Politecnico di Milano

Financial Engineering

A.Y. 2022/2023

Assignment 6

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1 Exercise 1

In this exercise we wanted to determine the participation coefficient of a certificate that pays a coupon corresponding to:

$$\alpha * (S(t) - P)^+ \tag{1}$$

at maturity date, where alpha is the participation coefficient of the performance and S(t) represents an equally weighted basket of ENEL S.p.A. and AXA S.A. knowing the spread over Libor (200 bps) and given an upfront of 2.5% at the certificate issue, in a single-curve interest rate modeling setting, knowing that the two counterparties have signed an ISDA with CSA that allows neglecting the counterparty risk.

After obtaining the discount factors and forward rates from the bootstrap, we initialized the stock values of ENEL and AXA with their closing prices on the 31^{st} of January 2023.
We then simulated with Monte Carlo 10^7 times their increments year by year and computed the mean of the coupon (before multiplying it by α), obtaining the following confidence interval: $[4.0678; 4.0683]$, with a length of 5 basis points. We also tried incrementing or decreasing the number of simulations, but we noticed that 10^7 was the first value with which the length of the interval reached the order of basis points. Moreover, passing from 10^7 to 10^8 simulations the confidence interval didn't change considerably (from 5 to 1 bps), while the computational time increased substantially (from 2 seconds to 160 seconds), so we decided that 10^7 was the best trade off between accuracy and computational cost.
We then computed the NPV of the certificate as a function of the participation coefficient:

$$NPV = X - 1 + B(t_0, t_{end}) * (coupon + P) - spol * BPV \tag{2}$$

and then found the value of α such that the NPV would be 0 with the Matlab solver, which was: **0.068650**.

2 Exercise 2

In this exercise we wanted to investigate the difference between the price of a digital option computed with Black Model considering or not the implied volatility smile. We have a closed formula to price a Digital Option with the Black Model, which is:

$$d_{CB}(K) = -\frac{\partial CB(K)}{\partial K} = B(t_0, t)N[d2] \tag{3}$$

In the computation we assumed K equal to S0 (2973.87398962681) since it is an ATM Spot option. To compute the price we still needed the volatility correspondent to our K, so we interpolated on the volatility surface to obtain the value for our strike (0.147746088233469).
At this point we had everything to use Black Formula and then derive it w.r.t. K to obtain the digital option price, which we then multiplied by it's payoff and obtained: **217606.289607**.
The next step was to compute the price considering the slope impact of the implied volatility curve. The slope impact is an additional term in the formula to price digital options due to the fact that also sigma is a function of K. If we consider this fact as well, we obtain a new result of the partial derivative:

$$d_{CB}(K) = -\frac{\partial CB(K, \sigma(K))}{\partial K} - \frac{\partial \sigma(K)}{\partial K}\frac{\partial CB(K, \sigma)}{\partial \sigma} = BlackTerm - SlopeImpact * Vega \tag{4}$$

At this point we needed to compute the derivative of the smile in the point of interest (S0) and we did it numerically using midpoint technique.
In the end, the price considering this new factor and the payoff was: **293559.943594**.
We can observe that this difference of price is a risk and it is present whenever we have a payoff with

discontinuities of first order. There are models that can handle this risk in a proper way.

3 Exercise 3

In this case we had to compute the price for a call option as a function of the log-moneyness using the Lewis formula.
Firstly, we were asked to use 3 different methods for a given α equal to zero. We report the results showing the graphical comparison between the 3 different methods and a zoomed version of it.

