

# MSc Project Notes

Zhiwei Zhou

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## 1 02.08.2022

### 1.1 Bidiagonal matrix

Let  $\tilde{U} \in \mathbb{R}^{n \times n}$  be a bidiagonal matrix with non-zero diagonal, *i.e.*  $\tilde{u}_{ii} \neq 0$  for  $i = 1, 2, \dots, n$ ,

$$\tilde{U} = \begin{bmatrix} \tilde{u}_{11} & \tilde{u}_{12} & & & \\ & \tilde{u}_{22} & \tilde{u}_{23} & & \\ & & \tilde{u}_{33} & \tilde{u}_{34} & \\ & & & \tilde{u}_{44} & \tilde{u}_{45} \\ & & & & \tilde{u}_{55} \end{bmatrix}$$

and  $D = \text{diag}(\tilde{u}_{ii})$  for  $i = 1, 2, \dots, n$ ,

$$D = \begin{bmatrix} \frac{1}{\tilde{u}_{11}} & & & & \\ & \frac{1}{\tilde{u}_{22}} & & & \\ & & \frac{1}{\tilde{u}_{33}} & & \\ & & & \frac{1}{\tilde{u}_{44}} & \\ & & & & \frac{1}{\tilde{u}_{55}} \end{bmatrix}$$

replace  $\frac{\tilde{u}_{i,i+1}}{\tilde{u}_{i+1,i+1}}$  by  $u_{i,i+1}$  for  $i = 1, 2, \dots, n-1$ , and let  $U = \tilde{U}D$ ,

$$\begin{aligned} U &= \begin{bmatrix} 1 & \frac{\tilde{u}_{12}}{\tilde{u}_{22}} & & & \\ & 1 & \frac{\tilde{u}_{23}}{\tilde{u}_{33}} & & \\ & & 1 & \frac{\tilde{u}_{34}}{\tilde{u}_{44}} & \\ & & & 1 & \frac{\tilde{u}_{45}}{\tilde{u}_{55}} \\ & & & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & u_{12} & & & \\ & 1 & u_{23} & & \\ & & 1 & u_{34} & \\ & & & 1 & u_{45} \\ & & & & 1 \end{bmatrix}. \end{aligned}$$

Let  $A = U^{-1}$  be the inverse of  $U$ , a upper triangular matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ & a_{22} & a_{23} & a_{24} & a_{25} \\ & & a_{33} & a_{34} & a_{35} \\ & & & a_{44} & a_{45} \\ & & & & a_{55} \end{bmatrix}$$

then  $I = UU^{-1} = UA$ ,

$$I = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} + a_{22}u_{12} & a_{13} + a_{23}u_{12} & a_{14} + a_{24}u_{12} & a_{15} + a_{25}u_{12} \\ & a_{22} & a_{23} + a_{33}u_{23} & a_{24} + a_{34}u_{23} & a_{25} + a_{35}u_{23} \\ & & a_{33} & a_{34} + a_{44}u_{34} & a_{35} + a_{45}u_{34} \\ & & & a_{44} & a_{45} + a_{55}u_{45} \\ & & & & a_{55} \end{bmatrix} = UA.$$

Note that  $UA = \text{diag}(a_{ii}) + \text{triu}(UA, 1)$  for  $i = 1, 2, \dots, n$ .

Find  $A$  such that  $\text{diag}(a_{ii}) = I$  and  $\text{triu}(UA, 1) = 0_{n \times n}$  for  $i = 1, 2, \dots, n$ .

$$xy^T = \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 & x_1y_4 & x_1y_5 \\ x_2y_1 & x_2y_2 & x_2y_3 & x_2y_4 & x_2y_5 \\ x_3y_1 & x_3y_2 & x_3y_3 & x_3y_4 & x_3y_5 \\ x_4y_1 & x_4y_2 & x_4y_3 & x_4y_4 & x_4y_5 \\ x_5y_1 & x_5y_2 & x_5y_3 & x_5y_4 & x_5y_5 \end{bmatrix}$$

