MSc Project Notes

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1.1 Bidiagonal matrix

Let $\tilde{U} \in \mathbb{R}^{n \times n}$ be a bidiagonal matrix with non-zero diagonal, *i.e.* $\tilde{u}_{ii} \neq 0$ for $i = 1, 2, \ldots, n$,

$$\tilde{U} = \begin{bmatrix} \tilde{u}_{11} & \tilde{u}_{12} & & & \\ & \tilde{u}_{22} & \tilde{u}_{23} & & \\ & & \tilde{u}_{33} & \tilde{u}_{34} & \\ & & & \tilde{u}_{44} & \tilde{u}_{45} \\ & & & & \tilde{u}_{55} \end{bmatrix}$$

and $D = \operatorname{diag}(\tilde{u}_{ii})$ for $i = 1, 2, \ldots, n$,

$$D = \begin{bmatrix} \frac{1}{\tilde{u}_{11}} & & & & \\ & \frac{1}{\tilde{u}_{22}} & & & \\ & & \frac{1}{\tilde{u}_{33}} & & \\ & & & \frac{1}{\tilde{u}_{44}} & \\ & & & & \frac{1}{\tilde{u}_{55}} \end{bmatrix}$$

replace $\frac{\tilde{u}_{i,i+1}}{\tilde{u}_{i+1,i+1}}$ by $u_{i,i+1}$ for $i = 1, 2, \ldots, n-1$, and let $U = \tilde{U}D$,

$$U = \begin{bmatrix} 1 & \frac{\tilde{u}_{12}}{\tilde{u}_{22}} & & & \\ & 1 & \frac{\tilde{u}_{23}}{\tilde{u}_{33}} & & \\ & & 1 & \frac{\tilde{u}_{34}}{\tilde{u}_{44}} & \\ & & & 1 & \frac{\tilde{u}_{45}}{\tilde{u}_{55}} \\ & & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & u_{12} & & & \\ & 1 & u_{23} & & \\ & & 1 & u_{34} & \\ & & & 1 & u_{45} \\ & & & & 1 \end{bmatrix}.$$

Let $A = U^{-1}$ be the inverse of U, a upper triangular matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ & a_{22} & a_{23} & a_{24} & a_{25} \\ & & a_{33} & a_{34} & a_{35} \\ & & & a_{44} & a_{45} \\ & & & & a_{55} \end{bmatrix}$$

then $I = UU^{-1} = UA$,

$$I = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12}+a_{22}u_{12} & a_{13}+a_{23}u_{12} & a_{14}+a_{24}u_{12} & a_{15}+a_{25}u_{12} \\ & a_{22} & a_{23}+a_{33}u_{23} & a_{24}+a_{34}u_{23} & a_{25}+a_{35}u_{23} \\ & & a_{33} & a_{34}+a_{44}u_{34} & a_{35}+a_{45}u_{34} \\ & & & a_{44} & a_{45}+a_{55}u_{45} \\ & & & & a_{55} \end{bmatrix} = UA.$$

Note that $UA = \operatorname{diag}(a_{ii}) + \operatorname{triu}(UA,1)$ for $i = 1, 2, \ldots, n$.
Find A such that $\operatorname{diag}(a_{ii}) = I$ and $\operatorname{triu}(UA,1) = 0_{n \times n}$ for $i = 1, 2, \ldots, n$.

$$xy^{\mathrm{T}} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 & x_1 y_4 & x_1 y_5 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 & x_2 y_4 & x_2 y_5 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 & x_3 y_4 & x_3 y_5 \\ x_4 y_1 & x_4 y_2 & x_4 y_3 & x_4 y_4 & x_4 y_5 \\ x_5 y_1 & x_5 y_2 & x_5 y_3 & x_5 y_4 & x_5 y_5 \end{bmatrix}$$

Express A in form of an outer product, *i.e.* $A = xy^{\mathrm{T}}$ for $\forall x, y \in \mathbb{R}^n$, then

$$\operatorname{triu}(UA,1) = \operatorname{triu}(Uxy^{\mathrm{T}},1) = 0_{n \times n}$$

