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Applied Maths V

* TUTORIAL 1 *

Q1) Find the eigen value and eigen vector of matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

The characteristic equation is given as:

$\therefore \lambda^3 - \lambda^2 (\text{Trace of } A) + \lambda(A_{11} + A_{22} + A_{33}) - |A| = 0$

$\therefore \lambda^3 - \lambda^2 (6) + \lambda(3+3+5) - 6 = 0$

$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

We obtain,

$\lambda_1 = 1$

$\lambda_2 = 3$

$\lambda_3 = 2$

$\therefore$ The eigen values are 1, 3, 2.

To find eigen vectors,

For λ_1 $\quad (A - \lambda_1 I) X = 0$

$$\left\{ \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} X = 0$$

$$\therefore \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore x_1 - x_2 + x_3 = 0$

$\therefore x_1 + x_2 - x_3 = 0$

$\therefore x_1 - x_2 + x_3 = 0$

$\therefore \cancel{x_1 =}$

$\cancel{x_2 =}$

$\cancel{x_3 =}$

Choosing eq.

$x_1 - x_2 + x_3 = 0$

$x_1 + x_2 - x_3 = 0$

$$\therefore \frac{x_1}{\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}} = K$$

$$\therefore \frac{x_1}{0} = \frac{-x_2}{-2} = \frac{x_3}{2} = K.$$

$\therefore$ The eigen vector is given as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

For λ_2 : $(A - \lambda_2 I) X = 0$

$$\therefore \left\{ \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore -x_1 - x_2 + x_3 = 0$

$\therefore x_1 - x_2 - x_3 = 0$

$\therefore x_1 - x_2 - x_3 = 0$

Choosing eq. (3)

$$-x_1 - x_2 + x_3 = 0$$
$$x_1 - x_2 - x_3 = 0$$

$$\therefore \frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix}} = k$$

$$\therefore \frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{2} = k$$

$\therefore$ The eigen vector is given as :

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

For λ_3 : $(A - \lambda_3 I)X = 0$

$$\therefore \left\{ \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore -x_2 + x_3 = 0$$
$$\therefore x_1 - x_3 = 0$$
$$\therefore x_1 - x_2 = 0$$

$$0x_1 - x_2 + x_3 = 0$$
$$x_1 + 0x_2 - x_3 = 0$$

$\therefore$ The eigen vector is given as: (consider 1^{st} 2^{nd} eq) 4

$$\therefore \frac{x_1}{\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}} = k$$

$$\therefore \frac{x_1}{1} = \frac{-x_2}{-1} = \frac{x_3}{1} = k$$

$\therefore$ Eigen vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ The eigen vectors X_1, X_2 and X_3 are:

$$\begin{array}{ccc} X_1 & X_2 & X_3 \\ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{array}$$

Q2) Find the eigen values and eigen vectors of matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic eq. is given as :

$\therefore \lambda^3 - \lambda^2(12) + \lambda(8+14+14) - 32 = 0$

$\therefore \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\therefore \lambda_1 = 8$

$\lambda_2 = 2$

$\lambda_3 = 2$

To find eigen vectors:

For $\lambda_1 = 8$ $\quad (A - \lambda_1 I)X = 0$

$$\left\{ \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore -2x_1 - 2x_2 + 2x_3 = 0$

$\therefore -2x_1 - 5x_2 - x_3 = 0$

$\therefore 2x_1 - x_2 - 5x_3 = 0$

$$\therefore \frac{x_1}{\begin{vmatrix} -5 & -1 \\ -1 & -5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & -1 \\ 2 & -5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -5 \\ 2 & -1 \end{vmatrix}} = k$$

$$\therefore \frac{x_1}{24} = \frac{-x_2}{12} = \frac{x_3}{12} = k$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ -12 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 2$: $(A - \lambda_2 I)X = 0$

$$\therefore \left\{ \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore 4x_1 - 2x_2 + 2x_3 = 0 \implies 2x_1 - x_2 + x_3 = 0$

$\therefore -2x_1 + x_2 - x_3 = 0$

$\therefore 2x_1 - x_2 + x_3 = 0$

Cramers rule fails since all the three equations are the same, the equation is $2x_1 - x_2 + x_3 = 0$.

