MSc Project Notes

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1 02.08.2022

1.1 Bidiagonal matrix

Let $\tilde{U} \in \mathbb{R}^{n\times n}$ be a bidiagonal matrix with non-zero diagonal, *i.e.* $\tilde{u}_{ii} \neq 0$ for $i = 1, 2, \dots, n$,

$$\tilde{U} = \begin{bmatrix} \tilde{u}_{11} & \tilde{u}_{12} & & & \\ & \tilde{u}_{22} & \tilde{u}_{23} & & \\ & & \tilde{u}_{33} & \tilde{u}_{34} & \\ & & & \tilde{u}_{44} & \tilde{u}_{45} \\ & & & & \tilde{u}_{55} \end{bmatrix}$$

and $D = \mathrm{diag}(\tilde{u}_{ii})$ for $i = 1, 2, \dots, n$,

$$D = \begin{bmatrix} \frac{1}{\tilde{u}_{11}} & & & & \\ & \frac{1}{\tilde{u}_{22}} & & & \\ & & \frac{1}{\tilde{u}_{33}} & & \\ & & & \frac{1}{\tilde{u}_{44}} & \\ & & & & \frac{1}{\tilde{u}_{55}} \end{bmatrix}$$

replace $\frac{\tilde{u}_{i,i+1}}{\tilde{u}_{i+1,i+1}}$ by $u_{i,i+1}$ for $i = 1, 2, \dots, n-1$, and let $U = \tilde{U}D$,

$$\begin{aligned} U &= \begin{bmatrix} 1 & \frac{\tilde{u}_{12}}{\tilde{u}_{22}} & & & \\ & 1 & \frac{\tilde{u}_{23}}{\tilde{u}_{33}} & & \\ & & 1 & \frac{\tilde{u}_{34}}{\tilde{u}_{44}} & \\ & & & 1 & \frac{\tilde{u}_{45}}{\tilde{u}_{55}} \\ & & & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & u_{12} & & & \\ & 1 & u_{23} & & \\ & & 1 & u_{34} & \\ & & & 1 & u_{45} \\ & & & & 1 \end{bmatrix}. \end{aligned}$$

Let $A = U^{-1}$ be the inverse of U, a upper triangular matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ & a_{22} & a_{23} & a_{24} & a_{25} \\ & & a_{33} & a_{34} & a_{35} \\ & & & a_{44} & a_{45} \\ & & & & a_{55} \end{bmatrix}$$

then $I = UU^{-1} = UA$,

$$I = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} + a_{22}u_{12} & a_{13} + a_{23}u_{12} & a_{14} + a_{24}u_{12} & a_{15} + a_{25}u_{12} \\ & a_{22} & a_{23} + a_{33}u_{23} & a_{24} + a_{34}u_{23} & a_{25} + a_{35}u_{23} \\ & & a_{33} & a_{34} + a_{44}u_{34} & a_{35} + a_{45}u_{34} \\ & & & a_{44} & a_{45} + a_{55}u_{45} \\ & & & & a_{55} \end{bmatrix} = UA.$$

Note that $UA = \mathrm{diag}(a_{ii}) + \mathrm{triu}(UA, 1)$ for $i = 1, 2, \ldots, n$.
Find A such that $\mathrm{diag}(a_{ii}) = I$ and $\mathrm{triu}(UA, 1) = 0_{n\times n}$ for $i = 1, 2, \ldots, n$.

$$xy^{\mathrm{T}} = \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 & x_1y_4 & x_1y_5 \\ x_2y_1 & x_2y_2 & x_2y_3 & x_2y_4 & x_2y_5 \\ x_3y_1 & x_3y_2 & x_3y_3 & x_3y_4 & x_3y_5 \\ x_4y_1 & x_4y_2 & x_4y_3 & x_4y_4 & x_4y_5 \\ x_5y_1 & x_5y_2 & x_5y_3 & x_5y_4 & x_5y_5 \end{bmatrix}$$

Express A in form of an outer product, *i.e.* $A = xy^{\mathrm{T}}$ for $\forall x, y \in \mathbb{R}^n$, then

$$\mathrm{triu}(UA, 1) = \mathrm{triu}(Uxy^{\mathrm{T}}, 1) = 0_{n\times n}$$

Note that $\mathrm{triu}(e_ny^{\mathrm{T}}, 1) = 0_{n\times n}$ as,

$$e_ny^{\mathrm{T}} = \begin{bmatrix} \\ \\ \\ \\ 1 \end{bmatrix} \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix} = \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix}$$

solve $Ux := e_n$ for x by backward substitution, then let $y_i := \frac{1}{x_i}$ or $x_iy_i = 1$ for $i = 1, 2, \ldots, n$.
Then,

$$\begin{aligned} UA &= \mathrm{diag}(a_{ii}) + \mathrm{triu}(UA, 1) \\ &= \mathrm{diag}(x_iy_i) + \mathrm{triu}(Uxy^{\mathrm{T}}, 1) \\ &= \mathrm{diag}(\underbrace{1, 1, \cdots, 1}_{n}) + \mathrm{triu}(e_ny^{\mathrm{T}}, 1) \\ &= I + 0_{n\times n} \\ &= I \end{aligned}$$

