Hypersphere Fitting: Model, Algorithms and Future Work

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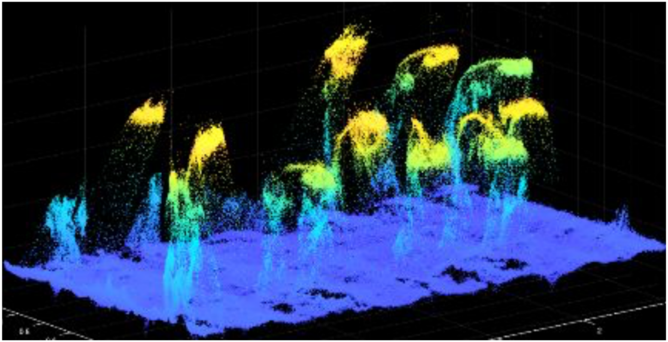
Outline

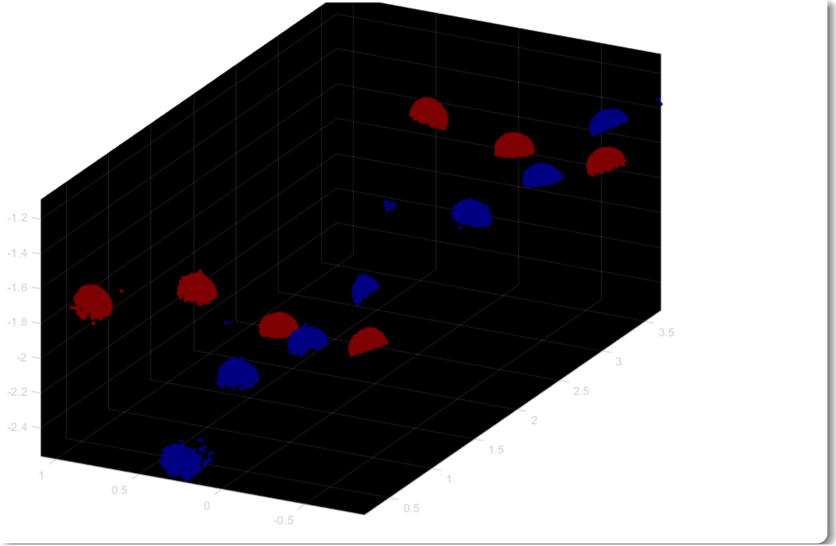
- Context
- Hypersphere Fitting
  - An EM Algorithm with a von-Mises Fisher prior
  - A robust version of the EM algorithm
- Future Work

Phenomobile

LiDAR Images



Calibration with Spheres



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Hypersphere Fitting - Review

Least squares

$$J_{\mathsf{LS}} = \sum_{i=1}^{N} \left[(x_i - c_1)^2 + (y_i - c_2)^2 - R^2 \right]^2$$

Analytical solutions for $\boldsymbol{c} = (c_1, c_2)^T$ and R.

References

I. Kasa, "A Circle Fitting Procedure and Its Error Analysis", IEEE Trans. Instrum. Meas., March 1976.

S. M. Thomas and Y. T. Chan, "A Simple Approach for the Estimation of Circular Arc Center and Its Radius", Computer Vision, Graphics and Image Process., vol. 45, pp. 362-370, 1989.

Hypersphere Fitting - Review

Algebraic Circle Fit

$$J_{\text{Pratt}} = \sum_{i=1}^{N} \frac{\left[A(x_i^2 + y_i^2) + Bx_i + Cy_i + D\right]^2}{B^2 + C^2 - 4AD}.$$

comes from the definition of a circle $A(x^2 + y^2) + Bx + Cy + D = 0$ with $B^2 + C^2 - 4AD > 0$.

Can be solved using a generalized eigenvalue problem

References

V. Pratt, "Direct least-squares fitting of algebraic surfaces", Computer Graphics, vol. 21, pp. 145-152, 1987.

Hypersphere Fitting - Review

Maximum Likelihood

Introducing latent variables $\boldsymbol{x}_i$ located on the sphere such that $\boldsymbol{z}_i = \boldsymbol{x}_i + \boldsymbol{n}_i$ leads to the following cost function

$$J_{\mathsf{ML}} = \sum_{i=1}^{N} \left(\|\boldsymbol{z}_i - \boldsymbol{c}\| - R \right)^2 .$$

Iterative optimization algorithm

References

W. Li et al., "Fitting Noisy Data to A Circle: A Simple Iterative Maximum Likelihood Approach", in Proc. Int. Conf. Commun. (ICC'11), Kyoto, Japan, June 2011.

