This Notebook belongs to :-

Riddhi B. Mehta

Contact :
riddhi.mehta1721@gmail.com

Date : ___/___/___

Math- Vector Integral. 1

Def: Let $\bar{f}(x,y,z)$ be a continuous vector field, then the line integral of $\bar{f}$ along a curve 'C' is defined as:

$$L.I. = \int \bar{f} \cdot d\bar{r} \quad \text{(dot product)}$$

where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of any point in 3D space.
and $\bar{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

For Example:

$$\bar{f}(x,y,z) = (x^2+y^2)\hat{i} + (3x-2y+z)\hat{j} + (z^2)\hat{k}$$

Mathematically:

$$d\bar{r} = d(\bar{r}) = d(x\hat{i}+y\hat{j}+z\hat{k}) = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\therefore \bar{f} \cdot d\bar{r} = f_1 dx + f_2 dy + f_3 dz$$

$$\boxed{\therefore \int_C \bar{f} \cdot d\bar{r} = \int_C f_1 dx + f_2 dy + f_3 dz.}$$

Date: __/__/__

Physicality:

We know that work done 'w' is a scalar quantity i.e. to move a particular object from point A to B along path 'C' using field $\bar{f}$ then,

$$w = \text{force} \cdot \text{displacement}$$

$$\therefore w = \int_c \bar{f} \cdot d\bar{r}$$

<u>Note</u>: If C is closed, then $\int_c \bar{f} \cdot d\bar{r}$ is called the 'circulation' of $\bar{f}$ along C.

<u>Also</u>: If $\bar{f} = \nabla \phi$, then such a field is called 'conservative'.

So, if field is conservative, then $\oint_c \bar{f} \cdot d\bar{r} = 0$, i.e. even the circulation is zero.

Date: __/__/__

Important Formula:

1) Vector differential operator:

$$\boxed{\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}}$$

2) Gradient of scalar function:

Let ϕ be a scalar function defined for region R, then

gradient of ϕ = vector function $\nabla\phi$

$$\therefore \boxed{\text{grad}(\phi) = \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}}$$

3) Standard results:

$$\boxed{\text{(i) } \nabla(\phi \pm \psi) = \nabla\phi \pm \nabla\psi}$$

$$\boxed{\text{(ii) } \nabla(\phi\psi) = \phi(\nabla\psi) + \psi(\nabla\phi)}$$

Date : __/__/____

4) Divergence of $\bar{f}$.

$$\nabla \cdot \bar{f} = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (i f_1 + j f_2 + k f_3)$$

$$\boxed{\nabla \cdot \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}}$$

$\downarrow\downarrow$

Scalar point function.

Note: If $\boxed{\nabla \cdot \bar{f} = 0}$, then vector field $\bar{f}(x, y, z)$ is Solenoidal.

⇨ Next Page

Date : __/__/____

5) Curl of $\bar{f}$. (5)

$$\boxed{\nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}}$$

$\downarrow\downarrow$

Vector point function

Note: If $\boxed{\nabla \times \bar{f} = \bar{0}}$, then vector field $\bar{f}(x, y, z)$ is Irrotational.

⇨ P.T.O.

Sums:

① $\bar{f} = (x^2+y^2)\hat{i} + (xy)\hat{j} + (z^2-3x)\hat{k}$

along $c: \; x = t, \; y = t^2, \; z = t^3$

joining $(0,0,0)$ and $(1,1,1)$

$\therefore \bar{f} = (t^2+t^4)\hat{i} + (t^3)\hat{j} + (t^6-3t)\hat{k}$.

$\therefore \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= t\hat{i} + t^2\hat{j} + t^3\hat{k}$

$\therefore d\bar{r} = \hat{i}\,dt + 2t\hat{j}\,dt + 3t^2\hat{k}\,dt$

$\therefore d\bar{r} = (\hat{i} + 2t\hat{j} + 3t^2\hat{k})\,dt$

$\therefore \bar{f}\cdot d\bar{r} = \left[(t^2+t^4) + (t^3\cdot 2t) + (t^6-3t)(3t^2)\right] dt$

$= (t^2+t^4+2t^4+3t^8-9t^3)\,dt$.

$= (3t^8+3t^4-9t^3+t^2)\,dt$

$\therefore \int_c \bar{f}\cdot d\bar{r} = \int_0^1 3t^8+3t^4-9t^3+t^2\,dt$

Limits: $0 < x < 1$; $0 < t < 1$ → We wanted limits of 't'

$0 < y < 1$; $0 < t^2 < 1$

$0 < z < 1$; $0 < t^3 < 1$ } → not reqd.

$\Rightarrow \left[\frac{3t^9}{9} + \frac{3t^5}{5} - \frac{9t^4}{4} + \frac{t^3}{3}\right]_0^1$

$\Rightarrow \left(\frac{3}{9} + \frac{3}{5} - \frac{9}{4} + \frac{1}{3}\right) - 0$

$\Rightarrow \boxed{\frac{-59}{60}}$

② Find line integral for $\bar{F} = x^2\hat{i} + (y^2-x^2)\hat{j}$ along an ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ (Circulation).

We can write, $x = r\cos\theta$; $y = r\sin\theta$

$\therefore x = 2\cos\theta \quad \therefore dx = -2\sin\theta\,d\theta$

$y = 3\sin\theta \quad \therefore dy = 3\cos\theta\,d\theta$

$\therefore d\bar{r} = \hat{i}\,dx + \hat{j}\,dy$

$\therefore d\bar{r} = \hat{i}(-2\sin\theta\,d\theta) + \hat{j}(3\cos\theta\,d\theta)$

Date : __/__/__

Calculating work done :

$$\Rightarrow \int_C \bar{F}\cdot d\bar{r} = \int_C d\phi = \int_C d\,(e^{xy}\cos z)$$

$$= \left[e^{xy}\cos z\right]_{(0,0,0)}^{(x,y,z)\;(-1,2,\pi)}$$

$$= e^{-2}\cdot\cos\pi - e^{0}\cos 0$$

$$= e^{-2}(-1) - 1$$

$$= \frac{-1}{e^2} - 1 = \frac{-1-e^2}{e^2} = \boxed{-\left(\frac{e^2+1}{e^2}\right)}$$

☞ Next Page

Date : __/__/__

⑥ Prove that $\bar{F}(x,y,z) = (y^2\cos x + z^3)\,\hat{i} + (2y\sin x - 4)\,\hat{j} + (3xz^2+2)\,\hat{k}$ is a conservative field. Also, find scalar potential and work done when the object is moved from $(0,1,-1)$ to $(\pi/2, -1, 2)$

i) Prove that $\nabla\times\bar{F} = \bar{0}$

$$\therefore \nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2+2 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(3xz^2+2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right)$$

$$- \hat{j}\left[\frac{\partial}{\partial x}(3xz^2+2) - \frac{\partial}{\partial z}(y^2\cos x + z^3)\right]$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x + z^3)\right)$$

$$= \hat{i}(0-0) - \hat{j}(3z^2 - 3z^2)$$

$$+ \hat{k}(2y\cos x - 2y\cos x)$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0) = \bar{0}$$

Hence, Proved !!

