Riddhi B. Mehta
3rd year Sem V
Roll. J030
Applied Maths V



# * TUTORIAL - 5 *

Q1) Prove that an analytic function f(z) having constant magnitude is always constant.

$\therefore f(z) = u + iv$ is analytic s.t. $|f(z)| = c$

Proof: $|f(z)| = \sqrt{u^2 + v^2} = c$

$\therefore u^2 + v^2 = c$ — ①

Diff. ① wrt $x$ & $y$.

→ $\therefore 2u \dfrac{\partial u}{\partial x} + 2v \dfrac{\partial v}{\partial x} = 0$

$\therefore u U_x + v V_x = 0$

→ $\therefore 2u \dfrac{\partial u}{\partial y} + 2v \dfrac{\partial v}{\partial y} = 0$

$\therefore u U_y + v V_y = 0$

Hence, C.R. eqs are getting satisfied as:

$\therefore U_x = V_y$

$\therefore U_y = -V_x$

Continuing,

1) $u U_x - v U_y = 0$ mult. $u$

2) $u U_y + v U_x = 0$ mult. $v$

$\therefore u^2 U_x - uv U_y = 0$ — ②

$\therefore uv U_y + v^2 U_x = 0$ — ③

Adding eq. ② & eq. ③:

$\therefore (u^2 + v^2) U_x = 0$

we have $u^2 + v^2 = c$

$\therefore U_x = 0$

→ $f'(z) = U_x - iU_y$ (OR) $V_y + iV_x$

$\therefore f'(z) = 0$ $\boxed{\text{Hence, } f(z) = \beta, \text{ constant no.}}$

Q2) If $f(z) = u + iv$ is analytic, then prove the foll:

1) $\det \begin{pmatrix} U_x & U_y \\ V_x & V_y \end{pmatrix} = |f'(z)|^2$

2) $[|f(z)|_x]^2 + [|f(z)|_y]^2 = |f'(z)|^2$

Proof:

1) Since C.R. equations are valid

$U_x = V_y$ & $U_y = -V_x$

R.H.S. $f'(z) = U_x - iU_y$

$|f'(z)| = \boxed{U_x^2 + U_y^2}$

L.H.S $\det \begin{pmatrix} U_x & U_y \\ -U_y & U_x \end{pmatrix} = \boxed{U_x^2 + U_y^2}$

Hence, Proved !!

2) L.H.S.

$f(z) = u + iv$

$|f(z)| = \sqrt{u^2 + v^2}$

$\therefore \dfrac{\partial |f(z)|}{\partial x} = \dfrac{1}{2\sqrt{u^2+v^2}} \cdot 2uU_x + 2vV_x = \dfrac{uU_x + vV_x}{\sqrt{u^2+v^2}}$

$\therefore \dfrac{\partial |f(z)|}{\partial y} = \dfrac{uU_y + vV_y}{\sqrt{u^2+v^2}}$

Squaring and Adding:

$\therefore \dfrac{(uU_x + vV_x)^2 + (uU_y + vV_y)^2}{u^2+v^2}$

$\therefore \dfrac{u^2U^2_x + v^2V^2_x + 2uvU_xV_x + u^2U^2_y + v^2V^2_y + 2uvU_yV_y}{u^2+v^2}$

$\therefore \dfrac{u^2(U^2_x + U^2_y) + v^2(V^2_x + V^2_y) + 4uv(U_xV_x + U_yV_y)}{u^2+v^2}$

Imp: $U_x = V_y$

$U_y = -V_x$ $\qquad \therefore U_xV_x + U_yV_y = 0$

$\therefore U_xU_y = -V_xV_y$

$\therefore -U_xV_x = U_yV_y$

$$\therefore \frac{U^2(U^2x + U^2y) + V^2(V^2x + V^2y) + 0}{U^2 + V^2}$$

$$\therefore \frac{U^2(U^2y + U^2x) + V^2(U^2y + U^2x)}{U^2 + V^2}$$

$$\therefore \frac{(U^2+V^2)(U^2x + U^2y)}{U^2 + V^2}$$

$$\therefore \boxed{U^2x + U^2y}$$

R.H.S

$f'(z) = Ux - iUy$

$\boxed{|f'(z)| = Ux^2 + Uy^2}$

L.H.S. = R.H.S.

Hence, Proved !!

Q3) Determine the constants a, b, c, d if $f(z) = x^2 + 2axy + by^2 + i(cx^2 + 2dxy + y^2)$ is analytic.

$\therefore u = x^2 + 2axy + by^2$

$\therefore v = cx^2 + 2dxy + y^2$

| | |
|---|---|
| $\therefore \frac{\partial u}{\partial x} = 2x + 2ay$ | $\therefore \frac{\partial v}{\partial x} = 2cx + 2dy$ |
| $\therefore \frac{\partial u}{\partial y} = 2ax + 2by$ | $\therefore \frac{\partial v}{\partial y} = 2dx + 2y$ |

Equating $Ux = Vy$ and $Uy = -Vx$

$\therefore 2x + 2ay = 2dx + 2y$

$\therefore x + ay = dx + y$

$\therefore d = 1$

$\therefore a = 1$

$\therefore 2ax + 2by = -2cx - 2dy$

$\therefore ax + by = -cx - dy$

$\therefore x + by = -cx - y$

$\therefore c = -1$

$\therefore b = -1$

Q4) Determine the constants if
$f(z) = ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2$
$+ i(4x^3y - exy^3 + 4xy)$ is analytic.

$\therefore u = ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2$
$\therefore v = 4x^3y + 4xy - exy^3$

| | |
|---|---|
| $\therefore \frac{\partial u}{\partial x} = 4ax^3 + 2bxy^2 + 2dx$ | $\therefore \frac{\partial v}{\partial x} = 12x^2y + 4y - ey^3$ |
| $\therefore \frac{\partial u}{\partial y} = 2bx^2y + 4cy^3 - 4y$ | $\therefore \frac{\partial v}{\partial y} = 4x^3 + 4x - 3exy^2$ |

$\therefore$ Equating $u_x = v_y$ & $u_y = -v_x$

1) $4ax^3 + 2bxy^2 + 2dx = 4x^3 + 4x - 3exy^2$

$\rightarrow 4ax^3 = 4x^3$ $\quad \therefore a = 1$
$\rightarrow 2bxy^2 = -3exy^2$ $\quad \therefore 2b = -3e$
$\rightarrow 2dx = 4x$ $\quad \therefore d = 2$

2) $2bx^2y + 4cy^3 - 4y = -12x^2y - 4y + ey^3$

$\rightarrow 2bx^2y = -12x^2y$ $\quad \therefore 2b = -12 \quad \therefore b = -6$
$\rightarrow 4cy^3 = ey^3$ $\quad \therefore 4c = e$
$\rightarrow -4y = -4y$ $\quad \therefore$

3) we have,

$a = 1$ $\qquad$ * $2b = -3e$
$d = 2$ $\qquad$ $\therefore e = -\frac{2}{3}b = -\frac{2}{3} \cdot -6 = 4$
$b = -6$
$e = 4$ $\qquad$ * $4c = e$
$c = 1$ $\qquad$ $\therefore c = \frac{e}{4} = \frac{4}{4} = 1$

$\therefore$
$a = 1$
$b = -6$
$c = 1$
$d = 2$
$e = 4$

Q5) Find k such that:

$f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{kx}{y}$ is analytic.

