

Thesis Contents

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Abstract

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[?]

12 Semiseparable

Define matrices $E_u(p)$ and $E_l(q)$ as following,

$$E_u(p) = \begin{cases} 1, & \text{if } i \leq j - p, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad E_l(q) = \begin{cases} 1, & \text{if } i \geq j - q, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

i.e.

$$\begin{aligned} E_u(-1) &= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & 1 \\ & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \quad E_u(0) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \quad E_u(1) = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}; \\ E_l(-1) &= \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 \end{bmatrix}, \quad E_l(0) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 \end{bmatrix}, \quad E_l(1) = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}. \end{aligned} \quad (2)$$

$$(3)$$

where $|p| \leq n - 1$ and $|q| \leq n - 1$.

Define the operations triu and tril,

$$\text{triu}(A, p) = A \circ E_u(p) \quad (4)$$

$$\text{tril}(A, q) = A \circ E_l(q) \quad (5)$$

i.e. $\text{triu}(A, p)$ retains the entries of A above the p -th super-diagonal (inclusive) and makes the entries of A below the p -th super-diagonal (exclusive) 0.

define symmetric $\{1,1\}$ -semiseparable matrix, S

$$S = \begin{cases} x_i y_j & \text{if } i \leq j; \\ x_j y_i & \text{if } i \geq j. \end{cases} \quad (6)$$

$$= \text{triu}(xy^T) + \text{tril}(yx^T, -1) \quad (7)$$

$$= \text{triu}(xy^T, 1) + \text{tril}(yx^T) \quad (8)$$

$$= \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_1 y_2 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1 y_n & x_2 y_n & \cdots & x_n y_n \end{bmatrix} \quad (9)$$

13 Matrix (basic def)

Matrix-Matrix Multiplication

$$AB = C \tag{10}$$

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} \tag{11}$$

Matrix-Vector Multiplication

$$Ax = b \tag{12}$$

$$b_i = \sum_{k=1}^n a_{ik} x_k \tag{13}$$

14 Algorithm (Appendix)

Backward Substitution

$$Ux = b \quad (14)$$

Triangular Matrix

$$\begin{bmatrix} d_1 & u_1^1 & u_1^2 & \cdots & \cdots & u_1^{n-1} \\ & d_2 & u_2^1 & u_2^2 & \cdots & u_2^{n-2} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & d_{n-2} & u_{n-2}^1 & u_{n-2}^2 \\ & & & & d_{n-1} & u_{n-1}^1 \\ & & & & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-2} \\ b_{n-1} \\ b_n \end{bmatrix} \quad (15)$$

$$x_n = b_n / u_{nn} \quad (16)$$

$$x_i = \left(b_i - \sum_{j=i+1}^n u_{ij} x_j \right) / u_{ii} \quad (17)$$

$$d_n x_n = b_n \quad \Rightarrow \quad x_n = b_n / d_n \quad (18)$$

$$d_{n-1} x_{n-1} + u_{n-1}^1 x_n = b_{n-1} \quad \Rightarrow \quad x_{n-1} = (b_{n-1} - u_{n-1}^1 x_n) / d_{n-1} \quad (19)$$

$$d_{n-2} x_{n-2} + u_{n-2}^1 x_{n-1} + u_{n-2}^2 x_n = b_{n-2} \quad \Rightarrow \quad x_{n-2} = (b_{n-2} - u_{n-2}^1 x_{n-1} - u_{n-2}^2 x_n) / d_{n-2} \quad (20)$$

$$\vdots \quad (21)$$

$$b_{n-k} = d_{n-k} x_{n-k} + u_{n-k}^1 x_{n-k+1} + u_{n-k}^2 x_{n-k+2} + \cdots + u_{n-k}^k x_{n-k+k} \quad (22)$$

$$\Rightarrow x_{n-k} = \left(b_{n-k} - u_{n-k}^1 x_{n-k+1} - u_{n-k}^2 x_{n-k+2} - \cdots - u_{n-k}^k x_{n-k+k} \right) / d_{n-k} \quad (23)$$

$$= \left(b_{n-k} - \sum_{j=1}^k u_{n-k}^j x_{n-k+j} \right) / d_{n-k} \quad (24)$$

$$(25)$$

$$x_i = \left(b_i - \sum_{j=1}^{n-i} u_i^j x_{i+j} \right) / d_i \quad (26)$$

$$x_i = \left(b_i - \sum_{j=i+1}^n u_{ij} x_j \right) / u_{ii} \quad (27)$$

Upper Banded Matrix

$$\begin{bmatrix} d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & \\ & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & \\ & & \ddots & \ddots & \ddots & & \ddots \\ & & & \ddots & \ddots & \ddots & b_{n-u}^u \\ & & & & \ddots & \ddots & \vdots \\ & & & & & d_{n-2} & b_{n-2}^1 & b_{n-2}^2 \\ & & & & & d_{n-1} & b_{n-1}^1 \\ & & & & & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ \vdots \\ b_{n-2} \\ b_{n-1} \\ b_n \end{bmatrix} \quad (28)$$

$$x_n = b_n / u_{nn} \quad (29)$$

$$x_i = \left(b_i - \sum_{j=i+1}^n u_{ij} x_j \right) / u_{ii} \quad (30)$$

$$d_n x_n = b_n \quad \Rightarrow \quad x_n = b_n / d_n \quad (31)$$

$$d_{n-1} x_{n-1} + u_{n-1}^1 x_n = b_{n-1} \quad \Rightarrow \quad x_{n-1} = (b_{n-1} - u_{n-1}^1 x_n) / d_{n-1} \quad (32)$$

$$d_{n-2} x_{n-2} + u_{n-2}^1 x_{n-1} + u_{n-2}^2 x_n = b_{n-2} \quad \Rightarrow \quad x_{n-2} = (b_{n-2} - u_{n-2}^1 x_{n-1} - u_{n-2}^2 x_n) / d_{n-2} \quad (33)$$

$$\vdots \quad (34)$$

$$b_{n-k} = d_{n-k} x_{n-k} + u_{n-k}^1 x_{n-k+1} + u_{n-k}^2 x_{n-k+2} + \cdots + u_{n-k}^k x_{n-k+k} \quad (35)$$

$$\Rightarrow x_{n-k} = \left(b_{n-k} - u_{n-k}^1 x_{n-k+1} - u_{n-k}^2 x_{n-k+2} - \cdots - u_{n-k}^k x_{n-k+k} \right) / d_{n-k} \quad (36)$$

$$= \left(b_{n-k} - \sum_{j=1}^k u_{n-k}^j x_{n-k+j} \right) / d_{n-k} \quad (37)$$

