Meeting Notes

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1 28.06.2022 Meeting draft

1.1 Notation

Vectors

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

Matrices and Columns

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{m\times n},\ a_i \in \mathbb{R}^m \end{aligned}$$

$$\begin{aligned} Ae_i &:= a_i \\ e_i &= i\text{-th column of } I \end{aligned}$$

Block Matrices and Block Columns

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1l} \\ A_{21} & A_{22} & \cdots & A_{1l} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kl} \end{bmatrix},\ A_{ij} \in \mathbb{R}^{m\times n} \\ &= \begin{bmatrix} A_1 & A_2 & \cdots & A_l \end{bmatrix} \in \mathbb{R}^{km\times ln},\ A_i \in \mathbb{R}^{km\times n} \end{aligned}$$

$$\begin{aligned} AE_i &\coloneqq A_i \\ E_i &= i\text{-th block column of } I \end{aligned}$$

1.2 Bidiagonal matrix (upper)

$$U = \begin{bmatrix} d_1 & b_1 & & & \\ & d_2 & b_2 & & \\ & & \ddots & \ddots & \\ & & & d_{n-1} & b_{n-1} \\ & & & & d_n \end{bmatrix} \in \mathbb{R}^{n\times n}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \in \mathbb{R}^{n\times 1} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} \in \mathbb{R}^{n\times 1} \qquad y^{\mathrm{T}} = \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \in \mathbb{R}^{1\times n}$$

$$\begin{aligned} xy^{\mathrm{T}} &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \\ &= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_{n-1} & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_{n-1} & x_2y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1}y_1 & x_{n-1}y_2 & \cdots & x_{n-1}y_{n-1} & x_{n-1}y_n \\ x_ny_1 & x_ny_2 & \cdots & x_ny_{n-1} & x_ny_n \end{bmatrix} \\ &= x\begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \\ &= \begin{bmatrix} xy_1 & xy_2 & \cdots & xy_{n-1} & xy_n \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \operatorname{triu}(xy^{\mathrm{T}}) &= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_{n-1} & x_1y_n \\ 0 & x_2y_2 & \cdots & x_2y_{n-1} & x_2y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_4y_{n-1} & x_4y_n \\ 0 & 0 & \cdots & 0 & x_5y_n \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} x_{1:1} \\ 0_{n-1} \end{bmatrix} y_1 & \begin{bmatrix} x_{1:2} \\ 0_{n-2} \end{bmatrix} y_2 & \cdots & \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} & xy_n \end{bmatrix} \end{aligned}$$

$$\begin{aligned} U^{-1} &= \operatorname{triu}(xy^{\mathrm{T}}) \\ I &= U \operatorname{triu}(xy^{\mathrm{T}}) \end{aligned}$$

n-th column,

$$\begin{aligned} e_n &= U \operatorname{triu}(xy^{\mathrm{T}}) e_n \\ &= U x y_n \end{aligned}$$

$$
\begin{aligned}
x &:= U^{-1} e_n \\
e_n &= U x y_n \\
&= U U^{-1} e_n y_n \\
&= e_n y_n \\
y_n &:= 1
\end{aligned}
$$

$(n-1)$-th column,

$$
\begin{aligned}
e_{n-1} &= U \operatorname{triu}(x y^{\mathrm{T}}) e_{n-1} \\
&= U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1}
\end{aligned}
$$

$(n-1)$-th row of $(n-1)$-th column,

$$
e_i^{\mathrm{T}} e_j = \delta_{ij}
$$

$$
\begin{aligned}
e_{n-1}^{\mathrm{T}} e_{n-1} = 1 &= e_{n-1}^{\mathrm{T}} U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \\
&= \begin{bmatrix} 0_{n-2} & d_{n-1} & b_{n-1} \end{bmatrix} \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \\
&= d_{n-1} x_{n-1} y_{n-1} \\
y_{n-1} &:= (d_{n-1} x_{n-1})^{-1}
\end{aligned}
$$

$$
\begin{aligned}
e_n &= U x \\
e_n^{\mathrm{T}} e_n = 1 &= e_n^{\mathrm{T}} U x \\
&= \begin{bmatrix} 0_{n-1} & d_n \end{bmatrix} x \\
&= d_n x_n \\
y_n &:= 1 \\
&:= (d_n x_n)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
x &:= U^{-1} e_n \\
y_i &:= (d_i x_i)^{-1}
\end{aligned}
$$

