Lab - 3

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CS302, Modeling and Simulation

In this lab we numerically and analytically analyze the different models of population growth. We consider different cases, with harvesting being constant and harvesting dependent on the present population. We also attempt to create a minimalist model, which simulates a more realistic scenario.

I. INTRODUCTION

Population growth (decay) can be modeled using differential equations. Theoretically, births are more than deaths and so the population undergoes exponential growth. Populations increase at first but soon, due to environmental constraints, they saturate at the carrying capacity for the population in a given environment.

II. MODELS AND RESULTS

A. Logistic Equation

The population growth can be modeled with the help of the logistic equation as given in Eq (1) which depicts the change in population with respect to time.

$$\frac{dP}{dt} = rP(1 - \frac{P}{k}) \tag{1}$$

Here,
r = growth rate
k = carrying capacity (maximum capacity that the total population can reach)
Here r and k are the free variables. Eliminating the free variables, we get Eq (2).

$$\frac{dn}{d\tau} = n(1 - n) \tag{2}$$

In Eq (2), $n = \frac{P}{k}$ and $\tau = rt$. We will use the logistic equation as a base to build on further models.

B. Population growth with constant harvesting

We consider constant harvesting at a rate, h. This can be modeled using Eq (3):

$$\frac{dP}{dt} = rP(1 - \frac{P}{k}) - h \tag{3}$$

Removing the free variables as in Eq (2) we get Eq (4):

$$\frac{dn}{d\tau} = n(1 - n) - \tilde{h} \tag{4}$$

Here $\tilde{h}$ is a constant which represents the ratio of instantaneous population decay to growth. Essentially, the higher it is, the slower is the growth, and it is equal to $\frac{h}{rk}$. As seen the value and nature of fixed point for Eq (4) depends on the value of $\tilde{h}$. We analyze three cases which arise due to the different values of $\tilde{h}$. Here, D is the discriminant of Eq (4).

CASE 1: When $D < 0$, $\tilde{h} < \frac{1}{4}$ hence $h > \frac{rk}{4}$
In this case, Eq. (4) has no real roots. Hence we do not have any fixed points. For any value of P_0, the population will go on decreasing as the downward facing parabolic curve of Eq. (4) is below the x axis. The population cannot exist in such a scenario, since the rate of harvesting is more than the rate at which the population grows, so the population perishes.Hence this value of $\tilde{h}$ is practically not possible.

CASE 2: When $D = 0$, $\tilde{h} = \frac{1}{4}$ hence $h = \frac{rk}{4}$
In this case, Eq. (4) has a double root at

$$n_1 = \frac{1}{2}$$

Hence we have one fixed point. The parabolic curve now touches the x-axis at n_1. As the parabola is downward facing, it has negative values for any value of n in Eq. (4). n_1 will be stable for any value of P_0 greater than n1 and unstable otherwise. Fig. 1 shows the plot for population saturating at half the carrying capacity. If we take a value for the initial population such that it is less than half the carrying capacity, the population perishes.

CASE 3: When $D > 0$, $\tilde{h} > \frac{1}{4}$ hence $h < \frac{rk}{4}$
In this case, Eq. (4) has two real roots:

$$n_1 = \frac{-1 + \sqrt{1 - 4\tilde{h}}}{-2}$$

$$n_2 = \frac{-1 - \sqrt{1 - 4\tilde{h}}}{-2}$$

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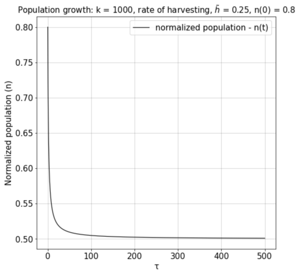

FIG. 1. k = 1000, $\tilde{h} = 0.25$ and initial population = 800. We see saturation at 0.5 (pop = 500, half the carrying capacity)

Hence we have two fixed points and based on the value of $\tilde{h}$, their position and nature of stability changes.
Here, we take $\tilde{h}$ as $\frac{0.51}{4}$. The parabolic curve now touches the x-axis at $n_1 = 0.15$ and $n_2 = 0.85$. As the parabola is downward facing, it has negative values for any value of n less than 0.15 and greater than 0.85. Hence here, n_1 is unstable and n_2 is a stable fixed point. Per Eq. (4), n_1 will be stable for any value of P_0 greater than n_1 and unstable otherwise. Fig. 2 and Fig. 3 are graphs we get for different values of P_0.

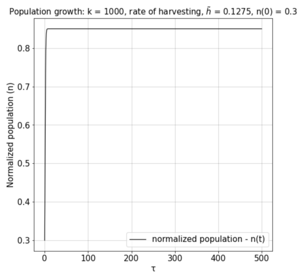

FIG. 2. k = 1000, $\tilde{h} = 0.1275$ (¡0.25) and initial population = 300. We see saturation at 0.85 (pop = 850).

