Probability and Statistics

Quiz 1

1 Question 1 Solution

Question: Let X be a Uniform $U[a, b]$ random variable. Let $Y = e^{2X}$. Find the pdf and cdf of Y.

1.1 Method 1: CDF to PDF

1.1.1 Step 1: PDF and CDF of X

Since $X \sim U[a, b]$:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

The CDF of X is:

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases}$$

1.1.2 Step 2: Range of Y

Since $Y = e^{2X}$ and $X \in [a, b]$:

- When $X = a$: $Y = e^{2a}$
- When $X = b$: $Y = e^{2b}$

Since e^{2x} is strictly increasing, $Y \in [e^{2a}, e^{2b}]$.

1.1.3 Step 3: CDF of Y

For $y \in [e^{2a}, e^{2b}]$:

$$\begin{align}
F_Y(y) &= \mathbb{P}(Y \leq y) \tag{1}\\
&= \mathbb{P}(e^{2X} \leq y) \tag{2}\\
&= \mathbb{P}(2X \leq \ln y) \tag{3}\\
&= \mathbb{P}\left(X \leq \frac{\ln y}{2}\right) \tag{4}\\
&= F_X\left(\frac{\ln y}{2}\right) \tag{5}
\end{align}$$

Since $\frac{\ln y}{2} \in [a, b]$ when $y \in [e^{2a}, e^{2b}]$:

$$F_Y(y) = \frac{\frac{\ln y}{2} - a}{b - a} = \frac{\ln y - 2a}{2(b - a)}$$

Therefore:

$$\boxed{F_Y(y) = \begin{cases} 0 & \text{if } y < e^{2a} \\ \frac{\ln y - 2a}{2(b-a)} & \text{if } e^{2a} \leq y \leq e^{2b} \\ 1 & \text{if } y > e^{2b} \end{cases}}$$

1.1.4 Step 4: PDF of Y

Taking the derivative of $F_Y(y)$ with respect to y:

For $y \in [e^{2a}, e^{2b}]$:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left(\frac{\ln y - 2a}{2(b - a)}\right) = \frac{1}{2(b - a)} \cdot \frac{1}{y}$$

Therefore:

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{2y(b-a)} & \text{if } e^{2a} \leq y \leq e^{2b} \\ 0 & \text{otherwise} \end{cases}}$$

1.2 Method 2: PDF to CDF

Since $Y = g(X) = e^{2X}$ where g is strictly increasing, we can use:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy}(g^{-1}(y))\right|$$

1.2.1 Step 1: Inverse function

From $Y = e^{2X}$, we get $X = \frac{\ln Y}{2}$, so $g^{-1}(y) = \frac{\ln y}{2}$.

1.2.2 Step 2: Derivative of the inverse

$$\frac{d}{dy}(g^{-1}(y)) = \frac{d}{dy}\left(\frac{\ln y}{2}\right) = \frac{1}{2y}$$

1.2.3 Step 3: Application of formula

For $y \in [e^{2a}, e^{2b}]$:

$$f_Y(y) = f_X\left(\frac{\ln y}{2}\right) \cdot \left|\frac{1}{2y}\right| = \frac{1}{b-a} \cdot \frac{1}{2y} = \frac{1}{2y(b-a)}$$

Therefore:

$$f_Y(y) = \begin{cases} \frac{1}{2y(b-a)} & \text{if } e^{2a} \le y \le e^{2b} \\ 0 & \text{otherwise} \end{cases}$$

