

Meeting Notes

Zhiwei Zhou

July 1, 2022

1 28.06.2022 Meeting draft

1.1 Tridiagonal matrix (Upper)

$n = 2k$

$$\begin{aligned}
 U &= \begin{bmatrix} d_1 & b_1^1 & b_1^2 & & & & \\ & d_2 & b_2^1 & b_2^2 & & & \\ & & d_3 & b_3^1 & b_3^2 & & \\ & & & d_4 & b_4^1 & b_4^2 & \\ & & & & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots \\ & & & & & & d_{n-3} & b_{n-3}^1 & b_{n-3}^2 \\ & & & & & & & d_{n-2} & b_{n-2}^1 & b_{n-2}^2 \\ & & & & & & & & d_{n-1} & b_{n-1}^1 \\ & & & & & & & & & d_n \end{bmatrix} \\
 &= \begin{bmatrix} \begin{bmatrix} d_1 & b_1^1 \\ & d_2 \end{bmatrix} & \begin{bmatrix} b_1^2 & \\ b_2^1 & b_2^2 \end{bmatrix} & & & & \\ & \begin{bmatrix} d_3 & b_3^1 \\ & d_4 \end{bmatrix} & \begin{bmatrix} b_3^2 & \\ b_4^1 & b_4^2 \end{bmatrix} & & & \\ & & \ddots & \ddots & \ddots & \\ & & & \begin{bmatrix} d_{2k-3} & b_{2k-3}^1 \\ & d_{2k-2} \end{bmatrix} & \begin{bmatrix} b_{2k-3}^2 & \\ b_{2k-2}^1 & b_{2k-2}^2 \end{bmatrix} \\ & & & & \begin{bmatrix} d_{2k-1} & b_{2k-1}^1 \\ & d_{2k} \end{bmatrix} \end{bmatrix} \\
 &= \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{2k \times 2k}
 \end{aligned}$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{k-1} \\ Y_k \end{bmatrix} \in \mathbb{R}^{2k \times 2} \quad Y^T = [Y_1^T \quad Y_2^T \quad \cdots \quad Y_{k-1}^T \quad Y_k^T] \in \mathbb{R}^{2 \times 2k}$$

$$\begin{aligned}
XY^T &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} [Y_1^T \ Y_2^T \ \cdots \ Y_{k-1}^T \ Y_k^T] \\
&= \begin{bmatrix} X_1 Y_1^T & X_1 Y_2^T & \cdots & X_1 Y_{k-1}^T & x_1 Y_k^T \\ X_2 Y_1^T & X_2 Y_2^T & \cdots & X_2 Y_{k-1}^T & x_2 Y_k^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{k-1} Y_1^T & X_{k-1} Y_2^T & \cdots & X_{k-1} Y_{k-1}^T & X_{k-1} Y_k^T \\ X_k Y_1^T & X_k Y_2^T & \cdots & X_k Y_{k-1}^T & X_k Y_k^T \end{bmatrix} \\
&= X [Y_1^T \ Y_2^T \ \cdots \ Y_{k-1}^T \ Y_k^T] \\
&= [XY_1^T \ XY_2^T \ \cdots \ XY_{k-1}^T \ XY_k^T]
\end{aligned}$$

$$\begin{aligned}
\text{triu}(X_k) &:= X_k \\
\text{triu}(Y_k^T) &:= Y_k^T \\
\text{triu}(X_k Y_k^T) &= X_k Y_k^T
\end{aligned}$$

$$\begin{aligned}
\text{triu}(XY^T) &= \begin{bmatrix} \text{triu}(X_1 Y_1^T) & X_1 Y_2^T & \cdots & X_1 Y_{k-1}^T & X_1 Y_k^T \\ 0_{2 \times 2} & \text{triu}(X_2 Y_2^T) & \cdots & X_2 Y_{k-1}^T & X_2 Y_k^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{2 \times 2} & 0_{2 \times 2} & \cdots & \text{triu}(X_{k-1} Y_{k-1}^T) & X_{k-1} Y_k^T \\ 0_{2 \times 2} & 0_{2 \times 2} & \cdots & 0_{2 \times 2} & X_k Y_k^T \end{bmatrix} \\
&= \begin{bmatrix} \text{triu}(X_1 Y_1^T) \\ 0_{2(k-1) \times 2} \end{bmatrix} \begin{bmatrix} X_1 Y_2^T \\ \text{triu}(X_2 Y_2^T) \\ 0_{2(k-2) \times 2} \end{bmatrix} \cdots \begin{bmatrix} X_{1:k-2} Y_{k-1}^T \\ \text{triu}(X_{k-1} Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix} XY_k^T
\end{aligned}$$

