Meeting Notes

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1.1 Tridiagonal matrix (Upper)

$n = 2k$

$$
\begin{aligned}
U &= \begin{bmatrix}
d_1 & b_1^1 & b_1^2 & & & & & & \\
 & d_2 & b_2^1 & b_2^2 & & & & & \\
 & & d_3 & b_3^1 & b_3^2 & & & & \\
 & & & d_4 & b_4^1 & b_4^2 & & & \\
 & & & & \ddots & \ddots & \ddots & & \\
 & & & & & \ddots & \ddots & \ddots & \\
 & & & & & & d_{n-3} & b_{n-3}^1 & b_{n-3}^2 \\
 & & & & & & & d_{n-2} & b_{n-2}^1 & b_{n-2}^2 \\
 & & & & & & & & d_{n-1} & b_{n-1}^1 \\
 & & & & & & & & & d_n
\end{bmatrix} \\
&= \begin{bmatrix}
\begin{bmatrix} d_1 & b_1^1 \\ & d_2 \end{bmatrix} & \begin{bmatrix} b_1^2 & \\ b_2^1 & b_2^2 \end{bmatrix} & & & \\
 & \begin{bmatrix} d_3 & b_3^1 \\ & d_4 \end{bmatrix} & \begin{bmatrix} b_3^2 & \\ b_4^1 & b_4^2 \end{bmatrix} & & \\
 & & \ddots & \ddots & \\
 & & & \begin{bmatrix} d_{2k-3} & b_{2k-3}^1 \\ & d_{2k-2} \end{bmatrix} & \begin{bmatrix} b_{2k-3}^2 & \\ b_{2k-2}^1 & b_{2k-2}^2 \end{bmatrix} \\
 & & & & \begin{bmatrix} d_{2k-1} & b_{2k-1}^1 \\ & d_{2k} \end{bmatrix}
\end{bmatrix} \\
&= \begin{bmatrix}
D_1 & B_1 & & & \\
 & D_2 & B_2 & & \\
 & & \ddots & \ddots & \\
 & & & D_{k-1} & B_{k-1} \\
 & & & & D_k
\end{bmatrix} \in \mathbb{R}^{2k \times 2k}
\end{aligned}
$$

$$
X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{k-1} \\ Y_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \qquad Y^{\mathrm{T}} = \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{2 \times 2k}
$$

$$
\begin{aligned}
XY^{\mathrm{T}} &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix} X_1Y_1^{\mathrm{T}} & X_1Y_2^{\mathrm{T}} & \cdots & X_1Y_{k-1}^{\mathrm{T}} & x_1Y_k^{\mathrm{T}} \\ X_2Y_1^{\mathrm{T}} & X_2Y_2^{\mathrm{T}} & \cdots & X_2Y_{k-1}^{\mathrm{T}} & x_2Y_k^{\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{k-1}Y_1^{\mathrm{T}} & X_{k-1}Y_2^{\mathrm{T}} & \cdots & X_{k-1}Y_{k-1}^{\mathrm{T}} & X_{k-1}Y_k^{\mathrm{T}} \\ X_kY_1^{\mathrm{T}} & X_kY_2^{\mathrm{T}} & \cdots & X_kY_{k-1}^{\mathrm{T}} & X_kY_k^{\mathrm{T}} \end{bmatrix} \\
&= X\begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix} XY_1^{\mathrm{T}} & XY_2^{\mathrm{T}} & \cdots & XY_{k-1}^{\mathrm{T}} & XY_k^{\mathrm{T}} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{triu}(X_k) &\coloneqq X_k \\
\operatorname{triu}(Y_k^{\mathrm{T}}) &\coloneqq Y_k^{\mathrm{T}} \\
\operatorname{triu}(X_kY_k^{\mathrm{T}}) &= X_kY_k^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{triu}(XY^{\mathrm{T}}) &= \begin{bmatrix} \operatorname{triu}(X_1Y_1^{\mathrm{T}}) & X_1Y_2^{\mathrm{T}} & \cdots & X_1Y_{k-1}^{\mathrm{T}} & X_1Y_k^{\mathrm{T}} \\ 0_{2\times 2} & \operatorname{triu}(X_2Y_2^{\mathrm{T}}) & \cdots & X_2Y_{k-1}^{\mathrm{T}} & X_2Y_k^{\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{2\times 2} & 0_{2\times 2} & \cdots & \operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) & X_{k-1}Y_k^{\mathrm{T}} \\ 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & X_kY_k^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} \operatorname{triu}(X_1Y_1^{\mathrm{T}}) \\ 0_{2(k-1)\times 2} \end{bmatrix} & \begin{bmatrix} X_1Y_2^{\mathrm{T}} \\ \operatorname{triu}(X_2Y_2^{\mathrm{T}}) \\ 0_{2(k-2)\times 2} \end{bmatrix} & \cdots & \begin{bmatrix} X_{1:k-2}Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} & XY_k^{\mathrm{T}} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
U^{-1} &= \operatorname{triu}(XY^{\mathrm{T}}) \\
I &= U \operatorname{triu}(XY^{\mathrm{T}})
\end{aligned}
$$

$$
E_i = \begin{bmatrix} 0 & \cdots & 0 & I_{2\times 2} & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}
$$

k-th block column,

$$
\begin{aligned}
E_k &= U \operatorname{triu}(XY^{\mathrm{T}})E_k \\
&= UXY_k^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{aligned}
X &\coloneqq U^{-1}E_k \\
E_k &= UXY_k^{\mathrm{T}} \\
&= UU^{-1}E_kY_k^{\mathrm{T}} \\
&= E_kY_k^{\mathrm{T}} \\
Y_k^{\mathrm{T}} &\coloneqq I_{2\times 2}
\end{aligned}
$$

