

CS 302.1 - Automata Theory

Lecture 11

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Quick Recap

The standard TM model is quite robust. It can simulate other seemingly “powerful” variants such as

- Lazy TM
- Multi-tape TM
- Two-way infinite tape TM
- Enumerator
- Non-deterministic TM

The set of problems that are decided by a standard TM = the set of problems decided by any of these variants

Turing Machines: some definitions

Total Turing Machines: A TM M is total if for all input strings $w \in \Sigma^*$, $M(w)$ accepts or rejects but never runs infinitely.

On every input, M halts

An **Algorithm** is nothing but a Total Turing Machine.

Recursive Language/Turing Decidable/Decidable: A language L is called Recursive or Turing decidable or Decidable if there exists a Total Turing Machine M for L , i.e.

$$\left. \begin{array}{l} \forall \omega \in L, M(\omega) \text{ accepts} \\ \forall \omega \notin L, M(\omega) \text{ rejects} \end{array} \right\} \text{Halts on all inputs}$$

Total TM M = On input w ,
If $M(w)$ reaches an accept state, ACCEPT
If $M(w)$ reaches a reject state, REJECT

Recursively Enumerable Language/Turing Recognizable (RE): A language L is called Recursively Enumerable (RE) or Turing Recognizable if

$$\left. \begin{array}{l} \forall \omega \in L, M(\omega) \text{ accepts} \\ \forall \omega \notin L, M(\omega) \text{ doesn't accept} \end{array} \right\} \text{(rejects or runs infinitely)}$$

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M = On input w ,
If $M(w)$ reaches an accept state, ACCEPT
If $M(w)$ reaches a reject state, REJECT
If $M(w)$ loops,

L is in RE if L is recognized by some Turing Machine M , i.e. $L(M) = L$. It halts for ALL the YES instances.

All Recursive Languages are Recursively Enumerable but not vice versa.

