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3rd year, Sem V
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Applied Maths V.

* TUTORIAL 2 *

Q1) Find the eigen values and vectors of :

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation is given as :

$\lambda^3 - \lambda^2(\text{Trace of } A) + \lambda(A_{11} + A_{22} + A_{33}) - |A| = 0$

$\therefore \lambda^3 - \lambda^2(3+3+3) + \lambda((9-1) + (9-1) + (9-1)) - 20 = 0$

$\therefore \lambda^3 - 9\lambda^2 + (24)\lambda - 20 = 0$

$\therefore \lambda^3 - 9\lambda^2 + 24\lambda^2 - 20 = 0$

$\therefore \lambda_1 = 5$

$\lambda_2 = 2$

$\lambda_3 = 2$

$\therefore$ Eigen values are 5, 2, 2.

$\therefore$ For Eigen vectors,

(i) $\lambda_1 = 5 \quad \therefore (A - \lambda I)X = 0$

$$\left\{ \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore -2x_1 - x_2 + x_3 = 0$

$\therefore -x_1 - 2x_2 - x_3 = 0$

$\therefore x_1 - x_2 - 2x_3 = 0$

We have 3 equations:

$$-2x_1 - x_2 + x_3 = 0$$
$$-x_1 - 2x_2 - x_3 = 0$$
$$x_1 - x_2 - 2x_3 = 0$$

$$\therefore \frac{x_1}{\begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix}} = K$$

$$\therefore \frac{x_1}{1+2} = \frac{-x_2}{2+1} = \frac{x_3}{4-1} = k$$

$$\therefore \frac{x_1}{3} = \frac{-x_2}{3} = \frac{x_3}{3} = K$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(ii) $\lambda_2 = 2$ $\quad \therefore (A - \lambda I)X = 0$

$$\left\{ \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore x_1 - x_2 + x_3 = 0$$
$$-x_1 + x_2 - x_3 = 0$$
$$x_1 - x_2 + x_3 = 0$$

$\therefore$ We have 3 same equations so Cramer's Rule fails.

Considering eq: $x_1 - x_2 + x_3 = 0$
and taking 2 cases:

$x_1 = 0$ $\qquad\qquad$ $x_2 = 0$

$\therefore x_2 = x_3$ $\qquad\qquad$ $\therefore x_1 = -x_3$

$\therefore$ Eigen vector is $[0 \quad 1 \quad 1]$ $\qquad\qquad$ $\therefore$ Eigen vector is $[1 \quad 0 \quad -1]$

$\therefore$ Both the eigenvectors are linearly independent

The matrix M is obtained as

$$M = [X_1 \quad X_2 \quad X_3]$$

$$\therefore M = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1/3 & -1/3 & 1/3 \\ 1/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & -1/3 \end{bmatrix}$$

$\therefore$ For the given matrix A to be diagonizable, the below equation needs to satisfy:

$$M^{-1}AM = D$$

Where D is eigen value matrix which looks like:

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Calculating $M^{-1}AM = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\therefore$ Hence, $M^{-1}AM = D$ Proven!

Q2) Calculate A^{10} for $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

from Q1, we have found

$\lambda_1 = 5 \quad ; \quad \lambda_2 = 2 \quad ; \quad \lambda_3 = 2$

Since, the eigen value 2 are repeating, we conclude the following:

There is a distinct eigen vector for $\lambda_1 = 5$.

'2' is repeated 2 times, it must be a root till $f'(x)$

The governing equation is given as:

$$A^{10} = a_0 A^2 + a_1 A + a_2 I$$

(because A is a 3x3 matrix and highest degree 2)

Replacing A by λ_1 for $\lambda_1 = 5$

$$\therefore \quad (5)^{10} = a_0 (5)^2 + a_1 (5) + a_2 \quad \text{——— (1)}$$

Replacing A by λ_2 for $\lambda_2 = 2$

$$\therefore \quad (2)^{10} = a_0 (2)^2 + a_1 (2) + a_2 \quad \text{——— (2)}$$

$\therefore$ Taking $\frac{d(A^{10})}{dA}$ on both sides, we have,

$$\therefore \quad 10A^9 = 2a_0 A + a_1$$

Replacing A by λ_3 for $\lambda_3 = 2$

$\therefore 10(2)^9 = 2a_0(2) + a_1$

$\therefore 10(2)^9 = 4a_0 + a_1$ —— (3)

We have 3 equations and we need to compute a_0, a_1 and a_2.