1.2 Upper triangular matrix

Let $\tilde{U} \in \mathbb{R}^{n\times n}$ be a banded matrix with non-zero diagonal, upper banded width u, and lower banded width 0 Similarly,

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} & & \\ & 1 & u_{23} & u_{24} & \\ & & 1 & u_{34} & u_{35} \\ & & & 1 & u_{45} \\ & & & & 1 \end{bmatrix}$$

$$UA = \begin{bmatrix} a_{11} & a_{12}+a_{22}u_{12} & a_{13}+a_{23}u_{12}+a_{33}u_{13} & a_{14}+a_{24}u_{12}+a_{34}u_{13} & a_{15}+a_{25}u_{12}+a_{35}u_{13} \\ & a_{22} & a_{23}+a_{33}u_{23} & a_{24}+a_{34}u_{23}+a_{44}u_{24} & a_{25}+a_{35}u_{23}+a_{45}u_{24} \\ & & a_{33} & a_{34}+a_{44}u_{34} & a_{35}+a_{45}u_{34}+a_{55}u_{35} \\ & & & a_{44} & a_{45}+a_{55}u_{45} \\ & & & & a_{55} \end{bmatrix}$$

$$= \operatorname{diag}(a_{ii}) + \operatorname{triu}(UA, 1) \text{ for } i = 1, 2, \dots, n$$

Find A such that $\operatorname{diag}(a_{ii}) = I$ and $\operatorname{triu}(UA, 1) = 0_{n\times n}$ for $i = 1, 2, \dots, n$.

$$XY^{\mathrm{T}} = \begin{bmatrix} x_{11}y_{11}+x_{12}y_{12} & x_{11}y_{21}+x_{12}y_{22} & x_{11}y_{31}+x_{12}y_{32} & x_{11}y_{41}+x_{12}y_{42} & x_{11}y_{51}+x_{12}y_{52} \\ x_{21}y_{11}+x_{22}y_{12} & x_{21}y_{21}+x_{22}y_{22} & x_{21}y_{31}+x_{22}y_{32} & x_{21}y_{41}+x_{22}y_{42} & x_{21}y_{51}+x_{22}y_{52} \\ x_{31}y_{11}+x_{32}y_{12} & x_{31}y_{21}+x_{32}y_{22} & x_{31}y_{31}+x_{32}y_{32} & x_{31}y_{41}+x_{32}y_{42} & x_{31}y_{51}+x_{32}y_{52} \\ x_{41}y_{11}+x_{42}y_{12} & x_{41}y_{21}+x_{42}y_{22} & x_{41}y_{31}+x_{42}y_{32} & x_{41}y_{41}+x_{42}y_{42} & x_{41}y_{51}+x_{42}y_{52} \\ x_{51}y_{11}+x_{52}y_{12} & x_{51}y_{21}+x_{52}y_{22} & x_{51}y_{31}+x_{52}y_{32} & x_{51}y_{41}+x_{52}y_{42} & x_{51}y_{51}+x_{52}y_{52} \end{bmatrix}$$

Express A in outer product, *i.e.* $A = XY^{\mathrm{T}}$, for $\forall X, Y \in \mathbb{R}^{n\times 2}$,

$$\operatorname{triu}(UA, 1) = \operatorname{triu}(UXY^{\mathrm{T}}, 1) = 0_{n\times n}$$

Note that $\operatorname{triu}(\begin{bmatrix} e_{n-1} & e_n \end{bmatrix} Y^{\mathrm{T}}, 2) = 0_{n\times n}$ as,

$$\begin{bmatrix} e_{n-1} & e_n \end{bmatrix} Y^{\mathrm{T}} = \begin{bmatrix} & \\ & \\ & \\ 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} y_{11} & y_{21} & y_{31} & y_{41} & y_{51} \\ y_{12} & y_{22} & y_{32} & y_{42} & y_{52} \end{bmatrix} = \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ y_{11} & y_{21} & y_{31} & y_{41} & y_{51} \\ y_{12} & y_{22} & y_{32} & y_{42} & y_{52} \end{bmatrix}$$

solve $UX = U\begin{bmatrix} x_1 & x_2 \end{bmatrix} := \begin{bmatrix} e_{n-1} & e_n \end{bmatrix}$ for x_1 and x_2 by backward substitution,

