

MSc Project Notes

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1 02.08.2022

1.1 Bidiagonal matrix

Let $\tilde{U} \in \mathbb{R}^{n \times n}$ be a bidiagonal matrix with non-zero diagonal, *i.e.* $\tilde{u}_{ii} \neq 0$ for $i = 1, 2, \dots, n$,

$$\tilde{U} = \begin{bmatrix} \tilde{u}_{11} & \tilde{u}_{12} & & & \\ & \tilde{u}_{22} & \tilde{u}_{23} & & \\ & & \tilde{u}_{33} & \tilde{u}_{34} & \\ & & & \tilde{u}_{44} & \tilde{u}_{45} \\ & & & & \tilde{u}_{55} \end{bmatrix}$$

and $D = \text{diag}(\tilde{u}_{ii})$ for $i = 1, 2, \dots, n$,

$$D = \begin{bmatrix} \frac{1}{\tilde{u}_{11}} & & & & \\ & \frac{1}{\tilde{u}_{22}} & & & \\ & & \frac{1}{\tilde{u}_{33}} & & \\ & & & \frac{1}{\tilde{u}_{44}} & \\ & & & & \frac{1}{\tilde{u}_{55}} \end{bmatrix}$$

replace $\frac{\tilde{u}_{i,i+1}}{\tilde{u}_{i+1,i+1}}$ by $u_{i,i+1}$ for $i = 1, 2, \dots, n-1$, and let $U = \tilde{U}D$,

$$\begin{aligned} U &= \begin{bmatrix} 1 & \frac{\tilde{u}_{12}}{\tilde{u}_{22}} & & & \\ & 1 & \frac{\tilde{u}_{23}}{\tilde{u}_{33}} & & \\ & & 1 & \frac{\tilde{u}_{34}}{\tilde{u}_{44}} & \\ & & & 1 & \frac{\tilde{u}_{45}}{\tilde{u}_{55}} \\ & & & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & u_{12} & & & \\ & 1 & u_{23} & & \\ & & 1 & u_{34} & \\ & & & 1 & u_{45} \\ & & & & 1 \end{bmatrix}. \end{aligned}$$

Let $A = U^{-1}$ be the inverse of U , a upper triangular matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ & a_{22} & a_{23} & a_{24} & a_{25} \\ & & a_{33} & a_{34} & a_{35} \\ & & & a_{44} & a_{45} \\ & & & & a_{55} \end{bmatrix}$$

then $I = UU^{-1} = UA$,

$$I = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} + a_{22}u_{12} & a_{13} + a_{23}u_{12} & a_{14} + a_{24}u_{12} & a_{15} + a_{25}u_{12} \\ & a_{22} & a_{23} + a_{33}u_{23} & a_{24} + a_{34}u_{23} & a_{25} + a_{35}u_{23} \\ & & a_{33} & a_{34} + a_{44}u_{34} & a_{35} + a_{45}u_{34} \\ & & & a_{44} & a_{45} + a_{55}u_{45} \\ & & & & a_{55} \end{bmatrix} = UA.$$

Note that $UA = \text{diag}(a_{ii}) + \text{triu}(UA, 1)$ for $i = 1, 2, \dots, n$.

Find A such that $\text{diag}(a_{ii}) = I$ and $\text{triu}(UA, 1) = 0_{n \times n}$ for $i = 1, 2, \dots, n$.

$$xy^T = \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 & x_1y_4 & x_1y_5 \\ x_2y_1 & x_2y_2 & x_2y_3 & x_2y_4 & x_2y_5 \\ x_3y_1 & x_3y_2 & x_3y_3 & x_3y_4 & x_3y_5 \\ x_4y_1 & x_4y_2 & x_4y_3 & x_4y_4 & x_4y_5 \\ x_5y_1 & x_5y_2 & x_5y_3 & x_5y_4 & x_5y_5 \end{bmatrix}$$