$\therefore (5)^{10} = 25a_0 + 5a_1 + a_2$

$\therefore (2)^{10} = 4a_0 + 2a_1 + a_2$

$\therefore 10(2)^9 = 4a_0 + a_1 + 0a_2$

$\therefore a_0 = 1083249$

$\therefore a_1 = -4327876$

$\therefore a_2 = 4323780$

Substituting a_0, a_1, a_2 to find A^{10}.

$$\therefore A^{10} = a_0 \begin{bmatrix} 11 & -7 & 7 \\ -7 & 11 & -7 \\ 7 & -7 & 11 \end{bmatrix} + a_1 \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(No need to solve further to get final answer.) (Just write a_0, a_1 and a_2 values and then directly calculate and write final answer)

Q3) Following with Q1 and Q2, check whether given matrix A is derogatory or not.

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

* Matrix is said to be derogatory if the degree of minimal polynomial p is STRICTLY less than characteristic polynomial $|A - \lambda I| = 0$ i.e. n. Else, matrix is non derogatory.

* If eigen values are DISTINCT then matrix is 'non-derogatory'.

* Eigen values are $\lambda_1 = 5$; $\lambda_2 = 2$; $\lambda_3 = 2$

$\therefore$ The characteristic eq. of A is

$$\lambda^3 - 9\lambda^2 + 24\lambda - 20 = 0$$

As eigen values are repeated, the possible minimal polynomial of A are $(A-5I)(A-2I)$; $(A-5I)(A-2I)^2$

* If $(A-5I)(A-2I) = 0$ then m.p. degree = 2, C.E. degree = 3, so A is derogatory. (Need to check this!)

Else if $(A-5I)(A-2I)^2 = 0$ then A is non derogatory (This is always TRUE)

Let us examine $(A-5I)(A-2I)$:

$$= \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

* Since, $(A-5I).(A-2I) = A^2 - 2A - 5A + 10I$
$= A^2 - 7A + 10I$ has degree '2' and also satisfied by A where degree $2 < 3$ (C.H.E. degree), the matrix is DEROGATORY.

Q4) Find $\tan A$ for $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$

Then prove that $3\tan A = A\tan 3$

Obtaining characteristic eq. as.

$\therefore \lambda^2 - (\text{Trace of } A)\lambda + |A| = 0$

$\therefore$ Using C.H.T. we obtain :

$\therefore \lambda^2 - 0\lambda + (-9) = 0$

$\therefore \lambda^2 - 9 = 0 \approx$ we get $\lambda = \pm 3$

$\therefore A^2 - 9I = 0$

We have,

$\tan A = a_0 A + a_1$ $\therefore$ Replacing A with λ.

$\therefore \tan\lambda = a_0\lambda + a_1$

To get 2 equations, we replace λ with ± 3.

$\therefore \tan 3 = 3a_0 + a_1$ —— ①

$\therefore \tan(-3) = -3a_0 + a_1$ —— ②

$\therefore \tan 3 = 3a_0 + a_1$ —— ①

$\therefore \tan(-3) = -3a_0 + a_1$ —— ②

$\therefore$ Adding eq. ① + eq. ②, we get

$\therefore \tan 3 + \tan(-3) = 2a_1$

$\therefore \tan 3 - \tan 3 = 2a_1$

$\therefore 2a_1 = 0$

$\therefore a_1 = 0$

$\therefore \tan 3 = 3a_0 + 0$

$\therefore a_0 = \dfrac{\tan 3}{3}$

$\therefore$ We have,

$\therefore \tan A = a_0 A + a_1$

$\therefore \tan A = \left(\dfrac{\tan 3}{3}\right) \cdot A + 0$

$\therefore \tan A = \dfrac{A \tan 3}{3}$

$\therefore 3 \tan A = A \tan 3 \implies$ Hence, Proved !!

