T-AM3: an introduction to bivariate signal processing

Polarization, quaternions and geometric representations

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Outline and tutorial objectives

Give an introduction to **bivariate signal processing (BSP)**,
focusing on **geometric** and **physical** properties

Today's agenda

I. Introduction to bivariate signals

II. Polarization: a key geometric concept for BSP

III. Building a quaternion-domain framework for BSP

IV. The framework in action: equipping BSP with new tools in
- spectral analysis
- linear filtering
- time-frequency representations

V. Conclusions, current challenges and open questions in BSP

Hands-on BSP with **BiSPy** `pip install bispy-polar`

$\rightarrow$ play around with jupyter notebooks during the tutorial!

References and supporting material

Journal papers

- J. Flamant, N. Le Bihan, P. Chainais. "Time-frequency analysis of bivariate signals", Appl. Comp. Harm. Anal., 2019
- J. Flamant, N. Le Bihan, P. Chainais. "Spectral analysis of stationary random bivariate signals", IEEE TSP, 2017
- J. Flamant, P. Chainais, N. Le Bihan. "A complete framework for linear filtering of bivariate signals", IEEE TSP, 2018

Manuscript

- J. Flamant, "A general approach for the analysis and filtering of bivariate signals", Ph.D. thesis, 2018

Tutorial material

- https://ricochet-anr.github.io/ressources/
- </> Link to Notebooks (google colab)

What is a bivariate signal?

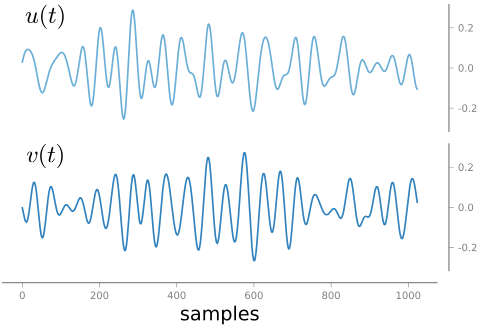
A **bivariate signal** is defined by 2 **components** $u(t)$ and $v(t)$

Different viewpoints ...

temporal evolution of the individual components $u(t)$ and $v(t)$

$\downarrow$

bivariate signal dynamics



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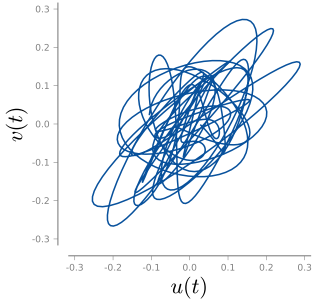

joint evolution of the components $u(t)$ and $v(t)$

$\downarrow$

bivariate signal geometry (or polarization)

What is a bivariate signal?

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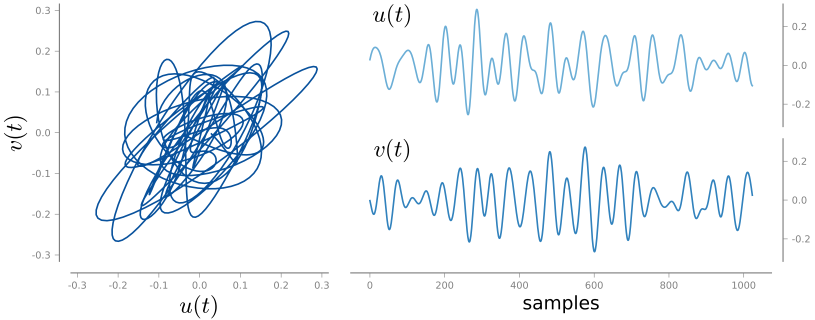
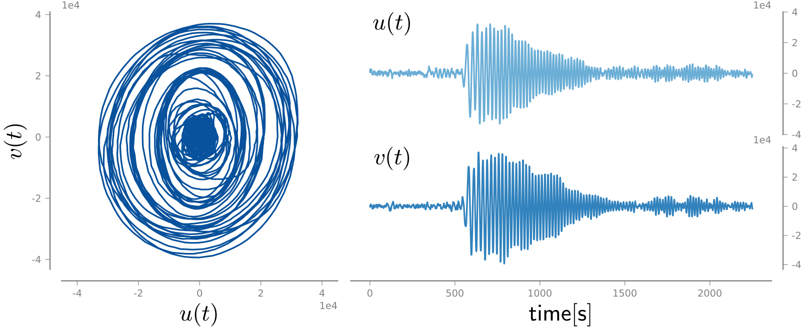
Different viewpoints ...



... must be considered simultaneously for a complete description

A tour of physical applications

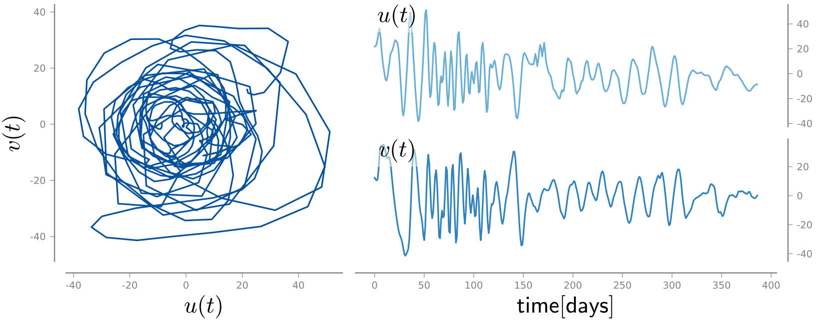
Example 1: polarized seismic waves



1991 Solomon Islands Earthquake data [Lilly et al., 1995]

- recorded on a 3-axes accelerometer at Pasadena Station
- $u(t)$ radial amplitude, $v(t)$ vertical amplitude
- elliptical $u - v$ motion $\rightarrow$ Rayleigh wave

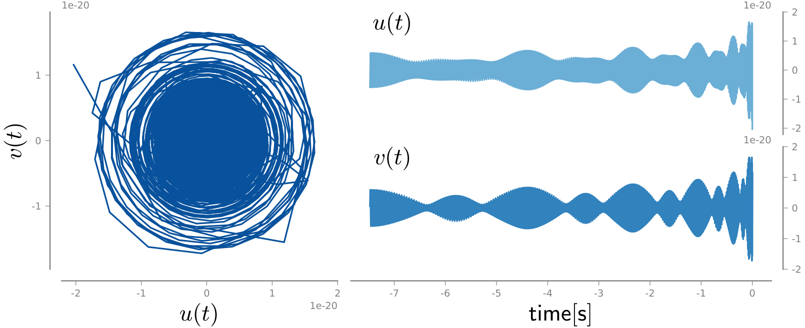
Example 2: ocean current velocities



Current velocities in Atlantic ocean [NOAA, available from JLab]

- recorded from a float deployed in Atlantic ocean
- $u(t)$ eastward velocity, $v(t)$ northward velocity
- tells us about eddys dynamics in oceans

Example 3: gravitational waves in precessing binaries



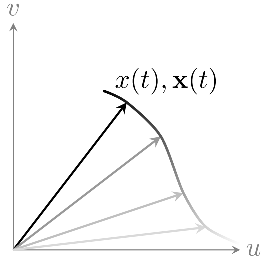
Synthetic data from GW simulations [Cano et al., 2023]

- Einstein's general relativity predicts two *modes* (called polarization) for GWs
- $u(t)$ "plus"-polarization, $v(t)$ "cross"- polarization
- polarizations encode the dynamics (e.g. precession) of the GW source

Bivariate signals and representations

Different but equivalent representations

$$\text{vector} \quad \mathbf{x}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \qquad \text{complex} \quad x(t) = u(t) + \boldsymbol{i}v(t)$$



bivariate signal processing

$\longleftrightarrow$

tools for the joint analysis / processing of 2 components $u(t)$ and $v(t)$

Vector representation of bivariate signals

$$\boxed{\mathbf{x}(t) = [u(t), v(t)]^\top \in \mathbb{R}^2}$$

- Special case of analysis of multivariate vector signals

  Hannan (1970), Priestley (1981)

- Polarization in optics

  Born and Wolf (1980), Goodman (1984), Mandel and Wolf (1995)

- Jones matrix-vector calculus

  Jones (1941), Azzam and Bashara (1978)

- Instantaneous polarization attributes in seismology

  Diallo et al. (2005), Roueff et al. (2006)

Complex representation of bivariate signals

$$\boxed{x(t) = u(t) + \boldsymbol{i}v(t) \in \mathbb{C}}$$

- Circularity of random complex signals (rotational invariance)

  Picinbono (1994), Amblard et al. (1996)

- Augmented representations ${}_{\uparrow}\mathbf{x}(t) = [x(t), \overline{x(t)}]^\top$

  Schreier and Scharf (2003, 2010), Adalı et al. (2011)

- Rotary components $x \equiv \sum \circlearrowleft + \sum \circlearrowright$

  Blanc-Lapierre and Fortet (1953), Gonella (1972), Walden (2013)

- Instantaneous ellipse description for nonstationary bivariate signals

  deterministic Lilly and Olhede (2010) random Schreier (2008)

  EMD Rilling et al. (2007)

Bivariate signals basics

Geometric and polarization parameters

What are "good" parameters for bivariate signals?

