Thesis Contents

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August 19, 2022

12 Semiseparable

Define matrices $E_u(p)$ and $E_l(q)$ as following,

$$E_u(p) = \begin{cases} 1 & \text{if } \ i \leqslant j - p \ ; \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad E_l(q) = \begin{cases} 1 & \text{if } \ i \geqslant j - q \ ; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

.

i.e.

$$E_u(-1) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & 1 \\ & & 1 & 1 \end{bmatrix}, \quad E_u(0) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix}, \quad E_u(1) = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \tag{2}$$

$$E_l(-1) = \begin{bmatrix} 1 & 1 & & \\ 1 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 1 \end{bmatrix}, \quad E_u(0) = \begin{bmatrix} 1 & & & \\ 1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ 1 & \cdots & 1 & 1 \end{bmatrix}, \quad E_u(1) = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ 1 & \cdots & 1 & 0 \end{bmatrix}, \tag{3}$$

where $|p| \leqslant n - 1$ and $|q| \leqslant n - 1$.

Define the operations triu and tril,

$$\operatorname{triu}(A, p) = A \circ E_u(p) \tag{4}$$

$$\operatorname{tril}(A, q) = A \circ E_l(q) \tag{5}$$

i.e. $\operatorname{triu}(A, p)$ retains the entries of A above the p-th super-diagonal (inclusive) and makes the entries of A below the p-th super-diagonal (exclusive) 0.

define symmetric {1,1}-semiseparable matrix, S

$$S = \begin{cases} x_i y_j & \text{if } i \leqslant j \ ; \\ x_j y_i & \text{if } i \geqslant j. \end{cases} \tag{6}$$

$$= \operatorname{triu}\left(xy^{\mathrm{T}}\right) + \operatorname{tril}\left(yx^{\mathrm{T}}, -1\right) \tag{7}$$

$$= \operatorname{triu}\left(xy^{\mathrm{T}}, 1\right) + \operatorname{tril}\left(yx^{\mathrm{T}}\right) \tag{8}$$

$$= \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_1 y_2 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1 y_n & x_2 y_n & \cdots & x_n y_n \end{bmatrix} \tag{9}$$

13 Algorithm

Backward Subsitution

$$Ux = b \tag{10}$$

$$\begin{bmatrix} d_1 & u_1^1 & u_1^2 & \cdots & \cdots & u_1^{n-1} \\ & d_2 & u_2^1 & u_2^2 & \cdots & u_1^{n-2} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & d_{n-2} & u_{n-2}^1 & u_{n-2}^2 \\ & & & & d_{n-1} & u_{n-1}^1 \\ & & & & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-2} \\ b_{n-1} \\ b_n \end{bmatrix} \tag{11}$$

$$\begin{gathered} d_n x_n = b_n \longrightarrow x_n = b_n / d_n \\ d_{n-1} x_{n-1} + u_{n-1}^1 x_n = b_{n-1} \longrightarrow x_{n-1} = \left(b_{n-1} - u_{n-1}^1 x_n \right) / d_{n-1} \\ d_{n-2} x_{n-2} + u_{n-2}' x_{n-1} + u_{n-2}^2 x_n = b_{n-2} \rightarrow x_{n-2} = \left(b_{n-2} - u_{12}^1 x_{n-1} - u_{n-2}^2 x_n \right) / d_{n-2} \\ x_{n-k} = \left(b_{n-k} - u_{n-k}^1 x_{n-k+1} - u_{n-k}^2 x_{n-k+2} \right. \\ i = n-1, n-2, \cdots - 1 \quad \left. \cdots - u_{n-k}^k u_{n-k+k} \right) \mid d_{n-k} \\ x_i = \left(b_i - \sum_{k=1}^{n-i} u_i^k x_{i+k} \right) (d_i \\ x_n = b_n / u_{nn} \\ x_i = \left(b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right) / u_{ii} \end{gathered}$$

14 Apply on symmetric positive definite tridiagonal matrices

$$\longleftarrow \text{ cholesky } \mathcal{O}\left(\frac{1}{3} n^3\right)$$

$$\begin{aligned} T &= U^{\mathrm{T}} U \\ U^{-1} &= \operatorname{triu}\left(x y^{\mathrm{T}}\right) \\ \left(U^{\mathrm{T}}\right)^{-1} &= \left(U^{-1}\right)^{\mathrm{T}} \\ &= \left(\operatorname{triu}\left(x y^{\mathrm{T}}\right)\right)^{\mathrm{T}} \\ &= \operatorname{tril}\left(y x^{\mathrm{T}}\right) \\ T^{-1} &= \left(U^{\mathrm{T}} U\right)^{-1} \\ &= \left(U^{-1}\right)\left(U^{\mathrm{T}}\right)^{-1} \\ &= \operatorname{triu}\left(x y^{\mathrm{T}}\right) \operatorname{tril}\left(y x^{\mathrm{T}}\right) \end{aligned}$$