$$\begin{aligned}
U^{-1} &= \text{triu}(XY^T) \\
I &= U \text{triu}(XY^T)
\end{aligned}$$

$$E_i = [0 \ \cdots \ 0 \ I_{2 \times 2} \ 0 \ \cdots \ 0]^T$$

k -th block column,

$$\begin{aligned}
E_k &= U \text{triu}(XY^T) E_k \\
&= U X Y_k^T
\end{aligned}$$

$$\begin{aligned}
X &:= U^{-1} E_k \\
E_k &= U X Y_k^T \\
&= U U^{-1} E_k Y_k^T \\
&= E_k Y_k^T \\
Y_k^T &:= I_{2 \times 2}
\end{aligned}$$

$(k-1)$ -th block column,

$$\begin{aligned} E_{k-1} &= U \operatorname{triu}(XY^T)E_{k-1} \\ &= U \begin{bmatrix} X_{1:k-2}Y_{k-1}^T \\ \operatorname{triu}(X_{k-1}Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix} \end{aligned}$$

$(k-1)$ -th block row of $(k-1)$ -th block column,

$$E_i^T E_j = \begin{cases} I_{2 \times 2}, & \text{if } i = j ; \\ 0_{2 \times 2}, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} E_{k-1}^T E_{k-1} &= I_{2 \times 2} = E_{k-1}^T U \begin{bmatrix} X_{1:k-2}Y_{k-1}^T \\ \operatorname{triu}(X_{k-1}Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix} \\ &= \begin{bmatrix} 0_{2 \times 2(k-2)} & D_{k-1} & B_{k-1} \end{bmatrix} \begin{bmatrix} X_{1:k-2}Y_{k-1}^T \\ \operatorname{triu}(X_{k-1}Y_{k-1}^T) \\ 0_{2 \times 2} \end{bmatrix} \\ &= D_{k-1} \operatorname{triu}(X_{k-1}Y_{k-1}^T) \\ Y_{k-1}^T &:= (D_{k-1}X_{k-1})^{-1} \end{aligned}$$

$$\begin{aligned} D_{k-1} &= \operatorname{triu}(D_{k-1}) \\ D_{k-1}^{-1} &= \operatorname{triu}(D_{k-1}^{-1}) \\ Y_{k-1}^T &:= (D_{k-1}X_{k-1})^{-1} \\ D_{k-1} \operatorname{triu}(X_{k-1}Y_{k-1}^T) &= D_{k-1} \operatorname{triu}(X_{k-1}(D_{k-1}X_{k-1})^{-1}) \\ &= D_{k-1} \operatorname{triu}(X_{k-1}X_{k-1}^{-1}D_{k-1}^{-1}) \\ &= D_{k-1} \operatorname{triu}(D_{k-1}^{-1}) \\ &= D_{k-1}D_{k-1}^{-1} \\ &= I_{2 \times 2} \end{aligned}$$

$$\begin{aligned} X &:= U^{-1}E_k \\ E_k &= UX \\ E_k^T E_k &= I_{2 \times 2} = E_k^T UX \\ &= \begin{bmatrix} 0_{2 \times 2(k-1)} & D_k \end{bmatrix} X \\ &= D_k X_k \\ Y_k^T &:= I_{2 \times 2} \\ &:= (D_k X_k)^{-1} \end{aligned}$$

