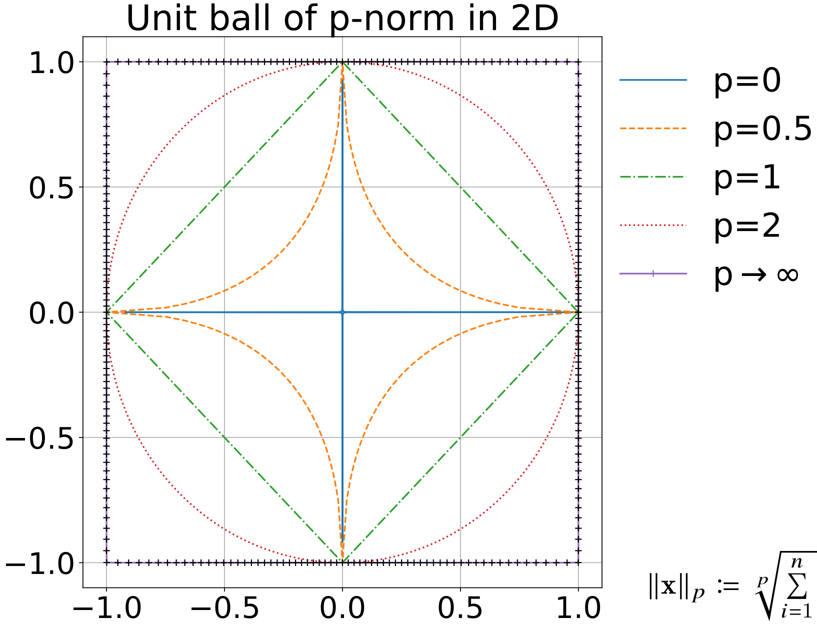
$$\{\mathbf{x} : \mathcal{M} \vDash d(\mathbf{x}, \mathbf{c}) = r\}$$

- 什么是半径为 1 的圆? — 基于什么距离? $\ell^2, \ell^1, \ell^\infty$ norm?

$$d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_i |x_i - y_i|^2} \quad d_1(\mathbf{x}, \mathbf{y}) = \sum_i |x_i - y_i| \quad d_\infty(\mathbf{x}, \mathbf{y}) = \max_i |x_i - y_i|$$

- 什么是圆周率 π? 在 "出租车世界"$(\mathbb{R}^2, d_1)$, $\pi = \frac{\text{周长}}{\text{直径}} = \frac{8}{2} = 4$

"We are often interested not just in whether or not something is true, but in where it is true." — *John Baez*



$$\|\mathbf{x}\|_p := \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$$

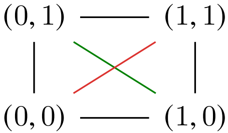
世界不仅比我们想象的更古怪, 甚至比我们能够想象的更古怪.

$\mathbb{F}_2^2$ 平面上的圆

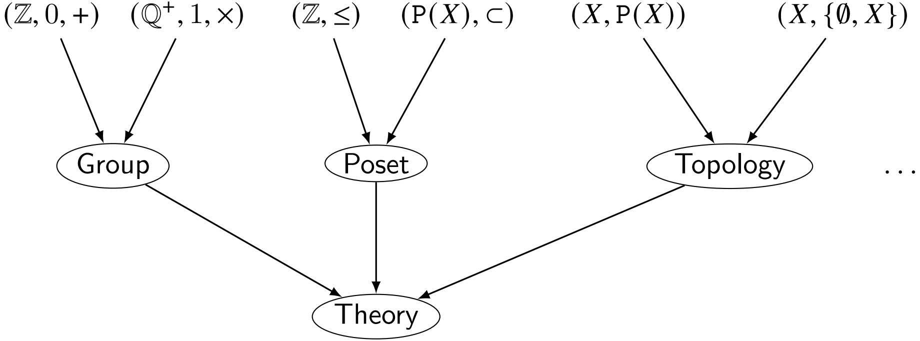
- $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} = (\{0,1\}, 0, 1, +, \cdot)$

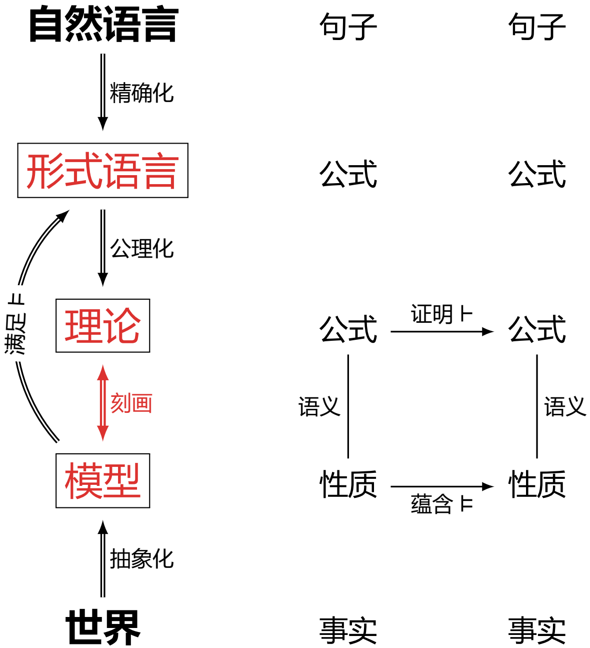
$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1



- $\mathbb{F}_2^2$ 平面只有四个点: $(0,0), (0,1), (1,0), (1,1)$
- $\mathbb{F}_2^2$ 平面有多少条直线? $ax + by + c = 0 \ \ (a, b, c \in \mathbb{F}_2)$, a, b 不同为 0
- $\mathbb{F}_2^2$ 平面有多少个圆? $(x-a)^2 + (y-b)^2 = r^2 \ \ (a, b, r \in \mathbb{F}_2)$
- 因为 $\forall x \in \mathbb{F}_2 : x^2 = x$, 所以
  $(x-a)^2 + (y-b)^2 = r^2 \iff x - a + y - b = r$
  即 $x + y = 0$ 或 $x + y = 1$.
  因此, 有且仅有 2 个圆, 都是直线!
- 例: $x + y = 0 \iff (x-0)^2 + (y-0)^2 = 0 \iff (x-1)^2 + (y-1)^2 = 0 \iff (x-0)^2 + (y-1)^2 = 1 \iff (x-1)^2 + (y-0)^2 = 1$
- 该圆周上有两个点: $(0,0), (1,1)$. 两个点都是圆心, 半径是 0.
- 该圆周外的两个点 $(0,1), (1,0)$ 也是圆心, 此时半径为 1.
- 因此, $\mathbb{F}_2^2$ 平面有 2 个圆, 每个圆都有 4 个圆心, 2 个半径.

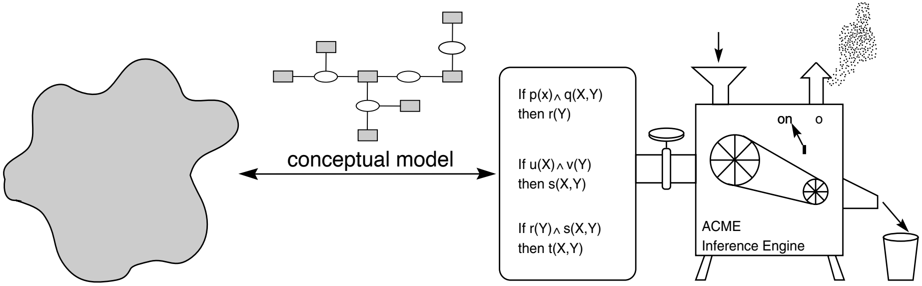

$(\mathbb{Z}, 0, +)$
$(\mathbb{Q}^+, 1, \times)$
$(\mathbb{Z}, \leq)$
$(\mathtt{P}(X), \subset)$
$(X, \mathtt{P}(X))$
$(X, \{\emptyset, X\})$
Group
Poset
Topology
...
Theory



自然语言
精确化
形式语言
公理化
理论
刻画
模型
抽象化
世界
满足 ⊨
句子
句子
公式
公式
公式
证明 ⊢
公式
语义
语义
性质
蕴含 ⊨
性质
事实
事实

Contents

Formal Systems



Expertise

Implemented System

- Tree Method
- Natural Deduction
- Sequent Calculus
- Hilbert System
- Resolution
- ...

命题逻辑的树形方法 ♡

$$\begin{array}{c} \neg\neg A \\ | \\ A \end{array}
\qquad
\begin{array}{ccc} & A \to B & \\ / & & \backslash \\ \neg A & & B \end{array}
\qquad
\begin{array}{c} \neg(A \to B) \\ | \\ A \\ \neg B \end{array}$$

$$\begin{array}{c} A \wedge B \\ | \\ A \\ B \end{array}
\qquad
\begin{array}{ccc} & \neg(A \wedge B) & \\ / & & \backslash \\ \neg A & & \neg B \end{array}
\qquad
\begin{array}{ccc} & A \vee B & \\ / & & \backslash \\ A & & B \end{array}
\qquad
\begin{array}{c} \neg(A \vee B) \\ | \\ \neg A \\ \neg B \end{array}$$

$$\begin{array}{ccc} & A \leftrightarrow B & \\ / & & \backslash \\ A & & \neg A \\ B & & \neg B \end{array}
\qquad
\begin{array}{ccc} & \neg(A \leftrightarrow B) & \\ / & & \backslash \\ A & & \neg A \\ \neg B & & B \end{array}$$

$$\begin{array}{c} \forall x A \\ | \\ A[t/x] \end{array} \qquad \qquad \begin{array}{c} \exists x A \checkmark \\ | \\ A(a) \end{array}$$

where a is a new constant which does not appear on the same branch as $\exists x A$.

$$\begin{array}{c} \neg\forall x A \checkmark \\ | \\ \exists x \neg A \end{array} \qquad \qquad \begin{array}{c} \neg\exists x A \checkmark \\ | \\ \forall x \neg A \end{array}$$

$$\begin{array}{c} \vdots \\ | \\ t = t \end{array} \qquad \qquad \begin{array}{c} s = t \\ A[s/x] \\ | \\ A[t/x] \end{array}$$

什么是 “证明”? ♡

Definition (证明)

$A_1, \ldots, A_n \vdash B$ 当且仅当, 存在一棵从 $\{A_1, \ldots, A_n, \neg B\}$ 开始的闭树.

小技巧

1. 少生枝节: 既能生一枝又能生两枝时, 优先生一枝.
2. 能闭掉的枝尽早闭掉.
3. 肯定量词和否定量词同时出现时, 优先处理否定量词.
4. 存在量词和全称量词同时出现时, 优先处理存在量词.

可靠性 & 完备性 ♡

Theorem (可靠性定理)

如果存在一棵从 Γ 开始的闭树, 那么 Γ 不可满足.

Theorem (完备性定理)

如果 Γ 不可满足, 那么存在一棵从 Γ 开始的闭树.

$$\frac{A_1, \ldots, A_n \vdash B}{\overline{A_1, \ldots, A_n \vDash B}}$$

$\boxed{\begin{array}{c}\vdash \text{ captures } \vDash \\ \text{No more, no less}\end{array}}$

可靠 $\vdash \implies \vDash$ 不多: 所有证明出来的论证都是有效的

完备 $\vDash \implies \vdash$ 不少: 所有有效的论证都能够证明出来

$$\overset{?}{\vDash} A \xrightarrow{\text{input}} \boxed{\cancel{\text{Machine}}} \xrightarrow{\text{output}} Y/N$$

Remark: 如果一个谓词逻辑的论证不是有效的, 且其反模型是无穷的, 那么, 我们无法通过树形方法找到它.

Example

$\boxed{\text{公式集 } \left\{\exists x(P(x) \vee P(f(x))), \neg\exists x P(x)\right\} \text{ 不可满足}}$

$$\exists x(P(x) \vee P(f(x))) \checkmark$$

$$\neg\exists x P(x) \checkmark$$

$$\forall x \neg P x$$

$$P(a) \vee P(f(a)) \checkmark$$

$$\begin{array}{cc} P(a) & P(f(a)) \\ \neg P(a) & \neg P(f(a)) \\ \times & \times \end{array}$$

$$\boxed{\dfrac{\exists x(P(x) \vee P(f(x)))}{\exists x P x}}$$

Example: 有一个人，如果他酗酒，那么所有人都酗酒

$\boxed{\vdash \exists x(A \to \forall x A)}$

$$\neg\exists x(A \to \forall x A)$$

$$|$$

$$\neg(Aa \to \forall x A) \checkmark$$

$$|$$

$$Aa$$

$$\neg\forall x A \checkmark$$

$$|$$

$$\neg Ab$$

$$|$$

$$\neg(Ab \to \forall x A) \checkmark$$

$$|$$

$$Ab$$

$$\neg\forall x A$$

$$\times$$

Remark: $\nvdash \exists x A \to \forall x A$

从“饱和开枝”到反模型

- 在谓词逻辑里，当一个论证无效时，通常无法通过闭不掉的树证明其无效.
- 偶尔，可以借助“饱和开枝”构造反模型.
- **饱和开枝 (Saturated open branch):** 一个开枝是饱和的，当且仅当，每个非文字公式 (不是原子公式或原子公式的否定) 都至少被拆过一次，并且，每个 $\forall$-公式都已经用该枝上的函数所能构造的项进行了实例化.
- 有“饱和开枝”的论证肯定无效.
- **Remark:** 但一个开枝只在特殊情况下才饱和. 毕竟，只要有一个一元函数和一个常元就可以构造出无穷多的项 $c, f(c), f(f(c))\ldots$

构造反模型

$$\forall x(A \vee B)$$
$$\neg(\forall x A \vee \forall x B)\ \checkmark$$
$$\neg\forall x A\ \checkmark$$
$$\neg\forall x B\ \checkmark$$
$$\neg Aa$$
$$\neg Bb$$
$$Aa \vee Ba$$

Aa	Ba
$\times$	$Ab \vee Bb$

Ab	Bb
	$\times$

$$\boxed{\dfrac{\forall x(A \vee B)}{\forall x A \vee \forall x B}\ ?}$$

令 $\mathcal{M} = (M, A^{\mathcal{M}}, B^{\mathcal{M}})$

$$M = \{a, b\}$$
$$A^{\mathcal{M}} = \{b\}$$
$$B^{\mathcal{M}} = \{a\}$$

则

$$\mathcal{M} \vDash \forall x(A \vee B)$$

但

$$\mathcal{M} \nvDash \forall x A \vee \forall x B$$

构造反模型

$$\begin{array}{l} \exists x(A \wedge B) \\ \exists x(B \wedge C) \\ \hline \exists x(A \wedge C) \end{array} \ {\color{red}?}$$

$$\begin{array}{c} \exists x(A \wedge B) \checkmark \\ \exists x(B \wedge C) \checkmark \\ \neg\exists x(A \wedge C) \\ | \\ Aa \\ Ba \\ | \\ Bb \\ Cb \end{array}$$

$$\begin{array}{ccc} & Cb & \\ \neg Aa & & \neg Ca \\ \times & & \end{array}$$

$$\begin{array}{ccc} & \neg Ca & \\ \neg Ab & & \neg Cb \\ & & \times \end{array}$$

令 $\mathcal{M} = (M, A^{\mathcal{M}}, B^{\mathcal{M}}, C^{\mathcal{M}})$

$$\begin{aligned} M &= \{a, b\} \\ A^{\mathcal{M}} &= \{a\} \\ B^{\mathcal{M}} &= \{a, b\} \\ C^{\mathcal{M}} &= \{b\} \end{aligned}$$

则

$$\mathcal{M} \vDash \exists x(A \wedge B)$$

$$\mathcal{M} \vDash \exists x(B \wedge C)$$

但

$$\mathcal{M} \nvDash \exists x(A \wedge C)$$

构造反模型

$$\boxed{\begin{array}{c} \exists x(A \to B) \\ \exists x(B \to A) \\ \hline \exists x(A \leftrightarrow B) \end{array} \;\color{red}{?}}$$

$$\begin{array}{l} \exists x(A \to B) \checkmark \\ \exists x(B \to A) \checkmark \\ \neg\exists x(A \leftrightarrow B) \end{array}$$

$$\begin{array}{c} Aa \to Ba \\ Bb \to Ab \end{array}$$

- Aa, $\neg Ba$
  - $\neg Aa$ — $\times$
  - Ba — $\times$
- $\neg Aa$, Ba
  - Ab, $\neg Bb$
    - …
    - …
  - $\neg Ab$, Bb
    - $\neg Bb$ — $\times$
    - Ab — $\times$

令 $\mathcal{M} = (M, A^{\mathcal{M}}, B^{\mathcal{M}})$

$$M = \{a, b\}$$

$$A^{\mathcal{M}} = \{b\}$$

$$B^{\mathcal{M}} = \{a\}$$

则

$$\mathcal{M} \vDash \exists x(A \to B)$$

$$\mathcal{M} \vDash \exists x(B \to A)$$

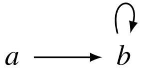
但

$$\mathcal{M} \nvDash \exists x(A \leftrightarrow B)$$

构造反模型

$$\boxed{\dfrac{\exists y \forall x Lxy}{\forall y \exists x Lxy} \;?}$$

$$
\begin{array}{c}
\exists y \forall x Lxy \checkmark \\
\neg \forall y \exists x Lxy \checkmark \\
| \\
\exists y \neg \exists x Lxy \checkmark \\
| \\
\neg \exists x Lxa \\
| \\
\forall x Lxb \\
| \\
Lab \\
Lbb \\
| \\
\neg Laa \\
\neg Lba
\end{array}
$$



令 $\mathcal{M} = (M, L^{\mathcal{M}})$

$$
\begin{aligned}
M &= \{a, b\} \\
L^{\mathcal{M}} &= \{(a, b), (b, b)\}
\end{aligned}
$$

则

$$\mathcal{M} \vDash \exists y \forall x Lxy$$

但

$$\mathcal{M} \nvDash \forall y \exists x Lxy$$

Examples

1. Everybody loves everybody
2. Everybody is loved by everybody
3. Somebody loves everybody
4. Somebody is loved by everybody
5. Everybody is loved by somebody
6. Everybody loves somebody
7. Somebody loves somebody
8. Somebody is loved by somebody

$$
\begin{tikzcd}
& \dfrac{\forall x \forall y Lxy}{\forall x \forall y Lyx} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\exists x \forall y Lxy \arrow[d, Rightarrow] \arrow[rr, dotted, no head, "\times" description] \arrow[drr, dotted, no head, "\times" description] & & \exists x \forall y Lyx \arrow[d, Rightarrow] \arrow[dll, dotted, no head] \\
\forall y \exists x Lxy \arrow[dr, Rightarrow] \arrow[rr, dotted, no head, "\times" description] & & \forall y \exists x Lyx \arrow[dl, Rightarrow] \\
& \dfrac{\exists x \exists y Lxy}{\exists x \exists y Lyx} &
\end{tikzcd}
$$

练习：有效性判定 ↯

$$\begin{array}{c}\forall x A \\ \hline A[t/x]\end{array} \qquad \begin{array}{c}A[t/x] \\ \hline \exists x A\end{array} \qquad\Big|\qquad \begin{array}{c}\neg\forall x A \\ \hline\hline \exists x \neg A\end{array} \qquad \begin{array}{c}\neg\exists x A \\ \hline\hline \forall x \neg A\end{array}$$

$$\begin{array}{c}\forall x(A \wedge B) \\ \hline\hline \forall x A \wedge \forall x B\end{array} \qquad \begin{array}{c}\forall x A \vee \forall x B \\ \hline \forall x(A \vee B)\end{array} \qquad\Big|\qquad \begin{array}{c}\exists x(A \vee B) \\ \hline\hline \exists x A \vee \exists x B\end{array} \qquad \begin{array}{c}\exists x(A \wedge B) \\ \hline \exists x A \wedge \exists x B\end{array}$$

$$\begin{array}{c}\forall x(A \to B) \\ \hline \forall x A \to \forall x B\end{array} \qquad \begin{array}{c}\forall x(A \to B) \\ \hline \exists x A \to \exists x B\end{array} \qquad\Big|\qquad \begin{array}{c}\forall x(A \leftrightarrow B) \\ \hline \forall x A \leftrightarrow \forall x B\end{array}$$

$$\begin{array}{c}\forall x \forall y A x y \\ \hline\hline \forall y \forall x A x y\end{array} \qquad \begin{array}{c}\exists x \exists y A x y \\ \hline\hline \exists y \exists x A x y\end{array} \qquad\Big|\qquad \begin{array}{c}\exists x \forall y A x y \\ \hline \forall y \exists x A x y\end{array}$$

$$\begin{array}{c}\forall x A \to \exists x B \\ \hline\hline \exists x(A \to B)\end{array} \qquad \begin{array}{c}\exists x A \to \forall x B \\ \hline\hline \forall x(A \to B)\end{array} \qquad\Big|\qquad \begin{array}{c} \\ \hline \exists x(A \to \forall x A)\end{array}$$

练习：有效性判定 ⚡

$x \notin \mathrm{Fv}(A)$:

$$\frac{A}{\forall x A} \qquad \frac{A}{\exists x A}$$

$$\frac{\forall x(A \vee B)}{A \vee \forall x B} \qquad \frac{\exists x(A \vee B)}{A \vee \exists x B}$$

$$\frac{\forall x(A \wedge B)}{A \wedge \forall x B} \qquad \frac{\exists x(A \wedge B)}{A \wedge \exists x B}$$

$$\frac{\forall x(A \to B)}{A \to \forall x B} \qquad \frac{\exists x(A \to B)}{A \to \exists x B}$$

$$\frac{\exists x B \to A}{\forall x(B \to A)} \qquad \frac{\forall x B \to A}{\exists x(B \to A)} \qquad \frac{\exists x(A \to B) \quad \exists x(B \to A)}{\exists x(A \leftrightarrow B)}$$

Exercise

$$\frac{\forall x B \to A}{\exists x (B \to A)}\ x \notin \mathrm{Fv}(A)$$

$$\mathsf{diam}(X) := \sup\left\{|x-y| : x, y \in X\right\}$$

$$\frac{\left(\forall x \in X\ |x| \leq 1\right) \to \mathsf{diam}(X) \leq 2}{\exists x \in X\ \left(|x| \leq 1 \to \mathsf{diam}(X) \leq 2\right)?}\ ?$$

1. 上述两公式是否等价?
2. 第二个公式是否成立?

$$\forall x (Cx \to Bx) \to A \quad \text{vs} \quad \exists x (Cx \wedge (Bx \to A))$$

有效性判定 ♡

$$\frac{}{t = t} \qquad \frac{}{s = t \to t = s} \qquad \frac{}{r = s \to s = t \to r = t}$$

$$\frac{s_1 = t_1, \ldots, s_n = t_n}{f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)} \qquad \frac{s_1 = t_1, \ldots, s_n = t_n}{P(s_1, \ldots, s_n) \leftrightarrow P(t_1, \ldots, t_n)}$$

$$\frac{s = t}{r[s/x] = r[t/x]} \qquad \frac{s = t}{A[s/x] \leftrightarrow A[t/x]}$$

有效性判定 ♡

$x \notin \mathrm{Var}(t)$:

$$\frac{}{\exists x(x = t)} \qquad \frac{A[t/x]}{\exists x(x = t \wedge A)} \qquad \frac{A[t/x]}{\forall x(x = t \longrightarrow A)}$$

$y \notin \mathrm{Var}(A)$:

$$\frac{}{\forall x(A \leftrightarrow \exists y(y = x \wedge A[y/x]))} \qquad \frac{}{\forall x(A \leftrightarrow \forall y(y = x \longrightarrow A[y/x]))}$$

构造性证明 vs 存在性证明

$$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A} \qquad \frac{\Gamma, \forall x \neg A \vdash \bot}{\Gamma \vdash \exists x A}$$

Example (There exist two irrational numbers x, y s.t. x^y is rational.)

$$\begin{cases} x := \sqrt{2} \\ y := \sqrt{2} \end{cases} \qquad \begin{cases} x := \sqrt{2}^{\sqrt{2}} \\ y := \sqrt{2} \end{cases}$$

- $R(x)$: x is rational
- $a = \sqrt{2}$
- $f(x, y) = x^y$

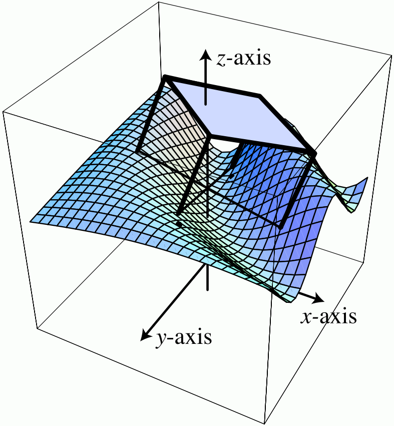
逻辑推理:

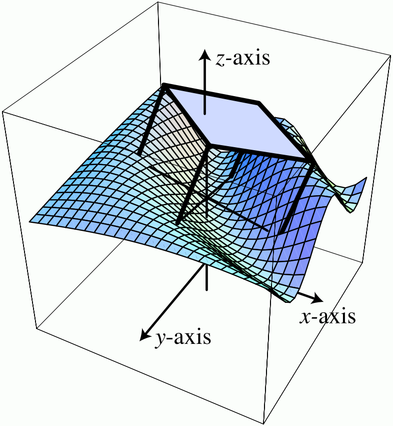
$$\frac{\neg R(a) \quad R(f(f(a, a), a))}{\exists x y(\neg R(x) \wedge \neg R(y) \wedge R(f(x, y)))}$$

$$\begin{cases} x := \sqrt{2} \\ y := \log_2 9 \end{cases}$$

存在性证明

Example (连续起伏的地面上摇晃的四腿方桌可以通过旋转放平稳)





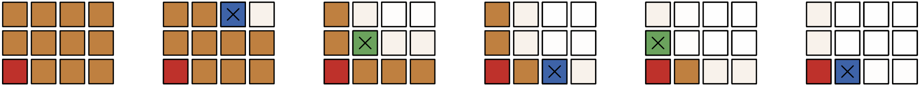
Example ($e+\pi$ 和 $e\pi$ 中至少有一个是超越数)

$$x^2-(e+\pi)x+e\pi=0$$

存在性证明

毒巧克力博弈

左下角格子的巧克力有毒. 玩家轮流选一个格子, 并吃掉格子里的以及格子右上方的所有巧克力.