Note that $\operatorname{triu}(e_n y^{\mathrm{T}},1) = 0_{n \times n}$ as,

$$e_n y^{\mathrm{T}} = \begin{bmatrix} \\ \\ \\ \\ 1 \end{bmatrix} \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix}$$

solve $Ux := e_n$ for x by backward substitution, then let $y_i := \frac{1}{x_i}$ or $x_i y_i = 1$ for $i = 1, 2, \ldots, n$.
Then,

$$\begin{aligned} UA &= \operatorname{diag}(a_{ii}) + \operatorname{triu}(UA,1) \\ &= \operatorname{diag}(x_i y_i) + \operatorname{triu}(Uxy^{\mathrm{T}},1) \\ &= \operatorname{diag}(\underbrace{1, 1, \cdots, 1}_{n}) + \operatorname{triu}(e_n y^{\mathrm{T}},1) \\ &= I + 0_{n \times n} \\ &= I \end{aligned}$$

1.2 Upper triangular matrix

Let $\tilde{U} \in \mathbb{R}^{n \times n}$ be a banded matrix with non-zero diagonal, upper banded width u, and lower banded width 0 Similarly,

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} & & \\ & 1 & u_{23} & u_{24} & \\ & & 1 & u_{34} & u_{35} \\ & & & 1 & u_{45} \\ & & & & 1 \end{bmatrix}$$

$$UA = \begin{bmatrix} a_{11} & a_{12} + a_{22}u_{12} & a_{13} + a_{23}u_{12} + a_{33}u_{13} & a_{14} + a_{24}u_{12} + a_{34}u_{13} & a_{15} + a_{25}u_{12} + a_{35}u_{13} \\ & a_{22} & a_{23} + a_{33}u_{23} & a_{24} + a_{34}u_{23} + a_{44}u_{24} & a_{25} + a_{35}u_{23} + a_{45}u_{24} \\ & & a_{33} & a_{34} + a_{44}u_{34} & a_{35} + a_{45}u_{34} + a_{55}u_{35} \\ & & & a_{44} & a_{45} + a_{55}u_{45} \\ & & & & a_{55} \end{bmatrix}$$

$$= \operatorname{diag}(a_{ii}) + \operatorname{triu}(UA, 1) \text{ for } i = 1, 2, \ldots, n$$

Find A such that $\operatorname{diag}(a_{ii}) = I$ and $\operatorname{triu}(UA, 1) = 0_{n \times n}$ for $i = 1, 2, \ldots, n$.

$$XY^{\mathrm{T}} = \begin{bmatrix} x_{11}y_{11} + x_{12}y_{12} & x_{11}y_{21} + x_{12}y_{22} & x_{11}y_{31} + x_{12}y_{32} & x_{11}y_{41} + x_{12}y_{42} & x_{11}y_{51} + x_{12}y_{52} \\ x_{21}y_{11} + x_{22}y_{12} & x_{21}y_{21} + x_{22}y_{22} & x_{21}y_{31} + x_{22}y_{32} & x_{21}y_{41} + x_{22}y_{42} & x_{21}y_{51} + x_{22}y_{52} \\ x_{31}y_{11} + x_{32}y_{12} & x_{31}y_{21} + x_{32}y_{22} & x_{31}y_{31} + x_{32}y_{32} & x_{31}y_{41} + x_{32}y_{42} & x_{31}y_{51} + x_{32}y_{52} \\ x_{41}y_{11} + x_{42}y_{12} & x_{41}y_{21} + x_{42}y_{22} & x_{41}y_{31} + x_{42}y_{32} & x_{41}y_{41} + x_{42}y_{42} & x_{41}y_{51} + x_{42}y_{52} \\ x_{51}y_{11} + x_{52}y_{12} & x_{51}y_{21} + x_{52}y_{22} & x_{51}y_{31} + x_{52}y_{32} & x_{51}y_{41} + x_{52}y_{42} & {\color{red}x_{51}}y_{51} + x_{52}y_{52} \end{bmatrix}$$