Hypersphere Fitting - Review

Corests

Coreset construction to reduce the dataset to few points and solve the median-sphere problem using these points leading to a solution such that

$$\sum_{i=1}^{N} \left| \|\boldsymbol{z}_i - \boldsymbol{c}\| - R \right| \leq (1+\epsilon) \sum_{i=1}^{N} \left| \|\boldsymbol{z}_i - \boldsymbol{c}^*\| - R^* \right|$$

References

D. Epstein and D. Feldman, "Sphere Fitting with Applications to Machine Tracking", Algorithms, vol. 13, July 2020.

An EM Algorithm for Hypersphere Fitting

Statistical Model

$$\boldsymbol{z}_i = \boldsymbol{y}_i + \boldsymbol{n}_i, \text{ avec } \boldsymbol{y}_i = \boldsymbol{c} + r\boldsymbol{x}_i \tag{1}$$

with $\boldsymbol{n}_i \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I}_d)$, $\boldsymbol{x}_i$ located on the hypersphere $\mathcal{H}_d$ of $\mathbb{R}^d$ ($\|\boldsymbol{x}_i\|_2 = 1$).

Hypothesis: Von Mises - Fisher distribution

$$\boldsymbol{x}_i \sim \mathsf{vMF}_d(\boldsymbol{x}_i; \boldsymbol{\mu}, \kappa) \tag{2}$$

Density

$$f_d(\boldsymbol{x}_i; \boldsymbol{\mu}, \kappa) = C_d(\kappa) \exp\left(\kappa \boldsymbol{\mu}^T \boldsymbol{x}_i\right) 1_{\mathcal{S}_d}(\boldsymbol{x}_i),$$

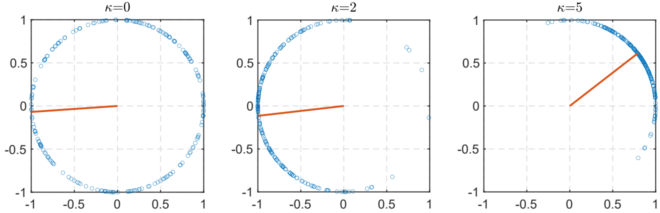
where $\boldsymbol{\mu} \in \mathbb{R}^d$ is a mean (directional) vector ($\|\boldsymbol{\mu}\| = 1$), $\kappa \geq 0$ and $C_d(\kappa)$ is a normalization constant

Distributions on the hypersphere $\mathcal{S}^d$

Von Mises-Fisher distribution

$$p(\boldsymbol{x}_i; \kappa, \mu) = C_d(\kappa) \exp\left(\kappa \boldsymbol{\mu}^T \boldsymbol{x}_i\right) 1_{\mathcal{S}^d}(\boldsymbol{x}_i)$$

where $\boldsymbol{\mu} \in \mathbb{R}^d$ is the mean direction with $\|\boldsymbol{\mu}\|_2 = 1$, and $\kappa \geq 0$ is the spread parameter



Uniform distribution ($\kappa = 0$)

$$p(\boldsymbol{x}_i) = C_d(0) 1_{\mathcal{S}^d}(\boldsymbol{x}_i)$$

Unknown parameters: $\boldsymbol{\theta} = (r, \boldsymbol{c}^T, \kappa, \boldsymbol{\mu}^T)^T$