Date : __/__/__

$\bar{F} = \hat{i}(4\cos^2\theta) + \hat{j}(9\sin^2\theta - 4\cos^2\theta)$

$\therefore \bar{F}\cdot d\bar{r} = (4\cos^2\theta)(-2\sin\theta\, d\theta)$
$+ (9\sin^2\theta - 4\cos^2\theta)(3\cos\theta\, d\theta)$

$= -8\sin\theta\cos^2\theta\, d\theta$
$+ 27\sin^2\theta\cos\theta\, d\theta$
$- 12\cos^3\theta\, d\theta$

Limits of 'θ', closed ellipse hence, $0 \to 2\pi$.

$$\therefore \int_c \bar{F}\cdot d\bar{r} = \int_0^{2\pi} \left(-8\sin\theta\cos^2\theta + 27\sin^2\theta\cos\theta - 12\cos^3\theta\right) d\theta$$

Remember:

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$$

V. Important ✱✱

$$\Rightarrow \int_0^{2\pi} -8\sin\theta\cos^2\theta\, d\theta + \int_0^{2\pi} 27\sin^2\theta\cos\theta\, d\theta + \int_0^{2\pi} -12\cos^3\theta\, d\theta$$

$$\Rightarrow 4\int_0^{\pi/2} -8\sin\theta\cos^2\theta\, d\theta + 4\int_0^{\pi/2} 27\sin^2\theta\cos\theta\, d\theta + 4\int_0^{\pi/2} -12\cos^3\theta\, d\theta$$

$$\Rightarrow -4.4.\int_0^{\pi/2} 2\sin\theta\cos^2\theta\, d\theta + 27.2\int_0^{\pi/2} 2\sin^2\theta\cos\theta\, d\theta + -4.6\int_0^{\pi/2} 2\cos^3\theta\, d\theta$$

Date : __/__/__

$$\Rightarrow -16\left(2\int_0^{\pi/2} \sin\theta \cos^2\theta \, d\theta\right)$$

$$+ 54\left(2\int_0^{\pi/2} \sin^2\theta \cos\theta \, d\theta\right)$$

$$- 24\left(2\int_0^{\pi/2} \cos^3\theta \, d\theta\right)$$

For: $2\int_0^{\pi/2} \sin\theta \cos^2\theta \, d\theta$

$\therefore 1 = 2m-1 \quad \therefore 2m = 2 \quad \therefore \boxed{m=1}$

$\therefore 2 = 2n-1 \quad \therefore 2n = 3 \quad \therefore \boxed{n=3/2}$

$$\therefore \beta\left(1, \frac{3}{2}\right) = \frac{\Gamma 1 \cdot \Gamma 3/2}{\Gamma 5/2}$$

$$= \frac{0! \cdot \cancel{\frac{2}{2}} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{\cancel{\frac{2}{2}} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}$$

$$\boxed{\therefore \beta\left(1, \frac{3}{2}\right) = \frac{2}{3}}$$ //

Date : __/__/__ 11

For: $2\int_0^{\pi/2} \sin^2\theta \cdot \cos\theta \, d\theta = \boxed{\frac{2}{3}.}$

For: $2\int_0^{\pi/2} \cos^3 d\theta$

$\therefore 0 = 2m-1 \quad \therefore 2m = 1 \quad \therefore \boxed{m=1/2}$

$\therefore 3 = 2n-1 \quad \therefore 2n = 4 \quad \therefore \boxed{n=2}$

$$\therefore \beta\left(\frac{1}{2}, 2\right) = \frac{\Gamma 1/2 \cdot \Gamma 2}{\Gamma 5/2} = \frac{\sqrt{\pi} \cdot 1!}{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}$$

$$\boxed{\therefore \beta\left(\frac{1}{2}, 2\right) = \frac{4}{3}}$$

$$\Rightarrow -16 \cdot \frac{2}{3} + 54 \cdot \frac{2}{3} - 24 \cdot \frac{4}{3}$$

$$\Rightarrow \boxed{\frac{-20}{3}}$$ //

Date : __/__/__

(3) $\bar{f}(x,y,z) = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ along the line joining $(0,0,0)$ and $(2,1,3)$

$\therefore$ Equation of line in 3D $\Rightarrow \dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1} = t$

$\therefore \dfrac{x}{2} = \dfrac{y}{1} = \dfrac{z}{3} = t$

$\therefore C: \; x = 2t \; ; \; y = t \; ; \; z = 3t.$

$\therefore \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore \bar{r} = 2t\,\hat{i} + t\,\hat{j} + 3t\,\hat{k}$

$\therefore d\bar{r} = (2\hat{i} + \hat{j} + 3\hat{k})\,dt.$

$\therefore \bar{f} = (4\cdot 2t\cdot 3t)\,\hat{i} - (t^2)\,\hat{j} + (3t^2)\,\hat{k} = 24t^2\,\hat{i} - t^2\,\hat{j} + 3t^2\,\hat{k}$

$\therefore \bar{f}\cdot d\bar{r} = 2\cdot 24t^2\,dt - t^2\,dt + 3\cdot 3t^2\,dt$

$= (48t^2 - t^2 + 9t^2)\,dt$

$\therefore \bar{f}\cdot d\bar{r} = 56t^2\,dt$

Date : __/__/__

$\therefore \int \bar{f}\cdot d\bar{r} = \int_0^1 56t^2\,dt$

Limits: $0 < x < 2 \; ; \; 0 < 2t < 2$

$0 < y < 1 \; ; \; 0 < t < 1$

$0 < z < 3 \; ; \; 0 < 3t < 3$

All verified ! $\boxed{0 < t < 1}$

$\Rightarrow \left[\dfrac{56t^3}{3}\right]_0^1 \Rightarrow \dfrac{56}{3} - 0 \Rightarrow \boxed{\dfrac{56}{3}}$

(4) $\phi = xyz + z^2(\sin xy)$

$\therefore \nabla\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$

$= \hat{i}\left(yz + z^2\cos(xy)\cdot y\right) +$

$\hat{j}\left(xz + z^2\cos(xy)\cdot x\right) +$

$\hat{k}\left(xy + 2z\sin(xy)\right)$

Let $\bar{f} = \nabla\phi$, then prove that $\nabla\times\bar{f} = \bar{0}$

$$\therefore \nabla \times \bar{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz + yz^2\cos xy & xz + xz^2\cos xy & xy + 2z\sin xy \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(xy + 2z\sin xy) - \frac{\partial}{\partial z}(xz + xz^2\cos xy)\right)$$

$$- \hat{j}\left(\frac{\partial}{\partial x}(xy + 2z\sin xy) - \frac{\partial}{\partial z}(yz + yz^2\cos xy)\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}(xz + xz^2\cos xy) - \frac{\partial}{\partial y}(yz + yz^2\cos xy)\right)$$

$$\Rightarrow \hat{i}\left(x + 2z\cos(xy)\cdot x - (x + x\cos xy\, 2z)\right)$$

$$- \hat{j}\left(y + 2z\cos(xy)\cdot y - (y + y\cos(xy)\cdot 2z)\right)$$

$$+ \hat{k}\left(z + z^2(x(-\sin xy)\cdot y + \cos xy) - (z + z^2(y(-\sin xy)\cdot x + \cos xy))\right)$$

$$\Rightarrow \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$$\Rightarrow \bar{0}$$

Hence, Proved !!

