MSc Project Notes

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1.1 Bidiagonal matrix

Let $\tilde{U} \in \mathbb{R}^{n\times n}$ be a bidiagonal matrix with non-zero diagonal, *i.e.* $\tilde{u}_{ii} \neq 0$ for $i = 1, 2, \ldots, n$,

$$\tilde{U} = \begin{bmatrix} \tilde{u}_{11} & \tilde{u}_{12} & & & \\ & \tilde{u}_{22} & \tilde{u}_{23} & & \\ & & \tilde{u}_{33} & \tilde{u}_{34} & \\ & & & \tilde{u}_{44} & \tilde{u}_{45} \\ & & & & \tilde{u}_{55} \end{bmatrix}$$

and $D = \operatorname{diag}(\tilde{u}_{ii})$ for $i = 1, 2, \ldots, n$,

$$D = \begin{bmatrix} \frac{1}{\tilde{u}_{11}} & & & & \\ & \frac{1}{\tilde{u}_{22}} & & & \\ & & \frac{1}{\tilde{u}_{33}} & & \\ & & & \frac{1}{\tilde{u}_{44}} & \\ & & & & \frac{1}{\tilde{u}_{55}} \end{bmatrix}$$

replace $\frac{\tilde{u}_{i,i+1}}{\tilde{u}_{i+1,i+1}}$ by $u_{i,i+1}$ for $i = 1, 2, \ldots, n-1$, and let $U = \tilde{U}D$,

$$\begin{aligned} U &= \begin{bmatrix} 1 & \frac{\tilde{u}_{12}}{\tilde{u}_{22}} & & & \\ & 1 & \frac{\tilde{u}_{23}}{\tilde{u}_{33}} & & \\ & & 1 & \frac{\tilde{u}_{34}}{\tilde{u}_{44}} & \\ & & & 1 & \frac{\tilde{u}_{45}}{\tilde{u}_{55}} \\ & & & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & u_{12} & & & \\ & 1 & u_{23} & & \\ & & 1 & u_{34} & \\ & & & 1 & u_{45} \\ & & & & 1 \end{bmatrix}. \end{aligned}$$

Let $A = U^{-1}$ be the inverse of U, a upper triangular matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ & a_{22} & a_{23} & a_{24} & a_{25} \\ & & a_{33} & a_{34} & a_{35} \\ & & & a_{44} & a_{45} \\ & & & & a_{55} \end{bmatrix}$$

then $I = UU^{-1} = UA$,

$$I = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} + a_{22}u_{12} & a_{13} + a_{23}u_{12} & a_{14} + a_{24}u_{12} & a_{15} + a_{25}u_{12} \\ & a_{22} & a_{23} + a_{33}u_{23} & a_{24} + a_{34}u_{23} & a_{25} + a_{35}u_{23} \\ & & a_{33} & a_{34} + a_{44}u_{34} & a_{35} + a_{45}u_{34} \\ & & & a_{44} & a_{45} + a_{55}u_{45} \\ & & & & a_{55} \end{bmatrix} = UA.$$

Note that $UA = \text{diag}(a_{ii}) + \text{triu}(UA, 1)$ for $i = 1, 2, \ldots, n$.
Find A such that $\text{diag}(a_{ii}) = I$ and $\text{triu}(UA, 1) = 0_{n\times n}$ for $i = 1, 2, \ldots, n$.

$$xy^{\mathrm{T}} = \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 & x_1y_4 & x_1y_5 \\ x_2y_1 & x_2y_2 & x_2y_3 & x_2y_4 & x_2y_5 \\ x_3y_1 & x_3y_2 & x_3y_3 & x_3y_4 & x_3y_5 \\ x_4y_1 & x_4y_2 & x_4y_3 & x_4y_4 & x_4y_5 \\ x_5y_1 & x_5y_2 & x_5y_3 & x_5y_4 & x_5y_5 \end{bmatrix}$$