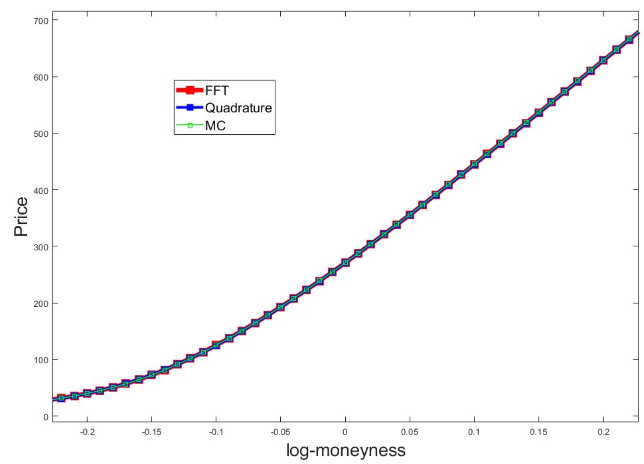

Figure 1: Price with three different methods

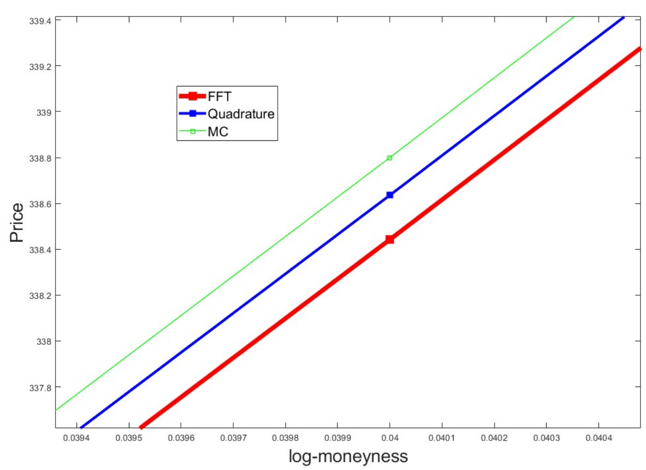

Figure 2: Zoom of price with three different methods

We want to point out the following steps in the procedure of FFT method:

1. when we use FastFourierTransform we have to pay attention to the function we give as argument: when we discretize using the double variate grid we end up with an additional term that must be considered as a new part of the function that will be the input of FFT algorithm.

2. to compute the characteristic function we use the formula with Laplace exponent (its limit for α going to zero)

3. the discretization parameters are 8: N,M, x_1, x_N, z_1, z_N, dx, dz; however, based on the following rules, only two of them are real free parameters (we chose x_1 and M): $N = 2^M$, $x_1 = -x_N$, $z_1 = -z_N$, $dx = \frac{x_N - x_1}{N}$, $dz = \frac{z_N - z_1}{N}$, $dx * dz = \frac{2*\pi}{N}$. By changing the parameters M and x1 we obtained the following results:

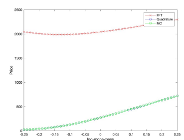

Figure 3: M=8

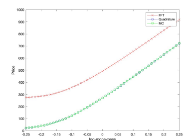

Figure 4: M=10

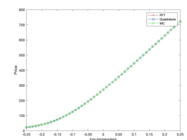

Figure 5: M=12

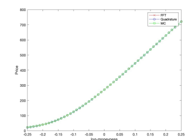

Figure 6: M=15

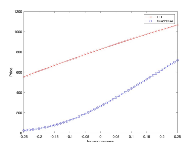

Figure 7: x_1=-2

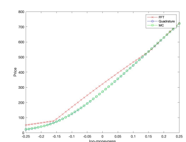

Figure 8: x_1=-10

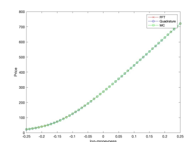

Figure 9: x_1=-100

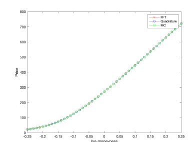

Figure 10: x_1=-500

We can observe that incrementing M, with x_1 fixed, the results improve and the viceversa holds as well (incrementing the absolute value of x_1 leads to an improvement). So the best results are obtained with M=15 and $x_1 = -500$, which were the values that we decided to use in the end.

Moreover, we want to analyze the performance of quadrature method procedure, which was:

1. easier to implement w.r.t. FFT
2. slightly slower than FFT

Elapsed Time	
FFT	Quadrature
0.022226	0.049269

Finally, we want to point out the following steps in the procedure of Monte Carlo method:

1. We simulated the price of the forward
2. We checked if numerical moments were equal to theoretical moments. To compute the theoretical moments we used the following result from probability theory which uses the characteristic function:

$$\frac{d\phi_x^{(n)}(\eta)}{d\eta}|_{\eta=0} = \frac{d\phi_x^{(n)}(E[e^{i\eta x}])}{d\eta}|_{\eta=0} = i^n E[x^n] \tag{5}$$

We obtained the following satisfying results:

	1^{st}moment	2^{nd}moment	3^{rd}moment	4^{th}moment
Theoretical	1.0000	2.1000	6.7200	28.8960
Numerical	0.9992	2.1005	6.7475	29.2869

3. we want to stress the fact that MonteCarlo method is way slower than the other two. The Elapsed Time for 1MIO simulation in the Montecarlo method is 0.9629, much higher than the other ones

Secondly, we were asked to repeat the calculation with a different α, equal to $\frac{1}{3}$, only for the first two methods. The results are the following:

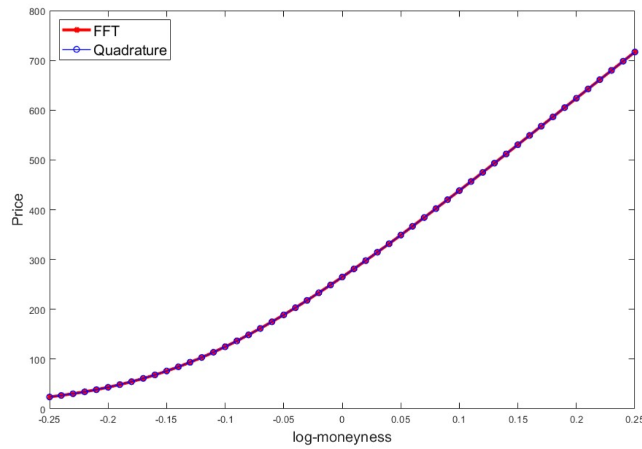

Figure 11: Prices with two different methods $\alpha = \frac{1}{3}$

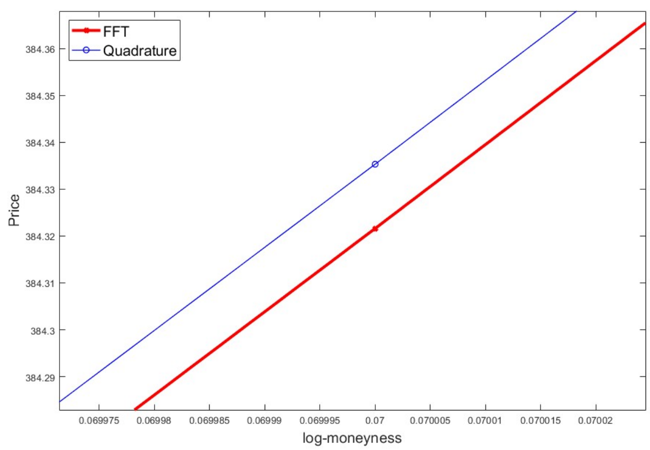

Figure 12: Zoom of prices with two different methods $\alpha = \frac{1}{3}$

In order to evaluate the difference between the results with respect to different values of alpha we considered the norm of the vector of the difference of prices computed via FFT and Quadrature. We obtained the following results:

	$\alpha = 0$	$\alpha = \frac{1}{3}$
Norm of the difference vector	2.0767	0.42234

4 Exercise 4

We were asked to calibrate a normal mean-variance mixture with $\alpha = 1/3$ model parameters considering SP 500 implied volatility surface via a global calibration with constant weights.
Starting from the given volatility smile we computed Black's prices. Then we computed prices using Lewis formula:

$$\frac{C(x)}{B(t_0,t)F_0} = 1 - e^{-\frac{x}{2}} \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{-i\xi x} \Phi(-\xi - \frac{i}{2}) \frac{1}{\xi^2 + \frac{1}{4}} \tag{6}$$

with:

$$\Phi(\xi) = \exp(-i\xi lnL(\eta))L(\frac{\xi^2 + i(1+2\eta)\xi}{2})$$

considering the Laplace Exponent with $\alpha = \frac{1}{3}$:

$$lnL[w] = 2\frac{\Delta t}{k}[1 - (1 + wk\sigma^2\frac{3}{2})^{\frac{1}{3}} \tag{7}$$

Then we minimized the $L^2 - norm$ of the difference between the two prices. We obtained the following values for σ: **0.122372853268195**, k: **1.567895840366799** and η: **7.744623162724620**.

We then computed the prices with the normal mean-variance model with the parameters found by minimization. Lastly, we computed the implied volatility obtained using these prices and compared it with the one of the market. The result can be observed in the figure below.

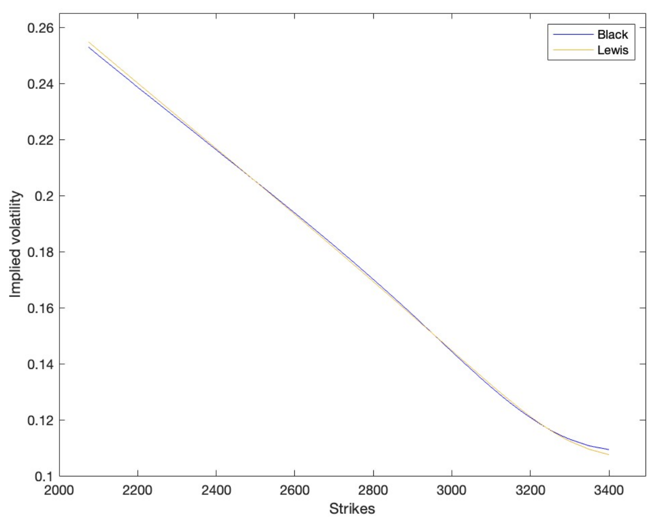

Figure 13: Comparison of model VS market implied volatility