$$\begin{aligned} X &:= U^{-1}E_k \\ Y_i^T &:= (D_i X_i)^{-1} \end{aligned}$$

$$n = 2k + 1$$

$$U = \begin{bmatrix} [d_1] & \begin{bmatrix} b_1^1 & b_1^2 \\ d_2 & b_2^1 \\ & d_3 \end{bmatrix} & \begin{bmatrix} b_2^2 & \\ b_3^1 & b_3^2 \end{bmatrix} & & \\ & & \ddots & \ddots & \\ & & & \begin{bmatrix} d_{2k-2} & b_{2k-2}^1 \\ & d_{2k-1} \end{bmatrix} & \begin{bmatrix} b_{2k-2}^2 & \\ b_{2k-1}^1 & b_{2k-1}^2 \\ d_{2k} & b_{2k}^1 \\ & d_{2k+1} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(2k+1) \times (2k+1)}$$

$$X := U^{-1} E_{k+1}$$

$$Y_i^T := (D_i X_i)^{-1} \text{ for } i = 2, \dots, k+1$$

$$Y_1^T := (D_1 X_1)^+ = (D_1 X_1)^T (D_1 X_1 (D_1 X_1)^T)^{-1}$$

1.2 Banded matrix (Upper)

$$n = ku + r, \quad 0 \leq r < u$$

$$U = \begin{bmatrix} d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & \\ & d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & \\ & & \ddots & \ddots & \ddots & \cdots & \ddots \\ & & & \ddots & \ddots & \ddots & b_{n-u}^u \\ & & & & \ddots & \ddots & \vdots \\ & & & & & \ddots & b_{n-2}^2 \\ & & & & & & \ddots & b_{n-1}^1 \\ & & & & & & & d_n \end{bmatrix}$$

$$r = 0, \quad n = ku,$$

$$U = \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{ku \times ku},$$

$$D_i \in \mathbb{R}^{u \times u}, \quad B_i \in \mathbb{R}^{u \times u} \text{ for } i = 1, 2, \dots, k.$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} \in \mathbb{R}^{ku \times u} \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{bmatrix} \in \mathbb{R}^{ku \times u} \quad Y^T = [Y_1^T \quad Y_2^T \quad \cdots \quad Y_k^T] \in \mathbb{R}^{u \times ku}$$

$$X_i, Y_i^T \in \mathbb{R}^{u \times u}, \text{ for } i = 1, 2, \dots, k$$

$$X := U^{-1} E_k$$

$$Y_i^T := (D_i X_i)^{-1}$$

$$r \neq 0, \quad n = ku + r,$$

$$U = \begin{bmatrix} \tilde{D}_1 & \tilde{B}_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_k & B_k \\ & & & & D_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times (ku+r)},$$

$$\tilde{D}_1 \in \mathbb{R}^{r \times r}, \quad \tilde{B}_1 \in \mathbb{R}^{r \times u};$$

$$D_i \in \mathbb{R}^{u \times u}, \quad B_i \in \mathbb{R}^{u \times u} \text{ for } i = 2, \dots, k+1.$$

$$X = \begin{bmatrix} \tilde{X}_1 \\ X_2 \\ \vdots \\ X_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times u} \quad Y = \begin{bmatrix} \tilde{Y}_1 \\ Y_2 \\ \vdots \\ Y_{k+1} \end{bmatrix} \in \mathbb{R}^{(ku+r) \times u} \quad Y^T = [\tilde{Y}_1^T \quad Y_2^T \quad \cdots \quad Y_{k+1}^T] \in \mathbb{R}^{u \times (ku+r)}$$

$$\begin{aligned}
&\tilde{X}_1 \in \mathbb{R}^{r \times u} \\
&\tilde{Y}_1^T \in \mathbb{R}^{u \times r} \\
&X_i, Y_i^T \in \mathbb{R}^{u \times u} \text{ for } i = 2, \dots, k+1 \\
&X := U^{-1}E_{k+1} \\
&\tilde{Y}_1^T := (D_1X_1)^+ = (D_1X_1)^T(D_1X_1(D_1X_1)^T)^{-1} \\
&\text{triu}(D_1X_1\tilde{Y}_1^T) = I_{r \times r} \\
&Y_i^T := (D_iX_i)^{-1} \text{ for } i = 2, \dots, k+1
\end{aligned}$$

2 28.06.2022 Meeting formal

2.1 Notation

This notes use Householder notation and start indexing at one.

