

Meeting Notes

Zhiwei Zhou

June 24, 2022

1 28.06.2022 Meeting

1.1 Notation

Vectors

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

Matrices and Columns

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ &= [a_1 \quad a_2 \quad \cdots \quad a_n] \in \mathbb{R}^{m \times n}, \quad a_i \in \mathbb{R}^m \end{aligned}$$

$$Ae_i := a_i$$

e_i = i -th column of I

Block Matrices and Block Columns

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1l} \\ A_{21} & A_{22} & \cdots & A_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kl} \end{bmatrix}, \quad A_{ij} \in \mathbb{R}^{m \times n} \\ &= [A_1 \quad A_2 \quad \cdots \quad A_l] \in \mathbb{R}^{km \times ln}, \quad A_i \in \mathbb{R}^{km \times n} \end{aligned}$$

$$AE_i := A_i$$

E_i = i -th block column of I

1.2 Bidiagonal matrix (upper)

$$U = \begin{bmatrix} d_1 & b_1 & & & \\ & d_2 & b_2 & & \\ & & \ddots & \ddots & \\ & & & d_{n-1} & b_{n-1} \\ & & & & d_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \in \mathbb{R}^{n \times 1} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} \in \mathbb{R}^{n \times 1} \quad y^T = [y_1 \quad y_2 \quad \cdots \quad y_{n-1} \quad y_n] \in \mathbb{R}^{1 \times n}$$

$$\begin{aligned} xy^T &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} [y_1 \quad y_2 \quad \cdots \quad y_{n-1} \quad y_n] \\ &= \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_{n-1} & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_{n-1} & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1} y_1 & x_{n-1} y_2 & \cdots & x_{n-1} y_{n-1} & x_{n-1} y_n \\ x_n y_1 & x_n y_2 & \cdots & x_n y_{n-1} & x_n y_n \end{bmatrix} \\ &= x [y_1 \quad y_2 \quad \cdots \quad y_{n-1} \quad y_n] \\ &= [x y_1 \quad x y_2 \quad \cdots \quad x y_{n-1} \quad x y_n] \end{aligned}$$

$$\begin{aligned} \text{triu}(xy^T) &= \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_{n-1} & x_1 y_n \\ 0 & x_2 y_2 & \cdots & x_2 y_{n-1} & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_4 y_{n-1} & x_4 y_n \\ 0 & 0 & \cdots & 0 & x_5 y_n \end{bmatrix} \\ &= \left[\begin{bmatrix} x_{1:1} \\ 0_{n-1} \end{bmatrix} y_1 \quad \begin{bmatrix} x_{1:2} \\ 0_{n-2} \end{bmatrix} y_2 \quad \cdots \quad \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \quad x y_n \right] \end{aligned}$$

$$\begin{aligned} U^{-1} &= \text{triu}(xy^T) \\ I &= U \text{triu}(xy^T) \end{aligned}$$

n -th column,

$$\begin{aligned} e_n &= U \text{triu}(xy^T) e_n \\ &= U x y_n \end{aligned}$$

$$\begin{aligned}
x &:= U^{-1}e_n \\
e_n &= Uxy_n \\
&= UU^{-1}e_ny_n \\
&= e_ny_n \\
y_n &:= 1
\end{aligned}$$

$(n-1)$ -th column,

$$\begin{aligned}
e_{n-1} &= U \operatorname{triu}(xy^T)e_{n-1} \\
&= U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1}
\end{aligned}$$

$(n-1)$ -th row of $(n-1)$ -th column,

$$e_i^T e_j = \delta_{ij}$$

$$\begin{aligned}
e_{n-1}^T e_{n-1} &= 1 = e_{n-1}^T U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \\
&= [0_{n-2} \quad d_{n-1} \quad b_{n-1}] \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} \\
&= d_{n-1} x_{n-1} y_{n-1} \\
y_{n-1} &:= (d_{n-1} x_{n-1})^{-1}
\end{aligned}$$

$$\begin{aligned}
e_n &= Ux \\
e_n^T e_n &= 1 = e_n^T Ux \\
&= [0_{n-1} \quad d_n] x \\
&= d_n x_n \\
y_n &:= 1 \\
&:= (d_n x_n)^{-1}
\end{aligned}$$