Express A in form of an outer product, *i.e.* $A = xy^{\mathrm{T}}$ for $\forall x, y \in \mathbb{R}^n$, then

$$\text{triu}(UA, 1) = \text{triu}(Uxy^{\mathrm{T}}, 1) = 0_{n\times n}$$

Note that $\text{triu}(e_ny^{\mathrm{T}}, 1) = 0_{n\times n}$ as,

$$e_ny^{\mathrm{T}} = \begin{bmatrix} \\ \\ \\ \\ 1 \end{bmatrix} \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix} = \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix}$$

solve $Ux := e_n$ for x by backward substitution, then let $y_i := \frac{1}{x_i}$ or $x_iy_i = 1$ for $i = 1, 2, \ldots, n$.
Then,

$$\begin{aligned} UA &= \text{diag}(a_{ii}) + \text{triu}(UA, 1) \\ &= \text{diag}(x_iy_i) + \text{triu}(Uxy^{\mathrm{T}}, 1) \\ &= \text{diag}(\underbrace{1, 1, \cdots, 1}_{n}) + \text{triu}(e_ny^{\mathrm{T}}, 1) \\ &= I + 0_{n\times n} \\ &= I \end{aligned}$$

1.2 Upper triangular matrix

Let $\tilde{U} \in \mathbb{R}^{n\times n}$ be a banded matrix with non-zero diagonal, upper banded width u, and lower banded width 0 Similarly,

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} & & \\ & 1 & u_{23} & u_{24} & \\ & & 1 & u_{34} & u_{35} \\ & & & 1 & u_{45} \\ & & & & 1 \end{bmatrix}$$

$$
UA = \begin{bmatrix}
a_{11} & a_{12} + a_{22}u_{12} & a_{13} + a_{23}u_{12} + a_{33}u_{13} & a_{14} + a_{24}u_{12} + a_{34}u_{13} & a_{15} + a_{25}u_{12} + a_{35}u_{13} \\
 & a_{22} & a_{23} + a_{33}u_{23} & a_{24} + a_{34}u_{23} + a_{44}u_{24} & a_{25} + a_{35}u_{23} + a_{45}u_{24} \\
 & & a_{33} & a_{34} + a_{44}u_{34} & a_{35} + a_{45}u_{34} + a_{55}u_{35} \\
 & & & a_{44} & a_{45} + a_{55}u_{45} \\
 & & & & a_{55}
\end{bmatrix}
$$

$$
= \operatorname{diag}(a_{ii}) + \operatorname{triu}(UA, 1) \text{ for } i = 1, 2, \ldots, n
$$

Find A such that $\operatorname{diag}(a_{ii}) = I$ and $\operatorname{triu}(UA, 1) = 0_{n \times n}$ for $i = 1, 2, \ldots, n$.

$$
XY^{\mathrm{T}} = \begin{bmatrix}
x_{11}y_{11} + x_{12}y_{12} & x_{11}y_{21} + x_{12}y_{22} & x_{11}y_{31} + x_{12}y_{32} & x_{11}y_{41} + x_{12}y_{42} & x_{11}y_{51} + x_{12}y_{52} \\
x_{21}y_{11} + x_{22}y_{12} & x_{21}y_{21} + x_{22}y_{22} & x_{21}y_{31} + x_{22}y_{32} & x_{21}y_{41} + x_{22}y_{42} & x_{21}y_{51} + x_{22}y_{52} \\
x_{31}y_{11} + x_{32}y_{12} & x_{31}y_{21} + x_{32}y_{22} & x_{31}y_{31} + x_{32}y_{32} & x_{31}y_{41} + x_{32}y_{42} & x_{31}y_{51} + x_{32}y_{52} \\
x_{41}y_{11} + x_{42}y_{12} & x_{41}y_{21} + x_{42}y_{22} & x_{41}y_{31} + x_{42}y_{32} & x_{41}y_{41} + x_{42}y_{42} & x_{41}y_{51} + x_{42}y_{52} \\
x_{51}y_{11} + x_{52}y_{12} & x_{51}y_{21} + x_{52}y_{22} & x_{51}y_{31} + x_{52}y_{32} & x_{51}y_{41} + x_{52}y_{42} & x_{51}y_{51} + x_{52}y_{52}
\end{bmatrix}
$$

Express A in outer product, *i.e.* $A = XY^{\mathrm{T}}$, for $\forall X, Y \in \mathbb{R}^{n \times 2}$,

$$
\operatorname{triu}(UA, 1) = \operatorname{triu}(UXY^{\mathrm{T}}, 1) = 0_{n \times n}
$$