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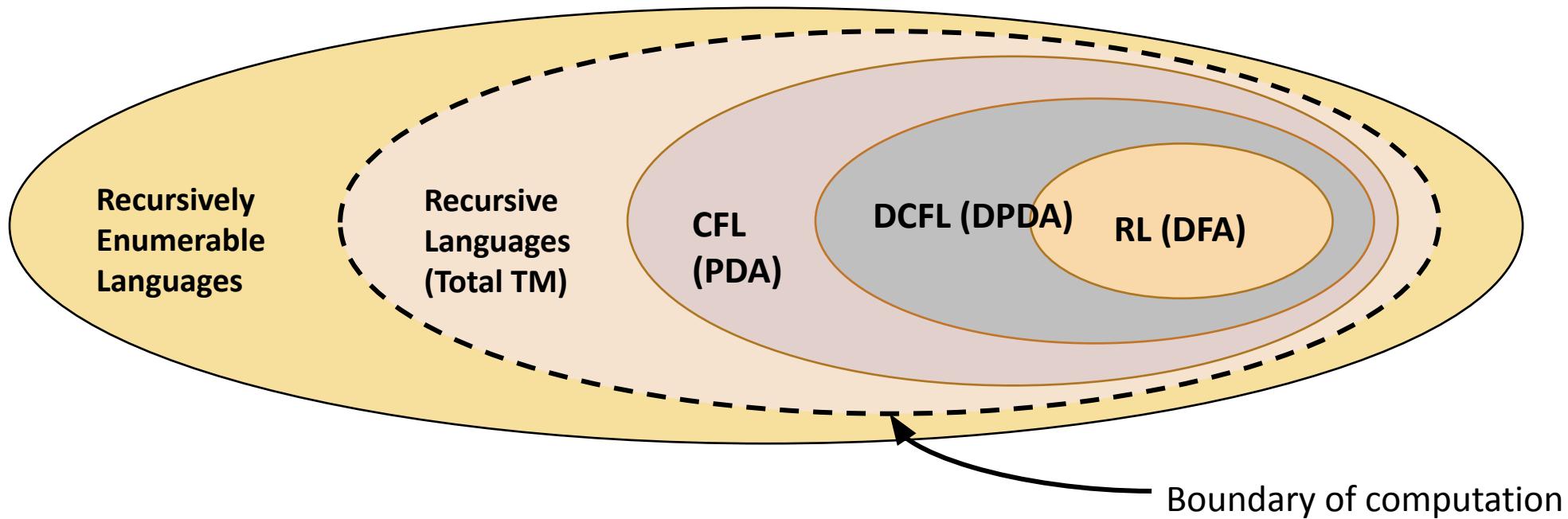
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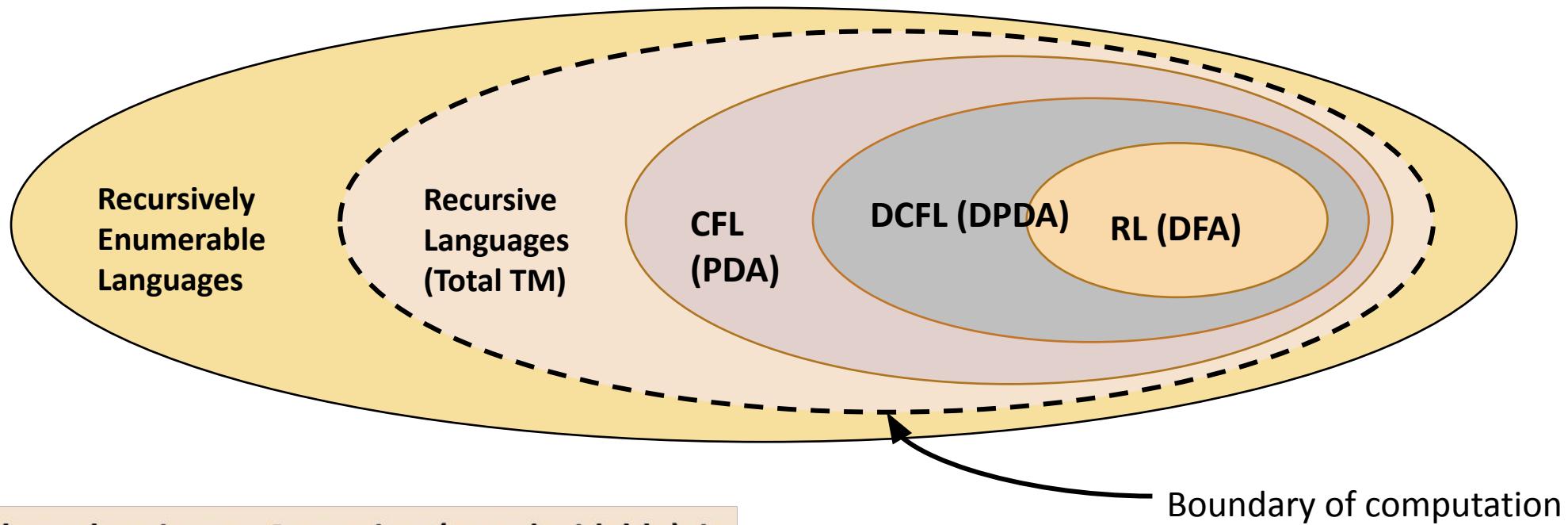
Co-Recursively Enumerable Language/co-Turing Recognizable (Co-RE/ \overline{RE} /nRE): A language L is Co-Recursively Enumerable (co-RE/ \overline{RE}) or Co-Turing Recognizable if

$$\begin{array}{ll} \forall \omega \in L, M(\omega) \text{ doesn't reject} & \text{(accepts or loops)} \\ \forall \omega \notin L, M(\omega) \text{ rejects} & \end{array}$$