$$(38)$$

$$x_i = \left(b_i - \sum_{j=1}^{n-i} u_i^j x_{i+j} \right) / d_i \quad (39)$$

$$x_i = \left(b_i - \sum_{j=i+1}^n u_{ij} x_j \right) / u_{ii} \quad (40)$$

15 Apply on expmod

16 Apply on symmetric positive definite tridiagonal matrices

$$\longleftarrow \text{cholesky } \mathcal{O}\left(\frac{1}{3}n^3\right)$$

$$\begin{aligned} T &= U^T U \\ U^{-1} &= \text{triu}(xy^T) \\ (U^T)^{-1} &= (U^{-1})^T \\ &= (\text{triu}(xy^T))^T \\ &= \text{tril}(yx^T) \\ T^{-1} &= (U^T U)^{-1} \\ &= (U^{-1})(U^T)^{-1} \\ &= \text{triu}(xy^T) \text{tril}(yx^T) \end{aligned}$$

$$\begin{aligned} (T^{-1})_{i,j} &= \sum_{k=1}^n [\text{triu}(xy^T)]_{i,k} [\text{tril}(yx^T)]_{k,j} \\ &= \sum_{k=1}^n \left(\begin{cases} x_i y_k, & \text{if } i \leq j; \\ 0, & \text{if } i > j. \end{cases} \right) \left(\begin{cases} 0, & \text{if } k < j; \\ y_k x_j, & \text{if } k \geq j. \end{cases} \right) \\ &= \sum_{k=1}^n (x_i y_k \mathbb{1}_{\mathbb{Z} \cap [i,n]}(k)) (y_k x_j \mathbb{1}_{\mathbb{Z} \cap [j,n]}(k)) \\ &= x_i x_j \sum_{k=1}^n y_k^2 (\mathbb{1}_{\mathbb{Z} \cap [i,n]}(k) \mathbb{1}_{\mathbb{Z} \cap [j,n]}(k)) \\ &= x_i x_j \sum_{k=1}^n y_k^2 \mathbb{1}_{\mathbb{Z} \cap [\max(i,j),n]}(k) \end{aligned}$$

$$(T^{-1})_{i,j} = \begin{cases} x_i x_j \sum_{k=j}^n y_k^2, & \text{if } i \leq j; \\ x_j x_i \sum_{k=i}^n y_k^2, & \text{if } i \geq j. \end{cases} \quad (41)$$

for $i = j$,

$$(T^{-1})_{i,i} = x_i^2 \sum_{k=i}^n y_k^2 = x_j^2 \sum_{k=j}^n y_k^2 = (T^{-1})_{j,j}$$

following the definition of $\{1,1\}$ -Semiseparable matrix

$$S = \begin{cases} x_i y_j & \text{if } i \leq j; \\ x_j y_i & \text{if } i \geq j. \end{cases} \quad (42)$$

$$= \text{triu}(xy^T) + \text{tril}(yx^T, -1) \quad (43)$$

$$(44)$$

$$(T^{-1})_{i,j} = \begin{cases} x_i \left(x_j \sum_{k=j}^n y_k^2 \right), & \text{if } i \leq j; \\ x_j \left(x_i \sum_{k=i}^n y_k^2 \right), & \text{if } i \geq j. \end{cases} \quad (45)$$

$$= \begin{cases} \tilde{x}_i \tilde{y}_j & \text{if } i \leq j; \\ \tilde{x}_j \tilde{y}_i & \text{if } i \geq j. \end{cases} \quad (46)$$

$$= \text{triu}(\tilde{x}\tilde{y}^T) + \text{tril}(\tilde{y}\tilde{x}^T, -1) \quad (47)$$

$$(48)$$

where $\tilde{x}_i = x_i$ and $\tilde{y}_i = x_i \sum_{k=i}^n y_k^2$.

$$\tilde{x}_i = x_i \quad (49)$$

$$\Rightarrow \tilde{x} = x \quad (50)$$

$$\tilde{y}_i = x_i \sum_{k=i}^n y_k^2 \quad (51)$$

$$\Rightarrow \tilde{y} = \begin{bmatrix} x_1 \left(\sum_{k=1}^n y_k^2 \right) \\ x_2 \left(\sum_{k=2}^n y_k^2 \right) \\ \vdots \\ x_n \left(\sum_{k=n}^n y_k^2 \right) \end{bmatrix} \quad (52)$$

$$= x \circ \begin{bmatrix} y^T y \\ y_{[2:n]}^T y_{[2:n]} \\ \vdots \\ y_n^2 \end{bmatrix} \quad (53)$$

$$x := U^{-1}e_n = \mathbf{backsub}(U, e_n);$$

$$y_i := (d_i x_i)^{-1} \quad \text{for } i = 1 : n.$$

$$x = U \setminus e_n$$

$$x_n = \frac{1}{d_n}$$

$$x_i = -\frac{u_i x_{i+1}}{d_i} \quad \text{for } i = n-1, n-2, \dots, 1$$

$$x_n = \frac{1}{d_n}$$

$$x_{n-1} = -\frac{u_{n-1}}{d_{n-1}} x_n = -\frac{u_{n-1}}{d_n d_{n-1}}$$

$$x_{n-2} = -\frac{u_{n-2}}{d_{n-2}} x_{n-1} = \frac{u_{n-1} u_{n-2}}{d_n d_{n-1} d_{n-2}}$$

$$x_1 = -\frac{u_{n-2}}{d_{n-2}} x_2 = (-1)^{n-1} \frac{u_{n-1} u_{n-2} \dots u_1}{d_n d_{n-1} d_{n-2} \dots d_1}$$

$$x_k = \frac{\prod_{i=k}^{n-1} -b_i}{\prod_{i=k}^n d_i} = \left(\prod_{i=k}^{n-1} -\frac{b_i}{d_i} \right) \Big/ d_n \quad \text{for } k = 1 : n$$

where $\prod_{i=k}^l a_i = 1$ for any $k > l$

$$\begin{aligned} x_k &= \left(\prod_{i=k}^{n-1} -\frac{b_i}{d_i} \right) \Big/ d_n \\ y_k &= \frac{1}{d_k x_k} = \left(\frac{d_n}{d_k} \right) \Big/ \prod_{i=k}^{n-1} -\frac{b_i}{d_i} = \frac{d_n}{d_k} \prod_{i=k}^{n-1} -\frac{d_i}{b_i} \\ y_k^2 &= \frac{d_n^2}{d_k^2} \prod_{i=k}^{n-1} \left(\frac{d_i}{b_i} \right)^2 \end{aligned}$$