15 28.06.2022 Meeting draft

15.1 Tridiagonal matrix (Upper)

$n = 2k$

$$
\begin{aligned}
U &= \begin{bmatrix}
d_1 & b_1^1 & b_1^2 & & & & & & & \\
 & d_2 & b_2^1 & b_2^2 & & & & & & \\
 & & d_3 & b_3^1 & b_3^2 & & & & & \\
 & & & d_4 & b_4^1 & b_4^2 & & & & \\
 & & & & \ddots & \ddots & \ddots & & & \\
 & & & & & \ddots & \ddots & \ddots & & \\
 & & & & & & d_{n-3} & b_{n-3}^1 & b_{n-3}^2 & \\
 & & & & & & & d_{n-2} & b_{n-2}^1 & b_{n-2}^2 \\
 & & & & & & & & d_{n-1} & b_{n-1}^1 \\
 & & & & & & & & & d_n
\end{bmatrix} \\
&= \begin{bmatrix}
\begin{bmatrix} d_1 & b_1^1 \\ & d_2 \end{bmatrix} & \begin{bmatrix} b_1^2 & \\ b_2^1 & b_2^2 \end{bmatrix} & & & \\
 & \begin{bmatrix} d_3 & b_3^1 \\ & d_4 \end{bmatrix} & \begin{bmatrix} b_3^2 & \\ b_4^1 & b_4^2 \end{bmatrix} & & \\
 & & \ddots & \ddots & \\
 & & & \begin{bmatrix} d_{2k-3} & b_{2k-3}^1 \\ & d_{2k-2} \end{bmatrix} & \begin{bmatrix} b_{2k-3}^2 & \\ b_{2k-2}^1 & b_{2k-2}^2 \end{bmatrix} \\
 & & & & \begin{bmatrix} d_{2k-1} & b_{2k-1}^1 \\ & d_{2k} \end{bmatrix}
\end{bmatrix} \\
&= \begin{bmatrix}
D_1 & B_1 & & & \\
 & D_2 & B_2 & & \\
 & & \ddots & \ddots & \\
 & & & D_{k-1} & B_{k-1} \\
 & & & & D_k
\end{bmatrix} \in \mathbb{R}^{2k\times 2k}
\end{aligned}
$$

$$
X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \in \mathbb{R}^{2k\times 2} \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{k-1} \\ Y_k \end{bmatrix} \in \mathbb{R}^{2k\times 2} \qquad Y^{\mathrm{T}} = \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{2\times 2k}
$$

$$
\begin{aligned}
XY^{\mathrm{T}} &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix}
X_1Y_1^{\mathrm{T}} & X_1Y_2^{\mathrm{T}} & \cdots & X_1Y_{k-1}^{\mathrm{T}} & x_1Y_k^{\mathrm{T}} \\
X_2Y_1^{\mathrm{T}} & X_2Y_2^{\mathrm{T}} & \cdots & X_2Y_{k-1}^{\mathrm{T}} & x_2Y_k^{\mathrm{T}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
X_{k-1}Y_1^{\mathrm{T}} & X_{k-1}Y_2^{\mathrm{T}} & \cdots & X_{k-1}Y_{k-1}^{\mathrm{T}} & X_{k-1}Y_k^{\mathrm{T}} \\
X_kY_1^{\mathrm{T}} & X_kY_2^{\mathrm{T}} & \cdots & X_kY_{k-1}^{\mathrm{T}} & X_kY_k^{\mathrm{T}}
\end{bmatrix} \\
&= X\begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix} XY_1^{\mathrm{T}} & XY_2^{\mathrm{T}} & \cdots & XY_{k-1}^{\mathrm{T}} & XY_k^{\mathrm{T}} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{triu}(X_k) &\coloneqq X_k \\
\operatorname{triu}(Y_k^{\mathrm{T}}) &\coloneqq Y_k^{\mathrm{T}} \\
\operatorname{triu}(X_k Y_k^{\mathrm{T}}) &= X_k Y_k^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{triu}(XY^{\mathrm{T}}) &= \begin{bmatrix}
\operatorname{triu}(X_1 Y_1^{\mathrm{T}}) & X_1 Y_2^{\mathrm{T}} & \cdots & X_1 Y_{k-1}^{\mathrm{T}} & X_1 Y_k^{\mathrm{T}} \\
0_{2\times 2} & \operatorname{triu}(X_2 Y_2^{\mathrm{T}}) & \cdots & X_2 Y_{k-1}^{\mathrm{T}} & X_2 Y_k^{\mathrm{T}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{2\times 2} & 0_{2\times 2} & \cdots & \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) & X_{k-1} Y_k^{\mathrm{T}} \\
0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & X_k Y_k^{\mathrm{T}}
\end{bmatrix} \\
&= \begin{bmatrix}
\begin{bmatrix} \operatorname{triu}(X_1 Y_1^{\mathrm{T}}) \\ 0_{2(k-1)\times 2} \end{bmatrix} &
\begin{bmatrix} X_1 Y_2^{\mathrm{T}} \\ \operatorname{triu}(X_2 Y_2^{\mathrm{T}}) \\ 0_{2(k-2)\times 2} \end{bmatrix} &
\cdots &
\begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} &
XY_k^{\mathrm{T}}
\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
U^{-1} &= \operatorname{triu}(XY^{\mathrm{T}}) \\
I &= U \operatorname{triu}(XY^{\mathrm{T}})
\end{aligned}
$$

$$
E_i = \begin{bmatrix} 0 & \cdots & 0 & I_{2\times 2} & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}
$$

k-th block column,

$$
\begin{aligned}
E_k &= U \operatorname{triu}(XY^{\mathrm{T}}) E_k \\
&= UXY_k^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{aligned}
X &\coloneqq U^{-1} E_k \\
E_k &= UXY_k^{\mathrm{T}} \\
&= UU^{-1} E_k Y_k^{\mathrm{T}} \\
&= E_k Y_k^{\mathrm{T}} \\
Y_k^{\mathrm{T}} &\coloneqq I_{2\times 2}
\end{aligned}
$$