Theorem

玩家 1 有必胜策略.

Proof.

玩家 1 开始可以选择最右上角的格子. 假如玩家 2 有必胜策略, 那么玩家 1 一开始就可以选择玩家 2 的动作. □

Theorem (Zermelo's Theorem 1913)

在两人的、完美信息的、没有平局的、有穷博弈中, 必有一方有必胜策略.

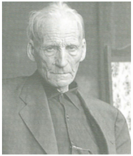
Proof.

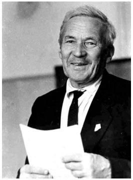
$$\exists x_1 \forall y_1 \ldots \exists x_n \forall y_n A \vee \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \neg A$$

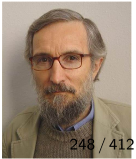
where A states that a final position is reached where player 1 wins. □

Remark: 两个上帝下围棋, 只需要猜先, 不用落子, 猜先完毕游戏结束.

直觉主义







- 非直谓主义 (*Poincaré*, Russell)
  禁止恶性循环原则: 任何一个实体都不能仅仅通过它所属的整体来定义.
- $\boxed{\text{直觉主义}}$ 数学是心灵的构造.
  (Kronecker, *Brouwer*, Heyting, *Kolmogorov*, Weyl)
  - 潜无穷 vs 实无穷
  - "存在就是被 (直觉) 构造" — Brouwer
  - 非构造性证明 ×
  - 反证法 ×
  - 双重否定消去律 ×
  - 排中律 ×
  - 选择公理 ×

  (There exist two irrational numbers x and y s.t. x^y is rational.)

  $$\sqrt{2} \quad \log_2 9$$

  "上帝创造了整数, 其余的都是人造的." — Kronecker
- 构造主义数学 (Bishop, *Martin-Löf*)

练习: 树形证明 — Now it's your turn ↯

$$\frac{\exists x \forall y (P(y) \to y = x) \quad \forall x P(f(x))}{\exists x (f(x) = x)}$$

$$\begin{array}{c} \exists x \big(Px \land \forall y (Py \to y = x) \land Qx\big) \\ \hline\hline \exists x \forall y \big((Py \leftrightarrow y = x) \land Qx\big) \end{array}$$

$$\frac{\exists x \big(Px \land \forall y (Py \to y = x)\big) \quad \exists x \big(Qx \land \forall y (Qy \to y = x)\big) \quad \neg \exists x (Px \land Qx)}{\exists xy \big(x \neq y \land (Px \lor Qx) \land (Py \lor Qy) \land \forall z (Pz \lor Qz \to z = x \lor z = y)\big)}$$

$*\mathbf{54 \cdot 43}.\ \vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

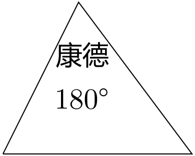
$$\begin{array}{lll}
\vdash . *54 \cdot 26 . \supset \vdash :. \alpha = \iota' x . \beta = \iota' y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y . & & \\
[*51 \cdot 231] & \equiv . \iota' x \cap \iota' y = \Lambda . & \\
[*13 \cdot 12] & \equiv . \alpha \cap \beta = \Lambda & (1) \\
\vdash . (1) . *11 \cdot 11 \cdot 35 . \supset & & \\
\quad \vdash :. (\exists x, y) . \alpha = \iota' x . \beta = \iota' y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda & & (2) \\
\vdash . (2) . *11 \cdot 54 . *52 \cdot 1 . \supset \vdash . Prop & &
\end{array}$$

From this proposition it will follow, when arithmetical addition has been defined, that $\mathbf{1} + \mathbf{1} = \mathbf{2}$.

$$\exists_1 x P \land \exists_1 x Q \land \neg \exists x (P \land Q) \to \exists_2 x (P \lor Q)$$

数学是“先验综合判断”吗?

- 苏格拉底: “我只知道我什么也不知道.”
- 笛卡尔: “我思故我在”. 我们可以通过内省获得部分知识的确定性, 但关于外部世界的知识呢?
- 休谟: 放弃吧. (i) 关于外部世界的知识是因果陈述. (ii) 因果陈述是综合判断, 所以只能后验的获知. (iii) 我们既无法直接观察因果本身, 又无法非循环的论证未来与过去的相似性, 所以我们无法后验的获知因果陈述.
- 康德: 虽然因果陈述是综合的, 但我们可以先验的获知因果. 因为因果结构既是关于外部世界的原始材料的, 又是内嵌在人类心灵的认知模式的一部分. “人为自然立法.”



康德 数学是先验综合判断.

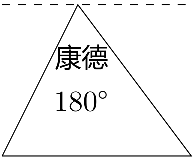
- 几何 — 空间的纯粹直观
- 算术 — 时间的纯粹直观

高斯 几何是后验的. (黎曼几何, 空间曲率)

弗雷格 算术是分析的.

$$\vdash 1 + 1 = 2$$

罗素[10]

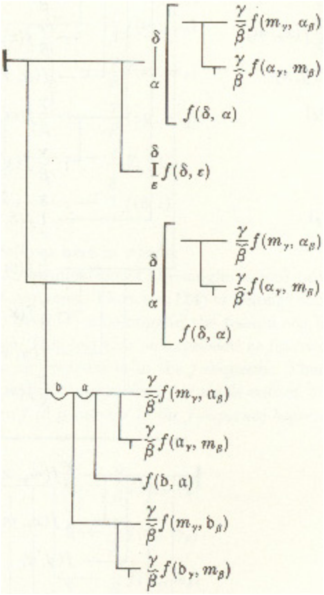


- 康德正确的认识到欧式几何的命题不能脱离图形的辅助直接从公理推出, 为此, 他不惜发明了一整套认识论.
- 康德发明的 “先验综合判断” 只不过是为了给一个有缺陷的逻辑 (亚里士多德三段论) 打补丁.
- 纯数学, 包括几何学, 不过是形式逻辑. 这对康德哲学是致命一击.

[10] Bertrand Russell: “Mathematics and the Metaphysicians”. Reprinted in *Mysticism and Logic, and Other Essays*.

Remark: 罗素的第一本书《论几何学的基础》贯彻的是康德的几何观, 把物质与空间分离, 用先验直观拒绝曲率不为 0 的空间, 结果被爱因斯坦打脸, 广义相对论恰恰连接了物质与时空曲率.

- 算术规律是分析判断，因此是先验的．算术不过是发展了的逻辑，算术定理是逻辑规律．
- 算术在自然科学上的应用不过是对观察现象的逻辑加工．计算即推理．
- 如果哲学的任务是破除语词对人类精神的支配，揭开由于语言的表达方式而造成的概念关系的假象，把思想从日常语言的迷雾中解放出来，那么，我的《概念文字》将成为哲学家手中的有用工具．
- 一个好的数学家至少是半个哲学家；一个好的哲学家至少是半个数学家．



	先验	后验
分析	数学	×
综合	×	物理

Table: 康德之前

	先验	后验
分析	逻辑	×
综合	算术 几何	

Table: 康德

	先验	后验
分析	×	×
综合	×	逻辑 算术 几何

Table: 蒯因、普特南

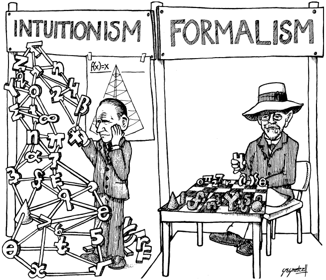
	先验	后验
分析	逻辑 算术 纯粹几何	×
综合	×	应用几何

Table: 逻辑实证主义

	先验	后验
分析		×
综合	逻辑 算术 纯粹几何	应用几何

Table: Martin-Löf

数学哲学: 逻辑主义/直觉主义/形式主义



逻辑主义	直觉主义	形式主义
数学 逻辑	逻辑 数学 心灵构造	数学 符号游戏
实在论	概念论	唯名论

何物存在?

唯名论:
抽象事物不存在

极端唯名论:
即使具体事物也不存在

柏拉图主义:
抽象事物存在

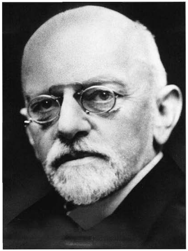
极端柏拉图主义:
就连具体事物也存在

- 名字叫“罗素”的那个人存在
- 人存在
- 电子存在
- 红存在
- 2 存在

“如果其他事物存在, 那数也存在.”

— 柏拉图《智者篇》

希尔伯特 David Hilbert 1862-1943



- 几何的**形式**公理化
  几何相对于算术的一致性
  (克莱因: 非欧几何相对于欧式几何的一致性)
  (natural/integer/rational/real/complex)
- 希尔伯特 23/24 问题 ($1^{st}, 2^{nd}, 10^{th}, 24^{th}, 6^{th}$)
- 元数学 — 证明论
- 形式主义 数学是符号游戏.
- 公理是初始概念的隐定义.
- 我们完全可以用 "桌子、椅子、啤酒瓶" 代替 "点、线、面" 而不影响推理的正确性.
- 数学是根据某些简单规则使用毫无意义的符号在纸上进行的游戏, 是制造快乐的游戏.
- 我们的内心响起了永恒的召唤: 那里有一个问题, 去找出它的答案! 你可以通过纯粹理性找到它 — 数学里没有不可知!
- 我们必须知道; 我们必将知道!

1. 形式化与公理化 (语言的丰富性): 形式化 — 形式与内涵分离; 公理化 — 公理化 (1) 逻辑 L, (2) 关于现实世界的 "有穷" 数学 F, (3) 关于理想世界的 "无穷" 数学 T. 所有数学命题都可以在 T 中表达.
2. 独立性: 证明公理之间彼此 "独立".
3. 完备性: (1) 所有有效的逻辑命题都可以在 L 中证明; (2) 所有真的数学命题都可以在 T 中证明. — 实现形式与内涵的统一.
4. **一致性**: 用 "有穷数学"F 的方法证明 "无穷数学"T 中推不出矛盾.
5. 保守性 ($\forall A \in \Pi_1 [\mathrm{T} \vdash A \implies \mathrm{F}, \mathrm{Con}_{\mathrm{T}} \vdash A]$ 一致性蕴含保守性): 任何关于 '实在对象' 的陈述, 如果可以在 T 中证明, 那么也可以在 F 中证明.
6. 可判定性 (能行性): 逻辑命题的有效性和数学命题的真理性都可以机械地判定.
7. 简单性: 证明某些证明的最大简单性.
8. 范畴性? 在同构的意义上, T 刻画了唯一一个结构.

[11]绕道理想世界迂回地获取关于现实世界的知识, 并基于现实世界证明理想世界的合理性.

希尔伯特规划

"当我们更仔细地考虑逻辑的公理化时，我们很快就会认识到，整数和集合的一致性问题不是孤立的，而是属于一个广阔的**认识论**领域[12]，这些问题具有特殊的数学色彩：例如，每个数学问题原则上是否可解的问题，结论是否可验证的问题，数学证明的简单性标准问题，数学与逻辑中形式与内涵的关系问题，以及一个数学问题是否在有限步骤内可判定的问题."

— 希尔伯特

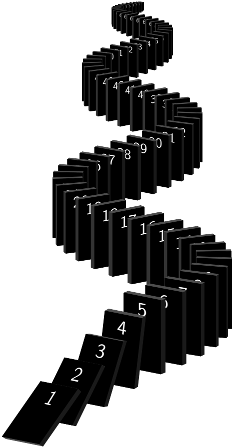
[12] Chaitin："今天正在发生的计算机接管世界、数字化、信息化，都是希尔伯特在 20 世纪初提出的一个**哲学问题**的结果."

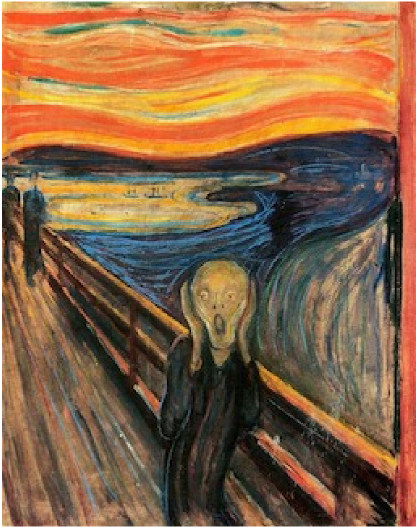
Example — 皮亚诺算术理论 ☺

1. 我是一个神
$$0 \in \mathbb{N}$$
2. 每个神的兽也是一个神
$$\forall n \in \mathbb{N} : s(n) \in \mathbb{N}$$
3. 我不是任何神的兽
$$\forall n \in \mathbb{N} : s(n) \neq 0$$
4. 不同的神有不同的兽
$$\forall mn \in \mathbb{N} : s(m) = s(n) \to m = n$$
5. 如果我有 X, 且每个神都把 X 递送给它的兽, 那么所有的神都有 X
$$\forall X \big[0 \in X \wedge \forall n (n \in X \to s(n) \in X) \to \forall n \in \mathbb{N} (n \in X) \big]$$

$+, \cdot$ 可以递归定义:

- $\forall n \in \mathbb{N} : n + 0 = n$
- $\forall mn \in \mathbb{N} : m + s(n) = s(m + n)$
- $\forall n \in \mathbb{N} : n \cdot 0 = 0$
- $\forall mn \in \mathbb{N} : m \cdot s(n) = m \cdot n + m$





理性可以对自身的有效性提出怀疑.

哥德尔句

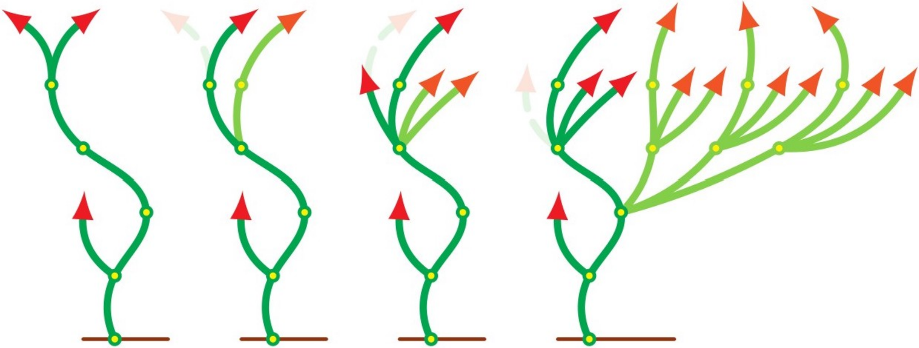
"我不可证"

Problem (哥德尔是什么人?)

- 一个岛上有"君子"、"小人"两类人. "君子"只说真话, "小人"只说假话.
- 岛上有人有身份证, 有人没有.
- 有身份证的都是君子.
- 你来岛上遇到了一个名字叫"哥德尔"的土著.
- 哥德尔说: "我没有身份证".

Hydra 九头蛇游戏 — “自然的” 不可证命题

- 每一步你砍掉它一个头
- 在第 n 步, 如果它的一个非根部的头被砍掉, 则从被砍掉头的下方节点处长出 n 个副本



Goodstein Theorem 不管你怎么砍都能杀死 “九头蛇”

Kirby-Paris Theorem 但你无法通过 PA 证明这一点

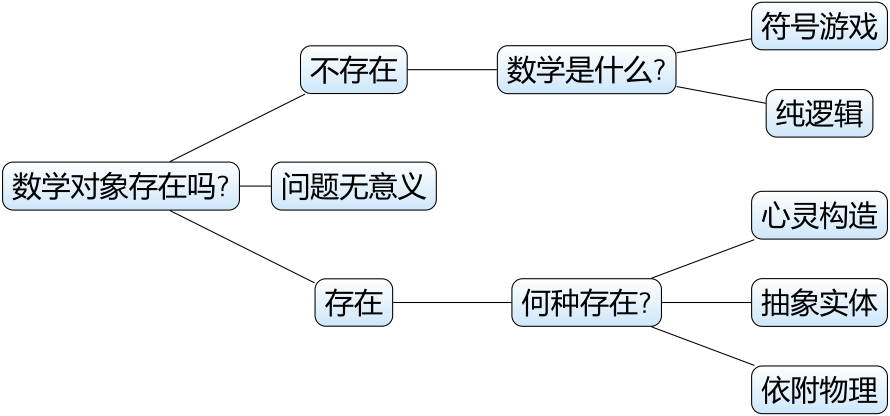


Figure: 形式主义/逻辑主义/直觉主义/柏拉图主义/物理主义

Is there more than one mathematical universe?[13]

[13] Penelope Maddy: What Do We Want a Foundation to Do?

有效论证?

$$\frac{\text{Whatever begins to exist, has a cause of its existence} \quad \text{The universe began to exist}}{\text{The universe has a cause of its existence}}$$

$$\frac{\forall x(Bx \to \exists y Cyx) \quad Bu}{\exists y Cyu}$$

1. 如果上帝存在, 那么不是真诚地相信上帝存在的人不会得到救赎.
2. 如果上帝不存在, 那么没有人会得到救赎.
3. 如果一个人相信上帝存在仅仅是由于帕斯卡赌, 那就不算真诚地相信上帝存在.
4. 帕斯卡仅仅由于帕斯卡赌才相信上帝存在.
5. 因此, 帕斯卡不会得到救赎.

$$\frac{Eg \to \forall x(\neg Bx \to \neg Sx) \quad \neg Eg \to \neg\exists x Sx \quad \forall x(Wx \to \neg Bx) \quad Wp}{\neg Sp}$$

有效论证?

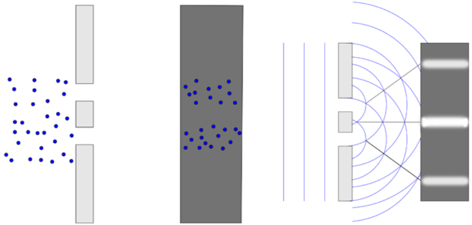
每个全善的东西都消灭了所有它能消灭的邪恶的东西
全能的东西能消灭所有东西
存在邪恶的东西
不存在被某个东西消灭了的东西

不存在全善且全能的东西

$$\forall x(Bx \to \forall y(Ey \land Cxy \to Dxy))$$
$$\forall x(Ox \to \forall y Cxy)$$
$$\exists x Ex$$
$$\neg\exists x\exists y Dyx$$

$$\neg\exists x(Bx \land Ox)$$

有效论证?



1. 假如发射光是粒子, 那么, 探测屏上的撞击图案就是一系列闪光, 并且会留下至多两条条纹.
2. 假如发射光是波, 那么, 探测屏上的撞击图案就不是一系列闪光, 并且会留下两条以上的条纹.
3. 假如撞击图案是一系列连续的闪光, 但留下了两条以上的条纹, 那么, 发射光既不是粒子也不是波.

$$\frac{\forall x(Px \to Fx \wedge Sx) \qquad \forall x(Wx \to \neg Fx \wedge \neg Sx)}{\forall x(Fx \wedge \neg Sx \to \neg Px \wedge \neg Wx)}$$

有效论证?

- 老公: “赌博时, 我押上了自己, 输了; 我又押上了你, 也输了.”
- 老婆: “当你输掉自己时, 你就已经不再拥有我了, 所以你无权押我.”

1. For all y there exists z such that z owns y.
2. For all x, y, z if z owns y and y owns x then z owns x.
3. For all x, y, z if y owns x and z owns x then $y = z$.

$$\frac{\begin{array}{c}\forall y \exists z Ozy \\ \forall xyz(Ozy \wedge Oyx \to Ozx) \\ \forall xyz(Oyx \wedge Ozx \to y = z)\end{array}}{\forall y(\neg Oyy \to \neg \exists x Oyx)}$$

有效论证?

It is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American. Theorefore, Colonel West is a criminal.

1. $\forall xyz(\mathsf{American}(x) \wedge \mathsf{Weapon}(y) \wedge \mathsf{Hostile}(z) \wedge \mathsf{Sell}(x, y, z) \to \mathsf{Criminal}(x))$
2. $\exists x(\mathsf{Own}(\mathsf{nono}, x) \wedge \mathsf{Missile}(x))$
3. $\forall x(\mathsf{Missile}(x) \wedge \mathsf{Own}(\mathsf{nono}, x) \to \mathsf{Sell}(\mathsf{west}, x, \mathsf{nono}))$
4. $\forall x(\mathsf{Missile}(x) \to \mathsf{Weapon}(x))$
5. $\forall x(\mathsf{Enemy}(x, \mathsf{america}) \to \mathsf{Hostile}(x))$
6. $\mathsf{American}(\mathsf{west})$
7. $\mathsf{Enemy}(\mathsf{nono}, \mathsf{america})$
8. $\mathsf{Criminal}(\mathsf{west})$

练习: 有效性判定 — Now it's your turn ⚡

1. 有些战争是正义的. 没有侵略战争是正义的. 因此, 有些战争是非侵略性的.
2. 所有中国学生和所有美国学生都爱 Anne. 中南大学只有中国学生和美国学生. 因此, 中南大学的学生都爱 Anne.
3. 非分析的有意义的命题均可证实或可证伪. 哲学命题既不是分析的又不可证实又不可证伪. 所以, 哲学命题无意义.
4. 如果狗是动物, 那么, 狗的头就是动物的头.
5. 唯有亚里士多德是哲学家. 柏拉图是抽烟的哲学家. 所以, 亚里士多德抽烟.
6. 小艾爱着小白, 小白爱着小菜, 小艾已婚, 小菜未婚. 因此, 某个已婚人士爱着某个未婚人士.
7. 人人都爱小艾. 小艾除我之外谁都不爱. 因此, 人人都爱我.
8. 凡爱人者人皆爱之. 罗密欧爱朱丽叶. 所以, 你爱我!
9. 没有理发师会给并且只给那些不给自己理发的人理发. *Russell*
10. 如果大鱼比小鱼游得快, 那么, 只要有最大的鱼就有游得最快的鱼.
11. 如果不知道爱情密码, 没有人能偷走小白的心. 小白的心被偷了. 只有小艾知道爱情密码. 所以, 小艾偷了小白的心.