$$\begin{aligned}
\frac{\tilde{y}_{k+1}}{\tilde{y}_k} &= \frac{x_{k+1} \sum_{\ell=k+1}^n y_\ell^2}{x_k \sum_{\ell=k}^n y_\ell^2} \\
&= \frac{x_{k+1}}{x_k} \left(1 - \frac{y_k^2}{\sum_{\ell=k}^n y_\ell^2} \right) \\
&= -\frac{d_k}{b_k} \left(1 - \frac{\frac{d_n^2}{d_k^2} \prod_{i=k}^{n-1} \left(\frac{d_i}{b_i} \right)^2}{\sum_{\ell=k}^n \frac{d_n^2}{d_\ell^2} \prod_{i=\ell}^{n-1} \left(\frac{d_i}{b_i} \right)^2} \right) \\
&= r_k \left(1 - \frac{\frac{1}{d_k^2} \prod_{i=k}^{n-1} r_i^2}{\sum_{\ell=k}^n \frac{1}{d_\ell^2} \prod_{i=\ell}^{n-1} r_i^2} \right) \\
&\leq r_{\max} \left(1 - \frac{\frac{1}{d_k^2} \prod_{i=k}^{n-1} r_{\min}^2}{\sum_{\ell=k}^n \frac{1}{d_\ell^2} \prod_{i=\ell}^{n-1} r_i^2} \right), \quad \text{as } r_{\min}^2 \leq r_i^2, \forall i, \\
&\leq r_{\max} \left(1 - \frac{\frac{1}{d_k^2} \prod_{i=k}^{n-1} r_{\min}^2}{\sum_{\ell=k}^n \frac{1}{d_\ell^2} \prod_{i=\ell}^{n-1} r_{\max}^2} \right), \quad \text{as } r_{\max}^2 \geq r_i^2, \forall i, \\
&= r_{\max} \left(1 - \frac{\frac{1}{d_{\max}^2} r_{\min}^{2(n-k)}}{\sum_{\ell=k}^n \frac{1}{d_{\min}^2} r_{\max}^{2(n-\ell)}} \right) \\
&= r_{\max} \left(1 + \frac{d_{\min}^2 (r_{\max}^2 - 1) r_{\max}^{2k} r_{\min}^{2(n-k)}}{d_{\max}^2 (r_{\max}^{2k} - r_{\max}^{2(n+1)})} \right) \\
&= r_{\max} \left(1 + \frac{(r_{\max}^2 - 1) r_{\max}^{2k} r_{\min}^{2(n-k)}}{r_{\max}^{2k} - r_{\max}^{2(n+1)}} \left(\frac{d_{\min}}{d_{\max}} \right)^2 \right) \\
&\leq r_{\max} \left(1 + \frac{(r_{\max}^2 - 1) r_{\max}^{2k} r_{\min}^{2(n-k)}}{r_{\max}^{2k} - r_{\max}^{2(n+1)}} \right), \quad \text{as } \left(\frac{d_{\min}}{d_{\max}} \right)^2 \leq 1, \\
&\leq r_{\max} \left(1 + \frac{(r_{\max}^2 - 1) r_{\max}^{2k} r_{\max}^{2(n-k)}}{r_{\max}^{2k} - r_{\max}^{2(n+1)}} \right), \quad \text{as } r_{\min}^2 \leq r_{\max}^2, \\
&= r \left(1 + \frac{(r^2 - 1) r^{2k} r^{2(n-k)}}{r^{2k} - r^{2(n+1)}} \right) \\
&= r \left(1 + \frac{(r^2 - 1) r^{2n}}{r^{2k} - r^{2(n+1)}} \right) \\
&= r \left(\frac{r^{2k} - r^{2(n+1)} + r^{2(n+1)} - r^{2n}}{r^{2k} - r^{2(n+1)}} \right) \\
&= r \left(\frac{r^{2k} - r^{2n}}{r^{2k} - r^{2(n+1)}} \right) \\
&= \frac{r^{2k+1} - r^{2n+1}}{r^{2k} - r^{2n+2}} \\
&= \frac{r^{2n+1} - r^{2k+1}}{r^{2n+2} - r^{2k}} \\
&< \frac{r^{2n+1} - r^{2k}}{r^{2n+2} - r^{2k}}, \quad \text{as } r^{2k+1} > r^{2k}, \\
&< \frac{r^{2n+1} - r^{2k}}{r^{2n+1} - r^{2k}}, \quad \text{as } r^{2n+1} < r^{2n+2}, \\
&= 1
\end{aligned}$$

where $d_{\max} = \max d_k$, $b_{\max} = \max b_k$, $r_k = -\frac{d_k}{b_k}$, $r_{\max} = -\frac{d_{\max}}{b_{\min}}$, and $r_{\min} = -\frac{d_{\min}}{b_{\max}}$, with $r_{\max} \geq r_k \geq r_{\min}, \forall k = 1 : n$.

$$= \frac{r^{2k+1} - r^{2n+1}}{r^{2k} - r^{2n+2}}$$

$$\begin{aligned} T^{-1} &= (U^{-1}) (U^T)^{-1} = \text{triu } xy^T \text{ tril } yx^T \\ (T^{-1})_{i,j} &= \sum_{k=1}^n [\text{triu } xy^T]_{i,k} [\text{tril } yx^T]_{k,j} \\ &= \sum_{k=1}^n \left(\begin{cases} x_i y_k, & \text{if } i \leq j; \\ 0, & \text{if } i > j. \end{cases} \right) \left(\begin{cases} 0, & \text{if } k < j; \\ y_k x_j, & \text{if } k \geq j. \end{cases} \right) \\ &= \sum_{k=1}^n (x_i y_k \mathbb{1}_{\{i:n\}}(k)) (y_k x_j \mathbb{1}_{\{j:n\}}(k)) \\ &= x_i x_j \sum_{k=1}^n y_k^2 (\mathbb{1}_{\{i:n\}}(k) \mathbb{1}_{\{j:n\}}(k)) \\ &= x_i x_j \sum_{k=1}^n y_k^2 \mathbb{1}_{\{\max(i,j):n\}}(k) \end{aligned} \tag{54}$$

$$\left. \begin{aligned} x &:= U^{-1} e_n = \text{backsub}(U, e_n), \\ y_i &:= (d_i x_i)^{-1} \quad \text{for } i = 1 : n. \end{aligned} \right\} \rightarrow (U^{-1})_{i,j} = x_i y_j \quad \text{for } j = i : n$$

$$T = \tilde{L}U = \tilde{U}^T U \quad \Rightarrow \quad T^{-1} = \left(\tilde{U}^T U \right)^{-1} = (U^{-1}) \left(\tilde{U}^T \right)^{-1} = (U^{-1}) \left(\tilde{U}^{-1} \right)^T$$

where $\tilde{L} = \tilde{U}^T$

Let $U^{-1} = \text{triu } uv^T$, $\tilde{U}^{-1} = \text{triu } qp^T$, then $\left(\tilde{U}^{-1} \right)^T = \text{tril } pq^T$ by Algorithm XXX and Lemma XXXXX

