Lecture 8

Angular momentum

8.1 Introduction

Now that we have introduced three-dimensional systems, we need to introduce into our quantum-mechanical framework the concept of *angular momentum*.

Recall that in classical mechanics angular momentum is defined as the vector product of position and momentum:

$$\underline{L} \equiv \underline{r} \times \underline{p} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} . \tag{8.1}$$

Note that the angular momentum is itself a *vector*. The three Cartesian components of the angular momentum are:

$$L_x = y\,p_z - z\,p_y, \qquad L_y = z\,p_x - x\,p_z, \qquad L_z = x\,p_y - y\,p_x\,. \tag{8.2}$$

8.2 Angular momentum operator

For a quantum system the angular momentum is an observable, we can *measure* the angular momentum of a particle in a given quantum state. According to the postulates that we have spelled out in previous lectures, we need to associate to each observable a Hermitean operator. We have already defined the operators $\underline{\hat{X}}$ and $\underline{\hat{P}}$ associated respectively to the position and the momentum of a particle. Therefore we can define the *operator*

$$\underline{\hat{L}} \equiv \underline{\hat{X}} \times \underline{\hat{P}}\,, \tag{8.3}$$

where $\underline{\hat{P}} = -i\hbar\underline{\nabla}$. Note that in order to define the angular momentum, we have used the definitions for the position and momentum operators *and* the expression for the angular momentum in classical mechanics. Eq. (8.3) yields explicit expressions for the components of the angular momentum as differential operators:

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\,, \qquad \hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\,, \qquad \hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\,. \tag{8.4}$$

Eq. (8.4) can be economically rewritten as:

$$\hat{L}_i = -i\hbar\,\varepsilon_{ijk}\,x_j\,\frac{\partial}{\partial x_k}\,, \tag{8.5}$$

where we have to sum over the repeated indices.

Mathematical aside

In Eq. (8.5) we have used the same convention introduced in Lecture 7; we use:

$$x_1 = x, \quad x_2 = y, \quad x_3 = z\,, \tag{8.6}$$

to denote the three components of the position vector. The same convention is also used for the partial derivatives:

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial x_2} = \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x_3} = \frac{\partial}{\partial z}\,. \tag{8.7}$$

In general the components of a vector $\underline{V}$ can labeled as:

$$V_1 = V_x, \quad V_2 = V_y, \quad V_3 = V_z\,. \tag{8.8}$$

The symbol ε_{ijk} denotes the totally antisymmetric unit tensor:

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \quad \text{cyclic indices} \tag{8.9}$$

$$\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1, \quad \text{anticyclic indices} \tag{8.10}$$

Out of twenty-seven components, only the six above are actually different from zero. Check that you understand Eq. (8.5).

The following relations are useful:

$$\varepsilon_{ikl}\varepsilon_{imn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}\,, \tag{8.11}$$

$$\varepsilon_{ikl}\varepsilon_{ikm} = 2\delta_{lm}\,, \tag{8.12}$$

$$\varepsilon_{ikl}\varepsilon_{ikl} = 6\,. \tag{8.13}$$

Using the canonical commutation relations, Eq. (7.17), we can easily prove that:

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \qquad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \qquad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y\,. \tag{8.14}$$

The proof of this statement is left as an exercise in problem sheet 4. Once again, it is useful to get familiar with the more compact notation:

$$\boxed{\left[\hat{L}_i, \hat{L}_j\right] = i\hbar\,\varepsilon_{ijk}\hat{L}_k}\,. \tag{8.15}$$

Example Instead of using the canonical commutation relations, we can derive the commutation relations bewteen the components L_i using their representation as differential operators.

