Riddhi B. Mehta
3rd year Sem V
Roll. J030
Applied Math V.

* TUTORIAL 4 *

Q1) $u(x,y) = \frac{x}{2} \log(x^2+y^2) - y \tan^{-1}(y/x) + \sin x \cdot \cosh y$

Find the analytic function $f(z) = u + iv$

Solution:

$f(z) = u + iv$ is analytic function of z.

given $u(x,y)$:

we differentiate $u(x,y)$ wrt 'x'

$$\therefore u_x = \frac{d}{dx}\left\{ \frac{x}{2} \log(x^2+y^2) - y \tan^{-1}(y/x) + \sin x \cosh y \right\}.$$

$$= \frac{x}{2} \cdot \frac{1}{x^2+y^2} \cdot 2x + \frac{1}{2}\log(x^2+y^2)$$

$$+ \frac{y}{(1+y^2/x^2)} \cdot \frac{y}{x^2} + (\cosh y)(\cos x)$$

$$= \frac{x^2}{x^2+y^2} + \frac{1}{2}\log(x^2+y^2) + \frac{y^2}{x^2} \cdot \frac{x^2}{x^2+y^2}$$

$$+ \cos x \cdot (\cosh y)$$

$$= \frac{x^2}{x^2+y^2} + \frac{1}{2}\log(x^2+y^2) + \frac{y^2}{x^2+y^2} + \cos x(\cosh y)$$

$$= \frac{1}{2}\log(x^2+y^2) + (\cos x)(\cosh y) + 1$$

$$\therefore U_y = \frac{d}{dy}\left(\frac{x}{2}\log(x^2+y^2) - y\tan^{-1}(y/x) + \sin x \cosh y\right)$$

$$= \frac{x}{\cancel{2}} \cdot \frac{1}{x^2+y^2} \cdot \cancel{2}y - \left[\frac{y}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} + \tan^{-1} y/x\right]$$

$$+ (\sin x)\left[\frac{e^y - e^{-y}}{2}\right]$$

$$= \frac{xy}{x^2+y^2} - \tan^{-1}(y/x) - \frac{y}{x}\left(\frac{x^{\cancel{2}}}{x^2+y^2}\right).$$

$$+ \sin x(\sin hy)$$

$$= \frac{xy}{x^2+y^2} - \tan^{-1}(y/x) - \frac{xy}{x^2+y^2}$$

$$+ \sin x(\sin hy)$$

$$= (\sin x)(\sin hy) - \tan^{-1}(y/x) \quad //$$

→ So, we have:

$$\phi_1(z) = U_x(z, 0)$$

$$= \frac{1}{2}\log(x^2+y^2) + (\cos x)(\cos hy) + 1$$

$$= \frac{1}{2}\log(z^2) + (\cos z)(\cos 0) + 1$$

$$= \frac{1}{2} \cdot 2\log z + \cos z + 1$$

$$= (\log z) + (\cos z) + 1$$

$$\phi_2(z) = U_y(z,0)$$
$$= (\sin x)(\sin hy) - \tan^{-1}(y/x)$$
$$= (\sin z)(\sin 0) - \tan^{-1}(0)$$
$$= 0$$

→ Using CR equations for derivative we have,

$$f'(z) = U_x - iU_y$$
$$= \phi_1(z) - i\,\phi_2(z)$$
$$= \log(z) + \cos(z) + 1 \;//$$

→ Integrating $f'(z)$ to obtain $f(z)$, we get:

$$f(z) = \int (\log z + \cos z + 1)\, dz$$

$$= z\log z - z + \sin z + z + c$$

$$f(z) = z\log z + \sin z + c \;//$$

Q2) $$U(x,y) = \frac{\sin 2x}{\cos 2y + \cos 2x}$$

∴ computing U_x as $\frac{d}{dx} U(x,y) = \frac{\partial}{\partial x}\left(\frac{\sin 2x}{\cos 2y + \cos 2x}\right)$

$$= \frac{2(\cos 2y + \cos 2x)\cos 2x - \sin 2x(-2\sin 2x)}{(\cos 2y + \cos 2x)^2}$$