1.3 Tridiagonal matrix (Upper)

$n = 2k$

$$
\begin{aligned}
U &= \begin{bmatrix}
d_1 & b_1^1 & b_1^2 & & & & & & \\
& d_2 & b_2^1 & b_2^2 & & & & & \\
& & d_3 & b_3^1 & b_3^2 & & & & \\
& & & d_4 & b_4^1 & b_4^2 & & & \\
& & & & \ddots & \ddots & \ddots & & \\
& & & & & \ddots & \ddots & \ddots & \\
& & & & & & d_{n-3} & b_{n-3}^1 & b_{n-3}^2 & \\
& & & & & & & d_{n-2} & b_{n-2}^1 & b_{n-2}^2 \\
& & & & & & & & d_{n-1} & b_{n-1}^1 \\
& & & & & & & & & d_n
\end{bmatrix} \\
&= \begin{bmatrix}
\begin{bmatrix} d_1 & b_1^1 \\ & d_2 \end{bmatrix} & \begin{bmatrix} b_1^2 & \\ b_2^1 & b_2^2 \end{bmatrix} & & & \\
& \begin{bmatrix} d_3 & b_3^1 \\ & d_4 \end{bmatrix} & \begin{bmatrix} b_3^2 & \\ b_4^1 & b_4^2 \end{bmatrix} & & \\
& & \ddots & \ddots & \\
& & & \begin{bmatrix} d_{2k-3} & b_{2k-3}^1 \\ & d_{2k-2} \end{bmatrix} & \begin{bmatrix} b_{2k-3}^2 & \\ b_{2k-2}^1 & b_{2k-2}^2 \end{bmatrix} \\
& & & & \begin{bmatrix} d_{2k-1} & b_{2k-1}^1 \\ & d_{2k} \end{bmatrix}
\end{bmatrix} \\
&= \begin{bmatrix}
D_1 & B_1 & & & \\
& D_2 & B_2 & & \\
& & \ddots & \ddots & \\
& & & D_{k-1} & B_{k-1} \\
& & & & D_k
\end{bmatrix} \in \mathbb{R}^{2k\times 2k}
\end{aligned}
$$

$$
X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \in \mathbb{R}^{2k\times 2} \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{k-1} \\ Y_k \end{bmatrix} \in \mathbb{R}^{2k\times 2} \qquad Y^{\mathrm{T}} = \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{2\times 2k}
$$