1.2.4 Step 4: CDF by Integration

For $y \in [e^{2a}, e^{2b}]$:

$$\begin{aligned} F_Y(y) &= \int_{e^{2a}}^{y} f_Y(t)\,dt && (6)\\ &= \int_{e^{2a}}^{y} \frac{1}{2t(b-a)}\,dt && (7)\\ &= \frac{1}{2(b-a)} \int_{e^{2a}}^{y} \frac{1}{t}\,dt && (8)\\ &= \frac{1}{2(b-a)} [\ln t]_{e^{2a}}^{y} && (9)\\ &= \frac{1}{2(b-a)} (\ln y - \ln e^{2a}) && (10)\\ &= \frac{1}{2(b-a)} (\ln y - 2a) && (11)\\ &= \frac{\ln y - 2a}{2(b-a)} && (12) \end{aligned}$$

$$F_Y(y) = \begin{cases} 0 & \text{if } y < e^{2a} \\ \frac{\ln y - 2a}{2(b-a)} & \text{if } e^{2a} \le y \le e^{2b} \\ 1 & \text{if } y > e^{2b} \end{cases}$$

2 Question 2 Solution

Question: Let X be a Binomial random variable with parameters n and p. Derive expression for the first two moments of X.

2.1 Method 1: Algebraic derivation

2.1.1 Step 1: Recall definition

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

2.1.2 Step 2: First moment $\mathbb{E}[X]$

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

For $k = 0$, the term vanishes, so:

$$\mathbb{E}[X] = \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

Using the identity $k\binom{n}{k} = n\binom{n-1}{k-1}$:

$$\mathbb{E}[X] = \sum_{k=1}^{n} n \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

Factor out np:

$$\mathbb{E}[X] = np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)}$$

Re-index with $j = k - 1$:

$$\mathbb{E}[X] = np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{(n-1)-j}$$

The summation is the binomial expansion of $(p + (1-p))^{n-1} = 1$. Hence:

$$\boxed{\mathbb{E}[X] = np}$$

2.1.3 Step 3: Second moment $\mathbb{E}[X^2]$

$$\mathbb{E}[X^2] = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}$$

Split $k^2 = k(k-1) + k$:

$$\mathbb{E}[X^2] = \sum_{k=0}^{n} k(k-1) \binom{n}{k} p^k (1-p)^{n-k} + \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

The second sum is just $\mathbb{E}[X] = np$.

Simplify the first sum: Using $k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}$:

$$\sum_{k=0}^{n} k(k-1)\binom{n}{k} p^k (1-p)^{n-k} = n(n-1)\sum_{k=2}^{n}\binom{n-2}{k-2} p^k (1-p)^{n-k}$$

Factor p^2:

$$= n(n-1)p^2 \sum_{k=2}^{n}\binom{n-2}{k-2} p^{k-2}(1-p)^{(n-2)-(k-2)}$$

Re-index with $j = k-2$:

$$= n(n-1)p^2 \sum_{j=0}^{n-2}\binom{n-2}{j} p^{j}(1-p)^{(n-2)-j}$$

This sum is $(p + (1-p))^{n-2} = 1$. Hence:

$$\sum_{k=0}^{n} k(k-1)\binom{n}{k} p^k (1-p)^{n-k} = n(n-1)p^2$$

Combine terms:

$$\boxed{\mathbb{E}[X^2] = n(n-1)p^2 + np}$$

2.1.4 Final Results

$$\boxed{\mathbb{E}[X] = np, \quad \mathbb{E}[X^2] = n(n-1)p^2 + np}$$

2.2 Method 2: Using sum of Bernoulli random variables

Note: This method relies on concepts not yet covered in the course and is provided for completeness.

Let $X \sim \text{Binomial}(n, p)$. Then X can be expressed as the sum of n independent Bernoulli random variables:

$$X = Y_1 + Y_2 + \cdots + Y_n$$

where $Y_1, \ldots, Y_n$ are independent and identically distributed Bernoulli random variables with:

$$p_{Y_i}(1) = p, \quad p_{Y_i}(0) = 1 - p$$

Explanation: Each Y_i represents a single trial in the binomial experiment, where the outcome is either success (1) with probability p or failure (0) with probability $1-p$. Having k out of n successes corresponds to the event $\{X = k\}$.