Recall the univariate case

- the simplest signal is the monochromatic signal with frequency $f_0 \geq 0$

$$x(t) = A\cos(2\pi f_0 t + \Phi)$$

2 key parameters: **amplitude** $A \geq 0$ and initial **phase** $\Phi \in (-\pi, \pi)$

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  2 key parameters: **amplitude** $A \geq 0$ and initial **phase** $\Phi \in (-\pi, \pi)$
- for an arbitrary $x(t)$, can be generalized to:
  - frequency-dependent amplitude and phase

$$X(f) = A(f)e^{\boldsymbol{i}\Phi(f)}, \text{ where } X(f) = \mathcal{F}\{x\}(f)$$

  - instantaneous amplitude and phase

$$x_+(t) = a(t)e^{\boldsymbol{i}\Phi(t)}, \text{ where } x_+(t) \text{ is the analytic signal of } x(t)$$

  - time-frequency-dependent amplitude and phase

$$V_g x(t,f) = a(t,f)e^{\boldsymbol{i}\Phi(t,f)}, \text{ where } V_g x(t,f) \text{ is the STFT of } x(t)$$

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how to generalize such parameters to the bivariate case?

Monochromatic bivariate signal

Definition for a frequency $f_0 \geq 0$,

$$\mathbf{x}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} A_u \cos(2\pi f_0 t + \Phi_u) \\ A_v \cos(2\pi f_0 t + \Phi_v) \end{bmatrix}$$

2 **amplitudes** $A_u, A_v \geq 0$ and 2 **phases** $\Phi_u, \Phi_v \in (-\pi, \pi)$

Issue: parameterization does not tell about joint structure of $\mathbf{x}(t)$

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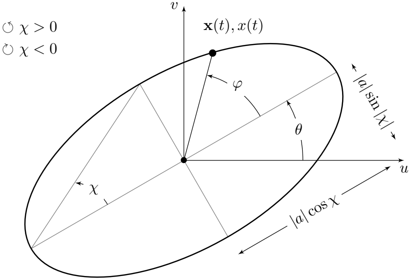
Issue: parameterization does not tell about joint structure of $\mathbf{x}(t)$

Physics-inspired solution: polarization ellipse parameters



4 parameters:

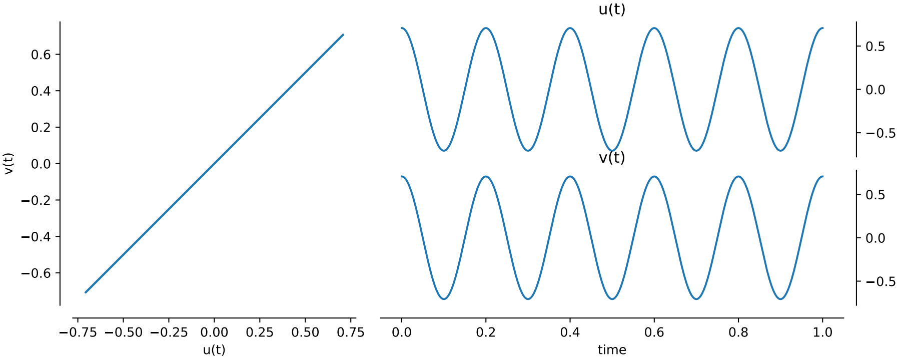
2 classical

- intensity $a \geq 0$ ($\equiv$ size)
- initial phase $\varphi \in (-\pi, \pi)$

2 geometric (new!)

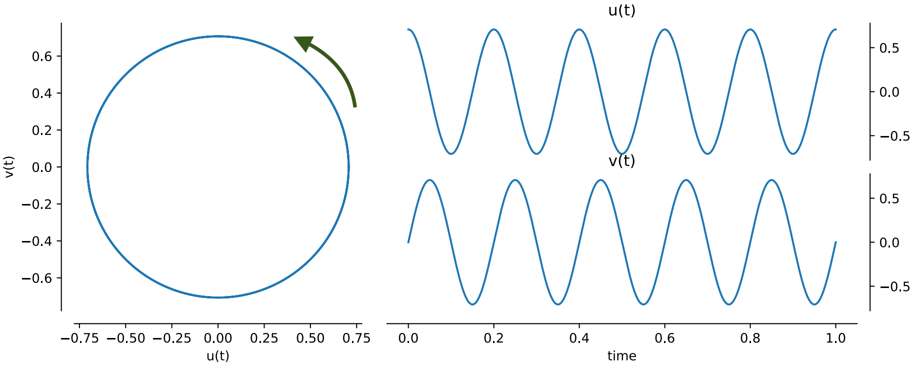
- orientation $\theta \in [-\pi/2, \pi/2]$
- ellipticity $\chi \in [-\pi/4, \pi/4]$

Some examples: linear polarization



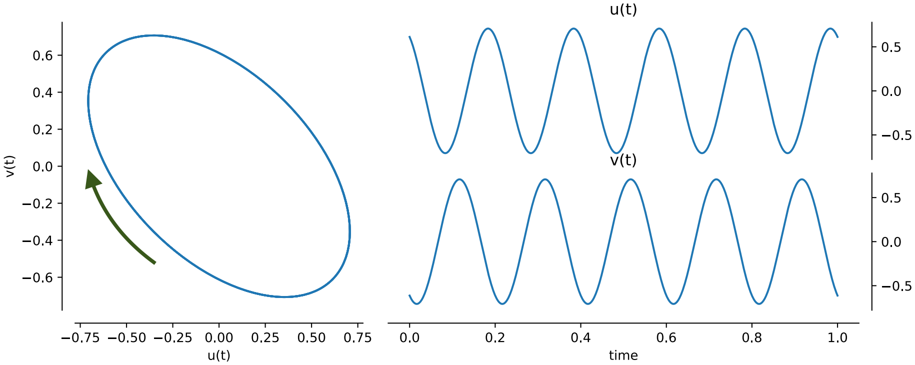
- intensity $a = 1$
- orientation $\theta = \frac{\pi}{4}$
- ellipticity $\chi = 0$
- initial phase $\varphi = 0$

Some examples: right-handed circular polarization



- intensity $a = 1$
- orientation $\theta = 0$ (by convention, or undefined)
- ellipticity $\chi = \frac{\pi}{4}$
- initial phase $\varphi = 0$

Some examples: elliptical polarization



- intensity $a = 1$
- orientation $\theta = -\frac{\pi}{4}$
- ellipticity, $\chi = -\frac{\pi}{6}$
- initial phase $\varphi = 0$

Stokes parameters and Poincaré sphere

aka measurable polarization parameters in optics

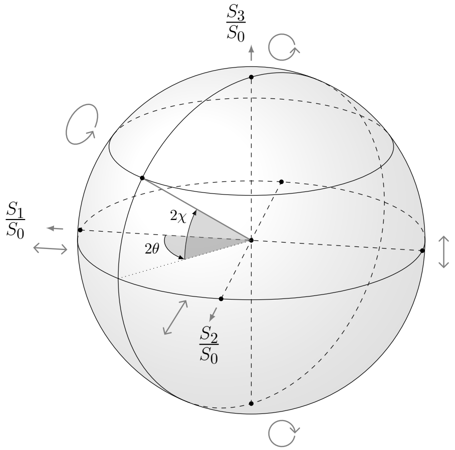
(deterministic) Stokes parameters

$$S_0 = a^2$$
$$S_1 = a^2 \cos 2\chi \cos 2\theta$$
$$S_2 = a^2 \cos 2\chi \sin 2\theta$$
$$S_3 = a^2 \sin 2\chi$$

Normalized Stokes parameters

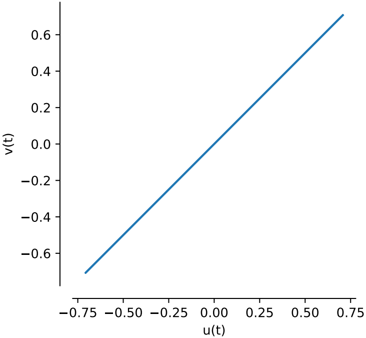
$$\left(\frac{S_1}{S_0}, \frac{S_2}{S_0}, \frac{S_3}{S_0}\right) \longleftrightarrow (2\theta, 2\chi)$$