$u = \frac{1}{2}\log(x^2+y^2)$

using only $U_x = V_y$

$v = \tan^{-1}\frac{kx}{y}$

$\therefore \frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2}$

$\therefore \frac{\partial v}{\partial y} = \frac{1}{\left(1+\frac{k^2x^2}{y^2}\right)}\cdot k.x.\left(-\frac{1}{y^2}\right)$

$= \frac{y^2}{(y^2+k^2x^2)}\cdot\frac{-kx}{y^2}$

$= \frac{-kx}{y^2+k^2x^2}$

Equating:

$$\frac{x}{x^2+y^2} = \frac{-kx}{k^2x^2+y^2}$$

$\therefore \quad x = -kx \quad \& \quad x^2 = k^2x^2$

$\therefore \quad k^2 = 1 \quad k = \pm 1$

But since, $x = -kx$, k has to be $-1$

$\boxed{\therefore \quad k = -1}$ //

Q6) Show that $w = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$ is an analytic func.

Also, obtain $\frac{dw}{dz}$ in terms of z.

$\therefore$ we have,

$u = \frac{x}{x^2+y^2} \quad ; \quad v = \frac{-y}{x^2+y^2}$

To prove its analytic, we verify the C.R. equations:

$$U_x = V_y \quad ; \quad U_y = -V_x$$

1) $\therefore \dfrac{\partial u}{\partial x} = \dfrac{(x^2+y^2)\cdot 1 - x(2x)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

2) $\therefore \dfrac{\partial u}{\partial y} = x \cdot \dfrac{-1}{(x^2+y^2)^2} \cdot 2y = \dfrac{-2xy}{(x^2+y^2)^2}$

3) $\therefore \dfrac{\partial v}{\partial x} = -y \cdot \dfrac{-1}{(x^2+y^2)^2} \cdot 2x = \dfrac{2xy}{(x^2+y^2)^2}$

4) $\therefore \dfrac{\partial v}{\partial y} = \dfrac{(x^2+y^2)\cdot(-1) - (-y)(2y)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

We observe that $U_x = V_y$ & $U_y = -V_x$

Hence, Proved !!

→ We know $w = f(z)$ $\therefore \dfrac{dw}{dz} = f'(z) = U_x - i U_y$

$$\therefore \frac{dw}{dz} = U_{x(z,0)} - i\, U_{y(z,0)}$$

$$U_{x(z,0)} = \frac{0 - z^2}{(z^2+0)^2} = \frac{-z^2}{z^4} = \frac{-1}{z^2}$$

$$U_{y(z,0)} = 0$$

$$\boxed{\therefore \frac{dw}{dz} = \frac{-1}{z^2}}$$

Q7) Show that $f(z) = \sqrt{|xy|}$ is NOT analytic @ origin although C.R. equations are satisfied @ $z = 0$

$\therefore \; u = \sqrt{|xy|} \quad ; \quad v = 0$

First, verify C.R. equations:

$\therefore \dfrac{\partial u}{\partial x} = \lim\limits_{x \to 0} \dfrac{u(x,0) - u(0,0)}{x} = \lim\limits_{x \to 0} \dfrac{0 - 0}{x} = 0$

$\therefore \dfrac{\partial u}{\partial y} = \lim\limits_{y \to 0} \dfrac{u(0,y) - u(0,0)}{y} = \lim\limits_{y \to 0} \dfrac{0-0}{y} = 0$

$\therefore \dfrac{\partial v}{\partial x} = 0$

$\therefore \dfrac{\partial v}{\partial y} = 0$

Hence, C.R. equations $U_x = V_y$ & $U_y = -V_x$ are getting satisfied.

Second, Partial derivatives $U_x, U_y, V_x, V_y$ must be continuous $\forall x, y \in \mathbb{R}$ ; $f(z)$ is differentiable then it could be expressed as $f'(z) = \frac{df}{dz} = U_x - iU_y = V_y + iV_x$

$\therefore f'(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$ consider $y = mx$

$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sqrt{|xy|} - 0}{x + iy}$

$= \lim_{x \to 0} \frac{\sqrt{|x^2 m|} - 0}{x + i}$

$= \lim_{x \to 0} \frac{x \sqrt{|m|}}{x(1 + im)}$

$= \frac{\sqrt{|m|}}{1 + im}$ ⇒ not unique since it is dependent on '$m$'

Hence, Proved !!

Q8) Show that $f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$ is not analytic @ origin although C.R. eqs. are satisfied @ $z = 0$

first, verify C.R. eq. at $z = 0$:

$\therefore f(z) = \frac{x^3 + ix^3 - y^3 + iy^3}{x^2+y^2}$

$\therefore u = \frac{x^3 - y^3}{x^2+y^2}$ ; $v = \frac{x^3 + y^3}{x^2+y^2}$

$\therefore \frac{\partial u}{\partial x} = \frac{(x^2+y^2)\cdot(3x^2) - (x^3-y^3)(2x)}{(x^2+y^2)^2}$

$\therefore U_x(x,0) = \frac{x^2 \cdot 3x^2 - x^3 \cdot 2x}{(x^2)^2} = \frac{3x^4 - 2x^4}{x^4} = \frac{x^4}{x^4} = 1$

$\therefore \frac{\partial u}{\partial y} = \frac{(x^2+y^2)(-3y^2) - (x^3-y^3)(2y)}{(x^2+y^2)^2}$