Express A in outer product, *i.e.* $A = XY^{\mathrm{T}}$, for $\forall X, Y \in \mathbb{R}^{n \times 2}$,

$$\operatorname{triu}(UA, 1) = \operatorname{triu}(UXY^{\mathrm{T}}, 1) = 0_{n \times n}$$

Note that $\operatorname{triu}(\begin{bmatrix} e_{n-1} & e_n \end{bmatrix} Y^{\mathrm{T}}, 2) = 0_{n \times n}$ as,

$$\begin{bmatrix} e_{n-1} & e_n \end{bmatrix} Y^{\mathrm{T}} = \begin{bmatrix} & \\ & \\ 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} y_{11} & y_{21} & y_{31} & y_{41} & y_{51} \\ y_{12} & y_{22} & y_{32} & y_{42} & y_{52} \end{bmatrix} = \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ y_{11} & y_{21} & y_{31} & y_{41} & y_{51} \\ y_{12} & y_{22} & y_{32} & y_{42} & y_{52} \end{bmatrix}$$

solve $UX = U \begin{bmatrix} x_1 & x_2 \end{bmatrix} := \begin{bmatrix} e_{n-1} & e_n \end{bmatrix}$ for x_1 and x_2 by backward substitution,

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2.1 Notation

Vectors

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

Matrices and Columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$
$$= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{m \times n}, \ a_i \in \mathbb{R}^m$$

$$Ae_i := a_i$$
$$e_i = i\text{-th column of } I$$

Block Matrices and Block Columns

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1l} \\ A_{21} & A_{22} & \cdots & A_{1l} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kl} \end{bmatrix}, \ A_{ij} \in \mathbb{R}^{m \times n}$$
$$= \begin{bmatrix} A_1 & A_2 & \cdots & A_l \end{bmatrix} \in \mathbb{R}^{km \times ln}, \ A_i \in \mathbb{R}^{km \times n}$$

$$AE_i := A_i$$
$$E_i = i\text{-th block column of } I$$

2.2 Bidiagonal matrix (upper)

$$U = \begin{bmatrix} d_1 & b_1 & & & \\ & d_2 & b_2 & & \\ & & \ddots & \ddots & \\ & & & d_{n-1} & b_{n-1} \\ & & & & d_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \in \mathbb{R}^{n \times 1} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} \in \mathbb{R}^{n \times 1} \qquad y^{\mathrm{T}} = \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix} \in \mathbb{R}^{1 \times n}$$

$$
xy^{\mathrm{T}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix}
$$

$$
= \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_{n-1} & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_{n-1} & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1} y_1 & x_{n-1} y_2 & \cdots & x_{n-1} y_{n-1} & x_{n-1} y_n \\ x_n y_1 & x_n y_2 & \cdots & x_n y_{n-1} & x_n y_n \end{bmatrix}
$$

$$
= x \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \end{bmatrix}
$$

$$
= \begin{bmatrix} xy_1 & xy_2 & \cdots & xy_{n-1} & xy_n \end{bmatrix}
$$

$$
\mathrm{triu}(xy^{\mathrm{T}}) = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_{n-1} & x_1 y_n \\ 0 & x_2 y_2 & \cdots & x_2 y_{n-1} & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_4 y_{n-1} & x_4 y_n \\ 0 & 0 & \cdots & 0 & x_5 y_n \end{bmatrix}
$$

$$
= \begin{bmatrix} \begin{bmatrix} x_{1:1} \\ 0_{n-1} \end{bmatrix} y_1 & \begin{bmatrix} x_{1:2} \\ 0_{n-2} \end{bmatrix} y_2 & \cdots & \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1} & xy_n \end{bmatrix}
$$

$$U^{-1} = \mathrm{triu}(xy^{\mathrm{T}})$$
$$I = U \, \mathrm{triu}(xy^{\mathrm{T}})$$

n-th column,

$$e_n = U \, \mathrm{triu}(xy^{\mathrm{T}}) e_n$$
$$= U x y_n$$

$$x := U^{-1} e_n$$
$$e_n = U x y_n$$
$$= U U^{-1} e_n y_n$$
$$= e_n y_n$$
$$y_n := 1$$