Note: $\bar{f}$ is Conservative and Irrotational.

Date : __/__/__

Also, since, $\bar{f} = \nabla \phi$

$\therefore f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$

$= \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$

$\therefore f_1 = \frac{\partial \phi}{\partial x} \; ; f_2 = \frac{\partial \phi}{\partial y} \; ; f_3 = \frac{\partial \phi}{\partial z}$

$\therefore \phi = xyz + z^2 (\sin xy)$

$\therefore f_1 = \frac{\partial \phi}{\partial x}$ = refer from previous page!

→ Next Page

Notes by : Riddhi Mehta

Contact : riddhi.mehta1721@gmail.com

1) Prove it is conservative field
2) Find scalar potential, ϕ
3) Calculate work done $(0,0,0)$ to $(-1,2,\pi)$

Date : __/__/__

(5) $\bar{f}(x,y,z) = (ye^{xy} \cos z)\hat{i} + (xe^{xy} \cos z)\hat{j} - (e^{xy} \sin z)\hat{k}$

Before moving onto this question, let's understand the concept of scalar potential.

* A vector field $\bar{f}(x,y,z)$ is irrotational if there exists some scalar potential $\phi(x,y,z) = \lambda$ such that $\bar{f} = \nabla \phi$

When we compare the $\hat{i}\,\hat{j}\,\hat{k}$ components of $\bar{f} = \nabla\phi$, we get

1) $f_1 = \frac{\partial \phi}{\partial x}$

2) $f_2 = \frac{\partial \phi}{\partial y}$

3) $f_3 = \frac{\partial \phi}{\partial z}$

We know, $\phi(x,y,z) = \lambda$

$\therefore d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$

Integrate this, to find SCALAR POTENTIAL !!!

Date : __/__/__

$\therefore \bar{F}(x,y,z) = (ye^{xy}\cos z)\hat{i} + (xe^{xy}\cos z)\hat{j} - (e^{xy}\sin z)\hat{k}$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{xy}\cos z & xe^{xy}\cos z & -e^{xy}\sin z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}\left(-e^{xy}\sin z\right) - \frac{\partial}{\partial z}\left(xe^{xy}\cos z\right)\right)$$
$$- \hat{j}\left(\frac{\partial}{\partial x}\left(-e^{xy}\sin z\right) - \frac{\partial}{\partial z}\left(ye^{xy}\cos z\right)\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}\left(xe^{xy}\cos z\right) - \frac{\partial}{\partial y}\left(ye^{xy}\cos z\right)\right)$$

$$= \hat{i}\left(-\sin z \cdot e^{xy} \cdot x + xe^{xy}\sin z\right)$$
$$- \hat{j}\left(-\sin z\, e^{xy} \cdot y + ye^{xy}\sin z\right)$$
$$+ \hat{k}\left(\cos z\left(xe^{xy} \cdot y + e^{xy}\right) - \cos z\left(ye^{xy} \cdot x + e^{xy}\right)\right)$$

$$= \bar{0}$$

Date : __/__/__

$\therefore \bar{f}$ is a conservative / irrotational field.

$\therefore$ Finding the scalar potential :

$\frac{\partial \phi}{\partial x} = f_1 = ye^{xy}\cos z$

$\frac{\partial \phi}{\partial y} = f_2 = xe^{xy}\cos z$

$\frac{\partial \phi}{\partial z} = f_3 = -e^{xy}\sin z$

Assume, $\phi = c \quad \therefore d\phi = 0$

$\therefore d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0$

$\therefore ye^{xy}\cos z\, dx + xe^{xy}\cos z\, dy - e^{xy}\sin z\, dz = 0$

Trick is to find such a func. when differentiated will yield the above result.

$\therefore \boxed{\phi(x,y,z) = e^{xy}\cos z}$

Scalar Potential ↑↑↑↑↑↑↑

* Scalar potential is obtained by:

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$\therefore d\phi = (y^2\cos x + z^3)\, dx + (2y\sin x - 4)\, dy + (3xz^2 + 2)\, dz.$$

$$\boxed{\therefore \phi = y^2 \sin x + xz^3 - 4y + 2z}$$

* Work done is obtained by:

$$\int_c \bar{F}\cdot d\bar{r} = \int_c d\phi$$

$$= \left[y^2\sin x + xz^3 - 4y + 2z\right]_{(0,1,-1)}^{(x,y,z)\ (\frac{\pi}{2},-1,2)}$$

$$= \left(1 \sin\frac{\pi}{2} + \frac{\pi}{2}\cdot 8 - 4(-1) + 4\right)$$

$$- \left(0 + 0 - 4(1) + 2(-1)\right)$$

$$= 1 + 4\pi + 4 + 4 + 4 + 2$$

$$= \underline{(4\pi + 15)} \text{ units.}$$

(7) Find work done round the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ in xy plane $(z=0)$

in the field of force given by:

$$\bar{F}(x,y,z) = (3x-2y)\hat{i} + (2x+3y)\hat{j} + y^2\hat{k}$$

$$\therefore x = 4\cos\theta \quad \therefore dx = -4\sin\theta\, d\theta$$

$$\therefore y = 3\sin\theta \quad \therefore dy = 3\cos\theta\, d\theta$$

$$\therefore d\bar{r} = \hat{i}(-4\sin\theta\, d\theta) + \hat{j}(3\cos\theta\, d\theta)$$

$$\therefore \bar{F} = (12\cos\theta - 6\sin\theta)\hat{i} + (8\cos\theta + 9\sin\theta)\hat{j} + (9\sin^2\theta)\hat{k}$$

$$\therefore \bar{F}\cdot d\bar{r} = -48\sin\theta\cos\theta\, d\theta + 24\sin^2\theta\, d\theta + 24\cos^2\theta\, d\theta + 27\sin\theta\cos\theta\, d\theta$$

$$= 24\, d\theta - 21\sin\theta\cos\theta\, d\theta$$

$$\therefore \int \bar{F}\cdot d\bar{r} = \int_c 24\, d\theta - 21\sin\theta\cos\theta\, d\theta$$

Date : __/__/__

Limits of $\theta \Rightarrow 0$ to 2π.