Express  $A$  in form of an outer product, i.e.  $A = xy^T$  for  $\forall x, y \in \mathbb{R}^n$ , then

$$\text{triu}(UA, 1) = \text{triu}(Uxy^T, 1) = 0_{n \times n}$$

Note that  $\text{triu}(e_n y^T, 1) = 0_{n \times n}$  as,

$$e_n y^T = \begin{bmatrix} \\ \\ \\ \\ 1 \end{bmatrix} \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix}$$

solve  $Ux := e_n$  for  $x$  by backward substitution, then let  $y_i := \frac{1}{x_i}$  or  $x_i y_i = 1$  for  $i = 1, 2, \dots, n$ . Then,

$$\begin{aligned} UA &= \text{diag}(a_{ii}) + \text{triu}(UA, 1) \\ &= \text{diag}(x_i y_i) + \text{triu}(Uxy^T, 1) \\ &= \text{diag}(\underbrace{1, 1, \dots, 1}_n) + \text{triu}(e_n y^T, 1) \\ &= I + 0_{n \times n} \\ &= I \end{aligned}$$

## 1.2 Upper triangular matrix

Let  $\tilde{U} \in \mathbb{R}^{n \times n}$  be a banded matrix with non-zero diagonal, upper banded width  $u$ , and lower banded width 0. Similarly,

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} & & \\ & 1 & u_{23} & u_{24} & \\ & & 1 & u_{34} & u_{35} \\ & & & 1 & u_{45} \\ & & & & 1 \end{bmatrix}$$

$$\begin{aligned}
UA &= \begin{bmatrix} a_{11} & a_{12} + a_{22}u_{12} & a_{13} + a_{23}u_{12} + a_{33}u_{13} & a_{14} + a_{24}u_{12} + a_{34}u_{13} & a_{15} + a_{25}u_{12} + a_{35}u_{13} \\ & a_{22} & a_{23} + a_{33}u_{23} & a_{24} + a_{34}u_{23} + a_{44}u_{24} & a_{25} + a_{35}u_{23} + a_{45}u_{24} \\ & & a_{33} & a_{34} + a_{44}u_{34} & a_{35} + a_{45}u_{34} + a_{55}u_{35} \\ & & & a_{44} & a_{45} + a_{55}u_{45} \\ & & & & a_{55} \end{bmatrix} \\
&= \text{diag}(a_{ii}) + \text{triu}(UA, 1) \text{ for } i = 1, 2, \dots, n
\end{aligned}$$

Find  $A$  such that  $\text{diag}(a_{ii}) = I$  and  $\text{triu}(UA, 1) = 0_{n \times n}$  for  $i = 1, 2, \dots, n$ .

$$XY^T = \begin{bmatrix} x_{11}y_{11} + x_{12}y_{12} & x_{11}y_{21} + x_{12}y_{22} & x_{11}y_{31} + x_{12}y_{32} & x_{11}y_{41} + x_{12}y_{42} & x_{11}y_{51} + x_{12}y_{52} \\ x_{21}y_{11} + x_{22}y_{12} & x_{21}y_{21} + x_{22}y_{22} & x_{21}y_{31} + x_{22}y_{32} & x_{21}y_{41} + x_{22}y_{42} & x_{21}y_{51} + x_{22}y_{52} \\ x_{31}y_{11} + x_{32}y_{12} & x_{31}y_{21} + x_{32}y_{22} & x_{31}y_{31} + x_{32}y_{32} & x_{31}y_{41} + x_{32}y_{42} & x_{31}y_{51} + x_{32}y_{52} \\ x_{41}y_{11} + x_{42}y_{12} & x_{41}y_{21} + x_{42}y_{22} & x_{41}y_{31} + x_{42}y_{32} & x_{41}y_{41} + x_{42}y_{42} & x_{41}y_{51} + x_{42}y_{52} \\ x_{51}y_{11} + x_{52}y_{12} & x_{51}y_{21} + x_{52}y_{22} & x_{51}y_{31} + x_{52}y_{32} & x_{51}y_{41} + x_{52}y_{42} & x_{51}y_{51} + x_{52}y_{52} \end{bmatrix}$$