$(n-1)$-th column,

$$e_{n-1} = U \operatorname{triu}(xy^{\mathrm{T}})e_{n-1}$$

$$= U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1}$$

$(n-1)$-th row of $(n-1)$-th column,

$$e_i^{\mathrm{T}} e_j = \delta_{ij}$$

$$e_{n-1}^{\mathrm{T}} e_{n-1} = 1 = e_{n-1}^{\mathrm{T}} U \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1}$$

$$= \begin{bmatrix} 0_{n-2} & d_{n-1} & b_{n-1} \end{bmatrix} \begin{bmatrix} x_{1:n-1} \\ 0 \end{bmatrix} y_{n-1}$$

$$= d_{n-1} x_{n-1} y_{n-1}$$

$$y_{n-1} := (d_{n-1} x_{n-1})^{-1}$$

$$e_n = Ux$$

$$e_n^{\mathrm{T}} e_n = 1 = e_n^{\mathrm{T}} Ux$$

$$= \begin{bmatrix} 0_{n-1} & d_n \end{bmatrix} x$$

$$= d_n x_n$$

$$y_n := 1$$

$$:= (d_n x_n)^{-1}$$

$$x := U^{-1} e_n$$

$$y_i := (d_i x_i)^{-1}$$

2.3 Tridiagonal matrix (Upper)

$n = 2k$

$$U = \begin{bmatrix} d_1 & b_1^1 & b_1^2 & & & & & & \\ & d_2 & b_2^1 & b_2^2 & & & & & \\ & & d_3 & b_3^1 & b_3^2 & & & & \\ & & & d_4 & b_4^1 & & b_4^2 & & \\ & & & & \ddots & \ddots & & \ddots & \\ & & & & & d_{n-3} & b_{n-3}^1 & b_{n-3}^2 & \\ & & & & & & d_{n-2} & b_{n-2}^1 & b_{n-2}^2 \\ & & & & & & & d_{n-1} & b_{n-1}^1 \\ & & & & & & & & d_n \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} d_1 & b_1^1 \\ & d_2 \end{bmatrix} & \begin{bmatrix} b_1^2 \\ b_2^1 & b_2^2 \end{bmatrix} & & & \\ & \begin{bmatrix} d_3 & b_3^1 \\ & d_4 \end{bmatrix} & \begin{bmatrix} b_3^2 \\ b_4^1 & b_4^2 \end{bmatrix} & & \\ & & \ddots & \ddots & \\ & & & \begin{bmatrix} d_{2k-3} & b_{2k-3}^1 \\ & d_{2k-2} \end{bmatrix} & \begin{bmatrix} b_{2k-3}^2 \\ b_{2k-2}^1 & b_{2k-2}^2 \end{bmatrix} \\ & & & & \begin{bmatrix} d_{2k-1} & b_{2k-1}^1 \\ & d_{2k} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{2k \times 2k}$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{k-1} \\ Y_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \qquad Y^{\mathrm{T}} = \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{2 \times 2k}$$

$$\begin{aligned} XY^{\mathrm{T}} &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \\ &= \begin{bmatrix} X_1 Y_1^{\mathrm{T}} & X_1 Y_2^{\mathrm{T}} & \cdots & X_1 Y_{k-1}^{\mathrm{T}} & x_1 Y_k^{\mathrm{T}} \\ X_2 Y_1^{\mathrm{T}} & X_2 Y_2^{\mathrm{T}} & \cdots & X_2 Y_{k-1}^{\mathrm{T}} & x_2 Y_k^{\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{k-1} Y_1^{\mathrm{T}} & X_{k-1} Y_2^{\mathrm{T}} & \cdots & X_{k-1} Y_{k-1}^{\mathrm{T}} & X_{k-1} Y_k^{\mathrm{T}} \\ X_k Y_1^{\mathrm{T}} & X_k Y_2^{\mathrm{T}} & \cdots & X_k Y_{k-1}^{\mathrm{T}} & X_k Y_k^{\mathrm{T}} \end{bmatrix} \\ &= X \begin{bmatrix} Y_1^{\mathrm{T}} & Y_2^{\mathrm{T}} & \cdots & Y_{k-1}^{\mathrm{T}} & Y_k^{\mathrm{T}} \end{bmatrix} \\ &= \begin{bmatrix} XY_1^{\mathrm{T}} & XY_2^{\mathrm{T}} & \cdots & XY_{k-1}^{\mathrm{T}} & XY_k^{\mathrm{T}} \end{bmatrix} \end{aligned}$$