Then, Equations XXXX and XXX for nonsymmetric bidiagonal T becomes,

$$\begin{aligned} T^{-1} &= \text{triu } uv^T \text{tril } pq^T, \\ (T^{-1})_{i,j} &= \sum_{k=1}^n [\text{triu } uv^T]_{i,k} [\text{tril } pq^T]_{k,j} \\ &= \sum_{k=1}^n (u_i v_k \mathbb{1}_{\{i:n\}}(k)) (p_k q_j \mathbb{1}_{\{j:n\}}(k)) \\ &= u_i q_j \sum_{k=1}^n v_k p_k \mathbb{1}_{\{\max(i,j):n\}}(k) \\ &= \begin{cases} u_i q_j \sum_{k=j}^n v_k p_k, & \text{if } i \leq j; \\ q_j u_i \sum_{k=i}^n v_k p_k, & \text{if } i \geq j. \end{cases} \\ &= \begin{cases} \tilde{u}_i \tilde{v}_j & \text{if } i \leq j; \\ \tilde{p}_j \tilde{q}_i & \text{if } i \geq j. \end{cases} \\ &= \text{triu } \tilde{u} \tilde{v}^T + \text{tril } \tilde{q} \tilde{p}^T, -1 \end{aligned}$$

where

$$\begin{cases} \tilde{u}_i = u_i; \\ \tilde{v}_j = q_j \sum_{k=j}^n v_k p_k. \end{cases}, \quad \text{and} \quad \begin{cases} \tilde{p}_j = q_j; \\ \tilde{q}_i = u_i \sum_{k=j}^n v_k p_k. \end{cases}.$$

The equality holds as, when $i = j$

$$\tilde{u}_i \tilde{v}_i = u_i q_i \sum_{k=i}^n v_k p_k = q_i u_i \sum_{k=i}^n v_k p_k = \tilde{p}_i \tilde{q}_i$$

$$\begin{aligned}
(\text{triu } xy^T b)_i &= \sum_{k=1}^n [\text{triu } xy^T]_{i,k} b_k \\
&= \sum_{k=1}^n \left(\begin{cases} x_i y_k, & \text{if } i \leq k; \\ 0, & \text{if } i > k. \end{cases} \right) b_k \\
&= \sum_{k=1}^n [x_i y_k \mathbb{1}_{\{i:n\}}(k)] b_k \\
&= x_i \sum_{k=i}^n y_k b_k \\
&= x \circ \left[y^T b \mid y_{[2:n]}^T b_{[2:n]} \mid \cdots \mid y_n b_n \right]^T \\
&= x \circ \text{normCoef}(y, b, \text{false})
\end{aligned}$$

$$\begin{aligned}
(\text{tril } yx^T, -1b)_i &= \sum_{k=1}^n [\text{triu } xy^T]_{i,k} b_k \\
&= \sum_{k=1}^n \left(\begin{cases} 0, & \text{if } i < k + 1; \\ y_i x_k, & \text{if } i \geq k + 1. \end{cases} \right) b_k \\
&= y_i \sum_{k=1}^{i-1} x_k b_k \\
&= y \circ \left[0 \mid x_1 b_1 \mid x_{[1:2]}^T b_{[1:2]} \mid \cdots \mid x_{[1:n-1]}^T b_{[1:n-1]} \right]^T \\
&= y \circ \left[0 \mid \text{normCoef}(x_{[1:n-1]}, b_{[1:n-1]}, \text{true}) \right]^T
\end{aligned}$$

$$x = T^{-1}b = \text{triu } \tilde{x}\tilde{y}^T + \text{tril } \tilde{y}\tilde{x}^T, -1b = \text{triu } \tilde{x}\tilde{y}^T b + \text{tril } \tilde{y}\tilde{x}^T, -1b$$

$$\begin{bmatrix} n & \tilde{E}_0 & E_1 & E_2 \\ & r & u & u \end{bmatrix}$$

17 28.06.2022 Meeting draft

17.1 Tridiagonal matrix (Upper)

$$n = 2k$$

$$\begin{aligned}
 U &= \begin{bmatrix} d_1 & b_1^1 & b_1^2 & & & & \\ & d_2 & b_2^1 & b_2^2 & & & \\ & & d_3 & b_3^1 & b_3^2 & & \\ & & & d_4 & b_4^1 & b_4^2 & \\ & & & & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots \\ & & & & & & d_{n-3} & b_{n-3}^1 & b_{n-3}^2 \\ & & & & & & & d_{n-2} & b_{n-2}^1 & b_{n-2}^2 \\ & & & & & & & & d_{n-1} & b_{n-1}^1 & b_{n-1}^2 \\ & & & & & & & & & d_n & \\ & & & & & & & & & & d_n \end{bmatrix} \\
 &= \begin{bmatrix} \begin{bmatrix} d_1 & b_1^1 \\ & d_2 \end{bmatrix} & \begin{bmatrix} b_1^2 \\ b_2^1 & b_2^2 \\ d_3 & b_3^1 \end{bmatrix} & & & & \\ & \begin{bmatrix} b_3^2 \\ b_4^1 & b_4^2 \end{bmatrix} & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & \begin{bmatrix} d_{2k-3} & b_{2k-3}^1 \\ & d_{2k-2} \end{bmatrix} & \begin{bmatrix} b_{2k-3}^2 \\ b_{2k-2}^1 & b_{2k-2}^2 \\ d_{2k-1} & b_{2k-1}^1 \end{bmatrix} & \\ & & & & \begin{bmatrix} b_{2k-1}^2 \\ d_{2k} \end{bmatrix} \end{bmatrix} \\
 &= \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{2k \times 2k}
 \end{aligned}$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{k-1} \\ Y_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \quad Y^T = [Y_1^T \ Y_2^T \ \cdots \ Y_{k-1}^T \ Y_k^T] \in \mathbb{R}^{2 \times 2k}$$

