A large deviation application to the securitized market

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Abstract

Asset-Backed-Security tranches can be priced in an elementary way in the Homogeneous reference Portfolio (HP) case and also when the central limit case conditions are satisfied, i.e. in the Large Homogeneous Portfolio (LHP) case. Unfortunately the first approach can be used in practice only when the number of obligor I is quite small, while the second approach is valid only when I is very large. In this note we get a closed formula in a simple integral form using large deviations theory for a reference portfolio with an arbitrary number of obligors. We show that the approximation is very good for a number of obligors larger than 10 when pricing all kind of tranches in the basic Vasicek model.

1 Problem description

We consider a Homogeneous reference Portfolio (hereinafter HP) in an Asset Backed Security tranche (hereinafter "the tranche") with detachment points K_d and K_u (in %). The reference portfolio has a notional $\mathfrak{N}$ and I obligors. Each obligor has a Loss Given Default $LGD = 1 - \pi$ with π its recovery (in %) of obligor's debt. The loss of the HP is equal to $L_{r.p.} := \mathfrak{N}\, LGD\, m/I$ with m the number of obligors defaulted in the reference portfolio.

HP tranche is characterized by a loss

$$L_t(m; I, \ldots) := \min\left\{\max(L_{r.p.} - \mathfrak{N}\, K_d, 0), \mathfrak{N}\,(K_u - K_d)\right\}$$

or equivalently

$$L_t(m; I, \ldots) = \mathfrak{N}_t\, \mathcal{L}\left(\frac{m}{I}\right)$$

with $\mathfrak{N}_t := \mathfrak{N}\,(K_u - K_d)$ tranche's notional and the loss function defined as

$$\mathcal{L}(z) := \frac{\min\{\max(z - d, 0), u - d\}}{u - d}\ , z \in [0, 1]$$

where $d := K_d/LGD$ and $u := K_u/LGD$. The loss function is a continuous function of the argument, equal to zero for $z \leq d$, to one for $z \geq u$ and it is a linear function of z between d and u.

The expected loss for the tranche is

$$\mathcal{E}_{HP}(K_d, K_u) := \mathbb{E}[L_t(m; I, \ldots)] = \mathfrak{N}_t \sum_{m=0}^{I} \mathcal{L}\left(\frac{m}{I}\right) f_{HP}(m; I) \tag{1}$$

with the probability density of having m defaults in the HP (see e.g. [4])

$$f_{HP}(m; I) := \int_{-\infty}^{\infty} dy\ \phi(y) \begin{pmatrix} I \\ m \end{pmatrix} p(y)^m [1 - p(y)]^{I-m}$$

where $\phi(y)$ is the probability density of the mixing variable y and $p(y)$ is obligor default probability for a given y: both depend on the particular model that is considered and $p(y)$ is a monotonous function of the argument. For example, in the Vasicek model [6] $p(y)$ is a decreasing function of y

$$p(y) = \mathcal{N}\left(\frac{\mathcal{N}^{-1}(p) - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)$$

and $\mathcal{N}(\cdot)$ is the st.n. Cumulative Distribution Function.

In the Large Homogeneous Portfolio (LHP) case we have

$$\mathcal{E}_{LHP}(K_d, K_u) = \mathfrak{N}_t \int_0^1 dz\ \mathcal{L}(z)\ f_{LHP}(z) \tag{2}$$

where

$$f_{LHP}(z) = \phi\left(p^{-1}(z)\right) \frac{\partial p^{-1}(z)}{\partial z}$$

is the density function for the fraction of defaulted obligors $z \equiv m/I$.

The two main criticalities in the above approaches are well known: on the one hand HP pricing can be used up to I quite small (it depends on the computer used but generally it is around one hundred), on the other hand we do not know the quality of LHP pricing formula and what is the error we commit for a finite I.

2 Large deviation approach

The main idea to solve both questions is to consider the large deviation approach proposed by Varadhan [5]; this is a very good approximation for I larger than 10. In the following proposition we show that a very simple expression is available in some cases.

Proposition 1 *For a mezzanine tranche, the Cumulative Distribution Function (CDF) of having a fraction of defaulted obligors up to x can be replaced via its large deviation approximation*

$$\mathbb{P}_{KL}\left[z \le x\right] = \int_{-\infty}^{\infty} dy\ \phi(y) \int_0^x dz\ C(z, y) \exp\left\{-I\ \mathcal{K}\left[z; p(y)\right]\right\} \qquad x \in (0, 1] \tag{3}$$

where the Kullback-Leibler entropy [2] is

$$\mathcal{K}(z; p) \equiv z \ln \frac{z}{p} + (1 - z) \ln \frac{1 - z}{1 - p}$$

and $C(z, y)$ is a normalization function of the density of z given y.