2.2.1 First moment

By linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = \sum_{i=1}^{n} p = np$$

2.2.2 Variance

Using the fact that the Y_i are independent, variance is additive:

$$\mathrm{Var}(X) = \mathrm{Var}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \mathrm{Var}(Y_i)$$

For a Bernoulli(p) random variable Y_i:

$$\mathrm{Var}(Y_i) = p(1-p)$$

Hence:

$$\mathrm{Var}(X) = np(1-p)$$

2.2.3 Second moment

Use the identity $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$. Rearranging:

$$\mathbb{E}[X^2] = \mathrm{Var}(X) + (\mathbb{E}[X])^2 = np(1-p) + (np)^2$$

Simplifying gives:

$$\mathbb{E}[X^2] = n(n-1)p^2 + np$$

3 Question 3 Solution

Question: Define the following: 1) A random variable, 2) Borel Sigma Algebra. For a random variable X, describe all possible relationships between $\mathbb{P}$, P_X and $F_X(\cdot)$.

3.1 Definitions

3.1.1 (1) Random variable

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable is a function $X : \Omega \to \Omega'$ that transforms the probability space to $(\Omega', \mathcal{F}', P_X)$ and is $(\mathcal{F}, \mathcal{F}')$-measurable, i.e., for every $B \in \mathcal{F}'$, the inverse image $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$ lies in $\mathcal{F}$.

This measurability is exactly what lets us push the original probability measure $\mathbb{P}$ forward to events about X, and P_X is called the induced probability measure.

The slides emphasize that working directly with $\omega \in \Omega$ can be hard, and X is the device that transports $(\Omega, \mathcal{F}, \mathbb{P})$ to a simpler space $(\Omega', \mathcal{F}', P_X)$; in practice $\Omega' = \mathbb{R}$ and $\mathcal{F}' = \mathcal{B}(\mathbb{R})$.

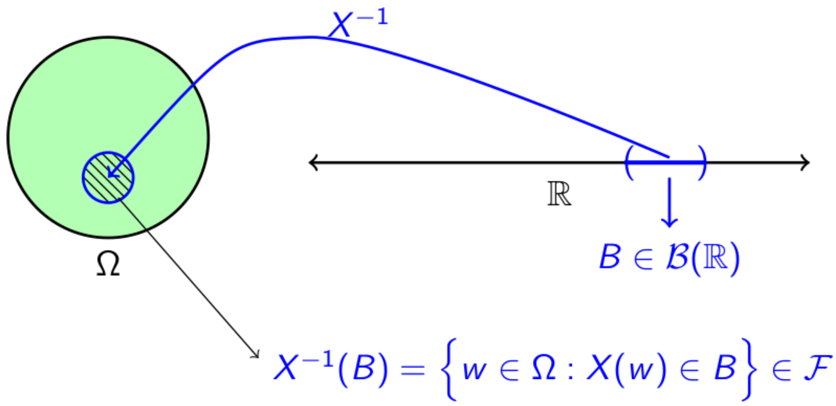

Figure 1: Random Variable

3.1.2 (2) Borel σ-algebra

A σ-algebra is a domain $\mathcal{F}$ such that if A_1 and A_2 belong to $\mathcal{F}$, so do $A_1 \cup A_2$ and $A_1 \cap A_2$.

The Borel σ-algebra on $\mathbb{R}$, denoted $\mathcal{B}(\mathbb{R})$, is the smallest σ-algebra containing all open intervals (a, b). Equivalently, it is the σ-algebra generated by open, closed, or half-open intervals. It contains sets such as (a, b), $[a, b]$, $(a, b]$, $[a, b)$, where $a \leq b$ and possibly $a = -\infty$ and/or $b = \infty$.

On $[0, 1]$ the Borel σ-algebra $\mathcal{B}([0, 1])$ is generated by intervals of the form $[a, b]$ (and hence also contains (a, b), $(a, b]$, $[a, b)$, etc.), where $a \leq b$ and $a, b \in [0, 1]$.