$(k-1)$-th block column,

$$\begin{aligned}
E_{k-1} &= U \operatorname{triu}(XY^{\mathrm{T}})E_{k-1} \\
&= U \begin{bmatrix} X_{1:k-2}Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix}
\end{aligned}$$

$(k-1)$-th block row of $(k-1)$-th block column,

$$E_i^{\mathrm{T}} E_j = \begin{cases} I_{2\times 2}, & \text{if } i = j \ ; \\ 0_{2\times 2}, & \text{otherwise.} \end{cases}$$

$$\begin{aligned}
E_{k-1}^{\mathrm{T}} E_{k-1} = I_{2\times 2} &= E_{k-1}^{\mathrm{T}} U \begin{bmatrix} X_{1:k-2}Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} \\
&= \begin{bmatrix} 0_{2\times 2(k-2)} & D_{k-1} & B_{k-1} \end{bmatrix} \begin{bmatrix} X_{1:k-2}Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix} \\
&= D_{k-1} \operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) \\
Y_{k-1}^{\mathrm{T}} &\coloneqq (D_{k-1}X_{k-1})^{-1}
\end{aligned}$$

$$\begin{aligned}
D_{k-1} &= \operatorname{triu}(D_{k-1}) \\
D_{k-1}^{-1} &= \operatorname{triu}(D_{k-1}^{-1}) \\
Y_{k-1}^{\mathrm{T}} &\coloneqq (D_{k-1}X_{k-1})^{-1} \\
D_{k-1} \operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) &= D_{k-1} \operatorname{triu}(X_{k-1}(D_{k-1}X_{k-1})^{-1}) \\
&= D_{k-1} \operatorname{triu}(X_{k-1}X_{k-1}^{-1}D_{k-1}^{-1}) \\
&= D_{k-1} \operatorname{triu}(D_{k-1}^{-1}) \\
&= D_{k-1}D_{k-1}^{-1} \\
&= I_{2\times 2}
\end{aligned}$$

$$\begin{aligned}
X &\coloneqq U^{-1}E_k \\
E_k &= UX \\
E_k^{\mathrm{T}} E_k = I_{2\times 2} &= E_k^{\mathrm{T}} U X \\
&= \begin{bmatrix} 0_{2\times 2(k-1)} & D_k \end{bmatrix} X \\
&= D_k X_k \\
Y_k^{\mathrm{T}} &\coloneqq I_{2\times 2} \\
&\coloneqq (D_k X_k)^{-1}
\end{aligned}$$

$$\begin{aligned}
X &\coloneqq U^{-1}E_k \\
Y_i^{\mathrm{T}} &\coloneqq (D_i X_i)^{-1}
\end{aligned}$$

$n = 2k+1$

$$U = \begin{bmatrix} \begin{bmatrix} d_1 \end{bmatrix} & \begin{bmatrix} b_1^1 & b_1^2 \end{bmatrix} & & & & \\ & \begin{bmatrix} d_2 & b_2^1 \\ & d_3 \end{bmatrix} & \begin{bmatrix} b_2^2 & \\ b_3^1 & b_3^2 \end{bmatrix} & & & \\ & & \ddots & \ddots & & \\ & & & \begin{bmatrix} d_{2k-2} & b_{2k-2}^1 \\ & d_{2k-1} \end{bmatrix} & \begin{bmatrix} b_{2k-2}^2 & \\ b_{2k-1}^1 & b_{2k-1}^2 \end{bmatrix} \\ & & & & \begin{bmatrix} d_{2k} & b_{2k}^1 \\ & d_{2k+1} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(2k+1)\times(2k+1)}$$

$$\begin{aligned} X &\coloneqq U^{-1} E_{k+1} \\ Y_i^{\mathrm{T}} &\coloneqq (D_i X_i)^{-1} \text{ for } i = 2,\ \cdots,\ k+1 \\ Y_1^{\mathrm{T}} &\coloneqq (D_1 X_1)^{+} = (D_1 X_1)^{\mathrm{T}} (D_1 X_1 (D_1 X_1)^{\mathrm{T}})^{-1} \end{aligned}$$

1.2 Banded matrix (Upper)

$n = ku + r,\ 0 \le r < u$

$$U = \begin{bmatrix} d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & & & \\ & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & & \\ & & \ddots & \ddots & \ddots & \cdots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \cdots & b_{n-u}^u \\ & & & & \ddots & \ddots & \ddots & \vdots \\ & & & & & \ddots & \ddots & b_{n-2}^2 \\ & & & & & & \ddots & b_{n-1}^1 \\ & & & & & & & d_n \end{bmatrix}$$