Express  $A$  in outer product, *i.e.*  $A = XY^T$ , for  $\forall X, Y \in \mathbb{R}^{n \times 2}$ ,

$$\text{triu}(UA, 1) = \text{triu}(UXY^T, 1) = 0_{n \times n}$$

Note that  $\text{triu}([e_{n-1} \ e_n]Y^T, 2) = 0_{n \times n}$  as,

$$[e_{n-1} \ e_n]Y^T = \begin{bmatrix} & \\ & \\ & \\ 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} y_{11} & y_{21} & y_{31} & y_{41} & y_{51} \\ y_{12} & y_{22} & y_{32} & y_{42} & y_{52} \end{bmatrix} = \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ y_{11} & y_{21} & y_{31} & y_{41} & y_{51} \\ y_{12} & y_{22} & y_{32} & y_{42} & y_{52} \end{bmatrix}$$

solve  $UX = U[x_1 \ x_2] := [e_{n-1} \ e_n]$  for  $x_1$  and  $x_2$  by backward substitution,

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### 2.1 Notation

#### Vectors

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

#### Matrices and Columns

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ &= [a_1 \ a_2 \ \cdots \ a_n] \in \mathbb{R}^{m \times n}, \ a_i \in \mathbb{R}^m \end{aligned}$$

$$Ae_i := a_i$$

$$e_i = i\text{-th column of } I$$

#### Block Matrices and Block Columns

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1l} \\ A_{21} & A_{22} & \cdots & A_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kl} \end{bmatrix}, \ A_{ij} \in \mathbb{R}^{m \times n} \\ &= [A_1 \ A_2 \ \cdots \ A_l] \in \mathbb{R}^{km \times ln}, \ A_i \in \mathbb{R}^{km \times n} \end{aligned}$$

$$AE_i := A_i$$

$$E_i = i\text{-th block column of } I$$

### 2.2 Bidiagonal matrix (upper)

$$U = \begin{bmatrix} d_1 & b_1 & & & \\ & d_2 & b_2 & & \\ & & \ddots & \ddots & \\ & & & d_{n-1} & b_{n-1} \\ & & & & d_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \in \mathbb{R}^{n \times 1} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} \in \mathbb{R}^{n \times 1} \quad y^T = [y_1 \quad y_2 \quad \cdots \quad y_{n-1} \quad y_n] \in \mathbb{R}^{1 \times n}$$

$$\begin{aligned} xy^T &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} [y_1 \quad y_2 \quad \cdots \quad y_{n-1} \quad y_n] \\ &= \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_{n-1} & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_{n-1} & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1} y_1 & x_{n-1} y_2 & \cdots & x_{n-1} y_{n-1} & x_{n-1} y_n \\ x_n y_1 & x_n y_2 & \cdots & x_n y_{n-1} & x_n y_n \end{bmatrix} \\ &= x [y_1 \quad y_2 \quad \cdots \quad y_{n-1} \quad y_n] \\ &= [xy_1 \quad xy_2 \quad \cdots \quad xy_{n-1} \quad xy_n] \end{aligned}$$

$$\begin{aligned} \text{triu}(xy^T) &= \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_{n-1} & x_1 y_n \\ 0 & x_2 y_2 & \cdots & x_2 y_{n-1} & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_4 y_{n-1} & x_4 y_n \\ 0 & 0 & \cdots & 0 & x_5 y_n \end{bmatrix} \\ &= \left[ \begin{bmatrix} x_{1:1} \\ 0_{n-1} \end{bmatrix} y_1 \quad \begin{bmatrix} x_{1:2} \\ 0_{n-2} \end{bmatrix} y_2 \quad \cdots \quad \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \quad xy_n \right] \end{aligned}$$

$$\begin{aligned} U^{-1} &= \text{triu}(xy^T) \\ I &= U \text{triu}(xy^T) \end{aligned}$$

