Thesis Contents

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[?]

12 Semiseparable

Define matrices $E_u(p)$ and $E_l(q)$ as following,

$$E_u(p) = \begin{cases} 1, & \text{if} \quad i \leqslant j - p, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad E_l(q) = \begin{cases} 1, & \text{if} \quad i \geqslant j - q, \\ 0, & \text{otherwise.} \end{cases}. \tag{1}$$

i.e.

$$E_u(-1) = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix}, \quad E_u(0) = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}, \quad E_u(1) = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}; \tag{2}$$

$$E_l(-1) = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{bmatrix}, \quad E_l(0) = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & 1 \end{bmatrix}, \quad E_l(1) = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 & 0 \end{bmatrix}. \tag{3}$$

where $|p| \leqslant n-1$ and $|q| \leqslant n-1$.

Define the operations triu and tril,

$$\operatorname{triu}(A, p) = A \circ E_u(p) \tag{4}$$
$$\operatorname{tril}(A, q) = A \circ E_l(q) \tag{5}$$

i.e. $\operatorname{triu}(A, p)$ retains the entries of A above the p-th super-diagonal (inclusive) and makes the entries of A below the p-th super-diagonal (exclusive) 0.

define symmetric {1,1}-semiseparable matrix, S

$$S = \begin{cases} x_i y_j & \text{if } i \leqslant j \ ; \\ x_j y_i & \text{if } i \geqslant j. \end{cases} \tag{6}$$
$$= \operatorname{triu}\left(xy^{\mathrm{T}}\right) + \operatorname{tril}\left(yx^{\mathrm{T}}, -1\right) \tag{7}$$
$$= \operatorname{triu}\left(xy^{\mathrm{T}}, 1\right) + \operatorname{tril}\left(yx^{\mathrm{T}}\right) \tag{8}$$
$$= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_1y_2 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1y_n & x_2y_n & \cdots & x_ny_n \end{bmatrix} \tag{9}$$

13 Matrix (basic def)

Matrix-Matrix Multiplication

$$AB = C \tag{10}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \tag{11}$$

Matrix-Vector Multiplication

$$Ax = b \tag{12}$$

$$b_i = \sum_{k=1}^{n} a_{ik} x_k \tag{13}$$

14 Algorithm (Appendix)

Backward Subsitution

$$Ux = b \tag{14}$$

Triangular Matrix

$$\begin{bmatrix} d_1 & u_1^1 & u_1^2 & \cdots & \cdots & u_1^{n-1} \\ & d_2 & u_2^1 & u_2^2 & \cdots & u_2^{n-2} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & d_{n-2} & u_{n-2}^1 & u_{n-2}^2 \\ & & & & d_{n-1} & u_{n-1}^1 \\ & & & & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-2} \\ b_{n-1} \\ b_n \end{bmatrix} \tag{15}$$

$$x_n = b_n / u_{nn} \tag{16}$$

$$x_i = \left(b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right) \Big/ u_{ii} \tag{17}$$

$$d_n x_n = b_n \qquad \Rightarrow \qquad x_n = b_n / \, d_n \tag{18}$$

$$d_{n-1} x_{n-1} + u_{n-1}^1 x_n = b_{n-1} \qquad \Rightarrow \qquad x_{n-1} = \left(b_{n-1} - u_{n-1}^1 x_n \right) / \, d_{n-1} \tag{19}$$

$$d_{n-2} x_{n-2} + u_{n-2}^1 x_{n-1} + u_{n-2}^2 x_n = b_{n-2} \qquad \Rightarrow \qquad x_{n-2} = \left(b_{n-2} - u_{n-2}^1 x_{n-1} - u_{n-2}^2 x_n \right) / \, d_{n-2} \tag{20}$$

$$\vdots \tag{21}$$

$$b_{n-k} = d_{n-k} x_{n-k} + u_{n-k}^1 x_{n-k+1} + u_{n-k}^2 x_{n-k+2} + \cdots u_{n-k}^k x_{n-k+k} \tag{22}$$

$$\Rightarrow \quad x_{n-k} = \left(b_{n-k} - u_{n-k}^1 x_{n-k+1} - u_{n-k}^2 x_{n-k+2} - \cdots u_{n-k}^k x_{n-k+k} \right) \Big/ \, d_{n-k} \tag{23}$$

$$= \left(b_{n-k} - \sum_{j=1}^{k} u_{n-k}^j x_{n-k+j} \right) \Big/ \, d_{n-k} \tag{24}$$

$$\tag{25}$$

$$x_i = \left(b_i - \sum_{j=1}^{n-i} u_i^j x_{i+j} \right) \Big/ \, d_i \tag{26}$$

$$x_i = \left(b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right) \Big/ \, u_{ii} \tag{27}$$

Upper Banded Matrix

$$
\begin{bmatrix}
d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & & & \\
& d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & & \\
& & \ddots & \ddots & \ddots & & & \ddots & \\
& & & \ddots & \ddots & \ddots & & & b_{n-u}^u \\
& & & & \ddots & \ddots & & \ddots & \vdots \\
& & & & & & d_{n-2} & b_{n-2}^1 & b_{n-2}^2 \\
& & & & & & & d_{n-1} & b_{n-1}^1 \\
& & & & & & & & d_n
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ \vdots \\ b_{n-2} \\ b_{n-1} \\ b_n \end{bmatrix}
\tag{28}
$$

$$x_n = b_n / u_{nn} \tag{29}$$

$$x_i = \left(b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right) \Bigg/ u_{ii} \tag{30}$$

$$d_n x_n = b_n \quad \Rightarrow \quad x_n = b_n / d_n \tag{31}$$

$$d_{n-1} x_{n-1} + u_{n-1}^1 x_n = b_{n-1} \quad \Rightarrow \quad x_{n-1} = \left(b_{n-1} - u_{n-1}^1 x_n \right) / d_{n-1} \tag{32}$$

$$d_{n-2} x_{n-2} + u_{n-2}^1 x_{n-1} + u_{n-2}^2 x_n = b_{n-2} \quad \Rightarrow \quad x_{n-2} = \left(b_{n-2} - u_{n-2}^1 x_{n-1} - u_{n-2}^2 x_n \right) / d_{n-2} \tag{33}$$

$$\vdots \tag{34}$$

$$b_{n-k} = d_{n-k} x_{n-k} + u_{n-k}^1 x_{n-k+1} + u_{n-k}^2 x_{n-k+2} + \cdots u_{n-k}^k x_{n-k+k} \tag{35}$$

$$\Rightarrow \quad x_{n-k} = \left(b_{n-k} - u_{n-k}^1 x_{n-k+1} - u_{n-k}^2 x_{n-k+2} - \cdots u_{n-k}^k x_{n-k+k} \right) \Big/ d_{n-k} \tag{36}$$

$$= \left(b_{n-k} - \sum_{j=1}^{k} u_{n-k}^j x_{n-k+j} \right) \Bigg/ d_{n-k} \tag{37}$$

$$\tag{38}$$

$$x_i = \left(b_i - \sum_{j=1}^{n-i} u_i^j x_{i+j} \right) \Bigg/ d_i \tag{39}$$

$$x_i = \left(b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right) \Bigg/ u_{ii} \tag{40}$$