Express A in form of an outer product, *i.e.* $A = xy^T$ for $\forall x, y \in \mathbb{R}^n$, then

$$\text{triu}(UA, 1) = \text{triu}(Uxy^T, 1) = 0_{n \times n}$$

Note that $\text{triu}(e_n y^T, 1) = 0_{n \times n}$ as,

$$e_n y^T = \begin{bmatrix} \\ \\ \\ \\ 1 \end{bmatrix} \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix}$$

solve $Ux := e_n$ for x by backward substitution, then let $y_i := \frac{1}{x_i}$ or $x_i y_i = 1$ for $i = 1, 2, \dots, n$. Then,

$$\begin{aligned} UA &= \text{diag}(a_{ii}) + \text{triu}(UA, 1) \\ &= \text{diag}(x_i y_i) + \text{triu}(Uxy^T, 1) \\ &= \text{diag}(\underbrace{1, 1, \dots, 1}_n) + \text{triu}(e_n y^T, 1) \\ &= I + 0_{n \times n} \\ &= I \end{aligned}$$

1.2 Upper triangular matrix

Let $\tilde{U} \in \mathbb{R}^{n \times n}$ be a banded matrix with non-zero diagonal, upper banded width u , and lower banded width 0. Similarly,

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} & & \\ & 1 & u_{23} & u_{24} & \\ & & 1 & u_{34} & u_{35} \\ & & & 1 & u_{45} \\ & & & & 1 \end{bmatrix}$$

$$\begin{aligned}
UA &= \begin{bmatrix} a_{11} & a_{12} + a_{22}u_{12} & a_{13} + a_{23}u_{12} + a_{33}u_{13} & a_{14} + a_{24}u_{12} + a_{34}u_{13} & a_{15} + a_{25}u_{12} + a_{35}u_{13} \\ & a_{22} & a_{23} + a_{33}u_{23} & a_{24} + a_{34}u_{23} + a_{44}u_{24} & a_{25} + a_{35}u_{23} + a_{45}u_{24} \\ & & a_{33} & a_{34} + a_{44}u_{34} & a_{35} + a_{45}u_{34} + a_{55}u_{35} \\ & & & a_{44} & a_{45} + a_{55}u_{45} \\ & & & & a_{55} \end{bmatrix} \\
&= \text{diag}(a_{ii}) + \text{triu}(UA, 1) \text{ for } i = 1, 2, \dots, n
\end{aligned}$$

Find A such that $\text{diag}(a_{ii}) = I$ and $\text{triu}(UA, 1) = 0_{n \times n}$ for $i = 1, 2, \dots, n$.

$$XY^T = \begin{bmatrix} x_{11}y_{11} + x_{12}y_{12} & x_{11}y_{21} + x_{12}y_{22} & x_{11}y_{31} + x_{12}y_{32} & x_{11}y_{41} + x_{12}y_{42} & x_{11}y_{51} + x_{12}y_{52} \\ x_{21}y_{11} + x_{22}y_{12} & x_{21}y_{21} + x_{22}y_{22} & x_{21}y_{31} + x_{22}y_{32} & x_{21}y_{41} + x_{22}y_{42} & x_{21}y_{51} + x_{22}y_{52} \\ x_{31}y_{11} + x_{32}y_{12} & x_{31}y_{21} + x_{32}y_{22} & x_{31}y_{31} + x_{32}y_{32} & x_{31}y_{41} + x_{32}y_{42} & x_{31}y_{51} + x_{32}y_{52} \\ x_{41}y_{11} + x_{42}y_{12} & x_{41}y_{21} + x_{42}y_{22} & x_{41}y_{31} + x_{42}y_{32} & x_{41}y_{41} + x_{42}y_{42} & x_{41}y_{51} + x_{42}y_{52} \\ x_{51}y_{11} + x_{52}y_{12} & x_{51}y_{21} + x_{52}y_{22} & x_{51}y_{31} + x_{52}y_{32} & x_{51}y_{41} + x_{52}y_{42} & x_{51}y_{51} + x_{52}y_{52} \end{bmatrix}$$