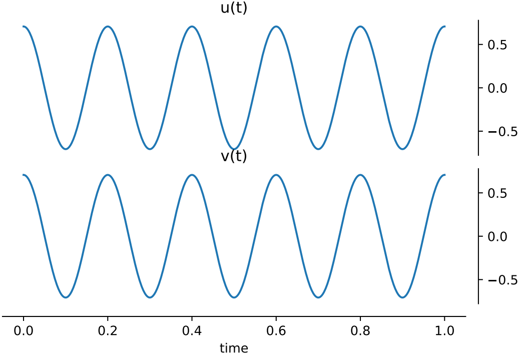
Stokes parameters are 2nd order quantities: $S_i \propto a^2$



Poincaré sphere

Some examples: linear polarization





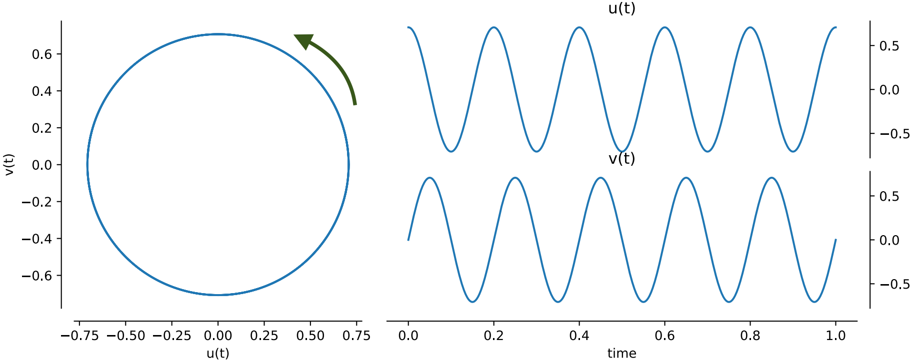
Geometric domain

- intensity $a = 1$
- orientation $\theta = \frac{\pi}{4}$
- ellipticity $\chi = 0$
- initial phase $\varphi = 0$

Stokes (energetic) domain

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Some examples: right-handed circular polarization



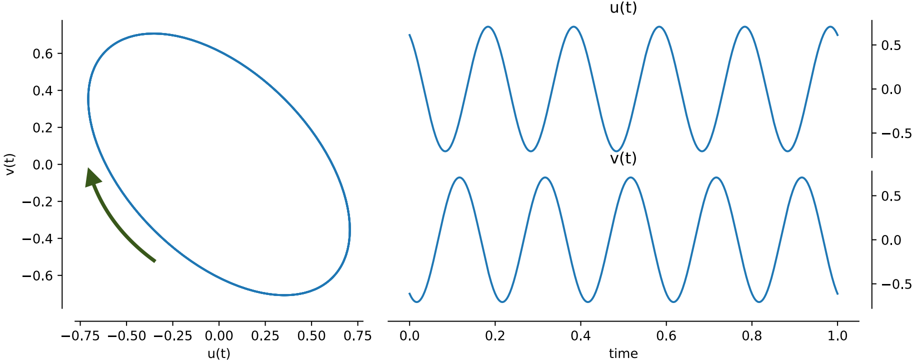
Geometric domain

- intensity $a = 1$
- orientation $\theta = 0$
- ellipticity $\chi = \frac{\pi}{4}$
- initial phase $\varphi = 0$

Stokes (energetic) domain

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Some examples: elliptical polarization



Geometric domain

- intensity $a = 1$
- orientation $\theta = -\frac{\pi}{4}$
- ellipticity, $\chi = -\frac{\pi}{6}$
- initial phase $\varphi = 0$

Stokes (energetic) domain

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

Recap: polarization parameters for bivariate signals

- geometric polarization ellipse parameters

  amplitude a, phase φ, orientation θ, ellipticity χ

- Stokes parameters: 2nd order (energetic) quantities

  $[S_0, S_1, S_2, S_3]^\top$ or S_0 and normalized Stokes $[\frac{S_1}{S_0}, \frac{S_2}{S_0}, \frac{S_3}{S_0}]^\top$

- geometric bridge: Poincaré sphere

Going beyond the monochromatic setting (signal processing after all)?

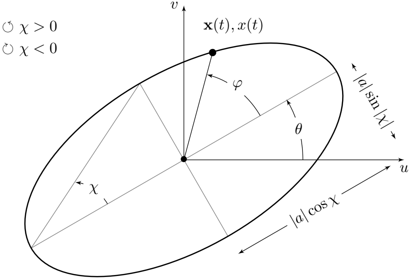
$\downarrow$

introduce dependence of polarization features
e.g. time t, frequency f, time-frequency (t, f), etc.

What about quaternions?

towards straightforward physical
and geometric interpretations

Monochromatic bivariate signal representations



Polarization ellipse parameters

- $a \geq 0$ intensity
- $\theta \in [-\pi/2, \pi/2]$ orientation
- $\chi \in [-\pi/4, \pi/4]$ ellipticity
- $\varphi(t) \in [0, 2\pi)$ phase

Jones vector: Vector representation (optics, seismology)

$$\mathbf{x}(t) = \begin{bmatrix} A_u \cos(2\pi\nu_0 t + \Phi_u) \\ A_v \cos(2\pi\nu_0 t + \Phi_v) \end{bmatrix} \quad \theta, \chi \leftarrow f(A_u, A_v, \Phi_u, \Phi_v)$$

Rotary components: Complex representation (oceanography, SPTM)

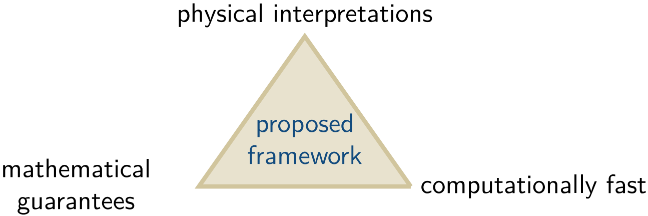
$$x(t) = A_+ e^{\boldsymbol{i}\theta_+} e^{\boldsymbol{i}2\pi\nu_0 t} \qquad \theta \leftarrow g_1(\theta_+, \theta_-)$$

$$+ A_- e^{-\boldsymbol{i}\theta_-} e^{-\boldsymbol{i}2\pi\nu_0 t} \qquad \chi \leftarrow g_2(A_+, A_-)$$

The need for interpretability

existing approaches: no straightforward physical descriptions

feature	$\mathbf{x}(t) \in \mathbb{R}^2$	$x(t) \in \mathbb{C}$	desired
direct ellipse parametrization	✗	✗	✓
positive frequencies	✓	✗	✓
interpretable filtering relations	✗	✗	✓



$\Downarrow$

efficient, relevant generalization of ubiquitous signal processing tools

Framework

Ingredient $\#1$ bivariate signal as complex signal embedded in $\mathbb{H}$

$$x(t) = u(t) + \boldsymbol{i}v(t) \in \mathbb{C}_{\boldsymbol{i}} \subset \mathbb{H}$$

alike embedding a signal $x(t) \in \mathbb{R}$ into $\mathbb{C}$

Quaternions

4D algebra $\quad \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1 \quad$ ⚠ $\boldsymbol{ij} = \boldsymbol{k}$, $\boldsymbol{ij} = -\boldsymbol{ji}$ ⚠

complex subfields of $\mathbb{H}$: $\mathbb{C}_{\boldsymbol{i}} = \mathrm{Span}\,\{1, \boldsymbol{i}\}$, $\mathbb{C}_{\boldsymbol{j}} = \mathrm{Span}\,\{1, \boldsymbol{j}\}$, ...

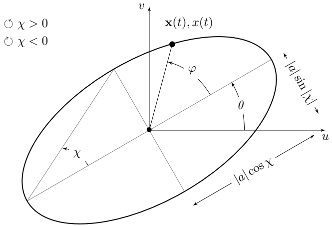
polar forms, 3D and 4D geometry, etc.