15 Apply on expmod

16 Apply on symmetric positive definite tridiagonal matrices

$$\longleftarrow \text{ cholesky } \mathcal{O}\left(\frac{1}{3}n^3\right)$$

$$\begin{aligned}
T &= U^{\mathrm{T}}U \\
U^{-1} &= \operatorname{triu}\left(xy^{\mathrm{T}}\right) \\
\left(U^{\mathrm{T}}\right)^{-1} &= \left(U^{-1}\right)^{\mathrm{T}} \\
&= \left(\operatorname{triu}\left(xy^{\mathrm{T}}\right)\right)^{\mathrm{T}} \\
&= \operatorname{tril}\left(yx^{\mathrm{T}}\right) \\
T^{-1} &= \left(U^{\mathrm{T}}U\right)^{-1} \\
&= \left(U^{-1}\right)\left(U^{\mathrm{T}}\right)^{-1} \\
&= \operatorname{triu}\left(xy^{\mathrm{T}}\right)\operatorname{tril}\left(yx^{\mathrm{T}}\right)
\end{aligned}$$

$$\begin{aligned}
\left(T^{-1}\right)_{i,j} &= \sum_{k=1}^{n}\left[\operatorname{triu}\left(xy^{\mathrm{T}}\right)\right]_{i,k}\left[\operatorname{tril}\left(yx^{\mathrm{T}}\right)\right]_{k,j} \\
&= \sum_{k=1}^{n}\left(\begin{cases} x_i y_k, & \text{if } i \leqslant j; \\ 0, & \text{if } i > j. \end{cases}\right)\left(\begin{cases} 0, & \text{if } k < j; \\ y_k x_j, & \text{if } k \geqslant j. \end{cases}\right) \\
&= \sum_{k=1}^{n}\left(x_i y_k \mathbb{1}_{\mathbb{Z}\cap[i,n]}(k)\right)\left(y_k x_j \mathbb{1}_{\mathbb{Z}\cap[j,n]}(k)\right) \\
&= x_i x_j \sum_{k=1}^{n} y_k^2\left(\mathbb{1}_{\mathbb{Z}\cap[i,n]}(k)\,\mathbb{1}_{\mathbb{Z}\cap[j,n]}(k)\right) \\
&= x_i x_j \sum_{k=1}^{n} y_k^2 \mathbb{1}_{\mathbb{Z}\cap[\max(i,j),n]}(k)
\end{aligned}$$

$$\left(T^{-1}\right)_{i,j} = \begin{cases} x_i x_j \sum_{k=j}^{n} y_k^2, & \text{if } i \leqslant j; \\ x_j x_i \sum_{k=i}^{n} y_k^2, & \text{if } i \geqslant j. \end{cases} \tag{41}$$

for $i = j$,

$$\left(T^{-1}\right)_{i,i} = x_i^2 \sum_{k=i}^{n} y_k^2 = x_j^2 \sum_{k=j}^{n} y_k^2 = \left(T^{-1}\right)_{j,j}$$

following the definition of {1,1}-Semiseparable matrix

$$S = \begin{cases} x_i y_j & \text{if } i \leqslant j \ ; \\ x_j y_i & \text{if } i \geqslant j. \end{cases} \tag{42}$$

$$= \operatorname{triu}\left(xy^{\mathrm{T}}\right) + \operatorname{tril}\left(yx^{\mathrm{T}}, -1\right) \tag{43}$$

$$\tag{44}$$

$$
\begin{aligned}
\left(T^{-1}\right)_{i,j} &= \begin{cases} x_i\left(x_j \sum_{k=j}^{n} y_k^2\right), & \text{if } i \leqslant j; \\ x_j\left(x_i \sum_{k=i}^{n} y_k^2\right), & \text{if } i \geqslant j. \end{cases} && (45)\\
&= \begin{cases} \tilde{x}_i\tilde{y}_j & \text{if } i \leqslant j\ ; \\ \tilde{x}_j\tilde{y}_i & \text{if } i \geqslant j. \end{cases} && (46)\\
&= \operatorname{triu}\left(\tilde{x}\tilde{y}^{\mathrm{T}}\right) + \operatorname{tril}\left(\tilde{y}\tilde{x}^{\mathrm{T}}, -1\right) && (47)\\
& && (48)
\end{aligned}
$$

where $\tilde{x}_i = x_i$ and $\tilde{y}_i = x_i \sum_{k=i}^{n} y_k^2$.

$$
\begin{aligned}
\tilde{x}_i &= x_i && (49)\\
\Rightarrow \tilde{x} &= x && (50)\\
\tilde{y}_i &= x_i \sum_{k=i}^{n} y_k^2 && (51)\\
\Rightarrow \tilde{y} &= \begin{bmatrix} x_1\left(\sum_{k=1}^{n} y_k^2\right) \\ x_2\left(\sum_{k=2}^{n} y_k^2\right) \\ \vdots \\ x_n\left(\sum_{k=n}^{n} y_k^2\right) \end{bmatrix} && (52)\\
&= x \circ \begin{bmatrix} y^{\mathrm{T}}y \\ y_{[2:n]}^{\mathrm{T}}y_{[2:n]} \\ \vdots \\ y_n^2 \end{bmatrix} && (53)
\end{aligned}
$$

$$
\begin{aligned}
x &:= U^{-1}e_n = \texttt{backsub}(U, e_n);\\
y_i &:= (d_i x_i)^{-1} \quad \text{for} \quad i = 1:n.
\end{aligned}
$$

$$
\begin{aligned}
x &= U \setminus e_n\\
x_n &= \frac{1}{d_n}\\
x_i &= -\frac{u_i x_{i+1}}{d_i} \quad \text{for} \quad i = n-1, n-2, \cdots, 1
\end{aligned}
$$