$$= \frac{(2\cos 2y + 2\cos 2x)\cos 2x + 2\sin^2 2x}{(\cos 2y + \cos 2x)^2}$$

$\therefore$ Computing U_y as $\frac{\partial}{\partial y}(U(x,y))$

$$= \frac{\partial}{\partial y}\left(\frac{\sin 2x}{\cos 2y + \cos 2x}\right)$$

$$= \frac{0 - \sin 2x(-2\sin 2y)}{(\cos 2y + \cos 2x)^2}$$

$$= \frac{2\sin 2x \sin 2y}{(\cos 2y + \cos 2x)^2}$$

$\rightarrow$ $\therefore \phi_1(z) = U_x(z,0)$

$$= \frac{(2\cos 2y + 2\cos 2x)\cos 2x + 2\sin^2 2x}{(\cos 2y + \cos 2x)^2}$$

$$= \frac{(2\cos 0 + 2\cos(2z))\cos(2z) + 2\sin^2 2(z)}{(\cos 0 + \cos(2z))^2}$$

$$= \frac{(2 + 2\cos 2z)\cos(2z) + 2\sin^2(2z)}{(1+\cos(2z))^2}$$

$$= \frac{2\cos(2z) + 2\cos^2(2z) + 2\sin^2(2z)}{(1+\cos(2z))^2}$$

$$= \frac{2 + 2\cos(2z)}{(1+\cos(2z))^2}$$

$$= \frac{2(1+\cos(2z))}{(1+\cos(2z))^2} = \frac{2}{(1+\cos(2z))}$$

$\longrightarrow$ $\therefore \phi_2(z) = U_y(z,0)$

$$= \frac{2\sin 2x \sin 2y}{(\cos 2y + \cos 2x)^2} = 0$$

→ Using CR equations for derivative, we have:

$$f'(z) = U_x - iU_y$$
$$= \phi_1(z) - i\,\phi_2(z)$$
$$= \frac{2}{1+\cos(2z)} - 0$$
$$= \frac{2}{1+\cos(2z)}$$

→ Integrating $f'(z)$ to obtain $f(z)$, we get:

$$f(z) = \int \frac{2}{1+\cos(2z)}\,dz$$
$$= \int \frac{2}{2\cos^2(z)}\,dz$$
$$= \int \sec^2(z)\,dz$$

$\therefore f(z) = \tan(z) + c$.

Q3) $v(x,y) = e^{-x}(y\sin y + x\cos y)$

Let $f(z) = u + iv$ be analytic function of z.

$\therefore$ Computing v_x as $\frac{\partial}{\partial x} v(x,y) = \frac{\partial}{\partial x}\left(e^{-x}(y\sin y + x\cos y)\right)$

$$= e^{-x}(\cos y) + (y\sin y + x\cos y)(-e^{-x})$$

$v_x = e^{-x}(\cos y - y\sin y - x\cos y)$.

$\therefore$ Computing v_y as $\frac{\partial}{\partial y} v(x,y)$

$$= \frac{\partial}{\partial y}\left(e^{-x}\left(y\sin y + x\cos y\right)\right)$$

$$= e^{-x}\left(y\cos y + \sin y + x(-\sin y)\right)$$

$$v_y = e^{-x}\left(y\cos y + \sin y - x\sin y\right)$$

$\rightarrow$ $\phi_1(z) = v_x(z,0)$

$$= e^{-x}(\cos y - y\sin y - x\cos y)$$

$$= e^{-z}(\cos 0 - 0 - x\cos 0)$$

$$= e^{-z}(1-z)$$

$\rightarrow$ $\phi_2(z) = v_y(z,0)$

$$= e^{-x}(y\cos y + \sin y - x\sin y)$$

$$= e^{-z}(0+0-0)$$

$$= 0$$

$\rightarrow$ Using CR equations for derivatives we have,

$$f'(z) = v_y + i v_x$$

$$= \phi_2(z) + i\phi_1(z)$$

$$= i e^{-z}(1-z)$$

→ Integrating $f'(z)$ to obtain $f(z)$, we get :

$$f(z) = \int i e^{-z} (1-z)\, dz$$

$$= i \int e^{-z} (1-z)\, dz$$

$$= i \int e^{-z} - z e^{-z}\, dz$$

$$= i \left[-e^{-z} - \left(-e^{-z} (z+1) \right) \right] + c$$

$$= i \left[-e^{-z} - \left(-e^{-z} \cdot z - e^{-z} \right) \right] + c$$

$$= i \left[-e^{-z} + z e^{-z} + e^{-z} \right] + c$$

$$f(z) = i \cdot z e^{-z} + c$$

* Using the formula :

$$\int uv\, dx = u \int v\, dx - \int \left(\frac{du}{dx} \int v\, dx \right) dx$$

$f(z) = u+iv$
$if(z) = iu - v$
$f(z)(1+i) = u+iv+iu-v$
$(1+i)f(z) = (u-v) + i(u+v)$

Q4) $u-v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$ where $f(\pi/2) = 0$

Since $f(z) = u+iv$, so we have
$$(1+i)f(z) = (u-v) + i(u+v)$$

$\therefore$ We assume that
$$(1+i)f(z) = U + iV$$

We have real part as:
$$U = u-v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$$

$\rightarrow$ Computing $\dfrac{\partial U}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}\right)$

$$= \frac{(2\cos x - e^{y} - e^{-y})(-\sin x + \cos x) - (\cos x + \sin x - e^{-y})(-2\sin x)}{(2\cos x - e^{y} - e^{-y})^2} \Rightarrow (1)$$

