Title of the problem:

Cristiano Ronaldo, arguably the best footballer in the world, is without a doubt the best spot-kick taker on the globe right now. With an average success rate of 95% in converting penalties into goals, he is a real nightmare for a goalkeeper when it comes to scoring from the 12 yards distance. Coaches and particularly goalkeepers give hours of thought in coming up with a strategy against him to face his penalty kick. The real question for a goalkeeper is what is Ronaldo going to do and what should be Ronaldo's best strategy. Although the game of penalty is a game of chance, it can be mathematically formulated and solved to come up with the answer to the above question.

Abstract:

We have undertaken the project of optimizing Ronaldo's penalty-kick. This involves closely looking into the penalty-kick direction, the velocity of the kick and the angle of elevation. The problem is divided into two parts.

i. The first part involves optimizing Ronaldo's direction of his penalty shot, i.e. we shall look at if he should shoot to his left, right or center.
ii. The second part goes more into the physics of the problem and attempts to find the maximum velocity of the kick and at which angle should the kick take place.

First part will be solved by Liner Programming using Game Theory And Second part will be solved by Non-liner Programming using Fundamental Physics Equations.

Introduction:

i. **Penalty-Kick Direction :**

The game of penalty shoot-out at the end of the day is a game in itself. Both the goalkeeper and the striker have to think of his or strategy before-hand. We can think of it as a game between two players where both of them have to strategize. Hence the concept of game theory needs to be applied. Game theory is a mathematical theory that deals with the general features of competitive situations. It places particular emphasis on the decision-making processes of the adversaries.

Description:

The first step in game theory is to create a payoff table. Payoff table shows the gain (positive or negative) for a particular player that would result from each combination of strategies for the two players. Two assumptions are involved in solving a game theory problem:

1. Both players are rational.

2. Both players choose their strategies solely to promote their own welfare.

The problem in hand is two-person, zero-sum game since the game involves only two adversaries or players. It is a zero-sum game because one player wins whatever the other one loses, so that the sum of their net winnings is zero.

After the payoff table has been formed, we need to look at the solving techniques. Basically first we need to search for dominating strategy. In a game player 1 will try to maximize his gain and player 2 will restrict him. So player 1's gain is always restricted by player 2's strategy. In other words, player 1 will actually try to maximize the minimum amount that player 2 will offer him. This is a maxi-min strategy. So at first for every choice that player 2 offers him, player 1 will choose his best strategy and see if there is one strategy that is the best solution for all the choices. That is the dominating strategy.

However, for the problem involved it is highly unlikely that we will get a dominating strategy. We will be obtaining a mixed strategy problem. When there is no dominating strategy, we try to mix the strategies in such a way so that our estimated gain is always maximised. In order to solve the mixed strategy problem, we looked at linear programming solution techniques. The objective is to find the probabilities of mixing in such a way that the player's gain is maximised.

ii. **The maximum velocity of the kick and angle at which should the kick take place:**

Once the direction of kick has been found, the velocity of kick and necessary angles are also of great concern. For this, we had to visit fundamental physics. The ball will surely experience a projectile motion and the equations of projectile motion need should be enough to find out these values. 'fmincon' is minimum of constrained nonlinear multivariable function. 'fmincon' attempts to find a constrained minimum of a scalar function of several variables starting at an initial estimate. This is generally referred to as constrained nonlinear optimization or nonlinear programming.

Methods Of Solving Penalty-Kick Direction:

For this problem, game theory is applied to find out the direction of penalty kick and the direction of the goalkeeper's jump. This is because, we believe, it is in this region that two rational players will have an arbitrary choice and need to find out the best possible direction. However, it is likely that the problem can be a mixed strategy problem since it is highly unlikely that we will be getting a dominating strategy since we know from experience that a goalkeeper does not shoot or the striker does not always kick in the same direction.