Ingredient #2 adapt Fourier transform

Quaternion Fourier Transform (QFT)

$$X(f) = \int \underbrace{x(t)}_{\in \mathbb{C}_{\boldsymbol{i}}} \underbrace{e^{-\boldsymbol{j}2\pi f t}}_{\in \mathbb{C}_{\boldsymbol{j}}} \mathrm{d}t \in \mathbb{H}$$

Monochromatic polarized signal



polar form by Bülow and Sommer (2001)

$$x(t) = \mathrm{Proj}_{\mathbb{C}_{\boldsymbol{i}}}\left\{ a e^{\boldsymbol{i}\theta} e^{-\boldsymbol{k}\chi} e^{\boldsymbol{j}(2\pi f_0 t + \varphi)} \right\}$$

$$\updownarrow \text{ QFT}$$

$$X(f) = a e^{\boldsymbol{i}\theta} e^{-\boldsymbol{k}\chi} e^{\boldsymbol{j}\varphi} \delta_{f_0}(f) + \text{sym.}$$

polar form $\leftrightarrow$ physical parameters

QFT properties

Existence for L^1, L^2 functions Jamison Ph.D. thesis (1970)

Easy to compute

$$x(t) = u(t) + \boldsymbol{i} v(t) \overset{\text{QFT}}{\longleftrightarrow} X(f) = \underbrace{U(f)}_{1,\boldsymbol{j}} + \underbrace{\boldsymbol{i} V(f)}_{\boldsymbol{i},\boldsymbol{k}}$$

For bivariate signals keep $f \geq 0$ only (*$\boldsymbol{i}$-Hermitian symmetry*)

$$X(-f) = -\boldsymbol{i} X(f) \boldsymbol{i}, \quad \text{for } x(t) \in \mathbb{C}_{\boldsymbol{i}}$$

2 invariants for finite energy signals (QFT Parseval theorem)

$$\int_{-\infty}^{+\infty} |x(t)|^2 \,\mathrm{d}t = \int_{-\infty}^{+\infty} |X(f)|^2 \,\mathrm{d}f \qquad \text{(energy)}$$

$$\boxed{\int_{-\infty}^{+\infty} x(t) \boldsymbol{j} \overline{x(t)} \,\mathrm{d}t = \int_{-\infty}^{+\infty} \underbrace{X(f) \boldsymbol{j} \overline{X(f)}}_{\in \operatorname{span}\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}} \,\mathrm{d}f \qquad \text{(geometry)}}$$

Basic quaternion calculus shows that:

$$|X(f)|^2 = S_0(f), \quad X(f)\boldsymbol{j}\overline{X(f)} = \boldsymbol{i}S_3(f) + \boldsymbol{j}S_1(f) + \boldsymbol{k}S_2(f)$$

QFT Parseval invariants $\longleftrightarrow$ Stokes parameters $\longleftrightarrow$ spectral densities

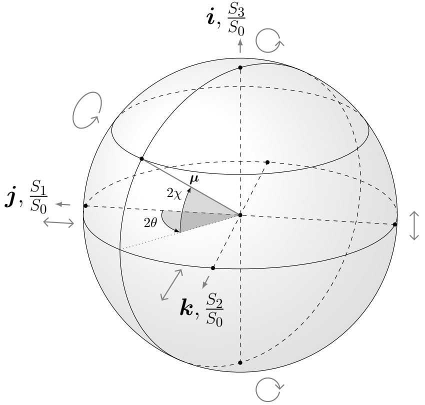
Polarization axis $\boldsymbol{\mu}$ ($\equiv$ vector in $\mathbb{R}^3$)

pure unit quaternion ($\boldsymbol{\mu}^2 = -1$) s.t.

$$X(f)\boldsymbol{j}\overline{X(f)} = S_0(f)\boldsymbol{\mu}(f)$$

Quaternion energy spectral density

$$\Gamma(f) = |X(f)|^2 + X(f)\boldsymbol{j}\overline{X(f)} \in \mathbb{H}$$



Summary: univariate vs bivariate signals

topic	univariate $x(t) \in \mathbb{R}$	bivariate $x(t) \in \mathbb{C}_{\boldsymbol{i}}$
Fourier	$X(f) = \int x(t) e^{-\boldsymbol{j}2\pi f t} \mathrm{d}t \in \mathbb{C}_{\boldsymbol{j}}$	$X(f) = \int x(t) e^{-\boldsymbol{j}2\pi f t} \mathrm{d}t \in \mathbb{H}$
Symmetry	$X(-f) = \overline{X(f)}$	$X(-f) = -\boldsymbol{i}X(f)\boldsymbol{i}$
Parameters	a, φ	a, θ, χ, φ
Parseval inv.	$\lvert X(f)\rvert^2$	$\lvert X(f)\rvert^2$, $X(f)\boldsymbol{j}\overline{X(f)}$
FT cost	1 FFT	2 FFTs

Play around!

See notebook 1: BiSPy basics and the monochromatic signal example

The framework in action

Spectral analysis and estimation

Spectral representation

Context

- second-order stationary (SOS) bivariate signals $x(t) \in \mathbb{C}$
- index t can be continuous or discrete

How to extend the quaternion spectral representation for such signals?

QFT spectral representation theorem

Let $x(t) \in \mathbb{C}$ be zero-mean SOS. Then

$$x(t) \overset{\text{m. s.}}{=} \int_{-\infty}^{+\infty} \mathrm{d}X(\nu) e^{\boldsymbol{j} 2\pi\nu t}$$

where $\mathrm{d}X(\nu)$ are spectral increments satisfying

1. $\forall \nu \in \mathbb{R}, \mathbf{E}\left\{\mathrm{d}X(\nu)\right\} = 0$
2. $\forall \nu, \nu' \in \mathbb{R}$ s.t. $\nu \neq \nu'$

$$\mathbf{E}\left\{\mathrm{d}X(\nu)\overline{\mathrm{d}X(\nu')}\right\} = \mathbf{E}\left\{\mathrm{d}X(\nu)\boldsymbol{j}\overline{\mathrm{d}X(\nu')}\right\} = 0$$

Quaternion Power Spectral Density: definition

Definition (with spectral increments)

$$\Gamma_{xx}(\nu)\mathrm{d}\nu = \underbrace{\mathbf{E}\left\{|\mathrm{d}X(\nu)|^2\right\}}_{\text{power } \in\mathbb{R}_+} + \underbrace{\mathbf{E}\left\{\mathrm{d}X(\nu)\boldsymbol{j}\overline{\mathrm{d}X(\nu)}\right\}}_{\text{geometry } \in \text{span}\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}}$$

This definition

- relies on the spectral representation theorem for the QFT
- contains the complete second-order information
- is interpretable as a density thanks to QFT Parseval theorem
- has a straightforward connection to frequency-dependent Stokes parameters

$\rightarrow$ Frequency-dependent polarization description of bivariate signals

$$\Gamma_{xx}(\nu) = \underbrace{S_0(\nu)}_{\text{power } \in \mathbb{R}^+} + \underbrace{S_0(\nu)\Phi_x(\nu)\boldsymbol{\mu}_x(\nu)}_{\text{geometry } \in \text{ span}\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}}$$

Poincaré sphere of polarization states

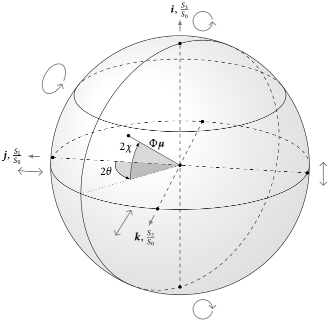
$\rightarrow$ geometric description of the covariance structure at every ν

- polarization axis $\boldsymbol{\mu}_x(\theta, \chi)$

  "average elliptical trajectory"

- degree of polarization $\Phi_x \in [0, 1]$

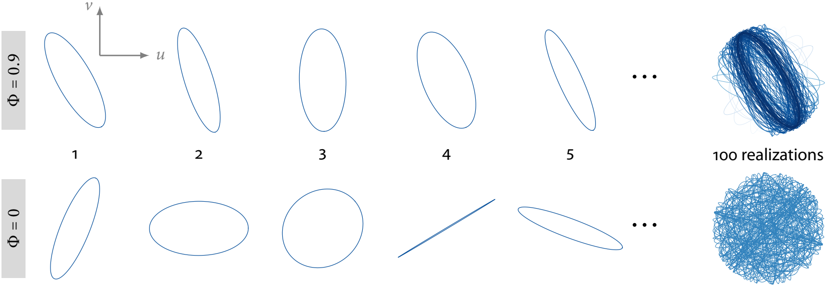
  "statistical dispersion of ellipses"