$r = 0,\ n = ku,$

$$U = \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{ku\times ku},$$

$$D_i \in \mathbb{R}^{u\times u},\ B_i \in \mathbb{R}^{u\times u} \text{ for } i = 1,\ 2,\ \cdots,\ k.$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} \in \mathbb{R}^{ku\times u} \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{bmatrix} \in \mathbb{R}^{ku\times u} \qquad Y^{\mathrm{T}} = \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_k^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{u\times ku}$$

$$\begin{aligned} X_i,\ Y_i^{\mathrm{T}} &\in \mathbb{R}^{u\times u},\ \text{for } i = 1,\ 2,\ \cdots,\ k \\ X &:= U^{-1}E_k \\ Y_i^{\mathrm{T}} &:= (D_i X_i)^{-1} \end{aligned}$$

$r \ne 0,\ n = ku + r,$

$$U = \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r)\times(ku+r)},$$

$$\begin{aligned} &\tilde{D}_1 \in \mathbb{R}^{r\times r},\ \tilde{B}_1 \in \mathbb{R}^{r\times u}; \\ &D_i \in \mathbb{R}^{u\times u},\ B_i \in \mathbb{R}^{u\times u} \text{ for } i = 2,\ \cdots,\ k+1. \end{aligned}$$

$$X = \begin{bmatrix} \tilde{X}_1 \\ X_2 \\ \vdots \\ X_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r)\times u} \quad Y = \begin{bmatrix} \tilde{Y}_1 \\ Y_2 \\ \vdots \\ Y_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r)\times u} \quad Y^{\mathrm{T}} = \begin{bmatrix} \tilde{Y}_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k+1}^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{u\times(ku+r)}$$

$$
\begin{aligned}
\tilde{X}_1 &\in \mathbb{R}^{r\times u} \\
\tilde{Y}_1^{\mathrm{T}} &\in \mathbb{R}^{u\times r} \\
X_i,\ Y_i^{\mathrm{T}} &\in \mathbb{R}^{u\times u} \text{ for } i = 2,\ \cdots,\ k+1 \\
X &\coloneqq U^{-1}E_{k+1} \\
\tilde{Y}_1^{\mathrm{T}} &\coloneqq (D_1X_1)^{+} = (D_1X_1)^{\mathrm{T}}(D_1X_1(D_1X_1)^{\mathrm{T}})^{-1} \\
\mathrm{triu}(D_1X_1\tilde{Y}_1^{\mathrm{T}}) &= I_{r\times r} \\
Y_i^{\mathrm{T}} &\coloneqq (D_iX_i)^{-1} \text{ for } i = 2,\ \cdots,\ k+1
\end{aligned}
$$

2 28.06.2022 Meeting formal

2.1 Notation

This notes use Householder notation and start indexing at one.

$$\begin{aligned} \text{Scalars, lower case Greek letters: } & \alpha,\ \beta,\ \gamma,\ \chi,\ etc. \\ \text{Vectors, lower case Roman letter: } & a,\ b,\ c,\ x,\ etc. \\ \text{Matrix, upper case Roman letter: } & A,\ B,\ C,\ X,\ etc. \end{aligned}$$

lower case Greek letters:	α, β, γ, χ, *etc.*	for Scalars;
lower case Roman letter:	a, b, c, x, *etc.*	for Vectors;
upper case Roman letter:	A, B, C, X, *etc.*	for Matrix.

2.2 Repersentation

Matrix & Vector

$$x = \left[\begin{array}{c} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_n \end{array}\right] \in \mathbb{R}^n$$

$$A = \left[\begin{array}{c|c|c|c} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{1n} \end{array}\right] = \left[\begin{array}{c|c|c|c} a_1 & a_2 & \cdots & a_n \end{array}\right] \in \mathbb{R}^{m\times n}$$

Block Matrix & Block Vector

$$x = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right] \in \mathbb{R}^{kn},\ x_i \in \mathbb{R}^k$$

$$A = \left[\begin{array}{c|c|c|c} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{1n} \end{array}\right] = \left[\begin{array}{c|c|c|c} A_1 & A_2 & \cdots & A_n \end{array}\right] \in \mathbb{R}^{lm\times kn},\ A_{ij} \in \mathbb{R}^{l\times k}$$

2.3 Algorithm

$$U^{-1} = \operatorname{triu}(XY^{\mathrm{T}})$$

For $U \in \mathbb{R}^{n\times n}$,
if $n = ku$

$$X \coloneqq U^{-1}E_k;$$
$$Y_i^{\mathrm{T}} \coloneqq (D_iX_i)^{-1} \text{ for } i = 1,\, 2,\, \ldots,\, k.$$

if $n = ku + r$

$$X \coloneqq U^{-1}E_k;$$
$$Y_i^{\mathrm{T}} \coloneqq (D_iX_i)^{-1} \text{ for } i = 2,\, \ldots,\, k+1;$$
$$Y_1^{\mathrm{T}} \coloneqq (D_1X_1)^{+}$$