$(k-1)$-th block column,

$$
\begin{aligned}
E_{k-1} &= U \operatorname{triu}(XY^{\mathrm{T}}) E_{k-1} \\
&= U \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix}
\end{aligned}
$$

$(k-1)$-th block row of $(k-1)$-th block column,

$$
E_i^{\mathrm{T}} E_j = \begin{cases} I_{2\times 2}, & \text{if } i = j \ ; \\ 0_{2\times 2}, & \text{otherwise.} \end{cases}
$$

$$
\begin{aligned}
E_{k-1}^{\mathrm{T}} E_{k-1} = I_{2\times 2} &= E_{k-1}^{\mathrm{T}} U \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} \\
&= \begin{bmatrix} 0_{2\times 2(k-2)} & D_{k-1} & B_{k-1} \end{bmatrix} \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} \\
&= D_{k-1} \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\
Y_{k-1}^{\mathrm{T}} &\coloneqq (D_{k-1} X_{k-1})^{-1}
\end{aligned}
$$

$$
\begin{aligned}
D_{k-1} &= \operatorname{triu}(D_{k-1}) \\
D_{k-1}^{-1} &= \operatorname{triu}(D_{k-1}^{-1}) \\
Y_{k-1}^{\mathrm{T}} &\coloneqq (D_{k-1} X_{k-1})^{-1} \\
D_{k-1} \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) &= D_{k-1} \operatorname{triu}(X_{k-1} (D_{k-1} X_{k-1})^{-1}) \\
&= D_{k-1} \operatorname{triu}(X_{k-1} X_{k-1}^{-1} D_{k-1}^{-1}) \\
&= D_{k-1} \operatorname{triu}(D_{k-1}^{-1}) \\
&= D_{k-1} D_{k-1}^{-1} \\
&= I_{2\times 2}
\end{aligned}
$$

$$
\begin{aligned}
X &\coloneqq U^{-1} E_k \\
E_k &= U X \\
E_k^{\mathrm{T}} E_k = I_{2\times 2} &= E_k^{\mathrm{T}} U X \\
&= \begin{bmatrix} 0_{2\times 2(k-1)} & D_k \end{bmatrix} X \\
&= D_k X_k \\
Y_k^{\mathrm{T}} &\coloneqq I_{2\times 2} \\
&\coloneqq (D_k X_k)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
X &\coloneqq U^{-1} E_k \\
Y_i^{\mathrm{T}} &\coloneqq (D_i X_i)^{-1}
\end{aligned}
$$

$n = 2k+1$

$$U = \begin{bmatrix} \begin{bmatrix} d_1 \end{bmatrix} & \begin{bmatrix} b_1^1 & b_1^2 \end{bmatrix} & & & & \\ & \begin{bmatrix} d_2 & b_2^1 \\ & d_3 \end{bmatrix} & \begin{bmatrix} b_2^2 & \\ b_3^1 & b_3^2 \end{bmatrix} & & & \\ & & \ddots & \ddots & & \\ & & & \begin{bmatrix} d_{2k-2} & b_{2k-2}^1 \\ & d_{2k-1} \end{bmatrix} & \begin{bmatrix} b_{2k-2}^2 & \\ b_{2k-1}^1 & b_{2k-1}^2 \end{bmatrix} \\ & & & & \begin{bmatrix} d_{2k} & b_{2k}^1 \\ & d_{2k+1} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(2k+1)\times(2k+1)}$$

$$\begin{aligned} X &\coloneqq U^{-1} E_{k+1} \\ Y_i^{\mathrm{T}} &\coloneqq (D_i X_i)^{-1} \text{ for } i = 2,\ \cdots,\ k+1 \\ Y_1^{\mathrm{T}} &\coloneqq (D_1 X_1)^{+} = (D_1 X_1)^{\mathrm{T}} (D_1 X_1 (D_1 X_1)^{\mathrm{T}})^{-1} \end{aligned}$$

15.2 Banded matrix (Upper)

$n = ku + r,\ 0 \le r < u$

$$U = \begin{bmatrix} d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & & \\ & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & \\ & & \ddots & \ddots & \ddots & \cdots & \ddots & \\ & & & \ddots & \ddots & \ddots & \cdots & b_{n-u}^u \\ & & & & \ddots & \ddots & \ddots & \vdots \\ & & & & & \ddots & \ddots & b_{n-2}^2 \\ & & & & & & \ddots & b_{n-1}^1 \\ & & & & & & & d_n \end{bmatrix}$$

$r = 0,\ n = ku,$

$$U = \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{ku \times ku},$$
$$D_i \in \mathbb{R}^{u\times u},\ B_i \in \mathbb{R}^{u\times u} \text{ for } i = 1,\ 2,\ \cdots,\ k.$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} \in \mathbb{R}^{ku\times u} \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{bmatrix} \in \mathbb{R}^{ku\times u} \qquad Y^{\mathrm{T}} = \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_k^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{u\times ku}$$

$$X_i,\ Y_i^{\mathrm{T}} \in \mathbb{R}^{u\times u},\ \text{for } i = 1,\ 2,\ \cdots,\ k$$
$$X := U^{-1}E_k$$
$$Y_i^{\mathrm{T}} := (D_i X_i)^{-1}$$