$$
\begin{aligned}
x_n &= \frac{1}{d_n}\\
x_{n-1} &= -\frac{u_{n-1}}{d_{n-1}}x_n = -\frac{u_{n-1}}{d_n d_{n-1}}\\
x_{n-2} &= -\frac{u_{n-2}}{d_{n-2}}x_{n-1} = \frac{u_{n-1}u_{n-2}}{d_n d_{n-1} d_{n-2}}\\
x_1 &= -\frac{u_{n-2}}{d_{n-2}}x_2 = (-1)^{n-1}\frac{u_{n-1}u_{n-2}\ldots u_1}{d_n d_{n-1} d_{n-2}\ldots d_1}
\end{aligned}
$$

$$x_k = \frac{\prod_{i=k}^{n-1} -b_i}{\prod_{i=k}^{n} d_i} = \left(\prod_{i=k}^{n-1} -\frac{b_i}{d_i}\right) \bigg/ d_n \quad \text{for} \quad k = 1 : n$$

where $\prod_{i=k}^{l} a_i = 1$ for any $k > l$

$$\begin{aligned}
x_k &= \left(\prod_{i=k}^{n-1} -\frac{b_i}{d_i}\right) \bigg/ d_n \\
y_k &= \frac{1}{d_k x_k} = \left(\frac{d_n}{d_k}\right) \bigg/ \prod_{i=k}^{n-1} -\frac{b_i}{d_i} = \frac{d_n}{d_k} \prod_{i=k}^{n-1} -\frac{d_i}{u_i} \\
y_k^2 &= \frac{d_n^2}{d_k^2} \prod_{i=k}^{n-1} \left(\frac{d_i}{b_i}\right)^2
\end{aligned}$$

$$
\begin{aligned}
\frac{\tilde{y}_{k+1}}{\tilde{y}_k} &= \frac{x_{k+1}\sum_{\ell=k+1}^{n} y_\ell^2}{x_k \sum_{\ell=k}^{n} y_\ell^2} \\
&= \frac{x_{k+1}}{x_k}\left(1 - \frac{y_k^2}{\sum_{\ell=k}^{n} y_\ell^2}\right) \\
&= -\frac{d_k}{b_k}\left(1 - \frac{\frac{d_n^2}{d_k^2}\prod_{i=k}^{n-1}\left(\frac{d_i}{b_i}\right)^2}{\sum_{\ell=k}^{n}\frac{d_n^2}{d_\ell^2}\prod_{i=\ell}^{n-1}\left(\frac{d_i}{b_i}\right)^2}\right) \\
&= r_k\left(1 - \frac{\frac{1}{d_k^2}\prod_{i=k}^{n-1} r_i^2}{\sum_{\ell=k}^{n}\frac{1}{d_\ell^2}\prod_{i=\ell}^{n-1} r_i^2}\right) \\
&\leqslant r_{\max}\left(1 - \frac{\frac{1}{d_k^2}\prod_{i=k}^{n-1} r_{\min}^2}{\sum_{\ell=k}^{n}\frac{1}{d_\ell^2}\prod_{i=\ell}^{n-1} r_i^2}\right), \quad \text{as } r_{\min}^2 \leqslant r_i^2, \forall i, \\
&\leqslant r_{\max}\left(1 - \frac{\frac{1}{d_k^2}\prod_{i=k}^{n-1} r_{\min}^2}{\sum_{\ell=k}^{n}\frac{1}{d_\ell^2}\prod_{i=\ell}^{n-1} r_{\max}^2}\right), \quad \text{as } r_{\max}^2 \geqslant r_i^2, \forall i, \\
&= r_{\max}\left(1 - \frac{\frac{1}{d_{\max}^2} r_{\min}^{2(n-k)}}{\sum_{\ell=k}^{n}\frac{1}{d_{\min}^2} r_{\max}^{2(n-\ell)}}\right) \\
&= r_{\max}\left(1 + \frac{d_{\min}^2\left(r_{\max}^2 - 1\right) r_{\max}^{2k} r_{\min}^{2(n-k)}}{d_{\max}^2\left(r_{\max}^{2k} - r_{\max}^{2(n+1)}\right)}\right) \\
&= r_{\max}\left(1 + \frac{\left(r_{\max}^2 - 1\right) r_{\max}^{2k} r_{\min}^{2(n-k)}}{r_{\max}^{2k} - r_{\max}^{2(n+1)}}\left(\frac{d_{\min}}{d_{\max}}\right)^2\right) \\
&\leqslant r_{\max}\left(1 + \frac{\left(r_{\max}^2 - 1\right) r_{\max}^{2k} r_{\min}^{2(n-k)}}{r_{\max}^{2k} - r_{\max}^{2(n+1)}}\right), \quad \text{as } \left(\frac{d_{\min}}{d_{\max}}\right)^2 \leqslant 1, \\
&\leqslant r_{\max}\left(1 + \frac{\left(r_{\max}^2 - 1\right) r_{\max}^{2k} r_{\max}^{2(n-k)}}{r_{\max}^{2k} - r_{\max}^{2(n+1)}}\right), \quad \text{as } r_{\min}^2 \leqslant r_{\max}^2, \\
&= r\left(1 + \frac{\left(r^2 - 1\right) r^{2k} r^{2(n-k)}}{r^{2k} - r^{2(n+1)}}\right) \\
&= r\left(1 + \frac{\left(r^2 - 1\right) r^{2n}}{r^{2k} - r^{2(n+1)}}\right) \\
&= r\left(\frac{r^{2k} - r^{2(n+1)} + r^{2(n+1)} - r^{2n}}{r^{2k} - r^{2(n+1)}}\right) \\
&= r\left(\frac{r^{2k} - r^{2n}}{r^{2k} - r^{2(n+1)}}\right) \\
&= \frac{r^{2k+1} - r^{2n+1}}{r^{2k} - r^{2n+2}} \\
&= \frac{r^{2n+1} - r^{2k+1}}{r^{2n+2} - r^{2k}} \\
&< \frac{r^{2n+1} - r^{2k}}{r^{2n+2} - r^{2k}}, \quad \text{as } r^{2k+1} > r^{2k}, \\
&< \frac{r^{2n+1} - r^{2k}}{r^{2n+1} - r^{2k}}, \quad \text{as } r^{2n+1} < r^{2n+2}, \\
&= 1
\end{aligned}
$$

where $d_{\max} = \max d_k$, $b_{\max} = \max b_k$, $r_k = -\frac{d_k}{b_k}$, $r_{\max} = -\frac{d_{\max}}{b_{\min}}$, and $r_{\min} = -\frac{d_{\min}}{b_{\max}}$, with $r_{\max} \geqslant r_k \geqslant r_{\min}, \forall k = 1 : n$.