$$
\begin{aligned}
XY^{\mathrm{T}} &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix}
X_1Y_1^{\mathrm{T}} & X_1Y_2^{\mathrm{T}} & \cdots & X_1Y_{k-1}^{\mathrm{T}} & x_1Y_k^{\mathrm{T}} \\
X_2Y_1^{\mathrm{T}} & X_2Y_2^{\mathrm{T}} & \cdots & X_2Y_{k-1}^{\mathrm{T}} & x_2Y_k^{\mathrm{T}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
X_{k-1}Y_1^{\mathrm{T}} & X_{k-1}Y_2^{\mathrm{T}} & \cdots & X_{k-1}Y_{k-1}^{\mathrm{T}} & X_{k-1}Y_k^{\mathrm{T}} \\
X_kY_1^{\mathrm{T}} & X_kY_2^{\mathrm{T}} & \cdots & X_kY_{k-1}^{\mathrm{T}} & X_kY_k^{\mathrm{T}}
\end{bmatrix} \\
&= X\begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix} XY_1^{\mathrm{T}} & XY_2^{\mathrm{T}} & \cdots & XY_{k-1}^{\mathrm{T}} & XY_k^{\mathrm{T}} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{triu}(X_k) &\coloneqq X_k \\
\operatorname{triu}(Y_k^{\mathrm{T}}) &\coloneqq Y_k^{\mathrm{T}} \\
\operatorname{triu}(X_k Y_k^{\mathrm{T}}) &= X_k Y_k^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{triu}(XY^{\mathrm{T}}) &= \begin{bmatrix}
\operatorname{triu}(X_1Y_1^{\mathrm{T}}) & X_1Y_2^{\mathrm{T}} & \cdots & X_1Y_{k-1}^{\mathrm{T}} & X_1Y_k^{\mathrm{T}} \\
0_{2\times 2} & \operatorname{triu}(X_2Y_2^{\mathrm{T}}) & \cdots & X_2Y_{k-1}^{\mathrm{T}} & X_2Y_k^{\mathrm{T}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{2\times 2} & 0_{2\times 2} & \cdots & \operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) & X_{k-1}Y_k^{\mathrm{T}} \\
0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & X_kY_k^{\mathrm{T}}
\end{bmatrix} \\
&= \begin{bmatrix}
\begin{bmatrix} \operatorname{triu}(X_1Y_1^{\mathrm{T}}) \\ 0_{2(k-1)\times 2} \end{bmatrix} &
\begin{bmatrix} X_1Y_2^{\mathrm{T}} \\ \operatorname{triu}(X_2Y_2^{\mathrm{T}}) \\ 0_{2(k-2)\times 2} \end{bmatrix} &
\cdots &
\begin{bmatrix} X_{1:k-2}Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} &
XY_k^{\mathrm{T}}
\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
U^{-1} &= \operatorname{triu}(XY^{\mathrm{T}}) \\
I &= U \operatorname{triu}(XY^{\mathrm{T}})
\end{aligned}
$$

$$
E_i = \begin{bmatrix} 0 & \cdots & 0 & I_{2\times 2} & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}
$$

k-th block column,

$$
\begin{aligned}
E_k &= U \operatorname{triu}(XY^{\mathrm{T}}) E_k \\
&= UXY_k^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{aligned}
X &\coloneqq U^{-1}E_k \\
E_k &= UXY_k^{\mathrm{T}} \\
&= UU^{-1}E_kY_k^{\mathrm{T}} \\
&= E_kY_k^{\mathrm{T}} \\
Y_k^{\mathrm{T}} &\coloneqq I_{2\times 2}
\end{aligned}
$$

$(k-1)$-th block column,

$$
\begin{aligned}
E_{k-1} &= U \operatorname{triu}(XY^{\mathrm{T}}) E_{k-1} \\
&= U \begin{bmatrix} X_{1:k-2}Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix}
\end{aligned}
$$

$(k-1)$-th block row of $(k-1)$-th block column,

$$
E_i^{\mathrm{T}}E_j = \begin{cases} I_{2\times 2}, & \text{if } i = j \ ; \\ 0_{2\times 2}, & \text{otherwise.} \end{cases}
$$

$$
\begin{aligned}
E_{k-1}^{\mathrm{T}} E_{k-1} = I_{2\times 2} &= E_{k-1}^{\mathrm{T}} U \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} \\
&= \begin{bmatrix} 0_{2\times 2(k-2)} & D_{k-1} & B_{k-1} \end{bmatrix} \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} \\
&= D_{k-1} \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\
Y_{k-1}^{\mathrm{T}} &\coloneqq (D_{k-1} X_{k-1})^{-1}
\end{aligned}
$$

