Chapter 10: Graph Theory

WMC CS CLUB

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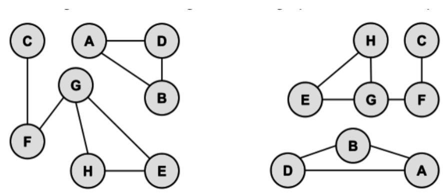
§1 Graph 101: Introduction

A **graph** is a collection of vertices and edges. An **edge** is a connection between two **vertices** (sometimes referred to as **nodes**). One can draw a graph by marking points for the vertices and drawing lines connecting them for the edges, but the graph is defined independently of the visual representation.



For the graph on the right, the precise way to represent this graph is to identify its set of vertices $\{A, B, C, D, E, F, G\}$, and its set of edges between these vertices $\{AB, AD, BD, CF, FG, GH, GE, HE\}$.

§2 Graph 102: Terminology

- **Undirected Graph:** The edges of the above graph have no directions meaning that the edge from one vertex A to another vertex B is the same as from vertex B to vertex A. Such a graph is called an undirected graph. Similarly, a graph having a direction associated with each edge is known as a directed graph.
- **Path & Simple Path:** A path from vertex x to y in a graph is a list of vertices, in which successive vertices are connected by edges in the graph. For example, $FGHE$ is path from F to E in the graph above. A simple path is a path with no vertex repeated. For example, $FGHEG$ is not a simple path.
- **Connected Graph & Connected Component:** A graph is connected if there is a path from every vertex to every other vertex in the graph. Intuitively, if the vertices were physical objects and the edges were strings connecting them, a connected graph would stay in one piece if picked up by any vertex. A graph which is not connected is made up of connected components. For example, the graph above has two connected components: $\{A, B, D\}$ and $\{C, E, F, G, H\}$.
- **Cycle:** A cycle is a path, which is simple except that the first and last vertex are the same (a path from a point back to itself). For example, the path $HEGH$ is a cycle in our example. Vertices must be listed in the order that they are traveled to make the path; any of the vertices may be listed first. Thus, $HEGH$ and $EHGE$ are different ways to identify the same cycle. For clarity, we list the start / end vertex twice: once at the start of the cycle and once at the end.

- **Sparse & Dense Graph:** Denote the number of vertices in a given graph by V and the number of edges by E. Note that E can range anywhere from V to V^2 (or $\frac{V(V-1)}{2}$ in an undirected graph). Graphs with all edges present are called complete graphs; graphs with relatively few edges present (say less than $V \log(V)$) are called sparse; graphs with relatively few edges missing are called dense.

- **Weighted Graph:** A weighted graph is a graph that has a weight (cost) associated with each edge.

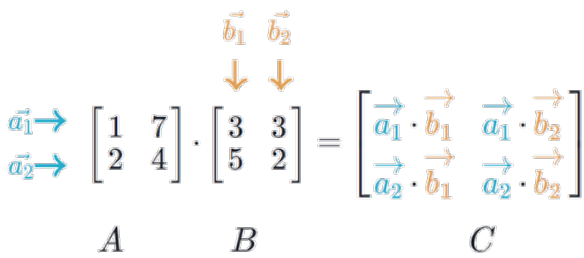
- **Tree and Forest:** A graph without cycles is called a tree. There is only one path between any two nodes in a tree. A tree with N vertices contains exactly $N1$ edges. A group of disconnected trees is called a forest.

- **Spanning Tree:** A spanning tree of a graph is a subgraph that contains all the vertices and forms a tree. A minimal spanning tree can be found for weighted graphs in order to minimize the cost across an entire network.

§3 Graph 103: Adjacency Matrices

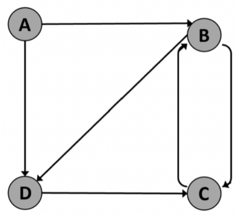
§3.1 Matrices

A matrix is a rectangular arrangement of numbers into rows and columns. Each number in a matrix is referred to as a matrix element or entry. For example, the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ has 2 rows and 3 columns. In order for two matrices A and B to multiply, the number of rows of A must equal to the number of columns of B. Multiplying matrices together is essentially taking the dot product ($\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$, where a_i and b_i are components of vectors $\vec{a}$ and $\vec{b}$) of vectors. The following diagram demonstrates matrix multiplication.



§3.2 Adjacency Matrices

It is frequently convenient to represent a graph by a matrix known as an adjacency matrix. Consider the following directed graph:



To draw the adjacency matrix, we create an N by N grid and label the rows and columns for each vertex. Then, place a 1 for each edge in the cell whose row and column correspond to the starting and ending vertices of the edge. Finally, place a 0 in all other cells.

M =

	A	B	C	D
A				
B				
C				
D				

=

	A	B	C	D
A		1		1
B			1	1
C		1		
D			1	

=

	A	B	C	D
A	0	1	0	1
B	0	0	1	1
C	0	1	0	0
D	0	0	1	0

By construction, cell (i, j) in the matrix with a value of 1 indicates a direct path from vertex i to vertex j. If we square the matrix, the value in cell (i, j) indicates the number of paths of length 2 from vertex i to vertex j.

M² =

	A	B	C	D
A	0	0	2	1
B	0	1	1	0
C	0	0	1	1
D	0	1	0	0

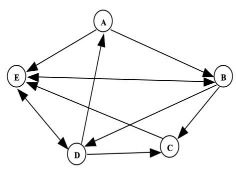
In general, if we raise M to the pth power, the resulting matrix indicates which paths of length p exist in the graph. The value in $M^p(i, j)$ is the number of paths from i to j.

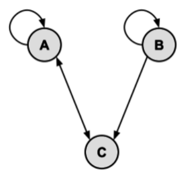
§4 Graph 104: Practice Problems

§4.1 Problems

1. Find the number of different cycles contained in the directed graph with vertices $\{A, B, C, D, E\}$ and edges $\{AB, BA, BC, CD, DC, DB, DE\}$.

2. How many paths of length 2 are in the directed graph shown in the first figure?





3. In the above directed graph (the second figure), find the total number of different paths from vertex A to vertex C of length 2 or 4.

4. Given the adjacency matrix, draw/describe the directed graph: $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

§4.2 Answer Key

1. The cycles are: ABA, $BCDB$, and CDC. Thus, there are 3 cycles in the graph.

2. To find the number of paths of length 2, add the entries in the square of the adjacency matrix. The sum is 24.

3. First construct M, then calculate M^2 and M^4. Observe that there is 1 path of length 2 from A to C (cell $[1, 3]$ in M^2); 3 paths of length 4 (cell $[1, 3]$ in M^4).

4. There must be exactly 4 vertices: $V = \{A, B, C, D\}$. There must be be exactly 7 edges: $E = \{AB, AD, BA, BD, CA, DB, DC\}$.