Q5) If $A = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$, then use C.H.T. to find

$$f(A) = A^6 - 5A^3 + 3A - 4I$$

$$f(A) = q(A) \cdot ch(A) + r(A)$$

By C.H.T, $q(A) = 0$ so $f(A) = r(A)$

characteristic eq. is

$\therefore \lambda^2 - 3\lambda + (-10) = 0$

$\therefore \lambda^2 - 3\lambda - 10 = 0$

Using CHT, we know that

$$f(\lambda) = \lambda^6 - 5\lambda^3 + 3\lambda - 4$$

→ Using long division method,

$$\begin{array}{r} \lambda^4 + 3\lambda^3 + 19\lambda^2 + 82\lambda + 436 \\ \lambda^2 - 3\lambda - 10 \overline{\big) \lambda^6 - 5\lambda^3 + 3\lambda - 4} \\ -\ \lambda^6 - 3\lambda^5 - 10\lambda^4 \\ \hline 3\lambda^5 + 10\lambda^4 - 5\lambda^3 + 3\lambda - 4 \\ -\ 3\lambda^5 - 9\lambda^4 - 30\lambda^3 \\ \hline 19\lambda^4 + 25\lambda^3 + 3\lambda - 4 \\ -\ 19\lambda^4 - 57\lambda^3 - 190\lambda^2 \\ \hline 82\lambda^3 + 190\lambda^2 + 3\lambda - 4 \\ -\ 82\lambda^3 - 246\lambda^2 - 820\lambda \\ \hline 436\lambda^2 + 823\lambda - 4 \\ -\ 436\lambda^2 - 1308\lambda - 4360 \\ \hline 2131\lambda + 4356 \end{array}$$

$\therefore$ Since $f(\lambda) = r(\lambda)$

Using CHT, we obtain $f(A) = r(A)$

$\therefore r(A) = 2131 \cdot A + 4356I$

→ Computing $f(A) = 2131A + 4356I$

$$= 2131\begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix} + 4356\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2131 & 6393 \\ 4262 & 8524 \end{bmatrix} + \begin{bmatrix} 4356 & 0 \\ 0 & 4356 \end{bmatrix}$$

$$= \begin{bmatrix} 2225 & 6393 \\ 4262 & 12880 \end{bmatrix}$$

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* TUTORIAL 3 *

Q1) Reduce $Q(X) = 2x_1^2 - 5x_2^2 + 3x_3^2 + 4x_1x_2 - 6x_2x_3 + 8x_1x_3$ to canonical form by congruent operations and hence find transforming matrix, rank, index and signature and also comment on the nature.

$$\therefore Q(x,y,z) = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 2 & -5 & -3 \\ 4 & -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = X^TAX$$

Hence, coeff. matrix A is given by

$$A = \begin{bmatrix} 2 & 2 & 4 \\ 2 & -5 & -3 \\ 4 & -3 & 3 \end{bmatrix}$$ which is also equal to $\underline{\underline{I_3 A I_3}}$

Hence we have:

$$\begin{bmatrix} 2 & 2 & 4 \\ 2 & -5 & -3 \\ 4 & -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overbrace{\begin{bmatrix} 2 & 2 & 4 \\ 2 & -5 & -3 \\ 4 & -3 & 3 \end{bmatrix}}^{A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now to convert $Q(x,y,z)$ to canonical form, let us apply congruent row and column elementary op. to pre and post factors shown by I of coefficient matrix A resp.

$\therefore R_2 \to R_2 - R_1$; $C_2 \to C_2 - C_1$; $R_3 \to R_3 - 2R_1$; $C_3 \to C_3 - 2C_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -7 & -7 \\ 0 & -7 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$; $C_3 \to C_3 - C_2$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, we must do the foll. operations:

$R_1 \to \frac{R_1}{\sqrt{2}}$; $R_2 \to \frac{R_2}{\sqrt{7}}$; $R_3 \to \frac{R_3}{\sqrt{2}}$ and

$C_1 \to \frac{C_1}{\sqrt{2}}$; $C_2 \to \frac{C_2}{\sqrt{7}}$; $C_3 \to \frac{C_3}{\sqrt{2}}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{7} & 1/\sqrt{7} & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{7}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{7}} & \frac{-1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$X^T \cdot A \cdot X$$

Hence, verified by calculator!