Express A in outer product, *i.e.* $A = XY^T$, for $\forall X, Y \in \mathbb{R}^{n \times 2}$,

$$\text{triu}(UA, 1) = \text{triu}(UXY^T, 1) = 0_{n \times n}$$

Note that $\text{triu}([e_{n-1} \ e_n]Y^T, 2) = 0_{n \times n}$ as,

$$[e_{n-1} \ e_n]Y^T = \begin{bmatrix} & \\ & \\ & \\ 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} y_{11} & y_{21} & y_{31} & y_{41} & y_{51} \\ y_{12} & y_{22} & y_{32} & y_{42} & y_{52} \end{bmatrix} = \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ y_{11} & y_{21} & y_{31} & y_{41} & y_{51} \\ y_{12} & y_{22} & y_{32} & y_{42} & y_{52} \end{bmatrix}$$

solve $UX = U[x_1 \ x_2] := [e_{n-1} \ e_n]$ for x_1 and x_2 by backward substitution,

2 09.06.2022

2.1 Notation

Vectors

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

Matrices

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ &= [a_1 \ a_2 \ \cdots \ a_n] \in \mathbb{R}^{m \times n}, \ a_i \in \mathbb{R}^m \end{aligned}$$

Columns

$$\begin{aligned} Ae_i &:= a_i \\ e_i &= i\text{-th column of } I \end{aligned}$$

2.2 Bidiagonal matrix

$$U = \begin{bmatrix} d_1 & u_1 & & & \\ & d_2 & u_1 & & \\ & & \ddots & \ddots & \\ & & & d_{n-1} & u_{n-1} \\ & & & & d_n \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \in \mathbb{R}^n \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} \in \mathbb{R}^n \qquad y^T = [y_1 \ y_2 \ \cdots \ y_{n-1} \ y_n]$$

$$\begin{aligned}
xy^T &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \\
&= \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_{n-1} & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_{n-1} & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_4 y_1 & x_4 y_2 & \cdots & x_4 y_{n-1} & x_4 y_n \\ x_5 y_1 & x_5 y_2 & \cdots & x_5 y_{n-1} & x_5 y_n \end{bmatrix} \\
&= x \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \\
&= \begin{bmatrix} xy_1 & xy_2 & \cdots & xy_{n-1} & xy_n \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{triu}(xy^T) &= \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_{n-1} & x_1 y_n \\ 0 & x_2 y_2 & \cdots & x_2 y_{n-1} & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_4 y_{n-1} & x_4 y_n \\ 0 & 0 & \cdots & 0 & x_5 y_n \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} x_{1:1} \\ 0_{n-1} \end{bmatrix} y_1 & \begin{bmatrix} x_{1:2} \\ 0_{n-2} \end{bmatrix} y_2 & \cdots & \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} & xy_n \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
U^{-1} &= \text{triu}(xy^T) \\
I &= U \text{triu}(xy^T)
\end{aligned}$$

n -th column,

$$\begin{aligned}
e_n &= U \text{triu}(xy^T) e_n \\
&= U xy_n
\end{aligned}$$

$$\begin{aligned}
x &:= U^{-1} e_n \\
e_n &= U xy_n \\
&= U U^{-1} e_n y_n \\
&= e_n y_n \\
y_n &:= 1
\end{aligned}$$

$(n-1)$ -th column,

$$\begin{aligned}
e_{n-1} &= U \text{triu}(xy^T) e_{n-1} \\
&= U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1}
\end{aligned}$$

$(n-1)$ -th row of $(n-1)$ -th column,

$$e_i^T e_j = \delta_{ij}$$

$$\begin{aligned} e_{n-1}^T e_{n-1} &= 1 = e_{n-1}^T U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \\ &= \begin{bmatrix} 0_{n-2} & d_{n-1} & u_{n-1} \end{bmatrix} \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \\ &= d_{n-1} x_{n-1} y_{n-1} \\ y_{n-1} &:= (d_{n-1} x_{n-1})^{-1} \end{aligned}$$

$$\begin{aligned} e_n &= Ux \\ e_n^T e_n &= 1 = e_n^T Ux \\ &= \begin{bmatrix} 0_{n-1} & d_n \end{bmatrix} x \\ &= d_n x_n \\ y_n &:= 1 \\ &:= (d_n x_n)^{-1} \end{aligned}$$

$$\begin{aligned} x &:= U^{-1} e_n \\ y_i &:= (d_i x_i)^{-1} \end{aligned}$$