$\therefore U_y(0,y) = \frac{y^2(-3y^2) + y^3 \cdot 2y}{y^4} = \frac{-3y^4 + 2y^4}{y^4} = -1$

$$\therefore \frac{\partial v}{\partial x} = \frac{(x^2+y^2)(3x^2) - (x^3+y^3)(2x)}{(x^2+y^2)^2}$$

$$\therefore v_x(x,0) = \frac{x^2 \cdot 3x^2 - x^3 \cdot 2x}{(x^2)^2} = \frac{3x^4 - 2x^4}{x^4} = \frac{x^4}{x^4} = 1$$

$$\therefore \frac{\partial v}{\partial y} = \frac{(x^2+y^2)(3y^2) - (x^3+y^3)(2y)}{(x^2+y^2)^2}$$

$$\therefore v_y(0,y) = \frac{y^2 \cdot 3y^2 - y^3 \cdot 2y}{(y^2)^2} = \frac{3y^4 - 2y^4}{y^4} = \frac{y^4}{y^4} = 1.$$

We see that $u_x = v_y$ & $u_y = -v_x$.

so, C-R eq. are getting satisfied @ $z = 0$

Second, Partial derivatives $u_x, u_y, v_x, v_y$ must be continuous $\forall x, y \in \mathbb{R}$; $f(z)$ is differentiable then it is expressed as

$$f'(z) = \frac{df}{dz} = u_x - iu_y = v_y + iv_x$$

$$\therefore f'(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$ Consider $y = mx$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}}{x+iy}$$

$$= \lim_{x \to 0} \frac{x^3(1+i) - x^3 m^3 (1-i)}{(x^2 + x^2 m^2) \cdot (x + imx)}$$

$$= \lim_{x \to 0} \frac{x^3[(1+i) - m^3(1-i)]}{x^3(1+m^2)(1+im)}$$

$$= \frac{(1+i) - m^3(1-i)}{(1+m^2)(1+im)}$$ $\Rightarrow$ not unique since it is dependent on 'm'.

Riddhi B. Mehta
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# * TUTORIAL - 6 *

Q1) Show that $u(x,y) = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ satisfies Laplace's equation and hence find analytic function $w = f(z) = u + iv$

$\therefore$ We need to check $U_{xx} + U_{yy} = 0$

$\therefore \frac{\partial u}{\partial x} = \cosh y \cos x + 2\sinh y(-\sin x) + 2x + 4y$

$\therefore \frac{\partial U_x}{\partial x} = \cosh y(-\sin x) + 2\sinh y(-\cos x) + 2$

$\therefore \frac{\partial u}{\partial y} = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$

$\therefore \frac{\partial U_y}{\partial y} = \sin x \cosh y + 2\cos x \sinh y - 2$

$\therefore U_{xx} + U_{yy} = 0$ Laplace's eq. is satisfied.

Now finding the analytic function,

$U_x(z,0) = (\cos 0)(\cos z) + 2\sin 0(-\sin z) + 2z + 0$
$= \cos z + 2z$

$U_y(z,0) = (\sin z)(\sin 0) + 2\cos z \cos 0 - 0 + 4z$
$= 2\cos z + 4z$

$\therefore f'(z) = U_x - iU_y$
$= (\cos z + 2z) - i(2\cos z + 4z)$

$\therefore f(z) = \int(\cos z + 2z)\,dz - i\int(2\cos z + 4z)\,dz$

$\therefore f(z) = \sin z + z^2 - i\left[2\sin z + 2z^2\right] + C$

Q2) Find the harmonic conjugate for $v(x,y) = 3x^2y + 6xy - y^3$

$\therefore V_x = 6xy + 6y$

$\therefore V_y = 3x^2 + 6x - 3y^2$

$\therefore V_x(z,0) = 0$

$\therefore V_y(z,0) = 3z^2 + 6z$

$\therefore f'(z) = V_y + iV_x$
$= 3z^2 + 6z + 0$

$\therefore f(z) = \int(3z^2 + 6z)\,dz$
$= z^3 + 3z^2 + c$

$\boxed{\therefore w = z^3 + 3z^2 + C}$

Q3) Find harmonic conjugate for $u(x,y) = \frac{1}{2}\log(x^2+y^2)$.

$\therefore$ Let $z = re^{i\theta}$

$\theta = \tan^{-1}(y/x)$

$\therefore \log z = \log(re^{i\theta})$

$\therefore \log z = \log r + i\theta \log e$

$\therefore \log z = \log r + i\theta$

$$\boxed{\therefore w = f'(z) = \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x)}$$ //

Q4) Find orthogonal trajectories corresponding to $x^3y - y^3x = c$

Assume given is $U(x,y) = x^3y - y^3x = C$

$\therefore U_x = 3x^2y - y^3$

$\therefore U_y = x^3 - 3y^2x$

$\therefore U_x(z,0) = 0$

$\therefore U_y(z,0) = z^3$

$\therefore f'(z) = U_x - iU_y = -iz^3$

$\therefore f(z) = \int -iz^3\,dz = \frac{-iz^4}{4}$

$i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

$i^6 = -1$

Substituting $z = x+iy$

$\therefore f(z) = \frac{-i}{4}(x+iy)^4$

$$= \frac{-i}{4}\left\{ {}^4C_0\cdot x^4(iy)^0 + {}^4C_1\, x^3(iy) + {}^4C_2\, x^2(iy)^2 + {}^4C_3\, x(iy)^3 + {}^4C_4\, x^0(iy)^4 \right\}$$

$$= \frac{-i}{4}\left\{ x^4 + 4x^3(iy) + 6x^2y^2(-1) + 4xy^3(i)^3 + (+1)y^4 \right\}$$

$$= \frac{-i}{4}\left\{ x^4 + 4x^3y(i) - 6x^2y^2 + 4xy^3-(i) + y^4 \right\}$$

$$= \frac{-i\cdot x^4}{4} + -x^3y(i)^2 + \frac{6x^2y^2\cdot i}{4} + xy^3(i)^2 - \frac{y^4 i}{4}$$

$$= x^3y - xy^3 + i\left[\frac{3}{2}x^2y^2 - \frac{x^4}{4} - \frac{y^4}{4}\right]$$ //

Q4.