12. 没有女孩会爱花心的人. 小艾是一个会爱上所有爱她的人的女孩. 小白爱小艾. 所以, 小白不花心.
13. 如果所有的思想都清楚, 那么没有思想需要解释; 如果所有的思想都不清楚, 那么没有思想能够解释清楚. 因此, 如果有的思想既需要解释又能够解释清楚, 那么说明有的思想清楚、有的思想不清楚.
14. 没有人相信并且只相信那些无法与人建立相互信任关系的人. *Quine*
15. 我是个玩摇滚的哲学家. 唯有哲学家才欣赏哲学家. 没有哲学家没有怪癖. 有怪癖的摇滚人都会有女孩欣赏. 有怪癖的人都自命不凡. 因此, 有女孩自命不凡.
16. 当今的那位皇上是秃子. 秃子都性感. 因此, 不管谁是当今的皇上, 都性感.
17. 只有小艾和小白欣赏小菜. 小艾和小白不是同一个人. 无论谁欣赏小菜都会爱上他, 也只有欣赏他的人才会爱他. 因此, 有且仅有两个人爱小菜.
18. If anyone speaks to anyone, then someone introduces them; no one introduces anyone to anyone unless he knows them both; everyone speaks to Alice; therefore everyone is introduced to Alice by someone who knows her.

Application — 扫雷

- There are exactly n mines in the game.
- If a cell contains the number 1, then there is exactly one mine in the adjacent cells.

  $\forall x(\text{Contain}(x, 1) \to$

  $\exists y(\text{Adjacent}(x, y) \wedge \text{Mine}(y) \wedge \forall z(\text{Adjacent}(x, z) \wedge \text{Mine}(z) \to z = y)))$

- ...

Application — “君子” 与 “抽烟”

Problem (“君子岛” 还是 “小人岛”? “抽烟” 还是 “不抽烟”?)

你想研究抽烟与说谎之间的相关性, 于是走访各个 “君子岛” 与 “小人岛”. “君子岛” 上的人只说真话, “小人岛” 上的人只说假话.

- 岛 *1:* 每个土著都说: “岛上君子都不抽烟”.
- 岛 *2:* 每个土著都说: “岛上有小人抽烟”.
- 岛 *3:* 每个土著都说: “如果我抽烟, 那么所有岛人都抽烟”.
- 岛 *4:* 每个土著都说: “如果有岛人抽烟, 那么我也抽烟”.
- 岛 *5:* 每个土著都说: “虽然有岛人抽烟, 但我不抽烟”.

岛 5:
$$\frac{\forall x(Tx \leftrightarrow \exists ySy \wedge \neg Sx) \quad \forall xTx \vee \forall x\neg Tx}{\forall x\neg Tx \wedge (\forall xSx \vee \forall x\neg Sx)}$$

1. T None
2. F None
3. T All or None
4. T All or None
5. F All or None

Application — “人有来生”?

1. 人的一生有太多可能性没有实现.
2. 如果只有一生没有来生, 没有实现的可能性将永远不可实现.
3. 永远不可实现的可能性没有意义.
4. 如果宇宙是有意义的, 那么其包含的对象的可能性都是有意义的.
5. 可被学习理解的东西是有意义的.
6. 有秩序的东西是可被学习理解的.
7. 宇宙是有秩序的.

1. $\forall xy(\mathsf{Man}(x) \wedge \mathsf{Life}(y,x) \to \exists z(\mathsf{Possible}(z,x,y) \wedge \neg\mathsf{Realize}(z)))$
2. $\forall xy(\mathsf{Man}(x) \wedge \mathsf{Life}(y,x) \wedge \neg\exists y'(\mathsf{Life}(y',x) \wedge y' \neq y) \to$ $\forall z(\mathsf{Possible}(z,x,y) \wedge \neg\mathsf{Realize}(z) \to \mathsf{UnRealizable}(z)))$
3. $\forall x(\mathsf{UnRealizable}(x) \to \neg\mathsf{Meaning}(x))$
4. $\mathsf{Meaning}(u) \to \forall xyz(\mathsf{Contain}(u,x) \wedge \mathsf{Possible}(z,x,y) \to \mathsf{Meaning}(z))$
5. $\forall x(\mathsf{Learnable}(x) \to \mathsf{Meaning}(x))$
6. $\forall x(\mathsf{Ordered}(x) \to \mathsf{Learnable}(x))$
7. $\mathsf{Ordered}(u)$
8. $\forall x(\mathsf{Man}(x) \to \mathsf{Contain}(u,x))$
9. $\forall xy(\mathsf{Man}(x) \wedge \mathsf{Life}(y,x) \to \exists y'(\mathsf{Life}(y',x) \wedge y' \neq y))$

怎么用一个 5 升和一个 7 升的桶去打 1 升的水?

- $S(x, y)$: x 和 y 分别是第一个和第二个桶中的水量.

Initial State $S(0, 0)$

Goal State $\exists x[S(1, x) \vee S(x, 1)]$

一共有 8 种可能的动作:

A_1. $\forall xy\big[S(x, y) \to S(5, y)\big]$ [Fill 1]

A_2. $\forall xy\big[S(x, y) \to S(0, y)\big]$ [Empty 1]

A_3. $\forall xy\big[S(x, y) \to S(x, 7)\big]$ [Fill 2]

A_4. $\forall xy\big[S(x, y) \to S(x, 0)\big]$ [Empty 2]

A_5. $\forall xy\big[S(x, y) \wedge x + y \leq 7 \to S(0, y + x)\big]$ [Empty 1 into 2]

A_6. $\forall xy\big[S(x, y) \wedge x + y > 7 \to S(x - (7 - y), 7)\big]$ [Pour 1 into 2]

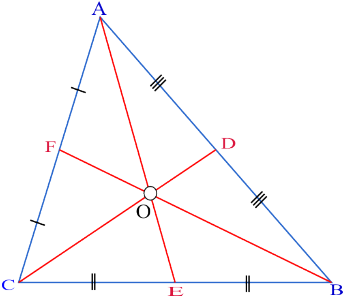
A_7. $\forall xy\big[S(x, y) \wedge x + y \leq 5 \to S(x + y, 0)\big]$ [Empty 2 into 1]

A_8. $\forall xy\big[S(x, y) \wedge x + y > 5 \to S(5, y - (5 - x))\big]$ [Pour 2 into 1]

$$\text{Initial State}, A_1, \ldots, A_8 \vdash \text{Goal State}$$

罗素的“摹状词理论”

1. The substitution of identicals.
   “The morning star is the evening star.”
2. The law of the excluded middle.
   “The present King of France is bald.” or
   “The present King of France is not bald.”
3. The problem of negative existentials.
   “The flying horse does not exist.”



$$B(\iota_x A) := \exists x\big(Ax \land \forall y(Ay \to y = x) \land Bx\big)$$

- Oedipus did not know the woman he married was his mother.
  $\neg\mathsf{K}(\iota_x A = m(o))$?
  $\iota_x A = m(o) \to \mathsf{K}(m(o) = m(o)) \to \mathsf{K}(\iota_x A = m(o))$?
  $$\exists x\big(Ax \land \forall y(Ay \to y = x) \land \neg\mathsf{K}(x = m(o))\big)$$
  $$\neg\mathsf{K}\left[\exists x\big(Ax \land \forall y(Ay \to y = x) \land x = m(o)\big)\right]$$
- The present King of France is bald. $B(\iota_x K) \lor (\neg B)(\iota_x K)$?
  $$\exists x\big(Kx \land \forall y(Ky \to y = x) \land \neg Bx\big) \quad (\neg B)(\iota_x K)$$
  $$\neg\exists x\big(Kx \land \forall y(Ky \to y = x) \land Bx\big) \quad \neg B(\iota_x K)$$
- The flying horse does not exist. $\neg E(\iota_x Fx)$
  $$\exists x\big(Fx \land \forall y(Fy \to y = x) \land \neg Ex\big) \text{ ?}$$
  $$\neg\exists x\big(Fx \land \forall y(Fy \to y = x)\big)$$
  $$Ex \overset{?}{:=} \exists P\big(Px \land \exists y \neg Py\big)$$
- Get rid of function symbols.
  $$\mathsf{Bald}(\iota_x \mathsf{Father}(x, \mathsf{alice})) \quad vs \quad \mathsf{Bald}(\mathsf{father}(\mathsf{alice}))$$
- Universal Instantiation. $\forall x B \to B(\iota_x A)$? $\quad \vdash \forall x B \to B(\iota_x^y A)$
  $$B(\iota_x^y A) := \big(\exists! x A \to \exists x(A \land B)\big) \land \big(\neg\exists! x A \to B[y/x]\big)$$

Translation

1. Every citizen of every country respects the King of that country.

$$\forall xy(Cy \land Zxy \to Rx\iota_z Kzy)$$

2. The daughter of the King of China is the person everyone respects.

$$\iota_y Dy\iota_x Kxc = \iota_x\big(Px \land \forall y(Py \to Ryx)\big)$$

3. The person everyone respects is a citizen of the country everyone respects.

$$Z\iota_x\big(Px \land \forall y(Py \to Ryx)\big)\iota_x\big(Cx \land \forall y(Py \to Ryx)\big)$$

罗素 Bertrand Russell 1872-1970

- 罗素悖论
  (第三次数学基础危机)
- 摹状词理论
  (当今的法国国王是秃子或不是秃子)
- 类型论
- 《数学原理》

没有理发师给并且只给那些不给自己理发的人理发. [14]

"哲学的特点：从简单得不值一提的东西开始，以荒谬得没人会相信的东西结束."

— 罗素

[14] Russell: On denoting. 1905.

分析哲学与数学分析 — 摹状词理论的由来

- 一个陈述的**逻辑形式**可能不等于它的**语法形式**.
- 弗雷格的**语境原则**: 不要孤立的问一个词的意义, 一个词只有包含在上下文语境中才有意义. [15]
- 罗素的语境定义的思想隐含在 19 世纪的数学分析严格化之中.
- 贝克莱: 第二次数学基础危机. "无穷小是不是 0?"
  对 $f(x) = x^2$,

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{f(x+\mathrm{d}x) - f(x)}{\mathrm{d}x} = \frac{(x+\mathrm{d}x)^2 - x^2}{\mathrm{d}x} = \frac{2x\mathrm{d}x + (\mathrm{d}x)^2}{\mathrm{d}x} = 2x + \mathrm{d}x = 2x$$

- 维尔斯特拉斯: $\frac{\mathrm{d}f(x)}{\mathrm{d}x}$ 不是 $\mathrm{d}f(x)$ 与 $\mathrm{d}x$ 的商, $\frac{\mathrm{d}}{\mathrm{d}x}$ 作为微分运算作用于 $f(x)$,

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

[15] Compositionality: The meaning of a whole (cf. sentence) should only depend on the meanings of its parts (cf. words) and how they are fitted together (cf. grammar).

三次数学基础危机

1. 万物皆 (有理) 数, $\sqrt{2}$ 是数吗?
2. 无穷小是数吗?
3. 什么是合法存在的 "集合"?

Remark

- 古巴比伦人和古埃及人的兴趣在于 5 个橘子而不是 "5".
- 正是古希腊人把数学变成一个抽象系统, 一种特殊的符号语言.
- 这使得人们不仅可以描述具体的现实世界, 而且可以解释它最深层次的模式和规律.
- 正因如此, 数学才得以从诸如 '如何辩护归纳法' 之类的迷宫问题中解放出来.

— 大卫·福斯特·华莱士

西方科学的发展是以两个伟大的成就为基础: 希腊哲学家发明的形式逻辑体系, 以及在文艺复兴时期发现通过系统的实验可能找出因果关系.

— 爱因斯坦

逻辑主义 & 逻辑实证主义

- 数学可以还原为逻辑.
- 科学可以还原为关于感觉材料的命题的逻辑复合.
- 只有可以被观察验证的或逻辑证明的命题才是有意义的.
- 弗雷格和罗素的新逻辑工具诱惑实证主义者做出了超出他们证明能力的猜想, 也让他们清楚的知道, 他们的猜想是不可证的.
- 在逻辑实证主义之前, 很少有哲学流派能够把自己的纲领陈述得足够清楚, 使得有可能看清其目标是不可达的.

卡尔纳普《通过语言的逻辑分析清除形而上学》

- 一个陈述的意义在于它的证实方法. 形而上学陈述不能被证实, 毫无意义.
- 那么留给哲学的还有什么呢? 一种方法: 逻辑分析法.
- 逻辑分析的消极应用是清除无意义的词和陈述, 积极应用是澄清有意义的概念和命题, 为经验科学和数学奠基.
- 形而上学家相信自己是在攸关真假的领域里前行, 却未断言任何东西. 他们只是试图表达一点儿人生态度.
- 艺术是表达人生态度的恰当手段. 抒情诗人并不企图在自己的诗里驳倒其他抒情诗人诗里的陈述, 但形而上学家却用论证维护他的陈述. 形而上学家是没有艺术才能的艺术家, 有的是在理论环境里工作的爱好, 却既不在科学领域里发挥这种爱好, 又不能满足艺术表达的要求, 倒是混淆了这两个方面, 创造出一种对知识既无贡献、对人生态度的表达又不相宜的东西.

休谟《人类理解研究》

如果我们拿起一本书，比如神学或经院哲学书，让我们问一下，其中包含着数和量方面的任何抽象推论么？没有．其中包含着事实和存在方面的任何经验推论么？没有．那就扔进火里吧，因为它所包含的没有别的，只有诡辩和幻想．

一阶逻辑语言表达力的局限性

- Most boys are funny.
- For every dog there is a cat. (There are more cats than dogs.)
- Some girls admire only one another.

$$\exists X\Big(\exists x Xx \wedge \forall x(Xx \to Gx) \wedge \forall x \forall y\big(Xx \wedge Axy \to Xy \wedge x \neq y\big)\Big)$$

- There are some gunslingers each of whom has shot the right foot of at least one of the others.

$$\exists X\Big(\exists x Xx \wedge \forall x(Xx \to Gx) \wedge \forall x\big(Xx \to \exists y(Xy \wedge y \neq x \wedge Sxy)\big)\Big)$$

- Least Number Principle.

$$\forall X\Big(\exists x Xx \wedge \forall x(Xx \to Nx) \to \exists x\big(Xx \wedge \forall y(Xy \wedge y \neq x \to x < y)\big)\Big)$$

- A linear order $(P, <)$ is *complete* iff every non-empty subset of P that is bounded above has a supremum in P.

$$\forall X\Bigg(\exists x Xx \wedge \exists y \forall x(Xx \to x \leq y) \to$$

$$\exists y\Big(\forall x(Xx \to x \leq y) \wedge \forall z\big(\forall x(Xx \to x \leq z) \to y \leq z\big)\Big)\Bigg)$$

Gentzen

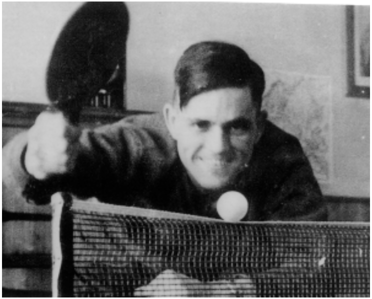


Figure: Gentzen 1909-1945

- Natural Deduction: one proposition on the right.
- Sequent Calculus: zero or more propositions on the right.

$$\Gamma \vdash \Delta \iff \vdash \bigwedge \Gamma \to \bigvee \Delta$$

- Consistency of PA (proof-theoretical strength of PA)

自然推演 Natural Deduction

$$\frac{A \quad B}{A \wedge B}\ \wedge^{+} \qquad \frac{A \wedge B}{A}\ \wedge^{-} \qquad \frac{A \wedge B}{B}\ \wedge^{-}$$

$$\frac{A}{A \vee B}\ \vee^{+} \qquad \frac{B}{A \vee B}\ \vee^{+} \qquad \frac{A \vee B \quad \begin{array}{c}[A]^n\\ \vdots\\ C\end{array} \quad \begin{array}{c}[B]^n\\ \vdots\\ C\end{array}}{C}\ \vee^{-n}$$

$$\frac{\begin{array}{c}[A]^n\\ \vdots\\ B\end{array}}{A \longrightarrow B}\ \longrightarrow^{+n} \qquad \frac{A \longrightarrow B \quad A}{B}\ \longrightarrow^{-}$$

$$\frac{\begin{array}{c}[A]^n\\ \vdots\\ \bot\end{array}}{\neg A}\ \neg^{+n} \qquad \frac{\begin{array}{c}[\neg A]^n\\ \vdots\\ \bot\end{array}}{A}\ \neg^{-n} \qquad \frac{\neg A \quad A}{\bot}\ \bot^{+} \qquad \frac{\bot}{A}\ \bot^{-}$$

自然推演 Natural Deduction

$$\frac{A(a)}{\forall x A}\ \forall^+ \qquad\qquad \frac{\forall x A}{A[t/x]}\ \forall^-$$

where $a \notin \mathrm{Cst}(\forall x A)$, and a is not in any assumption which is uncanceled in the derivation ending with $A(a)$.

$$\frac{A[t/x]}{\exists x A}\ \exists^+ \qquad\qquad \frac{\exists x A \qquad \begin{array}{c}[A(a)]^n \\ \vdots \\ B\end{array}}{B}\ \exists^{-n}$$

where $a \notin \mathrm{Cst}(\exists x A, B)$, and a is not in any assumption which is uncanceled in the derivations ending with $\exists x A, B$ except in $A(a)$.

$$\frac{}{t = t}\ =^+ \qquad\qquad \frac{s = t \qquad A[s/x]}{A[t/x]}\ =^-$$

Remark: 怎么证明全称命题?

- 从给定集合中选择一个任意 (arbitrary) 对象, 并证明该对象具有所需的性质.

$$\frac{A(a)}{\forall x A}\ \forall^+$$

  where $a \notin \mathrm{Cst}(\forall x A)$, and a is not in any assumption which is uncanceled in the derivation ending with $A(a)$.
  Remark: a 不依赖任何特定的假设. 不预设任何信息确保了 a 的任意性.

- **"arbitrary" $\neq$ "random"**
  - 任意对象是一个占位符, 它可以代表任何其他对象.
  - 如果我们随机选择一个对象, 这意味着选择它的概率与选择任何其他对象的概率相等.

Remarks: 违反 “可代入” 条件导致的错误

$$\boxed{\dfrac{\forall x \exists y Rxy}{\exists y Ryy}\ \forall^- \quad \textbf{incorrect}\ \ \exists y Rxy[y/x] \times}$$

$$\boxed{\dfrac{\forall y Ryy}{\exists x \forall y Rxy}\ \exists^+ \quad \textbf{incorrect}\ \ \forall y Rxy[y/x] \times}$$

$$\boxed{\dfrac{x = y \quad \exists y Rxy}{\exists y Ryy}\ =^- \quad \textbf{incorrect}\ \ \exists y Rxy[y/x] \times}$$

Remarks: 违反 $\forall^+$ 的约束条件导致的错误

$$\boxed{\frac{\forall x Rxx}{\forall x Rxa}\ ?}$$

$$\dfrac{\dfrac{\forall x Rxx}{Raa}\ \forall^-}{\forall x Rxa}\ \forall^+$$

incorrect

$$\boxed{\dfrac{}{\forall x(Ax \to \forall x Ax)}\ ?}$$

$$\dfrac{\dfrac{\dfrac{[Aa]^1}{\forall x Ax}\ \forall^+}{Aa \to \forall x Ax}\ \to^{+1}}{\forall x(Ax \to \forall x Ax)}\ \forall^+$$

incorrect

$$\boxed{\dfrac{\exists x A}{Aa}\ ?}$$

$$\boxed{\dfrac{\forall x \exists y Rxy}{\exists z Rzz}\ ?}$$

$$\dfrac{\exists x A \qquad \begin{array}{c}[Aa]^1\\ \vdots\\ Aa\end{array}}{Aa}\ \exists^{-1} \qquad \textbf{incorrect}$$

$$\dfrac{\dfrac{\forall x \exists y Rxy}{\exists y Ray}\ \forall^- \qquad \dfrac{[Raa]^1}{\exists z Rzz}\ \exists^+}{\exists z Rzz}\ \exists^{-1} \qquad \textbf{incorrect}$$

$$\boxed{\dfrac{\exists x A \qquad \exists x B}{\exists x (A \wedge B)}\ ?}$$

$$\dfrac{\exists x B \qquad \dfrac{\exists x A \qquad \dfrac{\dfrac{[Aa]^1 \qquad [Ba]^2}{Aa \wedge Ba}\ \wedge^+}{\exists x (A \wedge B)}\ \exists^+}{\exists x (A \wedge B)}\ \exists^{-1}}{\exists x (A \wedge B)}\ \exists^{-2} \qquad \textbf{incorrect}$$

Example

$$\boxed{\dfrac{\forall x(A \to B)}{\exists x A \to \exists x B}}$$

Proof.

$$\dfrac{\dfrac{[\exists x A]^2 \qquad \dfrac{\dfrac{\dfrac{\forall x(A \to B)}{Aa \to Ba}\,\forall^- \qquad [Aa]^1}{Ba}\,\to^-}{\exists x B}\,\exists^+}{\exists x B}\,\exists^{-\,1}}{\exists x A \to \exists x B}\,\to^{+\,2}$$

□

$$\boxed{\dfrac{\forall x(A \to B)}{A \to \forall x B} \quad \text{where} \ \ x \notin \mathrm{Fv}(A)}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x(A \to B)}{A \to Ba}\,\forall^- \qquad [A]^1}{Ba}\,\to^-}{\forall x B}\,\forall^+}{A \to \forall x B}\,\to^{+1}$$

$\square$

$$\boxed{\dfrac{A \to \forall x B}{\forall x(A \to B)} \quad \text{where} \ \ x \notin \mathrm{Fv}(A)}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to \forall x B \qquad [A]^1}{\forall x B}\,\to^-}{Ba}\,\forall^-}{A \to Ba}\,\to^{+1}}{\forall x(A \to B)}\,\forall^+$$

$\square$

Examples

$$\boxed{\dfrac{\forall x(A \to B)}{\exists x A \to B} \quad \text{where } x \notin \mathrm{Fv}(B)}$$

Proof.

$$\dfrac{\dfrac{[\exists x A]^1 \qquad \dfrac{[Aa]^2 \qquad \dfrac{\forall x(A \to B)}{Aa \to B}\,\forall^-}{B}\,\to^-}{B}\,\exists^{-2}}{\exists x A \to B}\,\to^{+3}$$