$r \neq 0,\ n = ku + r,$

$$U = \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r)\times(ku+r)},$$
$$\tilde{D}_1 \in \mathbb{R}^{r\times r},\ \tilde{B}_1 \in \mathbb{R}^{r\times u};$$
$$D_i \in \mathbb{R}^{u\times u},\ B_i \in \mathbb{R}^{u\times u} \text{ for } i = 2,\ \cdots,\ k+1.$$

$$X = \begin{bmatrix} \tilde{X}_1 \\ X_2 \\ \vdots \\ X_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r)\times u} \quad Y = \begin{bmatrix} \tilde{Y}_1 \\ Y_2 \\ \vdots \\ Y_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r)\times u} \quad Y^{\mathrm{T}} = \begin{bmatrix} \tilde{Y}_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k+1}^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{u\times(ku+r)}$$

$$
\begin{aligned}
\tilde{X}_1 &\in \mathbb{R}^{r\times u} \\
\tilde{Y}_1^{\mathrm{T}} &\in \mathbb{R}^{u\times r} \\
X_i,\ Y_i^{\mathrm{T}} &\in \mathbb{R}^{u\times u} \text{ for } i = 2,\ \cdots,\ k+1 \\
X &\coloneqq U^{-1}E_{k+1} \\
\tilde{Y}_1^{\mathrm{T}} &\coloneqq (D_1X_1)^{+} = (D_1X_1)^{\mathrm{T}}(D_1X_1(D_1X_1)^{\mathrm{T}})^{-1} \\
\mathrm{triu}(D_1X_1\tilde{Y}_1^{\mathrm{T}}) &= I_{r\times r} \\
Y_i^{\mathrm{T}} &\coloneqq (D_iX_i)^{-1} \text{ for } i = 2,\ \cdots,\ k+1
\end{aligned}
$$

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16.1 Notation

This notes use Householder notation and start indexing at one.

$$
\begin{aligned}
\text{Scalars, lower case Greek letters: } & \alpha,\ \beta,\ \gamma,\ \chi,\ \textit{etc.} \\
\text{Vectors, lower case Roman letter: } & a,\ b,\ c,\ x,\ \textit{etc.} \\
\text{Matrix, upper case Roman letter: } & A,\ B,\ C,\ X,\ \textit{etc.}
\end{aligned}
$$

16.2 Repersentation

Matrix & Vector

$$
\begin{aligned}
x &= \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_n \end{bmatrix} \in \mathbb{R}^n \\
A &= \left[\begin{array}{c|c|c|c} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{1n} \end{array}\right] \\
&= \left[\begin{array}{c|c|c|c} a_1 & a_2 & \cdots & a_n \end{array}\right] \in \mathbb{R}^{m\times n}
\end{aligned}
$$

Block Matrix & Block Vector

$$
\begin{aligned}
x &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{kn},\ x_i \in \mathbb{R}^k \\
A &= \left[\begin{array}{c|c|c|c} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{1n} \end{array}\right] \\
&= \left[\begin{array}{c|c|c|c} A_1 & A_2 & \cdots & A_n \end{array}\right] \in \mathbb{R}^{lm\times kn},\ A_{ij} \in \mathbb{R}^{l\times k}
\end{aligned}
$$

16.3 Content

Consider an upper banded matrix $U \in \mathbb{R}^{n\times n}$,

$$U = \begin{bmatrix} \alpha_1 & \beta_1^1 & \beta_1^2 & \cdots & \beta_1^u & & & \\ & \alpha_2 & \beta_2^1 & \beta_2^2 & \cdots & \beta_2^u & & \\ & & \ddots & \ddots & & \ddots & \ddots & \\ & & & & & & \beta_{n-u}^u \\ & & & & & & \vdots \\ & & & & & \ddots & \beta_{n-2}^2 \\ & & & & & \ddots & \beta_{n-1}^1 \\ & & & & & & \alpha_n \end{bmatrix}. \tag{12}$$

Then try to show that its inverse can be repersent as the upper triangular part of an outer product, *i.e.* ,

$$U^{-1} = \mathrm{triu}(XY^{\mathrm{T}}) \tag{13}$$

Let $X,\ Y \in \mathbb{R}^{n\times u}$, consider the following cases of n,

Case 1, $n = ku$

then (12) can be repersent as a block upper bidiagonal matrix,

$$U = \begin{bmatrix} \alpha_1 \cdots \beta_1^{u-1} & \beta_1^u & & & \\ \ddots \quad \vdots & \vdots \quad \ddots & & & \\ \alpha_u & \beta_u^1 \cdots \beta_u^u & & & \\ & \alpha_{u+1} \cdots \beta_{u+1}^{u-1} & \beta_{u+1}^u & & \\ & \ddots \quad \vdots & \vdots \quad \ddots & & \\ & \alpha_{2u} & \beta_{2u}^1 \cdots \beta_{2u}^u & & \\ & & \ddots & \ddots & \\ & & & & \alpha_{(k-1)u+1} \cdots \beta_{(k-1)u+1}^{u-1} \\ & & & & \ddots \quad \vdots \\ & & & & \alpha_{ku} \end{bmatrix}$$

$$= \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{ku\times ku},\ D_i,\ B_i \in \mathbb{R}^{u\times u} \text{ for } i = 1,\ 2,\ \ldots,\ k. \tag{14}$$