$$\begin{aligned}
x &:= U^{-1}e_n \\
y_i &:= (d_i x_i)^{-1}
\end{aligned}$$

1.3 Tridiagonal matrix (Upper)

$$n = 2k$$

$$\begin{aligned}
 U &= \begin{bmatrix} d_1 & b_1^1 & b_1^2 & & & \\ & d_2 & b_2^1 & b_2^2 & & \\ & & d_3 & b_3^1 & b_3^2 & \\ & & & d_4 & b_4^1 & b_4^2 \\ & & & & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & & d_{n-3} & b_{n-3}^1 & b_{n-3}^2 \\ & & & & & & & d_{n-2} & b_{n-2}^1 & b_{n-2}^2 \\ & & & & & & & & d_{n-1} & b_{n-1}^1 \\ & & & & & & & & & d_n \end{bmatrix} \\
 &= \begin{bmatrix} \begin{bmatrix} d_1 & b_1^1 \end{bmatrix} & \begin{bmatrix} b_1^2 \\ b_2^1 & b_2^2 \end{bmatrix} & & & \\ & \begin{bmatrix} d_3 & b_3^1 \\ b_4^1 & b_4^2 \end{bmatrix} & \begin{bmatrix} b_3^2 \\ b_4^1 & b_4^2 \end{bmatrix} & & \\ & & \ddots & \ddots & \\ & & & \begin{bmatrix} d_{2k-3} & b_{2k-3}^1 \\ & d_{2k-2} \end{bmatrix} & \begin{bmatrix} b_{2k-3}^2 \\ b_{2k-2}^1 & b_{2k-2}^2 \end{bmatrix} \\ & & & & \begin{bmatrix} d_{2k-1} & b_{2k-1}^1 \\ & d_{2k} \end{bmatrix} \end{bmatrix} \\
 &= \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{2k \times 2k} \\
 X &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{k-1} \\ Y_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \quad Y^T = [Y_1^T \ Y_2^T \ \cdots \ Y_{k-1}^T \ Y_k^T] \in \mathbb{R}^{2 \times 2k}
 \end{aligned}$$

$$\begin{aligned}
 XY^T &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} [Y_1^T \ Y_2^T \ \cdots \ Y_{k-1}^T \ Y_k^T] \\
 &= \begin{bmatrix} X_1 Y_1^T & X_1 Y_2^T & \cdots & X_1 Y_{k-1}^T & X_1 Y_k^T \\ X_2 Y_1^T & X_2 Y_2^T & \cdots & X_2 Y_{k-1}^T & X_2 Y_k^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{k-1} Y_1^T & X_{k-1} Y_2^T & \cdots & X_{k-1} Y_{k-1}^T & X_{k-1} Y_k^T \\ X_k Y_1^T & X_k Y_2^T & \cdots & X_k Y_{k-1}^T & X_k Y_k^T \end{bmatrix} \\
 &= X [Y_1^T \ Y_2^T \ \cdots \ Y_{k-1}^T \ Y_k^T] \\
 &= [XY_1^T \ XY_2^T \ \cdots \ XY_{k-1}^T \ XY_k^T]
 \end{aligned}$$

$$\begin{aligned}
\text{triu}(X_k) &:= X_k \\
\text{triu}(Y_k^T) &:= Y_k^T \\
\text{triu}(X_k Y_k^T) &= X_k Y_k^T
\end{aligned}$$

$$\begin{aligned}
\text{triu}(XY^T) &= \begin{bmatrix} \text{triu}(X_1 Y_1^T) & X_1 Y_2^T & \cdots & X_1 Y_{k-1}^T & X_1 Y_k^T \\ 0_{2 \times 2} & \text{triu}(X_2 Y_2^T) & \cdots & X_2 Y_{k-1}^T & X_2 Y_k^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{2 \times 2} & 0_{2 \times 2} & \cdots & \text{triu}(X_{k-1} Y_{k-1}^T) & X_{k-1} Y_k^T \\ 0_{2 \times 2} & 0_{2 \times 2} & \cdots & 0_{2 \times 2} & X_k Y_k^T \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} \text{triu}(X_1 Y_1^T) \\ 0_{2(k-1) \times 2} \end{bmatrix} & \begin{bmatrix} X_1 Y_2^T \\ \text{triu}(X_2 Y_2^T) \\ 0_{2(k-2) \times 2} \end{bmatrix} & \cdots & \begin{bmatrix} X_{1:k-2} Y_{k-1}^T \\ \text{triu}(X_{k-1} Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix} & X Y_k^T \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
U^{-1} &= \text{triu}(XY^T) \\
I &= U \text{triu}(XY^T)
\end{aligned}$$

$$E_i = [0 \quad \cdots \quad 0 \quad I_{2 \times 2} \quad 0 \quad \cdots \quad 0]^T$$

k -th block column,

$$\begin{aligned}
E_k &= U \text{triu}(XY^T) E_k \\
&= U X Y_k^T
\end{aligned}$$

$$\begin{aligned}
X &:= U^{-1} E_k \\
E_k &= U X Y_k^T \\
&= U U^{-1} E_k Y_k^T \\
&= E_k Y_k^T \\
Y_k^T &:= I_{2 \times 2}
\end{aligned}$$