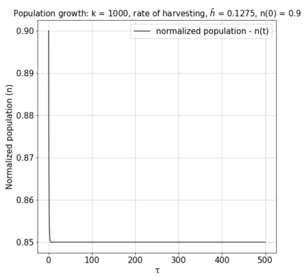

FIG. 3. k = 1000, $\tilde{h} = 0.1275$ (¡0.25) and initial population = 900. We see saturation at 0.85 (pop = 850).

The population growth will saturate at the fixed point n_2 if we consider any value of $P_0 > n_1$, if $\tilde{h} < \frac{1}{4}$.

C. Population growth with harvesting that depends on instantaneous population

Let ϵ_0 be the rate at which we harvest. Then the rate of change of population can be given by Eq. (5)

$$\frac{dP}{dt} = rP(1 - \frac{P}{k}) - \epsilon_0 P \tag{5}$$

Removing the free variables as in Eq. (2), we get Eq. (6)

$$\frac{dn}{d\tau} = n(1-n) - \tilde{\epsilon_0} n \tag{6}$$

$\tilde{\epsilon_0}$ in Eq. (6) is equal to $\frac{\epsilon_0}{r}$, which represents the ratio of instantaneous population decay to growth. Essentially, the higher it is, the slower is the growth. This is a quadratic equation and we have two fixed points, one at 0 and the other at $1 - \tilde{\epsilon_0}$. Based on the value of $\tilde{\epsilon_0}$, the nature of the fixed point changes. We see the two cases that arise as given below:

CASE 1: When $\tilde{\epsilon_0} < 1$ and hence, $\epsilon_0 < r$.

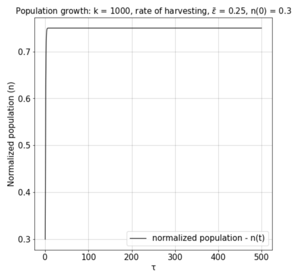

FIG. 4. k = 1000, $\tilde{\epsilon_0} < 1$ and = 0.25 and initial population = 300

In this case, since the rate of harvesting is less, we see that the population saturates to 1 - $\tilde{\epsilon_0}$, which is 0.75. This is shown in Fig. 4.

CASE 2: When $\tilde{\epsilon_0} > 1$ and hence, $\epsilon_0 > r$.

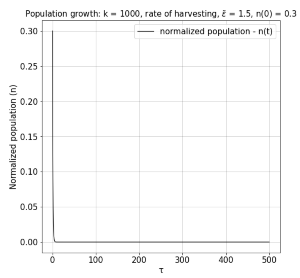

FIG. 5. k = 1000, $\tilde{\epsilon_0} > 1$ and = 1.5 and initial population = 300

In this case, since the rate of harvesting is more, the population perishes over time. This is shown in Fig. 5.

D. Realistic population growth model

If we follow the logistic equation, we see that for any value of population above zero, we get a non-zero population after a period of time, if there is no harvesting. We try to model a scenario where a non-zero population occurs only when the initial population is above some threshold value.

Inspecting the conditions, we wish to have:

- A fixed point at A, which is our threshold value and $0 < A < k$.
- The fixed point at A needs to be unstable as any initial population value(P_0) value above A saturates to k and for $P_0 < A$, the population decays to 0 with time.
- Based on the above two points, we need to have a stable fixed point at 0 and k.

Hence we get the Eq. (7) which represents the realistic population growth model.

$$\frac{dP}{dt} = rP(1 - \frac{P}{k})(\frac{P}{A} - 1) \tag{7}$$

Making the equation dimensionless, we get:

$$\frac{dn}{d\tau} = n(1-n)(\frac{n}{\tilde{A}} - 1) \tag{8}$$

where $n = \frac{P}{k}$, $\tilde{A} = \frac{A}{k}$ and $\tau = rt$.

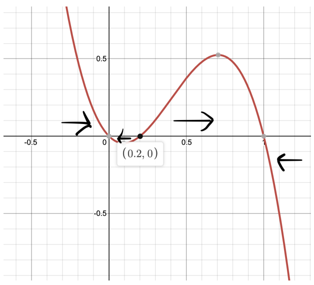

FIG. 6. Plot of Eq. (8) when $\tilde{A} = 0.2$ - stable and unstable fixed points

As seen in Fig. 6 we have fixed points at 0 and $\tilde{k}$ (normalised k), where $\tilde{k} = 1 = \frac{k}{k}$. The graphs for Eq. (8) with $\tilde{A} = 0.2$ and $\frac{P_0}{k} = 0.3$ and $\frac{P_0}{k} = 0.1$ are as given in Fig. (7) and Fig. (8) respectively. The results fit well with our conditions.

Now, let us try to examine this realistic model by incorporating harvesting.