Repersent the n-dimensional identity matrix $I_{n\times n}$ in block form,

$$I_{n\times n} = I_{ku\times ku} = \mathrm{diag}(\underbrace{I_{u\times u},\ I_{u\times u},\ \cdots,\ I_{u\times u}}_{k}) = \left[\ E_1 \ \middle|\ E_2 \ \middle|\ \cdots \ \middle|\ E_k\ \right].$$

then, its block columns E_i for $i = 1,\ 2,\ \ldots,\ k$ satisfies

$$
\begin{aligned}
E_i &\in \mathbb{R}^{ku\times u}\\
E_i^{\mathrm{T}} E_j &= \begin{cases} I_{u\times u}, & \text{if } i=j\ ;\\ 0_{u\times u}, & \text{otherwise.}\end{cases}.
\end{aligned}
$$

To find X and Y such that (13) holds, let

$$
X = \begin{bmatrix} X_1\\ X_2\\ \vdots\\ X_{k-1}\\ X_k \end{bmatrix} \in \mathbb{R}^{ku\times u},\ Y = \begin{bmatrix} Y_1\\ Y_2\\ \vdots\\ Y_{k-1}\\ Y_k \end{bmatrix} \in \mathbb{R}^{ku\times u},\ X_i,\, Y_i \in \mathbb{R}^{u\times u} \text{ for } i=1,\,2,\,\ldots,\,k,
$$

and let X_k and Y_k^{T} be upper triangular matrix, then $\mathrm{triu}(X_k Y_k^{\mathrm{T}}) = X_k Y_k^{\mathrm{T}}$.
For the outer product form,

$$
\begin{aligned}
Y^{\mathrm{T}} &= \left[\begin{array}{c|c|c|c|c} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{array}\right] \in \mathbb{R}^{u\times ku}\\
XY^{\mathrm{T}} &= \begin{bmatrix} X_1\\ X_2\\ \vdots\\ X_{k-1}\\ X_k \end{bmatrix} \left[\begin{array}{c|c|c|c|c} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{array}\right]\\
&= \left[\begin{array}{c|c|c|c|c}
X_1Y_1^{\mathrm{T}} & X_1Y_2^{\mathrm{T}} & \cdots & X_1Y_{k-1}^{\mathrm{T}} & x_1Y_k^{\mathrm{T}}\\
X_2Y_1^{\mathrm{T}} & X_2Y_2^{\mathrm{T}} & \cdots & X_2Y_{k-1}^{\mathrm{T}} & x_2Y_k^{\mathrm{T}}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
X_{k-1}Y_1^{\mathrm{T}} & X_{k-1}Y_2^{\mathrm{T}} & \cdots & X_{k-1}Y_{k-1}^{\mathrm{T}} & X_{k-1}Y_k^{\mathrm{T}}\\
X_kY_1^{\mathrm{T}} & X_kY_2^{\mathrm{T}} & \cdots & X_kY_{k-1}^{\mathrm{T}} & X_kY_k^{\mathrm{T}}
\end{array}\right]\\
&= X\left[\begin{array}{c|c|c|c|c} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{array}\right]\\
&= \left[\begin{array}{c|c|c|c|c} XY_1^{\mathrm{T}} & XY_2^{\mathrm{T}} & \cdots & XY_{k-1}^{\mathrm{T}} & XY_k^{\mathrm{T}} \end{array}\right] \in \mathbb{R}^{ku\times ku},
\end{aligned}
$$

and its the upper triangular componnent,

$$
\begin{aligned}
\mathrm{triu}(XY^{\mathrm{T}}) &= \left[\begin{array}{c|c|c|c|c}
\mathrm{triu}(X_1Y_1^{\mathrm{T}}) & X_1Y_2^{\mathrm{T}} & \cdots & X_1Y_{k-1}^{\mathrm{T}} & X_1Y_k^{\mathrm{T}}\\
0_{u\times u} & \mathrm{triu}(X_2Y_2^{\mathrm{T}}) & \cdots & X_2Y_{k-1}^{\mathrm{T}} & X_2Y_k^{\mathrm{T}}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0_{u\times u} & 0_{u\times u} & \cdots & \mathrm{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) & X_{k-1}Y_k^{\mathrm{T}}\\
0_{u\times u} & 0_{u\times u} & \cdots & 0_{u\times u} & X_kY_k^{\mathrm{T}}
\end{array}\right]\\
&= \left[\begin{array}{c|c|c|c|c}
\mathrm{triu}(X_1Y_1^{\mathrm{T}}) & X_1Y_2^{\mathrm{T}} & & X_{1:k-2}Y_{k-1}^{\mathrm{T}} & \\
0_{(k-2)u\times u} & \mathrm{triu}(X_2Y_2^{\mathrm{T}}) & \cdots & \mathrm{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) & XY_k^{\mathrm{T}}\\
0_{u\times u} & 0_{(k-2)u\times u} & & 0_{u\times u} &
\end{array}\right].zz
\end{aligned}
\tag{15}
$$