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi} 24\, d\theta - \int_0^{2\pi} 21 \sin\theta\cos\theta\, d\theta$$

$$\Rightarrow [24\,\theta]_0^{2\pi} - \frac{21}{2}\int_0^{2\pi} \sin 2\theta\, d\theta$$

$$\Rightarrow 24(2\pi - 0) + \frac{21}{2}\left[\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$\Rightarrow 48\pi + \frac{21}{4}(\cos 4\pi - \cos 0)$$

$$\Rightarrow 48\pi + \frac{21}{4}(1-1)$$

$$\Rightarrow \boxed{48\pi \text{ units}}$$

$+21\times 2\left[\cos 2\theta\right]_0^{\pi/2}$

$42\,(\cos\pi - \cos 0)$

$42(-1-1)$

$-21\times 4\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$

$= -21\times 2\times 2\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta$

$2m-1=1$ $\boxed{m=1}$

$2n-1=1$ $\boxed{n=1}$

$= -21\times 2\times \frac{\pi\cdot\pi}{\frac{\pi}{2}}$ $= -21\times 2$ $(n-1)!$

$= [-42]_0^{\pi/2}$

$[24\theta]_0^{2\pi} - 42 =$

Date : __/__/__

GREEN'S THEOREM :

→ If P and Q are 2 func. of x and y, and their partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous single valued func. over closed region bounded by C then:

$$\boxed{\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy}$$

→ $\bar{F} = P\hat{i} + Q\hat{j}$ and $\bar{r} = x\hat{i} + y\hat{j}$

$$\boxed{\therefore \int_C \bar{F}\cdot d\bar{r} = \iint_R \bar{N}\cdot(\nabla\times\bar{F})\, ds.}$$

where $\bar{N}$ = unit vector along z-axis.

* Identify P and Q from given vector
* Obtain $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$
* Sketch curve C and find limiting values of x & y
* Sub. in RHS of Green's theorem
* For verification, values in LHS and RHS should match.

Date : __/__/__

$$\therefore \int_0^1 \int_{x^2}^{\sqrt{x}} (-6y + 16y)\, dx\, dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y \; dx\, dy$$

$$= 10 \int_0^1 \int_{x^2}^{\sqrt{x}} y \; dx\, dy$$

$$= \frac{10}{2} \int_0^1 \left[y^2 \right]_{x^2}^{\sqrt{x}} dx$$

$$= 5 \int_0^1 (x - x^4)\, dx$$

$$= 5 \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1$$

$$= 5 \left\{ \frac{1}{2} - \frac{1}{5} \right\}$$

$$= 5 \left\{ \frac{5-2}{10} \right\} = \frac{5 \times 3}{10} = \frac{15}{10} = \boxed{\frac{3}{2}}$$

Hence, Verified !!

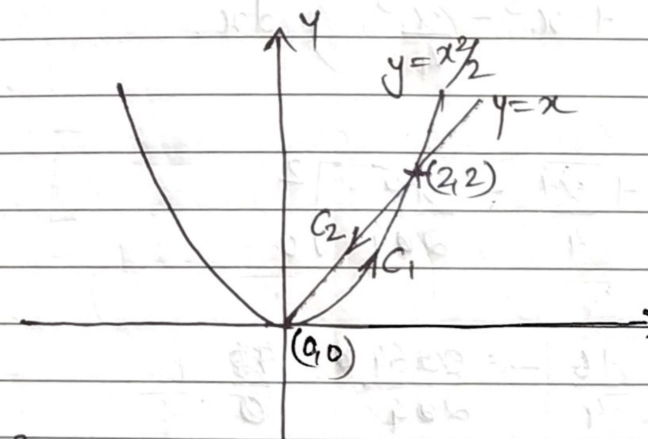
Date : __/__/__

③ Verify Green's theorem for $\bar{F} = (x^2 - xy)\hat{i} + (x^2 - y^2)\hat{j}$ over $2y = x^2$ and $y = x$.

$P : x^2 - xy$ $\qquad$ $Q : x^2 - y^2$

$\therefore \frac{\partial P}{\partial y} = -x$ $\qquad$ $\frac{\partial Q}{\partial x} = 2x$

$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x + x = 3x$



$x = \frac{x^2}{2}$

$x^2 - 2x = 0$

$x(x-2) = 0$

$\boxed{x = 2}$

$$\therefore \int_0^2 \int_{\frac{x^2}{2}}^{x} 3x \; dx\, dy$$

$$= \int_0^2 \left[3xy \right]_{\frac{x^2}{2}}^{x} dx$$

$$= \int_0^2 3x \left\{ x - \frac{x^2}{2} \right\} dx$$

$$= \int_0^2 3x^2 - \frac{3x^3}{2}\, dx$$

$$= \left[\frac{3x^3}{3} - \frac{3}{2} \cdot \frac{x^4}{4} \right]_0^2$$

$$= 8 - \frac{3}{2^3} \cdot 2^4$$

$$= 8 - 3 \cdot 2$$

$$= 8 - 6 = \boxed{2}$$

Date : __/__/__

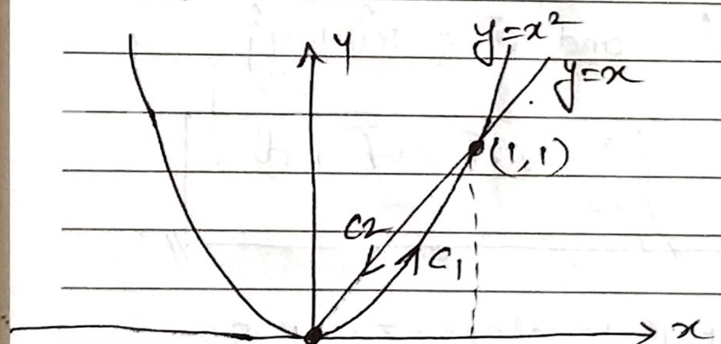
Sums :

① Verify green's theorem for $\bar{F}(x,y) = (xy + y^2)\hat{i} + x^2(\hat{j})$ over a closed region bounded by the curves $y = x^2$ and $y = x$.