$n$ -th column,

$$\begin{aligned} e_n &= U \text{triu}(xy^T) e_n \\ &= U xy_n \end{aligned}$$

$$\begin{aligned} x &:= U^{-1} e_n \\ e_n &= U xy_n \\ &= U U^{-1} e_n y_n \\ &= e_n y_n \\ y_n &:= 1 \end{aligned}$$

$(n-1)$ -th column,

$$\begin{aligned}
e_{n-1} &= U \operatorname{triu}(xy^T)e_{n-1} \\
&= U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1}
\end{aligned}$$

$(n-1)$ -th row of  $(n-1)$ -th column,

$$e_i^T e_j = \delta_{ij}$$

$$\begin{aligned}
e_{n-1}^T e_{n-1} &= 1 = e_{n-1}^T U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \\
&= \begin{bmatrix} 0_{n-2} & d_{n-1} & b_{n-1} \end{bmatrix} \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \\
&= d_{n-1} x_{n-1} y_{n-1} \\
y_{n-1} &:= (d_{n-1} x_{n-1})^{-1}
\end{aligned}$$

$$\begin{aligned}
e_n &= Ux \\
e_n^T e_n &= 1 = e_n^T Ux \\
&= \begin{bmatrix} 0_{n-1} & d_n \end{bmatrix} x \\
&= d_n x_n \\
y_n &:= 1 \\
&:= (d_n x_n)^{-1}
\end{aligned}$$

$$\begin{aligned}
x &:= U^{-1} e_n \\
y_i &:= (d_i x_i)^{-1}
\end{aligned}$$

### 2.3 Tridiagonal matrix (Upper)

$$n = 2k$$

$$\begin{aligned}
 U &= \begin{bmatrix} d_1 & b_1^1 & b_1^2 & & & \\ & d_2 & b_2^1 & b_2^2 & & \\ & & d_3 & b_3^1 & b_3^2 & \\ & & & d_4 & b_4^1 & b_4^2 \\ & & & & \ddots & \ddots & \ddots \\ & & & & & d_{n-3} & b_{n-3}^1 & b_{n-3}^2 \\ & & & & & & d_{n-2} & b_{n-2}^1 & b_{n-2}^2 \\ & & & & & & & d_{n-1} & b_{n-1}^1 \\ & & & & & & & & d_n \end{bmatrix} \\
 &= \begin{bmatrix} \begin{bmatrix} d_1 & b_1^1 \end{bmatrix} & \begin{bmatrix} b_1^2 \\ b_2^1 & b_2^2 \end{bmatrix} & & & \\ & \begin{bmatrix} d_3 & b_3^1 \\ b_4^1 & b_4^2 \end{bmatrix} & \begin{bmatrix} b_3^2 \\ b_4^1 & b_4^2 \end{bmatrix} & & \\ & & \ddots & \ddots & \\ & & & \begin{bmatrix} d_{2k-3} & b_{2k-3}^1 \\ & d_{2k-2} \end{bmatrix} & \begin{bmatrix} b_{2k-3}^2 \\ b_{2k-2}^1 & b_{2k-2}^2 \\ d_{2k-1} & b_{2k-1}^1 \\ & d_{2k} \end{bmatrix} \end{bmatrix} \\
 &= \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{2k \times 2k}
 \end{aligned}$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{k-1} \\ Y_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \quad Y^T = [Y_1^T \ Y_2^T \ \dots \ Y_{k-1}^T \ Y_k^T] \in \mathbb{R}^{2 \times 2k}$$