Understanding the degree of polarization

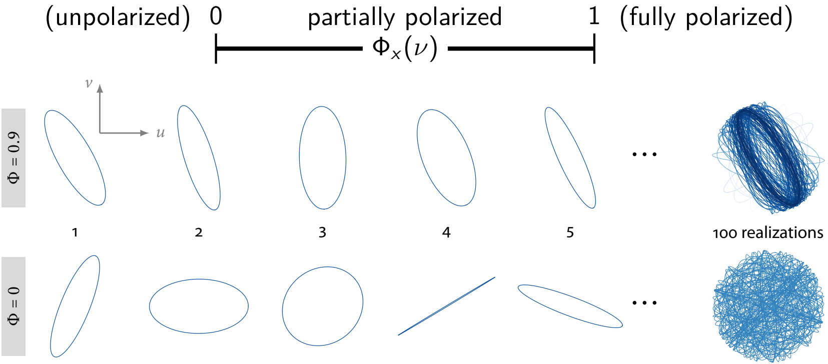
(unpolarized) 0 partially polarized 1 (fully polarized)

$\Phi_x(\nu)$



$$\Phi_x(\nu) = \frac{\text{intensity of the polarized part}}{\text{total intensity}} = \frac{\sqrt{S_1^2(\nu) + S_2^2(\nu) + S_3^2(\nu)}}{S_0(\nu)}$$

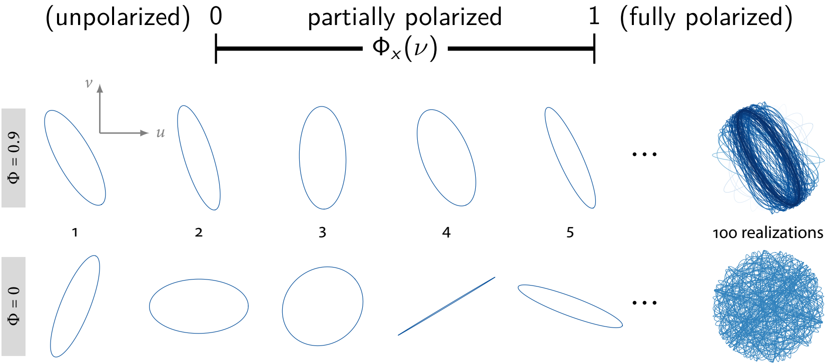
Understanding the degree of polarization



Decomposition into Unpolarized and Polarized parts

$$\Gamma_{xx}(\nu) = \underbrace{\Gamma^{\mathrm{u}}_{xx}(\nu)}_{\text{unpolarized part}} + \underbrace{\Gamma^{\mathrm{p}}_{xx}(\nu)}_{\text{polarized part}}$$

Understanding the degree of polarization



Decomposition into Unpolarized and Polarized parts

$$x(t) = x^{\mathrm{u}}(t) + x^{\mathrm{p}}(t) \qquad ?$$

Bivariate white Gaussian noise

Definition

$$w[n] = u[n] + \boldsymbol{i} v[n] \text{ biv. wGn} \iff \begin{cases} u[n] \\ v[n] \end{cases} \text{ correlated univariate wGn}$$

Quaternion power spectral density

$$\Gamma_{ww}(\nu) = \underbrace{\sigma_u^2 + \sigma_v^2}_{\text{total power}} + \underbrace{\boldsymbol{j}(\sigma_u^2 - \sigma_v^2) + 2\boldsymbol{k}\rho_{uv}\sigma_u\sigma_v}_{\text{geometric part}}$$

Quick facts

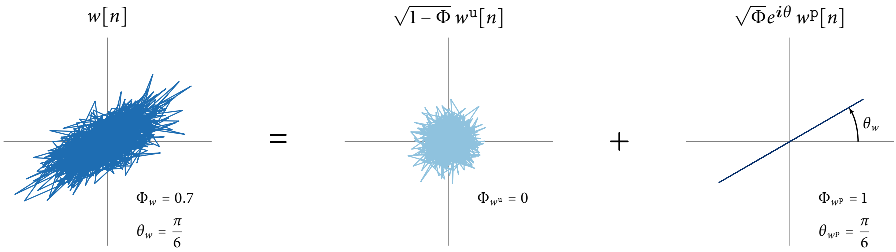
- PSD is constant
- no $\boldsymbol{i}$-component: $w[n]$ is *partially linearly polarized* for all ν
- unpolarized iff $\sigma_u = \sigma_v$ and $\rho_{uv} = 0$
- fully polarized iff $|\rho_{uv}| = 1$

Bivariate wGn synthesis with UP decomposition

Unpolarized-Polarized decomposition of the quaternion PSD

$$\forall \nu, \quad \Gamma_{xx}(\nu) = \underbrace{\Gamma^{\mathrm{u}}_{xx}(\nu)}_{\text{unpolarized part}} + \underbrace{\Gamma^{\mathrm{p}}_{xx}(\nu)}_{\text{polarized part}}$$

Example: bivariate wGn $\Phi = 0.7$, $\theta = \pi/6$



bivariate white noise — unpolarized white noise — polarized white noise

can be extended to more sophisticated processes, e.g. fGn [Lefèvre et al., 2018]

Nonparametric spectral estimation

Estimate Γ_{xx} given a realization $x[1], x[2], \cdots, x[N]$

Polarization periodogram

$$\hat{\Gamma}_{xx}^{(p)}(\nu) = \frac{\Delta_t}{N} \left| \sum_{n=1}^{N} x[n] e^{-\boldsymbol{j} 2\pi \nu n \Delta_t} \right|^2 + \frac{\Delta_t}{N} \left(\sum_{n=1}^{N} x[n] e^{-\boldsymbol{j} 2\pi \nu n \Delta_t} \right) \boldsymbol{j} \left(\overline{\sum_{n=1}^{N} x[n] e^{-\boldsymbol{j} 2\pi \nu n \Delta_t}} \right)$$

also: tapered and multitaper estimates

usual estimators and tradeoffs from spectral analysis apply

estimation of polarization parameters require special care

Degree of polarization estimation

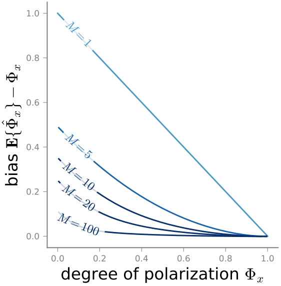
Naive estimator

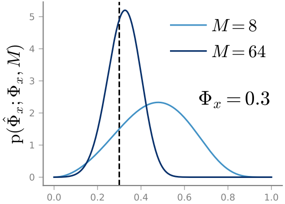
almost systematically biased

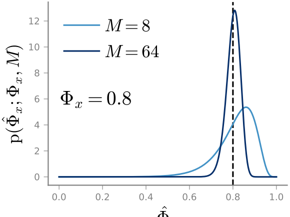
$$\hat{\Phi}_x^{(p)}(\nu) = \frac{|\mathcal{V}(\hat{\Gamma}_{xx}^{(p)}(\nu))|}{\mathcal{S}(\hat{\Gamma}_{xx}^{(p)}(\nu))} = 1$$

Estimation by averaging

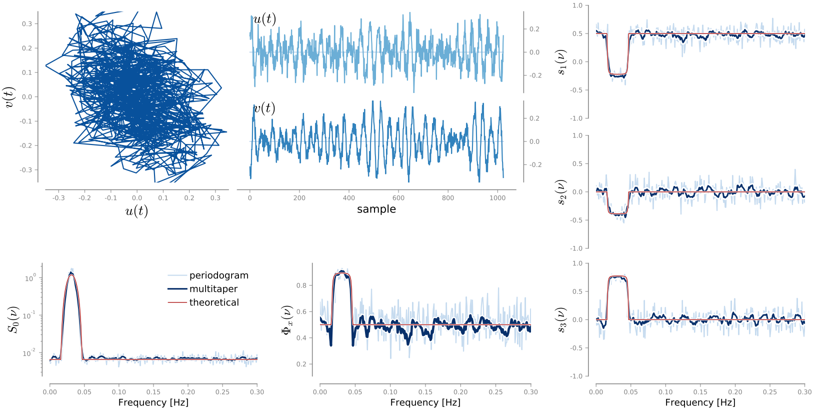
$$\hat{\Phi}_x^{M}(\nu) = \frac{|\sum_{m=1}^{M} \mathcal{V}(\hat{\Gamma}_{xx}^{m}(\nu))|}{\sum_{m=1}^{M} \mathcal{S}(\hat{\Gamma}_{xx}^{m}(\nu))}$$







Nonparametric spectral estimation example (see notebook)



observations $y = x + w$ with

- x SOS narrow-band signal, elliptically polarized ($\Phi_x = 0.9$)
- w partially horizontally polarized wGn ($\Phi_w = 0.4$)