EM algorithm

Expectation maximization is standard when dealing with missing variables

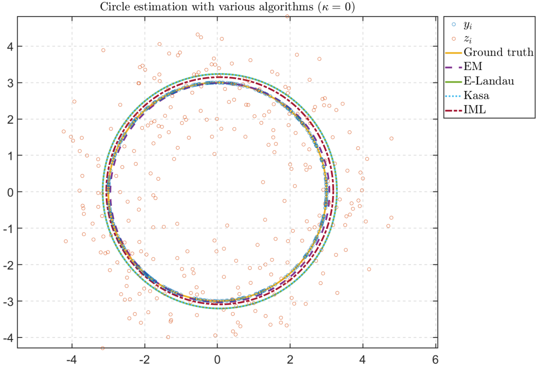
- EM algorithm for estimating $\boldsymbol{\theta}$ when observing $\boldsymbol{Z}$ and $\boldsymbol{X}$ is missing
  - E-step: compute $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)}) = \mathbb{E}_{\boldsymbol{X}|\boldsymbol{Z},\boldsymbol{\theta}^{(t)}}[\log \mathcal{L}_c(\boldsymbol{\theta}; \boldsymbol{Z}, \boldsymbol{X})]$
  - M-step: solve $\boldsymbol{\theta}^{(t+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)})$
- $\mathcal{L}_c$ is the complete likelihood of observed and missing variables computed using the prior distribution for $\boldsymbol{x}_i$

Reference

J. Lesouple, B. Pilastre, Y. Altmann and J.-Y. Tourneret, "Hyperpshere Fitting from Noisy Data Using an EM Algorithm", IEEE Signal Process. Lett., vol. 28, no. 1, pp. 314-318, Jan. 2021.

Results

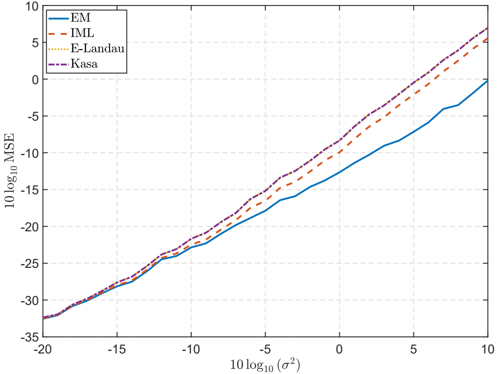
2D



Uniform distribution for latent variables with $\boldsymbol{c} = (0,0)^T$, $r = 3$ and $\sigma = 0.8$.

Results

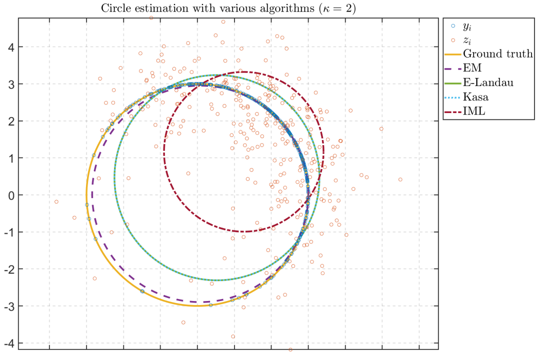
2D



MSE of $\widehat{\boldsymbol{\theta}} = (\hat{r}, \hat{\boldsymbol{c}}^T)^T$ for E-Landau, Kasa, IML and EM versus σ^2 $(500$ Monte Carlo runs).

Results

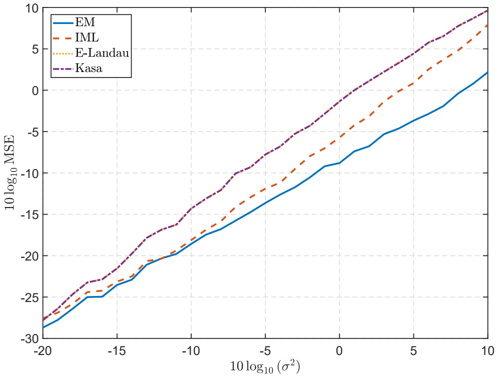
2D



von Mises-Fisher distribution with parameters $\kappa = 2$ and $\boldsymbol{\mu}$ with an angle $\pi/4$, for $\boldsymbol{c} = 0, 0)^T$, $r = 3$ and $\sigma = 0.8$.

Results

2D



MSE of $\widehat{\boldsymbol{\theta}} = (\hat{r}, \hat{\boldsymbol{c}}^T)^T$ for E-Landau, Kasa, IML and EM versus σ^2 (500 Monte Carlo runs).