2.3.1 Example

Let

$$U = \begin{bmatrix} D_1 & B_1 & \\ & D_2 & B_2 \\ & & D_3 \end{bmatrix} \in \mathbb{R}^{(2u+r)\times(2u+r)}$$

where $D_1 \in \mathbb{R}^{r\times r}$, $B_1 \in \mathbb{R}^{r\times u}$, and D_2, B_2, D_3, B_3, $\in \mathbb{R}^{u\times u}$.
By definitions, $X \coloneqq U^{-1}E_3$ or $UX = E_3$, that is:

$$\begin{array}{c c} & \begin{array}{ccc} r & u & u \end{array} \\ \begin{array}{c} r \\ u \\ u \end{array} & \left[\begin{array}{ccc} D_1 & B_1 & 0 \\ \hline 0 & D_2 & B_2 \\ \hline 0 & 0 & D_3 \end{array}\right] \end{array} \begin{array}{c c} & u \\ \begin{array}{c} r \\ u \\ u \end{array} & \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \end{array} = \begin{array}{c c} & u \\ \begin{array}{c} r \\ u \\ u \end{array} & \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \end{array}.$$

Using back-subsitution to find X as following:

$$\left[\begin{array}{c|c|c} 0 & 0 & D_3 \end{array}\right] \left[\begin{array}{c} X_1 \\ \hline X_2 \\ \hline X_3 \end{array}\right] = D_3X_3 = I_{u\times u} \qquad X_3 = D_3^{-1}$$

$$\left[\begin{array}{c|c|c} 0 & D_2 & B_2 \end{array}\right] \left[\begin{array}{c} X_1 \\ \hline X_2 \\ \hline X_3 \end{array}\right] = D_2X_2 + B_2X_3 = 0_{u\times u} \qquad X_2 = -D_2^{-1}B_2X_3 = -D_2^{-1}B_2D_3^{-1}$$

$$\left[\begin{array}{c|c|c} D_1 & B_1 & 0 \end{array}\right] \left[\begin{array}{c} X_1 \\ \hline X_2 \\ \hline X_3 \end{array}\right] = D_1X_1 + B_1X_2 = 0_{r\times u} \qquad X_1 = -D_1^{-1}B_1X_2 = D_1^{-1}B_1D_2^{-1}B_2D_3^{-1}$$

Following the definition of Y_i^{T},

$$Y_3^{\mathrm{T}} = (D_3X_3)^{-1} = X_3^{-1}D_3^{-1} = D_3D_3^{-1} = I_{u\times u}$$
$$Y_2^{\mathrm{T}} = (D_2X_2)^{-1} = X_2^{-1}D_2^{-1} = -D_3B_2^{-1}D_2D_2^{-1} = -D_3B_2^{-1}$$
$$Y_1^{\mathrm{T}} = (D_1X_1)^{+} = X_1^{+}D_1^{+}$$

Note that X_3 and Y_3^{T} are upper triangular matrices, then $\operatorname{triu}(X_3Y_3^{\mathrm{T}}) = X_3Y_3^{\mathrm{T}}$.

$$
\begin{aligned}
U\operatorname{triu}(XY^{\mathrm{T}}) &= \left[\begin{array}{ccc} D_1 & B_1 & 0 \\ \hline 0 & D_2 & B_2 \\ \hline 0 & 0 & D_3 \end{array}\right]\left[\begin{array}{c|c|c} \operatorname{triu}(X_1Y_1^{\mathrm{T}}) & X_1Y_2^{\mathrm{T}} & X_1Y_3^{\mathrm{T}} \\ 0 & \operatorname{triu}(X_2Y_2^{\mathrm{T}}) & X_2Y_3^{\mathrm{T}} \\ 0 & 0 & X_3Y_3^{\mathrm{T}} \end{array}\right] \\
&= \left[\begin{array}{ccc} D_1 & B_1 & 0 \\ \hline 0 & D_2 & B_2 \\ \hline 0 & 0 & D_3 \end{array}\right]\left[\begin{array}{c|c|c} \operatorname{triu}(D_1^{+}) & -D_1^{-1}B_1D_2^{-1} & D_1^{-1}B_1D_2^{-1}B_2D_3^{-1} \\ 0 & \operatorname{triu}(D_2^{-1}) & -D_2^{-1}B_2D_3^{-1} \\ 0 & 0 & D_3^{-1} \end{array}\right] \\
&= \left[\begin{array}{ccc} D_1 & B_1 & 0 \\ \hline 0 & D_2 & B_2 \\ \hline 0 & 0 & D_3 \end{array}\right]\left[\begin{array}{c|c|c} D_1^{-1} & -D_1^{-1}B_1D_2^{-1} & D_1^{-1}B_1D_2^{-1}B_2D_3^{-1} \\ 0 & D_2^{-1} & -D_2^{-1}B_2D_3^{-1} \\ 0 & 0 & D_3^{-1} \end{array}\right] \\
&= \left[\begin{array}{c|c|c} D_1D_1^{-1} & -B_1D_2^{-1} + B_1D_2^{-1} & B_1D_2^{-1}B_2D_3^{-1} - B_1D_2^{-1}B_2D_3^{-1} \\ 0 & D_2D_2^{-1} & -B_2D_3^{-1} + B_2D_3^{-1} \\ 0 & 0 & D_3D_3^{-1} \end{array}\right] \\
&= \left[\begin{array}{c|c|c} I_{r\times r} & 0 & 0 \\ 0 & I_{u\times u} & 0 \\ 0 & 0 & I_{u\times u} \end{array}\right] \\
&= I_{n\times n}
\end{aligned}
$$