$$
\begin{aligned}
D_{k-1} &= \operatorname{triu}(D_{k-1}) \\
D_{k-1}^{-1} &= \operatorname{triu}(D_{k-1}^{-1}) \\
Y_{k-1}^{\mathrm{T}} &\coloneqq (D_{k-1} X_{k-1})^{-1} \\
D_{k-1} \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) &= D_{k-1} \operatorname{triu}(X_{k-1} (D_{k-1} X_{k-1})^{-1}) \\
&= D_{k-1} \operatorname{triu}(X_{k-1} X_{k-1}^{-1} D_{k-1}^{-1}) \\
&= D_{k-1} \operatorname{triu}(D_{k-1}^{-1}) \\
&= D_{k-1} D_{k-1}^{-1} \\
&= I_{2\times 2}
\end{aligned}
$$

$$
\begin{aligned}
X &\coloneqq U^{-1} E_k \\
E_k &= U X \\
E_k^{\mathrm{T}} E_k = I_{2\times 2} &= E_k^{\mathrm{T}} U X \\
&= \begin{bmatrix} 0_{2\times 2(k-1)} & D_k \end{bmatrix} X \\
&= D_k X_k \\
Y_k^{\mathrm{T}} &\coloneqq I_{2\times 2} \\
&\coloneqq (D_k X_k)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
X &\coloneqq U^{-1} E_k \\
Y_i^{\mathrm{T}} &\coloneqq (D_i X_i)^{-1}
\end{aligned}
$$

$n = 2k+1$

$$U = \begin{bmatrix} \begin{bmatrix} d_1 \end{bmatrix} & \begin{bmatrix} b_1^1 & b_1^2 \end{bmatrix} & & & & \\ & \begin{bmatrix} d_2 & b_2^1 \\ & d_3 \end{bmatrix} & \begin{bmatrix} b_2^2 & \\ b_3^1 & b_3^2 \end{bmatrix} & & & \\ & & \ddots & \ddots & & \\ & & & \begin{bmatrix} d_{2k-2} & b_{2k-2}^1 \\ & d_{2k-1} \end{bmatrix} & \begin{bmatrix} b_{2k-2}^2 & \\ b_{2k-1}^1 & b_{2k-1}^2 \end{bmatrix} \\ & & & & \begin{bmatrix} d_{2k} & b_{2k}^1 \\ & d_{2k+1} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(2k+1)\times(2k+1)}$$

$$\begin{aligned} X &\coloneqq U^{-1} E_{k+1} \\ Y_i^{\mathrm{T}} &\coloneqq (D_i X_i)^{-1} \text{ for } i = 2, \ \cdots, \ k+1 \\ Y_1^{\mathrm{T}} &\coloneqq (D_1 X_1)^{+} = (D_1 X_1)^{\mathrm{T}} (D_1 X_1 (D_1 X_1)^{\mathrm{T}})^{-1} \end{aligned}$$

1.4 Banded matrix (Upper)

$n = ku + r,\ 0 \le r < u$

$$U = \begin{bmatrix} d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & & \\ & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & \\ & & \ddots & \ddots & \ddots & \cdots & \ddots & \\ & & & \ddots & \ddots & \ddots & \cdots & b_{n-u}^u \\ & & & & \ddots & \ddots & \ddots & \vdots \\ & & & & & \ddots & \ddots & b_{n-2}^2 \\ & & & & & & \ddots & b_{n-1}^1 \\ & & & & & & & d_n \end{bmatrix}$$

$r = 0,\ n = ku,$

$$U = \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{ku \times ku},$$

$$D_i \in \mathbb{R}^{u \times u},\ B_i \in \mathbb{R}^{u \times u} \text{ for } i = 1,\ 2,\ \cdots,\ k.$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} \in \mathbb{R}^{ku \times u} \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{bmatrix} \in \mathbb{R}^{ku \times u} \qquad Y^{\mathrm{T}} = \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_k^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{u \times ku}$$

$$\begin{aligned} X_i,\ Y_i^{\mathrm{T}} &\in \mathbb{R}^{u \times u},\ \text{for } i = 1,\ 2,\ \cdots,\ k \\ X &:= U^{-1} E_k \\ Y_i^{\mathrm{T}} &:= (D_i X_i)^{-1} \end{aligned}$$