$$= \frac{r^{2k+1} - r^{2n+1}}{r^{2k} - r^{2n+2}}$$

$$\begin{aligned}
T^{-1} &= \left(U^{-1}\right)\left(U^{\mathrm{T}}\right)^{-1} = \operatorname{triu} xy^{\mathrm{T}} \operatorname{tril} yx^{\mathrm{T}} \\
\left(T^{-1}\right)_{i,j} &= \sum_{k=1}^{n} \left[\operatorname{triu} xy^{\mathrm{T}}\right]_{i,k} \left[\operatorname{tril} yx^{\mathrm{T}}\right]_{k,j} \\
&= \sum_{k=1}^{n} \left(\begin{cases} x_i y_k, & \text{if } i \leqslant j; \\ 0, & \text{if } i > j. \end{cases}\right) \left(\begin{cases} 0, & \text{if } k < j; \\ y_k x_j, & \text{if } k \geqslant j. \end{cases}\right) \\
&= \sum_{k=1}^{n} \left(x_i y_k \mathbb{1}_{\{i:n\}}(k)\right)\left(y_k x_j \mathbb{1}_{\{j:n\}}(k)\right) \\
&= x_i x_j \sum_{k=1}^{n} y_k^2 \left(\mathbb{1}_{\{i:n\}}(k)\, \mathbb{1}_{\{j:n\}}(k)\right) \\
&= x_i x_j \sum_{k=1}^{n} y_k^2 \mathbb{1}_{\{\max(i,j):n\}}(k) \qquad (54)
\end{aligned}$$

$$\left.\begin{aligned} x &\coloneqq U^{-1} e_n = \texttt{backsub}(U, e_n), \\ y_i &\coloneqq (d_i x_i)^{-1} \quad \text{for} \quad i = 1 : n. \end{aligned}\right\} \to (U^{-1})_{i,j} = x_i y_j \quad \text{for} \quad j = i : n$$

$$T = \tilde{L}U = \tilde{U}^{\mathrm{T}}U \quad \Rightarrow \quad T^{-1} = \left(\tilde{U}^{\mathrm{T}}U\right)^{-1} = \left(U^{-1}\right)\left(\tilde{U}^{\mathrm{T}}\right)^{-1} = \left(U^{-1}\right)\left(\tilde{U}^{-1}\right)^{\mathrm{T}}$$

where $\tilde{L} = \tilde{U}^{\mathrm{T}}$

Let $U^{-1} = \operatorname{triu} uv^{\mathrm{T}}$, $\tilde{U}^{-1} = \operatorname{triu} qp^{\mathrm{T}}$, then $\left(\tilde{U}^{-1}\right)^{\mathrm{T}} = \operatorname{tril} pq^{\mathrm{T}}$ by Algorithm XXX and Lemma XXXXX

Then, Equations XXXX and XXX for nonsymmetric bidiagonal T becomes,

$$\begin{aligned}
T^{-1} &= \operatorname{triu} uv^{\mathrm{T}} \operatorname{tril} pq^{\mathrm{T}}, \\
\left(T^{-1}\right)_{i,j} &= \sum_{k=1}^{n} \left[\operatorname{triu} uv^{\mathrm{T}}\right]_{i,k} \left[\operatorname{tril} pq^{\mathrm{T}}\right]_{k,j} \\
&= \sum_{k=1}^{n} \left(u_i v_k \mathbb{1}_{\{i:n\}}(k)\right)\left(p_k q_j \mathbb{1}_{\{j:n\}}(k)\right) \\
&= u_i q_j \sum_{k=1}^{n} v_k p_k \mathbb{1}_{\{\max(i,j):n\}}(k) \\
&= \begin{cases} u_i q_j \sum_{k=j}^{n} v_k p_k, & \text{if } i \leqslant j; \\ q_j u_i \sum_{k=i}^{n} v_k p_k, & \text{if } i \geqslant j. \end{cases} \\
&= \begin{cases} \tilde{u}_i \tilde{v}_j & \text{if } i \leqslant j\ ; \\ \tilde{p}_j \tilde{q}_i & \text{if } i \geqslant j. \end{cases} \\
&= \operatorname{triu} \tilde{u}\tilde{v}^{\mathrm{T}} + \operatorname{tril} \tilde{q}\tilde{p}^{\mathrm{T}}, -1
\end{aligned}$$

where

$$\begin{cases} \tilde{u}_i = u_i; \\ \tilde{v}_j = q_j \sum_{k=j}^{n} v_k p_k. \end{cases}, \quad \text{and} \quad \begin{cases} \tilde{p}_j = q_j; \\ \tilde{q}_i = u_i \sum_{k=j}^{n} v_k p_k. \end{cases}.$$

The equality holds as, when $i = j$

$$\tilde{u}_i \tilde{v}_i = u_i q_i \sum_{k=i}^{n} v_k p_k = q_i u_i \sum_{k=i}^{n} v_k p_k = \tilde{p}_i \tilde{q}_i$$

$$
\begin{aligned}
\left(\operatorname{triu} xy^{\mathrm{T}} b\right)_i &= \sum_{k=1}^{n} \left[\operatorname{triu} xy^{\mathrm{T}}\right]_{i,k} b_k \\
&= \sum_{k=1}^{n} \left(\begin{cases} x_i y_k, & \text{if } i \leqslant k; \\ 0, & \text{if } i > k. \end{cases} \right) b_k \\
&= \sum_{k=1}^{n} \left[x_i y_k \mathbb{1}_{\{i:n\}}(k) \right] b_k \\
&= x_i \sum_{k=i}^{n} y_k b_k \\
&= x \circ \left[\begin{array}{c|c|c|c} y^{\mathrm{T}} b & y_{[2:n]}^{\mathrm{T}} b_{[2:n]} & \cdots & y_n b_n \end{array} \right]^{\mathrm{T}} \\
&= x \circ \texttt{normCoef}(y, b, \texttt{false})
\end{aligned}
$$

$$
\begin{aligned}
\left(\operatorname{tril} yx^{\mathrm{T}}, -1 b\right)_i &= \sum_{k=1}^{n} \left[\operatorname{triu} xy^{\mathrm{T}}\right]_{i,k} b_k \\
&= \sum_{k=1}^{n} \left(\begin{cases} 0, & \text{if } i < k+1; \\ y_i x_k, & \text{if } i \geqslant k+1. \end{cases} \right) b_k \\
&= y_i \sum_{k=1}^{i-1} x_k b_k \\
&= y \circ \left[\begin{array}{c|c|c|c|c} 0 & x_1 b_1 & x_{[1:2]}^{\mathrm{T}} b_{[1:2]} & \cdots & x_{[1:n-1]}^{\mathrm{T}} b_{[1:n-1]} \end{array} \right]^{\mathrm{T}} \\
&= y \circ \left[\begin{array}{c|c} 0 & \texttt{normCoef}(x_{[1:n-1]}, b_{[1:n-1]}, \texttt{true}) \end{array} \right]^{\mathrm{T}}
\end{aligned}
$$