Here, $P = xy + y^2$

$Q = x^2$

$\therefore \dfrac{\partial P}{\partial y} = x + 2y \quad ; \quad \dfrac{\partial Q}{\partial x} = 2x$



$\therefore$ green's theorem :

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

FOR LHS

Date : __/__/__

Along C_1, x varies from $0 \to 1$

$\therefore C_1 : y = x^2 \quad \therefore dy = 2x\,dx$

$\therefore P : xy + y^2 = x \cdot x^2 + x^4 = x^3 + x^4$

$\therefore Q : x^2$

$$\therefore \int_C (x^3 + x^4)\,dx + \int_C x^2 \cdot 2x\,dx$$

$$= \left[\frac{x^4}{4} + \frac{x^5}{5} + \frac{2x^4}{4}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{5} + \frac{2}{4} = \frac{5 + 4 + 10}{20} = \frac{19}{20}$$

Along C_2, x varies from $1 \to 0$

$\therefore C_2 : y = x \quad \therefore dy = dx$

$\therefore P : xy + y^2 = x^2 + x^2 = 2x^2$

$\therefore Q = x^2$

$$\therefore \int_C 2x^2\,dx + \int_C x^2\,dx$$

$$= \left[\frac{2x^3}{3} + \frac{x^3}{3}\right]_1^0$$

$$= -\left(\frac{2}{3} + \frac{1}{3}\right) = -1$$

$$\therefore \text{Final} = \frac{19}{20} - 1 = \boxed{\frac{-1}{20}}$$

Date : __/__/__

$$\therefore \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} (2x - x - 2y)\, dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)\, dx\,dy$$

$$= \int_0^1 \left[xy - \frac{2y^2}{2}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_0^1 (x^2 - x^2) - (x^3 - x^4)\, dx$$

$$= \int_0^1 x^4 - x^3\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = \boxed{\frac{-1}{20}}$$

Green's Theorem is Verified !!

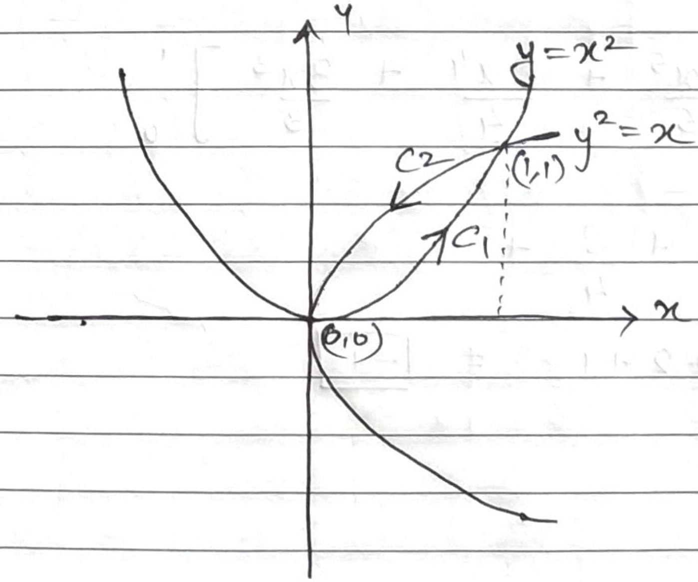
Date : __/__/__

② Verify Green's theorem for
$\bar{F} = (3x^2 - 8y^2)\,\hat{i} + (4y - 6xy)\,\hat{j}$
over closed region bounded by
$y = x^2$ and $y^2 = x$

$\therefore P = 3x^2 - 8y^2$

$Q = 4y - 6xy$

$\therefore \frac{\partial P}{\partial y} = -16y \;;\; \therefore \frac{\partial Q}{\partial x} = -6y$



$x = x^4$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0$

$x^3 = 1$

$\boxed{x = 1}$

$\therefore$ Green's theorem :

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Along C_1 : $y = x^2$, x varies $0 \to 1$

$P: 3x^2 - 8y^2 = 3x^2 - 8x^4 \qquad dy = 2x\,dx$

$Q: 4y - 6xy = 4x^2 - 6x^3$

$\therefore \int_0^1 (3x^2 - 8x^4)\,dx + \int_0^1 (4x^2 - 6x^3) \cdot 2x\,dx$

$= \int_0^1 3x^2 - 8x^4 + 8x^3 - 12x^4 \; dx$

$= \int_0^1 -20x^4 + 8x^3 + 3x^2\,dx$

$= \left[-20\frac{x^5}{5} + \frac{8x^4}{4} + \frac{3x^3}{3} \right]_0^1$

$= \frac{-20}{5} + \frac{8}{4} + 1$

$= -4 + 2 + 1 = \boxed{-1}$

Along C_2 : $y^2 = x$, y ~~x~~ varies $1 \to 0$

$P: 3x^2 - 8y^2 = 3y^4 - 8y^2 \qquad 2y\,dy = dx$

$Q: 4y - 6xy = 4y - 6y^3$

$\therefore \int_1^0 (3y^4 - 8y^2)\,2y\,dy + \int_1^0 (4y - 6y^3)\,dy$

$= \int_1^0 \left[(3y^4 - 8y^2) \cdot 2y + 4y - 6y^3 \right] dy$

$= \int_1^0 (6y^5 - 16y^3 + 4y - 6y^3)\,dy$

$= \int_1^0 (6y^5 - 22y^3 + 4y)\,dy$

$= \left[\frac{6y^6}{6} - \frac{22y^4}{4} + \frac{4y^2}{2} \right]_1^0$

$= -\left(1 - \frac{22}{4} + 2 \right) = \frac{22}{4} - 1 - 2$

$= \frac{22}{4} - 3 = \frac{5}{2} = \boxed{+\frac{5}{2}}$

Final $\Rightarrow -1 + \frac{5}{2} = \cancel{\frac{7}{2}}$ //

$\Rightarrow \frac{3}{2}$ ///

Date : __/__/__

$$dy = \frac{1}{2} \cdot 2x\,dx.$$

Along C_1: $y = \frac{x^2}{2}$, x varies $0 \to 2$

$P: x^2 - xy = x^2 - x \cdot \frac{x^2}{2} = x^2 - \frac{x^3}{2}$

$Q: x^2 - y^2 = x^2 - \left(\frac{x^2}{2}\right)^2 = x^2 - \frac{x^4}{4}$

$$\therefore \int_0^2 \left(x^2 - \frac{x^3}{2}\right) dx + \int_0^2 \left(x^2 - \frac{x^4}{4}\right) \cdot x\,dx$$

$$= \int_0^2 \left(x^2 - \frac{x^3}{2} + x^3 - \frac{x^5}{4}\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{8} + \frac{x^4}{4} - \frac{x^6}{24}\right]_0^2$$

$$= \frac{8}{3} - \frac{16}{8} + \frac{16}{4} - \frac{64}{24}$$

$$= \frac{8}{3} - 2 + 4 - \frac{\cancel{64}\ \cancel{32}\ \cancel{16}\ 8}{\cancel{24}\ \cancel{12}\ \cancel{6}\ 3}.$$

$$= \frac{8}{3} - \frac{8}{3} - 2 + 4$$

$$= 4 - 2 \Rightarrow 2.$$

Date : __/__/__

$$dy = dx$$

Along C_2: $y = x$, x varies $2 \to 0$

$\therefore P: x^2 - xy = x^2 - x^2 = 0$

$\therefore Q: x^2 - y^2 = x^2 - x^2 = 0.$

$$\therefore \int_2^0 0 + 0 = 0$$

$\therefore$ Final $= 2 + 0 \Rightarrow \boxed{2}$.