2.4 Content

Consider an upper banded matrix $U \in \mathbb{R}^{n\times n}$,

$$
U = \begin{bmatrix}
\alpha_1 & \beta_1^1 & \beta_1^2 & \cdots & \beta_1^u & & \\
& \alpha_2 & \beta_2^1 & \beta_2^2 & \cdots & \beta_2^u & \\
& & \ddots & \ddots & & \ddots & \\
& & & & & & \beta_{n-u}^u \\
& & & & & & \vdots \\
& & & & \ddots & \ddots & \beta_{n-2}^2 \\
& & & & & \ddots & \beta_{n-1}^1 \\
& & & & & & \alpha_n
\end{bmatrix}. \tag{1}
$$

Then try to show that its inverse can be repersent as the upper triangular part of an outer product, *i.e.* ,

$$
U^{-1} = \mathrm{triu}(XY^{\mathrm{T}}) \tag{2}
$$

Let $X,\ Y \in \mathbb{R}^{n\times u}$, consider the following cases of n,

Case 1, $n = ku$

then (1) can be repersent as a block upper bidiagonal matrix,

$$
\begin{array}{c c}
& \begin{array}{cccc} \overset{u}{\longleftrightarrow} & \overset{u}{\longleftrightarrow} & \cdots\cdots & \overset{u}{\longleftrightarrow} \end{array} \\
U = \begin{array}{c} u\updownarrow \\ u\updownarrow \\ \vdots \\ u\updownarrow \end{array} &
\left[\begin{array}{cccccccccc}
\alpha_1 & \cdots & \beta_1^{u-1} & \beta_1^u & & & & & & \\
& \ddots & \vdots & \vdots & \ddots & & & & & \\
& & \alpha_u & \beta_u^1 & \cdots & \beta_u^u & & & & \\
& & & \alpha_{u+1} & \cdots & \beta_{u+1}^{u-1} & \beta_{u+1}^u & & & \\
& & & & \ddots & \vdots & \vdots & \ddots & & \\
& & & & & \alpha_{2u} & \beta_{2u}^1 & \cdots & \beta_{2u}^u & \\
& & & & & & \ddots & & \ddots & \\
& & & & & & & \alpha_{(k-1)u+1} & \cdots & \beta_{(k-1)u+1}^{u-1} \\
& & & & & & & & \ddots & \vdots \\
& & & & & & & & & \alpha_{ku}
\end{array}\right]
\end{array}
$$

$$
= \begin{bmatrix}
D_1 & B_1 & & & \\
& D_2 & B_2 & & \\
& & \ddots & \ddots & \\
& & & D_{k-1} & B_{k-1} \\
& & & & D_k
\end{bmatrix} \in \mathbb{R}^{ku\times ku},\ D_i,\ B_i \in \mathbb{R}^{u\times u} \text{ for } i = 1,\ 2,\ \ldots,\ k. \tag{3}
$$

Repersent the n-dimensional identity matrix $I_{n\times n}$ in block form,

$$
I_{n\times n} = I_{ku\times ku} = \mathrm{diag}(\underbrace{I_{u\times u},\ I_{u\times u},\ \cdots,\ I_{u\times u}}_{k}) = \left[\ E_1 \mid E_2 \mid \cdots \mid E_k\ \right].
$$

then, its block columns E_i for $i = 1,\ 2,\ \ldots,\ k$ satisfies

$$
E_i \in \mathbb{R}^{ku\times u}
$$
$$
E_i^{\mathrm{T}} E_j = \begin{cases} I_{u\times u}, & \text{if } i = j \ ; \\ 0_{u\times u}, & \text{otherwise.} \end{cases} .
$$

To find X and Y such that (2) holds, let

$$
X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \in \mathbb{R}^{ku\times u},\ Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{k-1} \\ Y_k \end{bmatrix} \in \mathbb{R}^{ku\times u},\ X_i,\ Y_i \in \mathbb{R}^{u\times u} \text{ for } i = 1,\ 2,\ \ldots,\ k,
$$

and let X_k and Y_k^{T} be upper triangular matrix, then $\mathrm{triu}(X_k Y_k^{\mathrm{T}}) = X_k Y_k^{\mathrm{T}}$.
For the outer product form,