$r \neq 0,\ n = ku + r,$

$$U = \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times (ku+r)},$$

$$\begin{aligned} &\tilde{D}_1 \in \mathbb{R}^{r \times r},\ \tilde{B}_1 \in \mathbb{R}^{r \times u}; \\ &D_i \in \mathbb{R}^{u \times u},\ B_i \in \mathbb{R}^{u \times u} \text{ for } i = 2,\ \cdots,\ k+1. \end{aligned}$$

$$X = \begin{bmatrix} \tilde{X}_1 \\ X_2 \\ \vdots \\ X_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times u} \quad Y = \begin{bmatrix} \tilde{Y}_1 \\ Y_2 \\ \vdots \\ Y_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times u} \quad Y^{\mathrm{T}} = \begin{bmatrix} \tilde{Y}_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k+1}^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{u \times (ku+r)}$$

$$
\begin{aligned}
\tilde{X}_1 &\in \mathbb{R}^{r\times u} \\
\tilde{Y}_1^{\mathrm{T}} &\in \mathbb{R}^{u\times r} \\
X_i,\ Y_i^{\mathrm{T}} &\in \mathbb{R}^{u\times u} \text{ for } i = 2,\ \cdots,\ k+1 \\
X &\coloneqq U^{-1}E_{k+1} \\
\tilde{Y}_1^{\mathrm{T}} &\coloneqq (D_1X_1)^{+} = (D_1X_1)^{\mathrm{T}}(D_1X_1(D_1X_1)^{\mathrm{T}})^{-1} \\
\mathrm{triu}(D_1X_1\tilde{Y}_1^{\mathrm{T}}) &= I_{r\times r} \\
Y_i^{\mathrm{T}} &\coloneqq (D_iX_i)^{-1} \text{ for } i = 2,\ \cdots,\ k+1
\end{aligned}
$$

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Consider an upper banded matrix $U \in \mathbb{R}^{n\times n}$,

$$
U = \begin{bmatrix}
d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & & \\
 & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & \\
 & & \ddots & \ddots & \ddots & & \ddots & \\
 & & & & & & & b_{n-u}^u \\
 & & & & & & & \vdots \\
 & & & & & & \ddots & b_{n-2}^2 \\
 & & & & & & \ddots & b_{n-1}^1 \\
 & & & & & & & d_n
\end{bmatrix}. \tag{1}
$$

Then try to show that its inverse can be repersent as the upper triangular part of an outer product, *i.e.* ,

$$
U^{-1} = \operatorname{triu}(XY^{\mathrm{T}}) \tag{2}
$$

Let $X,\ Y \in \mathbb{R}^{n\times u}$, consider the following cases of n,

$n = ku$

then (1) can be repersent as a block upper bidiagonal matrix,

$$
\begin{array}{cc}
 & \begin{array}{cccc} \overleftrightarrow{\;u\;} & \overleftrightarrow{\;u\;} & \cdots\cdots & \overleftrightarrow{\;u\;} \end{array} \\
\begin{array}{c} u \\ u \\ \vdots \\ u \end{array} &
U = \begin{bmatrix}
\begin{matrix} d_1 & \cdots & b_1^{u-1} \\ & \ddots & \vdots \\ & & d_u \end{matrix} & \begin{matrix} b_1^u & & \\ \vdots & \ddots & \\ b_u^1 & \cdots & b_u^u \end{matrix} & & \\
 & \begin{matrix} d_{u+1} & \cdots & b_{u+1}^{u-1} \\ & \ddots & \vdots \\ & & d_{2u} \end{matrix} & \begin{matrix} b_{u+1}^u & & \\ \vdots & \ddots & \\ b_{2u}^1 & \cdots & b_{2u}^u \end{matrix} & \\
 & & \ddots & \ddots \\
 & & & \begin{matrix} d_{(k-1)u} & \cdots & b_{(k-1)u}^u \\ & \ddots & \vdots \\ & & d_{ku} \end{matrix}
\end{bmatrix}
\end{array}
$$

$$
= \begin{bmatrix}
D_1 & B_1 & & & \\
 & D_2 & B_2 & & \\
 & & \ddots & \ddots & \\
 & & & D_{k-1} & B_{k-1} \\
 & & & & D_k
\end{bmatrix} \tag{3}
$$