$$
x = T^{-1} b = \operatorname{triu} \tilde{x}\tilde{y}^{\mathrm{T}} + \operatorname{tril} \tilde{y}\tilde{x}^{\mathrm{T}}, -1 b = \operatorname{triu} \tilde{x}\tilde{y}^{\mathrm{T}} b + \operatorname{tril} \tilde{y}\tilde{x}^{\mathrm{T}}, -1 b
$$

$$
[n \quad E_0 \quad E_1 \quad E_2]
$$

$r \quad u \quad u$

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17.1 Tridiagonal matrix (Upper)

$n = 2k$

$$
\begin{aligned}
U &= \begin{bmatrix}
d_1 & b_1^1 & b_1^2 & & & & & & & \\
& d_2 & b_2^1 & b_2^2 & & & & & & \\
& & d_3 & b_3^1 & b_3^2 & & & & & \\
& & & d_4 & b_4^1 & b_4^2 & & & & \\
& & & & \ddots & \ddots & \ddots & & & \\
& & & & & \ddots & \ddots & \ddots & & \\
& & & & & & d_{n-3} & b_{n-3}^1 & b_{n-3}^2 & \\
& & & & & & & d_{n-2} & b_{n-2}^1 & b_{n-2}^2 \\
& & & & & & & & d_{n-1} & b_{n-1}^1 \\
& & & & & & & & & d_n
\end{bmatrix} \\
&= \begin{bmatrix}
\begin{bmatrix} d_1 & b_1^1 \\ & d_2 \end{bmatrix} & \begin{bmatrix} b_1^2 & \\ b_2^1 & b_2^2 \end{bmatrix} & & & \\
& \begin{bmatrix} d_3 & b_3^1 \\ & d_4 \end{bmatrix} & \begin{bmatrix} b_3^2 & \\ b_4^1 & b_4^2 \end{bmatrix} & & \\
& & \ddots & \ddots & \\
& & & \begin{bmatrix} d_{2k-3} & b_{2k-3}^1 \\ & d_{2k-2} \end{bmatrix} & \begin{bmatrix} b_{2k-3}^2 & \\ b_{2k-2}^1 & b_{2k-2}^2 \end{bmatrix} \\
& & & & \begin{bmatrix} d_{2k-1} & b_{2k-1}^1 \\ & d_{2k} \end{bmatrix}
\end{bmatrix} \\
&= \begin{bmatrix}
D_1 & B_1 & & & \\
& D_2 & B_2 & & \\
& & \ddots & \ddots & \\
& & & D_{k-1} & B_{k-1} \\
& & & & D_k
\end{bmatrix} \in \mathbb{R}^{2k \times 2k}
\end{aligned}
$$

$$
X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{k-1} \\ Y_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \qquad Y^{\mathrm{T}} = \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{2 \times 2k}
$$

$$
\begin{aligned}
XY^{\mathrm{T}} &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix} X_1Y_1^{\mathrm{T}} & X_1Y_2^{\mathrm{T}} & \cdots & X_1Y_{k-1}^{\mathrm{T}} & x_1Y_k^{\mathrm{T}} \\ X_2Y_1^{\mathrm{T}} & X_2Y_2^{\mathrm{T}} & \cdots & X_2Y_{k-1}^{\mathrm{T}} & x_2Y_k^{\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{k-1}Y_1^{\mathrm{T}} & X_{k-1}Y_2^{\mathrm{T}} & \cdots & X_{k-1}Y_{k-1}^{\mathrm{T}} & X_{k-1}Y_k^{\mathrm{T}} \\ X_kY_1^{\mathrm{T}} & X_kY_2^{\mathrm{T}} & \cdots & X_kY_{k-1}^{\mathrm{T}} & X_kY_k^{\mathrm{T}} \end{bmatrix} \\
&= X\begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix} XY_1^{\mathrm{T}} & XY_2^{\mathrm{T}} & \cdots & XY_{k-1}^{\mathrm{T}} & XY_k^{\mathrm{T}} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{triu}(X_k) &\coloneqq X_k \\
\operatorname{triu}(Y_k^{\mathrm{T}}) &\coloneqq Y_k^{\mathrm{T}} \\
\operatorname{triu}(X_kY_k^{\mathrm{T}}) &= X_kY_k^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{triu}(XY^{\mathrm{T}}) &= \begin{bmatrix} \operatorname{triu}(X_1Y_1^{\mathrm{T}}) & X_1Y_2^{\mathrm{T}} & \cdots & X_1Y_{k-1}^{\mathrm{T}} & X_1Y_k^{\mathrm{T}} \\ 0_{2\times 2} & \operatorname{triu}(X_2Y_2^{\mathrm{T}}) & \cdots & X_2Y_{k-1}^{\mathrm{T}} & X_2Y_k^{\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{2\times 2} & 0_{2\times 2} & \cdots & \operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) & X_{k-1}Y_k^{\mathrm{T}} \\ 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & X_kY_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} \operatorname{triu}(X_1Y_1^{\mathrm{T}}) \\ 0_{2(k-1)\times 2} \end{bmatrix} & \begin{bmatrix} X_1Y_2^{\mathrm{T}} \\ \operatorname{triu}(X_2Y_2^{\mathrm{T}}) \\ 0_{2(k-2)\times 2} \end{bmatrix} & \cdots & \begin{bmatrix} X_{1:k-2}Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} & XY_k^{\mathrm{T}} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
U^{-1} &= \operatorname{triu}(XY^{\mathrm{T}}) \\
I &= U\operatorname{triu}(XY^{\mathrm{T}})
\end{aligned}
$$

$$
E_i = \begin{bmatrix} 0 & \cdots & 0 & I_{2\times 2} & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}
$$

k-th block column,

$$
\begin{aligned}
E_k &= U\operatorname{triu}(XY^{\mathrm{T}})E_k \\
&= UXY_k^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{aligned}
X &\coloneqq U^{-1}E_k \\
E_k &= UXY_k^{\mathrm{T}} \\
&= UU^{-1}E_kY_k^{\mathrm{T}} \\
&= E_kY_k^{\mathrm{T}} \\
Y_k^{\mathrm{T}} &\coloneqq I_{2\times 2}
\end{aligned}
$$