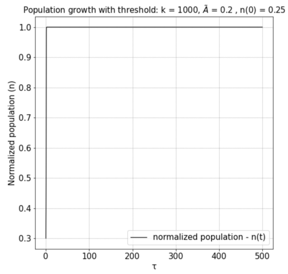

FIG. 7. Population growth when $\tilde{A} = 0.2$, threshold A= 200, initial population = 300, r=0.2 and $k = 1000$

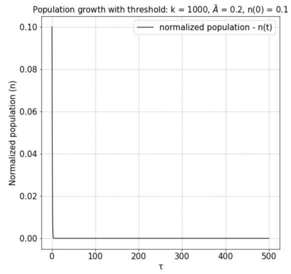

FIG. 8. Decaying Population growth when $\tilde{A} = 0.2$, threshold A=200, initial population = 100, r=0.2 and $k = 1000$

E. Realistic population growth with constant harvesting

Equation of the model with no free variables is as given in Eq. (9)

$$\frac{dn}{d\tau} = n(1-n)(\frac{n}{\tilde{A}} - 1) - \tilde{h} \tag{9}$$

We have a cubic equation for population growth. It will have either 1 fixed point or 3 fixed points. The cubic equation has a $-n^3$ term, so at $n = -\infty$, the value of function will go to ∞ and vice versa. We will get one stable fixed point for sure. The value and nature of other fixed points depend on the value of $\tilde{h}$.

For a value of $\tilde{h}$ greater than a particular value (around 0.526) we will stop getting three fixed points. That rate of harvesting will be too large and the population will perish for any value of P_0.

For a value of $\tilde{h}$ less than 0.526, we will get three fixed points, two of which will be stable.

The results are shown in Fig. (9). For $\frac{P_0}{k}$ greater than the unstable fixed point we saturate at the third stable fixed point (around 0.77).

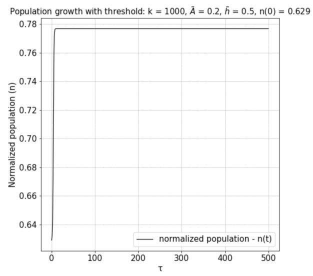

FIG. 9. Population growth when $\tilde{A} = 0.2$, threshold A=200, $\tilde{h} = 0.5$, initial population = 600, r = 0.2 and $k = 1000$

F. Realistic population growth with harvesting that depends on instantaneous population

Equation of the model with no free variables is as given in Eq. (10)

$$\frac{dn}{d\tau} = n(1-n)(\frac{n}{\tilde{A}} - 1) - \tilde{\epsilon_0} n \tag{10}$$

In this cubic, we get a fixed point at zero. And depending on value of $\tilde{\epsilon_0}$ we get zero or two other fixed points. As seen in Fig. (10) for $\tilde{\epsilon_0} = 0.5$ we get three fixed points.

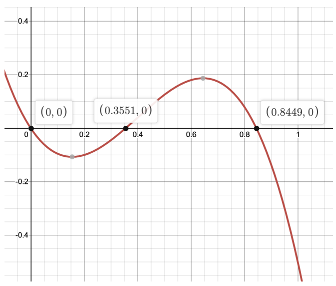

FIG. 10. Plot of Eq. (10) for $\tilde{A} = 0.2$ and $\tilde{\epsilon_0} = 0.5$ - stable and unstable fixed points.

We get Fig. (11) and Fig. (12) for different values of $\frac{P_0}{k}$. We observe that when we have harvesting, the unstable fixed point acts as the new increased threshold and any value of P_0 greater than the threshold, the population saturates at the next stable fixed point (in this case, 0.84 which behaves like a new carrying capacity). And any value of P_0 less than threshold, the population perishes.

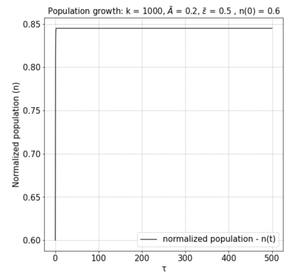

FIG. 11. k = 1000, $\tilde{A} = 0.2$, $\tilde{\epsilon} = 0.5$ and initial population = 600. We can see that the population saturates at n = 0.8449, which is $P = 845$

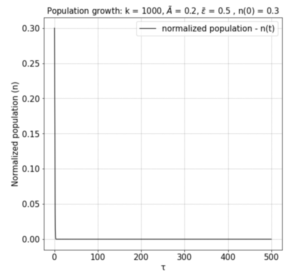

FIG. 12. k = 1000, $\tilde{A} = 0.2$, $\tilde{\epsilon} = 0.5$ and initial population = 300. We can see that the population perishes.

III. CONCLUSIONS

We draw the following conclusions:

- The logistic equation acts as a base model for every other population model that we develop.
- We go on to include harvesting in that equation which serves as a better model for the population. We also consider a case where the rate at which harvesting (essentially, deaths) occurs is proportional to the instantaneous population.
- For more realistic modeling, we consider a scenario where the population perishes if the initial population is less than a particular threshold.
- We consider cases of harvesting - both constant and instantaneous, with this as the base model as well. We see that it models populations better because it considers environmental factors (through the threshold), that is, cases where the environment cannot support a population unless it has a minimum initial organisms.
- In all of the above cases, we see that as population starts moving closer to the carrying capacity, the rate of population change decreases until it reaches saturation (in cases where populations do not perish).