Scalars, lower case Greek letters: $\alpha, \beta, \gamma, \chi$, *etc.*

Vectors, lower case Roman letter: a, b, c, x , *etc.*

Matrix, upper case Roman letter: A, B, C, X , *etc.*

2.2 Repersentation

Matrix & Vector

$$x = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_n \end{bmatrix} \in \mathbb{R}^n$$
$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{1n} \end{bmatrix}$$
$$= [a_1 \mid a_2 \mid \cdots \mid a_n] \in \mathbb{R}^{m \times n}$$

Block Matrix & Block Vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{kn}, x_i \in \mathbb{R}^k$$
$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{1n} \end{bmatrix}$$
$$= [A_1 \mid A_2 \mid \cdots \mid A_n] \in \mathbb{R}^{lm \times kn}, A_{ij} \in \mathbb{R}^{l \times k}$$

2.3 Content

Consider an upper banded matrix $U \in \mathbb{R}^{n \times n}$,

$$U = \begin{bmatrix} \alpha_1 & \beta_1^1 & \beta_1^2 & \dots & \beta_1^u & & \\ & \alpha_2 & \beta_2^1 & \beta_2^2 & \dots & \beta_2^u & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & \beta_{n-u}^u & \\ & & & & & \vdots & \\ & & & & & \beta_{n-2}^2 & \\ & & & & & \beta_{n-1}^1 & \\ & & & & & & \alpha_n \end{bmatrix}. \quad (1)$$

Then try to show that its inverse can be represented as the upper triangular part of an outer product, *i.e.*,

$$U^{-1} = \text{triu}(XY^T) \quad (2)$$

Let $X, Y \in \mathbb{R}^{n \times u}$, consider the following cases of n ,

Case 1, $n = ku$

then (1) can be represented as a block upper bidiagonal matrix,

$$U = \begin{bmatrix} \xrightarrow{u} & \xrightarrow{u} & \dots & \xrightarrow{u} \\ \begin{matrix} \alpha_1 & \dots & \beta_1^{u-1} \\ & \ddots & \vdots \\ & & \alpha_u \end{matrix} & \begin{matrix} \beta_1^u \\ \vdots \\ \beta_u^1 \end{matrix} & & \\ & \begin{matrix} \alpha_{u+1} & \dots & \beta_{u+1}^{u-1} \\ & \ddots & \vdots \\ & & \alpha_{2u} \end{matrix} & \begin{matrix} \beta_{u+1}^u \\ \vdots \\ \beta_{2u}^1 \end{matrix} & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \begin{matrix} \alpha_{(k-1)u+1} & \dots & \beta_{(k-1)u+1}^{u-1} \\ & \ddots & \vdots \\ & & \alpha_{ku} \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} D_1 & B_1 & & & \\ & D_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & D_{k-1} & B_{k-1} \\ & & & & D_k \end{bmatrix} \in \mathbb{R}^{ku \times ku}, D_i, B_i \in \mathbb{R}^{u \times u} \text{ for } i = 1, 2, \dots, k. \quad (3)$$

Represent the n -dimensional identity matrix $I_{n \times n}$ in block form,

$$I_{n \times n} = I_{ku \times ku} = \text{diag}(\underbrace{I_{u \times u}, I_{u \times u}, \dots, I_{u \times u}}_k) = [E_1 \mid E_2 \mid \dots \mid E_k].$$

then, its block columns E_i for $i = 1, 2, \dots, k$ satisfies

$$E_i \in \mathbb{R}^{ku \times u}$$

$$E_i^T E_j = \begin{cases} I_{u \times u}, & \text{if } i = j; \\ 0_{u \times u}, & \text{otherwise.} \end{cases}$$