3 Test

$$
\begin{bmatrix}
d_1 & \overbrace{b_1^1 & b_1^2 & \cdots & b_1^u}^{\text{bandwidth} = u} & & & \\
& d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & \\
& & \ddots & \ddots & \ddots & \cdots & \ddots & \\
& & & \ddots & \ddots & \ddots & \cdots & b_{n-u}^u \\
& & & & \ddots & \ddots & \ddots & \vdots \\
& & & & & \ddots & \ddots & b_{n-2}^2 \\
& & & & & & \ddots & b_{n-1}^1 \\
& & & & & & & d_n
\end{bmatrix}
$$

$$
\begin{array}{cc}
\left(\begin{array}{ccc|ccc}
a & \cdots & a & b & \cdots & b \\
& \ddots & \vdots & \vdots & \ddots & \\
& & a & b & & \\
\hline
& & & c & \cdots & c \\
& \text{\Huge 0} & & \vdots & & \vdots \\
& & & & &
\end{array}\right)
&
\begin{array}{l}
\left.\vphantom{\begin{array}{c}a\\ \vdots\\ a\end{array}}\right\} p \\
\left.\vphantom{\begin{array}{c}c\\ \vdots\\ c\end{array}}\right\} q
\end{array}
\\
\underbrace{\hphantom{a \cdots a}}_{m}\ \underbrace{\hphantom{b \cdots b}}_{m} &
\end{array}
$$

$$
\begin{array}{cc}
& \overset{n \text{ columns}}{\longleftrightarrow} \\
\updownarrow{\scriptstyle n \text{ rows}} &
\begin{pmatrix}
1 & 1 & 1 \ldots\ldots & 1 \\
1 & 1 & 1 & 1 \\
& & & \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 \ldots\ldots & 1
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{cc}
\begin{pmatrix}
1 & & & & & & & \\
& \ddots & & & & & & \\
& & 1 & & & & & \\
& & & 0 & & 1 & & \\
& & & & 1 & & & \\
& & & & & \ddots & & \\
& & & & & 1 & & \\
& & & 1 & & 0 & & \\
& & & & & & 1 & \\
& & & & & & & \ddots \\
& & & & & & & 1
\end{pmatrix}
&
\begin{array}{l}
\\ \\ \\ \leftarrow i \\ \\ \\ \\ \leftarrow j \\ \\ \\ \\
\end{array}
\\
\begin{array}{cc}
\uparrow & \uparrow \\
i & j
\end{array} &
\end{array}
$$

$$
\begin{array}{cc}
& \overset{n \text{ columns}}{\longleftrightarrow} \\
\updownarrow{\scriptstyle n \text{ rows}} &
\begin{pmatrix}
1 & 1 & 1 \ldots\ldots & 1 \\
1 & 1 & 1 & 1 \\
& & & \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 \ldots\ldots & 1
\end{pmatrix}
\end{array}
$$

$$
\left(\begin{array}{cc:cc:cc}
1 & 2 & 0 & 0 & 0 & 0 \\
4 & 5 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 7 & 1 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 3 & 4 \\
0 & 0 & 0 & 0 & 1 & 4
\end{array}\right)
$$