From (13),

$$
I_{n\times n} = U\,\mathrm{triu}(XY^{\mathrm{T}}). \tag{16}
$$

For X and Y_k, from (15), the k-th block column of (16) can be written as,

$$\begin{aligned}E_k &= U\,\mathrm{triu}(XY^{\mathrm{T}})E_k\\ E_k &= UXY_k^{\mathrm{T}} \end{aligned} \tag{17}$$

Define that

$$X \coloneqq U^{-1}E_k; \tag{18}$$
$$Y_k^{\mathrm{T}} \coloneqq I_{u\times u}, \tag{19}$$

which can be shown that (17) holds for the defined (18) and (19) through

$$\begin{aligned}UXY_k^{\mathrm{T}} &= UU^{-1}E_kI_{u\times u}\\ &= I_{n\times n}E_kI_{u\times u}\\ &= E_kI_{u\times u}\\ &= E_k,\ \text{as required.}\end{aligned}$$

Then, for Y_{k-1}, forming the $(k-1)$-th block column of (16) from (15) first,

$$\begin{aligned}E_{k-1} &= U\,\mathrm{triu}(XY^{\mathrm{T}})E_{k-1}\\ &= U\begin{bmatrix} X_{1:k-2}Y_{k-1}^{\mathrm{T}}\\ \mathrm{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}})\\ 0_{u\times u}\end{bmatrix}.\end{aligned}$$

Focusing on the $(k-1)$-th block row of $(k-1)$-th block column,

$$\begin{aligned}E_{k-1}^{\mathrm{T}}E_{k-1} &= E_{k-1}^{\mathrm{T}}U\begin{bmatrix} X_{1:k-2}Y_{k-1}^{\mathrm{T}}\\ \mathrm{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}})\\ 0_{u\times u}\end{bmatrix}\\ I_{u\times u} &= \left[\begin{array}{c|c|c} 0_{u\times(k-2)u} & D_{k-1} & B_{k-1}\end{array}\right]\left[\begin{array}{c} X_{1:k-2}Y_{k-1}^{\mathrm{T}}\\ \hline \mathrm{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}})\\ \hline 0_{u\times u}\end{array}\right]\\ I_{u\times u} &= D_{k-1}\,\mathrm{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}).\end{aligned} \tag{20}$$

Define that

$$Y_{k-1}^{\mathrm{T}} \coloneqq (D_{k-1}X_{k-1})^{-1}, \tag{21}$$

and notice that $D_{k-1} \in \mathbb{R}^{u\times u}$ is a upper triangular, therefore

$$\begin{aligned}D_{k-1} &= \mathrm{triu}(D_{k-1});\\ D_{k-1}^{-1} &= \mathrm{triu}(D_{k-1}^{-1}).\end{aligned} \tag{22}$$

With the fact (22), (20) holds for the defined (21) as,

$$
\begin{aligned}
D_{k-1}\,\mathrm{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) &= D_{k-1}\,\mathrm{triu}(X_{k-1}(D_{k-1}X_{k-1})^{-1}) \\
&= D_{k-1}\,\mathrm{triu}(X_{k-1}X_{k-1}^{-1}D_{k-1}^{-1}) \\
&= D_{k-1}\,\mathrm{triu}(D_{k-1}^{-1}) \\
&= D_{k-1}D_{k-1}^{-1} \\
&= I_{u\times u},\ \text{as required.}
\end{aligned}
$$

Similar ideas can be use to redefine Y_k^{T} in (19), from (18),

$$
\begin{aligned}
E_k &= UX \\
E_k^{\mathrm{T}}E_k &= E_k^{\mathrm{T}}UX \\
I_{u\times u} &= E_k^{\mathrm{T}}UX \\
&= \left[\begin{array}{c|c} 0_{u\times(k-1)u} & D_k \end{array}\right]\left[\begin{array}{c} X_{1:k-1} \\ \hline X_k \end{array}\right] \\
&= D_kX_k \\
I_{u\times u} &= (D_kX_k)^{-1},
\end{aligned}
$$

following that

$$
\begin{aligned}
Y_k^{\mathrm{T}} &\coloneqq I_{u\times u} \\
&\coloneqq (D_kX_k)^{-1}.
\end{aligned}
$$

In summary, for $U \in \mathbb{R}^{ku\times ku}$, (13) holds the following definitions of X and Y:

$$
\begin{aligned}
X &\coloneqq U^{-1}E_k \\
Y_i^{\mathrm{T}} &\coloneqq (D_iX_i)^{-1}
\end{aligned}
$$

17 Test

$$\begin{array}{c} \qquad\qquad\overbrace{\hphantom{b_1^1 \quad b_1^2 \quad \cdots \quad b_1^u}}^{\text{bandwidth} = u} \qquad\qquad\qquad \\ \begin{bmatrix} d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & & \\ & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & \\ & & \ddots & \ddots & \ddots & \cdots & \ddots & \\ & & & \ddots & \ddots & \ddots & \cdots & b_{n-u}^u \\ & & & & \ddots & \ddots & \ddots & \vdots \\ & & & & & \ddots & \ddots & b_{n-2}^2 \\ & & & & & & \ddots & b_{n-1}^1 \\ & & & & & & & d_n \end{bmatrix} \end{array}$$

