Probability and Statistics: MA6.101
Tutorial 4

Topics Covered: Continuous Random Variable, Functions of Random Variable

Q1: A continuous random variable X is given distribution with parameters $n, \gamma > 0$ with PDF given as:

$$f_X(x) = \begin{cases} cx^{n-1}e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find c.

(b) Find $\mathbb{E}[X]$ and $\mathrm{Var}[X]$.

(c) If I write the values calculated above as a sum of n i.i.d. random variables, then what **could be a potential continuous random variable** that can sum to this random variable?

Hint: Simplify in terms of the gamma function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$$

Q2: Tail CDF Summation for a Discrete-Like Continuous Variable
Let X be a continuous random variable with PDF $f_X(x) = e^{-x}$ for $x > 0$ (Standard Exponential).

Define a new discrete random variable $N = \lceil X \rceil$, which is the smallest integer greater than or equal to X.

(a) Find the probability mass function (PMF) of N, $P(N = n)$ for $n = 1, 2, 3, \ldots$.

(b) Calculate the expected value of N, $E[N]$, using the PMF.

(c) Use the "tail CDF summation" formula $E[N] = \sum_{n=0}^{\infty} P(N > n)$ to calculate $E[N]$.

Q3: Tail-Integral Formula

Let $X \geq 0$ be a continuous random variable with PDF $f_X(x)$ and let $p > 0$ with $\mathbb{E}[X^p] < \infty$. Derive the tail-integral identity:

$$\boxed{\mathbb{E}[X^p] \;=\; \int_0^\infty p\, s^{p-1}\, \mathbb{P}(X \geq s)\, ds}$$

Q4: Gaussian Sensor Noise

A sensor records $X \sim N(\mu, \sigma^2)$. Measurements are accepted if they lie in $[\mu - \sigma, \mu + \sigma]$.

Find the probability that a measurement is accepted. You may use a standard normal table or calculator.

Q5: Let $X_1, X_2, \ldots, X_n$ be independent random variables, where X_j represents the time that a particular student takes to complete an exam. Assume that

$$X_j \sim \text{Exp}(\lambda), \quad j = 1, 2, \ldots, n.$$

(a) Find the distribution of

$$L = \min(X_1, X_2, \ldots, X_n),$$

the time until the *first* student completes the exam.

(b) Find the distribution of

$$T = \max(X_1, X_2, \ldots, X_n),$$

the time until *all* students have completed the exam.

(c) Specialize to the case $n = 3$. What is the expected value of T, i.e., the expected time at which all three students have completed the exam? Then extend the pattern to general n.

Q6: **Generalizing Transformations to Non-monotonic functions**

Let $Y = g(X)$, where g is a real-valued differentiable function and X is a continuous random variable with density $p_X(x)$. Denote the real roots of the equation

$$y = g(x)$$

by $x_1, x_2, \ldots, x_n, \ldots$. Show that the probability density function of Y is given by

$$p_Y(y) = \sum_i \frac{p_X(x_i)}{|g'(x_i)|},$$

where the sum is over all roots x_i for a given y, and $g'(x)$ denotes the derivative of $g(x)$.

Q7: Mixed waiting time: taxi and bus

A taxi stand and a bus stop are at the same location. When Shubham arrives there is a taxi already waiting with probability 1/2 (in which case he boards immediately). If there is no taxi waiting (probability 1/2), then the next taxi arrival time T (measured in minutes from now) is distributed uniformly on $[0, 8]$, while the next bus will arrive exactly in 5 minutes. Shubham takes whichever (taxi or bus) comes first. Let X denote Shubham's waiting time (in minutes).

(a) Find the CDF $F_X(x)$ for all $x \geq 0$, identify any point masses, and write down the PDF / mixed representation of X.

(b) Compute $\mathbb{E}[X]$.

Q8: Start with an initial score of 0. In each turn, you generate a random number uniformly from (0,1) and add it to your score. What is the expected number of turns to reach a score > 2?

Hints:

(a) Try to model this in a recursive form, where you make a recursion based on what score you have, and how many turns do you need after seeing that score.
(b) Split this problem into 2 cases, when your sum is between $[0, 1)$ and when it is between $[1, 2)$.
(c) First solve the problem when you have sum ≥ 1, then use it to solve the case we are looking for.