Optimization Application: Sprinkling Problem

We have a set of sprinklers, fixed on a plane field, with their location and spatial function for water distribution known. The goal is to achieve optimum balance between the water reaching the given region G and the energy consumed, by varying the time for which each sprinkler is on.

1. FIXED PARAMETERS
   - The amount of water given by i^{th} sprinkler, per unit area in unit time at coordinates (x,y). Given by $\mu_i(x, y)$.
   - Power P_i supplied to i^{th} sprinkler for keeping it on.
2. INPUT PARAMETERS
   - The region G over which sprinkling is required.
   - The optimum amount of water required per unit area (called O) at each point (x,y) $\in$ G.
   - The cost function of deviation from the optimum amount of water to be supplied to G (numerical model of which to be suggested later).
3. OUTPUT PARAMETERS
   - The period t_i(it's vector represented by $\mathbf{T}$) for which the i^{th} sprinkler is to be switched on so that we achieve the desired optimum.

MODELING OF THE PROBLEM
The deviation from the optimum water level O (given by the user) is modeled by the following equation.

$$\eta = \int_G (\mu(x, y) - O)^2 dxdy ...(i)$$

Where $\mu(x, y)$ is the volume of water per unit area finally received by the field at (x,y) due to the sprinklers, which in our case

$$\mu(x, y) = \sum_{i=0}^{n} (t_i.\mu_i(x, y)) ...(ii)$$

In addition to this we spend some energy E, given by

$$E = \sum_{i=0}^{n} (t_i.P_i(x,y))$$

Minimizing both η and E simultaneously may not be possible hence we define a cost function for deviation η as, $g(\eta)$, which depicts the cost incurred due to deviating from optimum by an amount η. Thus, our problem reduces to minimizing the expression

$$g(\eta) + \sum_{i=0}^{n} (t_i.P_i(x,y))$$

APPROACH
On the expression given above we add an additional simplifying assumption that the function $g(/eta)$ is linear and is given by;

$$g(\eta) = \eta/\lambda$$

Where k is positive and hence the expression to be minimized becomes

$$\eta + /lamda(\sum_{i=0}^{n} (t_i.P_i(x,y)))$$

Now to simplify η, expand equation (i) using (ii)

$$\eta = \int_G ((\sum_{i=0}^{n} (t_i.\mu_i(x,y))) - O)^2 dxdy$$

Expanding which we get;

$$\eta = \int_G ((\sum_{i,j} (t_i.t_j.\mu_i(x,y).\mu_j(x,y))) - 2O(\sum_i (t_i.\mu_i(x,y))) + O^2) dxdy$$

After interchanging summation and integration we rewrite the above expression as;

$$\eta = t^T Ht - 2v^T t + O^2 ...(iii)$$

Where,

$$[H]_{ij} = \int_G (\mu_i(x,y).\mu_j(x,y)) dxdy ...(iv)$$

and

$$[v]_i = \int_G (\mu_i(x,y))dxdy...(v)$$

Ignoring the constant O^2 term and dividing by 2, the expression to minimize becomes

$$0.5t^T Ht + f^T t$$

where,$[f]_i = \lambda P_i/2 - [v]_i...(vi)$ Which can be minimized using quadratic programming algorithms. Note that all the components of the vector t are to be positive which gives us a set of linear constraints given by

$$t_i >= 0$$

MATLAB GUI
A GUI in matlab has been developed to implement these steps. In the program implementation, The sprinkler's distribution function $\mu_i(x,y)$ has been simplified as, $\mu_i(x,y) = A$ if $(x-x_c)^2 + (y-y_c)^2 < r^2$ and 0 otherwise with the parameters A and r of each sprinkler is set in the GUI
The region G is simplified as a circle whose radius is set in the GUI.
The GUI also allows for marking the various sprinklers.