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} a & \cdots & a & b & \cdots & b \\ & \ddots & \vdots & \vdots & \ddots & \\ & & a & b & & \\ \hline & & & c & \cdots & c \\ & \text{\Huge 0} & & \vdots & & \vdots \\ & & & & & \end{array}\right) & \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} p \\ \left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} q \end{array} \\ \underbrace{\hphantom{a \quad \cdots \quad a}}_{m}\ \underbrace{\hphantom{b \quad \cdots \quad b}}_{m} & \end{array}$$

$$\begin{array}{cc} & \overset{n \text{ columns}}{\longleftrightarrow} \\ \updownarrow {\scriptstyle n \text{ rows}} & \begin{pmatrix} 1 & 1 & 1 \ldots\ldots & 1 \\ 1 & 1 & 1 & 1 \\ \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 \ldots\ldots & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & \cdots & 1 & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & 1 & \cdots & 0 & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{array} \\ \qquad\qquad \uparrow \atop i \qquad\qquad \uparrow \atop j & \end{array}$$

$$\begin{array}{cc} & \overset{n \text{ columns}}{\longleftrightarrow} \\ \updownarrow {\scriptstyle n \text{ rows}} & \begin{pmatrix} 1 & 1 & 1 \ldots\ldots & 1 \\ 1 & 1 & 1 & 1 \\ \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 \ldots\ldots & 1 \end{pmatrix} \end{array}$$

$$\left(\begin{array}{cc:cc:cc}
1 & 2 & 0 & 0 & 0 & 0 \\
4 & 5 & 0 & 0 & 0 & 0 \\ \hdashline
0 & 0 & 7 & 1 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0 & 3 & 4 \\
0 & 0 & 0 & 0 & 1 & 4
\end{array}\right)$$

$$\begin{bmatrix}
d_1 & b_1^1 & b_1^2 & & & & & & & \\
 & d_2 & b_2^1 & b_2^2 & & & & & & \\
 & & d_3 & b_3^1 & b_3^2 & & & & & \\
 & & & d_4 & b_4^1 & b_4^2 & & & & \\
 & & & & \ddots & & \ddots & & & \\
 & & & & & \ddots & & \ddots & & \\
 & & & & & & d_{2k-3} & b_{2k-3}^1 & b_{2k-3}^2 & \\
 & & & & & & & d_{2k-2} & b_{2k-2}^1 & b_{2k-2}^2 \\
 & & & & & & & & d_{2k-1} & b_{2k-1}^1 \\
 & & & & & & & & & d_{2k}
\end{bmatrix}$$

$$= \begin{bmatrix}
\begin{bmatrix} d_1 & b_1^1 \\ & d_2 \end{bmatrix} & \begin{bmatrix} b_1^2 & \\ b_2^1 & b_2^2 \end{bmatrix} & & & \\
 & \begin{bmatrix} d_3 & b_3^1 \\ & d_4 \end{bmatrix} & \begin{bmatrix} b_3^2 & \\ b_4^1 & b_4^2 \end{bmatrix} & & \\
 & & \ddots & \ddots & \\
 & & & \begin{bmatrix} d_{2k-3} & b_{2k-3}^1 \\ & d_{2k-2} \end{bmatrix} & \begin{bmatrix} b_{2k-3}^2 & \\ b_{2k-2}^1 & b_{2k-2}^2 \end{bmatrix} \\
 & & & & \begin{bmatrix} d_{2k-1} & b_{2k-1}^1 \\ & d_{2k} \end{bmatrix}
\end{bmatrix}$$

$$\begin{bmatrix}
\begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\
\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 2 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 3 & 4 \\ 1 & 4 \end{matrix}
\end{bmatrix}$$

$$\begin{bmatrix}
\begin{bmatrix}
d_1 & b_1^1 & \cdots & b_1^{r-2} & b_1^{r-1} \\
 & d_2 & & & b_2^{r-2} \\
 & & \ddots & & \vdots \\
 & & & d_{r-1} & b_{r-1}^1 \\
 & & & & d_r
\end{bmatrix} &
\begin{bmatrix}
b_1^r & \cdots & b_1^u & \\
b_2^{r-1} & b_2^r & b_2^u & \\
\vdots & & \ddots & \\
b_{r-1}^2 & & & \\
b_r^1 & b_r^2 & \cdots & b_r^u
\end{bmatrix} \\
 & \begin{bmatrix}
d_1 & b_1^1 & \cdots & b_1^{r-2} & b_1^{r-1} \\
0 & d_2 & & & b_2^{r-2} \\
\vdots & \ddots & \ddots & & \vdots \\
 & & & d_{r-1} & b_{r-1}^1 \\
0 & \cdots & & 0 & d_r
\end{bmatrix}
\end{bmatrix}$$