The framework in action

Linear time-invariant filtering

LTI filtering

2 types of LTI filters $\longleftrightarrow$ 2 fundamental physical properties

$$\begin{cases} \text{Unitary filters} & \text{(birefringence)} \\ \text{Hermitian filters} & \text{(diattenuation or dichroism)} \end{cases}$$

QFT framework features

- spectral domain expressions
- explicit control of eigenproperties
- compact and straightforward expressions
- geometric handling of polarization effects (Poincaré sphere)

✓easy to design / prescribe ✓standard SP tasks

✓original decompositions

Main reference

J. Flamant, P. Chainais, N. Le Bihan. "A complete framework for linear filtering of bivariate signals", IEEE TSP, 2018

Unitary filters (birefringence)

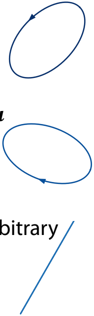
interpretation: 3D rotation of $\boldsymbol{\mu}_x$ by $(\boldsymbol{\mu}, \beta)$

eigenpolarizations

$\boldsymbol{\mu}_x(\nu) = \boldsymbol{\mu}$

$\boldsymbol{\mu}_x(\nu) = -\boldsymbol{\mu}$

$\boldsymbol{\mu}_x(\nu) =$ arbitrary



$x(t)$ → Unitary filter (phase φ) → $y(t)$

birefringence $\begin{cases} \text{axis} & \boldsymbol{\mu} \\ \text{angle} & \beta \in [0, 2\pi] \end{cases}$

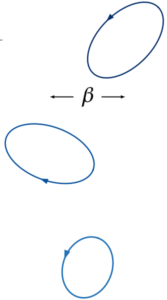
Filtering relation

$$Y(\nu) = e^{\boldsymbol{\mu}(\nu)\frac{\beta(\nu)}{2}} X(\nu) e^{\boldsymbol{j}\varphi(\nu)}$$

Quaternion PSD

$$S_{0,y}(\nu) = S_{0,x}(\nu) \quad \text{and} \quad \Phi_y(\nu) = \Phi_x(\nu)$$

$$\boldsymbol{\mu}_y(\nu) = e^{\boldsymbol{\mu}(\nu)\frac{\beta(\nu)}{2}} \boldsymbol{\mu}_x(\nu) e^{-\boldsymbol{\mu}(\nu)\frac{\beta(\nu)}{2}}$$



$(\varphi(\nu) = 0)$

$\pm\boldsymbol{\mu}(\nu) \leftrightarrow$ eigenpolarizations $\qquad \varphi(\nu) \pm \beta(\nu)/2 \leftrightarrow$ eigenvalues

Hermitian filters (diattenuation)

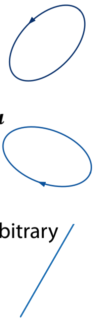
gain depending on the alignment $\langle \boldsymbol{\mu}_x, \boldsymbol{\mu} \rangle$

eigenpolarizations

$\boldsymbol{\mu}_x(\nu) = \boldsymbol{\mu}$

$\boldsymbol{\mu}_x(\nu) = -\boldsymbol{\mu}$

$\boldsymbol{\mu}_x(\nu) =$ arbitrary



$x(t)$ — gain K — Hermitian filter — $y(t)$

diattenuation axis $\boldsymbol{\mu}$

polarizing power $\eta \in [0,1]$

Filtering relation

$$Y(\nu) = K(\nu)\left[X(\nu) - \eta(\nu)\boldsymbol{\mu}(\nu)X(\nu)\boldsymbol{j}\right]$$

Quaternion PSD

$$S_{0,y} = S_{0,x}K^2\left[1 + \eta^2 + 2\eta\langle \boldsymbol{\mu}_x, \boldsymbol{\mu} \rangle\right]$$

$$\Phi_y S_{0,y}\boldsymbol{\mu}_y = S_{0,x}K^2\left[2\eta\boldsymbol{\mu} + \Phi_x(\boldsymbol{\mu}_x - \eta^2\boldsymbol{\mu}\boldsymbol{\mu}_x\boldsymbol{\mu})\right]$$

$\times \frac{1+\eta(\nu)}{2}$

$\times \frac{1-\eta(\nu)}{2}$

$\left(K(\nu) = \frac{1}{2}\right)$

$\pm\boldsymbol{\mu}(\nu) \leftrightarrow$ eigenpolarizations $\qquad K(\nu)\,(1 \pm \eta(\nu)) \leftrightarrow$ eigenvalues

Decompositions by linear filtering

Polarized and Unpolarized parts of Γ_{xx}

$$\Gamma_{xx}(\nu) = S_0(\nu) + S_0(\nu)\Phi_x(\nu)\boldsymbol{\mu}_x(\nu)$$

Decompositions by linear filtering

Polarized and Unpolarized parts of Γ_{xx}

$$\Gamma_{xx}(\nu) = \underbrace{\Phi_x(\nu) S_0(\nu)[1 + \boldsymbol{\mu}_x(\nu)]}_{\text{polarized part}} + \underbrace{[1 - \Phi_x(\nu)] S_0(\nu)}_{\text{unpolarized part}}$$

Decompositions by linear filtering

Polarized and Unpolarized parts of Γ_{xx}

$$\Gamma_{xx}(\nu) = \underbrace{\Gamma^{\mathtt{p}}_{xx}(\nu)}_{\text{polarized part}} + \underbrace{\Gamma^{\mathtt{u}}_{xx}(\nu)}_{\text{unpolarized part}}$$

Goal find $x(t) = x_a(t) + x_b(t)$ s.t. $\Phi_{x_a} = 1$ and $\boldsymbol{\mu}_{x_a} = \boldsymbol{\mu}_x$ and

1. $\Gamma_{x_a x_a}(\nu) = \Gamma^{\mathtt{p}}_{xx}(\nu)$
2. $\Gamma_{x_b x_b}(\nu) = \Gamma^{\mathtt{u}}_{xx}(\nu)$
3. x_a, x_b are uncorrelated

Decompositions by linear filtering

Polarized and Unpolarized parts of Γ_{xx}

$$\Gamma_{xx}(\nu) = \underbrace{\Gamma_{xx}^{\mathtt{p}}(\nu)}_{\text{polarized part}} + \underbrace{\Gamma_{xx}^{\mathtt{u}}(\nu)}_{\text{unpolarized part}}$$

Goal find $x(t) = x_a(t) + x_b(t)$ s.t. $\Phi_{x_a} = 1$ and $\boldsymbol{\mu}_{x_a} = \boldsymbol{\mu}_x$ and
impossible to fulfill 1-2-3 all together

1. $\Gamma_{x_a x_a}(\nu) = \Gamma_{xx}^{\mathtt{p}}(\nu)$
2. $\Gamma_{x_b x_b}(\nu) = \Gamma_{xx}^{\mathtt{u}}(\nu)$
3. x_a, x_b are uncorrelated

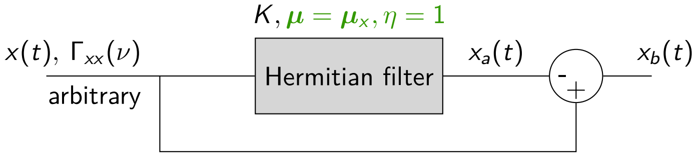
Decompositions by linear filtering

Polarized and Unpolarized parts of Γ_{xx}

$$\Gamma_{xx}(\nu) = \underbrace{\Gamma^{\mathtt{p}}_{xx}(\nu)}_{\text{polarized part}} + \underbrace{\Gamma^{\mathtt{u}}_{xx}(\nu)}_{\text{unpolarized part}}$$

Goal find $x(t) = x_a(t) + x_b(t)$ s.t. $\Phi_{x_a} = 1$ and $\boldsymbol{\mu}_{x_a} = \boldsymbol{\mu}_x$ and
impossible to fulfill 1-2-3 all together

