· Interpolation

· Lagrange

$$l_i(x) = \frac{\prod_{j=0, j\neq i}^{n} (x - x_j)}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)}$$

$$L_n(x) = \sum_{i=0} y_i \cdot l_i(x)$$

$$L_1(x) = \sum_{i=0}^{1} y_i \cdot l_i(x)$$

$$= \frac{(x-x_1)}{(x_0-x_1)} y_0 + \frac{(x-x_0)}{(x_1-x_0)} y_1$$

Example. Approximate $\sin(50^\circ)$ using Lagrange Interpolation.

Given $x_0 = \frac{\pi}{6}$, $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{3}$

$f(x_0) = \frac{1}{2}$, $f(x_1) = \frac{1}{\sqrt{2}}$, $f(x_3) = \frac{\sqrt{3}}{2}$.

Sol: $\sin(50^\circ) = \sin(\frac{\pi}{180} \cdot 50) = \sin(\frac{5}{18}\pi)$.

$n=1$, $L_1(x) = \frac{(x-x_1)}{(x_0-x_1)} y_0 + \frac{(x-x_0)}{(x_1-x_0)} y_1$

$$\Rightarrow L_1(\tfrac{5}{18}\pi) = \frac{(\frac{5\pi}{18} - \frac{\pi}{4})}{(\frac{\pi}{6} - \frac{\pi}{4})} \cdot \frac{1}{2} + \frac{(\frac{5}{18}\pi - \frac{\pi}{6})}{(\frac{\pi}{4} - \frac{\pi}{6})} \cdot \frac{1}{\sqrt{2}} \approx 0.77614$$

$f(x) = \sin(x)$, $f''(\sin x) = -\sin x$, $f''(\sin \xi_x) = -\sin \xi_x$

and $\xi_x \in (\frac{\pi}{6}, \frac{\pi}{3})$, $\frac{1}{2} \leq \sin(\xi_x) \leq \frac{\sqrt{3}}{2}$,

$$\Rightarrow R_1(x) = \frac{f''(\xi_x)}{2!}(x-x_0)(x-x_1) = \frac{f''(\xi_x)}{2!}(x - \tfrac{\pi}{6})(x - \tfrac{\pi}{4})$$

$$R_1(\tfrac{5}{18}\pi) = \frac{f''(\xi_x)}{2!}(\tfrac{5}{18}\pi - \tfrac{\pi}{6})(\tfrac{5}{18}\pi - \tfrac{\pi}{4})$$

$$\Rightarrow -0.013190321198 \leq R_1(\tfrac{5}{18}\pi) \leq -0.007615435$$

$n=2$, $\sin 50^\circ \approx 0.78543$

$$R_2(x) = \frac{f^{(3)}(x)}{3!}(x-x_0)(x-x_1)(x-x_2),$$

$0.00044 < R_2 < 0.00077$

$-0.01319 < R_1 < -0.00762$

Runge Phenomenon

· Runge function: $f(x) = \frac{1}{1+25x^2}$

The interpolation error tends toward infinity when degree of the polynomial increases.

· Error of $P_n(x)$: $\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)\cdots(x-x_n)$, $f(x) = \frac{1}{1+25x^2}$

$|f'(1)| \approx 0.0740$

$|f''(1)| \approx 0.2105$

· Practical Difficulty, $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 6$

$f(x) = e^x$

$$P_{1,2,4}(x) = \frac{(x-x_2)(x-x_4)}{(x_1-x_2)(x_1-x_4)}e^2 + \frac{(x-x_1)(x-x_4)}{(x_2-x_1)(x_2-x_4)}e^3 + \frac{(x-x_1)(x-x_2)}{(x_4-x_1)(x_4-x_2)}e^6$$

$$\Rightarrow P(x) = \frac{(x-x_j)P_{0,1,\dots,j-1,j+1,\dots,k}(x) - (x-x_i)P_{0,1,\dots,i-1,i+1,\dots,k}(x)}{(x_i - x_j)}$$

k^{th} Lagrange polynomial interpolate f at $k+1$ points, $(0,\dots,k)$.

Here comes the proof, hold on...

$$P(x) = \frac{(x-x_j)\hat{Q}(x) - (x-x_i)Q(x)}{(x_i-x_j)}$$

$$r \neq i,j: P(x_r) = \frac{(x_r-x_j)\hat{Q}(x_r) - (x_r-x_i)Q(x_r)}{(x_i-x_j)}$$

$$= Q(x_r)(x_r-x_j-x_r+x_i)/(x_i-x_j)$$

$$= f(x_r)$$

$$r = i : \quad P(x_i) = \frac{(x_i - x_j)\hat{Q}(x_i) - (x_i - x_i)Q(x_i)}{x_i - x_j}$$
$$= f(x_i)$$

Neville's Method

	$f(x)$	$L_1(x)$	$L_2(x)$	$L_3(x)$	$L_4(x)$
x_0	$P_0 = Q_{0,0}$				
x_1	$P_1 = Q_{1,0}$	$P_{0,1} = Q_{1,1}$			
x_2	$P_2 = Q_{2,0}$	$1,2$ $Q_{2,1}$	$P_{0,1,2}$ Q_{22}		
x_3	$P_3 = Q_{3,0}$	$Q_{3,1}$	$1,2,3$ Q_{32}	$P_{0,1,2,3}$ Q_{33}	
x_4	$P_4 = Q_{4,0}$	$Q_{4,1}$	Q_{42}	$1,2,4$ Q_{34}	$P_{0,1,2,3,4}$ Q_{44}