Note that $\operatorname{triu}(\begin{bmatrix} e_{n-1} & e_n \end{bmatrix} Y^{\mathrm{T}}, 2) = 0_{n \times n}$ as,

$$
\begin{bmatrix} e_{n-1} & e_n \end{bmatrix} Y^{\mathrm{T}} = \begin{bmatrix} & \\ & \\ & \\ 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} y_{11} & y_{21} & y_{31} & y_{41} & y_{51} \\ y_{12} & y_{22} & y_{32} & y_{42} & y_{52} \end{bmatrix} = \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ y_{11} & y_{21} & y_{31} & y_{41} & y_{51} \\ y_{12} & y_{22} & y_{32} & y_{42} & y_{52} \end{bmatrix}
$$

solve $UX = U\begin{bmatrix} x_1 & x_2 \end{bmatrix} := \begin{bmatrix} e_{n-1} & e_n \end{bmatrix}$ for x_1 and x_2 by backward substitution,

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2.1 Notation

Vectors

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

Matrices and Columns

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{m \times n}, \ a_i \in \mathbb{R}^m \end{aligned}$$

$$\begin{aligned} Ae_i &:= a_i \\ e_i &= i\text{-th column of } I \end{aligned}$$

Block Matrices and Block Columns

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1l} \\ A_{21} & A_{22} & \cdots & A_{1l} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kl} \end{bmatrix}, \ A_{ij} \in \mathbb{R}^{m \times n} \\ &= \begin{bmatrix} A_1 & A_2 & \cdots & A_l \end{bmatrix} \in \mathbb{R}^{km \times ln}, \ A_i \in \mathbb{R}^{km \times n} \end{aligned}$$

$$\begin{aligned} AE_i &\coloneqq A_i \\ E_i &= i\text{-th block column of } I \end{aligned}$$

2.2 Bidiagonal matrix (upper)

$$U = \begin{bmatrix} d_1 & b_1 & & & \\ & d_2 & b_2 & & \\ & & \ddots & \ddots & \\ & & & d_{n-1} & b_{n-1} \\ & & & & d_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \in \mathbb{R}^{n\times 1} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} \in \mathbb{R}^{n\times 1} \qquad y^{\mathrm{T}} = \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \in \mathbb{R}^{1\times n}$$

$$\begin{aligned} xy^{\mathrm{T}} &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \\ &= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_{n-1} & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_{n-1} & x_2y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1}y_1 & x_{n-1}y_2 & \cdots & x_{n-1}y_{n-1} & x_{n-1}y_n \\ x_ny_1 & x_ny_2 & \cdots & x_ny_{n-1} & x_ny_n \end{bmatrix} \\ &= x\begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \\ &= \begin{bmatrix} xy_1 & xy_2 & \cdots & xy_{n-1} & xy_n \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathrm{triu}(xy^{\mathrm{T}}) &= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_{n-1} & x_1y_n \\ 0 & x_2y_2 & \cdots & x_2y_{n-1} & x_2y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_4y_{n-1} & x_4y_n \\ 0 & 0 & \cdots & 0 & x_5y_n \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} x_{1:1} \\ 0_{n-1} \end{bmatrix} y_1 & \begin{bmatrix} x_{1:2} \\ 0_{n-2} \end{bmatrix} y_2 & \cdots & \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} & xy_n \end{bmatrix} \end{aligned}$$

$$\begin{aligned} U^{-1} &= \mathrm{triu}(xy^{\mathrm{T}}) \\ I &= U\,\mathrm{triu}(xy^{\mathrm{T}}) \end{aligned}$$

n-th column,

$$\begin{aligned} e_n &= U\,\mathrm{triu}(xy^{\mathrm{T}})e_n \\ &= Uxy_n \end{aligned}$$

$$\begin{aligned} x &:= U^{-1}e_n \\ e_n &= Uxy_n \\ &= UU^{-1}e_ny_n \\ &= e_ny_n \\ y_n &:= 1 \end{aligned}$$