Green's theorem is getting verified!

P.T.O.

STOKES THEOREM

$$\boxed{\iint_S \bar{N}\cdot(\nabla\times\bar{F})\,ds = \int_c \bar{F}\cdot d\bar{r}}$$

$\bar{N}$: outward normal vector to element ds.

Procedure:

1) Find dir^n ratios, and dir^n cosines of $\bar{N}$ and then write unit vector $\bar{N}$.
2) Calculate $\nabla\times\bar{F}$
3) Calculate $\bar{N}\cdot(\nabla\times\bar{F})$
4) Find reqd integral.

⇒ Next Page

Note: If $\bar{F}$ is considered in xy plane, then stokes = greens given $\bar{N}=\hat{k}$ and z dir^n component is not counted at all.

Sums:

① Use Stokes theorem to evaluate $\int \bar{F}\cdot d\bar{r}$ where $\bar{F} = x^2\hat{i} + xy\hat{j}$ and c is boundary of rectangle $x=0,\ y=0,\ x=a,\ y=b$.

$$\therefore \nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$\Rightarrow \hat{i}\left(-\frac{\partial}{\partial z}(xy)\right) - \hat{j}\left(-\frac{\partial}{\partial z}(x^2)\right) + \hat{k}\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2)\right)$$

$$\Rightarrow \hat{i}(0) - \hat{j}(0) + \hat{k}(y) = y\hat{k}$$

In xy plane, $ds = dx\,dy$; $\bar{N} = \hat{k}$

$$\therefore \int_0^a\int_0^b \hat{k}\cdot y\hat{k}\,dx\,dy$$

$$= \int_0^a\int_0^b y\,dx\,dy = \int_0^a \left[\frac{y^2}{2}\right]_0^b dx$$

$$= \int_0^a \frac{b^2}{2}\,dx = \frac{b^2}{2}[x]_0^a \Rightarrow \boxed{\frac{ab^2}{2}} \Rightarrow \int_c \bar{F}\cdot d\bar{r}$$

Date : / /

(2). Find the line integral of
$\bar{F} = x^2\hat{i} + xy\hat{j}$ round square
bounded by $z=0,\ x=0,\ x=a,$
$y=0,\ y=a$
by Stokes theorem.

From prev. Q, we know $\nabla\times\bar{F} = y\hat{k}$
$\therefore$ In xy plane, $\bar{N} = \hat{k}$, $ds = dx\,dy$

$$\therefore \int_0^a\int_0^a \hat{k}\cdot y\hat{k}\ dx\,dy$$

$$= \int_0^a\int_0^a [y]\ dx\,dy = \int_0^a \left[\frac{y^2}{2}\right]_0^a dx$$

$$= \int_0^a \frac{a^2}{2}\,dx = \frac{a^2}{2}[x]_0^a \Rightarrow \boxed{\frac{a^3}{2}}$$

⇨ Next Page

Date

(3) Find line integral of $\bar{F} = (2x-y)\hat{i} - (yz^2)\hat{j} - (y^2z)\hat{k}$ along
boundary of upper half of sphere $x^2+y^2+z^2=1$
using stokes theorem.

Let the figure rest on $z=0$ (xy plane)

$$\therefore \nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$\Rightarrow \hat{i}\left(\frac{\partial}{\partial y}(-y^2z) - \frac{\partial}{\partial z}(-yz^2)\right)$$
$$-\hat{j}\left(\frac{\partial}{\partial x}(-y^2z) - \frac{\partial}{\partial z}(2x-y)\right)$$
$$+\hat{k}\left(\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x-y)\right)$$

$$\Rightarrow \hat{i}\left(-z2y - (-y2z)\right)$$
$$-\hat{j}(0-0) + \hat{k}(0-(-1))$$

$$\Rightarrow \hat{i}(-2zy+2zy) + \hat{k}$$

$$\Rightarrow \hat{k}$$

Date : __/__/__

Since, the hemisphere base lies on xy plane, $\bar{N} = \hat{k}$ and $ds = dx\,dy$.

z=0
xy plane

Acc. to Stokes theorem, $\iint_S$ means area of closed curve.

$\Rightarrow \pi r^2 = \pi$ since $r=1$.

$$\therefore \iint_S \hat{k}\cdot\hat{k}\,dx\,dy = \iint_S dx\,dy \Rightarrow \boxed{\pi}$$

P.T.O.

Date : __/__/__

(4) Use stokes theorem to evaluate the line integral of $\bar{F} = 4xz\,\hat{i} - y^2\,\hat{j} - yz\,\hat{k}$ along $z=0$ (xy plane), $x=0$, $y=0$ and $x^2+y^2=1$

$$\therefore \nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xz & -y^2 & -yz \end{vmatrix}$$

$$\Rightarrow \hat{i}\left(\frac{\partial}{\partial y}(-yz) - \frac{\partial}{\partial z}(-y^2)\right)$$

$$-\hat{j}\left(\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial z}(4xz)\right)$$

$$+\hat{k}\left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(4xz)\right)$$

$$\Rightarrow \hat{i}(-z-0) - \hat{j}(0-4x) + \hat{k}(0)$$

$$\Rightarrow -z\hat{i} + 4x\hat{j}$$

Using stokes: $\bar{N} = \hat{k}$; $ds = dx\,dy$

$$\therefore \iint_{S=\pi} \hat{k}\cdot(-z\hat{i}+4x\hat{j})\,dx\,dy \Rightarrow \boxed{0}$$

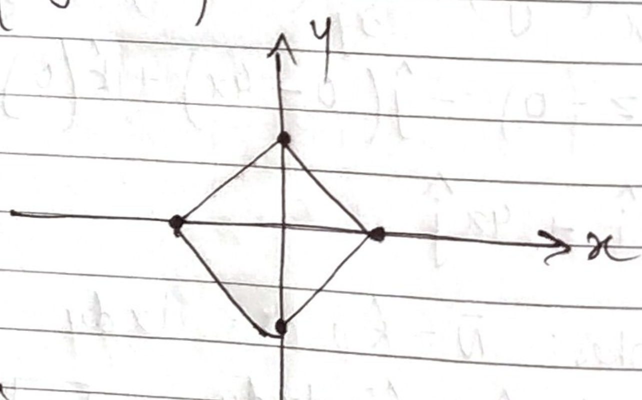
P.T.O.