□

$$\boxed{\dfrac{\exists x A \to B}{\forall x(A \to B)} \quad \text{where } x \notin \mathrm{Fv}(B)}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{[Aa]^1}{\exists x A}\,\exists^+ \qquad \exists x A \to B}{B}\,\to^-}{Aa \to B}\,\to^{+1}}{\forall x(A \to B)}\,\forall^+$$

□

Examples

$$\boxed{\dfrac{\exists x \forall y Rxy}{\forall y \exists x Rxy}}$$

Proof.

$$\dfrac{\exists x\forall y Rxy \qquad \dfrac{\dfrac{\dfrac{[\forall y Ray]^1}{Rab}\,\forall^-}{\exists x Rxb}\,\exists^+}{\forall y\exists x Rxy}\,\forall^+}{\forall y \exists x Rxy}\,\exists^{-1}$$

□

$$\frac{\text{Everybody loves anybody who loves somebody}}{\text{If somebody loves somebody, everybody loves everybody}}$$

$$\boxed{\frac{\forall xyz(Lyz \to Lxy)}{\exists xyLxy \to \forall xyLxy}}$$

$$\begin{tikzcd}
u \arrow[dr] \arrow[dd, dashed] & & \\
 & r \arrow[r] & j \\
m \arrow[ur] & &
\end{tikzcd}$$

Proof.

$$\dfrac{\dfrac{[\exists xyLxy]^2 \qquad \dfrac{\dfrac{\dfrac{\forall xyz(Lyz \to Lxy)}{Lrj \to Lmr} \qquad [Lrj]^1}{Lmr} \qquad \dfrac{\forall xyz(Lyz \to Lxy)}{Lmr \to Lum}}{Lum}}{\dfrac{Lum}{\forall xyLxy}} \; \exists^{-1}}{\exists xyLxy \to \forall xyLxy} \; \to^{+2}$$

□

Example

If anyone speaks to anyone, then someone introduces them
No one introduces anyone to anyone unless he knows them both
Everyone speaks to Alice

Everyone is introduced to Alice by someone who knows her

$$\begin{array}{c} \forall x \forall y (Sxy \to \exists z Izxy) \\ \forall x \forall y \forall z (Izxy \to Kzx \land Kzy) \\ \forall x Sxa \\ \hline \forall x \exists y (Iyxa \land Kya) \end{array}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall xy(Sxy \to \exists z Izxy)}{Sba \to \exists z Izba} \quad \dfrac{\forall x Sxa}{Sba}}{\exists z Izba} \quad \dfrac{\dfrac{\dfrac{[Icba]^1 \quad \dfrac{\forall xyz(Izxy \to Kzx \land Izy)}{Icba \to Kcb \land Kca}}{Kca}}{Icba \land Kca}}{\exists y(Iyba \land Kya)}}{\exists y(Iyba \land Kya)}\ \exists^{-1}}{\forall x \exists y (Iyxa \land Kya)}$$

Example

$$\frac{\text{If dogs are animals}}{\text{Every head of a dog is the head of an animal}}$$

$$\boxed{\dfrac{\forall x(Dx \to Ax)}{\forall x(Dx \to \exists y(Ay \wedge hx = hy))}}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x(Dx \to Ax)}{Da \to Aa} \qquad [Da]^1}{Aa} \qquad \dfrac{}{ha = ha}}{Aa \wedge ha = ha}}{\exists y(Ay \wedge ha = hy)}\;{\longrightarrow^{+1}}}{Da \to \exists y(Ay \wedge ha = hy)}}{\forall x(Dx \to \exists y(Ay \wedge hx = hy))}$$

$\square$

$$\frac{\text{如果大鱼比小鱼游得快}}{\text{只要有最大的鱼就有游得最快的鱼}}$$

$$\boxed{\dfrac{\forall x \forall y(Fx \land Fy \land Bxy \to Sxy)}{\exists x(Fx \land \forall y(Fy \to Bxy)) \to \exists x(Fx \land \forall y(Fy \to Sxy))}}$$

Proof.

$$\dfrac{\dfrac{[\exists x(Fx \land \forall y(Fy \to Bxy))]^3 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[Fa \land \forall y(Fy \to Bay)]^1}{Fa \quad \forall y(Fy \to Bay)}}{Fb \to Bab} \quad [Fb]^2}{Bab}}{Fa \land Fb \land Bab} \quad \dfrac{\forall x \forall y(Fx \land Fy \land Bxy \to Sxy)}{Fa \land Fb \land Bab \to Sab}}{Sab}}{Fb \to Sab}\ {\to}^{+2}}{\forall y(Fy \to Say)}}{Fa \land \forall y(Fy \to Say)}}{\exists x(Fx \land \forall y(Fy \to Sxy))}}{\exists x(Fx \land \forall y(Fy \to Sxy))}\ \exists^{-1}}{\exists x(Fx \land \forall y(Fy \to Bxy)) \to \exists x(Fx \land \forall y(Fy \to Sxy))}\ {\to}^{+3}$$

Example: 偶数的平方是偶数

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{[E(a)]^1}{\exists y(a = 2y)}
\qquad
\dfrac{\dfrac{[a = 2b]^2}{a^2 = 2(2b^2)}}{\exists y(a^2 = 2y)}
}{\exists y(a^2 = 2y)}\;\exists^{-2}
}{E(a^2)}
}{E(a) \to E(a^2)}\;\to^{+1}
}{\forall x(E(x) \to E(x^2))}
$$

归谬法 — 比萨斜塔思想实验

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}[\forall xy(m_x < m_y \to v_x < v_y)]^1 \\ \vdots \\ v_{\text{重}} < v_{\text{轻+重}}\end{matrix} \qquad \begin{matrix}\forall xy(v_x < v_y \to v_x < v_{x+y} < v_y) \\ \vdots \\ v_{\text{轻}} < v_{\text{轻+重}} < v_{\text{重}}\end{matrix}}{v_{\text{重}} < v_{\text{重}}}}{\bot}}{\neg\forall xy(m_x < m_y \to v_x < v_y)}\ \neg^{+1}
$$

哲学写在宇宙这本大书上.
这本书永远向我们敞开.
但除非先学会它的语言,
否则, 我们将一个字都理解不了,
从而只能在黑暗的迷宫中徒劳地游荡.
这本书是用数学语言写成的.

— 伽利略

$$P \wedge O \wedge (\neg L \to \neg U) \wedge (\neg U \to W) \wedge M$$

Hilbert System = Axiom + Inference Rule

公理模式

1. $A \to B \to A$
2. $(A \to B \to C) \to (A \to B) \to A \to C$
3. $(\neg A \to \neg B) \to (\neg A \to B) \to A$
4. $\forall x(A \to B) \to \forall x A \to \forall x B$
5. $\forall x A \to A[t/x]$ where t is substitutable for x in A.
6. $A \to \forall x A$ where $x \notin \mathrm{Fv}(A)$.
7. $x = x$
8. $x = y \to A \to A[y /\!\!/ x]$ where A is atomic and $A[y /\!\!/ x]$ is obtained from A by replacing x in one or more places by y.
9. $\forall x_1 \dots x_n A$ where $n \geq 0$ and A is any axiom of the preceding groups.

推理规则

$$\frac{A \quad A \to B}{B}\ \text{MP}$$

Example

Theorem

$$A \vdash \exists x A$$

Proof.

1. $(\forall x \neg A \rightarrow \neg A) \rightarrow A \rightarrow \neg \forall x \neg A$ — Tautology
2. $\forall x \neg A \rightarrow \neg A$ — A5
3. $A \rightarrow \neg \forall x \neg A$ — 1,2 MP
4. A — Premise
5. $\neg \forall x \neg A$ — 3,4 MP
6. $\exists x\, A$ — Definition of $\exists$

$\square$

什么是一个理论？— 推演封闭的句子集

一个公理化的理论由哪几部分组成？

一：基本概念（初始符号）
二：逻辑公理（有效式/普遍真理）
三：非逻辑公理（相对真理）
四：推理规则

面对一个理论，我们关心什么？

1. 语言表达力够丰富吗？
2. 可递归公理化吗？
3. 一致吗？
4. 完备吗？
5. 公理独立吗？
6. 能证明自身的一致性吗？
7. 可判定吗？
8. 有范畴性吗？

一些理论的例子：

- 欧几里得几何理论
- 皮亚诺算术理论
- 集合论
- 群论
- 概率论
- 牛顿力学
- 狭义相对论
- 广义相对论
- 量子力学
- 进化论
- 博弈论
- 马克思主义理论
- 五行生克理论

欧几里得《几何原本》

公理 1 从一点到另一点可以作一条直线.

公理 2 一条线段可以延伸成一条直线.

公理 3 以线段的一个端点为圆心, 该线段为半径, 可以作一个圆.

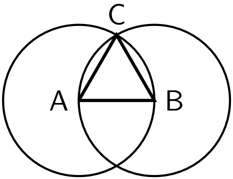
公理 4 所有直角都相等.

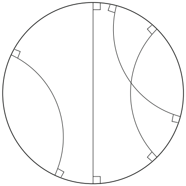
公理 5 过直线外一点, 有且只有一条直线与之平行.

定理

给定一条线段, 可以作一个以其为边的等边三角形.

1. $\odot A$ 公理 3
2. $\odot B$ 公理 3
3. AC 公理 1
4. BC 公理 1





群 $\mathscr{L} = \{e, \cdot\}$

公理 1 $\forall xyz : x\cdot(y\cdot z) = (x\cdot y)\cdot z$

公理 2 $\forall x : e \cdot x = x \cdot e = x$

公理 3 $\forall x \exists y : x \cdot y = y \cdot x = e$

模型

- $(\mathbb{Z}, 0, +)$
- $(\mathbb{Q}^+, 1, \times)$
- Klein group: $(\{\diamondsuit, \heartsuit, \spadesuit, \clubsuit\}, \diamondsuit, \cdot)$

$\cdot$	$\diamondsuit$	$\heartsuit$	$\spadesuit$	$\clubsuit$	permutation
$\diamondsuit$	$\diamondsuit$	$\heartsuit$	$\spadesuit$	$\clubsuit$	$\diamondsuit$
$\heartsuit$	$\heartsuit$	$\diamondsuit$	$\clubsuit$	$\spadesuit$	$(1,2)(3,4)$
$\spadesuit$	$\spadesuit$	$\clubsuit$	$\diamondsuit$	$\heartsuit$	$(1,3)(2,4)$
$\clubsuit$	$\clubsuit$	$\spadesuit$	$\heartsuit$	$\diamondsuit$	$(1,4)(2,3)$

双脚并拢, 跳过计算.
对运算按照复杂性而不是其表象加以群分类聚.

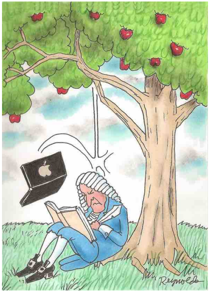
— 伽罗瓦

数学是给不同的事物起同一个名字的艺术.

— 庞加莱

牛顿力学: 苹果落地, 月亮咋就不落地?

"力是**保持**物体运动状态的原因."
— 亚里士多德



- 什么是 "运动"?
- 什么是 "运动状态"?
- 什么是 "物体"?
- 怎么 "保持"? 还是 "改变"?

牛顿力学理论

- 基本概念: 质量, 动量, 惯性, 外力
- 非逻辑公理:

  公理 1 惯性定律
  公理 2 动量变化定律
  公理 3 作用与反作用定律
  公理 4 万有引力定律

$$F = 0 \iff \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = 0$$

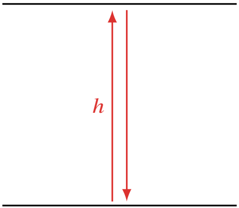
$$F = m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

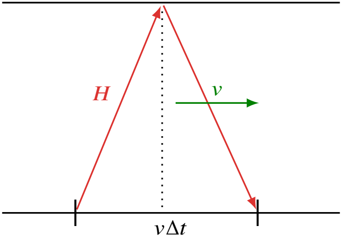
$$F_{12} = -F_{21}$$

$$F = \frac{Gm_1m_2}{r^2}$$

爱因斯坦的狭义相对论

- “力学的目的是描述物体在空间中的位置如何随时间变化.”
- 什么是“位置”? 什么是“空间”? 什么是“时间”? 是绝对的吗?
- 不同惯性参照系的观察者对同一事件的描述相同吗?





$$\left.\begin{aligned}\Delta t' &= \frac{2h}{c} \\ \Delta t &= \frac{2H}{c} = \frac{2}{c}\sqrt{h^2 + \left(\frac{v\Delta t}{2}\right)^2}\end{aligned}\right\} \Longrightarrow \Delta t' = \Delta t\sqrt{1 - \frac{v^2}{c^2}}$$

- 假如以光速飞行, 你还能从镜子中看到自己的脸吗?
- 列车头尾同时被闪电劈中. 怎么判断“同时”? 怎么测量“时间”?

Definition (时钟同步)

在同一个惯性参照系中，处在不同位置的处于静止状态的观察者各自拥有一个时钟 A 和 B. 时钟 A 和 B 同步，当且仅当，如果在 A 的时刻 t_A 从 A 朝 B 发出的光在 B 的时刻 t_B 到达 B 后立刻反射回 A，并在 A 的时刻 t'_A 到达 A，下述等式成立:

$$t_B - t_A = t'_A - t_B$$

Theorem

在同一个惯性参照系中，处在不同位置的处于静止状态的时钟之间的同步关系是等价关系.

Remark: 同时性的相对性 — 在一个惯性参照系的观察者眼中同时发生的两个事件，在相对匀速运动的另一个惯性参照系的观察者眼中不再是同时发生的.

站台上的观察者: "同时劈中." 列车中间的观察者: "先劈中头后劈中尾."

狭义相对论

公理 1 狭义相对性原理: 自然规律在所有惯性参照系中都相同.

公理 2 光速不变原理: 光在真空中的传播速度不变, 既与惯性参照系的选择无关, 也与光源的运动状态无关.

Remark: 光速上限假设. (因果作用只能在光锥内传播 $\Delta s \le c\Delta t$)

洛伦兹变换:

$$\begin{bmatrix} x \\ ct \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} x' \\ ct' \end{bmatrix} \qquad t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}} \qquad y = y' \qquad z = z'$$

时空间隔与惯性系的选择无关.

$$c^2\mathrm{d}t^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) = c^2\mathrm{d}t'^2 - (\mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2)$$

时间与长度的相对性 — 钟慢尺缩

$$\Delta t' = \Delta t\sqrt{1-\frac{v^2}{c^2}} \qquad l' = l\sqrt{1-\frac{v^2}{c^2}}$$

公孙大娘剑术精, 出刺迅捷如流星,
由于空间收缩性, 长剑变成短铁钉.

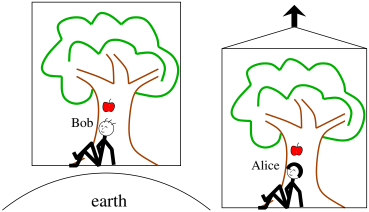
$$w = \frac{u+v}{1+\frac{uv}{c^2}} \qquad m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad p = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}v \qquad E = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}c^2$$

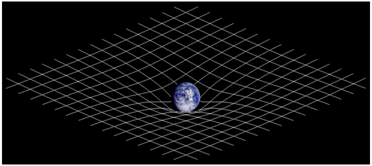
广义相对论

公理 1 广义相对性原理: 自然规律在所有参照系中都相同.

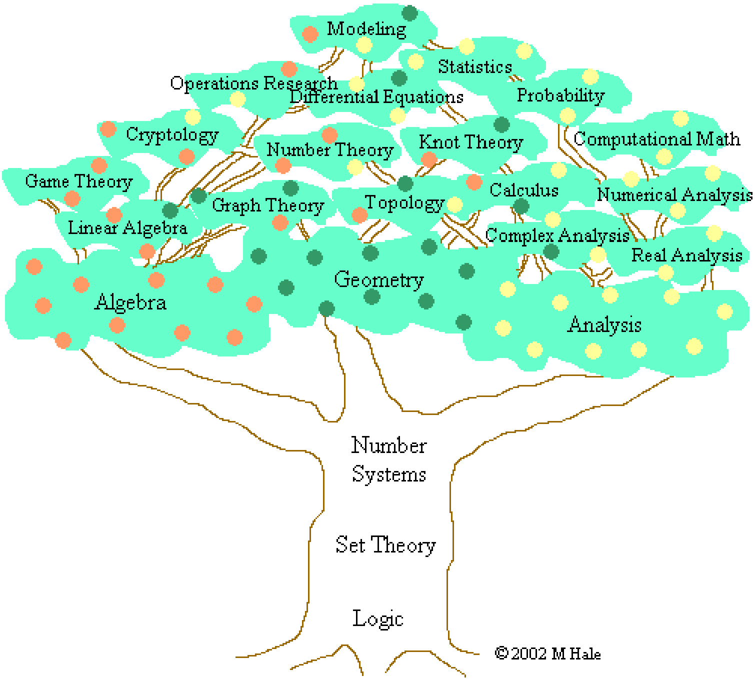
公理 2 等效原理: 引力场与惯性力场局域不可区分.

- 在大质量物体上感受到的引力, 与在加速参照系中感受到的惯性力局域不可区分.
- 惯性质量等于引力质量 $m_I = m_G$.
- 引力是由质量/能量分布不均导致的时空弯曲通过测地线原理作用的结果.
- 时空告诉物质如何运动; 物质告诉时空如何弯曲.





$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$


Modeling
Statistics
Operations Research
Differential Equations
Probability
Cryptology
Number Theory
Knot Theory
Computational Math
Game Theory
Graph Theory
Topology
Calculus
Numerical Analysis
Linear Algebra
Complex Analysis
Real Analysis
Geometry
Algebra
Analysis
Number Systems
Set Theory
Logic
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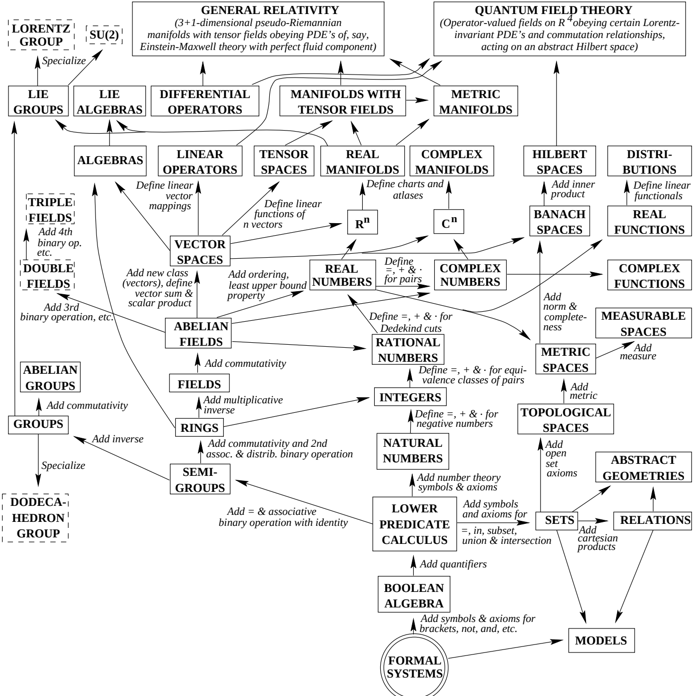

GENERAL RELATIVITY
(3+1-dimensional pseudo-Riemannian manifolds with tensor fields obeying PDE's of, say, Einstein-Maxwell theory with perfect fluid component)
QUANTUM FIELD THEORY
(Operator-valued fields on R 4 obeying certain Lorentz-invariant PDE's and commutation relationships, acting on an abstract Hilbert space)
LORENTZ GROUP
SU(2)
Specialize
LIE GROUPS
LIE ALGEBRAS
DIFFERENTIAL OPERATORS
MANIFOLDS WITH TENSOR FIELDS
METRIC MANIFOLDS
ALGEBRAS
LINEAR OPERATORS
TENSOR SPACES
REAL MANIFOLDS
COMPLEX MANIFOLDS
HILBERT SPACES
DISTRIBUTIONS
Define linear vector mappings
Define linear functions of n vectors
Define charts and atlases
Add inner product
Define linear functionals
TRIPLE FIELDS
Add 4th binary op. etc.
R n
C n
BANACH SPACES
REAL FUNCTIONS
DOUBLE FIELDS
VECTOR SPACES
REAL NUMBERS
Define =, + & · for pairs
COMPLEX NUMBERS
COMPLEX FUNCTIONS
Add new class (vectors), define vector sum & scalar product
Add ordering, least upper bound property
Add 3rd binary operation, etc.
Add norm & complete-ness
MEASURABLE SPACES
Define =, + & · for Dedekind cuts
ABELIAN FIELDS
RATIONAL NUMBERS
METRIC SPACES
Add measure
Add commutativity
Define =, + & · for equi-valence classes of pairs
ABELIAN GROUPS
FIELDS
Add metric
Add commutativity
Add multiplicative inverse
INTEGERS
GROUPS
RINGS
Define =, + & · for negative numbers
TOPOLOGICAL SPACES
Add inverse
Add commutativity and 2nd assoc. & distrib. binary operation
NATURAL NUMBERS
Add open set axioms
ABSTRACT GEOMETRIES
Specialize
SEMI-GROUPS
Add number theory symbols & axioms
DODECA-HEDRON GROUP
Add = & associative binary operation with identity
LOWER PREDICATE CALCULUS
Add symbols and axioms for =, in, subset, union & intersection
SETS
RELATIONS
Add cartesian products
Add quantifiers
BOOLEAN ALGEBRA
Add symbols & axioms for brackets, not, and, etc.
MODELS
FORMAL SYSTEMS

一、选择题. (8 道题, 每题 5 分, 共 40 分)

1、以下哪个公式是有效的?

A. $\exists y \forall x Rxy \to \forall y \exists x Rxy$　B. $\forall x \exists y Rxy \to \exists x \forall y Rxy$

C. $\forall x \exists y Rxy \to \exists y \forall x Rxy$　D. $\exists x \forall y Rxy \to \forall y \exists x Rxy$

二、下面公式是否有效? 若有效, 请证明, 若无效, 给出反模型. (10 分)

$$(p \to r) \vee (q \to r) \to (p \vee q) \to r$$

三、请将如下问题翻译成合适的命题逻辑语言并用逻辑学方法求解. (20 分)

已知一起凶杀案有三个嫌疑人: 小艾、小白和小菜. 至少有一人是凶手, 但不可能三人同时犯罪. 如果小艾是凶手, 那么小菜是同犯. 如果小白不是凶手, 那么小菜也不是. 请问, 谁肯定是凶手?

四、下面论证是否有效? 若有效, 请证明, 若无效, 给出反模型. (20 分)

人人都怕小艾. 小艾只怕我. 因此, 我就是小艾.

五、论述题. (10 分)

谈谈你对命题逻辑与布尔代数之间的关系的理解.