Proof

We consider the CDF of observing a number of defaulted obligors up to n in the HP case

$$\mathbb{P}(m \le n) = \sum_{m=0}^{n} f_{HP}(m; I)$$

and expand the Newton coefficient using the Stirling Formula

$$m! \simeq \sqrt{2\pi m}\, m^m\, e^{-m}\ .$$

Using the Stirling formula we get

$$\left(\begin{array}{c} I \\ m \end{array}\right) \simeq \frac{1}{I}\, C^{(1)}(z)\ \exp\{-I\,[z\ln z + (1-z)\ln(1-z)]\} \qquad z = \frac{m}{I}$$

with

$$C^{(1)}(z) := \sqrt{\frac{I}{2\pi\ (1-z)z}} \qquad .$$

We obtain

$$\mathbb{P}(m \le n) \simeq \int_{-\infty}^{\infty} dy\ \phi(y) \sum_{m=0}^{n} \frac{1}{I}\, C^{(1)}\left(\frac{m}{I}\right)\ \exp\left\{-I\,\mathcal{K}\left[\frac{m}{I}; p(y)\right]\right\}$$

with the above approximation very precise even for quite small I.

For a value of I sufficiently large, the sum $\sum_{m=0}^{n}$ can be approximated with the corresponding integral, i.e. defining

$$dz = \frac{\Delta m}{I}$$

we get

$$\mathbb{P}(z \le x) \simeq \int_{-\infty}^{\infty} dy\ \phi(y) \int_0^x dz\ C^{(1)}(z)\ \exp\{-I\,\mathcal{K}\,[z; p(y)]\}$$

with $x := n/I$.

Finally, from a numerical point of view, for every given value of y, one can consider a more precise normalization function $C(z,y) = C^{(1)}(z)/D(y)$, where

$$D(y) = \int_0^1 dz\ C^{(1)}(z) \exp\{-I\,\mathcal{K}\,[z; p(y)]\} \qquad ,$$

proving that

$$\mathbb{P}(z \le x) \simeq \mathbb{P}_{KL}(z \le x) \qquad \clubsuit$$

For a given value of p $\mathcal{K}(z;p)$ as a function of z is very simple. Its first derivative is

$$\frac{\partial \mathcal{K}(z;p)}{\partial z} = \ln \frac{z}{1-z}\,\frac{1-p}{p}$$

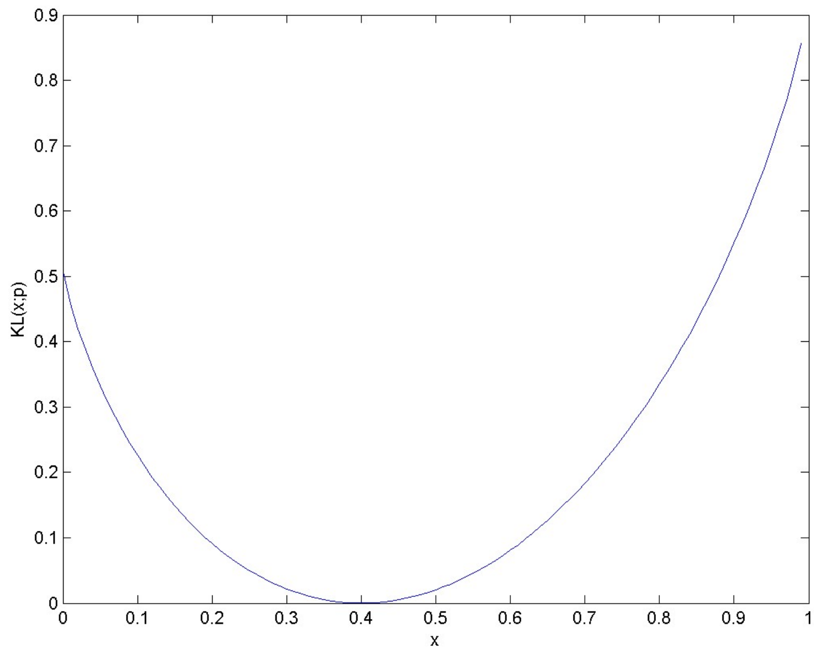

Figure 1: *Kullback Leibler entropy* $\mathcal{K}(x;p)$ *for* $p = 0.4$.

which is lower than zero for $z < p$, equal to zero in $z = p$ and greater than zero otherwise; then $\mathcal{K}(z;p)$ is always non-negative and equal to zero for $z = p$ where it reaches the minimum. Its second derivative is

$$\frac{\partial^2 \mathcal{K}(z;p)}{\partial z^2} = \frac{1}{z(1-z)}$$

which is always greater than zero for $z \in (0,1)$, i.e. $\mathcal{K}(z;p)$ is concave upward.