Final equation:

$$f(z) = x^3y - xy^3 + i\left[\frac{3}{2}x^2y^2 - \frac{x^4}{4} - \frac{y^4}{4}\right]$$

$$\boxed{\begin{aligned} \therefore\ u &= x^3y - xy^3 \\ \therefore\ v &= \frac{3}{2}x^2y^2 - \frac{x^4}{4} - \frac{y^4}{4} \end{aligned}}\ //$$

Q5) Find orthogonal trajectories corresponding to $e^{-x}\cos y + xy = \alpha$

$\therefore$ Assume $u = e^{-x}\cos y + xy$

$\therefore\ U_x = -e^{-x}\cos y + y$

$\therefore\ U_y = e^{-x}(-\sin y) + x$

$f'(z) = U_x(z,0) - i\,U_y(z,0)$

$f'(z) = -e^{-z} - i$

$\therefore f(z) = \int -e^{-z}\,dz - i\int z\,dz$

$= e^{-z} - \frac{iz^2}{2}$

Euler's formula
$e^{ix} = \cos x + i\sin x$
$e^{-ix} = \cos x - i\sin x$

Sub. $z = x + iy$

$f(z) = e^{-(x+iy)} - \frac{i}{2}(x+iy)^2$

$= e^{-x-iy} - \frac{i}{2}\left[x^2 + 2xyi + i^2y^2\right]$

$= e^{-x}\cdot e^{-iy} - \frac{i}{2}\left[x^2 + 2xyi - y^2\right]$

$= e^{-x}\cdot e^{-iy} - \frac{ix^2}{2} - xyi^2 + \frac{iy^2}{2}$

$= e^{-x}\cdot(\cos y - i\sin y) + xy + \frac{i}{2}(y^2 - x^2)$

$= e^{-x}\cdot\cos y - ie^{-x}\sin y + xy + \frac{i}{2}(y^2 - x^2)$

$= e^{-x}\cos y + xy - ie^{-x}\sin y + \frac{i}{2}(y^2 - x^2)$

Only consider the imaginary part: $-e^{-x}\sin y + \frac{1}{2}y^2 - \frac{1}{2}x^2$

$\therefore$ The required orthogonal trajectories are:

$$v = -e^{-x}\sin y + \frac{1}{2}y^2 - \frac{1}{2}x^2$$

$$\boxed{\therefore\ e^{-x}\sin y + \frac{1}{2}(x^2 - y^2) = c}$$

# Questions on Polar Form:

Q6) Find $p$ if $f(z) = r^2\cos 2\theta + ir^2\sin p\theta$ is analytic.

In polar form, we have C.R. eq. as:

$$U_r = \frac{1}{r}V_\theta \quad ; \quad V_\theta = -rV_r$$

$u = r^2\cos 2\theta$

$v = r^2\sin p\theta$

$\therefore U_r = 2r\cos 2\theta$

$\therefore V_\theta = r^2 . p\cos p\theta$

$\therefore U_r = \frac{1}{r}V_\theta \quad \therefore 2r\cos 2\theta = \frac{1}{r}r^2 p\cos p\theta$

$\therefore 2r\cos 2\theta = rp\cos p\theta$

$\boxed{\therefore p = 2}$

Q7) Find $k$ if $f(z) = r^3\cos k\theta + ir^k\sin 3\theta$ is analytic.

$u = r^3\cos k\theta \qquad \therefore U_r = 3r^2\cos k\theta$

$v = r^k\sin 3\theta \qquad \therefore V_\theta = 3r^k\cos 3\theta$

$\therefore U_r = \frac{1}{r}V_\theta$

$\therefore 3r^2\cos k\theta = \frac{1}{r}3r^k\cos 3\theta$

$\therefore 3r^2\cos k\theta = 3r^{k-1}\cos 3\theta$

$\therefore r^2 = r^{k-1}$

$\therefore k-1 = 2$

$\boxed{\therefore k = 3}$

Q8) Verify laplace equation for $u(r,\theta) = \left(r + \frac{a^2}{r}\right)\cos\theta$.
Find $v(r,\theta)$ and $f(z)$.

$$u = r\cos\theta + \frac{a^2\cos\theta}{r}$$

Laplace equation in polar form:

$$\boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$$

$$\Rightarrow \frac{d}{dr}(-r^{-2})$$

$$\Rightarrow 2r^{-3}$$

Verifying Laplace equation:

$$\frac{\partial u}{\partial r} = \cos\theta + a^2\cos\theta\left(-\frac{1}{r^2}\right)$$

$$\therefore \frac{\partial^2 u}{\partial r^2} = a^2\cos\theta \cdot \frac{2}{r^3} \quad \text{—— (1)}$$

$$\therefore \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{r}\cos\theta - \frac{1}{r^3}a^2\cos\theta \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial \theta} = -r\sin\theta - \frac{a^2}{r}\sin\theta$$

$$\therefore \frac{\partial^2 u}{\partial \theta^2} = -r\cos\theta - \frac{a^2}{r}\cos\theta$$

$$\therefore \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\cos\theta - \frac{a^2}{r^3}\cos\theta \quad \text{—— (3)}$$

Adding eq. (1) eq (2) eq (3) :

$$\frac{2a^2\cos\theta}{r^3} + \frac{\cos\theta}{r} - \frac{a^2\cos\theta}{r^3} - \frac{\cos\theta}{r} - \frac{a^2\cos\theta}{r^3} = 0$$

Hence, Verified !!

To find $v(r,\theta)$ and $f(z)$.

$$\therefore \frac{\partial u}{\partial r} = \cos\theta - \frac{a^2\cos\theta}{r^2}$$

$$\therefore U_r = \frac{1}{r}V_\theta$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\therefore r\cos\theta - \frac{a^2\cos\theta}{r} = \frac{\partial v}{\partial \theta}$$

$$\therefore \partial v = \left(r\cos\theta - \frac{a^2\cos\theta}{r}\right)\partial\theta$$

$$\therefore \int dv = \int\left(r\cos\theta - \frac{a^2\cos\theta}{r}\right)d\theta$$

$$\boxed{z = re^{i\theta}}$$



$$\therefore v = \int \left( r\cos\theta - \frac{a^2\cos\theta}{r} \right) d\theta$$

$$\therefore v = \int r\cos\theta \, d\theta - \int \frac{a^2\cos\theta}{r} \, d\theta$$

$$\therefore v = r\sin\theta - \frac{a^2}{r}\sin\theta + c$$

$$\therefore v = r\sin\theta - \frac{a^2}{r}\sin\theta = \boxed{\left( r - \frac{a^2}{r} \right)\sin\theta}$$

$$f(z) = u + iv$$

$$\boxed{\therefore f(z) = \left( r + \frac{a^2}{r} \right)\cos\theta + i\left( r - \frac{a^2}{r} \right)\sin\theta}$$

Q9) Verify laplace eq. for $u(r,\theta) = -r^3\sin 3\theta$. Find $v(r,\theta)$ and $f(z)$.