Contents

$$\frac{\begin{array}{c}\text{It is possible that it is raining.}\\ \text{It is certain that we will get wet if it is raining.}\end{array}}{\text{It is possible that we will get wet.}}$$

$$\frac{\begin{array}{c}\Diamond R\\ \Box(R \rightarrow W)\end{array}}{\Diamond W}$$

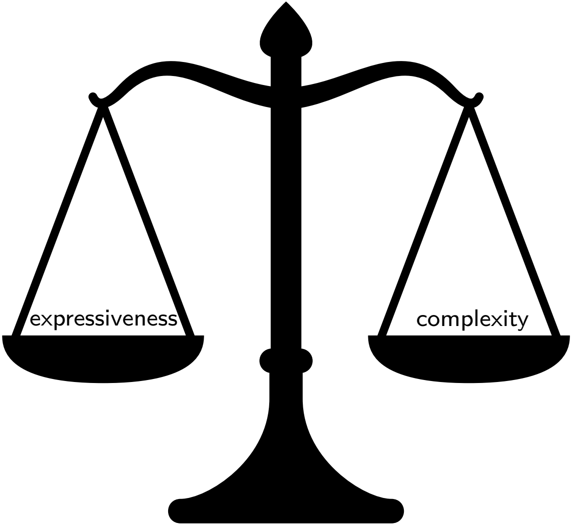
Paradox of Material Implication

- If God does not exist, then it's not the case that if I believe in God, I will have eternal life;
- and I don't believe in God;
- so God exists!

$$\frac{\neg G \rightarrow \neg(B \rightarrow E) \quad \neg B}{G} \qquad \frac{\neg G \rightarrow \neg\Diamond(B \rightarrow E) \quad \neg B}{G}\ ?$$

Why Study Modal Logic?

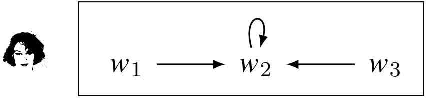
- Modal languages are simple yet expressive languages for talking about relational models.
- Modal languages are not isolated formal systems.
  - Modal vs classical (FOL,SOL), internal vs external perspective.
    In FOL, models are described from the top point of view. Each object and relation can be named. In modal logic, relational models are described from an internal perspective, there is no way to mention objects and relations.
  - Relational models vs Boolean algebra with operators.
    (Jónsson and Tarski's representation theorem.)
- Decidability.
  (seeking a balance between expressiveness and efficiency/complexity)



The more you can say, the less you can effectively/efficiently do.

Relational structures in first order and modal logic

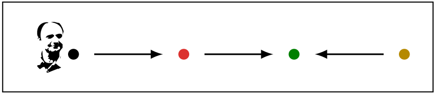
- In first order logic, relational structures are described from the top point of view. Each point of W and the relation R can be named.



Alice (in first order logic):

$$Rw_1w_2, Rw_2w_2, Rw_3w_2, \neg Rw_1w_3, \neg Rw_2w_1, \neg Rw_2w_3, \neg Rw_1w_1, \ldots$$

- In modal logic, relational structures are described from an internal, local perspective.



Bob (in modal logic): "I can reach a green point in 2 steps. There is no yellow point I can reach."

Contents

什么是 “模态”?

_____ 她有男朋友

- 可能 (basic modal logic)
- 过去 (temporal logic)
- 准许 (deontic logic)
- 我知道/相信 (epistemic logic)
- 事实证明 (provability logic)
- 相亲了 3 次之后 (dynamic logic)
- 父母确保 (coalition logic)

Figure: Saul Kripke: 1920-2022

Language

$$\mathscr{L} := \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \Box, \Diamond, (,)\} \cup \mathrm{Var}$$

where $\mathrm{Var} := \{p_1, p_2, p_3, \dots\}$.

Definition (Well-Formed Formula Wff)

$$A := p \mid \neg A \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid A \leftrightarrow A \mid \Box A \mid \Diamond A$$

- It will always be A. $\mathsf{G}A$
- You ought to do A. $\mathsf{O}A$
- I know A. $\mathsf{K}_i A$
- I believe A. $\mathsf{B}_i A$
- A is provable in T. $\Box_{\mathrm{T}} A$
- After the execution of the program α, A holds. $[\alpha]A$

Examples

1. "Ought" implies "can", but it does not imply "is".
$$(\Box p \to \Diamond p) \wedge \neg(\Box p \to p)$$
2. What must be is, and what is, is possible.
$$(\Box p \to p) \wedge (p \to \Diamond p)$$
3. Just because it happened that doesn't make it acceptable.
$$\neg(p \to \Diamond p)$$
4. Just because it happened that doesn't mean it ought to be permitted.
$$\neg(p \to \Box\Diamond p)$$
5. If it is raining, it is necessarily possible that it is raining.
$$p \to \Box\Diamond p$$
6. If it is possible that it is raining then it is necessarily possible that the corners are slippery.
$$\Diamond p \to \Box\Diamond q$$
7. Necessarily, if it is raining then it is possible that the corners are slippery.
$$\Box(p \to \Diamond q)$$

Contents

克里普克 Kripke 可能世界语义学[16]

克里普克框架 $\mathcal{F} := (W, R)$ 由非空可能世界集 W 及其上的可及关系 R 构成.

- $W \neq \emptyset$
- $R \subset W \times W$

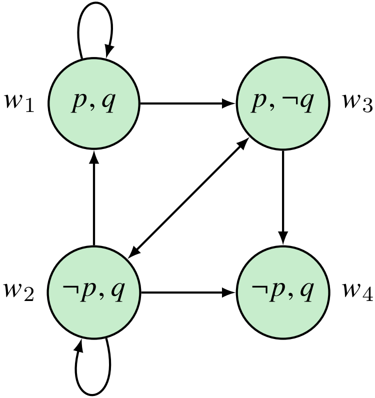
克里普克模型 $\mathcal{M} := (\mathcal{F}, V) = (W, R, V)$ 由克里普克框架 $\mathcal{F}$ 及其上的赋值 $V : \mathrm{Var} \to \mathcal{P}(W)$ 构成:

- $\mathcal{M}, w \vDash p$ iff $w \in V(p)$
- $\mathcal{M}, w \vDash \neg A$ iff $\mathcal{M}, w \nvDash A$
- $\mathcal{M}, w \vDash A \wedge B$ iff $\mathcal{M}, w \vDash A$ and $\mathcal{M}, w \vDash B$
- $\mathcal{M}, w \vDash \Box A$ iff $\forall v \in W \Big(Rwv \implies \mathcal{M}, v \vDash A \Big)$
- $\mathcal{M}, w \vDash \Diamond A$ iff $\exists v \in W \Big(Rwv \;\;\&\;\; \mathcal{M}, v \vDash A \Big)$

[16] Leibniz:

$$\mathcal{M}, w \vDash \Box A \text{ iff } \forall v \in W : \mathcal{M}, v \vDash A$$
$$\mathcal{M}, w \vDash \Diamond A \text{ iff } \exists v \in W : \mathcal{M}, v \vDash A$$

Example



$\mathcal{M}, w_1 \models p \wedge \Box p$

$\mathcal{M}, w_1 \models q \wedge \Diamond q$

$\mathcal{M}, w_1 \models \neg \Box q$

$\mathcal{M}, w_2 \models q \wedge \Diamond \neg q$

$\mathcal{M}, w_3 \models p$

$\mathcal{M}, w_3 \models \Box \neg p$

$\mathcal{M}, w_4 \models \Box p \wedge \neg \Diamond p$

可满足 & 有效

- 公式 A 是可满足的, 当且仅当, 存在 $\mathcal{M}, w$ 使得 $\mathcal{M}, w \vDash A$.
- 公式 A 在模型 $\mathcal{M}$ 上为真 ($\mathcal{M} \vDash A$), 当且仅当, 对任意 $w \in W$ 都有 $\mathcal{M}, w \vDash A$.
- 公式 A 在点框架 $\mathcal{F}, w$ 上有效 ($\mathcal{F}, w \vDash A$), 当且仅当, 对框架 $\mathcal{F}$ 上的任意模型 $\mathcal{M}$ 都有 $\mathcal{M}, w \vDash A$.
- 公式 A 在框架 $\mathcal{F}$ 上有效 ($\mathcal{F} \vDash A$), 当且仅当, 对框架 $\mathcal{F}$ 上的任意模型 $\mathcal{M}$ 都有 $\mathcal{M} \vDash A$.
- 公式 A 在框架类 $\mathcal{C}$ 上有效 ($\mathcal{C} \vDash A$), 当且仅当, 对任意框架 $\mathcal{F} \in \mathcal{C}$ 都有 $\mathcal{F} \vDash A$.
- 公式 A 是有效的 $\vDash A$, 当且仅当, 对任意框架 $\mathcal{F}$ 都有 $\mathcal{F} \vDash A$.

Truth is in the eye of the beholder.

Example

$$\vDash \Box(A \to B) \to \Box A \to \Box B$$

- 保点

$$\Gamma \vDash A \ := \ \forall \mathcal{M} \forall w \in W \left(\mathcal{M}, w \vDash \Gamma \implies \mathcal{M}, w \vDash A \right)$$

- 保模型

$$\Gamma \vDash_M A \ := \ \forall \mathcal{M} \left(\mathcal{M} \vDash \Gamma \implies \mathcal{M} \vDash A \right)$$

- 保框架

$$\Gamma \vDash_F A \ := \ \forall \mathcal{F} \left(\mathcal{F} \vDash \Gamma \implies \mathcal{F} \vDash A \right)$$

- 保有效

$$\vDash \Gamma \implies \vDash A$$

Remark: “保点” 强于 “保模型” 强于 “保框架” 强于 “保有效”.
分离规则保点有效, 必然化规则保模型有效, 代入规则保框架有效.

Example

- $p \nvDash \Box p$
- $p \vDash_M \Box p$

Accessibility

serial	$\forall x \exists y : Rxy$
reflexive	$\forall x : Rxx$
symmetric	$\forall xy : Rxy \rightarrow Ryx$
transitive	$\forall xyz : Rxy \wedge Ryz \rightarrow Rxz$
euclidean	$\forall xyz : Rxy \wedge Rxz \rightarrow Ryz$
total	$\forall xy : Rxy \vee Ryx$
isolation	$\exists x \forall y : \neg Rxy \wedge \neg Ryx$
successor reflexive	$\forall x \exists y : Rxy \wedge Ryy$
asymmetric	$\forall xy : Rxy \rightarrow \neg Ryx$
antisymmetric	$\forall xy : Rxy \wedge Ryx \rightarrow x = y$

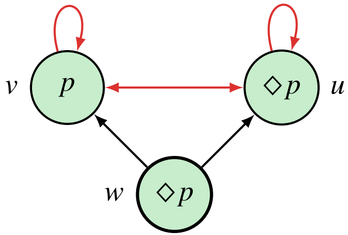
Correspondence Theorem

Theorem (Correspondence Theorem)

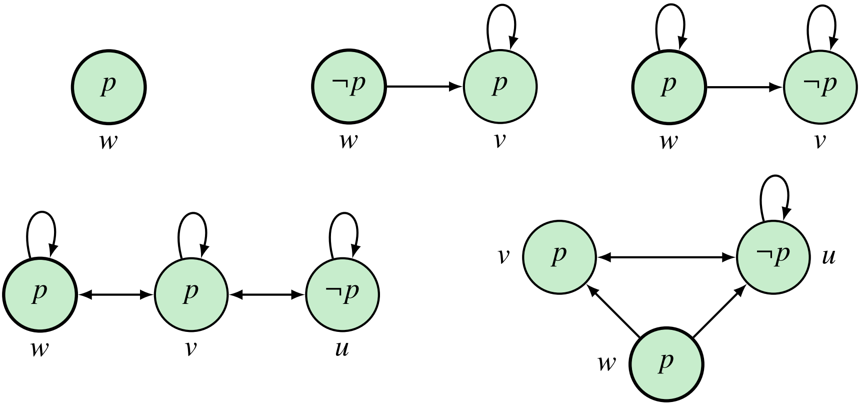
D	$W, R \vDash \Box p \to \Diamond p$	$\iff$	R *is serial*	$\forall x \exists y : Rxy$
T	$W, R \vDash \Box p \to p$	$\iff$	R *is reflexive*	$\forall x : Rxx$
B	$W, R \vDash p \to \Box\Diamond p$	$\iff$	R *is symmetric*	$\forall xy : Rxy \to Ryx$
4	$W, R \vDash \Box p \to \Box\Box p$	$\iff$	R *is transitive*	$\forall xyz : Rxy \wedge Ryz \to Rxz$
5	$W, R \vDash \Diamond p \to \Box\Diamond p$	$\iff$	R *is euclidean*	$\forall xyz : Rxy \wedge Rxz \to Ryz$

Remark: B is for Brouwer: if we regard "negation" as "necessarily negative", then we get $p \to \Box\neg\Box\neg p$ from $p \to \neg\neg p$.

Proof 5 ($\Longleftarrow$): Assume $W, R \nvDash 5$.
Then $w \vDash \Diamond p$ but $w \nvDash \Box\Diamond p$.
Now $w \vDash \Diamond p \implies \exists v : Rwv \;\&\; v \vDash p$,
and $w \nvDash \Box\Diamond p \implies w \vDash \Diamond\neg\Diamond p \implies \exists u : Rwu \;\&\; u \vDash \neg\Diamond p$.
However, $Ruv \implies u \vDash \Diamond p$.
So we have $Rwv \wedge Rwu$ but $\neg Ruv$.



Proof ($\Longrightarrow$): Counter-model for $D, T, B, 4, 5$



More Correspondence

D	$W, R \models \Box p \to \Diamond p$	$\Longleftrightarrow$	R is serial	$\forall x \exists y : Rxy$
T	$W, R \models \Box p \to p$	$\Longleftrightarrow$	R is reflexive	$\forall x : Rxx$
B	$W, R \models p \to \Box\Diamond p$	$\Longleftrightarrow$	R is symmetric	$\forall xy : Rxy \to Ryx$
4	$W, R \models \Box p \to \Box\Box p$	$\Longleftrightarrow$	R is transitive	$\forall xyz : Rxy \land Ryz \to Rxz$
5	$W, R \models \Diamond p \to \Box\Diamond p$	$\Longleftrightarrow$	R is euclidean	$\forall xyz : Rxy \land Rxz \to Ryz$
CD	$W, R \models \Diamond p \to \Box p$	$\Longleftrightarrow$	R is partially functional	$\forall xyz : Rxy \land Rxz \to y = z$
	$W, R \models \Diamond p \leftrightarrow \Box p$	$\Longleftrightarrow$	R is functional	$\forall x \exists! y : Rxy$
$\Box M$	$W, R \models \Box(\Box p \to p)$	$\Longleftrightarrow$	R is shift reflexive	$\forall xy : Rxy \to Ryy$
$C4$	$W, R \models \Box\Box p \to \Box p$	$\Longleftrightarrow$	R is dense	$\forall xy : Rxy \to \exists z : Rxz \land Rzy$
C	$W, R \models \Diamond\Box p \to \Box\Diamond p$	$\Longleftrightarrow$	R is convergent	$\forall xyz : Rxy \land Rxz \to \exists w : Ryw \land Rzw$
	$W, R \models \Box(\Box p \to q) \lor \Box(\Box q \to p)$	$\Longleftrightarrow$	R is connected	$\forall xyz : Rxy \land Rxz \to Ryz \lor Rzy$
	$W, R \models \Box(p \land \Box p \to q) \lor \Box(q \land \Box q \to p)$	$\Longleftrightarrow$	R is weakly connected	$\forall xyz : Rxy \land Rxz \to Ryz \lor y = z \lor Rzy$
	$W, R \models p \to \Box p$	$\Longleftrightarrow$		$\forall xy : Rxy \to y = x$
	$W, R \models \Box\bot$	$\Longleftrightarrow$		$\forall x \neg\exists y : Rxy$
	$W, R \models \Diamond p \to p \lor \Box\Diamond p$	$\Longleftrightarrow$		$\forall xyz : Rxy \land Rxz \to y = x \lor Ryz$

Contents

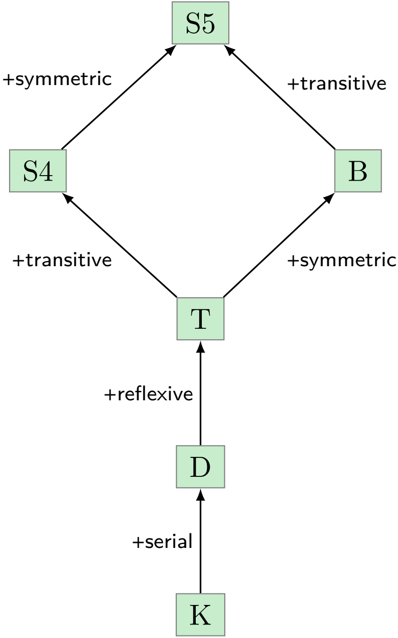
Formal System = Axiom + Inference Rule

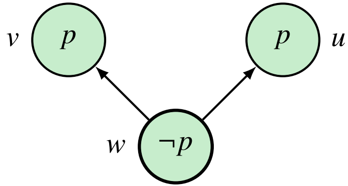
Axiom Schema

	tautologies
Dual	$\Diamond A \leftrightarrow \neg\Box\neg A$
K	$\Box(A \to B) \to \Box A \to \Box B$
D	$\Box A \to \Diamond A$
T	$\Box A \to A$
B	$A \to \Box\Diamond A$
4	$\Box A \to \Box\Box A$
5	$\Diamond A \to \Box\Diamond A$
L	$\Box(\Box A \to A) \to \Box A$

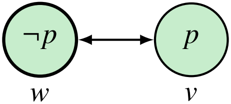
Inference Rule

$$\frac{A \quad A \to B}{B}\ \text{MP} \qquad\qquad \frac{\vdash A}{\vdash \Box A}\ \text{N}$$


S5
+symmetric
+transitive
S4
B
+transitive
+symmetric
T
+reflexive
D
+serial
K

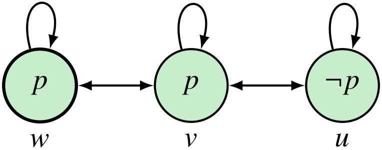


$p \rightarrow \Diamond p \not\models \Box p \rightarrow p$



$\mathrm{KB} \nvdash 4$

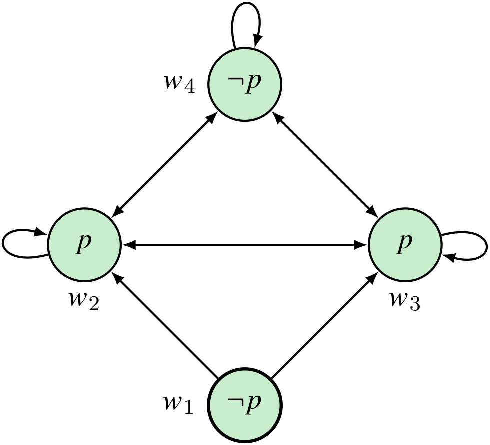
$w \not\models \Box p \rightarrow \Box\Box p$



$\mathrm{KTB} \nvdash 4 \qquad w \not\models \Box p \rightarrow \Box\Box p$

$\mathrm{KTB} \nvdash 5 \qquad v \not\models \Diamond\neg p \rightarrow \Box\Diamond\neg p$

KD5 $\nvdash$ 4



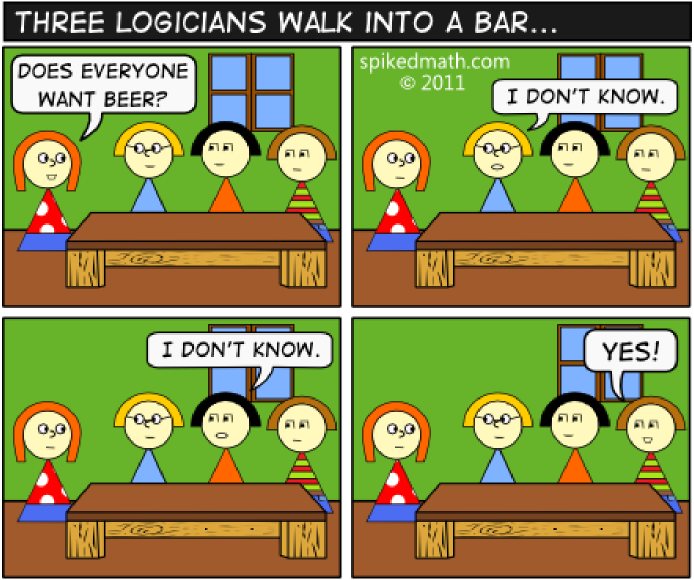
$$w_1 \nvDash \Box p \rightarrow \Box\Box p$$

Contents

Jaakko Hintikka

Figure: Jaakko Hintikka: 1929–2015. Epistemic Logic / Game Semantics / Independence-Friendly Logic

Information Update



知识的视角看问题

- 什么是密码? 你知我知, 但别人不知.
- 微信群是干嘛的? 制造公共知识.
- 邮件密送是干嘛的? 你知他知, 他不知你知, 且这是你我的公共知识.
- "代我问他好" 是干嘛的? 让你知道我尊重他.
- 送什么礼物给太太? 我知道她也知道对她有用的.
- 广告语的意义? 制造带意义的动作传递知识.
- 《三体》中的黑暗森林法则: 爱好和平的公共知识难以达成.
- 如何建设健康学术环境: 让他知道你知道学术规范.
- 狼人杀? 理性利用别人的不理性.
- 付费知识分享平台: 让你相信你知道.
- Would you like to come up to my apartment to see my etchings?
  阿 Q: 我想和你困觉!
  Nash: Could we just go straight to the sex? ✓̸

测试

从 $0 \sim 100$ 之间选一个自然数, 谁的数字最接近平均数的 $\frac{2}{3}$ 谁赢.

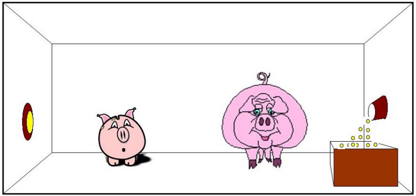
从逻辑的视角看“公共知识”

Problem (大女子主义村出轨问题)

1. 大女子主义村村规: 每个发现其老公出轨的女子必须当天枪杀之!
2. 村里住着 100 对夫妇.
3. 男人全都出轨了.
4. 虽然每个女子都知道别人的老公是否出轨, 但不知道自己老公是否出轨. 村里生活和谐幸福.
5. 某天, 女王来访, 临别提醒: “村里至少有一个男人出轨了”.

After a date, one says to the other:
“Would you like to come up to my apartment to see my etchings?”

智猪博弈 — 反复剔除劣策略均衡 ≠ 占优均衡



- 小猪是理性的

$$\text{Rational}(s)$$

- 大猪知道小猪是理性的

$$\text{K}_b\text{Rational}(s)$$

	按	等
按	$4,0$	$3,3$
等	$7,-1$	$0,0$

	按	等
按		$3,3$
等		$0,0$

	按	等
按		$3,3$
等		

反复剔除劣策略与 n-阶理性

	c_1	c_2	c_3	c_4
r_1	$5, 10$	$0, 11$	$1, 20$	$10, 10$
r_2	$4, 0$	$1, 1$	$2, 0$	$20, 0$
r_3	$3, 2$	$0, 4$	$4, 3$	$50, 1$
r_4	$2, 93$	$0, 92$	$0, 91$	$100, 90$

	c_1	c_2	c_3
r_1	$5, 10$	$0, 11$	$1, 20$
r_2	$4, 0$	$1, 1$	$2, 0$
r_3	$3, 2$	$0, 4$	$4, 3$
r_4	$2, 93$	$0, 92$	$0, 91$

	c_1	c_2	c_3
r_1	$5, 10$	$0, 11$	$1, 20$
r_2	$4, 0$	$1, 1$	$2, 0$
r_3	$3, 2$	$0, 4$	$4, 3$

	c_2	c_3
r_1	$0, 11$	$1, 20$
r_2	$1, 1$	$2, 0$
r_3	$0, 4$	$4, 3$

	c_2	c_3
r_2	$1, 1$	$2, 0$
r_3	$0, 4$	$4, 3$

	c_2
r_2	$1, 1$
r_3	$0, 4$

	c_2
r_2	$1, 1$

0-order	$c_4\times$	$\mathsf{Rational}(c)$
1-order	$r_4\times$	$\mathsf{K}_r\mathsf{Rational}(c)$
2-order	$c_1\times$	$\mathsf{K}_c\mathsf{K}_r\mathsf{Rational}(c)$
3-order	$r_1\times$	$\mathsf{K}_r\mathsf{K}_c\mathsf{K}_r\mathsf{Rational}(c)$
4-order	$c_3\times$	$\mathsf{K}_c\mathsf{K}_r\mathsf{K}_c\mathsf{K}_r\mathsf{Rational}(c)$
5-order	$r_3\times$	$\mathsf{K}_r\mathsf{K}_c\mathsf{K}_r\mathsf{K}_c\mathsf{K}_r\mathsf{Rational}(c)$

诸葛亮的《空城计》何以有效?