2 09.06.2022

2.1 Notation

Vectors

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

Matrices

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{m \times n},\ a_i \in \mathbb{R}^m \end{aligned}$$

Columns

$$\begin{aligned} Ae_i &:= a_i \\ e_i &= i\text{-th column of } I \end{aligned}$$

2.2 Bidiagonal matrix (upper)

$$U = \begin{bmatrix} d_1 & b_1 & & & \\ & d_2 & b_2 & & \\ & & \ddots & \ddots & \\ & & & d_{n-1} & b_{n-1} \\ & & & & d_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \in \mathbb{R}^{n \times 1} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} \in \mathbb{R}^{n \times 1} \qquad y^{\mathrm{T}} = \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \in \mathbb{R}^{1 \times n}$$

$$
\begin{aligned}
xy^{\mathrm{T}} &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \\
&= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_{n-1} & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_{n-1} & x_2y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_4y_1 & x_4y_2 & \cdots & x_4y_{n-1} & x_4y_n \\ x_5y_1 & x_5y_2 & \cdots & x_5y_{n-1} & x_5y_n \end{bmatrix} \\
&= x\begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \\
&= \begin{bmatrix} xy_1 & xy_2 & \cdots & xy_{n-1} & xy_n \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{triu}(xy^{\mathrm{T}}) &= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_{n-1} & x_1y_n \\ 0 & x_2y_2 & \cdots & x_2y_{n-1} & x_2y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_4y_{n-1} & x_4y_n \\ 0 & 0 & \cdots & 0 & x_5y_n \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} x_{1:1} \\ 0_{n-1} \end{bmatrix} y_1 & \begin{bmatrix} x_{1:2} \\ 0_{n-2} \end{bmatrix} y_2 & \cdots & \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} & xy_n \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
U^{-1} &= \operatorname{triu}(xy^{\mathrm{T}}) \\
I &= U \operatorname{triu}(xy^{\mathrm{T}})
\end{aligned}
$$

n-th column,

$$
\begin{aligned}
e_n &= U \operatorname{triu}(xy^{\mathrm{T}}) e_n \\
&= Uxy_n
\end{aligned}
$$

$$
\begin{aligned}
x &:= U^{-1}e_n \\
e_n &= Uxy_n \\
&= UU^{-1}e_ny_n \\
&= e_ny_n \\
y_n &:= 1
\end{aligned}
$$

$(n-1)$-th column,

$$
\begin{aligned}
e_{n-1} &= U \operatorname{triu}(xy^{\mathrm{T}}) e_{n-1} \\
&= U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1}
\end{aligned}
$$

$(n-1)$-th row of $(n-1)$-th column,

$$e_i^{\mathrm{T}} e_j = \delta_{ij}$$

$$\begin{aligned}
e_{n-1}^{\mathrm{T}} e_{n-1} = 1 &= e_{n-1}^{\mathrm{T}} U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \\
&= \begin{bmatrix} 0_{n-2} & d_{n-1} & b_{n-1} \end{bmatrix} \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \\
&= d_{n-1} x_{n-1} y_{n-1} \\
y_{n-1} &:= (d_{n-1} x_{n-1})^{-1}
\end{aligned}$$

$$\begin{aligned}
e_n &= Ux \\
e_n^{\mathrm{T}} e_n = 1 &= e_n^{\mathrm{T}} U x \\
&= \begin{bmatrix} 0_{n-1} & d_n \end{bmatrix} x \\
&= d_n x_n \\
y_n &:= 1 \\
&:= (d_n x_n)^{-1}
\end{aligned}$$

