Politecnico di Milano

Financial Engineering

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Assignment 4

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1 Question A

Under the Large Homogeneous Portfolio assumption, the price of a mezzanine Tranche was computed using a double t-Student model with $\nu = 4$ (degrees of freedom).

- Starting by computing the value of K:

  - Probability of default given y:

$$P(v_i \leq K \mid y) = t_\nu(\frac{k - \sqrt{\rho}y}{\sqrt{1-\rho}}) \tag{1}$$

  - Probability of default:

$$P(v_i \leq K) = \int_{-\infty}^{+\infty} t_\nu(\frac{k - \sqrt{\rho}y}{\sqrt{1-\rho}})\phi(y)dy \tag{2}$$

  where $\phi(y)$ is the t-student pdf

- The price of the tranche:

$$Price_{tranche}(t_0) = \eta B(t_0, T)[(k_u - k_d) - \mathbb{E}[l_{tr}(z)]] \tag{3}$$

  Where:

  – Discount factor: $B(t_0, T)$
  – Notional: η
  – Loss of the tranche: $l_{tr}(z) = min[max(l_{rp} - k_d, 0), k_u - k_d]$
  – Loss of the reference portfolio: $l_{rp} = (1 - \pi)z$
  – Frequency: $z = \frac{m}{I}$

  We need to compute the $\mathbb{E}[l_{tr}(z)]$ and for this we will need the pdf of z.

$$\mathbb{E}[l_{tr}(z)] = \int_0^1 l_{tr}(z)f(z)dz \tag{4}$$

- The pdf of z

$$\begin{aligned} f(x) &= \frac{\partial P(z \leq x)}{\partial x} \\ &= pdf_{t_v}(-y^*)\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{1}{pdf_{t_v}(cdf_{t_v}^{-1}(x))} \end{aligned} \tag{5}$$

By inverting formula (2) to calibrate K and applying the described procedure to price the tranche, we obtained the following numerical result:

$$Price_{tranche} = 4.866948997726329e + 07$$

2 Question B

In this section the goal is to verify the result of the price of the mezzanine tranche obtained using the t-Student model with a large number of degrees of freedom, namely $\nu = 200$, is the same as the price obtained using Vasicek Model. Since we know that when the numer of degrees of freedom is large we have $t_\nu \approx \mathcal{N}$.

Vasicek Model:
The Vasicek model is implemented following the same chart as for the double t-student model but keeping in mind that the random variables follow a Normal distribution.

- Starting by computing the value of K:

$$K = \mathcal{N}^{-1}(P) \tag{6}$$

where P is the probability of default ($P = P(v_i \leq K)$)

- The pdf of z

$$f(x) = pdf_{\mathcal{N}}(-y^*)\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{1}{pdf_{\mathcal{N}}(cdf_{\mathcal{N}}^{-1}(x))} \tag{7}$$

Price of mezzanine tranche		
double t-student($v = 200$)	Vasicek Model	Relative difference
4.604382481411368e+07	4.599298437440572e+07	0.001105395538026

As expected before we get the same value of the price of mezzanine tranche. The Vasicek and the double t-student models were compared and the results can be considered indistinguishable since the relative difference is approximately of 0.1%.

3 Question C

In this section we used an approximation of the expectation of the loss of the tranche using Kullback-Leibler entropy and Stirling formula in order to cope with the heavy innate computations of the factorial.

1. **Exact Expectation:**

$$\mathbb{E}[l_{tr}(z)] = \int_{-\infty}^{+\infty} dy\phi(y)\sum_{m=0}^{I} P(m \mid y) \tag{8}$$

   - $P(m \mid y) = \binom{n}{k} p(y)^m (1-p(y))^{I-m}$

2. **Approximated Expectation:**

$$\mathbb{E}[l_{tr}(z)] \approx \int_{-\infty}^{+\infty} dy\phi(y)\int_0^1 dzC^{(1)}(z)\exp\left(-Ik(z;p(y))\right)l_{tr}(z) \tag{9}$$

   where:

   - **Kullback-Leibler entropy**: $k(z;p(y)) = z\log\frac{z}{p(y)} + (1-z)\log\frac{1-z}{1-p(y)}$
   - $C^{(1)}(z) = \sqrt{\frac{I}{2\pi z(1-z)}}$

Namely, we used these approximations (KL and LHP described in the first section) to compute the price of the mezzanine tranche for different values of I, ranging between 10 and $2 * 10^4$. We also computed the price with the exact formula, but had to stop at $I = 50$, the maximum value for which our computer enabled us to obtain a price.
The results of the prices in percentage in log-lin scale can be observed in Figure 1.