It can be solved like: $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

~~Case 1~~

$x_1 = 0$

$x_2 = x_3$

$\therefore [0, 1, 1]$

~~Case 2~~

$\cancel{x_3 = 0}$

$\cancel{2x_1 = x_2}$

$\therefore \cancel{[1, 2, 0]}$

3 vector have to be orthogonal

$X_1 \perp^r X_2 \perp^r X_3$.

$\therefore$ ~~Both the eigen vectors~~ are linearly independent.

$X_3 = X_1 \cdot X_2 = 0$

$2z_1 - z_2 + z_3 = 0$

$0z_1 + z_2 + z_3 = 0$

$$\therefore \frac{z_1}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{-z_2}{\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}} = \frac{z_3}{\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}}$$

$$\therefore \frac{z_1}{-2} = \frac{-z_2}{2} = \frac{+z_3}{2}$$

$$\therefore X_3 = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Q3) Verify Cayley Hamilton Theorem (CHT) for matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and find A^{-1}, A^{-2}, A^{+4}

$\therefore$ The characteristic eq. is

$\lambda^3 - \lambda^2(6) + \lambda(3+3+3) - 4 = 0$

$\therefore \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

Replacing λ with A : (Using CHT)

$\therefore A^3 - 6A^2 + 9A - 4I = 0$

To verify CHT : lets compute A^2 and A^3

$A^2 = A \times A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

$\therefore A^3 = A^2 \times A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$

Hence, substituting in the equation :

$A^3 - 6A^2 + 9A - 4I = 0$

$$\therefore \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

is equal to $\equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ //

We observe that

LHS = RHS.

Hence CHT is proven !!

→ To find A^{-1}

$\therefore A^{-1}(A^3 - 6A^2 + 9A - 4I) = 0$

$\therefore A^2 - 6A + 9I - 4A^{-1} = 0$

$\therefore A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$

$$= \frac{1}{4}\left\{ \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \right\} = \begin{bmatrix} 3/4 & 1/4 & -1/4 \\ 1/4 & 3/4 & 1/4 \\ -1/4 & 1/4 & 3/4 \end{bmatrix}$$ //

∴ Verified with calculator !

→ To find A^{-2}

$\therefore A^{-2}(A^3-6A^2+9A-4I)=0$

$\therefore A-6I+9A^{-1}-4A^{-2}=0$

$\therefore A^{-2}=\frac{1}{4}(A-6I+9A^{-1})$

$$=\frac{1}{4}\left\{\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}-\begin{bmatrix}6 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 6\end{bmatrix}+\begin{bmatrix}27/4 & 9/4 & -9/4\\ 9/4 & 27/4 & 9/4\\ -9/4 & 9/4 & 27/4\end{bmatrix}\right.$$

$$=\begin{bmatrix}0.69 & 0.31 & -0.31\\ 0.31 & 0.69 & 0.31\\ -0.31 & 0.31 & 0.69\end{bmatrix}\Rightarrow\begin{bmatrix}0.69 & 0.31 & -0.31\\ 0.31 & 0.69 & 0.31\\ -0.31 & 0.31 & 0.69\end{bmatrix}$$

→ To find A^{+4}

$\therefore A(A^3-6A^2+9A-4I)=0$

$\therefore A^4-6A^{+3}+9A^2-4A=0$

$\therefore A^4=6A^3-9A^2+4A$

$$=6\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}-9\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}$$

$$+4\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}=\begin{bmatrix}86 & -85 & 85\\ -85 & 86 & -85\\ 85 & -85 & 86\end{bmatrix}$$

X