$$
\begin{aligned}
Y^{\mathrm{T}} &= \left[\begin{array}{c|c|c|c|c} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{array}\right] \in \mathbb{R}^{u\times ku} \\
XY^{\mathrm{T}} &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \left[\begin{array}{c|c|c|c|c} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{array}\right] \\
&= \left[\begin{array}{c|c|c|c|c} X_1Y_1^{\mathrm{T}} & X_1Y_2^{\mathrm{T}} & \cdots & X_1Y_{k-1}^{\mathrm{T}} & x_1Y_k^{\mathrm{T}} \\ X_2Y_1^{\mathrm{T}} & X_2Y_2^{\mathrm{T}} & \cdots & X_2Y_{k-1}^{\mathrm{T}} & x_2Y_k^{\mathrm{T}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{k-1}Y_1^{\mathrm{T}} & X_{k-1}Y_2^{\mathrm{T}} & \cdots & X_{k-1}Y_{k-1}^{\mathrm{T}} & X_{k-1}Y_k^{\mathrm{T}} \\ X_kY_1^{\mathrm{T}} & X_kY_2^{\mathrm{T}} & \cdots & X_kY_{k-1}^{\mathrm{T}} & X_kY_k^{\mathrm{T}} \end{array}\right] \\
&= X\left[\begin{array}{c|c|c|c|c} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{array}\right] \\
&= \left[\begin{array}{c|c|c|c|c} XY_1^{\mathrm{T}} & XY_2^{\mathrm{T}} & \cdots & XY_{k-1}^{\mathrm{T}} & XY_k^{\mathrm{T}} \end{array}\right] \in \mathbb{R}^{ku\times ku},
\end{aligned}
$$

and its the upper triangular componnent,

$$
\begin{aligned}
\mathrm{triu}(XY^{\mathrm{T}}) &= \left[\begin{array}{c|c|c|c|c} \mathrm{triu}(X_1Y_1^{\mathrm{T}}) & X_1Y_2^{\mathrm{T}} & \cdots & X_1Y_{k-1}^{\mathrm{T}} & X_1Y_k^{\mathrm{T}} \\ 0_{u\times u} & \mathrm{triu}(X_2Y_2^{\mathrm{T}}) & \cdots & X_2Y_{k-1}^{\mathrm{T}} & X_2Y_k^{\mathrm{T}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{u\times u} & 0_{u\times u} & \cdots & \mathrm{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) & X_{k-1}Y_k^{\mathrm{T}} \\ 0_{u\times u} & 0_{u\times u} & \cdots & 0_{u\times u} & X_kY_k^{\mathrm{T}} \end{array}\right] \\
&= \left[\begin{array}{c|c|c|c|c} \mathrm{triu}(X_1Y_1^{\mathrm{T}}) & X_1Y_2^{\mathrm{T}} & & X_{1:k-2}Y_{k-1}^{\mathrm{T}} & \\ 0_{(k-2)u\times u} & \mathrm{triu}(X_2Y_2^{\mathrm{T}}) & \cdots & \mathrm{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) & XY_k^{\mathrm{T}} \\ 0_{u\times u} & 0_{(k-2)u\times u} & & 0_{u\times u} & \end{array}\right].
\end{aligned} \tag{4}
$$

From (2),

$$
I_{n\times n} = U\,\mathrm{triu}(XY^{\mathrm{T}}). \tag{5}
$$

For X and Y_k, from (4), the k-th block column of (5) can be written as,

$$
\begin{aligned}
E_k &= U \operatorname{triu}(XY^{\mathrm{T}}) E_k \\
E_k &= UXY_k^{\mathrm{T}}
\end{aligned} \tag{6}
$$

Define that

$$
X \coloneqq U^{-1} E_k; \tag{7}
$$

$$
Y_k^{\mathrm{T}} \coloneqq I_{u\times u}, \tag{8}
$$

which can be shown that (6) holds for the defined (7) and (8) through

$$
\begin{aligned}
UXY_k^{\mathrm{T}} &= UU^{-1}E_k I_{u\times u} \\
&= I_{n\times n} E_k I_{u\times u} \\
&= E_k I_{u\times u} \\
&= E_k, \text{ as required.}
\end{aligned}
$$

Then, for Y_{k-1}, forming the $(k-1)$-th block column of (5) from (4) first,

$$
\begin{aligned}
E_{k-1} &= U \operatorname{triu}(XY^{\mathrm{T}}) E_{k-1} \\
&= U \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{u\times u} \end{bmatrix}.
\end{aligned}
$$

Focusing on the $(k-1)$-th block row of $(k-1)$-th block column,

$$
\begin{aligned}
E_{k-1}^{\mathrm{T}} E_{k-1} &= E_{k-1}^{\mathrm{T}} U \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{u\times u} \end{bmatrix} \\
I_{u\times u} &= \left[\begin{array}{c|c|c} 0_{u\times(k-2)u} & D_{k-1} & B_{k-1} \end{array}\right] \left[\begin{array}{c} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \hline \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ \hline 0_{u\times u} \end{array}\right] \\
I_{u\times u} &= D_{k-1} \operatorname{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}).
\end{aligned} \tag{9}
$$

Define that

$$
Y_{k-1}^{\mathrm{T}} \coloneqq (D_{k-1} X_{k-1})^{-1}, \tag{10}
$$

and notice that $D_{k-1} \in \mathbb{R}^{u\times u}$ is a upper triangular, therefore

$$
\begin{aligned}
D_{k-1} &= \operatorname{triu}(D_{k-1}); \\
D_{k-1}^{-1} &= \operatorname{triu}(D_{k-1}^{-1}).
\end{aligned} \tag{11}
$$

