Black scholes derivation -

→ We have assumed that the Japanese Government rate follows a Geometric Brownian motion.

- risk free rate is known
- volatility is known & constant
- European option.

⇒ Since it follows a GBM

$$R_t = R_0 e^{(r-\sigma^2/2)(t) + \sigma dW(t)}$$

$$\& \quad R_T = R_0 e^{(r-\sigma^2/2)T + \sigma dW(T)}$$

$$\therefore \frac{R_T}{R_t} = e^{(r-\sigma^2/2)(T-t) + \sigma(dW(T)-dW(t))}$$

$$R_T = R_t e^{(r-\sigma^2/2)(T-t) + \sigma(dW(T)-dW(t))}$$

assuming $\dfrac{-(dW(T)-dW(t))}{\sqrt{T-t}} = y$.

$$R_T = R_t e^{(r-\sigma^2/2)(T-t) - \sigma\sqrt{T-t}\, y} \quad \text{— } ①$$

We have the option

$$V(t) = E[e^{-r(T-t)} \underbrace{X(R_T \geq R_0 + 0.25)'}_{\text{Payoff}}]$$

$$V(t) = E[e^{-r(T-t)}(R_t e^{(r-\sigma^2/2)(T-t) - \sigma\sqrt{T-t}\, y} \geq R_0 + 0.25)']$$

using density function $f(y) = \dfrac{1}{\sqrt{2\pi}} e^{-y^2/2}$

$$V(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-r(T-t)}(R_t e^{(r-\sigma^2/2)(T-t) - \sigma\sqrt{T-t}\, y} \geq \underbrace{R_0 + 0.25}_{k \to \text{strike}})' e^{-y^2/2} dy$$

iff $R_t e^{(r-\sigma^2/2)(T-t) - \sigma\sqrt{T-t}\, y} \geq k$

taking ln (logarithm) on both sides.

$$\ln R_t + \left(r - \frac{\sigma^2}{2}\right)(T-t) - \sigma y \sqrt{T-t} \geq \ln k$$

$$\left(r - \frac{\sigma^2}{2}\right)(T-t) - \sigma y \sqrt{T-t} \geq \ln \frac{k}{R_t}.$$

$$\left(r - \frac{\sigma^2}{2}\right)(T-t) + \ln \frac{R_t}{k} \geq \sigma y \sqrt{T-t}$$

$$y \leq \boxed{\frac{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \ln\left(\frac{R_t}{k}\right)}{\sigma\sqrt{T-t}}} \quad d_-$$

This is the condition for which the option would give the payoff i.e. \$1.

$$V(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-} e^{-r(T-t)} \, 1 \cdot e^{-y^2/2} \, dy.$$

limits adjusted

$$= e^{-r(T-t)} \int_{-\infty}^{d_-} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy.$$

$$V(t) = e^{-r(T-t)} \cdot N(d_-)$$

Therefore, price of an option with \$1 payoff, in case the underlying passes the strike price is.

$$\boxed{V(t) = e^{-r(T-t)} \cdot N(d_-)}$$