$$\begin{aligned}
XY^T &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} [Y_1^T \ Y_2^T \ \cdots \ Y_{k-1}^T \ Y_k^T] \\
&= \begin{bmatrix} X_1 Y_1^T & X_1 Y_2^T & \cdots & X_1 Y_{k-1}^T & x_1 Y_k^T \\ X_2 Y_1^T & X_2 Y_2^T & \cdots & X_2 Y_{k-1}^T & x_2 Y_k^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{k-1} Y_1^T & X_{k-1} Y_2^T & \cdots & X_{k-1} Y_{k-1}^T & X_{k-1} Y_k^T \\ X_k Y_1^T & X_k Y_2^T & \cdots & X_k Y_{k-1}^T & X_k Y_k^T \end{bmatrix} \\
&= X [Y_1^T \ Y_2^T \ \cdots \ Y_{k-1}^T \ Y_k^T] \\
&= [XY_1^T \ XY_2^T \ \cdots \ XY_{k-1}^T \ XY_k^T]
\end{aligned}$$

$$\begin{aligned}
\text{triu}(X_k) &:= X_k \\
\text{triu}(Y_k^T) &:= Y_k^T \\
\text{triu}(X_k Y_k^T) &= X_k Y_k^T
\end{aligned}$$

$$\begin{aligned}
\text{triu}(XY^T) &= \begin{bmatrix} \text{triu}(X_1 Y_1^T) & X_1 Y_2^T & \cdots & X_1 Y_{k-1}^T & X_1 Y_k^T \\ 0_{2 \times 2} & \text{triu}(X_2 Y_2^T) & \cdots & X_2 Y_{k-1}^T & X_2 Y_k^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{2 \times 2} & 0_{2 \times 2} & \cdots & \text{triu}(X_{k-1} Y_{k-1}^T) & X_{k-1} Y_k^T \\ 0_{2 \times 2} & 0_{2 \times 2} & \cdots & 0_{2 \times 2} & X_k Y_k^T \end{bmatrix} \\
&= \begin{bmatrix} \text{triu}(X_1 Y_1^T) \\ 0_{2(k-1) \times 2} \end{bmatrix} \begin{bmatrix} X_1 Y_2^T \\ \text{triu}(X_2 Y_2^T) \\ 0_{2(k-2) \times 2} \end{bmatrix} \cdots \begin{bmatrix} X_{1:k-2} Y_{k-1}^T \\ \text{triu}(X_{k-1} Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix} XY_k^T
\end{aligned}$$

$$\begin{aligned}
U^{-1} &= \text{triu}(XY^T) \\
I &= U \text{triu}(XY^T)
\end{aligned}$$

$$E_i = [0 \ \cdots \ 0 \ I_{2 \times 2} \ 0 \ \cdots \ 0]^T$$

k -th block column,

$$\begin{aligned}
E_k &= U \text{triu}(XY^T) E_k \\
&= U X Y_k^T
\end{aligned}$$

$$\begin{aligned}
X &:= U^{-1} E_k \\
E_k &= U X Y_k^T \\
&= U U^{-1} E_k Y_k^T \\
&= E_k Y_k^T \\
Y_k^T &:= I_{2 \times 2}
\end{aligned}$$

$(k-1)$ -th block column,

$$\begin{aligned} E_{k-1} &= U \operatorname{triu}(XY^T)E_{k-1} \\ &= U \begin{bmatrix} X_{1:k-2}Y_{k-1}^T \\ \operatorname{triu}(X_{k-1}Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix} \end{aligned}$$

$(k-1)$ -th block row of $(k-1)$ -th block column,

$$E_i^T E_j = \begin{cases} I_{2 \times 2}, & \text{if } i = j ; \\ 0_{2 \times 2}, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} E_{k-1}^T E_{k-1} &= I_{2 \times 2} = E_{k-1}^T U \begin{bmatrix} X_{1:k-2}Y_{k-1}^T \\ \operatorname{triu}(X_{k-1}Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix} \\ &= \begin{bmatrix} 0_{2 \times 2(k-2)} & D_{k-1} & B_{k-1} \end{bmatrix} \begin{bmatrix} X_{1:k-2}Y_{k-1}^T \\ \operatorname{triu}(X_{k-1}Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix} \\ &= D_{k-1} \operatorname{triu}(X_{k-1}Y_{k-1}^T) \\ Y_{k-1}^T &:= (D_{k-1}X_{k-1})^{-1} \end{aligned}$$

$$\begin{aligned} D_{k-1} &= \operatorname{triu}(D_{k-1}) \\ D_{k-1}^{-1} &= \operatorname{triu}(D_{k-1}^{-1}) \\ Y_{k-1}^T &:= (D_{k-1}X_{k-1})^{-1} \\ D_{k-1} \operatorname{triu}(X_{k-1}Y_{k-1}^T) &= D_{k-1} \operatorname{triu}(X_{k-1}(D_{k-1}X_{k-1})^{-1}) \\ &= D_{k-1} \operatorname{triu}(X_{k-1}X_{k-1}^{-1}D_{k-1}^{-1}) \\ &= D_{k-1} \operatorname{triu}(D_{k-1}^{-1}) \\ &= D_{k-1}D_{k-1}^{-1} \\ &= I_{2 \times 2} \end{aligned}$$

$$\begin{aligned} X &:= U^{-1}E_k \\ E_k &= UX \\ E_k^T E_k &= I_{2 \times 2} = E_k^T UX \\ &= \begin{bmatrix} 0_{2 \times 2(k-1)} & D_k \end{bmatrix} X \\ &= D_k X_k \\ Y_k^T &:= I_{2 \times 2} \\ &:= (D_k X_k)^{-1} \end{aligned}$$

$$\begin{aligned} X &:= U^{-1}E_k \\ Y_i^T &:= (D_i X_i)^{-1} \end{aligned}$$

$$n = 2k + 1$$

$$U = \begin{bmatrix} [d_1] & \begin{bmatrix} b_1^1 & b_1^2 \\ d_2 & b_2^1 \\ & d_3 \end{bmatrix} & \begin{bmatrix} b_2^2 & \\ b_3^1 & b_3^2 \end{bmatrix} & & \\ & & \ddots & \ddots & \\ & & & \begin{bmatrix} d_{2k-2} & b_{2k-2}^1 \\ & d_{2k-1} \end{bmatrix} & \begin{bmatrix} b_{2k-2}^2 & \\ b_{2k-1}^1 & b_{2k-1}^2 \\ d_{2k} & b_{2k}^1 \\ & d_{2k+1} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(2k+1) \times (2k+1)}$$