$Q_{i,j} \Rightarrow$ index

$P_{i,j,k\ldots}$

对角线递增

P_0

P_1 $\quad P_{0,1}$

$P_{1,2}$ $\quad P_{0,1,2}$

Let's see whats happening.

$$Q_{0,0} = P_0(x) = \frac{(x - x_1)}{x_0 - x_1} f(x), \quad P_0(x_0) = \frac{(x_0 - x_1)}{(x_0 - x_1)} f(x_0) = f(x_0)\ !!$$

that's why $P_0, P_1, \ldots, P_4 = f(x_i)$ $i: 0 \sim 4$

Let's see $Q_{11} = P_{0,1}$.

$$P_{0,1}(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1) = Q_{11}.$$

$$\Downarrow \quad \frac{(x - x_0)Q_{10} - (x - x_1)Q_{00}}{\underset{x_j}{}\ (x_1 - x_0)\ \underset{x_i}{}}, \quad \text{for } Q \begin{cases} (x - x_j) : i, j-1 \\ (x - x_i) : i-1, j-1 \end{cases}$$

$Q_{2,1} = P_{1,2}$,

$$P_{1,2}(x) = \frac{(x - x_1)Q_{2,0} - (x - x_2)Q_{10}}{x_2 - x_1}$$

Example:

$f(x)$	$L_1(x)$	$L_2(x)$
$P_0 = Q_{00} = a$		
$P_1 = Q_{10} = b$	$P_{0,1} = Q_{1,1}$	
$P_2 = Q_{20} = c$	$P_{1,2} = Q_{2,1}$	$P_{0,1,2} = Q_{2,2} = 0.7420$

$x = 2.1$

i	x_i	$\ln x_i$
0	2.0	0.6931 a
1	2.2	0.7885 b
2	2.3	0.8329 c

$$Q_{11} = \frac{(x-x_0)Q_{10} - (x-x_1)Q_{00}}{x_1 - x_0}$$

$$|f(2.1) - P_2(2.1)| = \frac{f'''(\xi_x)}{3!}(x-x_0)(x-x_1)(x-x_2)$$

Devided Difference

0-th: $f[x_i] = f(x_i)$

1-th: $f[x_i, x_{i+1}] = \dfrac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$

2-th: $f[x_i, x_{i+1}, x_{i+2}] = \dfrac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$

k-th:

$$f[x_i, x_{i+1}, \cdots, x_{i+k}] = \frac{f[x_{i+1}, \cdots, x_{i+k}] - f[x_i, \cdots, x_{i+k-1}]}{x_{i+k} - x_i}$$

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)\cdots(x-x_n)$$

$P_n(x_0) = a_0 = f(x_0)$

$P_n(x_1) = a_0 + a_1(x-x_0)$

$a_1 = \dfrac{f(x_1) - f(x_0)}{(x-x_0)}$

$$P_n(x) = f(x_0) + \sum_{i=1}^{n} f[x_0, \cdots, x_k] \cdot (x-x_0)\cdots(x-x_{k-1})$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

$$f[x_3, x_4] = \frac{f(x_4) - f(x_3)}{x_4 - x_3}$$

$$f[x_4, x_5] = \frac{f(x_5) - f(x_4)}{x_5 - x_4}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

$$f[x_2, x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$$

$$f[x_3, x_4, x_5] = \frac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}.$$

$$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$$

$$f[x_2, x_3, x_4, x_5] = \frac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2}.$$

Formulae with equal spacing.

$x_i = x_0 + ih \quad (i = 0, \cdots, n)$

Forward diff:

- $\Delta f_i = f_{i+1} - f_i$
- $\Delta^k f_i = \Delta^{k-1} (\Delta f_i) = \Delta^{k-1} (f_{i+1}) - \Delta^{k-1} (f_i)$

Backward diff:

- $\nabla f_i = f_i - f_{i-1}$
- $\nabla^k f_i = \nabla^{k-1} f_i - \nabla^{k-1} f_{i-1}$

Newton's Forward diff Formular.

$$P_n(x) = f(x_0) + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$$

$$+ \cdots + f[x_0,\cdots,x_n](x-x_0)\cdots(x-x_{n-1})$$

$$x = x_0 + th \quad , \quad P_n(x) = P_n(x_0 + th)$$

$$= \sum_{k=0}^{n} \binom{t}{k} \Delta^k f(x_0) \quad , \quad R_n(t) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} t(t-1)\cdots(t-n) h^{n+1}$$

Backward formular

$$P_n(x) = f(x_n) + f[x_{n,n-1}](x-x_n) + f[x_{n,n-1,n-2}](x-x_n)(x-x_{n-1})$$

$$+ \cdots + f[x_n,\cdots,x_0](x-x_n)\cdots(x-x_1)$$

$$x = x_n + th \quad , \quad P_n(t) = P_n(x_n+th) = \sum_{k=0}^{n} (-1)^k \binom{-t}{k} \nabla^k f(x_n)$$

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x	-2	-1	0	1	2
$f(x)$	-1	3	1	-1	3

Backward: Degree 3.

$$P(x) = f(x_4) + f[x_4,x_3](x-x_4) + f[x_4,x_3,x_2](x-x_4)(x-x_3) + f[x_4,x_3,x_2,x_1](x-x_4)(x-x_3)(x-x_2)$$

$$\to 3 + \frac{f(3)-f(4)}{x_3-x_4}(x-3)$$

?

Hermite Interpolation.