Problem Details:

Our problem was to calculate the probability of mixing. This was solved first by constructing the payoff table. The major problem was constructing this payoff table. No direct data were obtained for any website. Hence we observed over 40 penalty videos and noted down the striker's direction of shot and the goalkeeper's jump direction. After collecting the data we had to convert them to probabilities and to translate the data in such a way so that it depicted Ronaldo's expected gain. The payoff table then was converted to a linear programming formulation which when solved should give the probabilities displaying where should the striker shoot with what chances of scoring.

Numerical Calculation :

Data Collection for Penalty-Kick Direction :

We will be examining both Cristiano Ronaldo's kick direction as well as the goal-keeper's direction of jump. For the purpose of mathematical modeling the problem is simplified and each of the two players (Cristiano Ronaldo and the goalkeeper) are provided with three strategies. From Ronaldo's perspective, he can shoot left, right or center and the goalkeeper can also jump to Ronaldo's left, right or center.

Therefore,

Ronaldo's strategies: L, C, R

Goalkeeper's strategies: L, C, R

L- Left C- Center R- Right

A random sample of 41 successful penalty kicks were observed and for each penalty kick, Ronaldo's shot direction and the goalkeeper's direction of jump were recorded.

Following is the data table showing Ronaldo's preferred direction in successful penalties as well as the goalkeeper's chosen direction to jump during those corresponding penalties.

Table 1: Data Table :

(a)

No of Observation	Cristiano Ronaldo	Goalkeeper
1	L	L
2	R	L
3	L	L
4	R	C
5	L	C
6	L	L
7	R	L
8	L	L
9	C	L
10	L	L
11	L	R
12	R	L
13	C	R
14	R	R
15	L	L
16	C	C
17	R	L
18	C	L
19	L	R
20	L	C
21	R	R
22	L	L
23	L	R

No of Observation	Cristiano Ronaldo	Goalkeeper
24	R	C
25	L	L
26	R	L
27	L	L
28	C	R
29	R	L
30	R	L
31	C	L
32	L	R
33	R	L
34	L	L
35	L	L
36	L	L
37	L	R
38	L	R
39	L	R
40	L	R
41	R	L

Mathematical formulation for penalty-kick direction:

These data are recorded in tabular format in the following way showing the number of occurrences of each combination

Table 2: Combined Data Table

Ronaldo's Kick Direction	Goalkeeper's jump direction			
		L	**C**	**R**
	L	12	2	8
	C	3	1	2
	R	9	2	2

Now, the probability of each combination is calculated. This is done by dividing the number of occurrence of each combination by the total number of observed penalties.

For example, the probability of Ronaldo shooting to his left and the goal keeper jumping to the left, P (LL) = 12/41 = 0.2927

In a similar way the probability of all the combinations are calculated and tabulated in the following way:

Table 3: Probability Table

Ronaldo's Kick Direction	Goalkeeper's jump direction			
		L	**C**	**R**
	L	0.2927	0.0488	0.1951
	C	0.0732	0.0244	0.0488
	R	0.2195	0.0488	0.0488

After calculating the probabilities, the next task is to look at the payoff of the player. Since we are looking at the problem from Ronaldo's viewpoint, only his payoff will be taken into consideration. Traditionally being a right-footed player, Ronaldo's preferred direction is left. Even the probability table shows us that he prefers to shoot more to his left than to the right since he is stronger on that side. Therefore he will have a higher level of satisfaction if he shoots and scores to the right. This presumption is based on the fact that he is weaker on the right side; since he already knows that his stronger side is left, therefore he is expected to score every time he shoots there. This has a lower level of satisfaction to him. However, the more he shoots and scores to the right the more he grows on his confidence level since that is his weaker side. Moreover, he will have lower level of satisfaction when he shoots to one side and the goalkeeper jumps in another direction. This is pretty obvious since when the goalkeeper jumps in another direction the goal is virtually empty and Ronaldo has only himself to blame if he is to miss. Based on these presumptions, an arbitrary utility point scale is set up: 1, 2, 3 and 4. Any number of increasing order can be chosen because we are interested in comparative results rather than the absolute value. This means our final decision will be more of a comparative form rather than demonstrating the absolute probabilities that Ronaldo will shoot with in the specified direction.