$$\therefore u = -r^3\sin 3\theta$$

$$\therefore \frac{\partial u}{\partial r} = -3r^2\sin 3\theta$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial \theta} = -3r^3\sin 3\theta$$

$$\therefore \partial v = -3r^3\sin 3\theta \, \partial\theta$$

$$\therefore \int dv = \int -3r^3\sin 3\theta \, d\theta$$

$$\therefore v = -3r^3 \left( \frac{-\cos 3\theta}{3} \right)$$

$$\boxed{\therefore v = r^3\cos 3\theta}$$

$$\therefore f(z) = u + iv$$
$$= -r^3\sin 3\theta + i(r^3\cos 3\theta) + c \qquad e^{i\theta} = \cos\theta + i\sin\theta$$
$$= i^2 r^3\sin 3\theta + i r^3\cos 3\theta + c$$
$$= i r^3 (i\sin 3\theta + \cos 3\theta) + c$$
$$= i r^3 e^{i3\theta} + c$$
$$= i(re^{i\theta})^3 + c$$
$$= iz^3 + c$$

$$\boxed{\therefore f(z) = iz^3 + c}$$

Riddhi B. Mehta
3rd year Sem V
Roll. J030
Applied Maths V.

1

# * TUTORIAL - 7 *

Q1) Find the image of the circle $|z| = 1$ under the transformation $w = z+3+2i$

we know $z = x+iy$ and $|z| = x^2+y^2 = 1$

$\therefore w = z+3+2i$

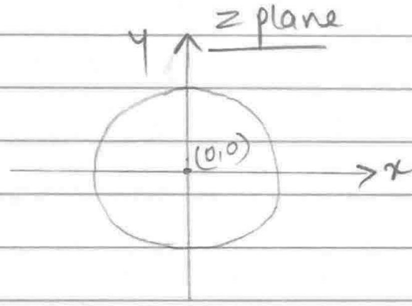
$\therefore z = w-3-2i$

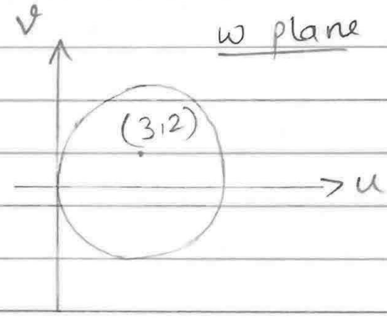
$\therefore z = (u+iv)-3-2i = (u-3)+i(v-2)$

$\therefore |z| = \sqrt{(u-3)^2+(v-2)^2}$

$\therefore 1 = (u-3)^2+(v-2)^2$

Since, the eq. above is in the form of a circle equation, we have new centres as $(3,2)$.




(Translation)

Q2) Find the image of the semi-infinite strip $x>0$, $0<y<2$ under the transformation $w = iz+1$

$\therefore w = iz+1$

$= i(x+iy)+1$

$= ix-y+1$

$= (1-y)+ix$

$u = 1-y$

$v = x$ since $x>0$ then even $\boxed{v>0}$

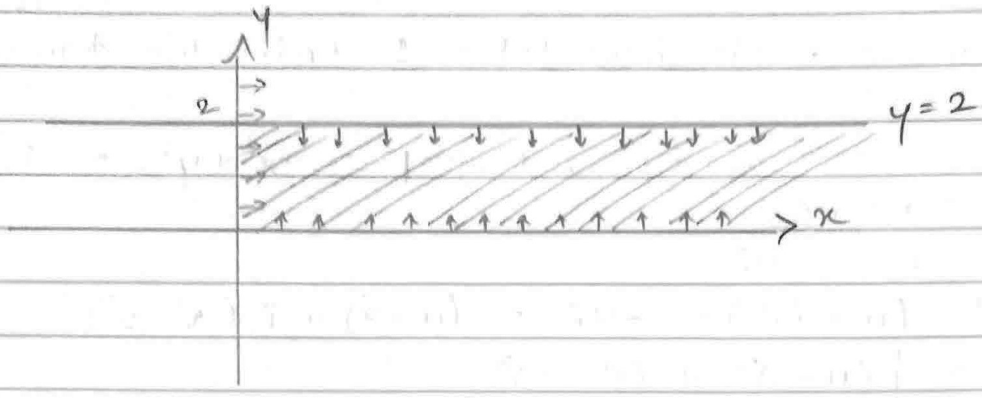
$\therefore y = 1-u$ (Magnification & Rotation)

$\therefore 0 < 1-u < 2$

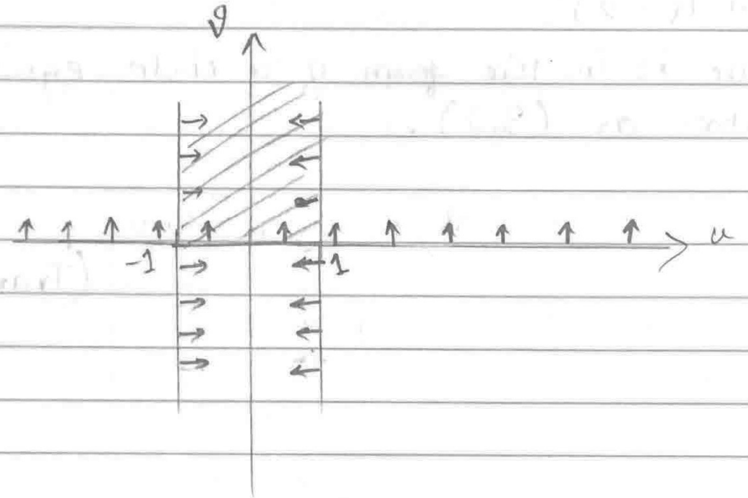
$\therefore -2 < u-1 < 0$

$\therefore \boxed{-1 < u < 1}$

* z plane $x > 0$ & $0 < y < 2$




* w plane $v > 0$ & $-1 < u < 1$




Q3) Find the image of $|z - ai| = a$ under transformation $w = \frac{1}{z}$

$\therefore z = \frac{1}{w}$

$\therefore \left|\frac{1}{w} - ai\right| = a$

$\therefore |1 - wai| = a|w|$

$\therefore |1 - (u+iv)ai| = a|u+iv|$

$\therefore |1 - uai - i^2 av| = a^2(u^2 + v^2)$

$\therefore |1 - uai + av| = a^2(u^2+v^2)$

$\therefore |1 + av - uai| = a^2(u^2+v^2)$

$\therefore (1+av)^2 + (ua)^2 = a^2(u^2+v^2)$

$\therefore 1 + 2av + a^2v^2 + a^2u^2 = a^2u^2 + a^2v^2$

$\therefore 1 + 2av + a^2v^2 + a^2u^2 - a^2u^2 - a^2v^2 = 0$

$\therefore 1 + 2av = 0$

$\therefore$ Final Ans: $\boxed{1 + 2av = 0}$ //

84) Find the image of $(x-3)^2+y^2 = 2$ under the transformation $w = \frac{1}{z}$.