$$X := U^{-1} E_{k+1}$$

$$Y_i^T := (D_i X_i)^{-1} \text{ for } i = 2, \dots, k+1$$

$$Y_1^T := (D_1 X_1)^+ = (D_1 X_1)^T (D_1 X_1 (D_1 X_1)^T)^{-1}$$

17.2 Banded matrix (Upper)

$$n = ku + r, \quad 0 \leq r < u$$

$$U = \begin{bmatrix} d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & \\ & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & \\ & & \ddots & \ddots & \ddots & \cdots & \ddots \\ & & & \ddots & \ddots & \ddots & b_{n-u}^u \\ & & & & \ddots & \ddots & \vdots \\ & & & & & \ddots & b_{n-2}^2 \\ & & & & & & \ddots & b_{n-1}^1 \\ & & & & & & & d_n \end{bmatrix}$$

$$r = 0, \quad n = ku,$$

$$U = \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{ku \times ku},$$

$$D_i \in \mathbb{R}^{u \times u}, \quad B_i \in \mathbb{R}^{u \times u} \text{ for } i = 1, 2, \dots, k.$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} \in \mathbb{R}^{ku \times u} \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{bmatrix} \in \mathbb{R}^{ku \times u} \quad Y^T = [Y_1^T \quad Y_2^T \quad \cdots \quad Y_k^T] \in \mathbb{R}^{u \times ku}$$

$$X_i, Y_i^T \in \mathbb{R}^{u \times u}, \text{ for } i = 1, 2, \dots, k$$

$$X := U^{-1} E_k$$

$$Y_i^T := (D_i X_i)^{-1}$$

$$r \neq 0, \quad n = ku + r,$$

$$U = \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times (ku+r)},$$

$$\tilde{D}_1 \in \mathbb{R}^{r \times r}, \quad \tilde{B}_1 \in \mathbb{R}^{r \times u};$$

$$D_i \in \mathbb{R}^{u \times u}, \quad B_i \in \mathbb{R}^{u \times u} \text{ for } i = 2, \dots, k+1.$$

$$X = \begin{bmatrix} \tilde{X}_1 \\ X_2 \\ \vdots \\ X_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times u} \quad Y = \begin{bmatrix} \tilde{Y}_1 \\ Y_2 \\ \vdots \\ Y_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times u} \quad Y^T = [\tilde{Y}_1^T \quad Y_2^T \quad \cdots \quad Y_{k+1}^T] \in \mathbb{R}^{u \times (ku+r)}$$

$$\begin{aligned}
&\tilde{X}_1 \in \mathbb{R}^{r \times u} \\
&\tilde{Y}_1^T \in \mathbb{R}^{u \times r} \\
&X_i, Y_i^T \in \mathbb{R}^{u \times u} \text{ for } i = 2, \dots, k+1 \\
&X := U^{-1}E_{k+1} \\
&\tilde{Y}_1^T := (D_1X_1)^+ = (D_1X_1)^T(D_1X_1(D_1X_1)^T)^{-1} \\
&\text{triu}(D_1X_1\tilde{Y}_1^T) = I_{r \times r} \\
&Y_i^T := (D_iX_i)^{-1} \text{ for } i = 2, \dots, k+1
\end{aligned}$$

18 28.06.2022 Meeting formal

18.1 Notation

This notes use Householder notation and start indexing at one.

Scalars, lower case Greek letters: $\alpha, \beta, \gamma, \chi$, *etc.*

Vectors, lower case Roman letter: a, b, c, x , *etc.*

Matrix, upper case Roman letter: A, B, C, X , *etc.*

18.2 Repersentation

Matrix & Vector

$$x = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_n \end{bmatrix} \in \mathbb{R}^n$$
$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{1n} \end{bmatrix}$$
$$= [a_1 \mid a_2 \mid \cdots \mid a_n] \in \mathbb{R}^{m \times n}$$

Block Matrix & Block Vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{kn}, x_i \in \mathbb{R}^k$$
$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{1n} \end{bmatrix}$$
$$= [A_1 \mid A_2 \mid \cdots \mid A_n] \in \mathbb{R}^{lm \times kn}, A_{ij} \in \mathbb{R}^{l \times k}$$

18.3 Content

Consider an upper banded matrix $U \in \mathbb{R}^{n \times n}$,

$$U = \begin{bmatrix} \alpha_1 & \beta_1^1 & \beta_1^2 & \dots & \beta_1^u & & \\ & \alpha_2 & \beta_2^1 & \beta_2^2 & \dots & \beta_2^u & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & & & \beta_{n-u}^u & \\ & & & & & \vdots & \\ & & & & & \beta_{n-2}^2 & \\ & & & & & \beta_{n-1}^1 & \\ & & & & & & \alpha_n \end{bmatrix}. \quad (55)$$

Then try to show that its inverse can be represented as the upper triangular part of an outer product, *i.e.* ,

$$U^{-1} = \text{triu}(XY^T) \quad (56)$$

Let $X, Y \in \mathbb{R}^{n \times u}$, consider the following cases of n ,

Case 1, $n = ku$

then (55) can be represented as a block upper bidiagonal matrix,

$$\begin{bmatrix} \xrightarrow{r} & \xleftarrow{u} & \dots & \xleftarrow{u} \\ \begin{array}{c} d_1 \dots b_1^{r-1} \\ \vdots \\ d_r \end{array} & \begin{array}{c} b_1^r \dots b_1^u \\ \vdots \\ b_r^1 \dots b_r^u \end{array} & & \\ & \begin{array}{c} d_{1+r} \dots b_{1+r}^{u-1} \\ \vdots \\ d_{u+r} \end{array} & \begin{array}{c} b_{r+1}^u \\ \vdots \\ b_{u+r}^1 \dots b_{u+r}^u \end{array} & \\ & & \ddots & \\ & & & \begin{array}{c} d_{n-u+1} \dots b_{n-u+1}^u \\ \vdots \\ d_n \end{array} \end{bmatrix}$$

$\begin{array}{l} \uparrow r \\ \uparrow u \\ \vdots \\ \uparrow u \end{array}$

$$\begin{aligned}
U &= \begin{bmatrix} \xrightarrow{u} & \xrightarrow{u} & \dots & \xrightarrow{u} \\ \alpha_1 \dots \beta_1^{u-1} & \beta_1^u & & \\ \vdots & \vdots & \ddots & \\ \alpha_u & \beta_u^1 \dots \beta_u^u & & \\ & \alpha_{u+1} \dots \beta_{u+1}^{u-1} & \beta_{u+1}^u & \\ & \vdots & \vdots & \ddots \\ & \alpha_{2u} & \beta_{2u}^1 \dots \beta_{2u}^u & \\ & & \ddots & \\ & & & \alpha_{(k-1)u+1} \dots \beta_{(k-1)u+1}^{u-1} \\ & & & \vdots \\ & & & \alpha_{ku} \end{bmatrix} \\
&= \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{ku \times ku}, D_i, B_i \in \mathbb{R}^{u \times u} \text{ for } i = 1, 2, \dots, k. \quad (57)
\end{aligned}$$