In figure 1 we have plotted the Kullback-Leibler entropy $\mathcal{K}(z;p)$ varying $z \in [0,1]$ with a constant value for p.

Corollary 1 *For a mezzanine tranche, the expected value of tranche can be computed using its large deviation approximation*

$$\mathcal{E}_{KL}(K_d, K_u) = \int_{-\infty}^{\infty} dy\ \phi(y) \int_0^1 dz\ \mathcal{L}(z)\ C(z,y) \exp\left\{-I\ \mathcal{K}\left[z; p(y)\right]\right\} \tag{4}$$

Proof

Straightforward given above proposition ♣.

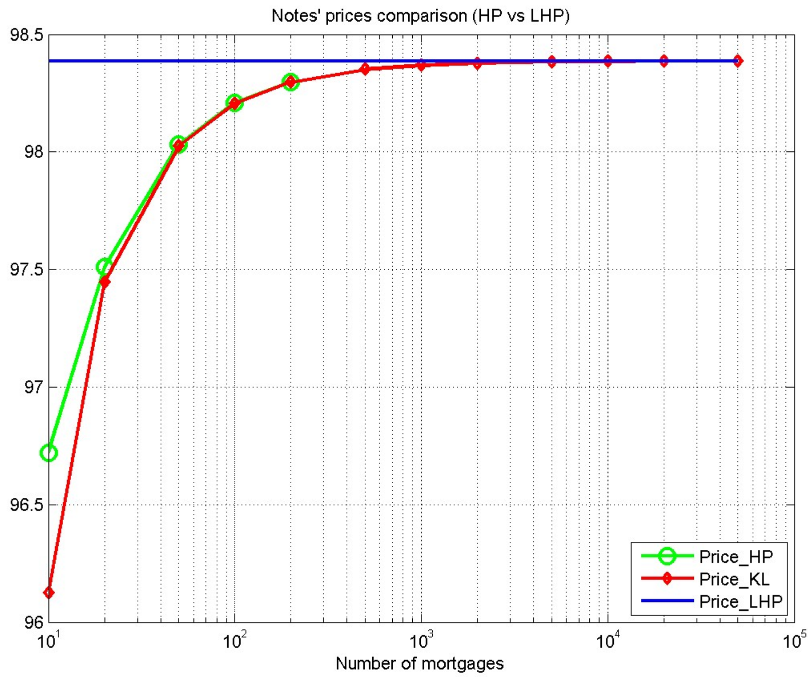

Figure 2: *Mezzanine tranche with* $K_d = 6\%$ *and* $K_u = 9\%$ *using the three different pricing methodologies. Model parameters are* $\pi = 70\%$, $p = 4\%$ *and* $\rho = 30\%$ *and discount factors are neglected.*

Remark

The above expansion cannot be used when pricing the Equity tranche case. Why? What can an alternative methodology be?

Hints

In order to price an equity tranche one can

- Price the equity trance as the expected total loss of the reference portfolio minus the price of a mezzanine tranche;
- For a $L_{r.p.}$ lower than $\mathfrak{N} K_u$ consider explicitly the loss function in the argument of the sum and, after the simplification, use the Stirling formula.

In figures 2 and 3 we show the results for the Vasicek model in the case of a Mezzanine tranche and an Equity tranche for several values of I.