Draw?

$z = x+iy$
$w = u+iv$

$\therefore \quad (x-3)^2 + y^2 = 2$

$\therefore \quad |(x-3) + iy| = \sqrt{2}$

$\therefore \quad |x+iy-3| = \sqrt{2}$

$\therefore \quad |z-3| = \sqrt{2}$

$\therefore \quad \left|\frac{1}{w} - 3\right| = \sqrt{2}$

$\therefore \quad |1-3w| = \sqrt{2}|w|$

$\therefore \quad |1-3(u+iv)| = \sqrt{2}\,|u+iv|$

$\therefore \quad |1-3u-3iv| = \sqrt{2}\,|u+iv|$

$\therefore \quad |(1-3u) - (3iv)| = \sqrt{2}\,|u+iv|$

$\therefore \quad (1-3u)^2 + (3v)^2 = 2(u^2+v^2)$

$\therefore \quad 1 + 9u^2 - 6u + 9v^2 = 2u^2 + 2v^2$

$\therefore \quad 1 + 7u^2 + 7v^2 - 6u = 0$

$\therefore \quad \frac{1}{7} + u^2 + v^2 - \frac{6}{7}u = 0$

$\therefore \quad \frac{1}{7} + \left(\frac{3}{7}\right)^2 - \left(\frac{3}{7}\right)^2 + u^2 + v^2 - \frac{6}{7}u = 0$

$\therefore \quad u^2 + \left(\frac{3}{7}\right)^2 - \frac{6u}{7} - \left(\frac{3}{7}\right)^2 + \frac{1}{7} + v^2 = 0$

$\therefore \quad \left(u - \frac{3}{7}\right)^2 - \frac{9}{49} + \frac{7}{49} + v^2 = 0$

$\therefore \quad \left(u - \frac{3}{7}\right)^2 - \frac{2}{49} + v^2 = 0$

$\therefore \quad \left(u - \frac{3}{7}\right)^2 + v^2 = \frac{2}{49}$

$\therefore$ Radius $= \frac{\sqrt{2}}{7}$

$\therefore$ Centre : $\left(\frac{3}{7}, 0\right)$

Q5) Find the image of the circle $|z| = k$ where $k \in \mathbb{R}$ under B.T. $w = \dfrac{5-4z}{4z-3}$

$\therefore \quad 4wz - 3w = 5 - 4z$

$\therefore \quad 4wz + 4z = 5 + 3w$

$\therefore \quad 4z(w+1) = 5 + 3w$

$\therefore \quad z = \dfrac{(5+3w)}{4(w+1)}$

$\therefore \left| \dfrac{5+3w}{4w+4} \right| = k$

$\therefore \left| \dfrac{5+3(u+iv)}{4(u+iv)+4} \right| = k$

$\therefore |5+3u+3iv| = k\,|4u+4iv+4|$

$\therefore (5+3u)^2 + (3v)^2 = k^2[(4u+4)^2 + (4v)^2]$

$\therefore 25 + 9u^2 + 30u + 9v^2 = k^2[16u^2 + 16 + 32u + 16v^2]$

$\therefore 25 + 9u^2 + 30u + 9v^2 - 16u^2k^2 - 16k^2 - 32uk - 16k^2v^2 = 0$

$\therefore k^2(-16u^2 - 16v^2) + 9(u^2+v^2) + 25 + 30u - 16k^2 - 32uk = 0$

$\therefore (u^2+v^2)(9-16k^2) + (30-32k)u + (25-16k^2) = 0$

$\boxed{\therefore (16k^2-9)(u^2+v^2) + (32k-30)u + (16k^2-25) = 0}$ //

Q6) Find the fixed points of the transformation $w = \dfrac{3z-4}{z-1}$. Also express it in normal form :

$\left(\dfrac{1}{w-\alpha}\right) = \left(\dfrac{1}{z-\alpha}\right) + \lambda$

$\therefore$ for fixed point transformation $w = z$.

$\therefore \quad z = \dfrac{3z-4}{z-1} \quad \therefore$

$\therefore \quad z^2 - z = 3z - 4$

$\therefore \quad z^2 - 4z + 4 = 0$

$\therefore \quad z^2 - 2z - 2z + 4 = 0$

$\therefore \quad (z-2)^2 = 0$

$\therefore \quad z - 2 = 0$

$\therefore \quad \boxed{z = 2}$ //

$\therefore$ Obtained value of $z = \alpha = 2$.

$\therefore$ To express in normal form, we deal with L.H.S. to reach & obtain R.H.S.

$$\therefore \frac{1}{w-\alpha} = \frac{1}{\left(\frac{3z-4}{z-1}\right)-2} = \frac{z-1}{3z-4-2z+2} = \frac{z-1}{z-2}$$

$$= \frac{z-1-1+1}{z-2} = \frac{z-2+1}{z-2} = \frac{z-2}{z-2} + \frac{1}{z-2}$$

$$= 1 + \frac{1}{z-2}$$

Comparing to R.H.S.

$$\frac{1}{z-\alpha} + \lambda = \frac{1}{z-2} + 1$$

$$\boxed{\begin{aligned} \therefore\ \lambda &= 1 \\ \therefore\ \alpha &= 2 \end{aligned}}$$

Q7) Show that the image of the rectangular hyperbola $x^2-y^2=1$ under the transformation $w=\frac{1}{z}$ is the leminscate $\rho^2 = \cos 2\phi$. Also, draw sketches

Hint: Use polar forms $z = re^{i\theta}$ and $w = \rho e^{i\phi}$

$$\therefore w = \frac{1}{z}$$

$$\therefore \rho e^{i\phi} = \frac{1}{re^{i\theta}} \quad \therefore \rho e^{i\phi} = \frac{e^{-i\theta}}{r}$$

$\boxed{\therefore \rho = \frac{1}{r}}$ & $\boxed{\phi = -\theta}$ Imp. formula for usage