$(k-1)$ -th block column,

$$\begin{aligned}
E_{k-1} &= U \text{triu}(XY^T) E_{k-1} \\
&= U \begin{bmatrix} X_{1:k-2} Y_{k-1}^T \\ \text{triu}(X_{k-1} Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix}
\end{aligned}$$

$(k-1)$ -th block row of $(k-1)$ -th block column,

$$E_i^T E_j = \begin{cases} I_{2 \times 2}, & \text{if } i = j ; \\ 0_{2 \times 2}, & \text{otherwise.} \end{cases}$$

$$\begin{aligned}
E_{k-1}^T E_{k-1} &= I_{2 \times 2} = E_{k-1}^T U \begin{bmatrix} X_{1:k-2} Y_{k-1}^T \\ \text{triu}(X_{k-1} Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix} \\
&= \begin{bmatrix} 0_{2 \times 2(k-2)} & D_{k-1} & B_{k-1} \end{bmatrix} \begin{bmatrix} X_{1:k-2} Y_{k-1}^T \\ \text{triu}(X_{k-1} Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix} \\
&= D_{k-1} \text{triu}(X_{k-1} Y_{k-1}^T) \\
Y_{k-1}^T &:= (D_{k-1} X_{k-1})^{-1}
\end{aligned}$$

$$\begin{aligned}
D_{k-1} &= \text{triu}(D_{k-1}) \\
D_{k-1}^{-1} &= \text{triu}(D_{k-1}^{-1}) \\
Y_{k-1}^T &:= (D_{k-1} X_{k-1})^{-1} \\
D_{k-1} \text{triu}(X_{k-1} Y_{k-1}^T) &= D_{k-1} \text{triu}(X_{k-1} (D_{k-1} X_{k-1})^{-1}) \\
&= D_{k-1} \text{triu}(X_{k-1} X_{k-1}^{-1} D_{k-1}^{-1}) \\
&= D_{k-1} \text{triu}(D_{k-1}^{-1}) \\
&= D_{k-1} D_{k-1}^{-1} \\
&= I_{2 \times 2}
\end{aligned}$$

$$\begin{aligned}
X &:= U^{-1} E_k \\
E_k &= U X \\
E_k^T E_k &= I_{2 \times 2} = E_k^T U X \\
&= \begin{bmatrix} 0_{2 \times 2(k-1)} & D_k \end{bmatrix} X \\
&= D_k X_k \\
Y_k^T &:= I_{2 \times 2} \\
&:= (D_k X_k)^{-1}
\end{aligned}$$

$$\begin{aligned}
X &:= U^{-1} E_k \\
Y_i^T &:= (D_i X_i)^{-1}
\end{aligned}$$

$$n = 2k + 1$$

$$U = \begin{bmatrix} [d_1] & \begin{bmatrix} b_1^1 & b_1^2 \\ d_2 & b_2^1 \\ & d_3 \end{bmatrix} & \begin{bmatrix} b_2^2 & \\ b_3^1 & b_3^2 \end{bmatrix} & & \\ & & \ddots & \ddots & \\ & & & \begin{bmatrix} d_{2k-2} & b_{2k-2}^1 \\ & d_{2k-1} \end{bmatrix} & \begin{bmatrix} b_{2k-2}^2 & \\ b_{2k-1}^1 & b_{2k-1}^2 \\ d_{2k} & b_{2k}^1 \\ & d_{2k+1} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(2k+1) \times (2k+1)}$$

$$X := U^{-1} E_{k+1}$$

$$Y_i^T := (D_i X_i)^{-1} \text{ for } i = 2, \dots, k+1$$

$$Y_1^T := (D_1 X_1)^+ = (D_1 X_1)^T (D_1 X_1 (D_1 X_1)^T)^{-1}$$

1.4 Banded matrix (Upper)

$$n = ku + r, \quad 0 \leq r < u$$

$$U = \begin{bmatrix} d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & \\ & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & \\ & & \ddots & \ddots & \ddots & \cdots & \ddots \\ & & & \ddots & \ddots & \ddots & b_{n-u}^u \\ & & & & \ddots & \ddots & \vdots \\ & & & & & \ddots & b_{n-2}^2 \\ & & & & & & \ddots & b_{n-1}^1 \\ & & & & & & & d_n \end{bmatrix}$$