- 诸葛亮谨慎. $C(z)$
- 司马懿知道诸葛亮谨慎. $\mathsf{K}_s C(z)$
- 诸葛亮知道司马懿知道诸葛亮谨慎. $\mathsf{K}_z \mathsf{K}_s C(z)$
- 司马懿不知道诸葛亮知道司马懿知道诸葛亮谨慎. $\neg \mathsf{K}_s \mathsf{K}_z \mathsf{K}_s C(z)$

老狐狸与小狐狸

- 老狐狸看到满载而归的渔夫驾车经过，于是躺在路边装病.
- 渔夫看老狐狸可怜，就让它搭个便车.
- 老狐狸悄悄地把鱼一条一条地扔到路边草丛里，然后一跃而下吃鱼去了.
- 小狐狸问老狐狸是如何得到这么多鱼的，老狐狸“如实相告”.
- 第二天，小狐狸也学着躺在路边装病，渔夫愤怒地把它打死了.

- $\neg \mathsf{K}_{\text{渔}} A$
- $\mathsf{K}_{\text{老}} \neg \mathsf{K}_{\text{渔}} A$
- $[A]\mathsf{K}_{\text{渔}} A$
- $\mathsf{K}_{\text{老}} [A]\mathsf{K}_{\text{渔}} A$
- $\neg \mathsf{K}_{\text{小}} A$
- $[A]\mathsf{K}_{\text{小}} A$
- $\neg \mathsf{K}_{\text{小}} [A]\mathsf{K}_{\text{渔}} A$
- $\mathsf{K}_{\text{老}} [A]\neg \mathsf{K}_{\text{小}} [A]\mathsf{K}_{\text{渔}} A$

知识就是力量

- Knowledge is power: act properly to achieve goals;
- Knowledge is time: to make decisions more efficiently;
- Knowledge is money: can be traded;
- Knowledge is responsibility: to prove someone is guilty;
- Knowledge is you: to identify oneself;
- Knowledge is an immune system: to protect you;
- Knowledge satisfies our curiosity.

The only good is knowledge and the only evil is ignorance.

— *Socrates*

ALL men by nature desire to know.

— *Aristotle*

The greatest enemy of knowledge is not ignorance, it is the illusion of knowledge.

— *Stephen Hawking*

$\boxed{\text{know the unknown from the known}}$

- There are things we know we know. There are things we know we don't know. There are things we don't know we don't know.

$$\exists x \mathsf{K}\mathsf{K}x \wedge \exists x \mathsf{K}\neg\mathsf{K}x \wedge \exists x \neg\mathsf{K}\neg\mathsf{K}x$$

$$\neg\mathsf{K}\neg\mathsf{K}p \rightsquigarrow \mathsf{K}\neg\mathsf{K}p \rightsquigarrow \neg\mathsf{K}\mathsf{K}p \rightsquigarrow \mathsf{K}\mathsf{K}p$$

- 知之为知之, 不知为不知, 是知也.

$$\mathsf{K}p \to \mathsf{K}\mathsf{K}p \quad \& \quad \neg\mathsf{K}p \to \mathsf{K}\neg\mathsf{K}p$$

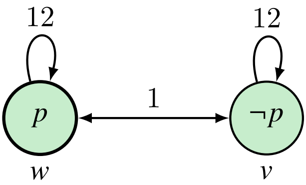
"Real knowledge is to know the extent of one's ignorance."

— 孔子

- 知其然, 知其所以然

$$\mathsf{K}\exists x Ax \to \exists x \mathsf{K} Ax$$

$$\mathcal{M}, w \vDash \mathsf{K}_i A \quad \text{iff} \quad \forall v \in W\Big(R_i wv \implies \mathcal{M}, v \vDash A\Big)$$



$$w \vDash p$$
$$w \vDash \neg \mathsf{K}_1 p$$
$$v \vDash \neg \mathsf{K}_1 p$$
$$w \vDash \mathsf{K}_2 p$$
$$v \vDash \mathsf{K}_2 \neg p$$
$$w \vDash \mathsf{K}_1(\mathsf{K}_2 p \vee \mathsf{K}_2 \neg p)$$
$$w \vDash \mathsf{K}_2 \neg \mathsf{K}_1 p$$
$$v \vDash \mathsf{K}_2 \neg \mathsf{K}_1 p$$
$$w \vDash \mathsf{K}_1 \mathsf{K}_2 \neg \mathsf{K}_1 p$$

$$w \vDash p \wedge \neg \mathsf{K}_1 p \wedge \mathsf{K}_2 p \wedge \mathsf{K}_1(\mathsf{K}_2 p \vee \mathsf{K}_2 \neg p) \wedge \mathsf{K}_2 \neg \mathsf{K}_1 p \wedge \mathsf{K}_1 \mathsf{K}_2 \neg \mathsf{K}_1 p$$

假设纽约在下雨 (p), 但 1 不知道, 而 2 知道, 不过 1 知道 2 知道纽约是否在下雨 (因为 2 住在纽约)……

知识的种类

- Mutual Knowledge:

  everybody in G knows p.

- Distributed Knowledge:

  everybody in G would know p
  if agents in G shared all their knowledge.

- Common Knowledge:

  everybody in G knows p,
  everybody knows that everybody knows,
  and so on.

Mutual Knowledge

Suppose a group $G \subset \{1 \ldots n\}$ of agents, everyone in G knows A:

$$\mathsf{E}_G A := \bigwedge_{i \in G} \mathsf{K}_i A$$

$$R_E := \bigcup_{i \in G} R_i$$

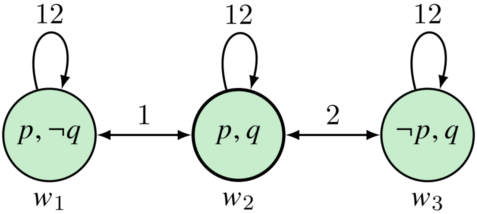
$$\mathcal{M}, w \vDash \mathsf{E}_G A \quad \text{iff} \quad \forall v \in W \Big(R_E w v \implies \mathcal{M}, v \vDash A \Big)$$

Distributed Knowledge

Intuition: what we know if we put all of our knowledge together.

$$R_D \;:=\; \bigcap_{i \in G} R_i$$

$$\mathcal{M}, w \vDash \mathsf{D}_G A \quad \text{iff} \quad \forall v \in W \Big(R_D w v \implies \mathcal{M}, v \vDash A \Big)$$



$$w_2 \vDash \mathsf{K}_1 p \wedge \neg \mathsf{K}_1 q \wedge \mathsf{K}_2 q \wedge \neg \mathsf{K}_2 p \wedge \mathsf{D}_{\{1,2\}}(p \wedge q)$$

$$\mathsf{D}_G A \not\equiv \bigvee_{i \in G} \mathsf{K}_i A$$

Common Knowledge

$$\mathsf{E}^1_G A := \mathsf{E}_G A$$
$$\mathsf{E}^{k+1}_G A := \mathsf{E}_G \mathsf{E}^k_G A$$
$$\mathsf{C}_G A := \bigwedge_{k=1}^{\infty} \mathsf{E}^k_G A$$

$$R^1 := R$$
$$R^{k+1} := R \circ R^k$$
$$R \circ S := \{(x, y) : \exists z (Rxz \land Szy)\}$$
$$R^* := \bigcup_{k=1}^{\infty} R^k$$

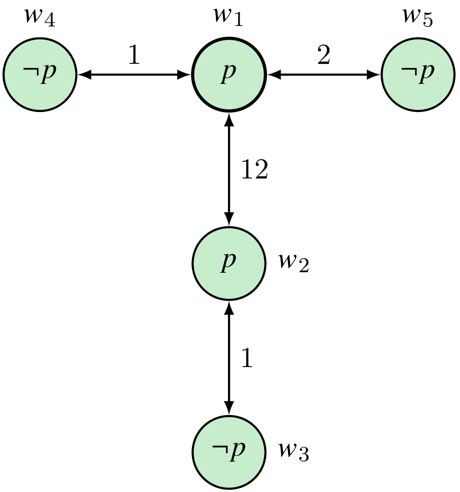
$$R_C := \left(\bigcup_{i \in G} R_i \right)^*$$

$$\mathcal{M}, w \vDash \mathsf{C}_G A \text{ iff } \forall v \in W \Big(R_C w v \implies \mathcal{M}, v \vDash A \Big)$$

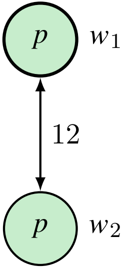
A Hierarchy of States of Knowledge

$$\mathsf{C}_G A \implies \cdots \mathsf{E}^k_G A \implies \cdots \mathsf{E}_G A \implies \bigvee_{i \in G} \mathsf{K}_i A \implies \mathsf{D}_G A \implies A$$

分布式知识的定义所面临的问题[17]



$w_1 \vDash \mathsf{D}_{12}(p \wedge \neg \mathsf{K}_1 p)$



"Communication Core"

$w_1 \vDash \mathsf{D}_{12}(p \wedge \mathsf{K}_1 p)$

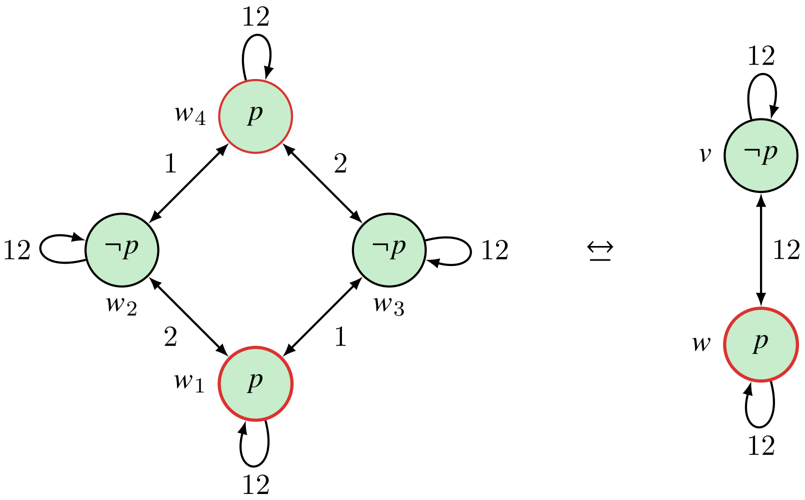
Remark: Communication turns distributed knowledge into common knowledge.

[17] "Communication Core" consists of the actual world plus all worlds linked to it by the intersection of all uncertainty relations.

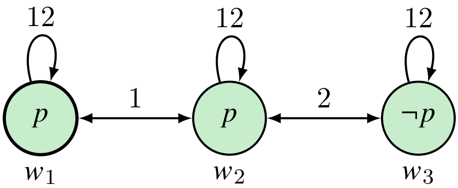
分布式知识的定义所面临的问题

- It is not invariant under bisimulation.
- It is not the case:

$$\{B : \mathcal{M}, w \vDash \mathsf{K}_i B \text{ for some } i \in G\} \vDash A \iff \mathcal{M}, w \vDash \mathsf{D}_G A$$



Can we easily have full common knowledge?



$$w_1 \vDash \mathsf{E}_{\{1,2\}} p \wedge \neg \mathsf{C}_{\{1,2\}} p$$

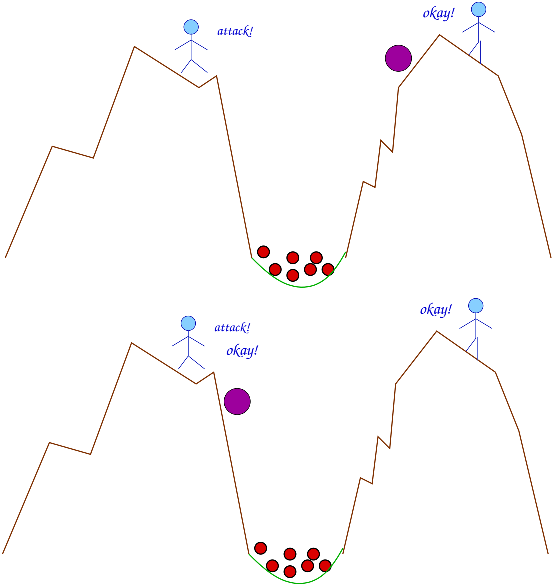
假设你秘密的分别给了 a 和 b 两个数字 2 和 3, 你只告诉他们俩这两个数字是相邻的自然数. 令 p 为 “两数字之和小于一千万”, 请问 p 是 a 和 b 的公共知识么?

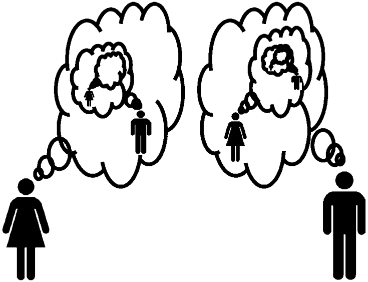
$$(0,1) \overset{b}{\longleftrightarrow} (2,1) \overset{a}{\longleftrightarrow} \underline{(2,3)} \overset{b}{\longleftrightarrow} (4,3) \overset{a}{\longleftrightarrow} (4,5) \overset{b}{\longleftrightarrow} (6,5) \cdots$$

$$(2,3) \vDash \neg \mathsf{K}_b \mathsf{K}_a \mathsf{K}_b (x + y \leq 10)$$

a and b commonly know that b's number is odd.

拜占庭将军协同进攻问题 Coordinated Attack

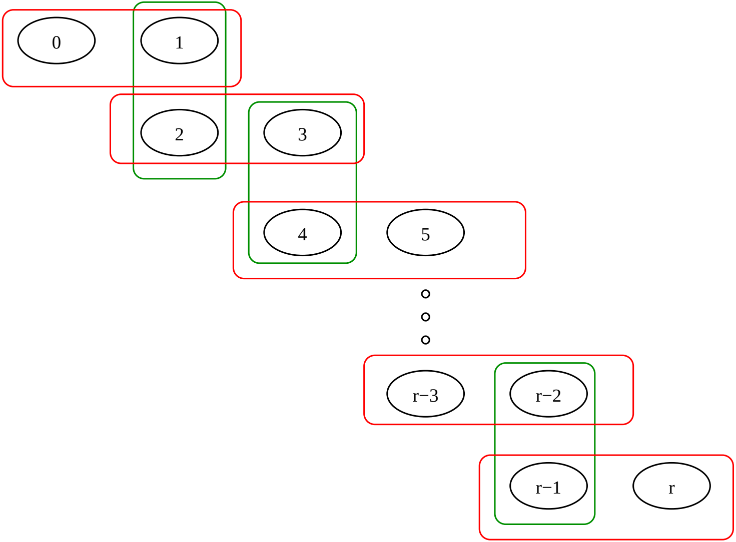




A: 如果我知道 B 一定进攻，我就进攻!

B: 如果我知道 A 一定进攻，我就进攻!

拜占庭将军协同进攻问题 Coordinated Attack



《三体》— 黑暗森林

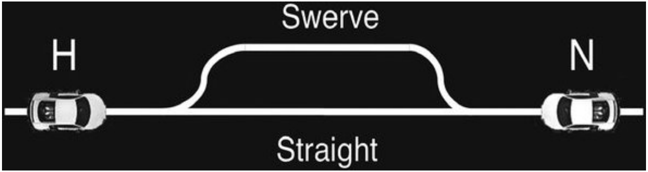
- 公理一：生存是文明的第一需要.
- 公理二：文明不断增长和扩张，但宇宙中的物质总量保持不变.

猜疑链

如果你认为我是善意的，这并不是你感到安全的理由，因为按照第一条公理，善意文明并不能预先把别的文明也想成善意的，所以，你现在还不知道我是怎么认为你的，你不知道我认为你是善意还是恶意；进一步，即使你知道我把你也想象成善意的，我也知道你把我想象成善意的，但是我不知道你是怎么想我怎么想你怎么想我的……

黑暗森林

宇宙就是一座黑暗森林，每个文明都是带枪的猎人，像幽灵般潜行于林间……如果他发现了别的生命，不管是不是猎人，不管是天使还是魔鬼，不管是娇嫩的婴儿还是步履蹒跚的老人，也不管是天仙般的少女还是天神般的男神，能做的只有一件事：开枪消灭之. 在这片森林中，他人就是地狱，就是永恒的威胁 (技术爆炸)，任何暴露自己存在的生命都将很快被消灭. 这就是宇宙文明的图景，这就是对费米悖论的解释.

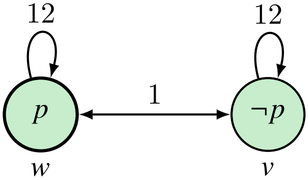


p: “XX 国机动车靠右行驶.”

- 什么样的关于 p 的知识状态能让你觉得往右躲是安全的?
- 你俩都知道 p 够吗 ($\mathsf{E}p$)?
- 你俩都知道你俩都知道 p 呢 ($\mathsf{EE}p$)?
- 也不行, 我能想象也许你以为我以为是靠左行驶 ($\neg\mathsf{K}_i\mathsf{K}_j\mathsf{K}_i p$).

$$\mathsf{C}A = \mathsf{E}A \wedge \mathsf{EE}A \wedge \mathsf{EEE}A \ldots$$

Example — A Question/Answer Scenario



$w \vDash p$

$w \vDash \mathsf{K}_2 p$

$w \vDash \neg\mathsf{K}_1 p \wedge \neg\mathsf{K}_1\neg p$

$w \vDash \mathsf{K}_1(\mathsf{K}_2 p \vee \mathsf{K}_2\neg p)$

$w \vDash \mathsf{K}_2(\mathsf{K}_2 p \vee \mathsf{K}_2\neg p)$

$w \vDash \mathsf{E}_{\{1,2\}}(\mathsf{K}_2 p \vee \mathsf{K}_2\neg p)$

$w \vDash \mathsf{K}_1(\neg\mathsf{K}_1 p \wedge \neg\mathsf{K}_1\neg p)$

$w \vDash \mathsf{K}_2(\neg\mathsf{K}_1 p \wedge \neg\mathsf{K}_1\neg p)$

$w \vDash \mathsf{E}_{\{1,2\}}(\neg\mathsf{K}_1 p \wedge \neg\mathsf{K}_1\neg p)$

$w \vDash \mathsf{C}_{\{1,2\}}(\mathsf{K}_2 p \vee \mathsf{K}_2\neg p)$

$w \vDash \mathsf{C}_{\{1,2\}}(\neg\mathsf{K}_1 p \wedge \neg\mathsf{K}_1\neg p)$

Remark: This is an excellent situation for 1 to ask 2 whether p is the case.

Epistemic Logic — Formal Systems

- Knowledge $\mathrm{S5} := K + T + 4 + 5$

  知之为知之(4), 不知为不知(5), 是知也

- Belief $K + D + 4 + 5$
- Common Knowledge

$$\mathrm{S5} \ \text{ for } \ \mathsf{C}_G$$

$$+$$

$$\mathsf{C}_G A \leftrightarrow A \wedge \mathsf{E}_G \mathsf{C}_G A$$

$$+$$

$$A \wedge \mathsf{C}_G(A \to \mathsf{E}_G A) \to \mathsf{C}_G A$$

- Distributed Knowledge

$$\mathrm{S5} \ \text{ for } \ \mathsf{D}_G$$

$$+$$

$$\mathsf{K}_i A \to \mathsf{D}_G A \ \text{ when } \ i \in G$$

$$+$$

$$\mathsf{D}_G A \to \mathsf{D}_{G'} A \ \text{ when } \ G \subset G'$$

知识 vs 信念 Knowledge vs Belief

1. $\mathsf{K}(p \to q) \to \mathsf{K}p \to \mathsf{K}q$
2. $\mathsf{K}p \to p$
3. $\mathsf{K}p \to \mathsf{KK}p$
4. $\neg\mathsf{K}p \to \mathsf{K}\neg\mathsf{K}p$
    1. $\mathsf{K}p \to \mathsf{B}p$
    2. $\mathsf{B}p \to \mathsf{KB}p$
    3. $\neg\mathsf{B}p \to \mathsf{K}\neg\mathsf{B}p$
    4. $\mathsf{B}p \to \mathsf{BK}p$ (strong belief)

1. $\mathsf{B}(p \to q) \to \mathsf{B}p \to \mathsf{B}q$
2. $\mathsf{B}p \to \neg\mathsf{B}\neg p$
3. $\mathsf{B}p \to \mathsf{BB}p$
4. $\neg\mathsf{B}p \to \mathsf{B}\neg\mathsf{B}p$

$$\boxed{\mathsf{B}p \overset{?}{\leftrightarrow} \neg\mathsf{K}\neg\mathsf{K}p}$$

摩尔悖论 Moore's Paradox

Are all truths knowable?

$$\boxed{\text{It's rainning but I don't know it's raining.}}$$

$$\vdash \neg \mathsf{K}(p \wedge \neg \mathsf{K}p)$$

1. $\mathsf{K}(p \wedge \neg \mathsf{K}p)$ Assumption
2. $\mathsf{K}p$
3. $\mathsf{K}\neg\mathsf{K}p$
4. $\neg\mathsf{K}p$ $\mathsf{K}p \to p$
5. $\mathsf{K}p \wedge \neg\mathsf{K}p$
6. $\neg\mathsf{K}(p \wedge \neg\mathsf{K}p)$

摩尔悖论 Moore's Paradox

Are all truths believable?

$\boxed{\text{It's raining but I don't believe it's raining.}}$

$$\vdash \neg\mathsf{B}(p \land \neg\mathsf{B}p)$$

1.	$\mathsf{B}(p \land \neg\mathsf{B}p)$	Assumption
2.	$\mathsf{B}p \land \mathsf{B}\neg\mathsf{B}p$	$\mathsf{B}(p \land q) \to \mathsf{B}p \land \mathsf{B}q$
3.	$\mathsf{B}p$	
4.	$\mathsf{B}\mathsf{B}p$	$\mathsf{B}p \to \mathsf{B}\mathsf{B}p$
5.	$\mathsf{B}\neg\mathsf{B}p$	
6.	$\mathsf{B}\mathsf{B}p \land \mathsf{B}\neg\mathsf{B}p$	
7.	$\neg(\mathsf{B}\mathsf{B}p \land \mathsf{B}\neg\mathsf{B}p)$	$(\mathsf{B}p \to \neg\mathsf{B}\neg p) \leftrightarrow \neg(\mathsf{B}p \land \mathsf{B}\neg p)$
8.	$\neg\mathsf{B}(p \land \neg\mathsf{B}p)$	

Against Negative Introspection?