$(n-1)$-th column,

$$
\begin{aligned}
e_{n-1} &= U \operatorname{triu}(xy^{\mathrm{T}})e_{n-1} \\
&= U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1}
\end{aligned}
$$

$(n-1)$-th row of $(n-1)$-th column,

$$
e_i^{\mathrm{T}} e_j = \delta_{ij}
$$

$$
\begin{aligned}
e_{n-1}^{\mathrm{T}} e_{n-1} = 1 &= e_{n-1}^{\mathrm{T}} U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \\
&= \begin{bmatrix} 0_{n-2} & d_{n-1} & b_{n-1} \end{bmatrix} \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \\
&= d_{n-1} x_{n-1} y_{n-1} \\
y_{n-1} &:= (d_{n-1} x_{n-1})^{-1}
\end{aligned}
$$

$$
\begin{aligned}
e_n &= Ux \\
e_n^{\mathrm{T}} e_n = 1 &= e_n^{\mathrm{T}} U x \\
&= \begin{bmatrix} 0_{n-1} & d_n \end{bmatrix} x \\
&= d_n x_n \\
y_n &:= 1 \\
&:= (d_n x_n)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
x &:= U^{-1} e_n \\
y_i &:= (d_i x_i)^{-1}
\end{aligned}
$$

2.3 Tridiagonal matrix (Upper)

$n = 2k$

$$
\begin{aligned}
U &= \begin{bmatrix}
d_1 & b_1^1 & b_1^2 & & & & & & \\
 & d_2 & b_2^1 & b_2^2 & & & & & \\
 & & d_3 & b_3^1 & b_3^2 & & & & \\
 & & & d_4 & b_4^1 & b_4^2 & & & \\
 & & & & \ddots & \ddots & \ddots & & \\
 & & & & & d_{n-3} & b_{n-3}^1 & b_{n-3}^2 & \\
 & & & & & & d_{n-2} & b_{n-2}^1 & b_{n-2}^2 \\
 & & & & & & & d_{n-1} & b_{n-1}^1 \\
 & & & & & & & & d_n
\end{bmatrix} \\
&= \begin{bmatrix}
\begin{bmatrix} d_1 & b_1^1 \\ & d_2 \end{bmatrix} & \begin{bmatrix} b_1^2 & \\ b_2^1 & b_2^2 \end{bmatrix} & & & \\
 & \begin{bmatrix} d_3 & b_3^1 \\ & d_4 \end{bmatrix} & \begin{bmatrix} b_3^2 & \\ b_4^1 & b_4^2 \end{bmatrix} & & \\
 & & \ddots & \ddots & \\
 & & & \begin{bmatrix} d_{2k-3} & b_{2k-3}^1 \\ & d_{2k-2} \end{bmatrix} & \begin{bmatrix} b_{2k-3}^2 & \\ b_{2k-2}^1 & b_{2k-2}^2 \end{bmatrix} \\
 & & & & \begin{bmatrix} d_{2k-1} & b_{2k-1}^1 \\ & d_{2k} \end{bmatrix}
\end{bmatrix} \\
&= \begin{bmatrix}
D_1 & B_1 & & & \\
 & D_2 & B_2 & & \\
 & & \ddots & \ddots & \\
 & & & D_{k-1} & B_{k-1} \\
 & & & & D_k
\end{bmatrix} \in \mathbb{R}^{2k \times 2k}
\end{aligned}
$$

$$
X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{k-1} \\ Y_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \qquad Y^{\mathrm{T}} = \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{2 \times 2k}
$$