$(k-1)$-th block column,

$$
\begin{aligned}
E_{k-1} &= U \operatorname{triu}(XY^{\mathrm{T}}) E_{k-1} \\
&= U \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix}
\end{aligned}
$$

$(k-1)$-th block row of $(k-1)$-th block column,

$$
E_i^{\mathrm{T}} E_j = \begin{cases} I_{2\times 2}, & \text{if } i = j \ ; \\ 0_{2\times 2}, & \text{otherwise.} \end{cases}
$$

$$
\begin{aligned}
E_{k-1}^{\mathrm{T}} E_{k-1} = I_{2\times 2} &= E_{k-1}^{\mathrm{T}} U \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} \\
&= \begin{bmatrix} 0_{2\times 2(k-2)} & D_{k-1} & B_{k-1} \end{bmatrix} \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} \\
&= D_{k-1} \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\
Y_{k-1}^{\mathrm{T}} &\coloneqq (D_{k-1} X_{k-1})^{-1}
\end{aligned}
$$

$$
\begin{aligned}
D_{k-1} &= \operatorname{triu}(D_{k-1}) \\
D_{k-1}^{-1} &= \operatorname{triu}(D_{k-1}^{-1}) \\
Y_{k-1}^{\mathrm{T}} &\coloneqq (D_{k-1} X_{k-1})^{-1} \\
D_{k-1} \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) &= D_{k-1} \operatorname{triu}(X_{k-1} (D_{k-1} X_{k-1})^{-1}) \\
&= D_{k-1} \operatorname{triu}(X_{k-1} X_{k-1}^{-1} D_{k-1}^{-1}) \\
&= D_{k-1} \operatorname{triu}(D_{k-1}^{-1}) \\
&= D_{k-1} D_{k-1}^{-1} \\
&= I_{2\times 2}
\end{aligned}
$$

$$
\begin{aligned}
X &\coloneqq U^{-1} E_k \\
E_k &= UX \\
E_k^{\mathrm{T}} E_k = I_{2\times 2} &= E_k^{\mathrm{T}} U X \\
&= \begin{bmatrix} 0_{2\times 2(k-1)} & D_k \end{bmatrix} X \\
&= D_k X_k \\
Y_k^{\mathrm{T}} &\coloneqq I_{2\times 2} \\
&\coloneqq (D_k X_k)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
X &\coloneqq U^{-1} E_k \\
Y_i^{\mathrm{T}} &\coloneqq (D_i X_i)^{-1}
\end{aligned}
$$

$n = 2k + 1$

$$
\begin{aligned}
U &= \begin{bmatrix}
\begin{bmatrix} d_1 \end{bmatrix} & \begin{bmatrix} b_1^1 & b_1^2 \end{bmatrix} & & & \\
& \begin{bmatrix} d_2 & b_2^1 \\ & d_3 \end{bmatrix} & \begin{bmatrix} b_2^2 & \\ b_3^1 & b_3^2 \end{bmatrix} & & \\
& & \ddots & \ddots & \\
& & & \begin{bmatrix} d_{2k-2} & b_{2k-2}^1 \\ & d_{2k-1} \end{bmatrix} & \begin{bmatrix} b_{2k-2}^2 & \\ b_{2k-1}^1 & b_{2k-1}^2 \end{bmatrix} \\
& & & & \begin{bmatrix} d_{2k} & b_{2k}^1 \\ & d_{2k+1} \end{bmatrix}
\end{bmatrix} \\
&= \begin{bmatrix}
\tilde{D}_1 & \tilde{B}_1 & & & \\
& D_2 & B_2 & & \\
& & \ddots & \ddots & \\
& & & D_k & B_k \\
& & & & D_{k+1}
\end{bmatrix} \in \mathbb{R}^{(2k+1)\times(2k+1)}
\end{aligned}
$$

$$
\begin{aligned}
X &\coloneqq U^{-1} E_{k+1} \\
Y_i^{\mathrm{T}} &\coloneqq (D_i X_i)^{-1} \text{ for } i = 2,\ \cdots,\ k+1 \\
Y_1^{\mathrm{T}} &\coloneqq (D_1 X_1)^{+} = (D_1 X_1)^{\mathrm{T}} (D_1 X_1 (D_1 X_1)^{\mathrm{T}})^{-1}
\end{aligned}
$$

17.2 Banded matrix (Upper)

$n = ku + r,\ 0 \le r < u$

$$U = \begin{bmatrix} d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & & & \\ & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & & \\ & & \ddots & \ddots & \ddots & \cdots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \cdots & b_{n-u}^u \\ & & & & \ddots & \ddots & \ddots & \vdots \\ & & & & & \ddots & \ddots & b_{n-2}^2 \\ & & & & & & \ddots & b_{n-1}^1 \\ & & & & & & & d_n \end{bmatrix}$$

$r = 0,\ n = ku,$

$$U = \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{ku \times ku},$$

$$D_i \in \mathbb{R}^{u \times u},\ B_i \in \mathbb{R}^{u \times u} \text{ for } i = 1,\ 2,\ \cdots,\ k.$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} \in \mathbb{R}^{ku \times u} \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{bmatrix} \in \mathbb{R}^{ku \times u} \qquad Y^{\mathrm{T}} = \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_k^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{u \times ku}$$

$$X_i,\ Y_i^{\mathrm{T}} \in \mathbb{R}^{u \times u},\ \text{for } i = 1,\ 2,\ \cdots,\ k$$
$$X \coloneqq U^{-1} E_k$$
$$Y_i^{\mathrm{T}} \coloneqq (D_i X_i)^{-1}$$