To find X and Y such that (2) holds, let

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} \in \mathbb{R}^{ku \times u}, Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{k-1} \\ Y_k \end{bmatrix} \in \mathbb{R}^{ku \times u}, X_i, Y_i \in \mathbb{R}^{u \times u} \text{ for } i = 1, 2, \dots, k,$$

and let X_k and Y_k^T be upper triangular matrix, then $\text{triu}(X_k Y_k^T) = X_k Y_k^T$.
For the outer product form,

$$\begin{aligned} Y^T &= [Y_1^T \mid Y_2^T \mid \dots \mid Y_{k-1}^T \mid Y_k^T] \in \mathbb{R}^{u \times ku} \\ XY^T &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{k-1} \\ X_k \end{bmatrix} [Y_1^T \mid Y_2^T \mid \dots \mid Y_{k-1}^T \mid Y_k^T] \\ &= \begin{bmatrix} X_1 Y_1^T & X_1 Y_2^T & \dots & X_1 Y_{k-1}^T & X_1 Y_k^T \\ X_2 Y_1^T & X_2 Y_2^T & \dots & X_2 Y_{k-1}^T & X_2 Y_k^T \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{k-1} Y_1^T & X_{k-1} Y_2^T & \dots & X_{k-1} Y_{k-1}^T & X_{k-1} Y_k^T \\ X_k Y_1^T & X_k Y_2^T & \dots & X_k Y_{k-1}^T & X_k Y_k^T \end{bmatrix} \\ &= X [Y_1^T \mid Y_2^T \mid \dots \mid Y_{k-1}^T \mid Y_k^T] \\ &= [XY_1^T \mid XY_2^T \mid \dots \mid XY_{k-1}^T \mid XY_k^T] \in \mathbb{R}^{ku \times ku}, \end{aligned}$$

and its the upper triangular component,

$$\begin{aligned} \text{triu}(XY^T) &= \begin{bmatrix} \text{triu}(X_1 Y_1^T) & X_1 Y_2^T & \dots & X_1 Y_{k-1}^T & X_1 Y_k^T \\ 0_{u \times u} & \text{triu}(X_2 Y_2^T) & \dots & X_2 Y_{k-1}^T & X_2 Y_k^T \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{u \times u} & 0_{u \times u} & \dots & \text{triu}(X_{k-1} Y_{k-1}^T) & X_{k-1} Y_k^T \\ 0_{u \times u} & 0_{u \times u} & \dots & 0_{u \times u} & X_k Y_k^T \end{bmatrix} \\ &= \begin{bmatrix} \text{triu}(X_1 Y_1^T) & X_1 Y_2^T & \dots & X_{1:k-2} Y_{k-1}^T & X_1 Y_k^T \\ 0_{(k-2)u \times u} & \text{triu}(X_2 Y_2^T) & \dots & \text{triu}(X_{k-1} Y_{k-1}^T) & X_2 Y_k^T \\ 0_{u \times u} & 0_{(k-2)u \times u} & \dots & 0_{u \times u} & X_k Y_k^T \end{bmatrix} .zz \end{aligned} \quad (4)$$

From (2),

$$I_{n \times n} = U \text{triu}(XY^T). \quad (5)$$

For X and Y_k , from (4), the k -th block column of (5) can be written as,

$$\begin{aligned}
E_k &= U \text{triu}(XY^T)E_k \\
E_k &= UXY_k^T
\end{aligned} \tag{6}$$

Define that

$$X := U^{-1}E_k; \tag{7}$$

$$Y_k^T := I_{u \times u}, \tag{8}$$

which can be shown that (6) holds for the defined (7) and (8) through

$$\begin{aligned}
UXY_k^T &= UU^{-1}E_k I_{u \times u} \\
&= I_{n \times n} E_k I_{u \times u} \\
&= E_k I_{u \times u} \\
&= E_k, \text{ as required.}
\end{aligned}$$

Then, for Y_{k-1} , forming the $(k-1)$ -th block column of (5) from (4) first,

$$\begin{aligned}
E_{k-1} &= U \text{triu}(XY^T)E_{k-1} \\
&= U \begin{bmatrix} X_{1:k-2}Y_{k-1}^T \\ \text{triu}(X_{k-1}Y_{k-1}^T) \\ 0_{u \times u} \end{bmatrix}.
\end{aligned}$$