With the fact (11), (9) holds for the defined (10) as,

$$
\begin{aligned}
D_{k-1}\operatorname{triu}(X_{k-1}Y_{k-1}^{\mathrm{T}}) &= D_{k-1}\operatorname{triu}(X_{k-1}(D_{k-1}X_{k-1})^{-1}) \\
&= D_{k-1}\operatorname{triu}(X_{k-1}X_{k-1}^{-1}D_{k-1}^{-1}) \\
&= D_{k-1}\operatorname{triu}(D_{k-1}^{-1}) \\
&= D_{k-1}D_{k-1}^{-1} \\
&= I_{u\times u}, \text{ as required.}
\end{aligned}
$$

Similar ideas can be use to redefine Y_k^{T} in (8), from (7),

$$
\begin{aligned}
E_k &= UX \\
E_k^{\mathrm{T}}E_k &= E_k^{\mathrm{T}}UX \\
I_{u\times u} &= E_k^{\mathrm{T}}UX \\
&= \left[\begin{array}{c|c} 0_{u\times(k-1)u} & D_k \end{array}\right]\left[\begin{array}{c} X_{1:k-1} \\ \hline X_k \end{array}\right] \\
&= D_kX_k \\
I_{u\times u} &= (D_kX_k)^{-1},
\end{aligned}
$$

following that

$$
\begin{aligned}
Y_k^{\mathrm{T}} &\coloneqq I_{u\times u} \\
&\coloneqq (D_kX_k)^{-1}.
\end{aligned}
$$

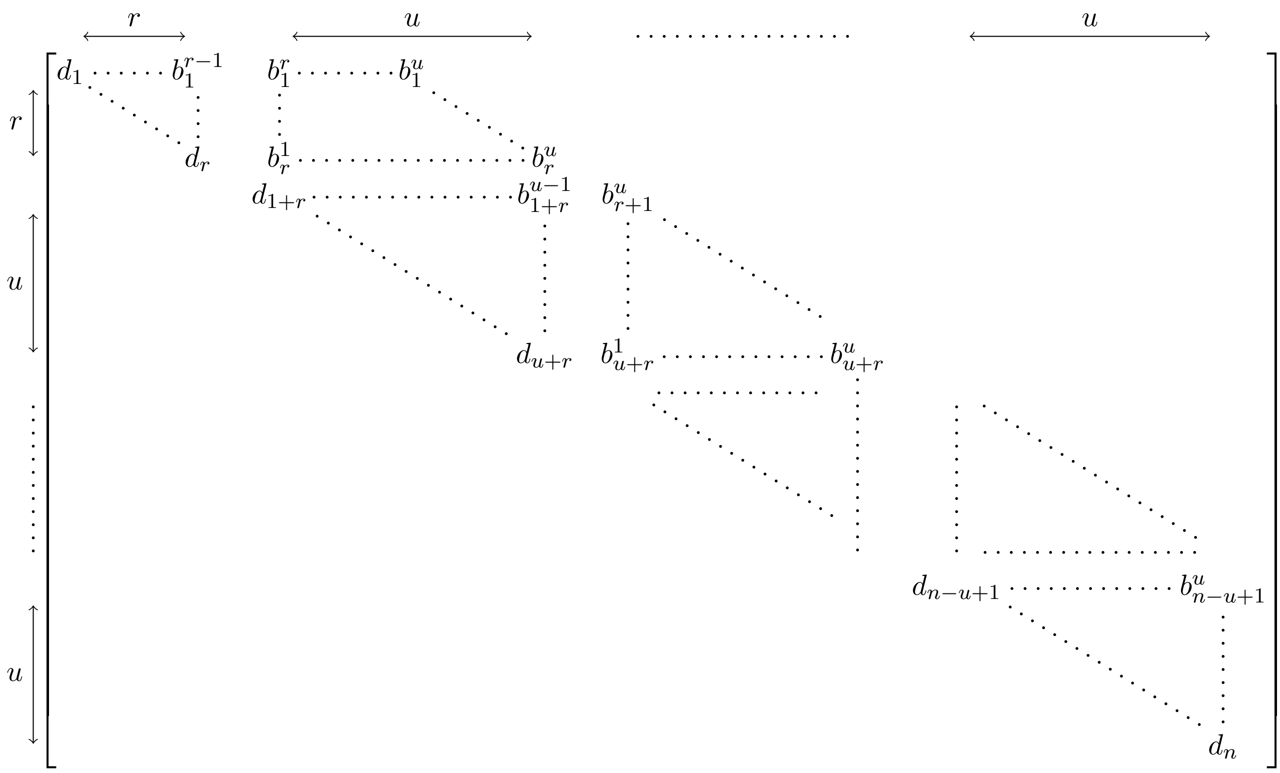
In summary, for $U \in \mathbb{R}^{ku\times ku}$, (2) holds the following definitions of X and Y:

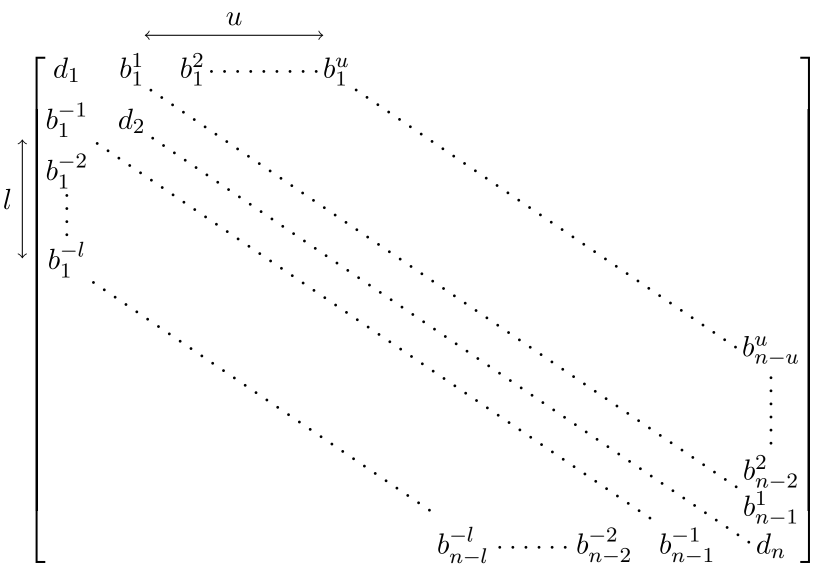
$$
\begin{aligned}
X &\coloneqq U^{-1}E_k \\
Y_i^{\mathrm{T}} &\coloneqq (D_iX_i)^{-1}
\end{aligned}
$$

3 Test

$$
\begin{array}{c}
\qquad\qquad\overbrace{\hspace{4.5cm}}^{\text{bandwidth} = u} \\
\begin{bmatrix}
d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & & \\
 & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & \\
 & & \ddots & \ddots & \ddots & \cdots & \ddots & \\
 & & & \ddots & \ddots & \ddots & \cdots & b_{n-u}^u \\
 & & & & \ddots & \ddots & \ddots & \vdots \\
 & & & & & \ddots & \ddots & b_{n-2}^2 \\
 & & & & & & \ddots & b_{n-1}^1 \\
 & & & & & & & d_n
\end{bmatrix}
\end{array}
$$

$$
\begin{bmatrix}
d_1 & b_1^1 & b_1^2 & & & & & & & \\
 & d_2 & b_2^1 & b_2^2 & & & & & & \\
 & & d_3 & b_3^1 & b_3^2 & & & & & \\
 & & & d_4 & b_4^1 & b_4^2 & & & & \\
 & & & & \ddots & & \ddots & & & \\
 & & & & & \ddots & & \ddots & & \\
 & & & & & & d_{2k-3} & b_{2k-3}^1 & b_{2k-3}^2 & \\
 & & & & & & & d_{2k-2} & b_{2k-2}^1 & b_{2k-2}^2 \\
 & & & & & & & & d_{2k-1} & b_{2k-1}^1 \\
 & & & & & & & & & d_{2k}
\end{bmatrix}
$$





Input : An upper banded matrix $U \in \mathbb{R}^{n\times n}$ and its bandwidth u.
Output: Its inverse U^{-1} and $X, Y \in \mathbb{R}^{n\times u}$ such that $U^{-1} = \mathrm{triu}(XY^{\mathrm{T}})$.

`// n = ku + r where 0 <= r < u`
$k \leftarrow$ quotient of n divided by u ; `// floor(n/u)`
$r \leftarrow$ remainder of n divided by u ; `// rem(n,u)`
$E_k \leftarrow$ last block column of $I_{n\times n}$; `// one(U)[:, n-u+1:n]`
$X \leftarrow$ `BackSubstitution`$(U,\ E_k)$
for $i \leftarrow k$ **to** 1 **do**
  $D_i \leftarrow i$-th diagonal block of U ; `// U[(i-1)*u+r+1:i*u+r, (i-1)*u+r+1:i*u+r]`
  $X_i \leftarrow i$-th block of X ; `// X[(i-1)*u+r+1:i*u+r, :]`
  $Y_i^{\mathrm{T}} \leftarrow (D_i X_i)^{-1}$, i-th block of Y^{T} ; `// Yt[:,(i-1)*u+r+1:i*u+r]`
end
if $r = 0$ **then return** $U^{-1} := \mathrm{triu}(XY^{\mathrm{T}})$, X, Y
else need find Y_0^{T} using Moore-Penrose right inverse
  $D_0 \in \mathbb{R}^{r\times r} \leftarrow$ first diagonal block of U ; `// U[1:r, 1:r]`
  $X_0 \in \mathbb{R}^{r\times u} \leftarrow$ first block of X ; `// X[1:r, :]`
  $Y_0^{\mathrm{T}} \in \mathbb{R}^{u\times r} \leftarrow (D_0X_0)^+ = (D_0X_0)^{\mathrm{T}}(D_0X_0(D_0X_0)^{\mathrm{T}})^{-1}$, first block of Y^{T}
end
return $U^{-1} := \mathrm{triu}(XY^{\mathrm{T}})$, X, Y

Algorithm 1: Inverse of banded matrix in outer product form