Politecnico di Milano

Financial Engineering

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Assignment 1

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1 A

After setting all the simulation parameters, we used a for loop to price the call option with three different approaches:

- via blkprice Matlab function,
- with a CRR tree approach
- with a Monte-Carlo (MC) approach.

In particular, in the CRR tree approach, we computed the discretization parameters, upward-downward movements and probabilities. We proceeded to compute the forward and the price at maturity - which coincides with the payoff of the call option.
We then performed a backward in time procedure to compute at each time step the expected value at each node using the upward and downward values from the previous time ("previous" in the sense of the procedure, so following in the actual timeline) and finally discounted the value at the initial node to obtain the price.

Concerning the Monte Carlo approach, instead, we built a function that simulated N standard normal random variables, used them to compute the forward and averaged the discounted payoff to obtain the price.

Using the previously described methods with $M = 100$ (both for the tree and MC since we are using the for loop) we get a reasonable price for the tree approach (comparing it to the closed formula), while the price obtained by the MC approach is quite off and is extremely dependant on the simulation. This is of course due to the fact that while $M = 100$ is reasonable as number of time steps in the CRR tree, it for sure isn't an optimal choice for the MC approach, which usually requires a much bigger value for simulations. Anyways, we present these first results for the prices in the following table and refer to the next question to investigate on more adequate values for M.

Call option prices		
Method	Unitary	Notional
Closed Formula	4.764227e-02	4.764227e+04
CRR tree	4.758248e-02	4.758248e+04
MC sim(100)	5.091409e-02	5.091409e+04
MC sim(10^6)	4.763317e-02	4.763317e+04

2 B

We proceeded with the evaluation of the errors of the numerical techniques: the CRR tree and the MC simulation.
For the CRR tree, we set 100 time steps and the abs of the difference between the price of this method and the one of the closed formula as the error.
For the MC simulation, instead, we set 10^6 simulations and performed two separate simulations to compute the price and it's standard deviation - our estimate for the error - to make it unbiased.
We can observe our results in the table below, in particular in both cases the error is below 1bp and, knowing that the bid-ask spread is always higher than this, we can say that the chosen values for the discretization parameter (M) can be considered adequate. Also, this choice of the discretization parameters isn't computationally expensive, allowing the code to run extremely fast, so we can further validate our previous conclusion.

Call option errors	
Method	Error
CRR tree	5.978511e-05
MC simulation	7.560748e-05

3 C

We also iterated over different values for the discretization parameter M both for the CRR tree and the MC simulation to compute the respective errors exactly as in the previous section. As we can see from the plots below, we can notice that the CRR tree error rescales with M as $\frac{1}{M}$, while the MC error rescales as $\frac{1}{\sqrt{M}}$.

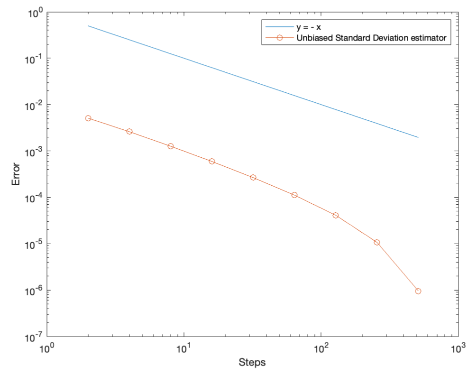

Figure 1: Error estimate for the CRR tree.

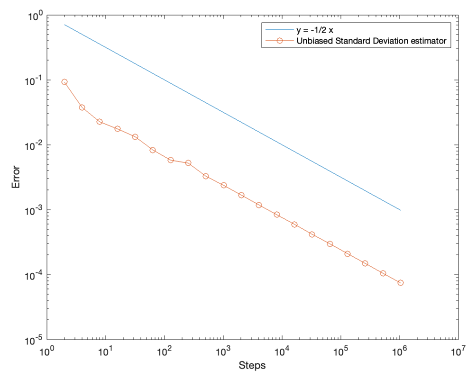

Figure 2: Error estimate for the MC simulation.