Represent the n -dimensional identity matrix $I_{n \times n}$ in block form,

$$I_{n \times n} = I_{ku \times ku} = \text{diag}(\underbrace{I_{u \times u}, I_{u \times u}, \dots, I_{u \times u}}_k) = [E_1 \mid E_2 \mid \dots \mid E_k].$$

then, its block columns E_i for $i = 1, 2, \dots, k$ satisfies

$$\begin{aligned}
E_i &\in \mathbb{R}^{ku \times u} \\
E_i^T E_j &= \begin{cases} I_{u \times u}, & \text{if } i = j; \\ 0_{u \times u}, & \text{otherwise.} \end{cases}
\end{aligned}$$

To find X and Y such that (56) holds, let

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \in \mathbb{R}^{ku \times u}, Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{k-1} \\ Y_k \end{bmatrix} \in \mathbb{R}^{ku \times u}, X_i, Y_i \in \mathbb{R}^{u \times u} \text{ for } i = 1, 2, \dots, k,$$

and let X_k and Y_k^T be upper triangular matrix, then $\text{triu}(X_k Y_k^T) = X_k Y_k^T$.
For the outer product form,

$$\begin{aligned}
Y^T &= [Y_1^T \mid Y_2^T \mid \cdots \mid Y_{k-1}^T \mid Y_k^T] \in \mathbb{R}^{u \times ku} \\
XY^T &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} [Y_1^T \mid Y_2^T \mid \cdots \mid Y_{k-1}^T \mid Y_k^T] \\
&= \begin{bmatrix} X_1 Y_1^T & X_1 Y_2^T & \cdots & X_1 Y_{k-1}^T & X_1 Y_k^T \\ X_2 Y_1^T & X_2 Y_2^T & \cdots & X_2 Y_{k-1}^T & X_2 Y_k^T \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{k-1} Y_1^T & X_{k-1} Y_2^T & \cdots & X_{k-1} Y_{k-1}^T & X_{k-1} Y_k^T \\ X_k Y_1^T & X_k Y_2^T & \cdots & X_k Y_{k-1}^T & X_k Y_k^T \end{bmatrix} \\
&= X [Y_1^T \mid Y_2^T \mid \cdots \mid Y_{k-1}^T \mid Y_k^T] \\
&= [XY_1^T \mid XY_2^T \mid \cdots \mid XY_{k-1}^T \mid XY_k^T] \in \mathbb{R}^{ku \times ku},
\end{aligned}$$

and its the upper triangular component,

$$\begin{aligned}
\text{triu}(XY^T) &= \begin{bmatrix} \text{triu}(X_1 Y_1^T) & X_1 Y_2^T & \cdots & X_1 Y_{k-1}^T & X_1 Y_k^T \\ 0_{u \times u} & \text{triu}(X_2 Y_2^T) & \cdots & X_2 Y_{k-1}^T & X_2 Y_k^T \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{u \times u} & 0_{u \times u} & \cdots & \text{triu}(X_{k-1} Y_{k-1}^T) & X_{k-1} Y_k^T \\ 0_{u \times u} & 0_{u \times u} & \cdots & 0_{u \times u} & X_k Y_k^T \end{bmatrix} \\
&= \begin{bmatrix} \text{triu}(X_1 Y_1^T) & X_1 Y_2^T & \cdots & X_{1:k-2} Y_{k-1}^T & X_1 Y_k^T \\ 0_{(k-2)u \times u} & \text{triu}(X_2 Y_2^T) & \cdots & \text{triu}(X_{k-1} Y_{k-1}^T) & X_2 Y_k^T \\ 0_{u \times u} & 0_{(k-2)u \times u} & \cdots & 0_{u \times u} & X_k Y_k^T \end{bmatrix} .zz \quad (58)
\end{aligned}$$

From (56),

$$I_{n \times n} = U \text{triu}(XY^T). \quad (59)$$

For X and Y_k , from (58), the k -th block column of (59) can be written as,

$$\begin{aligned}
E_k &= U \text{triu}(XY^T) E_k \\
E_k &= UXY_k^T \quad (60)
\end{aligned}$$

Define that

$$X := U^{-1} E_k; \quad (61)$$

$$Y_k^T := I_{u \times u}, \quad (62)$$

which can be shown that (60) holds for the defined (61) and (62) through

$$\begin{aligned}
UXY_k^T &= UU^{-1} E_k I_{u \times u} \\
&= I_{n \times n} E_k I_{u \times u} \\
&= E_k I_{u \times u} \\
&= E_k, \text{ as required.}
\end{aligned}$$

Then, for Y_{k-1} , forming the $(k-1)$ -th block column of (59) from (58) first,

$$\begin{aligned} E_{k-1} &= U \text{triu}(XY^T)E_{k-1} \\ &= U \begin{bmatrix} X_{1:k-2}Y_{k-1}^T \\ \text{triu}(X_{k-1}Y_{k-1}^T) \\ 0_{u \times u} \end{bmatrix}. \end{aligned}$$

Focusing on the $(k-1)$ -th block row of $(k-1)$ -th block column,

$$\begin{aligned} E_{k-1}^T E_{k-1} &= E_{k-1}^T U \begin{bmatrix} X_{1:k-2}Y_{k-1}^T \\ \text{triu}(X_{k-1}Y_{k-1}^T) \\ 0_{u \times u} \end{bmatrix} \\ I_{u \times u} &= \left[\begin{array}{c|c|c} 0_{u \times (k-2)u} & D_{k-1} & B_{k-1} \end{array} \right] \begin{bmatrix} X_{1:k-2}Y_{k-1}^T \\ \text{triu}(X_{k-1}Y_{k-1}^T) \\ 0_{u \times u} \end{bmatrix} \\ I_{u \times u} &= D_{k-1} \text{triu}(X_{k-1}Y_{k-1}^T). \end{aligned} \tag{63}$$

Define that

$$Y_{k-1}^T := (D_{k-1}X_{k-1})^{-1}, \tag{64}$$

and notice that $D_{k-1} \in \mathbb{R}^{u \times u}$ is a upper triangular, therefore

$$\begin{aligned} D_{k-1} &= \text{triu}(D_{k-1}); \\ D_{k-1}^{-1} &= \text{triu}(D_{k-1}^{-1}). \end{aligned} \tag{65}$$