$$
\begin{aligned}
XY^{\mathrm{T}} &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix}
X_1 Y_1^{\mathrm{T}} & X_1 Y_2^{\mathrm{T}} & \cdots & X_1 Y_{k-1}^{\mathrm{T}} & x_1 Y_k^{\mathrm{T}} \\
X_2 Y_1^{\mathrm{T}} & X_2 Y_2^{\mathrm{T}} & \cdots & X_2 Y_{k-1}^{\mathrm{T}} & x_2 Y_k^{\mathrm{T}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
X_{k-1} Y_1^{\mathrm{T}} & X_{k-1} Y_2^{\mathrm{T}} & \cdots & X_{k-1} Y_{k-1}^{\mathrm{T}} & X_{k-1} Y_k^{\mathrm{T}} \\
X_k Y_1^{\mathrm{T}} & X_k Y_2^{\mathrm{T}} & \cdots & X_k Y_{k-1}^{\mathrm{T}} & X_k Y_k^{\mathrm{T}}
\end{bmatrix} \\
&= X \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix} XY_1^{\mathrm{T}} & XY_2^{\mathrm{T}} & \cdots & XY_{k-1}^{\mathrm{T}} & XY_k^{\mathrm{T}} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{triu}(X_k) &\coloneqq X_k \\
\operatorname{triu}(Y_k^{\mathrm{T}}) &\coloneqq Y_k^{\mathrm{T}} \\
\operatorname{triu}(X_k Y_k^{\mathrm{T}}) &= X_k Y_k^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{triu}(XY^{\mathrm{T}}) &= \begin{bmatrix}
\operatorname{triu}(X_1 Y_1^{\mathrm{T}}) & X_1 Y_2^{\mathrm{T}} & \cdots & X_1 Y_{k-1}^{\mathrm{T}} & X_1 Y_k^{\mathrm{T}} \\
0_{2\times 2} & \operatorname{triu}(X_2 Y_2^{\mathrm{T}}) & \cdots & X_2 Y_{k-1}^{\mathrm{T}} & X_2 Y_k^{\mathrm{T}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{2\times 2} & 0_{2\times 2} & \cdots & \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) & X_{k-1} Y_k^{\mathrm{T}} \\
0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & X_k Y_k^{\mathrm{T}}
\end{bmatrix} \\
&= \begin{bmatrix}
\begin{bmatrix} \operatorname{triu}(X_1 Y_1^{\mathrm{T}}) \\ 0_{2(k-1)\times 2} \end{bmatrix} &
\begin{bmatrix} X_1 Y_2^{\mathrm{T}} \\ \operatorname{triu}(X_2 Y_2^{\mathrm{T}}) \\ 0_{2(k-2)\times 2} \end{bmatrix} &
\cdots &
\begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} &
XY_k^{\mathrm{T}}
\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
U^{-1} &= \operatorname{triu}(XY^{\mathrm{T}}) \\
I &= U \operatorname{triu}(XY^{\mathrm{T}})
\end{aligned}
$$

$$
E_i = \begin{bmatrix} 0 & \cdots & 0 & I_{2\times 2} & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}
$$

k-th block column,

$$
\begin{aligned}
E_k &= U \operatorname{triu}(XY^{\mathrm{T}}) E_k \\
&= UXY_k^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{aligned}
X &\coloneqq U^{-1} E_k \\
E_k &= UXY_k^{\mathrm{T}} \\
&= UU^{-1} E_k Y_k^{\mathrm{T}} \\
&= E_k Y_k^{\mathrm{T}} \\
Y_k^{\mathrm{T}} &\coloneqq I_{2\times 2}
\end{aligned}
$$

$(k-1)$-th block column,

$$
\begin{aligned}
E_{k-1} &= U \operatorname{triu}(XY^{\mathrm{T}}) E_{k-1} \\
&= U \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix}
\end{aligned}
$$

$(k-1)$-th block row of $(k-1)$-th block column,

$$
E_i^{\mathrm{T}} E_j = \begin{cases} I_{2\times 2}, & \text{if } i = j \ ; \\ 0_{2\times 2}, & \text{otherwise.} \end{cases}
$$

$$
\begin{aligned}
E_{k-1}^{\mathrm{T}} E_{k-1} = I_{2\times 2} &= E_{k-1}^{\mathrm{T}} U \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} \\
&= \begin{bmatrix} 0_{2\times 2(k-2)} & D_{k-1} & B_{k-1} \end{bmatrix} \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} \\
&= D_{k-1} \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\
Y_{k-1}^{\mathrm{T}} &\coloneqq (D_{k-1} X_{k-1})^{-1}
\end{aligned}
$$