3.2 Relationships among $\mathbb{P}$, P_X, and $F_X(\cdot)$

- **Original measure $\mathbb{P}$:** The probability measure on $(\Omega, \mathcal{F})$ that assigns probabilities to events about sample points $\omega \in \Omega$. It forms the foundation of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- **Induced measure P_X:** Given a random variable $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, the *induced* measure P_X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is defined by:
$$P_X(B) := \mathbb{P}\big(X^{-1}(B)\big), \quad B \in \mathcal{B}(\mathbb{R})$$
- **Cumulative distribution function F_X:** The CDF of X is:
$$F_X(x) := \mathbb{P}(\omega \in \Omega : X(\omega) \leq x) = P_X\big((-\infty, x]\big), \quad x \in \mathbb{R}$$

3.3 Discrete/continuous specializations

- **Discrete X:** If X takes countably many values, P_X is determined by the PMF $p_X(x) = P_X(\{x\}) = \mathbb{P}(X = x)$; the CDF is $F_X(x) = \sum_{u \leq x} p_X(u)$.

- **Continuous X:** If X has a density f_X, then:

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du, \quad P_X(B) = \int_{u\in B} f_X(u)\,du$$

and $P_X(\{a\}) = 0$ for any point a.

4 Question 4 Solution

Question: Derive the expression for the mean of an exponential random variable with parameter λ. Also prove the memory-less property for the exponential random variable.

4.1 Setup

Let X be a random variable with density:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0, \quad \lambda > 0$$

and $f_X(x) = 0$ for $x < 0$.

4.2 Mean

$$\mathbb{E}[X] = \int_0^{\infty} x\,\lambda e^{-\lambda x}\,dx$$

Integrate by parts with $u = x$ and $dv = \lambda e^{-\lambda x}\,dx$ (so $du = dx$ and $v = -e^{-\lambda x}$):

$$\mathbb{E}[X] = \Big[-xe^{-\lambda x}\Big]_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx = 0 + \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^{\infty} = \frac{1}{\lambda}$$

Hence:

$$\boxed{\mathbb{E}[X] = \frac{1}{\lambda}}$$

4.3 Derivation of the CDF

For $x \geq 0$ the cumulative distribution function is obtained by integrating the pdf:

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_0^{x} \lambda e^{-\lambda t}\,dt$$

Evaluate the integral (anti-derivative of $\lambda e^{-\lambda t}$ is $-e^{-\lambda t}$):

$$F_X(x) = \Big[-e^{-\lambda t}\Big]_{t=0}^{t=x} = -e^{-\lambda x} - \big(-e^{0}\big) = 1 - e^{-\lambda x}$$

For $x < 0$ clearly $F_X(x) = 0$. Thus the CDF is:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

The tail CDF is therefore:

$$\mathbb{P}(X > x) = 1 - F_X(x) = e^{-\lambda x}, \quad x \geq 0$$

4.4 Memoryless property

Let X be a nonnegative random variable and fix $a, h \geq 0$. Define the events

$$A = \{X > a + h\}, \qquad B = \{X > a\}, \qquad C = \{X > h\}.$$

The memoryless property can be stated in these events as

$$\mathbb{P}(A \mid B) = \mathbb{P}(C).$$

Observe the set inclusion: if $X > a + h$ then certainly $X > a$, hence

$$A \subseteq B.$$

Therefore $A \cap B = A$, and by the definition of conditional probability

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)}.$$

Now assume $X \sim \mathrm{Exp}(\lambda)$ so that the tail CDF is $\mathbb{P}(X > x) = e^{-\lambda x}$ for $x \geq 0$. Using this tail CDF,

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(X > a + h)}{\mathbb{P}(X > a)} = \frac{e^{-\lambda(a+h)}}{e^{-\lambda a}} = e^{-\lambda h} = \mathbb{P}(X > h) = \mathbb{P}(C).$$

Thus,

$$\boxed{\mathbb{P}(A \mid B) = \mathbb{P}(C)},$$

so the exponential distribution satisfies the memoryless property.