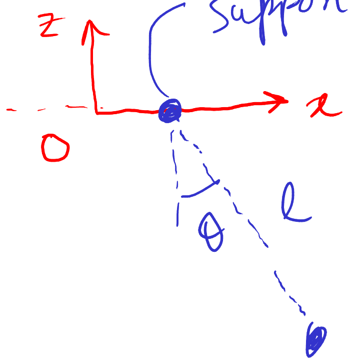
1. **(5 points) A simple pendulum of mass m and length l has its point of support moving horizontally given by $x(t) = a\sin(\omega t)$ (assume that string remains "straight" at all times). Find the equation of motion of the bob, and then for small oscillations, comment on the expected solution.**

support $(x, z) = (a\sin(\omega t), 0)$



$(x, z) = ((a\sin(\omega t) + l\sin\theta), -l\cos\theta)$

- Generalised Coordinates (l, θ)
- KE of bob $= \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) =$

$$= \frac{1}{2}m\left[(-a\omega\cos\omega t + l\cos\theta\,\dot\theta) + (l\sin\theta\,\dot\theta)^2\right]$$

$$= \frac{1}{2}m\left(a^2\omega^2\cos^2\omega t - 2a\omega l\cos\omega t\cdot\cos\theta\,\dot\theta + l^2\dot\theta^2\right)$$

- PE $= mgz = -mgl\cos\theta$
- $L = KE - PE = \frac{1}{2}m\left[a^2\omega^2\cos^2\omega t - 2a\omega l\cos\omega t\cos\theta\,\dot\theta + l^2\dot\theta^2\right) + mgl\cos\theta$

$\boxed{L(\theta, \dot\theta, t)}$

$$\frac{\partial L}{\partial\theta} = \frac{1}{2}m\left[+2a\omega l\cos\omega t\sin\theta\,\dot\theta\right] - mgl\sin\theta$$

$$\frac{\partial L}{\partial\dot\theta} = \frac{1}{2}m\left[-2a\omega l\cos\omega t\cos\theta + l^2 2\dot\theta\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = \frac{1}{2}m\left[2a\omega l\left(\cancel{+\cos\omega t\sin\theta\,\dot\theta} + \omega\sin\omega t\cos\theta\right) + l^2 2\ddot\theta\right]$$

$$= \frac{\partial L}{\partial\theta} = \frac{1}{2}m\left(2a\omega \cancel{l\cos\omega t\sin\theta\,\dot\theta}\right] - mgl\sin\theta$$

$$\Rightarrow \boxed{l^2\ddot\theta = a\omega^2 l\sin\omega t\cos\theta - gl\sin\theta} \Rightarrow \boxed{\ddot\theta = -\frac{g}{l}\theta + \frac{a\omega^2}{l}\sin\omega t}$$

2. **(5 points) On a particle the force is the following form $\vec{F}(\vec{r}) = (-ze^{-x}, \ln z, e^{-x} + y/z)$, where position vector $\vec{r} = (x, y, z)$. Does the work done when particle is moved from $(1, 1, 1)$ to $(2, 2, 2)$ depend on the path taken? Why or why not? Find the work for at least one such path by doing the line integral for the work done.**

$\frac{\partial}{\partial y} F_x = 0$ | $\frac{\partial}{\partial z} F_x = -e^{-x}$

$\frac{\partial}{\partial x} F_y = 0$ | $\frac{\partial}{\partial z} F_y = \frac{1}{z}$

$\frac{\partial}{\partial x} F_z = -e^{-x}$ | $\frac{\partial}{\partial y} F_z = \frac{1}{z}$

Clearly $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$, $\frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}$

and similarly for other cross derivatives.

So F is conservative.

$\boxed{\text{PATH INDEPENDENT WORK!}}$

$F(x,y,z) = -\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$

$-\frac{\partial u}{\partial x} = F_x = -ze^{-x} \Rightarrow -u = ze^{-x} + c_1(y,z)$

$F_y = \ln z \Rightarrow -u = y\ln z + c_2(x,z)$

$F_z = \frac{y}{z} \Rightarrow -u = y\ln z + c_3(x,y)$

$-u = ze^{-x} + y\ln z + c$

$\Rightarrow u(x,y,z) = -ze^{-x} - y\ln z + c_2$

$$W = \int_A^B dr\cdot F = \int_A^B dx(-ze^{-x}) + dy(\ln z) + dz(e^{-x} + y/z)$$

$A = (1,1,1) \rightarrow B = (2,2,2)$, $dx = dy = dz$

$dx = dy = 0$, $dz = 1$: $(1,1,2) \longrightarrow (1,2,2)$, $dx = dz = 0$, $dy = 1$

$dx = 1$, $dy = 0$, $dz = 0$

$W = -(u_2 - u_1)$

For conservative

3. **(5 points) A rod of length L_0 in its rest frame is lying at an angle θ with respect to x-axis. What is the length and orientation of this rod as measured by an observer moving along x-axis with speed v?**

Length contraction along x-axis; gives length $\Delta x' = \sqrt{1-\frac{v^2}{c^2}}\,(L_0 \cos\theta)$

$\Delta y' = L_0 \sin\theta$.

so length of rod will be $\sqrt{\Delta x'^2 + \Delta y'^2} = \sqrt{\left(1-\frac{v^2}{c^2}\right) L_0^2 \cos^2\theta + L_0^2 \sin^2\theta}$

$$\boxed{L = L_0 \sqrt{1 - \frac{v^2}{c^2}\cos^2\theta}}$$

Angle will be $\theta' = \tan^{-1}\left(\frac{\Delta y'}{\Delta x'}\right) = \tan^{-1}\left(\frac{L_0 \sin\theta}{L_0 \cos\theta \sqrt{1-\frac{v^2}{c^2}}}\right)$

i.e. $$\boxed{\tan\theta' = \frac{\tan\theta}{\sqrt{1-\frac{v^2}{c^2}}}}$$

4. **(5 points) Rest mass energy of electron is 0.5 MeV, and of proton is 938 MeV. Anti-particles has same mass but opposite charge. In a particular nuclear reaction, electron and its anti-particle positron collide to give proton and its antiparticle anti-proton. What is the minimum kinetic energy of each particle to produce this reaction?**

Reaction is $\underbrace{e^- + e^+}_{\text{Reac}} \longrightarrow \underbrace{p + p^-}_{\text{Prod.}}$

Energy Conservation $E(\text{React}) = E(\text{prod})$

$$\gamma_e m_e c^2 + \gamma_{e+} m_{e+} c^2 = \gamma_p m_p c^2 + \gamma_{p+} m_{p+} c^2$$

$$m_e c^2 [\gamma_e + \gamma_{e+}] = m_p c^2 (\gamma_p + \gamma_{p+})$$

clearly for minimum KE, γ has to be minimum.

"Obviously": $\gamma_p = \gamma_{p+} = 1$ (ie $v_p = v_{p+} = 0$)

This gives $v_e = -v_{e+}$ (by conservation of momentum)

and hence $(0.5\text{ MeV})\, 2\gamma_e = (938\text{ MeV})(1+1)$

$\Rightarrow \gamma_e = 2 \times 938\text{ MeV} = 1876\text{ MeV}$

and hence
$$\left\{ \begin{aligned} KE_e = KE_{e+} &= (\gamma - 1) m_e c^2 \\ &= 1875 \times 0.5\text{ MeV} \\ &= 937.5\text{ MeV} \\ KE_p = KE_{p+} &= 0. \end{aligned} \right.$$

ROUGH WORK