$$\begin{aligned}
 XY^T &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} [Y_1^T \ Y_2^T \ \dots \ Y_{k-1}^T \ Y_k^T] \\
 &= \begin{bmatrix} X_1 Y_1^T & X_1 Y_2^T & \dots & X_1 Y_{k-1}^T & x_1 Y_k^T \\ X_2 Y_1^T & X_2 Y_2^T & \dots & X_2 Y_{k-1}^T & x_2 Y_k^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{k-1} Y_1^T & X_{k-1} Y_2^T & \dots & X_{k-1} Y_{k-1}^T & X_{k-1} Y_k^T \\ X_k Y_1^T & X_k Y_2^T & \dots & X_k Y_{k-1}^T & X_k Y_k^T \end{bmatrix} \\
 &= X [Y_1^T \ Y_2^T \ \dots \ Y_{k-1}^T \ Y_k^T] \\
 &= [XY_1^T \ XY_2^T \ \dots \ XY_{k-1}^T \ XY_k^T]
 \end{aligned}$$

$$\begin{aligned}
\text{triu}(X_k) &:= X_k \\
\text{triu}(Y_k^T) &:= Y_k^T \\
\text{triu}(X_k Y_k^T) &= X_k Y_k^T
\end{aligned}$$

$$\begin{aligned}
\text{triu}(XY^T) &= \begin{bmatrix} \text{triu}(X_1 Y_1^T) & X_1 Y_2^T & \cdots & X_1 Y_{k-1}^T & X_1 Y_k^T \\ 0_{2 \times 2} & \text{triu}(X_2 Y_2^T) & \cdots & X_2 Y_{k-1}^T & X_2 Y_k^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{2 \times 2} & 0_{2 \times 2} & \cdots & \text{triu}(X_{k-1} Y_{k-1}^T) & X_{k-1} Y_k^T \\ 0_{2 \times 2} & 0_{2 \times 2} & \cdots & 0_{2 \times 2} & X_k Y_k^T \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} \text{triu}(X_1 Y_1^T) \\ 0_{2(k-1) \times 2} \end{bmatrix} & \begin{bmatrix} X_1 Y_2^T \\ \text{triu}(X_2 Y_2^T) \\ 0_{2(k-2) \times 2} \end{bmatrix} & \cdots & \begin{bmatrix} X_{1:k-2} Y_{k-1}^T \\ \text{triu}(X_{k-1} Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix} & X Y_k^T \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
U^{-1} &= \text{triu}(XY^T) \\
I &= U \text{triu}(XY^T)
\end{aligned}$$

$$E_i = [0 \quad \cdots \quad 0 \quad I_{2 \times 2} \quad 0 \quad \cdots \quad 0]^T$$

$k$ -th block column,

$$\begin{aligned}
E_k &= U \text{triu}(XY^T) E_k \\
&= U X Y_k^T
\end{aligned}$$

$$\begin{aligned}
X &:= U^{-1} E_k \\
E_k &= U X Y_k^T \\
&= U U^{-1} E_k Y_k^T \\
&= E_k Y_k^T \\
Y_k^T &:= I_{2 \times 2}
\end{aligned}$$

$(k-1)$ -th block column,

$$\begin{aligned}
E_{k-1} &= U \text{triu}(XY^T) E_{k-1} \\
&= U \begin{bmatrix} X_{1:k-2} Y_{k-1}^T \\ \text{triu}(X_{k-1} Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix}
\end{aligned}$$

$(k-1)$ -th block row of  $(k-1)$ -th block column,

$$E_i^T E_j = \begin{cases} I_{2 \times 2}, & \text{if } i = j ; \\ 0_{2 \times 2}, & \text{otherwise.} \end{cases}$$



$$\begin{aligned}
E_{k-1}^T E_{k-1} &= I_{2 \times 2} = E_{k-1}^T U \begin{bmatrix} X_{1:k-2} Y_{k-1}^T \\ \text{triu}(X_{k-1} Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix} \\
&= \begin{bmatrix} 0_{2 \times 2(k-2)} & D_{k-1} & B_{k-1} \end{bmatrix} \begin{bmatrix} X_{1:k-2} Y_{k-1}^T \\ \text{triu}(X_{k-1} Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix} \\
&= D_{k-1} \text{triu}(X_{k-1} Y_{k-1}^T) \\
Y_{k-1}^T &:= (D_{k-1} X_{k-1})^{-1}
\end{aligned}$$