(5) Using stokes theorem for vector field $\bar{f} = xy\hat{i} + xy^2\hat{j}$ along C whose end points are (0,1) (1,0) (-1,0) (0,-1)

$$\nabla \times \bar{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}$$

$$\Rightarrow \hat{i}\left(-\frac{\partial}{\partial z}(xy^2)\right) - \hat{j}\left(-\frac{\partial}{\partial z}(xy)\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(xy)\right)$$

$$\Rightarrow \hat{k}\left(y^2 - x\right)$$



$\bar{N} = \hat{k}$; $dx\,dy = ds$

$$\Rightarrow \int_{-1}^{1}\int_{-1}^{1} \hat{k} \cdot (y^2 - x)\hat{k}\; dx\,dy$$

$$\Rightarrow \int_{-1}^{1}\int_{-1}^{1} (y^2 - x)\; dx\,dy$$

$$\Rightarrow \int_{-1}^{1} \left[\frac{y^3}{3} - xy\right]_{-1}^{1} dx$$

$$\Rightarrow \int_{-1}^{1} \left(\frac{1}{3} - x\right) - \left(\frac{-1}{3} + x\right) dx$$

$$\Rightarrow \int_{-1}^{1} \left(\frac{1}{3} - x + \frac{1}{3} - x\right) dx$$

$$\Rightarrow \int_{-1}^{1} \left(\frac{2}{3} - 2x\right) dx$$

$$\Rightarrow \left[\frac{2}{3}x - \frac{2x^2}{2}\right]_{-1}^{1}$$

$$\Rightarrow \left[\frac{2}{3}x - x^2\right]_{-1}^{1}$$

$$\Rightarrow \left(\frac{2}{3} - 1\right) - \left(\frac{-2}{3} - 1\right)$$

$$\Rightarrow \frac{2}{3} - 1 + \frac{2}{3} + 1$$

$$\Rightarrow \boxed{\frac{4}{3}}$$

Date : __/__/__

GAUSS DIVERGENCE THEOREM

$$\boxed{\iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = \iiint_V (\nabla \cdot \bar{F})\, dV}$$

dot prod.

Procedure :

1) Calculate $\nabla \cdot \bar{F}$
2) $dV = dx\,dy\,dz$
3) $\iiint (\nabla \cdot \bar{F})\, dx\,dy\,dz.$

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

$$= \iint_S P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy$$

//

Date : __/__/__

Sums

① Use gauss divergence theorem to evaluate $\iint \bar{F} \cdot \hat{n}\, ds$ where $\bar{F} = x^3 \hat{i} + y^3 \hat{j} + z^3 \hat{k}$ over the surfaces of the sphere $x^2 + y^2 + z^2 = a^2$

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial x} x^3 + \frac{\partial}{\partial y} y^3 + \frac{\partial}{\partial z} z^3$$

$$= 3x^2 + 3y^2 + 3z^2$$

$$= 3(x^2 + y^2 + z^2)$$

$$= 3r^2$$

In 3D space :

$x = r \sin\theta \cos\phi \qquad 0 < r < a$

$y = r \sin\theta \sin\phi \qquad 0 < \theta < \pi$

$z = r \cos\theta \qquad 0 < \phi < 2\pi$

$dV = dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$$\therefore \iiint_V (\nabla \cdot \bar{F})\, dV = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} 3r^2 \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 3\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^4 \sin\theta \, dr\, d\theta\, d\phi$$

$$= 3\int_0^{2\pi}\int_0^{\pi} \left[\frac{r^5}{5}\sin\theta\right]_0^a d\theta\, d\phi$$

$$= 3\int_0^{2\pi}\int_0^{\pi} \frac{a^5}{5}\sin\theta \, d\theta\, d\phi$$

$$= -\frac{3a^5}{5}\int_0^{2\pi} [\cos\theta]_0^{\pi}\, d\phi$$

$$= -\frac{3a^5}{5}\int_0^{2\pi} (\cos\pi - \cos 0)\, d\phi$$

$$= -\frac{3a^5}{5}\int_0^{2\pi} -1-1 \, d\phi$$

$$= -\frac{3a^5}{5}(-2)\int_0^{2\pi} d\phi$$

$$= \frac{6a^5}{5}[\phi]_0^{2\pi} = \frac{6a^5}{5}\cdot 2\pi$$

$$\Rightarrow \boxed{\frac{12a^5\pi}{5}}$$

2) Apply GDT for $\bar{F} = 3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}$ over surface of cube bounded by planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$

$dV = dx\,dy\,dz$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}\cdot 3x^2 + \frac{\partial}{\partial y}(2xz-y) + \frac{\partial}{\partial z}\cdot z$$

$$= 6x + 0 - 1 + 1 = 6x$$

$$\therefore \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (\nabla\cdot\bar{F})\, dV$$

$$= \int_0^1\int_0^1\int_0^1 6x\, dx\,dy\,dz$$

$$= 6\int_0^1\int_0^1 [xz]_0^1\, dx\,dy$$

$$= 6\int_0^1\int_0^1 x\, dx\,dy$$

$$= 6\int_0^1\int_0^1 x\, dx\,dy$$

$$= 6\int_0^1 [xy]_0^1\, dx$$

$$= 6\int_0^1 x\, dx$$

$$= 6\cdot\left[\frac{x^2}{2}\right]_0^1$$

$$= 3[x^2]_0^1$$

$$\Rightarrow \boxed{3}$$

Date : __/__/__

*. IMPORTANT FORMULAE. *

1) $\sin x = \frac{e^{ix} - e^{-ix}}{2}$

2) $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

3) $\sinh x = \frac{e^{x} - e^{-x}}{2}$

4) $\cosh x = \frac{e^{x} + e^{-x}}{2}$

5) $\sin(x) = -i\sinh(ix)$

6) $\sinh(x) = -i\sin(ix)$

7) $\cos(x) = \cosh(ix)$

8) $\cosh(x) = \cos(ix)$

9) $\cos^2 h(x) - \sinh^2(x) = 1$

Date : __/__/__

10) $\cosh(x) + \sinh(x) = +e^{x}$

11) $\cosh(x) - \sinh(x) = +e^{-x}$

12) $\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$

13) $\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$

14) $e^{ix} = \cos x + i\sin x$

15) $e^{-ix} = \cos x - i\sin x$

— Riddhi Mehta

Date : __/__/__

Sums for Vectors Practice.