$$
\begin{aligned}
\hat{L}_x \hat{L}_y &= -\hbar^2 \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \\
&= -\hbar^2 \left\{ y\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial z \partial x} - yx\frac{\partial^2}{\partial z^2} - z^2\frac{\partial^2}{\partial y \partial x} + zx\frac{\partial^2}{\partial y \partial z} \right\} ,
\end{aligned}
$$

whilst

$$
\begin{aligned}
\hat{L}_y \hat{L}_x &= -\hbar^2 \left((z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \left((y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \\
&= -\hbar^2 \left\{ zy\frac{\partial^2}{\partial x \partial z} - z^2\frac{\partial^2}{\partial x \partial y} - xy\frac{\partial^2}{\partial z^2} + xz\frac{\partial^2}{\partial z \partial y} + x\frac{\partial}{\partial y} \right\}
\end{aligned}
$$

Noting the usual properties of partial derivatives

$$
\frac{\partial^2}{\partial x \partial z} = \frac{\partial^2}{\partial z \partial x} , \quad \text{etc} \tag{8.16}
$$

we obtain on subtraction the desired result:

$$
\left[\hat{L}_x, \hat{L}_y \right] = \hbar^2 \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) = i\hbar \hat{L}_z . \tag{8.17}
$$

Note that the Cartesian components of the angular momentum **do not** commute with each other. Following our previous discussion on compatible observables, this means that the components are **not** compatible observables. We cannot measure, for instance, L_x and L_y simultaneously, and we do not have a basis of common eigenfunctions of the two operators. Physically, this also implies that measuring one component of the angular momentum modifies the probability of finding a given result for the other two.

Angular momentum plays a central role in discussing *central potentials*, i.e. potentials that only depend on the radial coordinate r. It will also prove useful to have expression for the operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ in spherical polar coordinates. Using the expression for the Cartesian coordinates as functions of the spherical ones, and the chain rule for the derivative, yields

$$
\begin{aligned}
\hat{L}_x &= i\hbar \left(\sin\phi \frac{\partial}{\partial \theta} + \cot\theta \, \cos\phi \frac{\partial}{\partial \phi} \right) \\
\hat{L}_y &= i\hbar \left(-\cos\phi \frac{\partial}{\partial \theta} + \cot\theta \, \sin\phi \frac{\partial}{\partial \phi} \right) \\
\hat{L}_z &= -i\hbar \frac{\partial}{\partial \phi} .
\end{aligned}
$$

8.3 Square of the angular momentum

Let us now introduce an operator that represents the square of the magnitude of the angular momentum:

$$\hat{L}^2 \equiv \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \sum_{i=1}^{3} \hat{L}_i^2\,, \tag{8.18}$$

or, in spherical polar coordinates

$$\hat{L}^2 \equiv -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]. \tag{8.19}$$

The importance of this observable is that *it is compatible with any of the Cartesian components of the angular momentum*;

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0\,. \tag{8.20}$$

Sample proof. Consider for instance the commutator $[\hat{L}^2, \hat{L}_z]$:

$$\begin{aligned}
[\hat{L}^2, \hat{L}_z] &= [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \hat{L}_z] \quad \text{from the definition of } \hat{L}^2 \\
&= [\hat{L}_x^2, \hat{L}_z] + [\hat{L}_y^2, \hat{L}_z] + [\hat{L}_z^2, \hat{L}_z] \\
&= [\hat{L}_x^2, \hat{L}_z] + [\hat{L}_y^2, \hat{L}_z] \quad \text{since } \hat{L}_z \text{ commutes with itself} \\
&= \hat{L}_x \hat{L}_x \hat{L}_z - \hat{L}_z \hat{L}_x \hat{L}_x + \hat{L}_y \hat{L}_y \hat{L}_z - \hat{L}_z \hat{L}_y \hat{L}_y\,.
\end{aligned}$$

We can use the commutation relation $[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$ to rewrite the first term on the RHS as

$$\hat{L}_x \hat{L}_x \hat{L}_z = \hat{L}_x \hat{L}_z \hat{L}_x - i\hbar \hat{L}_x \hat{L}_y\,,$$

and the second term as

$$\hat{L}_z \hat{L}_x \hat{L}_x = \hat{L}_x \hat{L}_z \hat{L}_x + i\hbar \hat{L}_y \hat{L}_x\,.$$