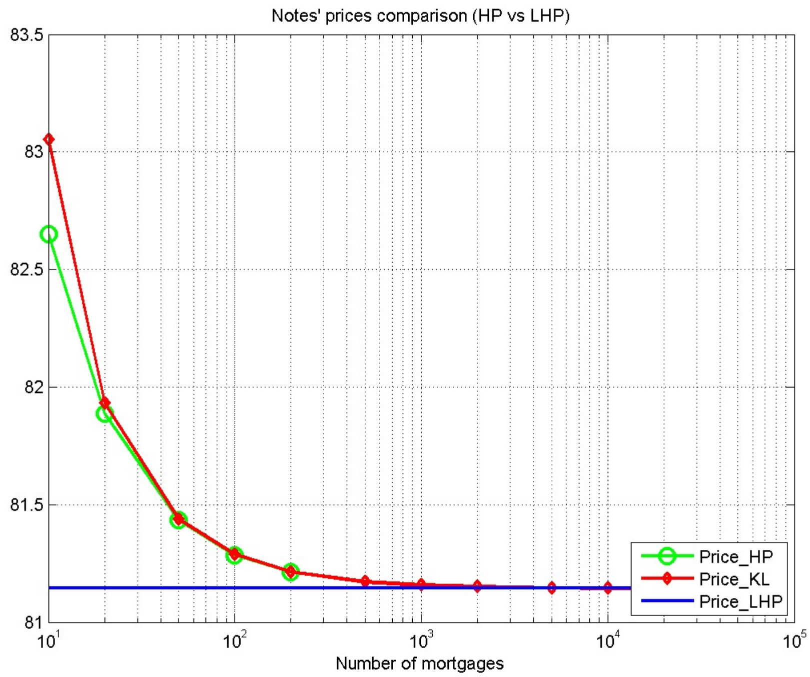

Figure 3: *Equity tranche with $K_u = 6\%$ using the three different pricing methodologies. Model parameters are $\pi = 70\%$, $p = 4\%$ and $\rho = 30\%$ and discount factors are neglected.*

3 LHP limit

In order to check equation (4) we can derive the LHP formula in the Vasicek case (see e.g. [4]), using the saddle point method for $I >> 1$ [1].

Lemma 1 *For $I >> 1$, the integral in z in equation (4) is equal to*

$$\mathcal{I}(x) := \int_0^x dz\; C^{(1)}(z)\; \exp\left\{-I\; \mathcal{K}\left[z; p(y)\right]\right\} = \begin{cases} 1 + O\left(\dfrac{1}{I}\right) & if\ p(y) \leq x \\ e^{-I\; \mathcal{Z}} \left[1 + O\left(\dfrac{1}{\sqrt{I}}\right)\right] & otherwise \end{cases}$$

with $\mathcal{Z}$ a positive number.

Proof

For $I >> 1$ the integral can be computed using the saddle point method. For large values of I the integrand has narrow sharp peak located in the minimum of $\mathcal{K}[z]$ and its width is $O(1/\sqrt{I})$. Let x_0 be the location of the minimum of $\mathcal{K}[z]$ in the domain of the integral, we can change

[1] The method is due to Debye [1], even if some ideas in the method were suggested earlier by Riemann [3].

the integration variable from z to a new variable w according to

$$z = z_0 + \frac{w}{\sqrt{I}} \qquad .$$

Then one can expand in powers of w the exponent

$$I\,\mathcal{K}\left[z\right] = I\,\mathcal{K}\left[z_0\right] + \sqrt{I}\,w\,\mathcal{K}'\left[z_0\right] + \frac{w^2}{2}\,\mathcal{K}''\left[z_0\right] + \frac{1}{\sqrt{I}}\frac{w^3}{6}\,\mathcal{K}'''\left[z_0\right] + O\left(\frac{1}{I}\right) \, ,$$

the prefactor $C^{(1)}(z)$

$$C^{(1)}\left[z\right] = C^{(1)}\left[z_0\right]\left\{1 + \frac{1}{\sqrt{I}}\,w\,\frac{C'^{(1)}\left[z_0\right]}{C^{(1)}\left[z_0\right]} + O\left(\frac{1}{I}\right)\right\}$$

and the integrand becomes

$$C^{(1)}(z)\;e^{-I\,\mathcal{K}[z]} = C^{(1)}(z_0)\;e^{-I\,\mathcal{K}[z_0]-\sqrt{I}\,w\,\mathcal{K}'[z_0]-\frac{w^2}{2}\,\mathcal{K}''[z_0]}\left\{1 + \frac{1}{\sqrt{I}}\left[w\,\frac{C'^{(1)}\left[z_0\right]}{C^{(1)}\left[z_0\right]} - \frac{w^3}{6}\,\mathcal{K}'''\left[z_0\right]\right] + O\left(\frac{1}{I}\right)\right\}$$

One may have two cases, depending on integral endpoints and in particular on the value of x that one is considering.