→ ∴ Rank of matrix $= r =$ total no. of 1's and −1's present in canonical form of coefficient matrix A $= 3$

→ ∴ Index of matrix $= p =$ total no. of 1's present in canonical form of coefficient matrix A $= 2$

→ ∴ Signature $= s = 2p - r = 2(2) - 3 = 4 - 3 = 1$

→ Canonical form is given by:

$$Q'(y_1, y_2, y_3) = y_1^2 - y_2^2 + y_3^2$$

$\therefore$ We have, $X = P \cdot Y$ so, $Y = P^{-1} \cdot X$

by orthogonality of P we have $P \cdot P^{T} = P \cdot P^{-1} = I$

so, $P^{T} = P^{-1}$

we have

$$P^{T} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{-1}{\sqrt{7}} & \frac{1}{\sqrt{7}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore \quad Y = P^{T} \cdot X$$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{-1}{\sqrt{7}} & \frac{1}{\sqrt{7}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore y_1 = \frac{1}{\sqrt{2}} x_1$$

$$\therefore y_2 = \frac{-1}{\sqrt{7}} x_1 + \frac{1}{\sqrt{7}} x_2$$

$$\therefore y_3 = \frac{-1}{\sqrt{2}} x_1 - \frac{1}{\sqrt{2}} x_2 + \frac{1}{\sqrt{2}} x_3$$

→ Since, we have mixed Q, the nature is '<u>Indefinite</u>'.

Q2) Reduce $Q(X) = 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_1x_3$ to canonical form by orthogonal operations and hence find the transforming matrix, rank, index, signature and also comment on the nature.

→ The transforming orthogonal matrix P is such that $P^T . A . P$ is a diagonal matrix.

→ We have coefficient matrix A as:

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

To find the eigen values as $(\lambda_1, \lambda_2, \lambda_3)$ by using the characteristic equation:

$\therefore \lambda^3 - \lambda^2(12) + \lambda(8+14+14) - 32 = 0$

$\therefore \lambda^3 - 12\lambda + 36\lambda - 32 = 0$

$\therefore \lambda_1 = 8$

$\lambda_2 = 2$

$\lambda_3 = 2$

For $\lambda_1 = 8 \therefore (A - \lambda_1 I)X = 0$

$$\therefore \left\{ \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore -2x_1 - 2x_2 + 2x_3 = 0$

$\therefore -2x_1 - 5x_2 - x_3 = 0$

$\therefore 2x_1 - x_2 - 5x_3 = 0$

→ Using Cramers rule for eq. 2 and eq. 3

$$\frac{x_1}{\begin{vmatrix} -5 & -1 \\ -1 & -5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & -1 \\ 2 & -5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -5 \\ 2 & -1 \end{vmatrix}} = K$$

$$\therefore \frac{x_1}{24} = \frac{-x_2}{12} = \frac{x_3}{12} = K$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ -12 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow 1^{st} \text{ eigen vector}$$

For $\lambda_2 = 2$ $(A - \lambda_2 I)\, x = 0$

$$\therefore \left\{ \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore \; 4x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 + x_2 - x_3 = 0$$
$$2x_1 - x_2 + x_3 = 0$$

$\therefore$ Since, all 3 eq. are same, cramer's rule FAILS.

Taking eq. 3 into consideration $2x_1 - x_2 + x_3 = 0$

$x_1 = 0$

$\therefore \; x_2 = x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

→ The transforming matrix P is given by

$$P = \begin{bmatrix} \frac{X_1}{|X_1|} & \frac{X_2}{|X_2|} & \frac{X_3}{|X_3|} \end{bmatrix} ; \quad X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} ; \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

→ As all 3 vectors have to be orthogonal, $X_3 \perp^r X_1, X_2$

$$\therefore X_3 \cdot X_1 = 0$$
$$\therefore X_3 \cdot X_2 = 0$$
$$\therefore +2z_1 - z_2 + z_3 = 0$$
$$\therefore 0z_1 + z_2 + z_3 = 0$$

$|X_1| = \sqrt{6}$

$|X_2| = \sqrt{2}$

$$\therefore \frac{z_1}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{-z_2}{\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}} = \frac{z_3}{\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}} = k$$

$$\therefore \frac{z_1}{-2} = \frac{-z_2}{2} = \frac{z_3}{2}$$

$$\therefore \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad \therefore |Z| = \sqrt{3}$$

→ P now looks like

$$P = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\therefore P^T A . P = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} A \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} //$$

$\therefore$ The quadratic form $Q(X) = 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_1x_3$ has now been reduced to $Q(Y) = 8y_1^2 + 2y_2^2 + 2y_3^2$

$$\therefore Y = P^T . X$$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore y_1 = \frac{2}{\sqrt{6}} x_1 - \frac{1}{\sqrt{6}} x_2 + \frac{1}{\sqrt{6}} x_3$$

$$\therefore y_2 = \frac{1}{\sqrt{2}} x_2 + \frac{1}{\sqrt{2}} x_3$$

$$\therefore y_3 = -\frac{1}{\sqrt{3}} x_1 - \frac{1}{\sqrt{3}} x_2 + \frac{1}{\sqrt{3}} x_3$$

→ Rank = 3 ; Index = 3 ; Signature = $2p - r = 3$

→ Since, $r = p = n$ Q is in Positive Definite quadratic form.