Constructing the payoff table:

All the cells with "mismatched" directions are given the "utility" rating of 1. This is because it is easiest to score on an empty side as discussed above.

Therefore, Ronaldo's satisfaction rating if he shoots to the left and the goalkeeper right, u(LR) = 1. Similarly, u(RC) = 1.

Out of the remaining cells, the "LL" cell is given a rating of 2 since as discussed he is stronger on that side and is expected to score there.

The "RR" cell is given a rating of 4 (highest) because it is a big deal for any footballer to beat the goalkeeper on his weaker side.

The remaining "CC" cell is given a moderate utility which lies between the two, i.e. 3.

u(LL) = 2;

u(CC) = 3;

u(RR) = 4

Table 4: Satisfaction Ratings Table

Ronaldo's Kick Direction	Goalkeeper's jump direction			
		L	**C**	**R**
	L	2	1	1
	C	1	3	1
	R	1	1	4

The next step is to find the expected payoff. This is done by taking the product of the probability and the utility rating of each combination.

Example, expected utility of Ronaldo shooting to his left and the goalkeeper jumping to the left, e.u. (LL) = 0.2927*2 = 0.5854

e.u. (RR) = 0.0488*4 = 0.1952

Similarly, all the expected payoffs are calculated and the expected payoff table is formed.

Table 5: Expected Payoff Table

Ronaldo's Kick Direction	Goalkeeper's jump direction			
		L	**C**	**R**
	L	0.5854	0.0488	0.1951
	C	0.0732	0.0732	0.0488
	R	0.2195	0.0488	0.1952

This is the expected payoff table, but in order to make the calculations easier, each of the cell is multiplied by 10 and the nearest whole number is recorded. This will have no effect on the comparison since all the cells are multiplied by the same number.

Expected payoff (LL) = 0.5854*10 = 5.854 = 6

The final payoff table is then constructed.

Table 6: Final Payoff Table

	Goalkeeper's jump direction			
Ronaldo's Kick Direction		**L**	**C**	**R**
	L	6	0	2
	C	1	1	0
	R	2	0	2

This payoff table shows the expected gain as well as Ronaldo's satisfaction level when he shoots in a preferred direction.

This has now become a game theory problem of a two person simultaneous move game. As we can see there is no dominating strategy. Given that the goalkeeper moves left, Ronaldo's best response is to shoot left since it has a higher payoff. Similarly, if goalkeeper stays center, Ronaldo's best response is to shoot center and if the goalkeeper moves right Ronaldo can either shoot right or left. Since there is no dominating strategy, the concept of mixed strategies has to be undertaken.

x_1, x_2, x_3 are the ratios that the strategies are mixed in. In other words, they are the probabilities that are assigned to each strategy and it is our task to find these ratios that will correspond to the optimal solution of the penalty kick.

Therefore, the linear programming formulation is as follows;

Maximize $z = x_4$

Subject to

$6x_1 + x_2 + 2x_3 - x_4 \geq 0$

$x_2 - x_4 \geq 0$

$2x_1 + 2x_3 - x_4 \geq 0$

$x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3, x_4 \geq 0$

Solve:

Using MATLAB, we obtained the following solution:

```
f= [0; 0; 0; -1]

A=[-6 -1 -2 1; 0 -1 0 1; -2 0 -2 1; 1 1 1 0; -1 -1 -1 0; -1 0 0 0; 0 -1 0 0; 0 0 -1 0; 0 0 0 -1]

b= [0; 0; 0; 1; -1; 0; 0; 0; 0]

[x, fval]=linprog(f,A,b)

f =


     0

     0

     0

    -1
```

A =

$$\begin{matrix} -6 & -1 & -2 & 1 \\ 0 & -1 & 0 & 1 \\ -2 & 0 & -2 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix}$$

b =

$$\begin{matrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

Optimization terminated.

x =

0.3010

0.6667

0.0324

0.6667

fval =

-0.6667

Result:

From the above calculation, we see that Cristiano Ronaldo should mix his strategies of striking the ball at left, center or right according to the following probabilities:

Left- 0.3010

Center- 0.6667

Right- 0.0324