1. $\neg p \wedge \mathsf{BK}p$ — suppose you falsely believes that you know p
2. $\neg \mathsf{K}p$ — knowledge implies truth
3. $\mathsf{K}\neg\mathsf{K}p$ — negative introspection
4. $\mathsf{B}\neg\mathsf{K}p$ — knowledge implies belief
5. $\mathsf{B}\bot$

$$\neg\mathsf{K}p \rightarrow \mathsf{K}\neg\mathsf{K}p \ ? \qquad (\text{Ax } 5)$$

$$\neg\mathsf{K}\neg\mathsf{K}p \rightarrow \mathsf{K}p \ ? \qquad (\text{Ax } 5)$$

$$\neg\mathsf{K}\neg\mathsf{K}p \rightarrow \mathsf{K}\neg\mathsf{K}\neg p \ ? \qquad (\text{Ax } 4.2)$$

Margin for Error and Against Positive Introspection?

p_n : n grains is a heap.

1. $\mathsf{K}\neg p_n$ — Assumption
2. $\mathsf{K}(p_{n+1} \to \neg\mathsf{K}\neg p_n)$ — Margin for Error
3. $\mathsf{K}(\mathsf{K}\neg p_n \to \neg p_{n+1})$
4. $\mathsf{K}\neg p_n \to \mathsf{KK}\neg p_n$ — Positive Introspection
5. $\mathsf{K}\neg p_{n+1}$ — ?

$$\mathsf{K}p \to \mathsf{KK}p \quad ?$$

Remark: 谷堆悖论是连锁悖论, 下面这个是连锁悖论吗: 不存在哺乳动物, 因为哺乳动物的母亲也必须是哺乳动物.

逻辑全知 vs 怀疑论

$$\frac{A}{\mathsf{K}A} \qquad \frac{A \to B}{\mathsf{K}A \to \mathsf{K}B} \qquad \frac{A \leftrightarrow B}{\mathsf{K}A \leftrightarrow \mathsf{K}B} \qquad \frac{\mathsf{K}(A \to B) \quad \mathsf{K}A}{\mathsf{K}B}$$

S_1 I know "I have hands". $\mathsf{K}p$

S_2 I know if "I have hands" then "I am not a brain in the vat". $\mathsf{K}(p \to q)$

S_3 I don't know that "I am not a brain in the vat". $\neg\mathsf{K}q$

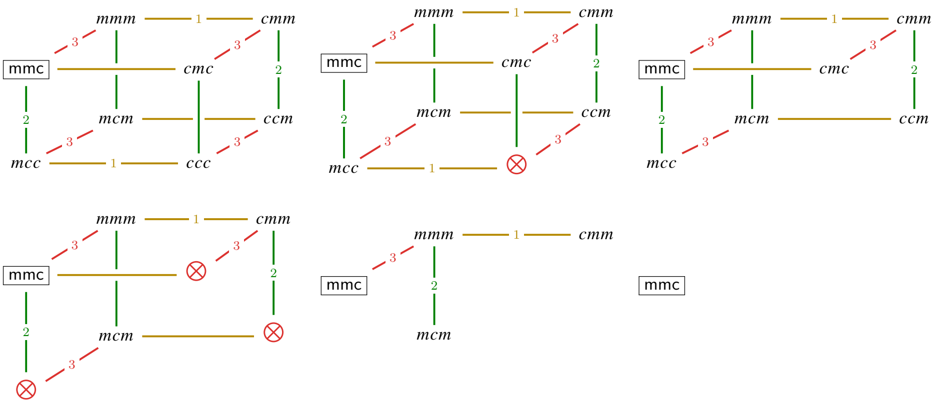
Possible "solutions":

- Impossible worlds: inconsistent alternatives
- Awareness: K = awareness + implicit knowledge
- Algorithmic knowledge: K = answer by algorithm
- Timed knowledge: reasoning takes time
- Neighbourhood semantics: still problematic
- Counterfactual implication:
  - Nozick: $\mathsf{K}p := p \wedge \mathsf{B}p \wedge (\neg p \mathrel{\Box\!\!\to} \neg\mathsf{B}p) \wedge (p \mathrel{\Box\!\!\to} \mathsf{B}p)$
  - Sosa: $\mathsf{K}p := p \wedge \mathsf{B}p \wedge (\mathsf{B}p \mathrel{\Box\!\!\to} p) \wedge (p \mathrel{\Box\!\!\to} \mathsf{B}p)$

knowing-whether	$\mathsf{K}A \vee \mathsf{K}\neg A$	我知道股票是否会涨
knowing-what	$\exists x\mathsf{K}(A \to x = c)$	我知道彩票的中奖号码是什么
knowing-how	$\exists \alpha \mathsf{K}[\alpha]A$	我知道如何实现一个小目标
knowing-why	$\exists t\mathsf{K}(t : A)$	我知道为什么她生气了

Problem (泥孩问题)

- n 个小朋友里的 k 个脸上弄上了泥巴.
- 不过他们只能看到别人的脸, 而不知道自己脸上是否有泥巴.
- 老师看到了很生气: "你们中间有人把泥巴弄到脸上了!"
- 然后老师命令道: "知道自己脸上有泥巴的给我站出来!"
- 没有人站出来.
- 老师重复道: "知道自己脸上有泥巴的给我站出来!"
- ……



1. “有人把泥巴弄到脸上了! 知道自己脸上有泥巴的给我站出来!”

$$\neg \mathsf{K}_1 m_1 \wedge \neg \mathsf{K}_1 \neg m_1 \wedge \neg \mathsf{K}_2 m_2 \wedge \neg \mathsf{K}_2 \neg m_2 \wedge \neg \mathsf{K}_3 m_3 \wedge \neg \mathsf{K}_3 \neg m_3$$

2. 没有人站出来. “知道自己脸上有泥巴的给我站出来!”

$$\mathsf{K}_1 m_1 \wedge \mathsf{K}_2 m_2 \wedge \neg \mathsf{K}_3 m_3 \wedge \neg \mathsf{K}_3 \neg m_3$$

3. 1 和 2 站了出来.

$$\mathsf{K}_3 \neg m_3$$

公开宣告逻辑 Public Announcement Logic

$$A := p \mid \neg A \mid A \wedge A \mid \mathsf{K}_i A \mid [A]A$$

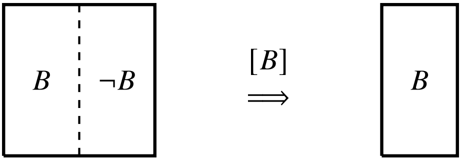
$$\mathcal{M}, w \vDash [B]A \text{ iff } \mathcal{M}, w \vDash B \Longrightarrow \mathcal{M}{\restriction_B}, w \vDash A$$
$$\mathcal{M}, w \vDash \langle B \rangle A \text{ iff } \mathcal{M}, w \vDash B \ \& \ \mathcal{M}{\restriction_B}, w \vDash A$$

where

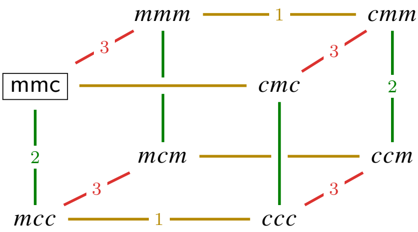
$$\mathcal{M}{\restriction_B} := \left(W', \{R'_i\}_{i \in G}, V'\right)$$

and

$$W' := \{w \in W : \mathcal{M}, w \vDash B\} \quad R'_i := R_i{\restriction_{W' \times W'}} \quad V'(p) := V(p) \cap W'$$



Remark: the meaning of an action is the change it brings to the states!



$\mathcal{M}, mmc \vDash m_1 \wedge m_2 \wedge \neg m_3$

$\mathcal{M}, mmc \vDash \mathsf{E}_{\{1,2,3\}} P$

$\mathcal{M}, mmc \vDash \neg\mathsf{C}_{\{1,2,3\}} P$

$\mathcal{M}, mmc \vDash \neg\mathsf{K}_1 m_1 \wedge \mathsf{K}_1 m_2$

$\mathcal{M}, mmc \vDash \mathsf{K}_1\mathsf{K}_3 m_2 \wedge \mathsf{K}_1 \neg\mathsf{K}_2 m_2$

$\mathcal{M}\restriction_P, mcc \vDash \mathsf{K}_1 m_1$

$\mathcal{M}\restriction_P, mmc \vDash \langle \neg Q_1 \wedge \neg Q_2 \wedge \neg Q_3 \rangle Q_1 \vee Q_2 \vee Q_3$

$\mathcal{M}\restriction_P, mmm \vDash \langle \neg Q_1 \wedge \neg Q_2 \wedge \neg Q_3 \rangle \neg Q_1 \wedge \neg Q_2 \wedge \neg Q_3$

$\mathcal{M}\restriction_P\restriction_{\neg Q_1 \wedge \neg Q_2 \wedge \neg Q_3}, mmm \vDash \langle \neg Q_1 \wedge \neg Q_2 \wedge \neg Q_3 \rangle Q_1 \vee Q_2 \vee Q_3$

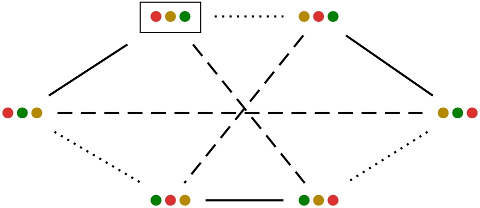
1. “有人把泥巴弄到脸上了!” $P := m_1 \vee m_2 \vee m_3$
2. “知道…站出来!” $\neg Q_1 \wedge \neg Q_2 \wedge \neg Q_3$ where $Q_i := \mathsf{K}_i m_i \vee \mathsf{K}_i \neg m_i$
3. “知道…站出来!” $Q_1 \wedge Q_2 \wedge \neg Q_3$

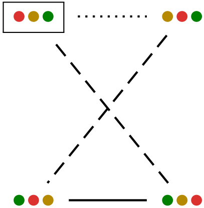
$\mathcal{M}, mmc \vDash [P][\neg Q_1 \wedge \neg Q_2 \wedge \neg Q_3][Q_1 \wedge Q_2 \wedge \neg Q_3](\mathsf{K}_1 m_1 \wedge \mathsf{K}_2 m_2 \wedge \mathsf{K}_3 \neg m_3)$

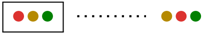
$\mathcal{M}, mmm \vDash [P][\neg Q_1 \wedge \neg Q_2 \wedge \neg Q_3][\neg Q_1 \wedge \neg Q_2 \wedge \neg Q_3](\mathsf{K}_1 m_1 \wedge \mathsf{K}_2 m_2 \wedge \mathsf{K}_3 m_3)$

三色卡

- “红”、“黄”、“绿” 三色卡依次被分给了 1, 2, 3 三个小朋友.
- 每个小朋友只能看到自己手里的卡.
- 2 问 1: “你有绿卡吗?”
- 1: “没有!”







孩子们的年龄分别多大?

一个人口普查员向一位母亲询问她的孩子的情况.

- 母亲: "我有三个孩子, 他们的年龄乘起来是 36, 加起来是今天的日期."
- 普查员: "我还是不知道你孩子多大啊."
- 母亲: "不好意思, 忘了说了, 我们家老大喜欢喝咸豆脑."
- 普查员: "那我知道了!"

$$
\begin{array}{ccc|c}
1 & 1 & 36 & \cancel{38} \\
1 & 2 & 18 & 21 \\
1 & 4 & 9 & 14 \\
1 & 6 & 6 & 13 \\
2 & 2 & 9 & 13 \\
2 & 3 & 6 & 11 \\
3 & 3 & 4 & 10
\end{array}
\Longrightarrow
\begin{array}{ccc|c}
1 & 6 & 6 & 13 \\
2 & 2 & 9 & 13
\end{array}
\Longrightarrow
\begin{array}{ccc|c}
2 & 2 & 9 & 13
\end{array}
$$

小菜的生日是哪天?

小艾和小白都想知道小菜的生日.
小菜给了他们 10 个可能的候选:

5.15	5.16	5.19
6.17	6.18	
7.14	7.16	
8.14	8.15	8.17

然后小菜分别告诉了小艾和小白她的生日的月份和日子.

- 小艾: “我不知道小菜的生日, 但我知道小白也不知道.”
- 小白: “之前我不知道, 但现在我知道了.”
- 小艾: “那我也知道了.”

小菜的生日是哪天?

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- 小白: “之前我不知道, 但现在我知道了.”
- 小艾: “那我也知道了.”

5.15	5.16	5.19
6.17	6.18	
7.14	7.16	
8.14	8.15	8.17

$\Longrightarrow$

7.14	7.16	
8.14	8.15	8.17

$\Longrightarrow$

7.16	
8.15	8.17

$\Longrightarrow$ 7.16

俄罗斯纸牌

- 从七张纸牌 “0123456” 里小艾和小白分别被秘密的发放了三张纸牌, 小菜拿走了剩下的一张.
- **问题:** 小艾和小白有没有可能通过公开的宣告使得彼此都知道对方的牌是什么, 但同时小菜还是不知道任何一张不在手里的牌的归属?

Solution

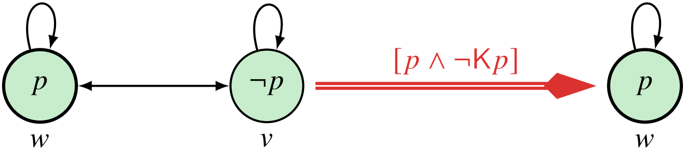
- 假设小艾手里的是 012, 剩下的几张是 3456. 从 012 里选一张, 然后从 3456 里选两张. 凑成 034, 056, 135, 146, 236, 245. 小艾公开宣告拥有七种组合里的某一种:
$$\begin{array}{ll} 012 & \\ 034 & 056 \\ 135 & 146 \\ 236 & 245 \end{array}$$
- 小白公开宣告小菜拥有的那张牌.

Unsuccessful Update

$$\vDash [p]\mathsf{C}_G p$$

$$\vDash [\mathsf{C}_G A]\mathsf{C}_G A$$

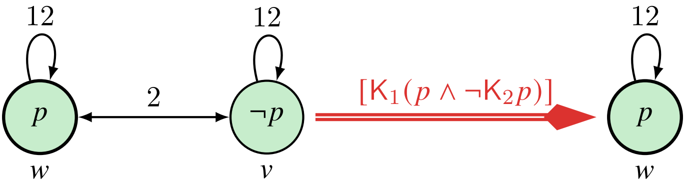
$$\overset{?}{\vDash} [A]\mathsf{C}_G A \qquad \overset{?}{\vDash} [A]\mathsf{K}A \qquad \overset{?}{\vDash} [A]A$$



$$\mathcal{M}, w \vDash (p \wedge \neg\mathsf{K}p) \wedge [p \wedge \neg\mathsf{K}p]\mathsf{K}p$$

Remark: If the goal of the announcing person was to "spread the truth of this formula," then this attempt was clearly unsuccessful.

Unsuccessful Update



$$\mathcal{M}, w \vDash (p \wedge \neg \mathsf{K}_2 p) \wedge \mathsf{K}_1(p \wedge \neg \mathsf{K}_2 p) \wedge [\mathsf{K}_1(p \wedge \neg \mathsf{K}_2 p)]\mathsf{K}_1\mathsf{K}_2 p$$

1. 哪些“公开宣告”可以使“真”的变成“假”的?
2. 哪些“公开宣告”可以使“假”的变成“真”的?
3. 哪些“公开宣告”可以保“真”?
4. 哪些“公开宣告”可以被“知道”?
5. 哪些“公开宣告”可以变成“公共知识”?

1. $\langle B\rangle A \leftrightarrow B \wedge [B]A$
2. $[B](A \to C) \leftrightarrow ([B]A \to [B]C)$
3. $[B]p \leftrightarrow (B \to p)$
4. $[B]\neg A \leftrightarrow (B \to \neg A)$?
5. $[B]\neg A \leftrightarrow \neg[B]A$?
6. $[B]\neg A \leftrightarrow (B \to \neg[B]A)$
7. $[B]\mathsf{K}_i A \leftrightarrow (B \to \mathsf{K}_i(B \to [B]A))$
8. $[B]\mathsf{K}_i A \leftrightarrow (B \to \mathsf{K}_i[B]A)$
9. $[A]\mathsf{K}_i A$?
10. $[B][C]A \leftrightarrow [B \wedge C]A$?
11. $[B][C]A \leftrightarrow [B \wedge [B]C]A$
12. $\dfrac{A}{[B]A} \qquad \dfrac{A(p)}{A[B/p]}$? $\qquad \dfrac{A \leftrightarrow B}{[A]C \leftrightarrow [B]C} \qquad \dfrac{A \leftrightarrow B}{[C]A \leftrightarrow [C]B}$

公开宣告逻辑 Public Announcement Logic (PAL)

Axiom Schema

1. Tautologies
2. $\mathsf{K}_i(A \to B) \to \mathsf{K}_i A \to \mathsf{K}_i B$
3. $[B]p \leftrightarrow (B \to p)$
4. $[B]\neg A \leftrightarrow (B \to \neg[B]A)$
5. $[B](A \wedge C) \leftrightarrow [B]A \wedge [B]C$
6. $[B]\mathsf{K}_i A \leftrightarrow (B \to \mathsf{K}_i[B]A)$
7. $[B][C]A \leftrightarrow [B \wedge [B]C]A$

Inference Rule

$$\frac{A \quad A \to B}{B}\ \text{MP} \qquad\qquad \frac{A}{\mathsf{K}_i A}\ \text{N}$$

$$\frac{P \to [Q]A \quad P \land Q \to \mathsf{E}_G P}{P \to [Q]\mathsf{C}_G A}$$

'Common knowledge induction' is a special case.
Take $P := A$ and $Q := \top$.

$$\mathsf{C}_G(A \to \mathsf{E}_G A) \to A \to \mathsf{C}_G A$$

$$A := \top \mid p \mid \neg A \mid A \wedge A \mid [\alpha]A$$
$$\alpha := a \mid A? \mid \alpha;\alpha \mid \alpha \cup \alpha \mid \alpha^*$$

$$\mathcal{M}, w \vDash [\alpha]A \;\text{ iff }\; \forall v \in W\Big(R_\alpha wv \implies \mathcal{M}, v \vDash A\Big)$$
$$\mathcal{M}, w \vDash \langle\alpha\rangle A \;\text{ iff }\; \exists v \in W\Big(R_\alpha wv \;\&\; \mathcal{M}, v \vDash A\Big)$$

where

$$\begin{aligned}
R_{A?} &:= \{(w,w) : \mathcal{M}, w \vDash A\}\\
R_{\alpha;\beta} &:= \Big\{(w,v) : \exists u\big(R_\alpha wu \wedge R_\beta uv\big)\Big\}\\
R_{\alpha\cup\beta} &:= R_\alpha \cup R_\beta\\
R_{\alpha^*} &:= \bigcup_{n=0}^{\infty} R_{\alpha^n}
\end{aligned}$$

命题动态逻辑 Propositional Dynamic Logic (PDL)

Axiom Schema

1. Tautologies
2. $[\alpha](A \to B) \to [\alpha]A \to [\alpha]B$
3. $[B?]A \leftrightarrow (B \to A)$
4. $[\alpha;\beta]A \leftrightarrow [\alpha][\beta]A$
5. $[\alpha \cup \beta]A \leftrightarrow [\alpha]A \wedge [\beta]A$
6. $[\alpha^*]A \leftrightarrow A \wedge [\alpha][\alpha^*]A$
7. $A \wedge [\alpha^*](A \to [\alpha]A) \to [\alpha^*]A$

Inference Rule

$$\frac{A \quad A \to B}{B}\ \text{MP} \qquad\qquad \frac{A}{[\alpha]A}\ \text{N}$$

PDL is sound and weak complete.
PDL is not compact. $\{\langle a^*\rangle p, \neg p, \neg\langle a\rangle p, \neg\langle a;a\rangle p, \neg\langle a;a;a\rangle p, \ldots\}$
Its satisfiability is decidable (in EXPTIME).

Axiom Schema

1. FOL
2. PDL
3. $\langle x := t \rangle A \leftrightarrow A[t/x]$

Inference Rule

$$\frac{A \quad A \to B}{B}\ \text{MP} \qquad \frac{A}{[\alpha]A}\ \text{N}$$

$$\frac{A \to [\alpha^n]B, \quad n \in \omega}{A \to [\alpha^*]B}\ \text{IC} \qquad \frac{A}{\forall x A}\ \text{G}$$

Application — Program Analysis

$$
\begin{aligned}
\mathbf{skip} &\coloneqq \top? \\
\mathbf{fail} &\coloneqq \bot? \\
\mathbf{if}\ B\ \mathbf{then}\ \alpha\ \mathbf{else}\ \beta &\coloneqq B?;\alpha \cup \neg B?;\beta \\
\mathbf{while}\ B\ \mathbf{do}\ \alpha &\coloneqq (B?;\alpha)^*;\neg B? \\
\mathbf{repeat}\ \alpha\ \mathbf{until}\ B &\coloneqq \alpha;(\neg B?;\alpha)^*;B? \\
\{A\}\ \alpha\ \{B\} &\coloneqq A \rightarrow [\alpha]B
\end{aligned}
$$

Algorithm GCD

while $x \neq y$ **do**
  if $x > y$ **then**
    $x \leftarrow x - y$
  else
    $y \leftarrow y - x$
  end if
end while

$$[(x = m \wedge y = n)?]\Big\langle (x \neq y?;(x > y?;x \leftarrow x - y) \cup (x < y?;y \leftarrow y - x))^*;x = y?\Big\rangle x = \gcd(m, n)$$

Hoare Logic

$$\frac{}{\{P\}\ \textbf{skip}\ \{P\}}$$

$$\frac{}{\{P[t/x]\}\ x := t\ \{P\}}$$

$$\frac{\{P\}\ \alpha\ \{Q\} \quad \{Q\}\ \beta\ \{R\}}{\{P\}\ \alpha;\beta\ \{R\}}$$

$$\frac{\{B \land P\}\ \alpha\ \{Q\} \quad \{\neg B \land P\}\ \beta\ \{Q\}}{\{P\}\ \textbf{if}\ B\ \textbf{then}\ \alpha\ \textbf{else}\ \beta\ \{Q\}}$$

$$\frac{P_1 \to P_2 \quad \{P_2\}\ \alpha\ \{Q_2\} \quad Q_2 \to Q_1}{\{P_1\}\ \alpha\ \{Q_1\}}$$

$$\frac{\{P \land B\}\ \alpha\ \{P\}}{\{P\}\ \textbf{while}\ B\ \textbf{do}\ \alpha\ \{\neg B \land P\}}$$

Theorem

The rules of Hoare Logic are derivable in Dynamic Logic.