Q3) Reduce the matrix $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ to find the transforming orthogonal matrix 'P', and the reduced canonical equation.

→ For eigen values, we need characteristic eq.

$\therefore \lambda^2 - \lambda(2a) + (a^2-b^2) = 0$

$\therefore \lambda^2 - 2a\lambda + (a^2-b^2) = 0$

We obtain,

$\lambda_1 = a+b$

$\lambda_2 = a-b$

So we get 2 equations as follows:

1) $\therefore (A - \lambda_1 I)X = 0$

$\therefore (A - (a+b)I)X = 0$ ——— ①

2) $\therefore (A - \lambda_2 I)X = 0$

$\therefore (A - (a-b)I)X = 0$ ——— ②

→ From eq. ①

$\therefore \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} - \begin{bmatrix} a+b & 0 \\ 0 & a+b \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\therefore \begin{bmatrix} -b & b \\ b & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ $\therefore -bx_1 + bx_2 = 0$

$\therefore bx_1 - bx_2 = 0$

$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ // means $x_1 = x_2$

→ from eq. 2

$$\left\{\begin{bmatrix} a & b \\ b & a \end{bmatrix} - \begin{bmatrix} a-b & 0 \\ 0 & a-b \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} b & b \\ b & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \therefore bx_1 + bx_2 = 0$$

$$\therefore x_1 = -x_2$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

→ Eigen matrix $= \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

→ Transforming orthogonal matrix $P = \begin{bmatrix} X_1/|X_1| & X_2/|X_2| \end{bmatrix}$

$$\therefore P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

→ We have

$$P^T.A.P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} . \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$$

→ $Y = P^T.X \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\therefore y_1 = \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2 \; ; \; y_2 = \frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{2}}x_2$$

Q4) Decompose $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$ using single vector decomposition.

→ A matrix of order 'm×n' can be decomposed to 3 matrices x, y, z such that :

$$A_{m\times n} = P_{m\times r} \cdot R_{r\times r} \cdot Q_{r\times n}$$

* Step 1 Obtain A^T

$$\therefore A^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

* Step 2 get $A^T \cdot A$

$$\Rightarrow \underset{2\times3}{\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}_{2\times2}$$

* Step 3 Eigen values of $A^T \cdot A$

$$\lambda_1 = 2 \ ; \ \lambda_2 = 3$$

* Step 4 Eigen vectors are : $\therefore X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$X_1 = \left((A^T \cdot A) - \lambda_1 I \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

* Step 5: Construct Q such as $Q = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|} \right]$

$\therefore Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

* Step 6: Square root of eigen values and compute R.

$\therefore \sqrt{\lambda_1} = \sqrt{2}$; $\sqrt{\lambda_2} = \sqrt{3}$ $\quad \therefore R = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$

* Step 7: Compute $\frac{AX_1}{\sqrt{\lambda_1}}$, $\frac{AX_2}{\sqrt{\lambda_2}}$ and $NS(A^T)$

$$\therefore \frac{AX_1}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}_{3\times2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{3\times1}$$

$$\therefore \frac{AX_2}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}_{3\times2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2\times1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{3\times1}$$

$NS(A^T)$ is null space of A^T, for that we need to compute $X_3 \neq 0$ such that

$$A^T \cdot X_3 = 0 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 + 0x_2 - x_3 = 0 \qquad \therefore 2x_1 + x_2 = 0 \qquad X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\therefore x_1 + x_2 + x_3 = 0 \qquad \therefore x_2 = -2x_1$

$\therefore x_1 = x_3$

* Step 8: Compute $NS(A^T)$

$$\therefore NS(A^T) = \frac{X_3}{\|X_3\|} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

* Step 9: Compute P such as $\begin{bmatrix} \frac{AX_1}{\sqrt{\lambda_1}} & \frac{AX_2}{\sqrt{\lambda_2}} & NS(A^T) \end{bmatrix}$

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

* Step 10: Final step.

$$\therefore A_{m\times n} = P_{m\times r} \, R_{r\times r} \cdot Q_{r\times n}$$

$$\therefore \underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}} = \underset{3\times 3}{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

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