A Robust version of the EM Algorithm

Statistical Model

$$p(\boldsymbol{z}_i|\boldsymbol{x}_i, \boldsymbol{\theta}) = \gamma \underbrace{\mathcal{U}_A(\boldsymbol{z}_i)}_{\text{outliers}} + (1-\gamma) \underbrace{\mathcal{N}(\boldsymbol{z}_i; \boldsymbol{c} + r\boldsymbol{x}_i, \sigma^2 I_d)}_{\text{hypersphere}}$$

where γ is the proportion of outliers, A is the observation domain with $|A| = a$.

Reparametrization

Hidden variables ν_i indicating the outliers with independent Bernoulli distributions

$$p(\nu_1, \ldots, \nu_n|\gamma) = \prod_{i=1}^{n} \gamma^{\nu_i} (1-\gamma)^{1-\nu_i}$$

Unknown parameters: $\boldsymbol{\theta} = (r, \boldsymbol{c}^T, \kappa, \boldsymbol{\mu}^T, \gamma)^T$, latent variables: $\boldsymbol{x}_i, \nu_i$

Reference

J. Lesouple, B. Pilastre, Y. Altmann and J.-Y. Tourneret, "Robust Hyperpshere Fitting from Noisy Data Using an EM Algorithm", in Proc. European Conf. on Sig. Proc. (EUSIPCO'21), Dublin, Ireland, Aug. 23-27, 2021.

Proposed model

Definition

- Observations $\boldsymbol{Z} = \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n\} \in \mathbb{R}^{d\times n}$
- Latent (missing) variables
$\boldsymbol{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \in \mathbb{R}^{d\times n}, \boldsymbol{\nu} = \{\nu_1, \ldots, \nu_n\} \in \mathbb{R}^n$
- Unknown parameters $\boldsymbol{\theta}^T = \{ \underbrace{r, \boldsymbol{c}^T}_{\text{hypersphere}}, \underbrace{\kappa, \boldsymbol{\mu}^T}_{\text{localization}}, \underbrace{\gamma}_{\text{outliers}}, \underbrace{\sigma^2}_{\text{noise}} \}$

The maximum likelihood estimator of $\boldsymbol{\theta}$ can be found with the EM algorithm

Complete likelihood

$\mathcal{L}_c(\boldsymbol{\theta}; \boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{\nu}) = \prod_{i=1}^n p(\boldsymbol{z}_i, \boldsymbol{x}_i, \nu_i|\boldsymbol{\theta})$ with

$$p(\boldsymbol{z}_i, \boldsymbol{x}_i, \nu_i|\boldsymbol{\theta}) = \left(\frac{\gamma}{a}\right)^{\nu_i} \left(\frac{1-\gamma}{(2\pi\sigma^2)^{d/2}} C_d(\kappa)\right)^{1-\nu_i}$$
$$\times \left(\exp\left\{-\frac{\|\boldsymbol{z}_i - \boldsymbol{c} - r\boldsymbol{x}_i\|_2^2}{2\sigma^2} + \kappa\boldsymbol{\mu}^T\boldsymbol{x}_i\right\}\right)^{1-z_i}$$

Expectation step: Q function

Definition

$$Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)}) = \mathbb{E}_{\boldsymbol{X},\boldsymbol{z}|\boldsymbol{Y},\boldsymbol{\theta}^{(t)}}[\log \mathcal{L}_c(\boldsymbol{\theta}; \boldsymbol{Y}, \boldsymbol{X}, \boldsymbol{z})]$$

Conditioning on $\boldsymbol{Z}$ and $\boldsymbol{\theta}$

$$\boldsymbol{x}_i|\boldsymbol{z}_i, \boldsymbol{\theta} \sim \text{vMF}_d\left(\boldsymbol{x}_i; \boldsymbol{\mu}_i, \kappa_i\right)$$

where

$$\boldsymbol{\mu}_i = \frac{r(\boldsymbol{z}_i - \boldsymbol{c}) + \sigma^2\kappa\boldsymbol{\mu}}{\|r(\boldsymbol{y}_i - \boldsymbol{c}) + (\sigma^2)\kappa\boldsymbol{\mu}\|_2}, \quad \kappa_i = \frac{\|r(\boldsymbol{z}_i - \boldsymbol{c}) + \sigma^2\kappa\boldsymbol{\mu}\|_2}{\sigma^2}$$