$$r = 0, \quad n = ku,$$

$$U = \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{ku \times ku},$$

$$D_i \in \mathbb{R}^{u \times u}, \quad B_i \in \mathbb{R}^{u \times u} \text{ for } i = 1, 2, \dots, k.$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} \in \mathbb{R}^{ku \times u} \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{bmatrix} \in \mathbb{R}^{ku \times u} \quad Y^T = [Y_1^T \quad Y_2^T \quad \cdots \quad Y_k^T] \in \mathbb{R}^{u \times ku}$$

$$X_i, Y_i^T \in \mathbb{R}^{u \times u}, \text{ for } i = 1, 2, \dots, k$$

$$X := U^{-1} E_k$$

$$Y_i^T := (D_i X_i)^{-1}$$

$$r \neq 0, \quad n = ku + r,$$

$$U = \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times (ku+r)},$$

$$\tilde{D}_1 \in \mathbb{R}^{r \times r}, \quad \tilde{B}_1 \in \mathbb{R}^{r \times u};$$

$$D_i \in \mathbb{R}^{u \times u}, \quad B_i \in \mathbb{R}^{u \times u} \text{ for } i = 2, \dots, k+1.$$

$$X = \begin{bmatrix} \tilde{X}_1 \\ X_2 \\ \vdots \\ X_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times u} \quad Y = \begin{bmatrix} \tilde{Y}_1 \\ Y_2 \\ \vdots \\ Y_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times u} \quad Y^T = [\tilde{Y}_1^T \quad Y_2^T \quad \cdots \quad Y_{k+1}^T] \in \mathbb{R}^{u \times (ku+r)}$$

$$\begin{aligned}
&\tilde{X}_1 \in \mathbb{R}^{r \times u} \\
&\tilde{Y}_1^T \in \mathbb{R}^{u \times r} \\
&X_i, Y_i^T \in \mathbb{R}^{u \times u} \text{ for } i = 2, \dots, k+1 \\
&X := U^{-1}E_{k+1} \\
&\tilde{Y}_1^T := (D_1X_1)^+ = (D_1X_1)^T(D_1X_1(D_1X_1)^T)^{-1} \\
&\text{triu}(D_1X_1\tilde{Y}_1^T) = I_{r \times r} \\
&Y_i^T := (D_iX_i)^{-1} \text{ for } i = 2, \dots, k+1
\end{aligned}$$

2 Test

$$\begin{bmatrix} d_1 & \overbrace{b_1^1 & b_1^2 & \cdots & b_1^u}^{\text{bandwidth} = u} & & & & \\ d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & & & \\ & \ddots & \ddots & \ddots & \cdots & \ddots & & & \\ & & \ddots & \ddots & \ddots & \cdots & b_{n-u}^u & & \\ & & & \ddots & \ddots & \ddots & \vdots & & \\ & & & & \ddots & \ddots & b_{n-2}^2 & & \\ & & & & & \ddots & b_{n-1}^1 & & \\ & & & & & & d_n & & \end{bmatrix}$$

$$\left(\begin{array}{ccc|ccc} a & \cdots & a & b & \cdots & b \\ & \ddots & \vdots & \vdots & \ddots & \\ & & a & b & & \\ \hline 0 & & & c & \cdots & c \\ & & & \vdots & & \vdots \end{array} \right) \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} p$$

$$\left. \begin{array}{l} \\ \\ \\ \end{array} \right\} q$$

$$\overbrace{\underbrace{\quad\quad\quad}_{m} \quad \underbrace{\quad\quad\quad}_{m}}^{n \text{ columns}}$$

$$\overbrace{\left(\begin{array}{cccccc} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{array} \right)}^{n \text{ rows}}$$

$$\left(\begin{array}{ccc|ccc} 1 & \cdots & 1 & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & 0 & \cdots & 1 & \\ & & & \ddots & & \\ & & & & 1 & \\ \hline & & 1 & \cdots & 0 & \\ & & & \ddots & & \\ & & & & 1 & \\ \hline & & & & & 1 \end{array} \right) \begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \end{array}$$

$$\begin{array}{c} \uparrow \\ i \end{array} \quad \begin{array}{c} \uparrow \\ j \end{array}$$

$$\overbrace{\left(\begin{array}{cccccc} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{array} \right)}^{n \text{ columns}}$$

$$\overbrace{\left(\begin{array}{cccccc} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{array} \right)}^{n \text{ rows}}$$

$$\left(\begin{array}{cc|cc|cc} 1 & 2 & 0 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 7 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right)$$