Focusing on the $(k-1)$ -th block row of $(k-1)$ -th block column,

$$\begin{aligned}
E_{k-1}^T E_{k-1} &= E_{k-1}^T U \begin{bmatrix} X_{1:k-2}Y_{k-1}^T \\ \text{triu}(X_{k-1}Y_{k-1}^T) \\ 0_{u \times u} \end{bmatrix} \\
I_{u \times u} &= \begin{bmatrix} 0_{u \times (k-2)u} & D_{k-1} & B_{k-1} \end{bmatrix} \begin{bmatrix} X_{1:k-2}Y_{k-1}^T \\ \frac{\text{triu}(X_{k-1}Y_{k-1}^T)}{0_{u \times u}} \end{bmatrix} \\
I_{u \times u} &= D_{k-1} \text{triu}(X_{k-1}Y_{k-1}^T).
\end{aligned} \tag{9}$$

Define that

$$Y_{k-1}^T := (D_{k-1}X_{k-1})^{-1}, \tag{10}$$

and notice that $D_{k-1} \in \mathbb{R}^{u \times u}$ is a upper triangular, therefore

$$\begin{aligned}
D_{k-1} &= \text{triu}(D_{k-1}); \\
D_{k-1}^{-1} &= \text{triu}(D_{k-1}^{-1}).
\end{aligned} \tag{11}$$

With the fact (11), (9) holds for the defined (10) as,

$$\begin{aligned}
D_{k-1} \text{triu}(X_{k-1} Y_{k-1}^T) &= D_{k-1} \text{triu}(X_{k-1} (D_{k-1} X_{k-1})^{-1}) \\
&= D_{k-1} \text{triu}(X_{k-1} X_{k-1}^{-1} D_{k-1}^{-1}) \\
&= D_{k-1} \text{triu}(D_{k-1}^{-1}) \\
&= D_{k-1} D_{k-1}^{-1} \\
&= I_{u \times u}, \text{ as required.}
\end{aligned}$$

Similar ideas can be use to redefine Y_k^T in (8), from (7),

$$\begin{aligned}
E_k &= UX \\
E_k^T E_k &= E_k^T UX \\
I_{u \times u} &= E_k^T UX \\
&= \begin{bmatrix} 0_{u \times (k-1)u} & D_k \end{bmatrix} \begin{bmatrix} X_{1:k-1} \\ X_k \end{bmatrix} \\
&= D_k X_k \\
I_{u \times u} &= (D_k X_k)^{-1},
\end{aligned}$$

following that

$$\begin{aligned}
Y_k^T &:= I_{u \times u} \\
&:= (D_k X_k)^{-1}.
\end{aligned}$$

In summary, for $U \in \mathbb{R}^{ku \times ku}$, (2) holds the following definitions of X and Y :

$$\begin{aligned}
X &:= U^{-1} E_k \\
Y_i^T &:= (D_i X_i)^{-1}
\end{aligned}$$

3 Test

$$\begin{bmatrix} d_1 & \overbrace{b_1^1 & b_1^2 & \cdots & b_1^u}^{\text{bandwidth} = u} & & & & \\ d_2 & b_2^1 & b_2^2 & \cdots & b_2^u & & & & \\ & \ddots & \ddots & \ddots & \cdots & \ddots & & & \\ & & \ddots & \ddots & \ddots & \cdots & b_{n-u}^u & & \\ & & & \ddots & \ddots & \ddots & \vdots & & \\ & & & & \ddots & \ddots & b_{n-2}^2 & & \\ & & & & & \ddots & b_{n-1}^1 & & \\ & & & & & & d_n & & \end{bmatrix}$$

$$\left(\begin{array}{ccc|ccc} a & \cdots & a & b & \cdots & b \\ & \ddots & \vdots & \vdots & \ddots & \\ & & a & b & & \\ \hline & & 0 & c & \cdots & c \\ & & & \vdots & & \vdots \end{array} \right) \left. \vphantom{\begin{array}{ccc|ccc} a & \cdots & a & b & \cdots & b \\ & \ddots & \vdots & \vdots & \ddots & \\ & & a & b & & \\ \hline & & 0 & c & \cdots & c \\ & & & \vdots & & \vdots \end{array}} \right\} p$$

$$\left. \vphantom{\begin{array}{ccc|ccc} a & \cdots & a & b & \cdots & b \\ & \ddots & \vdots & \vdots & \ddots & \\ & & a & b & & \\ \hline & & 0 & c & \cdots & c \\ & & & \vdots & & \vdots \end{array}} \right\} q$$