$$\text{triu}(X_k) := X_k$$
$$\text{triu}(Y_k^{\text{T}}) := Y_k^{\text{T}}$$
$$\text{triu}(X_k Y_k^{\text{T}}) = X_k Y_k^{\text{T}}$$

$$
\text{triu}(XY^{\text{T}}) =
\begin{bmatrix}
\text{triu}(X_1 Y_1^{\text{T}}) & X_1 Y_2^{\text{T}} & \cdots & X_1 Y_{k-1}^{\text{T}} & X_1 Y_k^{\text{T}} \\
0_{2\times 2} & \text{triu}(X_2 Y_2^{\text{T}}) & \cdots & X_2 Y_{k-1}^{\text{T}} & X_2 Y_k^{\text{T}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{2\times 2} & 0_{2\times 2} & \cdots & \text{triu}(X_{k-1} Y_{k-1}^{\text{T}}) & X_{k-1} Y_k^{\text{T}} \\
0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & X_k Y_k^{\text{T}}
\end{bmatrix}
$$
$$
=
\begin{bmatrix}
\begin{bmatrix} \text{triu}(X_1 Y_1^{\text{T}}) \\ 0_{2(k-1)\times 2} \end{bmatrix} &
\begin{bmatrix} X_1 Y_2^{\text{T}} \\ \text{triu}(X_2 Y_2^{\text{T}}) \\ 0_{2(k-2)\times 2} \end{bmatrix} &
\cdots &
\begin{bmatrix} X_{1:k-2} Y_{k-1}^{\text{T}} \\ \text{triu}(X_{k-1} Y_{k-1}^{\text{T}}) \\ 0_{2\times 2} \end{bmatrix} &
XY_k^{\text{T}}
\end{bmatrix}
$$

$$U^{-1} = \text{triu}(XY^{\text{T}})$$
$$I = U\,\text{triu}(XY^{\text{T}})$$

$$E_i = \begin{bmatrix} 0 & \cdots & 0 & I_{2\times 2} & 0 & \cdots & 0 \end{bmatrix}^{\text{T}}$$

k-th block column,

$$E_k = U\,\text{triu}(XY^{\text{T}})E_k$$
$$= UXY_k^{\text{T}}$$

$$X := U^{-1} E_k$$
$$E_k = UXY_k^{\text{T}}$$
$$= UU^{-1} E_k Y_k^{\text{T}}$$
$$= E_k Y_k^{\text{T}}$$
$$Y_k^{\text{T}} := I_{2\times 2}$$

$(k-1)$-th block column,

$$E_{k-1} = U\,\text{triu}(XY^{\text{T}})E_{k-1}$$
$$= U\begin{bmatrix} X_{1:k-2} Y_{k-1}^{\text{T}} \\ \text{triu}(X_{k-1} Y_{k-1}^{\text{T}}) \\ 0_{2\times 2} \end{bmatrix}$$

$(k-1)$-th block row of $(k-1)$-th block column,

$$E_i^{\text{T}} E_j = \begin{cases} I_{2\times 2}, & \text{if } i = j \;; \\ 0_{2\times 2}, & \text{otherwise.} \end{cases}$$

$$E_{k-1}^{\mathrm{T}} E_{k-1} = I_{2\times 2} = E_{k-1}^{\mathrm{T}} U \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \mathrm{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix}$$

$$= \begin{bmatrix} 0_{2\times 2(k-2)} & D_{k-1} & B_{k-1} \end{bmatrix} \begin{bmatrix} X_{1:k-2} Y_{k-1}^{\mathrm{T}} \\ \mathrm{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) \\ 0_{2\times 2} \end{bmatrix}$$

$$= D_{k-1} \, \mathrm{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}})$$