$$\begin{aligned}
D_{k-1} &= \text{triu}(D_{k-1}) \\
D_{k-1}^{-1} &= \text{triu}(D_{k-1}^{-1}) \\
Y_{k-1}^T &:= (D_{k-1} X_{k-1})^{-1} \\
D_{k-1} \text{triu}(X_{k-1} Y_{k-1}^T) &= D_{k-1} \text{triu}(X_{k-1} (D_{k-1} X_{k-1})^{-1}) \\
&= D_{k-1} \text{triu}(X_{k-1} X_{k-1}^{-1} D_{k-1}^{-1}) \\
&= D_{k-1} \text{triu}(D_{k-1}^{-1}) \\
&= D_{k-1} D_{k-1}^{-1} \\
&= I_{2 \times 2}
\end{aligned}$$

$$\begin{aligned}
X &:= U^{-1} E_k \\
E_k &= U X \\
E_k^T E_k &= I_{2 \times 2} = E_k^T U X \\
&= \begin{bmatrix} 0_{2 \times 2(k-1)} & D_k \end{bmatrix} X \\
&= D_k X_k \\
Y_k^T &:= I_{2 \times 2} \\
&:= (D_k X_k)^{-1}
\end{aligned}$$

$$\begin{aligned}
X &:= U^{-1} E_k \\
Y_i^T &:= (D_i X_i)^{-1}
\end{aligned}$$

$$n = 2k + 1$$

$$U = \begin{bmatrix} [d_1] & \begin{bmatrix} b_1^1 & b_1^2 \\ d_2 & b_2^1 \\ d_3 & b_3^1 \end{bmatrix} & \begin{bmatrix} b_2^2 & \\ b_3^1 & b_3^2 \end{bmatrix} & & & \\ & & \ddots & \ddots & & \\ & & & \begin{bmatrix} d_{2k-2} & b_{2k-2}^1 \\ & d_{2k-1} \end{bmatrix} & \begin{bmatrix} b_{2k-2}^2 & \\ b_{2k-1}^1 & b_{2k-1}^2 \end{bmatrix} & \\ & & & & \begin{bmatrix} d_{2k} & b_{2k}^1 \\ & d_{2k+1} \end{bmatrix} & \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(2k+1) \times (2k+1)}$$

$$X := U^{-1} E_{k+1}$$

$$Y_i^T := (D_i X_i)^{-1} \text{ for } i = 2, \dots, k+1$$

$$Y_1^T := (D_1 X_1)^+ = (D_1 X_1)^T (D_1 X_1 (D_1 X_1)^T)^{-1}$$

## 2.4 Banded matrix (Upper)

$$n = ku + r, \quad 0 \leq r < u$$

$$U = \begin{bmatrix} d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & \\ & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & \\ & & \ddots & \ddots & \ddots & \cdots & \ddots \\ & & & \ddots & \ddots & \ddots & b_{n-u}^u \\ & & & & \ddots & \ddots & \vdots \\ & & & & & \ddots & b_{n-2}^2 \\ & & & & & & \ddots & b_{n-1}^1 \\ & & & & & & & d_n \end{bmatrix}$$

$$r = 0, \quad n = ku,$$

$$U = \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{ku \times ku},$$

$$D_i \in \mathbb{R}^{u \times u}, \quad B_i \in \mathbb{R}^{u \times u} \text{ for } i = 1, 2, \dots, k.$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} \in \mathbb{R}^{ku \times u} \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{bmatrix} \in \mathbb{R}^{ku \times u} \quad Y^T = [Y_1^T \quad Y_2^T \quad \cdots \quad Y_k^T] \in \mathbb{R}^{u \times ku}$$