4 D

We now consider a European Call Option with European barrier at €1.2 (up&out).
We start by pricing it with the CRR tree method, namely we build a new function which is almost identical to the one described in the first section for the European call, the only difference is that in the last nodes we set as payoff the one of the call multiplied by the indicator function which ensures that we are below the knock out value at maturity. This is enough since the barrier is European and therefore only checked at maturity.
We then priced it with the Monte Carlo simulation, building a function that is again identical to the one in section 1 except for the payoff.
Furthermore, the exact price can be computed by recalling the fact that (as can be easily seen looking at the payoff) the U&O call is equivalent to a bear spread (with strikes corresponding to the one of the call and the barrier) minus a digital option at the barrier multiplied by the difference between the barrier and the strike of the call:

$$Call\_UO(K, B) = Bear\_spread(K, B) - (B - K) * Digital(B). \tag{1}$$

The results for the three methods can be observed in the following table:

U&O Call option prices	
Method	Price
Closed Formula	3.204977e+04
CRR tree	3.196405e+04
MC simulation	3.205599e+04

All the values are close to each other (the difference in price with respect to the closed formula is approximately 1bps in both cases) and they are lower than the European call without the barrier, as we expect for every barrier option because of the additional constraints they entail.

5 E

An array of Spot prices S_0, was created going from 0.7 to 1.35. The gamma was calculated based on three methods:

- Closed formula Method,
- with a CRR tree approach
- with a Monte-Carlo (MC) approach.

During each one, the prices of the up and out call were calculated using the functions that were created before, namely, using the closed formula, the tree method and the MC method to form an array of prices corresponding to each spot, then the second derivative was calculated using a numerical method

$$f''(x_i) = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} \qquad i = 2, ..., N-1 \tag{2}$$

Below is the plot of gamma as a function of S_0, we can notice that all methods give same results up to a certain point. However, it's worthy to note that for an increment of 0.025, we reached the best results for the convergence of gamma, in the tree method, other values led to uneven diverging plots.

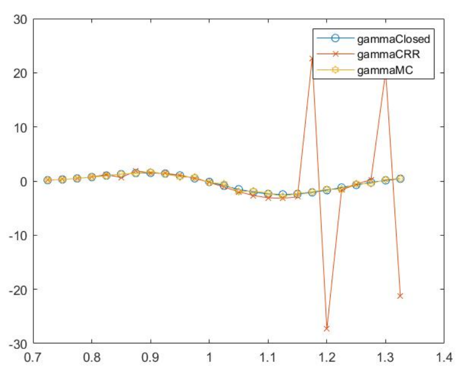

Figure 3: Gamma

6 F

At this stage, we went back and tried to see if a variance reduction technique, namely Antithetic Variables, would reduce the error in the MC simulation of section 2.
In order to do this, we estimated the error by simulating both the futures and an antithetic version

(obtained by switching the standard normal with minus the same standard normal, such that the correlation between the two of them would be exactly -1).
In this way we obtained two payoffs (related to the futures and the antithetic ones) and averaged them before computing the standard deviation (our estimate for the error).
In particular, we obtained an error of $4.153744e-05$, which was an improvement (approximately of a factor 2) with respect to the value we obtained in section 2, namely: $7.571947e-05$.

7 G

Lastly, we computed the price of a Bermudan option, where the holder has also the right to exercise the option after one month, obtaining the stock at the strike price.
In particular, we used again the CRR tree method with the same structure as the one described in the first section, with just a few changes.
Namely, in the backward procedure, when we reached the early exercise date (in this case the midpoint of the procedure) we replaced the continuation value with the maximum between the latter and the exercise value, while in all the other time steps we computed the discounted expected value as in the European case.
With 100 time steps the result that we obtained is: $4.758951e+04$.
The price is larger than the one of the European case ($4.758248e+04$) as we expected. Moreover, we noticed that the difference is not very large, but we tried increasing the number of dates in which is possible to exercise the option and the difference got wider, also as we expected. We therefore attributed this fact to only having one exercise date before the maturity, and expect the price to reach the value of the American option in the limit as the number of early exercise dates increases.

8 Appendix

For further verification, we tried to calculate the gamma in part E, using a full formula of gamma by deriving the Gamma of the two European calls and the digital call and summing them all together, below is the plot of Gamma as a function of S0, using both the numerical approximation of the second derivative and the full formula of the second derivative (instead of calculating the prices of the calls at each spot) and they both seemed to be very similar.