$$
\begin{bmatrix}
d_1 & b_1^1 & b_1^2 & & & & & & & \\
 & d_2 & b_2^1 & b_2^2 & & & & & & \\
 & & d_3 & b_3^1 & b_3^2 & & & & & \\
 & & & d_4 & b_4^1 & b_4^2 & & & & \\
 & & & & \ddots & & \ddots & & & \\
 & & & & & \ddots & & \ddots & & \\
 & & & & & & d_{2k-3} & b_{2k-3}^1 & b_{2k-3}^2 & \\
 & & & & & & & d_{2k-2} & b_{2k-2}^1 & b_{2k-2}^2 \\
 & & & & & & & & d_{2k-1} & b_{2k-1}^1 \\
 & & & & & & & & & d_{2k}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\begin{bmatrix} d_1 & b_1^1 \\ & d_2 \end{bmatrix} & \begin{bmatrix} b_1^2 & \\ b_2^1 & b_2^2 \end{bmatrix} & & & \\
 & \begin{bmatrix} d_3 & b_3^1 \\ & d_4 \end{bmatrix} & \begin{bmatrix} b_3^2 & \\ b_4^1 & b_4^2 \end{bmatrix} & & \\
 & & \ddots & \ddots & \\
 & & & \begin{bmatrix} d_{2k-3} & b_{2k-3}^1 \\ & d_{2k-2} \end{bmatrix} & \begin{bmatrix} b_{2k-3}^2 & \\ b_{2k-2}^1 & b_{2k-2}^2 \end{bmatrix} \\
 & & & & \begin{bmatrix} d_{2k-1} & b_{2k-1}^1 \\ & d_{2k} \end{bmatrix}
\end{bmatrix}
$$

$$
\begin{bmatrix}
\begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\
\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 2 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 3 & 4 \\ 1 & 4 \end{matrix}
\end{bmatrix}
$$

$$
\begin{bmatrix}
d_1 & b_1^1 & \cdots & b_1^{r-2} & b_1^{r-1} \\
 & d_2 & & & b_2^{r-2} \\
 & & \ddots & & \vdots \\
 & & & d_{r-1} & b_{r-1}^1 \\
 & & & & d_r
\end{bmatrix}
\begin{bmatrix}
b_1^r & \cdots & b_1^u & \\
b_2^{r-1} & b_2^r & b_2^u & \\
\vdots & & & \ddots \\
b_{r-1}^2 & & & \\
b_r^1 & b_r^2 & \cdots & b_r^u
\end{bmatrix}
$$

$$
\begin{bmatrix}
d_1 & b_1^1 & \cdots & b_1^{r-2} & b_1^{r-1} \\
0 & d_2 & & & b_2^{r-2} \\
\vdots & \ddots & \ddots & & \vdots \\
 & & & d_{r-1} & b_{r-1}^1 \\
0 & \cdots & & 0 & d_r
\end{bmatrix}
$$

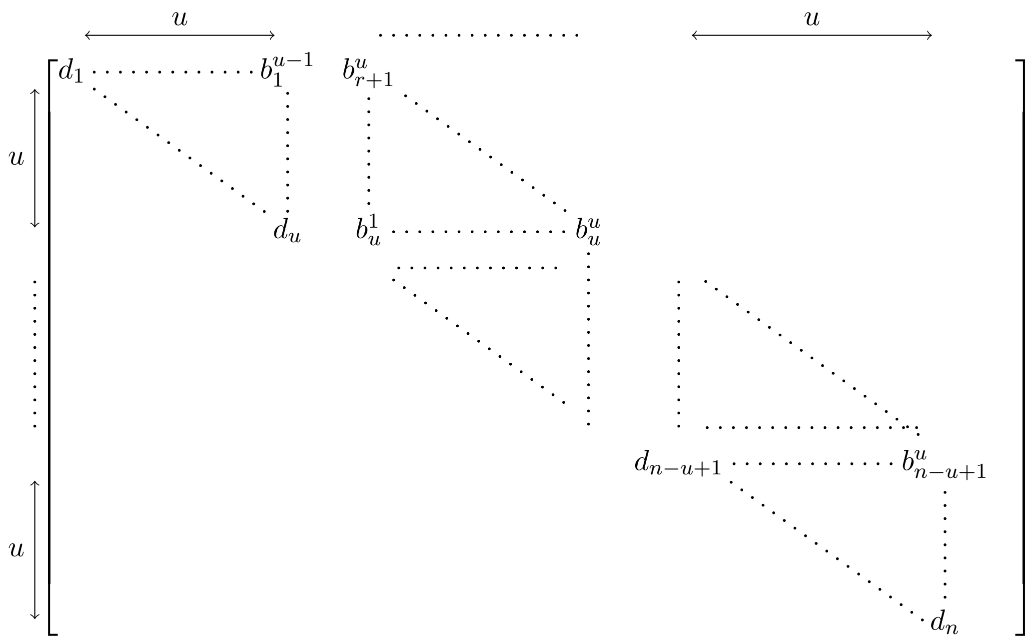
$$
\begin{bmatrix}
d_1 & b_1^1 & \cdots & b_1^{r-2} & b_1^{r-1} \\
 & d_2 & & & b_2^{r-2} \\
 & & \ddots & & \vdots \\
 & & & d_{r-1} & b_{r-1}^1 \\
 & & & & d_r
\end{bmatrix}
\begin{bmatrix}
b_1^r & b_1^{r+1} & \cdots & b_1^{u-1} & b_1^u & & & \\
b_2^{r-1} & & & & & b_2^u & & \\
\vdots & & & & & & \ddots & \\
b_{r-1}^2 & & & & & & & b_{r-1}^u \\
b_r^1 & b_r^2 & \cdots & & & & b_r^{u-1} & b_r^u
\end{bmatrix}
$$

