Chapter 7: Boolean Algebra

WMC CS Club

March 3, 2022

§1 Revisit Boolean Operators

Operators used in Boolean algebra is the same as the ones we mentioned in bits manipulation. These operators are NOT ($\overline{x}$), AND (xy), OR ($x + y$), XOR ($x \oplus y$), XNOR ($x \odot y$).

- NOT is a unary operator that performs logical negation. The NOT of a value is its opposite; that is, the not of a true value is false whereas the not of a false value is true. For example, if $x = 0$, $\overline{x} = 1$.
- AND is a binary operator. The AND of two values is true only whenever both values are true. For example, if $x = 0$, $y = 1$, $xy = 0$.
- OR is a binary operator. The OR of two values is true whenever either or both values are true. For example, if $x = 1$, $y = 1$, $x + y = 1$.
- XOR is a binary operator. The XOR of two values is true whenever the values are different. For example, if $x = 0$, $y = 1$, $x \oplus y = 1$.
- XNOR is a binary operator. The XNOR of two values is true whenever the values are the same. It is the NOT of the XOR function. For example, if $x = 0$, $y = 1$, $x \oplus y = 0$.

§2 Laws of Boolean Algebra

A law of Boolean algebra is an identity such as $x + (y + z) = (x + y) + z between$ two Boolean terms, where a Boolean term is defined as an expression built up from variables, the constants 0 and 1, and operations and, or, not, xor, and xnor.

Like ordinary algebra, parentheses are used to group terms. When a not is represented with an overhead horizontal line, there is an implicit grouping of the terms under the line. That is, $x \cdot \overline{y + z}$ is evaluated as if it were written $x \cdot \overline{(y + z)}$.

> **Remark 2.1 (Order of Precedence).** The order of operator precedence is not; then and; then xor and xnor; and finally or. Operators with the same level of precedence are evaluated from left-to-right.

Fundamental Identities

Commutative Law – The order of application of two separate terms is not important.	$x+y=y+x$	$x \cdot y = y \cdot x$
Associative Law – Regrouping of the terms in an expression doesn't change the value of the expression.	$(x+y)+z=x+(y+z)$	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
Idempotent Law – A term that is *or*'ed or *and*'ed with itself is equal to that term.	$x+x=x$	$x \cdot x = x$
Annihilator Law – A term that is *or*'ed with 1 is 1; a term *and*'ed with 0 is 0.	$x+1=1$	$x \cdot 0 = 0$
Identity Law – A term *or*'ed 0 or *and*'ed with a 1 will always equal that term.	$x+0=x$	$x \cdot 1 = x$
Complement Law – A term *or*'ed with its complement equals 1 and a term *and*'ed with its complement equals 0.	$x+\overline{x}=1$	$x \cdot \overline{x} = 0$
Absorptive Law – Complex expressions can be reduced to a simpler ones by absorbing like terms.	$x+xy=x$ $x+\overline{x}y=x+y$ $x(x+y)=x$	
Distributive Law – It's OK to multiply or factor-out an expression.	$x \cdot (y+z) = xy+xz$ $(x+y)\cdot(p+q)=xp+xq+yp+yq$ $(x+y)(x+z)=x+yz$	
DeMorgan's Law – An *or* (*and*) expression that is negated is equal to the *and* (*or*) of the negation of each term.	$\overline{x+y}=\overline{x}\cdot\overline{y}$	$\overline{x \cdot y}=\overline{x}+\overline{y}$
Double Negation – A term that is inverted twice is equal to the original term.	$\overline{\overline{x}}=x$	
Relationship between XOR and XNOR	$x \odot y = \overline{x \oplus y} = x \oplus \overline{y} = \overline{x} \oplus y$	

Example 2.2

Simplify: $\overline{A(A+B)}+A$.

Example 2.3

How many ordered pairs (A, B) satisfy $\overline{\overline{(A+B)}+\overline{A}B}=1$?

§3 Practice Problems

Problems

1. Simplify $AB+A\overline{B}$
2. Simplify the following Boolean expression: $(A+B)(A+C)$.
3. Simplify $(A+C)(AD+A\overline{D})+AC+C$.
4. Find all ordered pairs (A, B) that make $\overline{AB}(\overline{A}+B)(\overline{B}+B)$ true.
5. Find all ordered pairs (A, B) that make $\overline{\overline{A(A+B)}+B\overline{A}}$ false.
6. Find the equivalent of $A \oplus B$ using only AND, OR, and NOT. Then, use the fundamental identities to find the equivalent of $A \odot B$.

Answer Key

1. A
2. $A+BC$
3. $A+C$
4. $(0,0),(0,1)$
5. $(0,0),(0,1)$
6. $A \oplus B = A\overline{B}+\overline{A}B$, $A \odot B = \overline{AB}+AB$