$$\overbrace{\begin{array}{cccccc} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{array}}^{n \text{ columns}}$$

$$\begin{array}{c} \uparrow \\ n \text{ rows} \\ \downarrow \end{array} \left(\begin{array}{cccccc} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc|ccc} 1 & \cdots & 1 & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & 0 & \cdots & 1 & \\ & & & \ddots & & \\ & & & & 1 & \\ \hline & & 1 & \cdots & 0 & \\ & & & \ddots & & \\ & & & & 1 & \\ \hline & & & & & 1 \end{array} \right) \begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \end{array}$$

$$\begin{array}{c} \uparrow \\ i \\ \uparrow \\ j \end{array}$$

$$\begin{array}{c} \uparrow \\ n \text{ rows} \\ \downarrow \end{array} \overbrace{\left(\begin{array}{cccccc} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{array} \right)}^{n \text{ columns}}$$

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\begin{aligned} & \begin{bmatrix} d_1 & b_1^1 & b_1^2 & & & \\ & d_2 & b_2^1 & b_2^2 & & \\ & & d_3 & b_3^1 & b_3^2 & \\ & & & d_4 & b_4^1 & b_4^2 \\ & & & & \ddots & \ddots \\ & & & & & \ddots \\ & & & & & & d_{2k-3} & b_{2k-3}^1 & b_{2k-3}^2 \\ & & & & & & & d_{2k-2} & b_{2k-2}^1 & b_{2k-2}^2 \\ & & & & & & & & d_{2k-1} & b_{2k-1}^1 \\ & & & & & & & & & d_{2k} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} d_1 & b_1^1 \end{bmatrix} & \begin{bmatrix} b_1^2 & \\ b_2^1 & b_2^2 \end{bmatrix} & & & \\ & \begin{bmatrix} d_3 & b_3^1 \end{bmatrix} & \begin{bmatrix} b_3^2 & \\ b_4^1 & b_4^2 \end{bmatrix} & & & \\ & & \begin{bmatrix} b_4^2 & \\ b_4^1 & b_4^2 \end{bmatrix} & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & \begin{bmatrix} d_{2k-3} & b_{2k-3}^1 \end{bmatrix} & \begin{bmatrix} b_{2k-3}^2 & \\ b_{2k-2}^1 & b_{2k-2}^2 \end{bmatrix} \\ & & & & & \begin{bmatrix} d_{2k-2} & b_{2k-2}^1 \end{bmatrix} & \begin{bmatrix} b_{2k-2}^2 & \\ b_{2k-1}^1 & b_{2k-1}^2 \end{bmatrix} \\ & & & & & & \begin{bmatrix} d_{2k-1} & b_{2k-1}^1 \end{bmatrix} & \begin{bmatrix} b_{2k-1}^2 & \\ d_{2k} & \end{bmatrix} \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 4 \\ 1 & 4 \end{bmatrix}$$

$$\begin{aligned} & \begin{bmatrix} d_1 & b_1^1 & \dots & b_1^{r-2} & b_1^{r-1} \\ & d_2 & & & b_2^{r-2} \\ & & \ddots & & \vdots \\ & & & d_{r-1} & b_{r-1}^1 \\ & & & & d_r \end{bmatrix} \begin{bmatrix} b_1^r & \dots & b_1^u \\ b_2^{r-1} & b_2^r & \dots & b_2^u \\ \vdots & \vdots & \ddots & \vdots \\ b_{r-1}^2 & b_r^2 & \dots & b_r^u \\ b_r^1 & b_r^2 & \dots & b_r^u \end{bmatrix} \\ & \begin{bmatrix} d_1 & b_1^1 & \dots & b_1^{r-2} & b_1^{r-1} \\ 0 & d_2 & & & b_2^{r-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & d_{r-1} & b_{r-1}^1 \\ 0 & \dots & \dots & 0 & d_r \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} d_1 & b_1^1 \cdots \cdots b_1^{r-2} & b_1^{r-1} \\ & d_2 & b_2^{r-2} \\ & & \vdots \\ & & d_{r-1} & b_{r-1}^1 \\ & & & d_r \end{bmatrix} \begin{bmatrix} b_1^r & b_1^{r+1} \cdots \cdots b_1^{u-1} & b_1^u \\ b_2^{r-1} & & b_2^u \\ \vdots & & \vdots \\ b_{r-1}^2 & & b_{r-1}^u \\ b_r^1 & b_r^2 \cdots \cdots \cdots b_r^{u-1} & b_r^u \end{bmatrix}$$