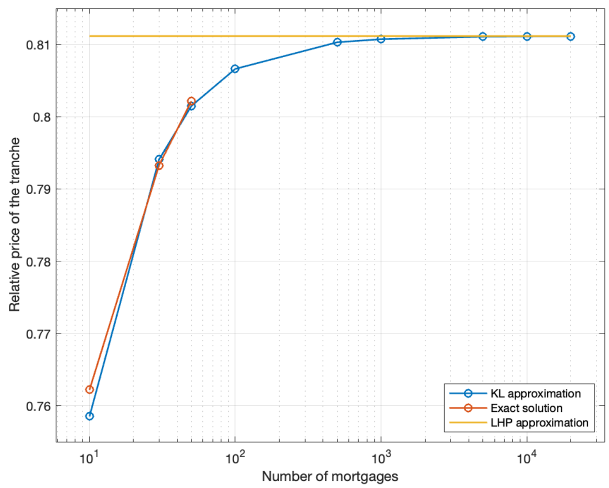

Figure 1: Price of the mezzanine tranche

Finally, we want to point out the fact that for the computation of the external integral we avoided the classical method since it lead us to bad result due to the shape of the density of y. To solve this problem we provided 3 different numerical methods to approximate the integral: midpoint, trapezoidal and Cavalier-Simpson formulas.

4 Question D

Using the same function that we described above to price the tranche, we computed the relative price of the equity tranche using the values $k_d = 0$ and $k_u = 0.03$. The result can be observed in Figure 2.

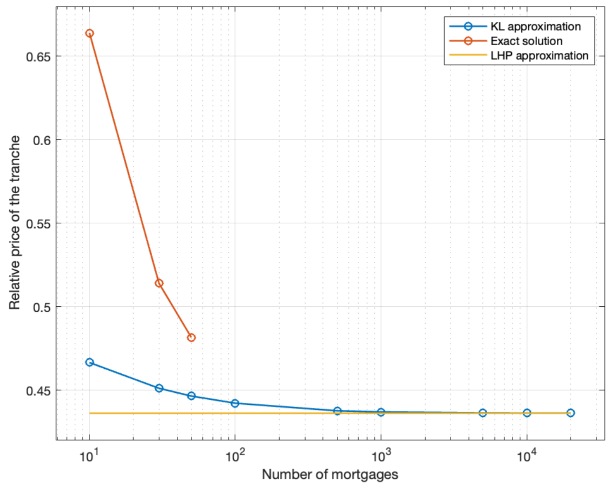

Figure 2: Price of Equity tranche

As we can notice, KL is not an adequate approximation, and this is due to the fact that Stirling approximation doesn't work in this range (namely if we reach $K_d = 0$). An adequate modification for the pricing function for the equity tranche is the following: since the price of the whole reference portfolio can computed in the following way

$$Price_{rp} = Price_{tranche} + Price_{Equitytranche}, \tag{10}$$

where:

- $Price_{rp} = \eta B(0, 3y)[1 - (1 - \pi)p]$ is the price of the reference portfolio
- $Price_{tranche}$: is the price of a tranche composed by all the mezzanine tranches from $k_d = 0.03$ and $k_u = 1$.

we can compute the price of the equity tranche by difference between the price of the portfolio and the one of the other tranche.
By applying this method we obtained the following, satisfying result:

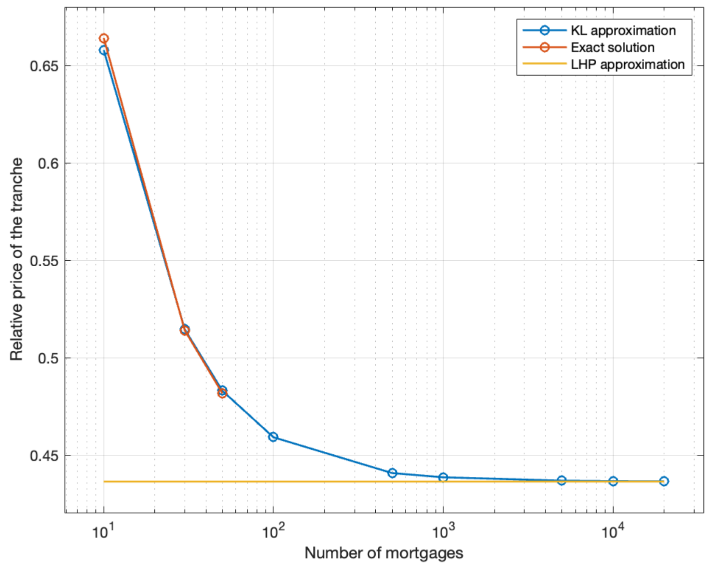

Figure 3: Price of Equity tranche

Lastly, we wanted to mention the fact that our whole code for this assignment takes between 1 and 2 minutes to run. By looking at the profiler report we noticed that the heaviest function are the one for numerical computation, like midpoint function, to approximate the integral, and tcdf.
Anyways, we can overall deem the computations required for this assignment to be quite intensive, so a 2 minutes long run is probably still an acceptable result according to us.

PROFILER

Profile Summary (Total time: 113.205 s)

Flame Graph

Flame graph is not available because the number of function calls exceeds the current profiler history size of 5000000. Increase the value for the 'historysize' profiling option and rerun the Profiler. For more information, see profile.

Generated 16-mar-2023 17:32:58 using performance time.

Function Name	Calls	Total Time (s)	Self Time* (s)	Total Time Plot (dark band = self time)
run_assignment4_group7	1	113.205	0.062	
midpointc	36	108.892	0.012	
tcdf	732544	76.117	29.071	
mezzanine_tranche_price_dts_exact	3	66.928	0.007	
mezzanine_tranche_price_dts_exact>@(y)int_inner(y).*tpdf(y,nu)	9	66.844	0.002	
mezzanine_tranche_price_dts_exact>@(y)arrayfun(@(y)sum(P(m,y).*loss_tranche(m./I(i))),y)	9	66.839	0.091	
mezzanine_tranche_price_dts_exact>@(y)sum(P(m,y).*loss_tranche(m./I(i)))	9000	66.749	0.081	
mezzanine_tranche_price_dts_exact>@(m,y)arrayfun(@(m)nchoosek(I(i),m).*p(y).^m.*(1-p(y)).^(I(i)-m),m)	9000	66.643	3.010	
mezzanine_tranche_price_dts_exact>@(m)nchoosek(I(i),m).*p(y).^m.*(1-p(y)).^(I(i)-m)	279000	63.633	2.045	
mezzanine_tranche_price_dts_exact>@(y)tcdf((K-sqrt(rho).*y)./sqrt(1-rho),nu)	558000	57.779	1.746	
mezzanine_tranche_price_dts_KL	3	42.189	0.010	
mezzanine_tranche_price_dts_KL>@(y)int_inner(y).*tpdf(y,nu)	27	42.032	0.002	
mezzanine_tranche_price_dts_KL>@(y)arrayfun(@(y)quadgk(@(z)C1(z).*exp(-I(i).*KL(z,y)).*loss_tranche(z),0,1),y)	27	42.019	0.420	
quadgk	27125	41.847	6.815	
mezzanine_tranche_price_dts_KL>@(y)quadgk(@(z)C1(z).*exp(-I(i).*KL(z,y)).*loss_tranche(z),0,1)	27000	41.599	0.302	
quadgk>vadapt	27125	35.032	8.102	
normcdf	732544	27.393	11.479	
quadgk>f1	87481	26.499	0.772	
quadgk>evalFun	87481	25.726	1.162	
mezzanine_tranche_price_dts_KL>@(z)C1(z).*exp(-I(i).*KL(z,y)).*loss_tranche(z)	87092	22.911	0.830	
mezzanine_tranche_price_dts_KL>@(z,y)z.*log(z./p(y))+(1-z).*log((1-z)./(1-p(y)))	87092	21.570	1.051	
mezzanine_tranche_price_dts_KL>@(y)tcdf((K-sqrt(rho).*y)./sqrt(1-rho),nu)	174184	20.519	0.622	
normcdf>localnormcdf	732544	15.915	15.915	
distchck	733032	12.776	12.776	
dominantType	733032	6.908	6.908	
nchoosek	279000	3.808	3.808	
semilogx_tranche_prices	3	2.757	0.762	
quadgk>checkSpacing	60356	0.884	0.884	
legend	3	0.868	0.012	

Figure 4: Profiler report