In a similar way, we can use $[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$ to rewrite the third term as

$$\hat{L}_y \hat{L}_y \hat{L}_z = \hat{L}_y \hat{L}_z \hat{L}_y + i\hbar \hat{L}_y \hat{L}_x\,,$$

and the fourth term

$$\hat{L}_z \hat{L}_y \hat{L}_y = \hat{L}_y \hat{L}_z \hat{L}_y - i\hbar \hat{L}_x \hat{L}_y\,,$$

thus, on substituting in we find that

$$[\hat{L}^2, \hat{L}_z] = -i\hbar \hat{L}_x \hat{L}_y - i\hbar \hat{L}_y \hat{L}_x + i\hbar \hat{L}_y \hat{L}_x + i\hbar \hat{L}_x \hat{L}_y = 0\,.$$

QED

8.4 Eigenfunctions

The compatibility theorem tells us that $\hat{L}^2$ and $\hat{L}_z$ thus have *simultaneous eigenfunctions*. These turn out to be the *spherical harmonics*, $Y_\ell^m(\theta,\phi)$. In particular, the eigenvalue equation for $\hat{L}^2$ is

$$\hat{L}^2\,Y_\ell^m(\theta,\phi) = \ell(\ell+1)\hbar^2\,Y_\ell^m(\theta,\phi)\,, \tag{8.21}$$

where $\ell = 0,\,1,\,2,\,3,\ldots$ and

$$Y_\ell^m(\theta,\phi) = (-1)^m\left[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos\theta)\exp{(im\phi)}\ , \tag{8.22}$$

with $P_\ell^m(\cos\theta)$ known as the *associated Legendre polynomials.* Some examples of spherical harmonics will be given below.

The eigenvalue $\ell(\ell+1)\hbar^2$ is *degenerate*; there exist $(2\ell+1)$ eigenfunctions corresponding to a given ℓ and they are distinguished by the label m which can take any of the $(2\ell+1)$ values

$$m = \ell,\,\ell-1,\,\ldots,\,-\ell\,, \tag{8.23}$$

In fact it is easy to show that m labels the eigenvalues of $\hat{L}_z$. Since

$$Y_\ell^m(\theta,\phi) \sim \exp{(im\phi)}\,, \tag{8.24}$$

we obtain directly that

$$\hat{L}_z\,Y_\ell^m(\theta,\phi) \equiv -i\hbar\frac{\partial}{\partial\phi}\,Y_\ell^m(\theta,\phi) = m\hbar\,Y_\ell^m(\theta,\phi), \tag{8.25}$$

confirming that the spherical harmonics are also eigenfunctions of $\hat{L}_z$ with eigenvalues $m\hbar$.

Mathematical aside

A few examples of spherical harmonics are

$$\begin{aligned}
Y_0^0(\theta,\phi) &= \frac{1}{\sqrt{4\pi}}\\
Y_1^0(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta\\
Y_1^1(\theta,\phi) &= -\sqrt{\frac{3}{8\pi}}\sin\theta\exp{(i\phi)}\\
Y_1^{-1}(\theta,\phi) &= \sqrt{\frac{3}{8\pi}}\sin\theta\exp{(-i\phi)}\,.
\end{aligned}$$

8.5 Physical interpretation

We have arrived at the important conclusion that *angular momentum is quantised*. The square of the magnitude of the angular momentum can only assume one of the discrete set of values

$$\ell(\ell+1)\hbar^2, \qquad \ell = 0,\, 1,\, 2,\, \ldots$$

and the z-component of the angular momentum can only assume one of the discrete set of values

$$m\hbar, \qquad m = \ell,\, \ell-1,\, \ldots,\, -\ell$$

for a given value of ℓ.

ℓ and m are called the *angular momentum quantum number* and the *magnetic quantum number* respectively.

Finally a piece of jargon: we refer to a particle in a state with angular momentum quantum number ℓ as *having angular momentum* ℓ, rather than saying, more clumsily but accurately, that it has angular momentum of magnitude $\sqrt{\ell(\ell+1)}\,\hbar$.