$$
\begin{aligned}
D_{k-1} &= \operatorname{triu}(D_{k-1}) \\
D_{k-1}^{-1} &= \operatorname{triu}(D_{k-1}^{-1}) \\
Y_{k-1}^{\mathrm{T}} &\coloneqq (D_{k-1} X_{k-1})^{-1} \\
D_{k-1} \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) &= D_{k-1} \operatorname{triu}(X_{k-1} (D_{k-1} X_{k-1})^{-1}) \\
&= D_{k-1} \operatorname{triu}(X_{k-1} X_{k-1}^{-1} D_{k-1}^{-1}) \\
&= D_{k-1} \operatorname{triu}(D_{k-1}^{-1}) \\
&= D_{k-1} D_{k-1}^{-1} \\
&= I_{2\times 2}
\end{aligned}
$$

$$
\begin{aligned}
X &\coloneqq U^{-1} E_k \\
E_k &= U X \\
E_k^{\mathrm{T}} E_k = I_{2\times 2} &= E_k^{\mathrm{T}} U X \\
&= \begin{bmatrix} 0_{2\times 2(k-1)} & D_k \end{bmatrix} X \\
&= D_k X_k \\
Y_k^{\mathrm{T}} &\coloneqq I_{2\times 2} \\
&\coloneqq (D_k X_k)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
X &\coloneqq U^{-1} E_k \\
Y_i^{\mathrm{T}} &\coloneqq (D_i X_i)^{-1}
\end{aligned}
$$

$n = 2k+1$

$$U = \begin{bmatrix} \begin{bmatrix} d_1 \end{bmatrix} & \begin{bmatrix} b_1^1 & b_1^2 \end{bmatrix} & & & & \\ & \begin{bmatrix} d_2 & b_2^1 \\ & d_3 \end{bmatrix} & \begin{bmatrix} b_2^2 & \\ b_3^1 & b_3^2 \end{bmatrix} & & & \\ & & \ddots & \ddots & & \\ & & & \begin{bmatrix} d_{2k-2} & b_{2k-2}^1 \\ & d_{2k-1} \end{bmatrix} & \begin{bmatrix} b_{2k-2}^2 & \\ b_{2k-1}^1 & b_{2k-1}^2 \end{bmatrix} \\ & & & & \begin{bmatrix} d_{2k} & b_{2k}^1 \\ & d_{2k+1} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(2k+1)\times(2k+1)}$$

$$X \coloneqq U^{-1} E_{k+1}$$
$$Y_i^{\mathrm{T}} \coloneqq (D_i X_i)^{-1} \text{ for } i = 2, \ \cdots, \ k+1$$
$$Y_1^{\mathrm{T}} \coloneqq (D_1 X_1)^{+} = (D_1 X_1)^{\mathrm{T}} (D_1 X_1 (D_1 X_1)^{\mathrm{T}})^{-1}$$

2.4 Banded matrix (Upper)

$n = ku + r,\ 0 \le r < u$

$$U = \begin{bmatrix} d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & & \\ & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & \\ & & \ddots & \ddots & \ddots & \cdots & \ddots & \\ & & & \ddots & \ddots & \ddots & \cdots & b_{n-u}^u \\ & & & & \ddots & \ddots & \ddots & \vdots \\ & & & & & \ddots & \ddots & b_{n-2}^2 \\ & & & & & & \ddots & b_{n-1}^1 \\ & & & & & & & d_n \end{bmatrix}$$

$r = 0,\ n = ku,$

$$U = \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{ku \times ku},$$

$$D_i \in \mathbb{R}^{u \times u},\ B_i \in \mathbb{R}^{u \times u} \text{ for } i = 1,\ 2,\ \cdots,\ k.$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} \in \mathbb{R}^{ku \times u} \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{bmatrix} \in \mathbb{R}^{ku \times u} \qquad Y^{\mathrm{T}} = \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_k^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{u \times ku}$$

$$\begin{aligned} X_i,\ Y_i^{\mathrm{T}} &\in \mathbb{R}^{u \times u},\ \text{for } i = 1,\ 2,\ \cdots,\ k \\ X &:= U^{-1} E_k \\ Y_i^{\mathrm{T}} &:= (D_i X_i)^{-1} \end{aligned}$$