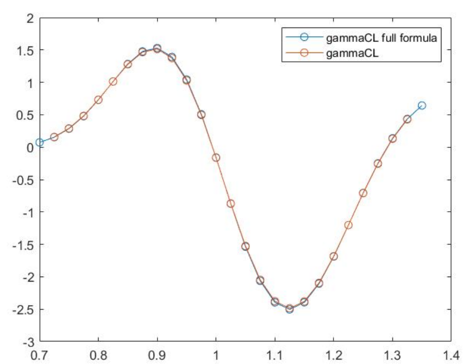

Figure 4: Gamma

GAMMA:

$$\begin{aligned}\Gamma &= \frac{\partial \Delta}{\partial S_0} \\ &= \frac{\partial^2 C}{\partial S_0\, \partial S_0}\end{aligned}$$

we will compute every term and then sum up them:

$$\Gamma = \Gamma_{Ck} - \Gamma_{Ck0} - \Gamma_{D0}$$

Before starting: Qantities that we need

$d_1 = \frac{ln(\frac{F_0}{K})}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$
$d_2 = \frac{ln(\frac{F_0}{K})}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}$
$F_0 = \frac{S_0 \exp{(-d(t-t_0)}}{B(t_0,t)}$
$C_k(t_0,t) = B(t_0,t)(F_0\mathcal{N}(d_1) - K\mathcal{N}(d_2))$
$\Delta = \frac{\partial C}{\partial S_0} = B(t_0,t)\mathcal{N}(d_1)\frac{\partial F_0}{\partial S_0}$
$D_0 = (k_0 - k)B(t_0,t)\mathcal{N}(d_2)$

Calculation: $\Gamma_{Call} = \frac{\exp{(-d(t-t_0)\phi(d_1)}}{F_0 B\sigma\sqrt{t-t_0}}$

We have: $F_0 = \frac{S_0 \exp{(-dT)}}{B}$
And:
$d_{1,2} = \frac{ln(\frac{F_0}{K})}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$

$\Delta = \frac{\partial C}{\partial S_0} = B(t_0,t)\mathcal{N}(d_1)\frac{F_0}{S_0} = B(t_0,t)\frac{\exp{-dT}}{B(t_0,t)}\mathcal{N}(d_1)$

$\frac{\partial\Delta}{\partial S_0} = B(t_0,t)\frac{\exp{(-dT)}}{B(t_0,t)}\frac{\partial(\mathcal{N}(d_1))}{\partial S_0}$

$\frac{\partial(\mathcal{N}(d_1))}{\partial S_0} = \phi(d_1)\frac{\partial d_1}{\partial S_0} = \phi(d_1)\frac{1}{S_0}\frac{1}{\sigma\sqrt{T}} = \phi(d_1)\frac{\exp{-dT}}{F_0 B}\frac{1}{\sigma\sqrt{T}}$

$\frac{\partial^2 C}{\partial S_0\,\partial S_0} = \frac{\exp{(-dT)}\phi(d_1)}{S_0\sigma\sqrt{T}} = \frac{\exp{(-2dT)}\phi(d_1)}{F_0 B\sigma\sqrt{T}}$

Calculation of GAMMA: $\Gamma = (k_0 - k)\frac{\exp{(-2dT)}}{B}\frac{\phi(d_2)}{F_0^2\sigma\sqrt{T}}(\frac{-d_2}{\sigma\sqrt{T}} - 1)$

We Know: $\Gamma = \frac{\partial^2 D_0}{\partial S_0\,\partial S_0}$

$\frac{\partial(\mathcal{N}(d_2))}{\partial S_0} = \phi(d_2)\frac{\partial d_2}{\partial S_0} = \phi(d_2)\frac{1}{\sigma\sqrt{T}}\frac{1}{S_0}$

$$\begin{aligned}
\frac{\partial(\phi(d_2)\frac{1}{\sigma\sqrt{T}}\frac{1}{S_0})}{\partial S_0} &= \frac{\partial(\phi(d_2))}{\partial S_0}\frac{1}{\sigma\sqrt{T}} + \phi(d_2)\frac{1}{\sigma\sqrt{T}}(\frac{-1}{S_0^2}) \\
&= \phi(d_2)(\frac{-1}{2}2d_2\frac{\partial(d_2)}{\partial S_0})\frac{1}{\sigma\sqrt{T}}\frac{1}{S_0} + \phi(d_2)\frac{1}{\sigma\sqrt{T}}(\frac{-1}{S_0^2}) \\
&= \phi(d_2)\frac{1}{\sigma\sqrt{T}}(-d_2\frac{1}{S_0^2}\frac{1}{\sigma\sqrt{T}} - \frac{1}{S_0^2}) \\
&= \phi(d_2)(\frac{-d_2}{\sigma\sqrt{T}} - 1)\frac{1}{S_0^2\sigma\sqrt{T}}
\end{aligned}$$

Final Γ:

$$\Gamma = \frac{\exp{(-2dT)}}{B\sigma\sqrt{T}}(\frac{\phi(d_1,k)}{F_0} - \frac{\phi(d_1,k_0)}{F_0} - (k_0 - k)\frac{\phi(d_2)}{F_0^2}(\frac{-d_2}{\sigma\sqrt{T}} - 1))$$