and

$$\nu_i|\boldsymbol{y}_i, \boldsymbol{\theta} \sim \mathcal{B}\left(\nu_i; \gamma_i\right)$$

where

$$\gamma_i = \frac{\tilde{\pi}_{i,1}}{\tilde{\pi}_{i,1} + \tilde{\pi}_{i,2}}$$

$$\tilde{\pi}_{i,1} = \frac{\gamma}{a}, \quad \tilde{\pi}_{i,2} = \frac{1-\gamma}{(2\pi\sigma^2)^{d/2}}\frac{C_d(\kappa)}{C_d(\kappa_i)}\exp\left(-\frac{\|\boldsymbol{z}_i - \boldsymbol{c}\|_2^2 + r^2}{2\sigma^2}\right)$$

$\rightarrow$ γ_i is the posterior probability of $\boldsymbol{z}_i$ being an outlier

Expectation step: Q function

Final expression

$$\begin{aligned}\mathbb{E}_{\boldsymbol{X},\boldsymbol{\nu}|\boldsymbol{Z},\boldsymbol{\theta}^{(t)}}\left[\log \mathcal{L}_c\ (\boldsymbol{\theta};\boldsymbol{Z},\boldsymbol{X},\boldsymbol{\nu})\right] = K + \log(\gamma)\sum_{i=1}^{n}\gamma_i^{(t)} \\ + \left[\log(1-\gamma) - \frac{d}{2}\log\left(\sigma^2\right) + \log C_d(\kappa)\right]\sum_{i=1}^{n}(1-\gamma_i^{(t)}) \\ - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\|\boldsymbol{z}_i - \boldsymbol{c}\|_2^2 + r^2\right)(1-\gamma_i^{(t)}) \\ + \sum_{i=1}^{n}\underbrace{\kappa_i A_d(\kappa_i^{(t)})}_{\text{not the same } \kappa}\ \underbrace{\boldsymbol{\mu}_i^T\boldsymbol{\mu}_i^{(t)}}_{\text{not the same } \boldsymbol{\mu}}\ (1-\gamma_i^{(t)}),\end{aligned}$$

to maximize with respect to $\boldsymbol{\theta}$

Maximization step: update equations

M Step

The maximization problem

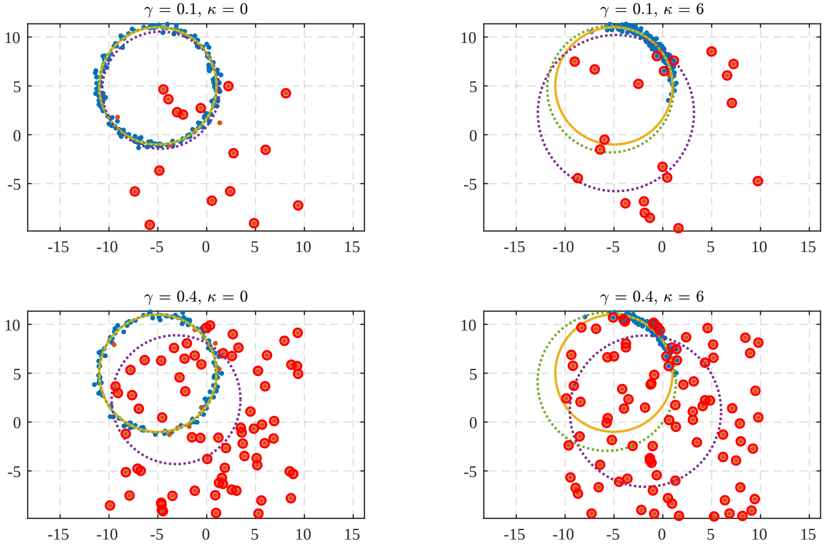
$$\boldsymbol{\theta}^{(t+1)} = \arg\max_{\boldsymbol{\theta}} \mathbb{E}_{\boldsymbol{X},\boldsymbol{\nu}|\boldsymbol{Z},\boldsymbol{\theta}^{(t)}} \left[\log \mathcal{L}_c \ (\boldsymbol{\theta}; \boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{\nu})\right]$$

leads to closed-form solutions (see EUSIPCO paper).