$\therefore$ Substituting, $x = r\cos\theta$ ; $y = r\sin\theta$

$$\therefore r^2\cos^2\theta - r^2\sin^2\theta = 1$$

$$\therefore r^2(\cos^2\theta - \sin^2\theta) = 1$$

$$\therefore r^2(\cos 2\theta) = 1$$

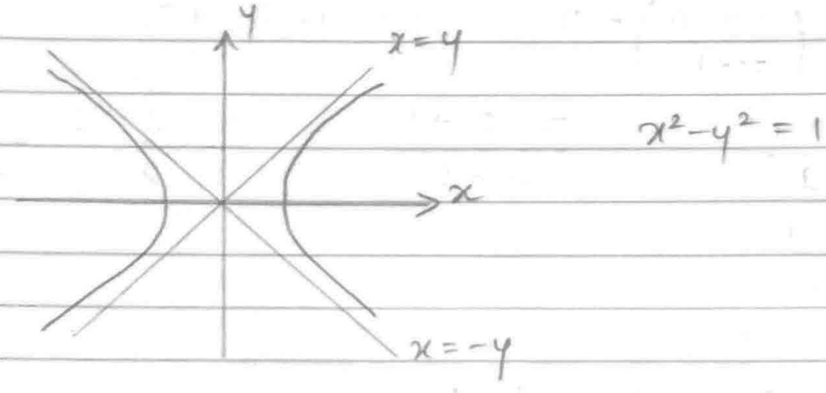
$$\therefore \frac{1}{\rho^2}\cos 2(-\phi) = 1$$

$$\therefore \cos 2\phi = \rho^2$$

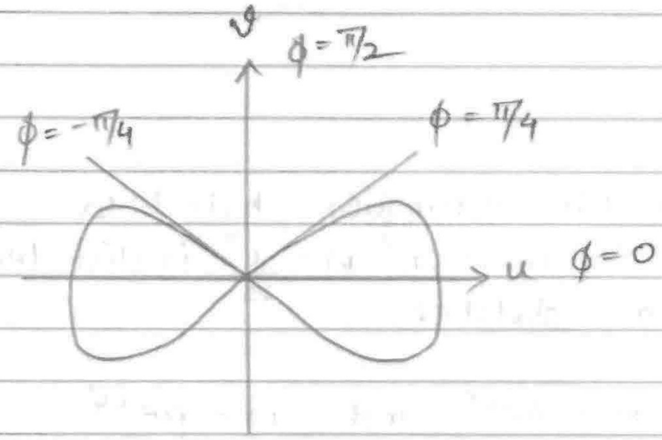
$\boxed{\therefore \rho^2 = \cos 2\phi}$ Hence, proved !!

→ The sketches look as follows:

z plane




w plane




Q8) Find the B.T. which maps the points $2, i, -2$ onto the points $1, i, -1$ by using cross ratio property.

z plane $\langle 2, i, -2 \rangle$ to w plane $\langle 1, i, -1 \rangle$
$\quad z_2\ z_3\ z_4 \qquad\qquad w_2\ w_3\ w_4$

We use the cross ratio formula:

$$\therefore \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)}$$

$$\therefore \frac{(z_1-2)(i+2)}{(2-i)(-2-z_1)} = \frac{(w-1)(i+1)}{(1-i)(-1-w_1)}$$

$$\therefore \frac{(z_1-2)(2+i)}{-(z_1+2)(2-i)} = \frac{(w-1)(1+i)}{-(w_1+1)(1-i)}$$

$$\therefore \frac{(z_1-2)(2+i)}{(z_1+2)(2-i)} = \frac{(w_1-1)(1+i)}{(w_1+1)(1-i)}$$

$\therefore \dfrac{(w_1-1)}{(w_1+1)} = \dfrac{(z_1-2)(2+i)(1-i)}{(z_1+2)(2-i)(1+i)}$

$\therefore \dfrac{w-1}{w+1} = \dfrac{(z-2)}{(z+2)} \dfrac{(2-2i+i-i^2)}{(2+2i-i-i^2)}$

$\therefore \dfrac{w-1}{w+1} = \dfrac{(z-2)}{(z+2)} \dfrac{(3-i)}{(3+i)}$

$\therefore$ Using componendo dividendo rule :

If $\dfrac{a}{b} = \dfrac{c}{d}$ then $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$

$\therefore \dfrac{(w-1)+(w+1)}{(w-1)-(w+1)} = \dfrac{(z-2)(3-i)+(z+2)(3+i)}{(z-2)(3-i)-(z+2)(3+i)}$

$\therefore \dfrac{2w}{-2} = \dfrac{3z-\cancel{zi}-\cancel{6}+2i+3z+\cancel{zi}+\cancel{6}+2i}{\cancel{3z}-zi-6+2i-(\cancel{3z}+2i+6+\cancel{2i})}$

$\therefore -w = \dfrac{6z+4i}{-2zi-12}$

$\therefore w = \dfrac{6z+4i}{2zi+12}$

$\therefore \boxed{w = \dfrac{3z+2i}{zi+6}}$

final ans ↑

Q9) Find the B.T. which maps the points $\infty, i, 0$ onto the points $0, i, \infty$ by using cross ratio property.

z plane $\langle \overset{2}{\infty}, \overset{3}{i}, \overset{4}{0} \rangle$ w plane $\langle \overset{2}{0}, \overset{3}{i}, \overset{4}{\infty} \rangle$

We use cross ratio property.

$\therefore \dfrac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} = \dfrac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)}$

$\therefore$ Basically we have to find $w = f(z)$

$\therefore$ Dividing LHS & RHS by $z_2$ and $w_4$ $(\infty)$ resp.

$\therefore \lim_{z_2\to\infty} \dfrac{\left(\dfrac{z_1}{z_2}-1\right)(z_3-z_4)}{\left(1-\dfrac{z_3}{z_2}\right)(z_4-z_1)} = \lim_{w_4\to\infty} \dfrac{(w_1-w_2)\left(\dfrac{w_3}{w_4}-1\right)}{(w_2-w_3)\left(1-\dfrac{w_1}{w_4}\right)}$

$\therefore \dfrac{(-1)(z_3-z_4)}{(1)(z_4-z_1)} = \dfrac{(w_1-w_2)(-1)}{(w_2-w_3)(1)}$

$\therefore \dfrac{(i-0)}{-z} = \dfrac{(w-0)}{(-i)}$

$\therefore \dfrac{i}{-z} = \dfrac{w}{-i}$

$\therefore \dfrac{i}{z} = \dfrac{w}{i}$

$\therefore w = \dfrac{i^2}{z}$

$\therefore \boxed{w = \dfrac{-1}{z}}$

final ans

Q10) Find the image of rect. region bounded by $x=0$, $x=3$, $y=0$, $y=2$ under transformation $w = z+(1+i)$. Hint: Use translation.