$$\begin{aligned} & \left[\begin{array}{cc|cc|cc|cc|cc} d_1 & b_1^1 & b_1^2 & & & & & & & \\ & d_2 & b_2^1 & b_2^2 & & & & & & \\ \hline & & d_3 & b_3^1 & b_3^2 & & & & & \\ & & & d_4 & b_4^1 & b_4^2 & & & & \\ \hline & & & & \ddots & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & d_{2k-3} & b_{2k-3}^1 & b_{2k-3}^2 \\ & & & & & & & d_{2k-2} & b_{2k-2}^1 & b_{2k-2}^2 \\ \hline & & & & & & & & d_{2k-1} & b_{2k-1}^1 \\ & & & & & & & & & d_{2k} \end{array} \right] \\ &= \left[\begin{array}{cc|cc|cc|cc|cc} \left[\begin{array}{cc} d_1 & b_1^1 \\ & d_2 \end{array} \right] & \left[\begin{array}{cc} b_1^2 & \\ b_2^1 & b_2^2 \\ d_3 & b_3^1 \end{array} \right] & \left[\begin{array}{cc} b_3^2 & \\ b_4^1 & b_4^2 \end{array} \right] & & & & & & \\ & & \ddots & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & \left[\begin{array}{cc} d_{2k-3} & b_{2k-3}^1 \\ & d_{2k-2} \end{array} \right] & \left[\begin{array}{cc} b_{2k-3}^2 & \\ b_{2k-2}^1 & b_{2k-2}^2 \\ d_{2k-1} & b_{2k-1}^1 \end{array} \right] & & & \\ & & & & & & & & & \end{array} \right] \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 4 \\ 1 & 4 \end{bmatrix}$$

$$\begin{aligned} & \left[\begin{array}{cccc|cccc} d_1 & b_1^1 & \dots & b_1^{r-2} & b_1^{r-1} & & & \\ & d_2 & & & b_2^{r-2} & & & \\ & & \ddots & & \vdots & & & \\ & & & d_{r-1} & b_{r-1}^1 & & & \\ & & & & d_r & & & \end{array} \right] \left[\begin{array}{cccc|cccc} b_1^r & \dots & \dots & b_1^u & & & & \\ b_2^{r-1} & & & b_2^r & & & b_2^u & \\ \vdots & & & \vdots & & & \vdots & \\ b_{r-1}^2 & & & b_r^2 & \dots & \dots & b_r^u & \\ b_r^1 & & & & & & & \end{array} \right] \\ & \left[\begin{array}{cccc|cccc} d_1 & b_1^1 & \dots & b_1^{r-2} & b_1^{r-1} & & & \\ 0 & d_2 & & & b_2^{r-2} & & & \\ \vdots & \vdots & \ddots & \vdots & \vdots & & & \\ 0 & \dots & \dots & d_{r-1} & b_{r-1}^1 & & & \\ 0 & \dots & \dots & 0 & d_r & & & \end{array} \right] \end{aligned}$$

$$\begin{bmatrix} d_1 & b_1^1 \cdots \cdots b_1^{r-2} & b_1^{r-1} \\ & d_2 & b_2^{r-2} \\ & & \vdots \\ & & d_{r-1} & b_{r-1}^1 \\ & & & d_r \end{bmatrix} \begin{bmatrix} b_1^r & b_1^{r+1} \cdots \cdots b_1^{u-1} & b_1^u \\ b_2^{r-1} & & b_2^u \\ \vdots & & \vdots \\ b_{r-1}^2 & & b_{r-1}^u \\ b_r^1 & b_r^2 \cdots \cdots \cdots b_r^{u-1} & b_r^u \end{bmatrix}$$

$$\begin{bmatrix} d_1 \cdots \cdots b_1^{r-1} \\ \vdots \\ d_r \end{bmatrix} \begin{bmatrix} b_1^r \cdots \cdots b_1^u \\ \vdots \\ b_r^1 \cdots \cdots b_r^u \end{bmatrix} \begin{bmatrix} b_1^r \cdots \cdots b_1^u \\ \vdots \\ b_r^1 \cdots \cdots b_r^u \end{bmatrix} \begin{bmatrix} b_1^r \cdots \cdots b_1^u \\ \vdots \\ b_r^1 \cdots \cdots b_r^u \end{bmatrix}$$

$$\begin{bmatrix} d_1 \cdots \cdots b_1^{r-1} & b_1^r \cdots \cdots b_1^u \\ \vdots & \vdots \\ d_r & b_r^1 \cdots \cdots b_r^u \\ & d_{r+1} \cdots \cdots b_{r+1}^{u-1} \\ & \vdots \\ & d_{r+u} \end{bmatrix}$$