$r \ne 0,\ n = ku + r,$

$$U = \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times (ku+r)},$$

$$\begin{aligned} &\tilde{D}_1 \in \mathbb{R}^{r \times r},\ \tilde{B}_1 \in \mathbb{R}^{r \times u}; \\ &D_i \in \mathbb{R}^{u \times u},\ B_i \in \mathbb{R}^{u \times u} \text{ for } i = 2,\ \cdots,\ k+1. \end{aligned}$$

$$X = \begin{bmatrix} \tilde{X}_1 \\ X_2 \\ \vdots \\ X_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times u} \quad Y = \begin{bmatrix} \tilde{Y}_1 \\ Y_2 \\ \vdots \\ Y_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times u} \quad Y^{\mathrm{T}} = \begin{bmatrix} \tilde{Y}_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k+1}^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{u \times (ku+r)}$$

$$
\begin{aligned}
\tilde{X}_1 &\in \mathbb{R}^{r\times u} \\
\tilde{Y}_1^{\mathrm{T}} &\in \mathbb{R}^{u\times r} \\
X_i,\ Y_i^{\mathrm{T}} &\in \mathbb{R}^{u\times u} \text{ for } i = 2,\ \cdots,\ k+1 \\
X &\coloneqq U^{-1}E_{k+1} \\
\tilde{Y}_1^{\mathrm{T}} &\coloneqq (D_1X_1)^{+} = (D_1X_1)^{\mathrm{T}}(D_1X_1(D_1X_1)^{\mathrm{T}})^{-1} \\
\mathrm{triu}(D_1X_1\tilde{Y}_1^{\mathrm{T}}) &= I_{r\times r} \\
Y_i^{\mathrm{T}} &\coloneqq (D_iX_i)^{-1} \text{ for } i = 2,\ \cdots,\ k+1
\end{aligned}
$$

3 Test

$$
\begin{array}{c}
\hphantom{d_1 \quad} \overbrace{\hphantom{b_1^1 \quad b_1^2 \quad \cdots \quad b_1^u}}^{\text{bandwidth} = u} \hphantom{\quad b_2^u \quad \cdots \quad b_{n-u}^u} \\
\begin{bmatrix}
d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & & \\
 & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & \\
 & & \ddots & \ddots & \ddots & \cdots & \ddots & \\
 & & & \ddots & \ddots & \ddots & \cdots & b_{n-u}^u \\
 & & & & \ddots & \ddots & \ddots & \vdots \\
 & & & & & \ddots & \ddots & b_{n-2}^2 \\
 & & & & & & \ddots & b_{n-1}^1 \\
 & & & & & & & d_n
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{cc}
\left(\begin{array}{ccc|ccc}
a & \cdots & a & b & \cdots & b \\
 & \ddots & \vdots & \vdots & \ddots & \\
 & & a & b & & \\
\hline
 & & & c & \cdots & c \\
 & 0 & & \vdots & & \vdots \\
 & & & & &
\end{array}\right)
&
\begin{array}{l}
\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} p \\
\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} q
\end{array}
\\
\underbrace{\hphantom{a \quad \cdots \quad a}}_{m} \; \underbrace{\hphantom{b \quad \cdots \quad b}}_{m} &
\end{array}
$$

$$
\begin{array}{cc}
 & \overset{n \text{ columns}}{\longleftrightarrow} \\
\updownarrow\,{\scriptstyle n \text{ rows}} &
\begin{pmatrix}
1 & 1 & 1 \ldots\ldots & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 \ldots\ldots & 1
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{cc}
\begin{pmatrix}
1 & & & & & & & \\
 & \ddots & & & & & & \\
 & & 1 & & & & & \\
 & & & 0 & & 1 & & \\
 & & & & 1 & & & \\
 & & & & & \ddots & & \\
 & & & & & & 1 & \\
 & & & 1 & & 0 & & \\
 & & & & & & 1 & \\
 & & & & & & & \ddots \\
 & & & & & & & 1
\end{pmatrix}
&
\begin{array}{l}
\\ \\ \\ \leftarrow i \\ \\ \\ \\ \leftarrow j \\ \\ \\ \\
\end{array}
\\
\begin{array}{cccccccc}
 & & & \uparrow & & \uparrow & & \\
 & & & i & & j & &
\end{array}
&
\end{array}
$$