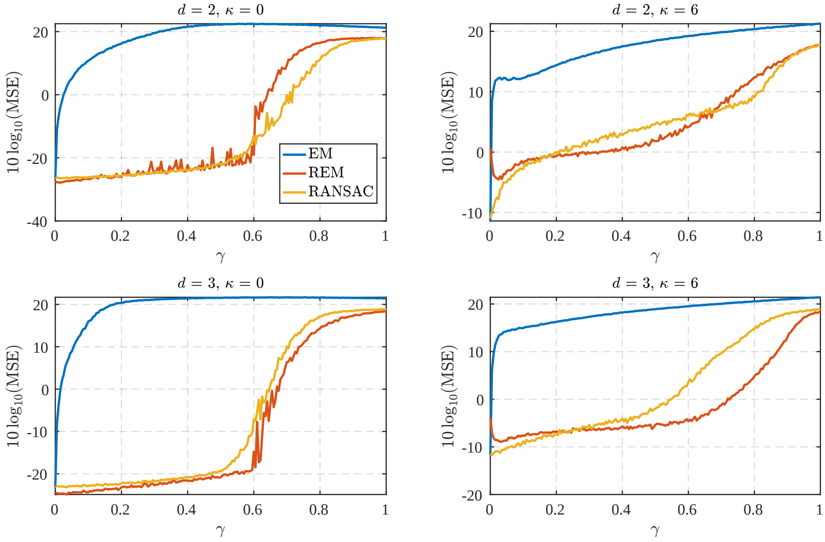
Reference

J. Lesouple, B. Pilastre, Y. Altmann and J.-Y. Tourneret, "Robust Hyperpshere Fitting from Noisy Data Using an EM Algorithm", in Proc. European Conf. on Sig. Proc. (EUSIPCO'21), Dublin, Ireland, Aug. 23-27, 2021.

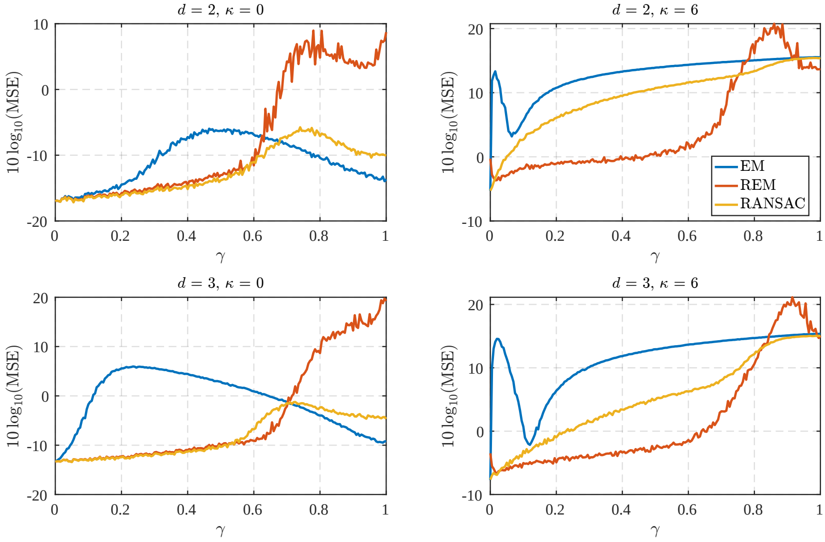
Performance versus κ and γ



MSE comparison for $(r, \boldsymbol{c}, \sigma^2)$



MSE comparison for $(\kappa, \boldsymbol{\mu})$



Computation times in seconds for γ=0.2

Algorithm	$\kappa = 0$ $d = 2$	$\kappa = 6$ $d = 2$	$\kappa = 0$ $d = 3$	$\kappa = 6$ $d = 3$
EM	0.014	0.014	0.015	0.017
REM	0.010	0.013	0.009	0.018
RANSAC+EM	0.530	0.529	2.171	3.537

Algorithms were ran with 40 iterations for the EM algorithm, 294 iterations for RANSAC with $d = 2$ and 1178 iterations for RANSAC for $d = 3$ (corresponding to a probability of 0.9 of drawing at least one subset with only normal data, with an outlier prior probability of 0.5).