$$Y_{k-1}^{\mathrm{T}} := (D_{k-1} X_{k-1})^{-1}$$

$$D_{k-1} = \mathrm{triu}(D_{k-1})$$

$$D_{k-1}^{-1} = \mathrm{triu}(D_{k-1}^{-1})$$

$$Y_{k-1}^{\mathrm{T}} := (D_{k-1} X_{k-1})^{-1}$$

$$D_{k-1} \, \mathrm{triu}(X_{k-1} Y_{k-1}^{\mathrm{T}}) = D_{k-1} \, \mathrm{triu}(X_{k-1} (D_{k-1} X_{k-1})^{-1})$$

$$= D_{k-1} \, \mathrm{triu}(X_{k-1} X_{k-1}^{-1} D_{k-1}^{-1})$$

$$= D_{k-1} \, \mathrm{triu}(D_{k-1}^{-1})$$

$$= D_{k-1} D_{k-1}^{-1}$$

$$= I_{2\times 2}$$

$$X := U^{-1} E_k$$

$$E_k = U X$$

$$E_k^{\mathrm{T}} E_k = I_{2\times 2} = E_k^{\mathrm{T}} U X$$

$$= \begin{bmatrix} 0_{2\times 2(k-1)} & D_k \end{bmatrix} X$$

$$= D_k X_k$$

$$Y_k^{\mathrm{T}} := I_{2\times 2}$$

$$:= (D_k X_k)^{-1}$$

$$X := U^{-1} E_k$$

$$Y_i^{\mathrm{T}} := (D_i X_i)^{-1}$$

$$n = 2k+1$$

$$
U =
\begin{bmatrix}
\begin{bmatrix} d_1 \end{bmatrix} & \begin{bmatrix} b_1^1 & b_1^2 \\ d_2 & b_2^1 \\ & d_3 \end{bmatrix} & \begin{bmatrix} b_2^2 \\ b_3^1 & b_3^2 \end{bmatrix} \\
& & \ddots & & \ddots \\
& & & \begin{bmatrix} d_{2k-2} & b_{2k-2}^1 \\ & d_{2k-1} \end{bmatrix} & \begin{bmatrix} b_{2k-2}^2 \\ b_{2k-1}^1 & b_{2k-1}^2 \\ \begin{bmatrix} d_{2k} & b_{2k}^1 \\ & d_{2k+1} \end{bmatrix} \end{bmatrix}
\end{bmatrix}
$$

$$
=
\begin{bmatrix}
\tilde{D}_1 & \tilde{B}_1 \\
& D_2 & B_2 \\
& & \ddots & \ddots \\
& & & D_k & B_k \\
& & & & D_{k+1}
\end{bmatrix}
\in \mathbb{R}^{(2k+1) \times (2k+1)}
$$

$$X := U^{-1} E_{k+1}$$

$$Y_i^{\mathrm{T}} := (D_i X_i)^{-1} \text{ for } i = 2, \cdots, k+1$$

$$Y_1^{\mathrm{T}} := (D_1 X_1)^{+} = (D_1 X_1)^{\mathrm{T}} (D_1 X_1 (D_1 X_1)^{\mathrm{T}})^{-1}$$

2.4 Banded matrix (Upper)

$n = ku + r, \ 0 \le r < u$

$$U = \begin{bmatrix} d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & & \\ & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & \\ & & \ddots & \ddots & \ddots & \cdots & \ddots & \\ & & & \ddots & \ddots & \ddots & \cdots & b_{n-u}^u \\ & & & & \ddots & \ddots & \ddots & \vdots \\ & & & & & \ddots & \ddots & b_{n-2}^2 \\ & & & & & & \ddots & b_{n-1}^1 \\ & & & & & & & d_n \end{bmatrix}$$

$$= \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{n \times n} = \mathbb{R}^{(ku+r) \times (ku+r)}$$

$r = 0, \ D_i \in \mathbb{R}^{u \times u}, B_i \in \mathbb{R}^{u \times u} \text{ for } i = 2, \ \cdots, \ k+1$

$r = 0, \ D_i \in \mathbb{R}^{u \times u}, B_i \in \mathbb{R}^{u \times u} \text{ for } r = 0$