① Find work done by a particle which moves along a parabola $y^2 = x$ from $(0,0)$ to $(1,1)$ under

$$\bar{f} = (x^2+y^2)\,\hat{i} + (x^2-y^2)\,\hat{j}$$

$$\therefore \int \bar{f}\cdot d\bar{r}$$

$$\bar{f} = (x^2+y^2)\,\hat{i} + (x^2-y^2)\,\hat{j} \qquad \therefore y^2 = x$$

$$= (x^2+x)\,\hat{i} + (x^2-x)\,\hat{j} \qquad \therefore 2y\,dy = dx$$

$$d\bar{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$= dx\,\hat{i} + \frac{dx}{2\sqrt{x}}\,\hat{j}$$

$$\therefore \bar{f}\cdot d\bar{r} = (x^2+x)\,dx + \frac{(x^2-x)\,dx}{2\sqrt{x}}$$

$$= \left(x^2 + x + \frac{x^2}{2\sqrt{x}} - \frac{x}{2\sqrt{x}}\right) dx$$

$$= \left(x^2 + x + \frac{1x^{3/2}}{2} - \frac{1x^{1/2}}{2}\right) dx$$

$$\therefore \int \bar{f}\cdot d\bar{r} = \int_0^1 x^2 + x + \frac{x^{3/2}}{2} - \frac{x^{1/2}}{2}\,dx$$

Date : __/__/__

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + \frac{x^{5/2}}{5} - \frac{x^{3/2}}{3}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{2} + \frac{1}{5} - \frac{1}{3}$$

$$= \frac{1}{2} + \frac{1}{5} = \frac{5+2}{10} = \boxed{\frac{7}{10}\text{ units.}}$$

② Verify green's for $\int_C (x^2-y)\,dx + (2y^2+x)\,dy$ around $y = x^2$ and $y = 4$

Green's theorem states that:

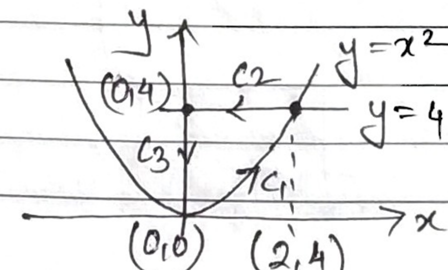
$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$P = x^2 - y \qquad Q = 2y^2 + x$$

$$\frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial x} = 1$$

$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 + 1 = 2.$$

$$4 = x^2$$
$$x^2 - 4 = 0$$
$$(x+2)(x-2) = 0$$
$$x = 2, -2$$



Along C_1 : $y = x^2$ $\qquad dy = 2x\,dx$

P: $x^2 - y = x^2 - x^2 = 0$.

Q: $2y^2 + x = 2x^4 + x$.

$\therefore \int_0^2 (2x^4 + x) \cdot 2x\,dx$

$= \int_0^2 4x^5 + 2x^2\,dx$

$= \left[\frac{4x^6}{6} + \frac{2x^3}{3}\right]_0^2$

$= \frac{4 \cdot 2^6}{6} + \frac{2 \cdot 8}{3} = 48$

Along C_2 : $y = 4$ $\qquad dy = 0$

P: $x^2 - y = x^2 - 4$

Q: $2y^2 + x = 2 \cdot 16 + x = x + 32$.

$\therefore \int_0^2 (x^2 - 4)\,dx + 0$

$= -\left[\frac{x^3}{3} - 4x\right]_0^2 = -\frac{8}{3} + 8 = +\frac{16}{3}$

Along C_3 : $x = 0$ $\qquad dx = 0$.

P: $x^2 - y = -y$ $\qquad$ y changes from 4, 0.

Q: $2y^2 + x = 2y^2$.

$\therefore \int_4^0 2y^2\,dy = \left[\frac{2y^3}{3}\right]_4^0 = -\frac{2}{3} \cdot (4)^3$

$= -\frac{128}{3}$

Total $= 48 + \frac{16}{3} - \frac{128}{3} = \frac{32}{3}$ //

<u>Verifying</u> $\int_{y=0}^{4} \int_{x=0}^{\sqrt{y}} 2\,dx\,dy$

$= \int_0^4 [2x]_0^{\sqrt{y}}\,dy = 2 \int_0^4 \sqrt{y}\,dy$

$= \left[\frac{2 \cdot 2y^{3/2}}{3}\right]_0^4 = \frac{4}{3}(4)^{3/2}$

$= \frac{4}{3}(2)^3$

$= \frac{8 \cdot 4}{3} \Rightarrow \frac{32}{3}$ //

Hence, Verified!!

Date : __/__/__

③ Evaluate $\int \frac{y}{x^2+y^2}dx - \int \frac{x}{x^2+y^2}dy$ using green's theorem

$P = \frac{y}{x^2+y^2} \quad Q = \frac{-x}{x^2+y^2}$

$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$\frac{\partial Q}{\partial x} = -\left[\frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}\right]$

$= \frac{-x^2+2x^2-y^2}{(x^2+y^2)^2} = \boxed{\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}}$

$\frac{\partial P}{\partial y} = \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$

$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$

$\boxed{\frac{\partial P}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}}$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ //

$\boxed{\text{Ans} = 0}$ //

Date : __/__/__

④ $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = (2x+y)\hat{i} - 4z^2\hat{j} - y^2z\hat{k}$ and C is hemisphere $x^2+y^2+z^2=a^2$, $z=0$ (xy plane) using stokes theorem.

$\int_C \bar{F}\cdot d\bar{r} = \iint_S \bar{N}\cdot(\nabla\times\bar{F})\,ds$

$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+y & -4z^2 & -y^2z \end{vmatrix}$

$\therefore \hat{i}\left(\frac{\partial}{\partial y}(-y^2z) + \frac{\partial}{\partial z}(4z^2)\right)$

$-\hat{j}\left(\frac{\partial}{\partial x}(-y^2z) - \frac{\partial}{\partial z}(2x+y)\right)$

$+\hat{k}\left(\frac{\partial}{\partial x}(-4z^2) - \frac{\partial}{\partial y}(2x+y)\right)$

$= \hat{i}(-2yz+8z) - \hat{j}(0)$

$+\hat{k}(-1) = \hat{i}(-2yz+8z) - \hat{k}$.

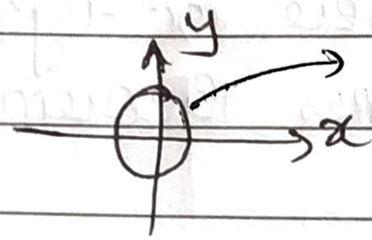
In xy plane, $ds = dx\,dy$.

Date : ____/____/

$\bar{N} = \hat{k}$

$\therefore \bar{N} \cdot (\nabla \times \bar{F}) = -1$

$z = 0$.
(xy plane)



base of hemisphere area is still the same.

$\therefore \int_C \bar{F} \cdot d\bar{r} = \iint_S -1 \, dx \, dy = \boxed{-\pi a^2}$ //

Thank - you

Notes by : Riddhi Mehta.

END OF NOTES!

Thank-You