$\therefore \phi_1(z) = U_x(z,0)$
$$= \frac{(2\cos(z) - 1 - 1)(-\sin(z) + \cos(z)) - \cancel{(1-1)}(-2\sin(z))(\cos z + \sin z - 1)}{(2\cos(z) - 1 - 1)^2}$$

$$= \frac{(2\cos z - 2)(\cos z - \sin z) - (-2\sin z)(\cos z + \sin z - 1)}{(2\cos z - 2)^2}$$

$$\therefore \phi_1(z) = \cancel{\frac{1-\sin z}{2\cos z - 2}} \Rightarrow \cancel{*} \boxed{\frac{1}{2} \frac{1}{(\cos z - 1)}}$$

$\rightarrow$ Computing $\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}\right)$

$$= \frac{(2\cos x - e^{y} - e^{-y})(e^{-y}) - (\cos x + \sin x - e^{-y})(-e^{y} + e^{-y})}{(2\cos x - e^{y} - e^{-y})^2} \Rightarrow ②$$

$$\therefore \phi_2(z) = U_y(z,0)$$

$$= \frac{(2\cos x - e^{y} - e^{-y})(e^{-y}) - (\cos x + \sin x - e^{-y})(-e^{y} + e^{-y})}{(2\cos x - e^{y} - e^{-y})^2}$$

$$= \frac{(2\cos z - 1 - 1)(1) - (\cos z + \sin z - 1)(-1+1)}{(2\cos z - 1 - 1)^2}$$

$$= \frac{(2\cos z - 2)}{(2\cos z - 2)^2} = \boxed{\frac{1}{(2\cos z - 2)}} \Rightarrow \cancel{*}$$

$$= \boxed{\frac{1}{2} \cdot \frac{1}{(\cos z - 1)}}$$

→ Using CR equations for derivative, we have

$$(1+i)\, f'(z) = U_x - i U_y$$
$$= \phi_1(z) - i\,\phi_2(z)$$
$$= \frac{-1}{2\cos z - 2} - i\left(\frac{1}{2\cos z - 2}\right)$$

→ Simplifying the same :

$$\therefore (1+i) f'(z) = \frac{-1}{2(\cos z - 1)} \{1 + i\}.$$

$$\therefore f'(z) = \frac{-1}{2(\cos z - 1)}$$

→ finding f(z) by integrating f'(z)

$$\therefore f(z) = -\int \frac{1}{2(\cos z - 1)}\, dz$$
$$= -\frac{1}{2}\int \frac{dz}{\cos z - 1}$$
$$= \frac{1}{2}\int \frac{dz}{1 - \cos z}$$
$$= \frac{1}{2}\int \frac{dz}{2\sin^2(z/2)}$$
$$= \frac{1}{4}\int \operatorname{cosec}^2(z/2)\, dz$$
$$= \frac{1}{\cancel{4}_2}\left(-\cot z/2\right)\cdot\left(\frac{\cancel{2}}{1}\right) + c$$
$$= -\frac{1}{2}\cot(z/2) + c \quad //$$

→ Computing value of constant using $f(\pi/2) = 0$.

$\therefore f(\pi/2) = -\frac{1}{2}\cot\left(\frac{\pi}{4}\right) + c = 0$

$\therefore c = \frac{1}{2}\cot(\pi/4) = \cancel{38.15}\ \frac{1}{2}$ //

$\boxed{\tan(45^\circ) = \frac{1}{\cot(45^\circ)}}$ //

→ Final answer:

$$\boxed{f(z) = -\frac{1}{2}\cot(z/2) + \frac{1}{2}}$$ //

Q5) $u+v = (x-y)(x^2+4xy+y^2)$

Let $(1+i)f(z) = (u-v) + i(u+v) = U + iV$

Imaginary part $V = (x-y)(x^2+4xy+y^2)$

$\therefore V_x = \frac{\partial}{\partial x}\left((x-y)(x^2+4xy+y^2)\right)$

$= (x-y)(2x+4y) + (x^2+4xy+y^2)$

$\therefore V_y = \frac{\partial}{\partial y}\left((x-y)(x^2+4xy+y^2)\right)$

$= (x-y)(4x+2y) - (x^2+4xy+y^2)$

→ $v_x(z,0) = (x-y)(2x+4y) + (x^2+4xy+y^2)$
$= (z)(2z) + (z^2)$
$= 2z^2 + z^2 = 3z^2$ //

→ $v_y(z,0) = (x-y)(4x+2y) - (x^2+4xy+y^2)$
$= (z)(4z) - (z^2)$
$= 4z^2 - z^2 = 3z^2$ //

→ We have,
$\therefore \quad (1+i)f'(z) = v_y + i v_x$
$\therefore \quad (1+i)f'(z) = 3z^2 + i\,3z^2$
$\therefore \quad (1+i)f'(z) = 3z^2(1+i)$
$\therefore \quad f'(z) = 3z^2$

→ Computing $f(z)$ by $\int f'(z)\,dz$

$\therefore f(z) = \int 3z^2\,dz$

$= \frac{3z^3}{3} + C$

$\boxed{\therefore f(z) = z^3 + C}$ //