$r \neq 0,\ n = ku + r,$

$$U = \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times (ku+r)},$$

$$\tilde{D}_1 \in \mathbb{R}^{r \times r},\ \tilde{B}_1 \in \mathbb{R}^{r \times u};$$
$$D_i \in \mathbb{R}^{u \times u},\ B_i \in \mathbb{R}^{u \times u} \text{ for } i = 2,\ \cdots,\ k+1.$$

$$X = \begin{bmatrix} \tilde{X}_1 \\ X_2 \\ \vdots \\ X_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times u} \quad Y = \begin{bmatrix} \tilde{Y}_1 \\ Y_2 \\ \vdots \\ Y_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times u} \quad Y^{\mathrm{T}} = \begin{bmatrix} \tilde{Y}_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k+1}^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{u \times (ku+r)}$$

$$
\begin{aligned}
\tilde{X}_1 &\in \mathbb{R}^{r\times u} \\
\tilde{Y}_1^{\mathrm{T}} &\in \mathbb{R}^{u\times r} \\
X_i,\ Y_i^{\mathrm{T}} &\in \mathbb{R}^{u\times u} \text{ for } i = 2,\ \cdots,\ k+1 \\
X &\coloneqq U^{-1}E_{k+1} \\
\tilde{Y}_1^{\mathrm{T}} &\coloneqq (D_1X_1)^{+} = (D_1X_1)^{\mathrm{T}}(D_1X_1(D_1X_1)^{\mathrm{T}})^{-1} \\
\mathrm{triu}(D_1X_1\tilde{Y}_1^{\mathrm{T}}) &= I_{r\times r} \\
Y_i^{\mathrm{T}} &\coloneqq (D_iX_i)^{-1} \text{ for } i = 2,\ \cdots,\ k+1
\end{aligned}
$$

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18.1 Notation

This notes use Householder notation and start indexing at one.

$$\begin{aligned}
\text{Scalars, lower case Greek letters: } & \alpha,\ \beta,\ \gamma,\ \chi,\ etc. \\
\text{Vectors, lower case Roman letter: } & a,\ b,\ c,\ x,\ etc. \\
\text{Matrix, upper case Roman letter: } & A,\ B,\ C,\ X,\ etc.
\end{aligned}$$

18.2 Repersentation

Matrix & Vector

$$x = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_n \end{bmatrix} \in \mathbb{R}^n$$

$$\begin{aligned}
A &= \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{1n} \end{bmatrix} \\
&= \left[\begin{array}{c|c|c|c} a_1 & a_2 & \cdots & a_n \end{array}\right] \in \mathbb{R}^{m\times n}
\end{aligned}$$

Block Matrix & Block Vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{kn},\ x_i \in \mathbb{R}^k$$

$$\begin{aligned}
A &= \left[\begin{array}{c|c|c} A_{11} & A_{12} & \cdots \quad A_{1n} \\ A_{21} & A_{22} & \cdots \quad A_{2n} \\ \vdots & \vdots & \\ A_{m1} & A_{m2} & \cdots \quad A_{1n} \end{array}\right] \\
&= \left[\begin{array}{c|c|c|c} A_1 & A_2 & \cdots & A_n \end{array}\right] \in \mathbb{R}^{lm\times kn},\ A_{ij} \in \mathbb{R}^{l\times k}
\end{aligned}$$

18.3 Content

Consider an upper banded matrix $U \in \mathbb{R}^{n\times n}$,

$$U = \begin{bmatrix} \alpha_1 \cdots \beta_1^1 & \beta_1^2 & & \beta_1^u & & & \\ & \alpha_2 \cdots \beta_2^1 & \beta_2^2 & & \beta_2^u & & \\ & & \ddots & & & \ddots & \\ & & & & & & \beta_{n-u}^u \\ & & & & & & \vdots \\ & & & & & & \beta_{n-2}^2 \\ & & & & & & \beta_{n-1}^1 \\ & & & & & & \alpha_n \end{bmatrix}. \tag{55}$$

Then try to show that its inverse can be repersent as the upper triangular part of an outer product, *i.e.* ,

$$U^{-1} = \mathrm{triu}(XY^{\mathrm{T}}) \tag{56}$$

Let $X,\ Y \in \mathbb{R}^{n\times u}$, consider the following cases of n,

Case 1, $n = ku$

then (55) can be repersent as a block upper bidiagonal matrix,

$$\begin{bmatrix} d_1 \cdots\cdots b_1^{r-1} & b_1^r & b_1^u & & & & & \\ \ddots & \vdots & & & & & & \\ & d_r & b_r^1 & & b_r^u & & & \\ & & d_{1+r} & & b_{1+r}^{u-1} & b_{r+1}^u & & \\ & & \vdots & & & & & \\ & & & & d_{u+r} & b_{u+r}^1 & b_{u+r}^u & \\ & & & & & \ddots & \vdots \quad \vdots & \ddots \\ & & & & & & d_{n-u+1} & b_{n-u+1}^u \\ & & & & & & & \ddots \\ & & & & & & & d_n \end{bmatrix}$$

$$U = \begin{bmatrix} \alpha_1 & \beta_1^{u-1} & \beta_1^{u} & & & & & \\ & \ddots & \ddots & & & & & \\ & & \alpha_u & \beta_u^{1} & \beta_u^{u} & & & \\ & & & \alpha_{u+1} & \beta_{u+1}^{u-1} & \beta_{u+1}^{u} & & \\ & & & & \ddots & & & \\ & & & & \alpha_{2u} & \beta_{2u}^{1} & \beta_{2u}^{u} & \\ & & & & & \ddots & \ddots & \\ & & & & & & \alpha_{(k-1)u+1} & \beta_{(k-1)u+1}^{u-1} \\ & & & & & & & \alpha_{ku} \end{bmatrix}$$

$$= \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{ku\times ku},\ D_i,\ B_i \in \mathbb{R}^{u\times u} \text{ for } i = 1,\ 2,\ \ldots,\ k. \tag{57}$$

Repersent the n-dimensional identity matrix $I_{n\times n}$ in block form,

$$I_{n\times n} = I_{ku\times ku} = \operatorname{diag}(\underbrace{I_{u\times u},\ I_{u\times u},\ \cdots,\ I_{u\times u}}_{k}) = \begin{bmatrix} E_1 & E_2 & \cdots & E_k \end{bmatrix}.$$

then, its block columns E_i for $i = 1,\ 2,\ \ldots,\ k$ satisfies

$$E_i \in \mathbb{R}^{ku\times u}$$
$$E_i^{\mathrm{T}} E_j = \begin{cases} I_{u\times u}, & \text{if } i = j\ ; \\ 0_{u\times u}, & \text{otherwise.} \end{cases}$$