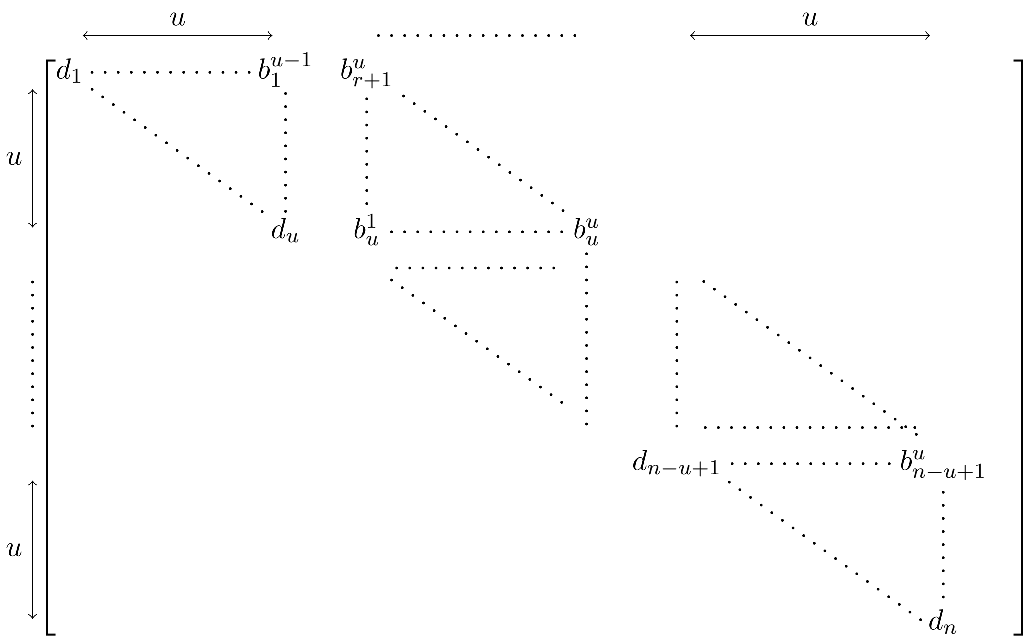
$$
\begin{bmatrix}
d_1 & b_1^1 & \cdots & b_1^{r-2} & b_1^{r-1} \\
 & d_2 & & & b_2^{r-2} \\
 & & \ddots & & \vdots \\
 & & & d_{r-1} & b_{r-1}^1 \\
 & & & & d_r
\end{bmatrix}
\begin{bmatrix}
b_1^r & b_1^{r+1} & \cdots & b_1^{u-1} & b_1^u & & & \\
b_2^{r-1} & & & & & b_2^u & & \\
\vdots & & & & & & \ddots & \\
b_{r-1}^2 & & & & & & & b_{r-1}^u \\
b_r^1 & b_r^2 & \cdots & & & & & b_r^{u-1} & b_r^u
\end{bmatrix}
$$

$$
\begin{bmatrix}
d_1 & \cdots & b_1^{r-1} \\
 & \ddots & \vdots \\
 & & d_r
\end{bmatrix}
\begin{bmatrix}
b_1^r & \cdots & b_1^u & \\
\vdots & & & \ddots \\
b_r^1 & \cdots & & b_r^u
\end{bmatrix}
$$

$$
\begin{bmatrix}
b_1^r & \cdots & b_1^u & \\
\vdots & & & \ddots \\
b_r^1 & \cdots & & b_r^u
\end{bmatrix}
\qquad
\begin{bmatrix}
b_1^r & \cdots & b_1^u & \\
\vdots & & & \ddots \\
b_r^1 & \cdots & & b_r^u
\end{bmatrix}
$$

$$
\begin{bmatrix}
d_1 & \cdots & b_1^{r-1} & b_1^r & \cdots & b_1^u & \\
 & \ddots & \vdots & \vdots & & & \ddots \\
 & & d_r & b_r^1 & \cdots & & b_r^u \\
 & & & d_{r+1} & \cdots & & b_{r+1}^{u-1} \\
 & & & & \ddots & & \vdots \\
 & & & & & & d_{r+u}
\end{bmatrix}
$$

$$
\begin{array}{c}
 \\ r \\ \\ u \\ \\ \vdots \\ \\ u
\end{array}
\begin{array}{c}
\overset{r}{\longleftrightarrow} \qquad \overset{u}{\longleftrightarrow} \qquad \cdots\cdots \qquad \overset{u}{\longleftrightarrow} \\
\begin{bmatrix}
d_1 & \cdots & b_1^{r-1} & b_1^r & \cdots & b_1^u & & & & & & \\
 & \ddots & \vdots & \vdots & & & \ddots & & & & & \\
 & & d_r & b_r^1 & \cdots & & b_r^u & & & & & \\
 & & & d_{1+r} & \cdots & & b_{1+r}^{u-1} & b_{r+1}^u & & & & \\
 & & & & \ddots & & \vdots & \vdots & \ddots & & & \\
 & & & & & & d_{u+r} & b_{u+r}^1 & \cdots & b_{u+r}^u & & \\
 & & & & & & & \cdots & & \vdots & \vdots & \ddots \\
 & & & & & & & & \ddots & & & \cdots \\
 & & & & & & & & & & d_{n-u+1} & \cdots & b_{n-u+1}^u \\
 & & & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & & & & d_n
\end{bmatrix}
\end{array}
$$



$$
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c}
\left[\begin{array}{c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}\right]
\begin{array}{c} 8 \\ 8 \\ 8 \\ 8 \\ 8 \end{array}
\end{array}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccccccccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \end{array} \\
\begin{array}{c} 0 \\ 0 \end{array}
\left[\begin{array}{ccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19
\end{array}\right]
\begin{array}{c} 20 \\ 20 \end{array} \\
\begin{array}{cccccccccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \end{array}
\end{array}
$$