$$\begin{bmatrix} d_1 \cdots \cdots b_1^{r-1} \\ \vdots \\ d_r \end{bmatrix} \begin{bmatrix} b_1^r \cdots \cdots b_1^u \\ \vdots \\ b_r^1 \cdots \cdots b_r^u \end{bmatrix} \begin{bmatrix} b_1^r \cdots \cdots b_1^u \\ \vdots \\ b_r^1 \cdots \cdots b_r^u \end{bmatrix} \begin{bmatrix} b_1^r \cdots \cdots b_1^u \\ \vdots \\ b_r^1 \cdots \cdots b_r^u \end{bmatrix}$$

$$\begin{bmatrix} d_1 \cdots \cdots b_1^{r-1} & b_1^r \cdots \cdots b_1^u \\ \vdots & \vdots \\ d_r & b_r^1 \cdots \cdots b_r^u \\ & d_{r+1} \cdots \cdots b_{r+1}^{u-1} \\ & \vdots \\ & d_{r+u} \end{bmatrix}$$

$$\begin{array}{c} \xleftarrow{r} \quad \quad \quad \xleftarrow{u} \quad \quad \quad \cdots \quad \quad \quad \xrightarrow{u} \\ \begin{array}{c} \uparrow r \\ \downarrow u \\ \vdots \\ \uparrow u \end{array} \left[\begin{array}{cccc} d_1 \cdots \cdots b_1^{r-1} & b_1^r \cdots \cdots b_1^u & & \\ & b_r^1 \cdots \cdots b_r^u & & \\ & d_{1+r} \cdots \cdots b_{1+r}^{u-1} & b_{r+1}^u & \\ & & b_{u+r}^1 \cdots \cdots b_{u+r}^u & \\ & & & \ddots \\ & & & d_{n-u+1} \cdots \cdots b_{n-u+1}^u \\ & & & & d_n \end{array} \right] \end{array}$$

$$\begin{array}{c}
\begin{array}{c} \xleftarrow{u} \qquad \qquad \qquad \xrightarrow{u} \end{array} \\
\left[\begin{array}{ccccccc}
d_1 & \cdots & b_1^{u-1} & b_{r+1}^u & \cdots & \cdots & \cdots \\
& \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
& & d_u & b_u^1 & \cdots & b_u^u & \vdots \\
& & & \ddots & \ddots & \ddots & \vdots \\
& & & & \ddots & \ddots & \vdots \\
& & & & & d_{n-u+1} & b_{n-u+1}^u \\
& & & & & & \vdots \\
& & & & & & d_n
\end{array} \right]
\end{array}$$

$$\begin{array}{c}
1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\
\left[\begin{array}{c|c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array} \right]
\end{array}$$

$$\begin{array}{cccccccccccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
0 & \left[\begin{array}{cccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19
\end{array} \right] & 20 \\
0 & \left[\begin{array}{cccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19
\end{array} \right] & 20 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20
\end{array}$$

$$\begin{array}{c}
\begin{array}{c} \xleftarrow{u} \end{array} \\
\left[\begin{array}{ccccccc}
d_1 & b_1^1 & b_1^2 & \cdots & b_1^u & & \\
b_1^{-1} & d_2 & & & & & \\
b_1^{-2} & & & & & & \\
\vdots & & & & & & \\
b_1^{-l} & & & & & & \\
& & & & & & b_{n-u}^u \\
& & & & & & \vdots \\
& & & & & & b_{n-2}^2 \\
& & & & & & b_{n-1}^1 \\
& & & & b_{n-l}^{-l} & \cdots & b_{n-2}^{-2} & b_{n-1}^{-1} & d_n
\end{array} \right]
\end{array}$$