To find X and Y such that (56) holds, let

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \in \mathbb{R}^{ku\times u},\ Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{k-1} \\ Y_k \end{bmatrix} \in \mathbb{R}^{ku\times u},\ X_i,\ Y_i \in \mathbb{R}^{u\times u} \text{ for } i = 1,\ 2,\ \ldots,\ k,$$

and let X_k and Y_k^{T} be upper triangular matrix, then $\operatorname{triu}(X_k Y_k^{\mathrm{T}}) = X_k Y_k^{\mathrm{T}}$.
For the outer product form,

$$
\begin{aligned}
Y^{\mathrm{T}} &= \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{u \times ku} \\
XY^{\mathrm{T}} &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix} X_1Y_1^{\mathrm{T}} & X_1Y_2^{\mathrm{T}} & \cdots & X_1Y_{k-1}^{\mathrm{T}} & x_1Y_k^{\mathrm{T}} \\ X_2Y_1^{\mathrm{T}} & X_2Y_2^{\mathrm{T}} & \cdots & X_2Y_{k-1}^{\mathrm{T}} & x_2Y_k^{\mathrm{T}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{k-1}Y_1^{\mathrm{T}} & X_{k-1}Y_2^{\mathrm{T}} & \cdots & X_{k-1}Y_{k-1}^{\mathrm{T}} & X_{k-1}Y_k^{\mathrm{T}} \\ X_kY_1^{\mathrm{T}} & X_kY_2^{\mathrm{T}} & \cdots & X_kY_{k-1}^{\mathrm{T}} & X_kY_k^{\mathrm{T}} \end{bmatrix} \\
&= X \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix} XY_1^{\mathrm{T}} & XY_2^{\mathrm{T}} & \cdots & XY_{k-1}^{\mathrm{T}} & XY_k^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{ku \times ku},
\end{aligned}
$$

and its the upper triangular componnent,

$$
\begin{aligned}
\operatorname{triu}(XY^{\mathrm{T}}) &= \begin{bmatrix} \operatorname{triu}(X_1Y_1^{\mathrm{T}}) & X_1Y_2^{\mathrm{T}} & \cdots & X_1Y_{k-1}^{\mathrm{T}} & X_1Y_k^{\mathrm{T}} \\ 0_{u\times u} & \operatorname{triu}(X_2Y_2^{\mathrm{T}}) & \cdots & X_2Y_{k-1}^{\mathrm{T}} & X_2Y_k^{\mathrm{T}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{u\times u} & 0_{u\times u} & \cdots & \operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) & X_{k-1}Y_k^{\mathrm{T}} \\ 0_{u\times u} & 0_{u\times u} & \cdots & 0_{u\times u} & X_kY_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix} \operatorname{triu}(X_1Y_1^{\mathrm{T}}) & X_1Y_2^{\mathrm{T}} & & X_{1:k-2}Y_{k-1}^{\mathrm{T}} & \\ 0_{(k-2)u\times u} & \operatorname{triu}(X_2Y_2^{\mathrm{T}}) & \cdots & \operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) & XY_k^{\mathrm{T}} \\ 0_{u\times u} & 0_{(k-2)u\times u} & & 0_{u\times u} & \end{bmatrix} .zz
\end{aligned} \tag{58}
$$

From (56),

$$
I_{n\times n} = U \operatorname{triu}(XY^{\mathrm{T}}). \tag{59}
$$

For X and Y_k, from (58), the k-th block column of (59) can be written as,

$$
\begin{aligned}
E_k &= U \operatorname{triu}(XY^{\mathrm{T}})E_k \\
E_k &= UXY_k^{\mathrm{T}}
\end{aligned} \tag{60}
$$

Define that

$$
X \coloneqq U^{-1}E_k; \tag{61}
$$

$$
Y_k^{\mathrm{T}} \coloneqq I_{u\times u}, \tag{62}
$$

which can be shown that (60) holds for the defined (61) and (62) through

$$
\begin{aligned}
UXY_k^{\mathrm{T}} &= UU^{-1}E_kI_{u\times u} \\
&= I_{n\times n}E_kI_{u\times u} \\
&= E_kI_{u\times u} \\
&= E_k, \quad \text{as required.}
\end{aligned}
$$

Then, for Y_{k-1}, forming the $(k-1)$-th block column of (59) from (58) first,

$$\begin{aligned} E_{k-1} &= U \operatorname{triu}(XY^{\mathrm{T}}) E_{k-1} \\ &= U \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{u \times u} \end{bmatrix}. \end{aligned}$$

Focusing on the $(k-1)$-th block row of $(k-1)$-th block column,

$$\begin{aligned} E_{k-1}^{\mathrm{T}} E_{k-1} &= E_{k-1}^{\mathrm{T}} U \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{u \times u} \end{bmatrix} \\ I_{u \times u} &= \begin{bmatrix} 0_{u \times (k-2)u} & D_{k-1} & B_{k-1} \end{bmatrix} \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{u \times u} \end{bmatrix} \\ I_{u \times u} &= D_{k-1} \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}). \end{aligned} \tag{63}$$

Define that

$$Y_{k-1}^{\mathrm{T}} \coloneqq (D_{k-1} X_{k-1})^{-1}, \tag{64}$$

and notice that $D_{k-1} \in \mathbb{R}^{u \times u}$ is a upper triangular, therefore

$$\begin{aligned} D_{k-1} &= \operatorname{triu}(D_{k-1}); \\ D_{k-1}^{-1} &= \operatorname{triu}(D_{k-1}^{-1}). \end{aligned} \tag{65}$$

With the fact (65), (63) holds for the defined (64) as,

$$\begin{aligned} D_{k-1} \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) &= D_{k-1} \operatorname{triu}(X_{k-1} (D_{k-1} X_{k-1})^{-1}) \\ &= D_{k-1} \operatorname{triu}(X_{k-1} X_{k-1}^{-1} D_{k-1}^{-1}) \\ &= D_{k-1} \operatorname{triu}(D_{k-1}^{-1}) \\ &= D_{k-1} D_{k-1}^{-1} \\ &= I_{u \times u}, \text{ as required.} \end{aligned}$$

Similar ideas can be use to redefine Y_k^{T} in (62), from (61),

$$\begin{aligned} E_k &= UX \\ E_k^{\mathrm{T}} E_k &= E_k^{\mathrm{T}} UX \\ I_{u \times u} &= E_k^{\mathrm{T}} UX \\ &= \begin{bmatrix} 0_{u \times (k-1)u} & D_k \end{bmatrix} \begin{bmatrix} X_{1:k-1} \\ X_k \end{bmatrix} \\ &= D_k X_k \\ I_{u \times u} &= (D_k X_k)^{-1}, \end{aligned}$$

following that

$$
\begin{aligned}
Y_k^{\mathrm{T}} &:= I_{u\times u} \\
&:= (D_k X_k)^{-1}.
\end{aligned}
$$

In summary, for $U \in \mathbb{R}^{ku\times ku}$, (56) holds the following definitions of X and Y:

$$
\begin{aligned}
X &:= U^{-1} E_k \\
Y_i^{\mathrm{T}} &:= (D_i X_i)^{-1}
\end{aligned}
$$