$$
\begin{bmatrix}
d_1 & \cdots & b_1^{r-1} \\
 & \ddots & \vdots \\
 & & d_r
\end{bmatrix}
\begin{bmatrix}
b_1^r & \cdots & b_1^u & \\
\vdots & & & \ddots \\
b_r^1 & \cdots & \cdots & b_r^u
\end{bmatrix}
$$

$$
\begin{bmatrix}
b_1^r & \cdots & b_1^u & \\
\vdots & & & \ddots \\
b_r^1 & \cdots & \cdots & b_r^u
\end{bmatrix}
\qquad\qquad
\begin{bmatrix}
b_1^r & \cdots & b_1^u & \\
\vdots & & & \ddots \\
b_r^1 & \cdots & \cdots & b_r^u
\end{bmatrix}
$$

$$
\begin{bmatrix}
d_1 & \cdots & b_1^{r-1} & b_1^r & \cdots & b_1^u & \\
 & \ddots & \vdots & \vdots & & & \ddots \\
 & & d_r & b_r^1 & \cdots & \cdots & b_r^u \\
 & & & d_{r+1} & \cdots & \cdots & b_{r+1}^{u-1} \\
 & & & & \ddots & & \vdots \\
 & & & & & & d_{r+u}
\end{bmatrix}
$$

$$
\begin{array}{c c}
 & \begin{array}{cccc} \overset{r}{\longleftrightarrow} & \overset{u}{\longleftrightarrow} & \cdots\cdots\cdots & \overset{u}{\longleftrightarrow} \end{array} \\
\begin{array}{c} r \\ u \\ \vdots \\ u \end{array} &
\begin{bmatrix}
d_1 & \cdots & b_1^{r-1} & b_1^r & \cdots & b_1^u & & & & & & \\
 & \ddots & \vdots & \vdots & & & \ddots & & & & & \\
 & & d_r & b_r^1 & \cdots & \cdots & b_r^u & & & & & \\
 & & & d_{1+r} & \cdots & \cdots & b_{1+r}^{u-1} & b_{r+1}^u & & & & \\
 & & & & \ddots & & \vdots & \vdots & \ddots & & & \\
 & & & & & & d_{u+r} & b_{u+r}^1 & \cdots & b_{u+r}^u & & \\
 & & & & & & & & \ddots & \vdots & \ddots & \\
 & & & & & & & & & d_{n-u+1} & \cdots & b_{n-u+1}^u \\
 & & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & & & d_n
\end{bmatrix}
\end{array}
$$



$$\begin{array}{c|c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}$$

$$\begin{array}{ccccccccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20
\end{array}$$