$\therefore \ w = z+(1+i)$

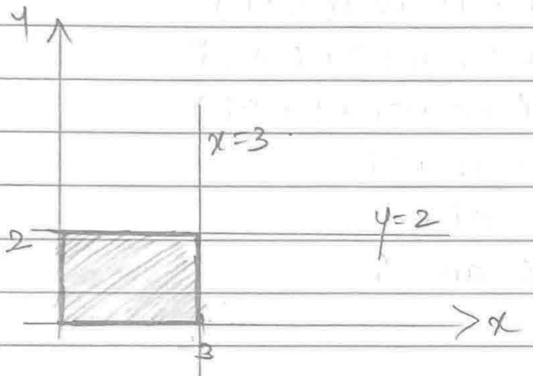
$\therefore \ u+iv = x+iy+1+i$

$\therefore \ u+iv = (x+1) + i(y+1)$

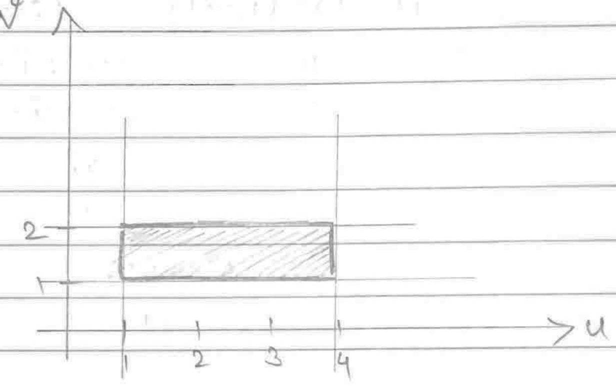
$\therefore \ u = x+1 \qquad$ New coordinates : $u=1 \ ; \ u=4$

$\quad v = y+1 \qquad\qquad\qquad\qquad\qquad v=1 \ ; \ v=2.$

z plane




w plane




Q11) Find the image of the $\Delta^r$ region whose vertices are $i$, $1 \pm i$ under transformation $w = z+4-2i$. Draw sketch.

$\therefore$ We have 3 vertices

$V_1 = x_1+iy_1 = i \quad \therefore x_1=0 \quad y_1=1$ Doubt : How to

$V_2 = x_2+iy_2 = 1+i \quad \therefore x_2=1 \quad y_2=1$ sketch !

$V_3 = x_3+iy_3 = 1-i \quad \therefore x_3=1 \quad y_3=-1$

Using translation :

$\therefore \ u+iv = x+iy+4-2i$

$\therefore \ u+iv = (x+4)+i(y-2)$

$\therefore \ u = x+4$

$\therefore \ v = y-2$

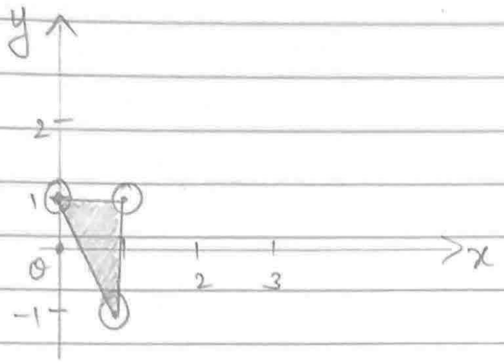
$\therefore$ New vertices are :

$\therefore \ V_1' = u_1+iv_1 = 4+i(-1) = 4-i$

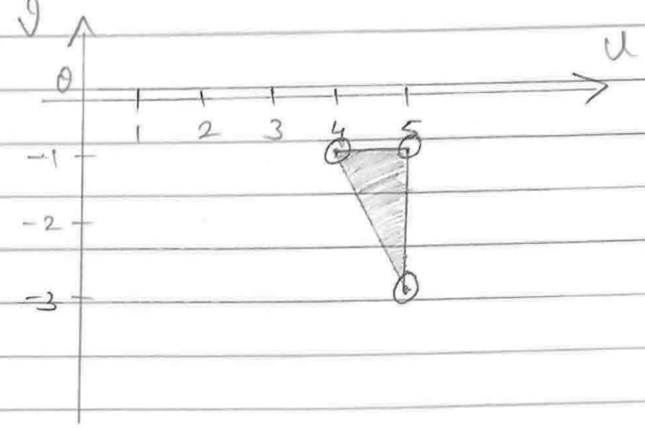
$V_2' = u_2+iv_2 = 5+i(-1) = 5-i$

$V_3' = u_3+iv_3 = 5+i(-3) = 5-3i$

z plane




w plane.




Q12) Find the image of the region bounded by $x=0$, $x=2$, $y=0$, $y=2$ under the transformation $w=z(1+i)$. Hint: Magnification & Rotation

$\therefore\ u+iv = (x+iy)(1+i)$

$\therefore\ u+iv = x+xi+iy+i^2y$     Doubt

$\therefore\ u+iv = x+xi+iy-y$     $(1+i)$

$\therefore\ u+iv = (x-y)+i(x+y)$     $a_1 = 1$

$\therefore\ u = x-y$     $a_2 = 1$

$\therefore\ v = x+y$

$u = a_1x - a_2y$

$v = a_2x + a_1y$.

When :

1) $x=0$ ; $u = x-y = -y$ ; $v = x+y = y$.

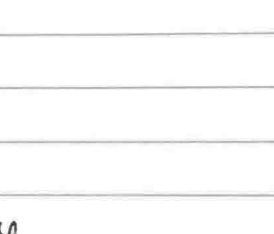
$\boxed{u=-y,\ v=y}$ $\therefore$ $\boxed{u=-v}$

2) $x=2$ ; $u = x-y = 2-y$ ; $v = x+y = 2+y$

$u+v = 2-y+2+y$ $\therefore$ $\boxed{u+v=4}$

3) $y=0$ ; $u = x-y = x$ ; $v = x+y = x$.

$\boxed{u=v}$



4) $y=2$ ; $u = x-y = x-2$ ; $v = x+y = x+2$

$\therefore\ u+v = x-2+x+2$

$\therefore\ u+v = 2x \rightarrow$ not reqd.

$\therefore\ u-v = x-2-x-2$

$\therefore\ \boxed{u-v=-4}$