With the fact (65), (63) holds for the defined (64) as,

$$\begin{aligned} D_{k-1} \text{triu}(X_{k-1}Y_{k-1}^T) &= D_{k-1} \text{triu}(X_{k-1}(D_{k-1}X_{k-1})^{-1}) \\ &= D_{k-1} \text{triu}(X_{k-1}X_{k-1}^{-1}D_{k-1}^{-1}) \\ &= D_{k-1} \text{triu}(D_{k-1}^{-1}) \\ &= D_{k-1}D_{k-1}^{-1} \\ &= I_{u \times u}, \text{ as required.} \end{aligned}$$

Similar ideas can be use to redefine Y_k^T in (62), from (61),

$$\begin{aligned} E_k &= UX \\ E_k^T E_k &= E_k^T UX \\ I_{u \times u} &= E_k^T UX \\ &= \left[\begin{array}{c|c} 0_{u \times (k-1)u} & D_k \end{array} \right] \begin{bmatrix} X_{1:k-1} \\ X_k \end{bmatrix} \\ &= D_k X_k \\ I_{u \times u} &= (D_k X_k)^{-1}, \end{aligned}$$

following that

$$\begin{aligned}
Y_k^{\text{T}} &:= I_{u \times u} \\
&:= (D_k X_k)^{-1}.
\end{aligned}$$

In summary, for $U \in \mathbb{R}^{ku \times ku}$, (56) holds the following definitions of X and Y :

$$\begin{aligned}
X &:= U^{-1} E_k \\
Y_i^{\text{T}} &:= (D_i X_i)^{-1}
\end{aligned}$$

Tridiagonal	invBanded				invSymT2SSexp			
	Absolute Error		Relative Error		Absolute Error		Relative	
Size	Left	Right	Left	Right	Left	Right	Left	
100	$2.18\text{E} - 15$	$2.18\text{E} - 15$	$2.18\text{E} - 16$	$2.18\text{E} - 16$	$3.75\text{E} - 13$	$3.75\text{E} - 13$	$3.75\text{E} - 14$	
200	$2.88\text{E} - 15$	$2.88\text{E} - 15$	$2.04\text{E} - 16$	$2.04\text{E} - 16$	$1.35\text{E} - 12$	$1.35\text{E} - 12$	$9.51\text{E} - 14$	
400	$4.27\text{E} - 15$	$4.27\text{E} - 15$	$2.13\text{E} - 16$	$2.13\text{E} - 16$	$4.42\text{E} - 12$	$4.42\text{E} - 12$	$2.21\text{E} - 13$	
800	$6.29\text{E} - 15$	$6.29\text{E} - 15$	$2.22\text{E} - 16$	$2.22\text{E} - 16$	$1.34\text{E} - 11$	$1.34\text{E} - 11$	$4.72\text{E} - 13$	
1600	$8.56\text{E} - 15$	$8.56\text{E} - 15$	$2.14\text{E} - 16$	$2.14\text{E} - 16$	$4.07\text{E} - 11$	$4.07\text{E} - 11$	$1.02\text{E} - 12$	
3200	$1.22\text{E} - 14$	$1.22\text{E} - 14$	$2.16\text{E} - 16$	$2.16\text{E} - 16$	$1.20\text{E} - 10$	$1.20\text{E} - 10$	$2.12\text{E} - 12$	
6400	$1.66\text{E} - 14$	$1.66\text{E} - 14$	$2.08\text{E} - 16$	$2.08\text{E} - 16$	$3.47\text{E} - 10$	$3.47\text{E} - 10$	$4.34\text{E} - 12$	
12800	$2.31\text{E} - 14$	$2.31\text{E} - 14$	$2.04\text{E} - 16$	$2.04\text{E} - 16$	$1.03\text{E} - 09$	$1.03\text{E} - 09$	$9.07\text{E} - 12$	

$$\begin{array}{c} r \\ u \\ u \\ \vdots \end{array} \begin{bmatrix} \overset{r}{\tilde{D}_0} & \overset{u}{\tilde{B}_0} & 0 & 0 & 0 \\ 0 & D_1 & B_1 & 0 & 0 \\ 0 & 0 & D_2 & B_2 & 0 \\ 0 & 0 & 0 & D_2 & B_2 & 0 \end{bmatrix}$$

$$\begin{array}{c} r \\ u \\ u \\ \vdots \\ \vdots \\ \vdots \\ u \end{array} \begin{bmatrix} \overset{r}{\tilde{D}_0} & \overset{u}{\tilde{B}_0} & u & \dots & \dots & u \\ & D_1 & B_1 & & & \\ & & D_2 & B_2 & & \\ & & & \ddots & \ddots & \\ & & & & D_{k-1} & B_{k-1} \\ & & & & & D_k \end{bmatrix}$$

$$\begin{array}{c} u \\ u \\ u \\ \vdots \\ \vdots \\ \vdots \\ u \end{array} \begin{bmatrix} \overset{u}{D_1} & \overset{u}{B_1} & u & \dots & \dots & u \\ & D_2 & B_2 & & & \\ & & \ddots & \ddots & & \\ & & & D_{k-1} & B_{k-1} & \\ & & & & D_k & \end{bmatrix}$$

19 Code

A, B, U, L, T, Δ	Matrix
$\alpha, \beta, \gamma, \chi$	Scalar
A^{-1}	The inverse matrix of the matrix A
A^+	The Moore-Penro pseudo inverse matrix of the matrix A
$a_{i,j}, (A)_{i,j}, A(i, j)$	The $(i, j)^{\text{th}}$ entry of the matrix A
$i : j$	All integer between i and j , i.e., $\mathbb{Z} \cap [i, j]$
$x_{(i:j)}, x(i : j)$	The subvector consisting of i^{th} to j^{th} entries of the vector x
\circ	Hadamard product
op	Floating point number operation
I_n	The identity matrix of size n
$0_{m \times n}$	The zero matrix of size $m \times n$
0_n	The zero matrix of size $n \times n$
$\mathbb{1}_\Omega(x)$	Indicator function, i.e., $\mathbb{1}_\Omega(x) = \begin{cases} 1, & \text{if } x \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$
$\delta_{i,j}$	Kronecker delta
e_i	The i^{th} column of the identity matrix I
E_i	The i^{th} block column of the block identity matrix I
$\text{triu } A$	The upper triangular part of the matrix A
$\text{tril } A$	The lower triangular part of the matrix A
$T_n(b^{-1}, d, b^1)$	The n-by-n tridiagonal Toeplitz matrix
$U_n(b, d)$	The n-by-n bidiagonal matrix