1. $\Gamma_{x_a x_a}(\nu) = \Gamma^{\mathtt{p}}_{xx}(\nu)$
2. $\Gamma_{x_b x_b}(\nu) = \Gamma^{\mathtt{u}}_{xx}(\nu)$
3. x_a, x_b are uncorrelated



decomposition is ruled by the gain K

Decompositions by linear filtering: condition 2

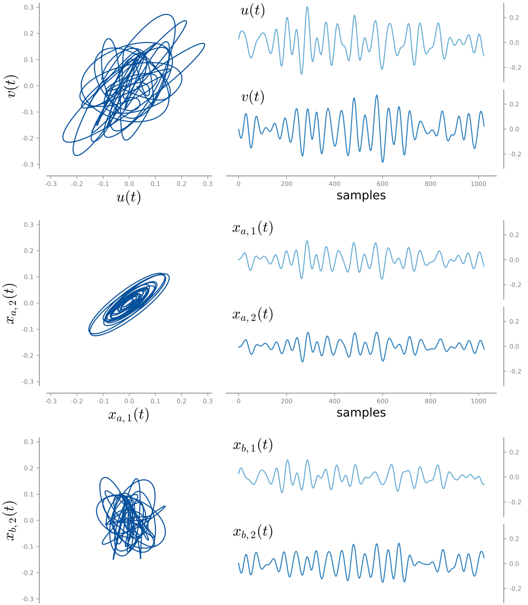
$K = f(\Phi_x)$ $\qquad x(t)$

$=$

polarized $\qquad x_a(t)$

(correlated) $\qquad +$

unpolarized $\qquad x_b(t)$



Decompositions by linear filtering: condition 3

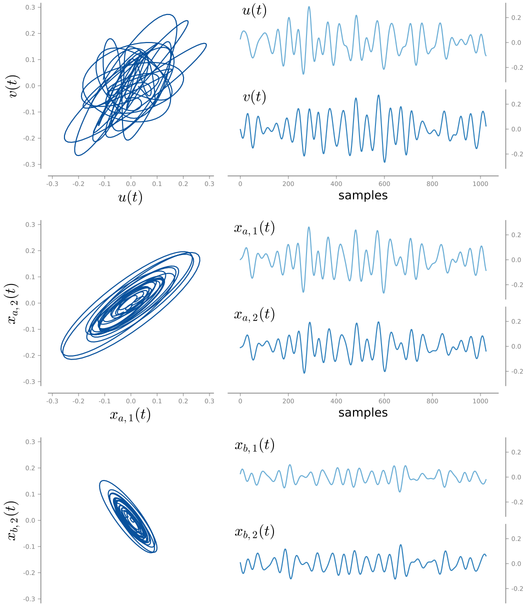
$K = 1/2$ $\quad x(t)$

$=$

polarized $\quad x_a(t)$

(uncorrelated) $\quad +$

polarized $\quad x_b(t)$



Recap: spectral analysis and LTI filtering

Spectral analysis

- definition of quaternion PSD for SOS bivariate signals
- the degree of polarization encodes statistical dispersion of ellipses
- Wiener-Khintchine theorem: quaternion autocovariance
- non-parametric PSD estimation methods extend nicely to the QFT

LTI filtering

- 2 types of filters: unitary and hermitian
- QFT-domain enables simple parameterization of eigenproperties
- Several applications
  - polarized-unpolarized decomposition
  - spectral synthesis from wGn
  - Wiener filtering

Play around!

See notebook 2: Spectral analysis and linear filtering

The framework in action

Time-frequency analysis

What are we after for bivariate signals ?

Key ingredients

- A concept similar to the *analytic signal*

  univariate: $\mathbb{R} \longrightarrow \mathbb{C}$
  bivariate: $\mathbb{R}^2 \longrightarrow \mathbb{C}^2$

- A *bivariate* version for *intantaneous phase* and *amplitude*

  more phases ?

- A bivariate theory that *generalizes* the univariate case

  Symetries, STFT, wavelets, etc.

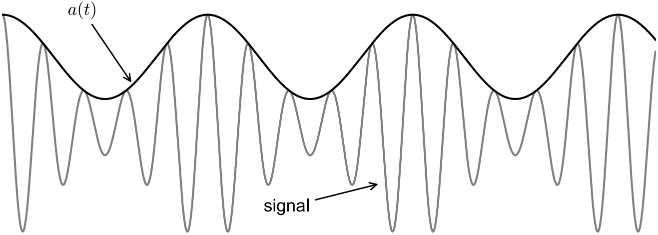
Good news: unified theory exists
we just need to drop the commutativity out !

Fundamental property: Hermitian symmetry of FT of real signals

Analytic signal of a real signal Gabor (1946), Ville (1948)

One to one corresp. between a real signal and its analytic signal

$$x(t) \in \mathbb{R} \longleftrightarrow x_+(t) \in \mathbb{C}$$
$$a(t)\cos[\varphi(t)] \longleftrightarrow a(t)e^{i\varphi(t)}$$



Amplitude Modulation

Quaternion embedding of bivariate signals

$x(t) = u(t) + \boldsymbol{i}v(t) \qquad X(-\nu) = -\boldsymbol{i}X(\nu)\boldsymbol{i} \quad$ ($\boldsymbol{i}$-Hermitian symmetry)

Quaternion embedding

One-to-one correspondence

$$\text{bivariate signal} \longleftrightarrow \text{quaternion embedding}$$
$$x(t) \in \mathbb{C}_{\boldsymbol{i}} \longleftrightarrow x_+(t) \in \mathbb{H}$$

Polar form: instantaneous attributes

$$x_+(t) = \underbrace{a(t)}_{\text{amplitude}} \times \underbrace{e^{\boldsymbol{i}\theta(t)}e^{-\boldsymbol{k}\chi(t)}}_{\text{geometry}} \times \underbrace{e^{\boldsymbol{j}\varphi(t)}}_{\text{phase}}$$

$a(t) \geq 0$

$\theta(t) \in [-\pi/2, \pi/2]$

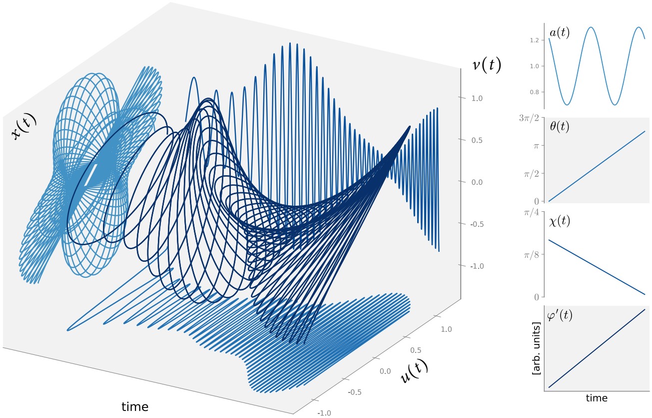
$\chi(t) \in [-\pi/4, \pi/4]$

$\varphi(t) \in [-\pi, \pi]$

Canonical quadruplet

Bivariate AM-FM signal model

$$x(t) = \mathrm{Proj}_{\mathbb{C}_{\boldsymbol{i}}}\{x_+(t)\} = a(t)e^{\boldsymbol{i}\theta(t)}\left[\cos\chi(t)\cos\varphi(t) + \boldsymbol{i}\sin\chi(t)\sin\varphi(t)\right]$$



bivariate linear chirp w/ amplitude, orientation and ellipticity modulation

Quaternion spectrogram

Quaternion Short Term Fourier Transform

Extend the STFT to the QFT setting

$$\mathrm{F}_x^g(\tau,\nu) = \int \underbrace{x(t)}_{\in\mathbb{C}_{\boldsymbol{i}}} \underbrace{g(t-\tau)}_{\in\mathbb{R}} \underbrace{\exp(-\boldsymbol{j}2\pi\nu t)}_{\in\mathbb{C}_{\boldsymbol{j}}} \mathrm{d}t$$

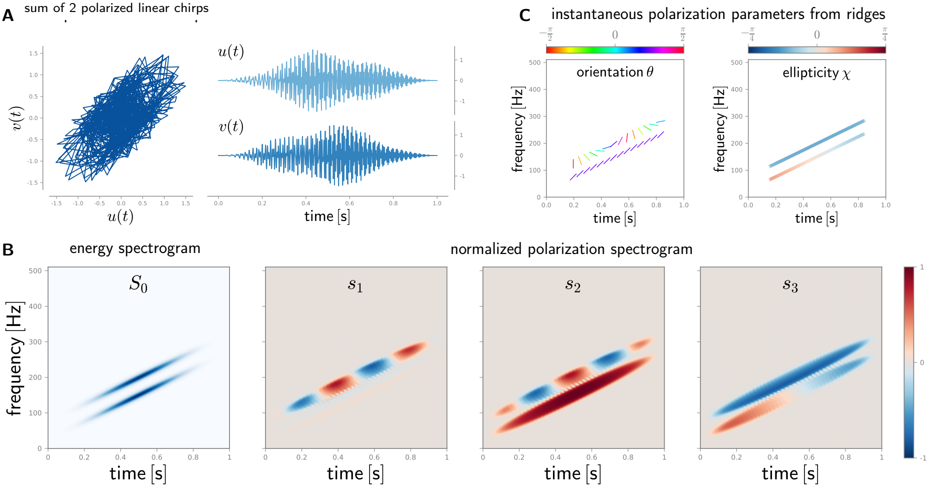
$$\text{Theorems} \quad \begin{cases} \text{inversion} \\ \text{conservation: energy} \quad \text{geometry/polarization} \end{cases}$$