$\{x = 4 \land y = 3\}\ \textbf{if}\ x < y\ \textbf{then}\ z := x; y := y + 1\ \textbf{else}\ z := y; z := z + 1\ \{x = 4 \land y = 3 \land z = 4\}$

Temporal Logic

some past / finally future / all past / globally future / next / since / until

$$A \coloneqq p \mid \bot \mid \neg A \mid A \wedge A \mid \mathsf{P}A \mid \mathsf{F}A \mid \mathsf{H}A \mid \mathsf{G}A \mid \mathsf{X}A \mid A\mathsf{S}A \mid A\mathsf{U}A$$

- $\mathcal{M}, n \vDash \mathsf{P}A$ iff $\exists m < n\big(\mathcal{M}, m \vDash A\big)$
- $\mathcal{M}, n \vDash \mathsf{F}A$ iff $\exists m > n\big(\mathcal{M}, m \vDash A\big)$
- $\mathcal{M}, n \vDash \mathsf{H}A$ iff $\forall m < n\big(\mathcal{M}, m \vDash A\big)$
- $\mathcal{M}, n \vDash \mathsf{G}A$ iff $\forall m > n\big(\mathcal{M}, m \vDash A\big)$
- $\mathcal{M}, n \vDash \mathsf{X}A$ iff $\mathcal{M}, n+1 \vDash A$
- $\mathcal{M}, n \vDash A\mathsf{S}B$ iff $\exists m < n\Big[\mathcal{M}, m \vDash B \;\;\&\;\; \forall t\big(m < t < n \implies \mathcal{M}, t \vDash A\big)\Big]$
- $\mathcal{M}, n \vDash A\mathsf{U}B$ iff $\exists m > n\Big[\mathcal{M}, m \vDash B \;\;\&\;\; \forall t\big(n < t < m \implies \mathcal{M}, t \vDash A\big)\Big]$

$$\mathsf{H}A \equiv \neg\mathsf{P}\neg A \qquad \mathsf{G}A \equiv \neg\mathsf{F}\neg A \qquad \mathsf{P}A \equiv \top\mathsf{S}A \qquad \mathsf{F}A \equiv \top\mathsf{U}A$$

Yablo Paradox in Linear Temporal Logic

Yablo Paradox

A_1: for all $k > 1$, A_k is false.

A_2: for all $k > 2$, A_k is false.

A_3: for all $k > 3$, A_k is false.

$\vdots$

$$\not\models A \leftrightarrow \mathsf{G}\neg A$$

Other Versions of Yablo Paradox

always	$A_n \leftrightarrow \forall m > n : \neg A_m$	$\models \neg\mathsf{G}(A \leftrightarrow \mathsf{G}\neg A)$
sometimes	$A_n \leftrightarrow \exists m > n : \neg A_m$	$\models \neg\mathsf{G}(A \leftrightarrow \mathsf{F}\neg A)$
almost always	$A_n \leftrightarrow \exists k > n \forall m > k : \neg A_m$	$\models \neg\mathsf{G}(A \leftrightarrow \mathsf{FG}\neg A)$
infinitely often	$A_n \leftrightarrow \forall k > n \exists m > k : \neg A_m$	$\models \neg\mathsf{G}(A \leftrightarrow \mathsf{GF}\neg A)$

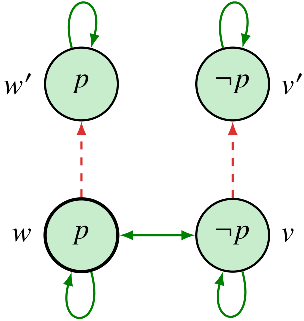
Temporal Logic — Formal System

Axiom Schema

- Tautologies
- $\mathsf{G}(A \to B) \to \mathsf{G}A \to \mathsf{G}B$
- $\mathsf{H}(A \to B) \to \mathsf{H}A \to \mathsf{H}B$
- $A \to \mathsf{GP}A$
- $A \to \mathsf{HF}A$

Inference Rule

$$\frac{A \quad A \to B}{B} \qquad\qquad \frac{A}{\mathsf{H}A} \qquad\qquad \frac{A}{\mathsf{G}A}$$



$$w \vDash \neg \mathsf{K}p \wedge \mathsf{F}\mathsf{K}p$$

Remark: two-dimensional semantics

宿命论论证

1. 如果明天有海战为真, 那么明天有海战就不可能为假, 即明天必然有海战.
2. 如果明天有海战为假, 那么明天有海战就不可能为真, 即明天必然没有海战.
3. 因此, 要么明天必然有海战, 要么明天必然没有海战.

$$\frac{\begin{array}{c} p \to \Box p \\ \neg p \to \Box \neg p \end{array}}{\Box p \vee \Box \neg p}$$

$$\frac{\begin{array}{c} \Box(p \to p) \\ \Box(\neg p \to \neg p) \end{array}}{\Box p \vee \Box \neg p} \times$$

庄子《秋水》

庄子与惠子游于濠梁之上.

1. 庄子: 鲦鱼出游从容, 是鱼之乐也.
2. 惠子: 子非鱼, 安知鱼之乐?

$$\forall xy(\mathsf{K}_x Hy \vee \mathsf{K}_x \neg Hy \to Fy \to Fx) \quad ?$$

$$\forall x(\mathsf{K}_x Hf \vee \mathsf{K}_x \neg Hf \to x = f) \quad \checkmark$$

3. 庄子: 子非我, 安知我不知鱼之乐?

$$\forall xy(\mathsf{K}_x \mathsf{K}_y Hf \vee \mathsf{K}_x \neg \mathsf{K}_y Hf \to x = y)$$

4. 惠子: 我非子, 固不知子矣; 子固非鱼也, 子之不知鱼之乐, 全矣.

For any **'subjective'** formula A,

$$\frac{\forall xy\big(\mathsf{K}_x Ay \vee \mathsf{K}_x \neg Ay \to x = y\big) \quad z \neq f \quad h \neq z}{\neg \mathsf{K}_z Hf \wedge \neg \mathsf{K}_z \neg Hf \wedge \neg \mathsf{K}_h \mathsf{K}_z Hf \wedge \neg \mathsf{K}_h \neg \mathsf{K}_z Hf} \text{ Moore's Paradox?}$$

5. 庄子: 请循其本. 子曰 '汝安知鱼乐' 云者, 既已知吾知之而问我. 我知之濠上也.

上帝存在的本体论论证?

1. God is a being which has every perfection.
2. Existence is a perfection.
3. Hence God exists.

$$E\left(\iota_x\left(\bigwedge_{i\in I} P_i x \wedge Ex\right)\right)$$

Problem: 这是否是循环论证?

上帝存在的本体论论证

- R: reality
- M: mind
- P: the positive qualities
- $x \in y$: x belongs to y

1. There exists a thing belonging to mind that has all the positive qualities and no negative quality.
$$\exists x[x \in M \land \forall y(y \in P \leftrightarrow x \in y)]$$
2. "Being real" is a positive quality.
$$R \in P$$
3. Two things belonging to mind that have exactly the same qualities are identical.
$$\forall xy[x \in M \land y \in M \to \forall z(x \in z \leftrightarrow y \in z) \to x = y]$$
4. God belongs to reality.
$$\iota_x[x \in M \land \forall y(y \in P \leftrightarrow x \in y)] \in R$$

Gödel's Proof of God's Existence

Ax.1 Either a property or its negation is positive, but not both. $\forall X[P(\neg X) \leftrightarrow \neg P(X)]$

Ax.2 A property necessarily implied by a positive property is positive.
$$\forall X \forall Y\, [P(X) \wedge \Box \forall x[X(x) \rightarrow Y(x)] \rightarrow P(Y)]$$

Th.1 Positive properties are possibly exemplified. $\forall X[P(X) \rightarrow \Diamond \exists x X(x)]$

Df.1 A *God-like* being possesses all positive properties. $G(x) \coloneqq \forall X[P(X) \rightarrow X(x)]$

Ax.3 The property of being God-like is positive. $P(G)$

Th.2 Possibly, God exists. $\Diamond \exists x G(x)$

Ax.4 Positive properties are necessarily positive. $\forall X[P(X) \rightarrow \Box P(X)]$

Df.2 An *essence* of an individual is a property necessarily implying any of its properties.
$$E(X,x) \coloneqq X(x) \wedge \forall Y(Y(x) \rightarrow \Box \forall y(X(y) \rightarrow Y(y)))$$

Th.3 Being God-like is an essence of any God-like being. $\forall x[G(x) \rightarrow E(G,x)]$

Df.3 *Necessary existence* of an individual is the necessary exemplification of all its essences. $N(x) \coloneqq \forall X[E(X,x) \rightarrow \Box \exists y X(y)]$

Ax.5 Necessary existence is a positive property. $P(N)$

Th.4 Necessarily, God exists. $\Box \exists x G(x)$

$$\frac{\text{任何经验都可能不可靠}}{\text{可能所有经验都不可靠}}\ ?$$

- $\forall x \Diamond A \to \Diamond \forall x A$?
- $\Diamond \forall x A \to \forall x \Diamond A$?

1. Barcan Formula

$$\forall x \Box A \to \Box \forall x A$$

holds in frame iff

$$Rwv \to D_v \subset D_w$$

2. The Converse of Barcan Formula

$$\Box \forall x A \to \forall x \Box A$$

holds in frame iff

$$Rwv \to D_w \subset D_v$$

Propositional Quantifiers

- I believe that everything I believe is true: $\mathsf{B}\forall p(\mathsf{B}p \to p)$.
- I know that there's a truth I don't know: $\mathsf{K}\exists p(p \wedge \neg \mathsf{K}p)$.
- a knows that b knows everything a knows: $\mathsf{K}_a\forall p(\mathsf{K}_a p \to \mathsf{K}_b p)$.
- There is a true proposition that necessarily implies every true proposition: $\exists p(p \wedge \forall q(q \to \Box(p \to q)))$.

$$[\![\forall p\varphi]\!]_V = \bigcap_{X \subset W} [\![\varphi]\!]_{V[X/p]}$$

- $[\![\forall p(\Box p \to p)]\!] = \{w \in W : Rww\}$
- $[\![\exists p(p \wedge \neg\Box p)]\!] = \{w \in W : \exists v \neq w : Rwv\}$
- $[\![\exists p(p \wedge \forall q(q \to \Box(p \to q)))]\!] = W$

Fitch's Paradox of Knowability

Theorem

If all truths are knowable, then all truths are known.

$$\forall p(p \to \Diamond \mathsf{K}p) \vdash \forall p(p \to \mathsf{K}p)$$

1. $\neg \mathsf{K}(p \wedge \neg \mathsf{K}p)$ 摩尔悖论
2. $\Box \neg \mathsf{K}(p \wedge \neg \mathsf{K}p)$ $\dfrac{\vdash A}{\vdash \Box A}$
3. $\neg \Diamond \mathsf{K}(p \wedge \neg \mathsf{K}p)$
4. $p \wedge \neg \mathsf{K}p$ 假设
5. $\Diamond \mathsf{K}(p \wedge \neg \mathsf{K}p)$ 前提 $p \to \Diamond \mathsf{K}p$
6. $\neg(p \wedge \neg \mathsf{K}p)$
7. $p \to \mathsf{K}p$

Is $\Diamond(\mathsf{K}p \vee \mathsf{K}\neg p)$?

- He who refuses to do arithmetic is doomed to talk nonsense.

  — *John McCarthy*

$$\forall x(\mathrm{Refuse}(x, \mathrm{arithmetic}) \to \Box\mathrm{TalkNonsense}(x))$$

- It is a truth universally acknowledged, that a single man in possession of a good fortune, must be in want of a wife.

  — *Jane Austen*

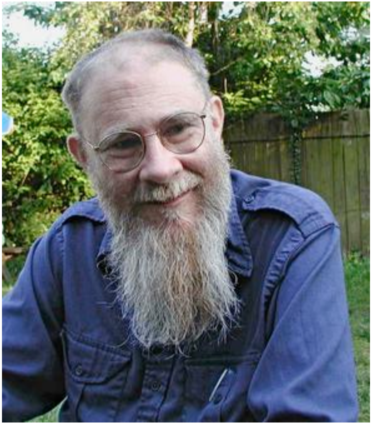
$$\mathrm{C_{Human}}\Bigg(\forall x\Bigg(\mathrm{Man}(x) \land \mathrm{Single}(x) \land \mathrm{Fortune}(x) \to \mathrm{Want}_x\Big(\exists y\big(\mathrm{Woman}(y) \land \mathrm{Wife}(y, x)\big)\Big)\Bigg)\Bigg)$$

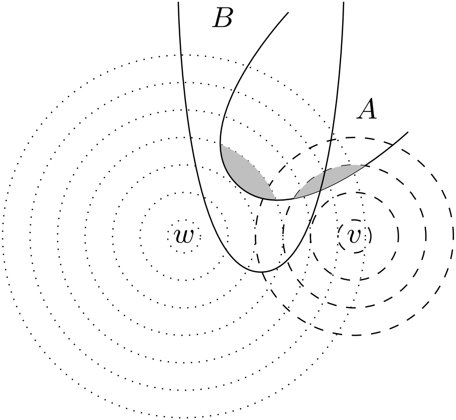
- 我们知道他们在说谎，他们也知道自己在说谎，他们也知道我们知道他们在说谎，我们也知道他们知道我们知道他们说谎，但是他们依然在说谎．

  — 索尔仁尼琴

Contents

Counterfactual Causation





$w \vDash A \;\Box\!\!\rightarrow B$

$v \nvDash A \;\Box\!\!\rightarrow B$

Indicative vs Counterfactual Conditional

- If Oswald <u>did</u> not kill Kennedy, someone else <u>did</u>.
- If Oswald <u>had</u> not kill<u>ed</u> Kennedy, someone else <u>would have</u>.

Antecedents and Consequents

$$\frac{\neg A}{A \rightarrow B} \checkmark \qquad \frac{\neg A}{A \mathrel{\Box\!\!\rightarrow} B} \times$$

$$\frac{\text{I did not strike the match}}{\text{if I had struck the match, it would have turned into a feather}} \ ?$$

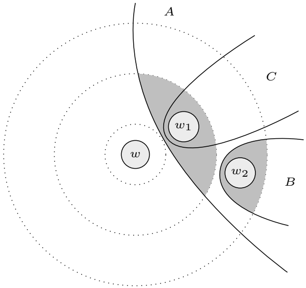
$$\frac{B}{A \rightarrow B} \checkmark \qquad \frac{B}{A \mathrel{\Box\!\!\rightarrow} B} \times$$

$$\frac{\text{George W. Bush won the } 2004 \text{ United States presidential election}}{\begin{array}{l}\text{If the newspapers had discovered beforehand that} \\ \text{Bush had an affair with Al Gore, he would still have won}\end{array}} \ ?$$

Antecedent Strengthening

$$\frac{A \rightarrow C}{A \wedge B \rightarrow C} \checkmark \qquad \frac{A \mathrel{\Box\!\!\rightarrow} C}{A \wedge B \mathrel{\Box\!\!\rightarrow} C} \times$$

$$\frac{\text{If I went to the beach, I would have a good time}}{\text{If I went to the beach and got attacked by a shark, I would have a good time}}$$



The closest world where I go to the beach and get attacked by a shark is much further removed from the actual world than the closest world where I go to the beach is.

Transitivity

$$\frac{A \to B \qquad B \to C}{A \to C} \ \checkmark \qquad \frac{A \mathrel{\Box\!\!\to} B \qquad B \mathrel{\Box\!\!\to} C}{A \mathrel{\Box\!\!\to} C} \ \times \qquad \frac{A \mathrel{\Box\!\!\to} B \qquad A \land B \mathrel{\Box\!\!\to} C}{A \mathrel{\Box\!\!\to} C} \ \checkmark$$

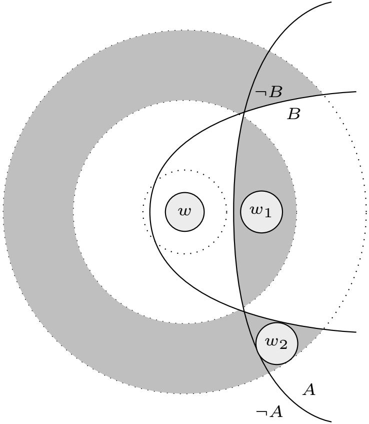
If I were king, I would wear a crown
If I wore a crown, people would find me ridiculous

If I were king, people would find me ridiculous ?

Contraposition

$$\frac{A \rightarrow B}{\neg B \rightarrow \neg A} \checkmark \qquad \frac{A \mathrel{\Box\!\!\rightarrow} B}{\neg B \mathrel{\Box\!\!\rightarrow} \neg A} \times$$

$$\frac{\text{If Goethe hadn't died in 1832, he would (still) be dead now}}{\text{If Goethe weren't dead now, he would have died in 1832}} \;?$$



Minimal Change Semantics

Definition (Sphere Model)

A *sphere model* is a triple $\mathcal{M} = \{W, O, V\}$, where $W \neq \emptyset$, V : Atom $\to \mathtt{P}(W)$, and $O : W \to \mathtt{P}(\mathtt{P}(W))$ assigns to each world w a *system of spheres* O_w. For each w, O_w is a set of sets of worlds such that:

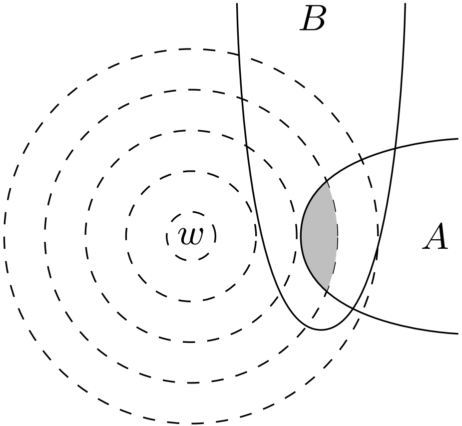
1. O_w is *centered* on w: $\{w\} \in O_w$.
2. O_w is *nested*: whenever $S_1, S_2 \in O_w$, $S_1 \subset S_2$ or $S_2 \subset S_1$.
3. O_w is closed under non-empty unions.
4. O_w is closed under non-empty intersections.

Definition (Counterfactual Conditional)

$\mathcal{M}, w \vDash A \;\Box\!\!\rightarrow B$ iff either

1. for all $v \in \bigcup O_w : \mathcal{M}, v \nvDash A$, or
2. for some $S \in O_w$,
   - 2.1 $\mathcal{M}, v \vDash A$ for some $v \in S$, and
   - 2.2 for all $v \in S : \mathcal{M}, v \vDash A \to B$.

Minimal Change Semantics



Problem

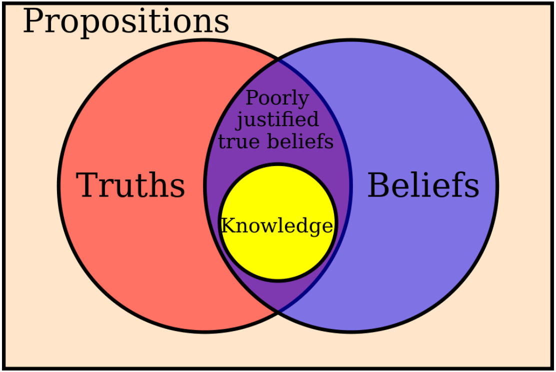
How do humans represent "possible worlds" in their minds and compute the closest one, when the number of possibilities is far beyond the capacity of the human brain?

Formal System

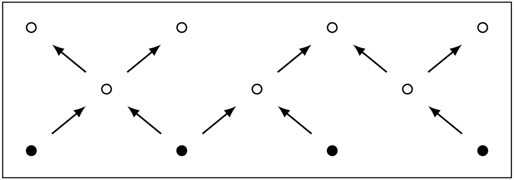
$$\frac{\bigwedge_{i=1}^{n} B_i \rightarrow C}{\bigwedge_{i=1}^{n} (A \mathrel{\Box\!\!\rightarrow} B_i) \rightarrow (A \mathrel{\Box\!\!\rightarrow} C)}$$

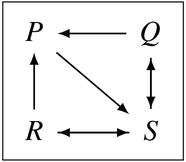
- $A \mathrel{\Box\!\!\rightarrow} A$
- $(A \mathrel{\Box\!\!\rightarrow} B) \wedge (B \mathrel{\Box\!\!\rightarrow} A) \rightarrow (A \mathrel{\Box\!\!\rightarrow} C) \leftrightarrow (B \mathrel{\Box\!\!\rightarrow} C)$
- $((A \vee B) \mathrel{\Box\!\!\rightarrow} A) \vee ((A \vee B) \mathrel{\Box\!\!\rightarrow} B) \vee (((A \vee B) \mathrel{\Box\!\!\rightarrow} C) \leftrightarrow (A \mathrel{\Box\!\!\rightarrow} C) \wedge (B \mathrel{\Box\!\!\rightarrow} C))$
- $(\neg A \mathrel{\Box\!\!\rightarrow} A) \rightarrow (B \mathrel{\Box\!\!\rightarrow} A)$
- $(A \mathrel{\Box\!\!\rightarrow} \neg B) \vee ((A \wedge B \mathrel{\Box\!\!\rightarrow} C) \leftrightarrow (A \mathrel{\Box\!\!\rightarrow} (B \rightarrow C)))$
- $(A \mathrel{\Box\!\!\rightarrow} B) \rightarrow (A \rightarrow B)$
- $A \wedge B \rightarrow (A \mathrel{\Box\!\!\rightarrow} B)$

Gettier Problem: Is Justified True Belief Knowledge?



1. Smith believes **“Bob owns a Ford”**.
2. He was told this by Bob.
3. Bob then sells his Ford.
4. Meanwhile Bob wins a Ford in a raffle.





- Internalist theory of epistemic justification.
  - Foundationalism
    — S's belief p is justified iff either
    (a) p is a basic belief; or
    (b) p is a non-basic belief justified inferentially by S's basic beliefs.
  - Coherentism
    — S's belief p is justified iff S has a coherent set of beliefs which includes p.
- Externalist theory of epistemic justification.
  - Reliabilism
    — S's belief p is justified iff p is produced by a reliable cognitive process.

证成难题

1. 一个信念只能由另一个信念证成.
2. 不允许循环证成.
3. 证成链条是有穷的.
4. 有被证成的信念.

- 上面四个命题不相容.
- 基础论者拒斥 1.
- 融贯论者拒斥 2.
- 皮尔士拒斥 3.
- 取消论者拒斥 4.

Nozick's Truth-Tracking Condition

Nozick's Truth-Tracking Condition

S knows that p iff:

1. p is true;
2. S believes that p;
3. if p were not true, S would not believe that p;
4. if p were true, S would believe that p.

$$\mathsf{K}p \coloneqq p \wedge \mathsf{B}p \wedge (\neg p \mathrel{\Box\!\!\rightarrow} \neg \mathsf{B}p) \wedge (p \mathrel{\Box\!\!\rightarrow} \mathsf{B}p)$$

Kripke's counter-example — Fake Barn Country

- Smith is driving in a country containing fake barns.
- The fake barns are painted green.
- In the midst of these fake barns is one real barn, which is painted red.
  1. Smith looks up and happens to see the real barn. **"I see a red barn."**
  2. What if Smith looks up and forms the belief **"I see a barn"**?
- Smith knows "there is a red barn", but doesn't know "there is a barn".

怀疑论 vs 反事实

S_1 I know "I have hands". $\mathsf{K}p$

S_2 I know if "I have hands" then "I am not a brain in the vat". $\mathsf{K}(p \to q)$

S_3 I don't know that "I am not a brain in the vat". $\neg\mathsf{K}q$

Remark: I fail to know that "I am not a brain in the vat" $\neg\mathsf{K}q$, since I would falsely believe "I was not a brain in the vat" in the closest world in which I am a brain in the vat.

- Nozick's definition of Knowledge

$$\mathsf{K}p := p \wedge \mathsf{B}p \wedge (\neg p \mathrel{\Box\!\!\to} \neg \mathsf{B}p) \wedge (p \mathrel{\Box\!\!\to} \mathsf{B}p)$$

then K is not closed under known entailment.

$$\mathsf{K}p, \mathsf{K}(p \to q) \nvDash \mathsf{K}q$$

- By Sosa's definition,

$$\mathsf{K}p := p \wedge \mathsf{B}p \wedge (\mathsf{B}p \mathrel{\Box\!\!\to} p) \wedge (p \mathrel{\Box\!\!\to} \mathsf{B}p)$$

then

$$\mathsf{K}p, \mathsf{K}(p \to q) \vDash \mathsf{K}q$$

Thank U