In the first case, the minimum of $\mathcal{K}\left[z\right]$ in the interval $[0,x]$ corresponds to the unique absolute minimum of $\mathcal{K}\left[z\right]$ with $z \in [0,1]$, that is attained in $z_0 = p(y)$, i.e. $p(y) < x$. In this case $\mathcal{K}\left[z_0\right] = \mathcal{K}'\left[z_0\right] = 0$. Moreover, in the large I limit the two end points for w go respectively to $-\infty$ and $+\infty$ and then the term $O(1/\sqrt{I})$ drops away due to symmetry reason (integral of an odd function on a symmetric domain). The integral becomes

$$\mathcal{I}(x) = C^{(1)}(z_0)\sqrt{\frac{2\pi}{I\,\mathcal{K}''\left[z_0\right]}}\left\{1 + O\left(\frac{1}{I}\right)\right\} = 1 + O\left(\frac{1}{I}\right) \, .$$

In the second case, the minimum of $\mathcal{K}\left[z\right]$ in the interval $[0,x]$ is attained on the border of the interval ($z_0 = x$) i.e. $p(y) > x$. As proven in previous section, the first derivative of $\mathcal{K}\left[x\right]$ is negative when $x < p(y)$. In this case the integral becomes

$$\mathcal{I}(x) = C^{(1)}(z_0)e^{-I\left[\mathcal{K}[z_0]+\frac{1}{2}\frac{(\mathcal{K}'[z_0])^2}{\mathcal{K}''[z_0]}\right]}\sqrt{\frac{2\pi}{I\,\mathcal{K}''\left[z_0\right]}}\left[1 + O\left(\frac{1}{\sqrt{I}}\right)\right] = e^{-I\,\mathcal{Z}}\left[1 + O\left(\frac{1}{\sqrt{I}}\right)\right] \, ,$$

with

$$\mathcal{Z} = \mathcal{K}\left[x\right] + \frac{1}{2}\frac{(\mathcal{K}'\left[x\right])^2}{\mathcal{K}''\left[x\right]}$$

which is a positive number since $\mathcal{K}\left[x\right], \mathcal{K}''\left[x\right] > 0$ for $x < p(y)$, as proven in previous section ♣

Proposition 2 *The limit for $I \to \infty$ of the large deviation approximation of the CDF* (4) *is the LHP CDF.*

Proof

It is sufficient to observe that

$$\lim_{I \to \infty} \mathbb{P}_{KL}\left[z \leq x\right] = \int_{-\infty}^{\infty} dy \; \phi(y) \; \mathbb{1}_{p(y)<x}$$

where we have used previous lemma and the regularity of the density $\phi(y)$ that allows to exchange the limit and the integral.

The proposition is proven after having defined y^* as the value s.t. $x = p(-y^*)$

$$\lim_{I \to \infty} \mathbb{P}_{KL}\left[z \leq x\right] = \mathcal{N}(y^*) = \mathbb{P}_{LHP}\left[z \leq x\right] \qquad \clubsuit$$

Notation

Symbol	Description
$C^{(1)}(z)$	normalization function of the dentity of z given y derived from Stirling formula
d (u)	normalized lower (upper) detachment point, equal to K_d/LGD (K_u/LGD)
$\mathcal{E}_{(\bullet)}$	expected tranche loss in the ($\bullet$) case
K_d (K_u)	lower (upper) detachment point in %
$\mathcal{K}(z;p)$	Kullback-Leibler entropy of z and p
$\phi(\cdot)$	st.n. density function
I	number of obligors in the reference portfolio
LGD	Loss Given Default, equal to $1-\pi$
$L_{r.p.}$	total loss in the reference portfolio
L_t	tranche loss
$\mathcal{L}(z)$	loss function as a function of the fraction $z \in [0,1]$, $\min\{\max(z-d,0)/(u-d),1\}$
n	number of defaulted obligors
x	fraction of defaulted obligors, equal to n/I
$\mathfrak{N}$	notional of the reference portfolio
$\mathfrak{N}_t$	notional of the tranche, equal to $\mathfrak{N}\,(K_u - K_d)$
$\mathcal{N}(\cdot)$	st.n. CDF

References

[1] P. Debye, *Näherungsformeln für die Zylinderfunktionen für große Werte des Arguments und unbeschränkt veränderliche Werte des Index* Mathematische Annalen **67** (1909) 535-558.

[2] S. Kullback and R.A. Leibler, *On information and sufficiency.* Annals of Mathematical Statistics **22** (1951) 7986.

[3] B. Riemann, *Gesammelte mathematische Werke und wissenschaftlicher Nachlass*, (1876), BG Teubner.

[4] P. J. Schönbucher, *Credit derivatives pricing models: models, pricing and implementation*, (2003), Wiley.

[5] S. R. S. Varadhan, *Large deviations*, The Annals of Probability **36** (2008) 397–419.

[6] O. Vasicek, *Probability of loss on loan portfolio*, Working paper (1987), KMV Corporation.