$|\mathrm{F}_x^g(\tau,\nu)|^2 \rightarrow$ Time-frequency energy density (S_0)

$\mathrm{F}_x^g(\tau,\nu)\boldsymbol{j}\overline{\mathrm{F}_x^g(\tau,\nu)} \rightarrow$ Time-frequency-polarization features (S_1, S_2, S_3)

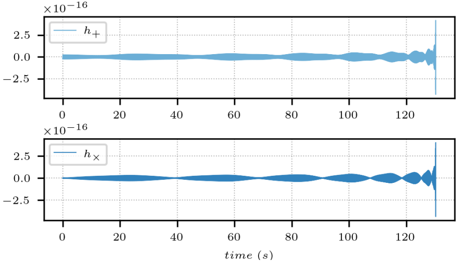
new interpretable time-frequency-polarization representation

Quaternion spectrogram: sum of two linear chirps



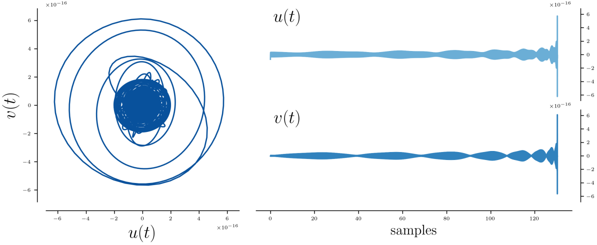
Polarization of gravitational waves (GW) in astronomy (I)

- Source: precessing Binary Black Hole (BBH)
- Specificity: polarized GWs emission during the merging phase

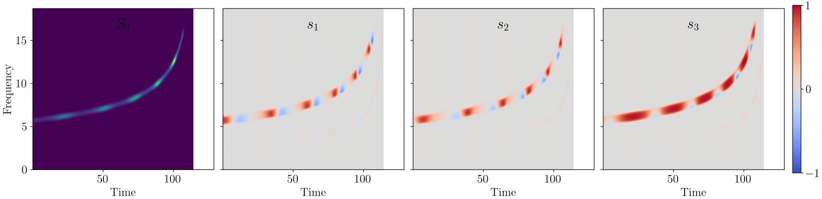


$\hookrightarrow$ $h_+(t) + \boldsymbol{i} h_\times(t)$ form a bivariate signal with time-varying polarization

Polarization of gravitational waves in astronomy (II)



Bivariate signal associated to the polarized gravitational wave



Quaternion spectrogram of the polarized gravitational wave

Polarization of normal modes in underwater acoustics

Particle velocity data recorded by IVAR sensor

impulsive source

$z_s \approx 20$ m, $r \approx 16$ km

IVAR detector

$z \approx 1$ m above seafloor

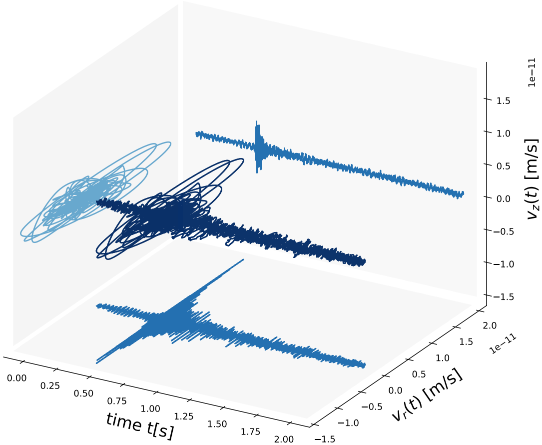
preprocessing

geometric projection

$$[v_x, v_y] \rightarrow v_r$$

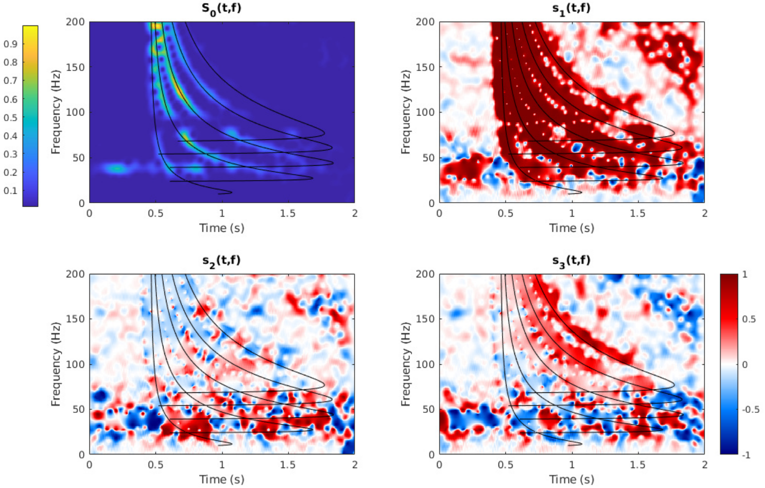
$+$ bandpass filtering

$+$ source deconvolution



use the polarization spectrogram to reveal normal modes !

Polarization of normal modes in underwater acoustics



particular velocity data from SBCEX17 experiment

Bonnel et al. 2021

See notebook 3: Time-frequency analysis

Conclusions and outlook

Conclusions

A physics-oriented framework for bivariate signals

- bivariate signals as complex signals embedded in quaternions $\mathbb{H}$
- quaternion spectral representation with the QFT

✓theoretical guarantees ✓physical interpretations ✓efficient implementations

Enables the construction of 3 building blocks of BSP

- spectral analysis and estimation
  quaternion DSP, white noise, degree of polarization, periodogram ...
- linear time-invariant filtering
  unitary/ hermitian filters, spectral synthesis, wiener filtering, UP decomposition
- time-frequency analysis
  quaternion embedding, spectrogram, cohen class ...

yields quaternion features for bivariate signals

$\rightarrow$ one can plug in their favorite quaternion SP tool:

- quaternion linear algebra such as QSVD;
- quaternion-domain optimization for solving inverse problems;
- quaternion statistical signal processing e.g. quaternion ICA

Ongoing research directions

ANR project RICOCHET (2022-2027)

Bivariate signal processing: a geometric approach to decipher polarization

Research directions

- new BSP theory: parametric and low-rank models, detection theory
- inverse problem in BSP: Bayesian and ML/DL approaches
- applications to physical sciences: GW, UA, seismics...

see ricochet-anr.github.io for more details

2 papers at Eusipco'24! TU2.PA1, Tue 27th, 2024

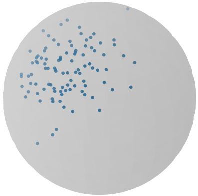
- "Denoising bivariate signals via smoothing and polarization priors" Y.Y. Pivlaci, J. Boulanger, P.-A. Thouvening, P. Chainais
- "The geometric phase of bivariate signals", NLB, JF, P.-O. Amblard

Perspectives

- Robust estimation of polarization parameters

  naive estimators of $\boldsymbol{\mu}_x$, Φ_x can be strongly biased

  ▶ better estimators using e.g. directional statistics? Mardia and Jupp (2000)

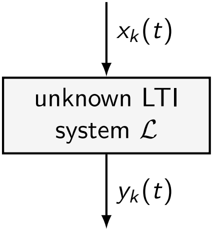


- Identification of LTI systems

  identify $\mathcal{L}$ given K inputs/outputs $\{x_k(t), y_k(t)\}$

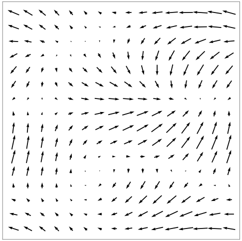
  ▶ systematic procedure borrowing ideas from *ellipsometry* Azzam and Bashara (1978)

also: design of causal, minimum-phase systems



Blind source separation of bivariate signals

- QFT domain mixing models
- new independence proxys — e.g. quaternion HOS
- optimization requires $\mathbb{HR}$-calculus — Xu et al. (2015)



2D QFT for bivariate vector fields

✓ formal construction

? interpretability and usefulness

and beyond, e.g. bivariate signals on graphs

Thank you for your attention!

Find all the material (pdf, notebooks, references) on GitHub:

see repository