$$\begin{aligned}
x &:= U^{-1} e_n \\
y_i &:= (d_i x_i)^{-1}
\end{aligned}$$

2.3 Tridiagonal matrix (Upper)

$n = 2k$

$$
\begin{aligned}
U &= \begin{bmatrix}
d_1 & b_1^1 & b_1^2 & & & & & & \\
 & d_2 & b_2^1 & b_2^2 & & & & & \\
 & & d_3 & b_3^1 & b_3^2 & & & & \\
 & & & d_4 & b_4^1 & b_4^2 & & & \\
 & & & & \ddots & \ddots & \ddots & & \\
 & & & & & d_{n-3} & b_{n-3}^1 & b_{n-3}^2 & \\
 & & & & & & d_{n-2} & b_{n-2}^1 & b_{n-2}^2 \\
 & & & & & & & d_{n-1} & b_{n-1}^1 \\
 & & & & & & & & d_n
\end{bmatrix} \\
&= \begin{bmatrix}
\begin{bmatrix} d_1 & b_1^1 \\ & d_2 \end{bmatrix} & \begin{bmatrix} b_1^2 & \\ b_2^1 & b_2^2 \end{bmatrix} & & & \\
 & \begin{bmatrix} d_3 & b_3^1 \\ & d_4 \end{bmatrix} & \begin{bmatrix} b_3^2 & \\ b_4^1 & b_4^2 \end{bmatrix} & & \\
 & & \ddots & \ddots & \\
 & & & \begin{bmatrix} d_{n-3} & b_{n-3}^1 \\ & d_{n-2} \end{bmatrix} & \begin{bmatrix} b_{n-3}^2 & \\ b_{n-2}^1 & b_{n-2}^2 \end{bmatrix} \\
 & & & & \begin{bmatrix} d_{n-1} & b_{n-1}^1 \\ & d_n \end{bmatrix}
\end{bmatrix} \\
&= \begin{bmatrix}
D_1 & B_1 & & & \\
 & D_2 & B_2 & & \\
 & & \ddots & \ddots & \\
 & & & D_{k-1} & B_{k-1} \\
 & & & & D_k
\end{bmatrix} \in \mathbb{R}^{n\times n} = \mathbb{R}^{2k\times 2k}
\end{aligned}
$$

$$
\begin{bmatrix} d_{n-3} & b_{n-3}^1 \\ & d_{n-2} \end{bmatrix}
$$

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \in \mathbb{R}^{n\times 1} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} \in \mathbb{R}^{n\times 1} \qquad y^{\mathrm{T}} = \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \in \mathbb{R}^{1\times n}
$$

$$
\begin{aligned}
xy^{\mathrm{T}} &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \\
&= \begin{bmatrix}
x_1y_1 & x_1y_2 & \cdots & x_1y_{n-1} & x_1y_n \\
x_2y_1 & x_2y_2 & \cdots & x_2y_{n-1} & x_2y_n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_4y_1 & x_4y_2 & \cdots & x_4y_{n-1} & x_4y_n \\
x_5y_1 & x_5y_2 & \cdots & x_5y_{n-1} & x_5y_n
\end{bmatrix} \\
&= x\begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \\
&= \begin{bmatrix} xy_1 & xy_2 & \cdots & xy_{n-1} & xy_n \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{triu}(xy^{\mathrm{T}}) &= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_{n-1} & x_1y_n \\ 0 & x_2y_2 & \cdots & x_2y_{n-1} & x_2y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_4y_{n-1} & x_4y_n \\ 0 & 0 & \cdots & 0 & x_5y_n \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} x_{1:1} \\ 0_{n-1} \end{bmatrix} y_1 & \begin{bmatrix} x_{1:2} \\ 0_{n-2} \end{bmatrix} y_2 & \cdots & \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} & xy_n \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
U^{-1} &= \operatorname{triu}(xy^{\mathrm{T}}) \\
I &= U \operatorname{triu}(xy^{\mathrm{T}})
\end{aligned}
$$

n-th column,

$$
\begin{aligned}
e_n &= U \operatorname{triu}(xy^{\mathrm{T}}) e_n \\
&= Uxy_n
\end{aligned}
$$

$$
\begin{aligned}
x &:= U^{-1} e_n \\
e_n &= Uxy_n \\
&= UU^{-1} e_n y_n \\
&= e_n y_n \\
y_n &:= 1
\end{aligned}
$$

$(n-1)$-th column,

$$
\begin{aligned}
e_{n-1} &= U \operatorname{triu}(xy^{\mathrm{T}}) e_{n-1} \\
&= U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1}
\end{aligned}
$$

$(n-1)$-th row of $(n-1)$-th column,

$$
e_i^{\mathrm{T}} e_j = \delta_{ij}
$$

$$
\begin{aligned}
e_{n-1}^{\mathrm{T}} e_{n-1} = 1 &= e_{n-1}^{\mathrm{T}} U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \\
&= \begin{bmatrix} 0_{n-2} & d_{n-1} & u_{n-1} \end{bmatrix} \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \\
&= d_{n-1} x_{n-1} y_{n-1} \\
y_{n-1} &:= (d_{n-1} x_{n-1})^{-1}
\end{aligned}
$$

$$
\begin{aligned}
e_n &= Ux \\
e_n^{\mathrm{T}} e_n = 1 &= e_n^{\mathrm{T}} Ux \\
&= \begin{bmatrix} 0_{n-1} & d_n \end{bmatrix} x \\
&= d_n x_n \\
y_n &:= 1 \\
&:= (d_n x_n)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
x &:= U^{-1} e_n \\
y_i &:= (d_i x_i)^{-1}
\end{aligned}
$$