$$X_i, Y_i^T \in \mathbb{R}^{u \times u}, \text{ for } i = 1, 2, \dots, k$$

$$X := U^{-1} E_k$$

$$Y_i^T := (D_i X_i)^{-1}$$

$$r \neq 0, \quad n = ku + r,$$

$$U = \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times (ku+r)},$$

$$\tilde{D}_1 \in \mathbb{R}^{r \times r}, \quad \tilde{B}_1 \in \mathbb{R}^{r \times u};$$

$$D_i \in \mathbb{R}^{u \times u}, \quad B_i \in \mathbb{R}^{u \times u} \text{ for } i = 2, \dots, k+1.$$

$$X = \begin{bmatrix} \tilde{X}_1 \\ X_2 \\ \vdots \\ X_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times u} \quad Y = \begin{bmatrix} \tilde{Y}_1 \\ Y_2 \\ \vdots \\ Y_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times u} \quad Y^T = [\tilde{Y}_1^T \quad Y_2^T \quad \cdots \quad Y_{k+1}^T] \in \mathbb{R}^{u \times (ku+r)}$$

$$\begin{aligned}
&\tilde{X}_1 \in \mathbb{R}^{r \times u} \\
&\tilde{Y}_1^T \in \mathbb{R}^{u \times r} \\
&X_i, Y_i^T \in \mathbb{R}^{u \times u} \text{ for } i = 2, \dots, k+1 \\
&X := U^{-1}E_{k+1} \\
&\tilde{Y}_1^T := (D_1X_1)^+ = (D_1X_1)^T(D_1X_1(D_1X_1)^T)^{-1} \\
&\text{triu}(D_1X_1\tilde{Y}_1^T) = I_{r \times r} \\
&Y_i^T := (D_iX_i)^{-1} \text{ for } i = 2, \dots, k+1
\end{aligned}$$

### 3 Test

$$\begin{bmatrix} d_1 & \overbrace{b_1^1 & b_1^2 & \cdots & b_1^u}^{\text{bandwidth} = u} & & & & \\ d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & & \\ & \ddots & \ddots & \ddots & \cdots & \ddots & & \\ & & \ddots & \ddots & \ddots & \cdots & b_{n-u}^u & \\ & & & \ddots & \ddots & \ddots & \vdots & \\ & & & & \ddots & \ddots & b_{n-2}^2 & \\ & & & & & \ddots & b_{n-1}^1 & \\ & & & & & & d_n & \end{bmatrix}$$

$$\left( \begin{array}{ccc|ccc} a & \cdots & a & b & \cdots & b \\ & \ddots & \vdots & \vdots & \ddots & \\ & & a & b & & \\ \hline & & 0 & c & \cdots & c \\ & & & \vdots & & \vdots \end{array} \right) \left. \vphantom{\begin{array}{ccc|ccc} a & \cdots & a & b & \cdots & b \\ & \ddots & \vdots & \vdots & \ddots & \\ & & a & b & & \\ \hline & & 0 & c & \cdots & c \\ & & & \vdots & & \vdots \end{array}} \right\} p$$

$$\left. \vphantom{\begin{array}{ccc|ccc} a & \cdots & a & b & \cdots & b \\ & \ddots & \vdots & \vdots & \ddots & \\ & & a & b & & \\ \hline & & 0 & c & \cdots & c \\ & & & \vdots & & \vdots \end{array}} \right\} q$$

$$\overbrace{\begin{array}{cccccc} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{array}}^{n \text{ columns}} \quad \overbrace{\begin{array}{cccccc} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{array}}^m$$

$$\begin{matrix} \uparrow \\ n \text{ rows} \\ \downarrow \end{matrix} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \\ & & 0 & \cdots & 1 \\ & & & \ddots & \\ & & & & 1 & \cdots & 0 \\ & & & & & \ddots & \\ & & & & & & 1 & \cdots & 1 \end{pmatrix} \begin{matrix} \leftarrow i \\ \leftarrow j \end{matrix}$$

$$\begin{matrix} \uparrow i \\ \uparrow j \end{matrix}$$