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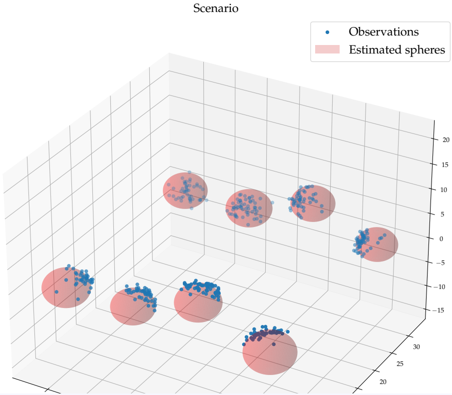
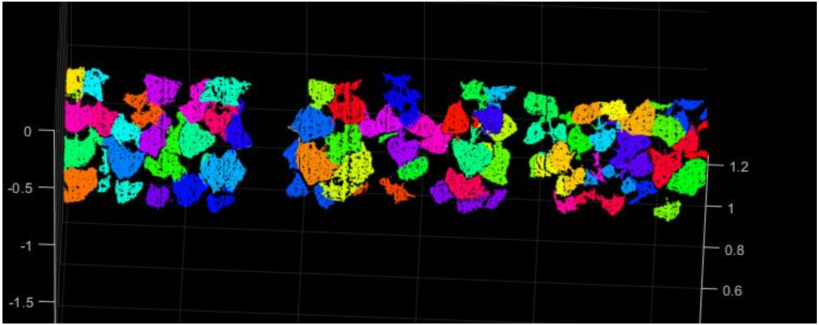
- Future Work

Mixture of several spheres

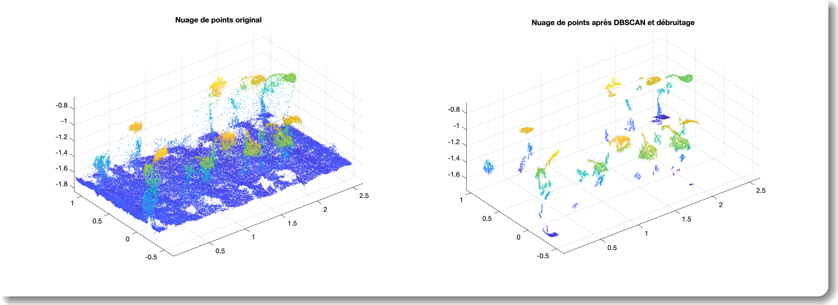
$$p(\boldsymbol{z}_i|\boldsymbol{x}_i, \boldsymbol{\theta}) = \pi_{K+1} \underbrace{\mathcal{U}_A(\boldsymbol{z}_i)}_{\text{outliers}} + \sum_{j=1}^{K} \frac{\pi_j}{(2\pi\sigma^2)^{d/2}} \exp\left\{-\frac{\|\boldsymbol{z}_i - \boldsymbol{c}_j - r_j \boldsymbol{x}_i\|_2^2}{2\sigma^2}\right\} \quad (3)$$



Point cloud segmentation (DBSCAN)

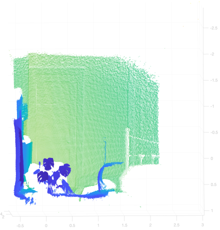


Point cloud denoising



Fusion of LiDAR and RGB images

Problem



Reference

J. Castorena, U. S. Kamilov, P. T. Boufounos, "Autocalibration of lidar and optical cameras via edge alignment," in Proc. Int. Conf. Acoustics, Speech and Signal Processing (ICASSP), Shanghai, China, Mar. 2016.

Conclusions

Summary

- A Bayesian approach with a von Mises-Fisher prior for hypersphere fiitng in presence of outliers
- The EM algorithm is used to solve the estimation problem
- Existence of a breakpoint close to $\gamma = 60\%$ for data uniformly distributed on the sphere
- Execution time much faster than the combination of EM and RANSAC

Future work

- Generalization to mixtures of hyperspheres
- Point cloud segmentation and denoising
- Fusion of LiDAR and RGB images