Hierarchy of Languages



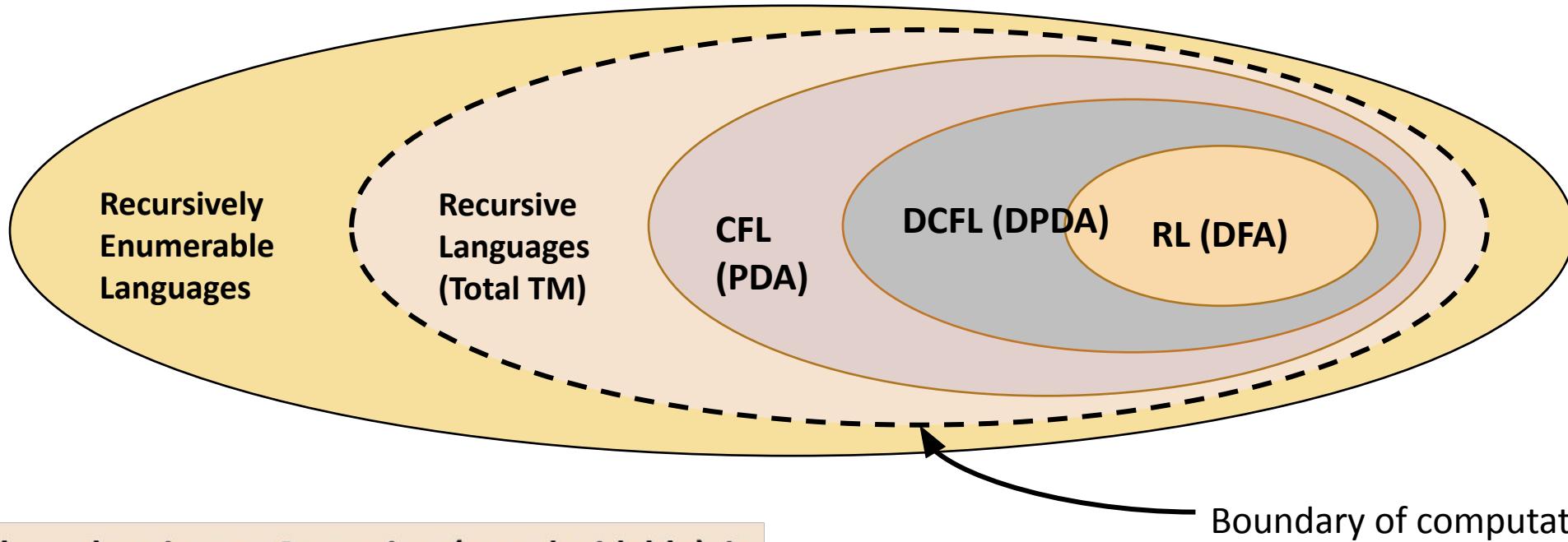
Hierarchy of Languages



Any problem that is not Recursive (not decidable) is called Undecidable. There exists some input w for which the Turing Machine loops forever and hence, cannot decide whether or not w belongs to the Language or not.

We cannot write Algorithms to decide the membership of undecidable problems

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We cannot write Algorithms to decide the membership of undecidable problems

There are problems in RE which are not Recursive. For such problems there exists some $\omega \notin L$, the TM never halts but rather loops forever. So such problems are undecidable.

However, they can recognize any $\omega \in L$, so these undecidable problems are also called partially decidable.

Turing Machines: some definitions

Undecidable language: A language L is undecidable if it is not decidable/recursiv. Any TM for L **will loop infinitely for some input ω** . You cannot write an algorithm to decide the membership of L .

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Undecidable languages can be of two kinds:

- **Partially decidable Language:** A language L is partially decidable if L is **Recursively Enumerable as well as Undecidable (not recursive)** (TM accepts all the YES instances and loops infinitely for at least one NO instance), i.e.

$\forall \omega \in L, M(\omega) \text{ accepts}$

$\forall \omega \notin L, M(\omega) \text{ doesn't accept}$ but \exists at least one instance where the program will loop forever.

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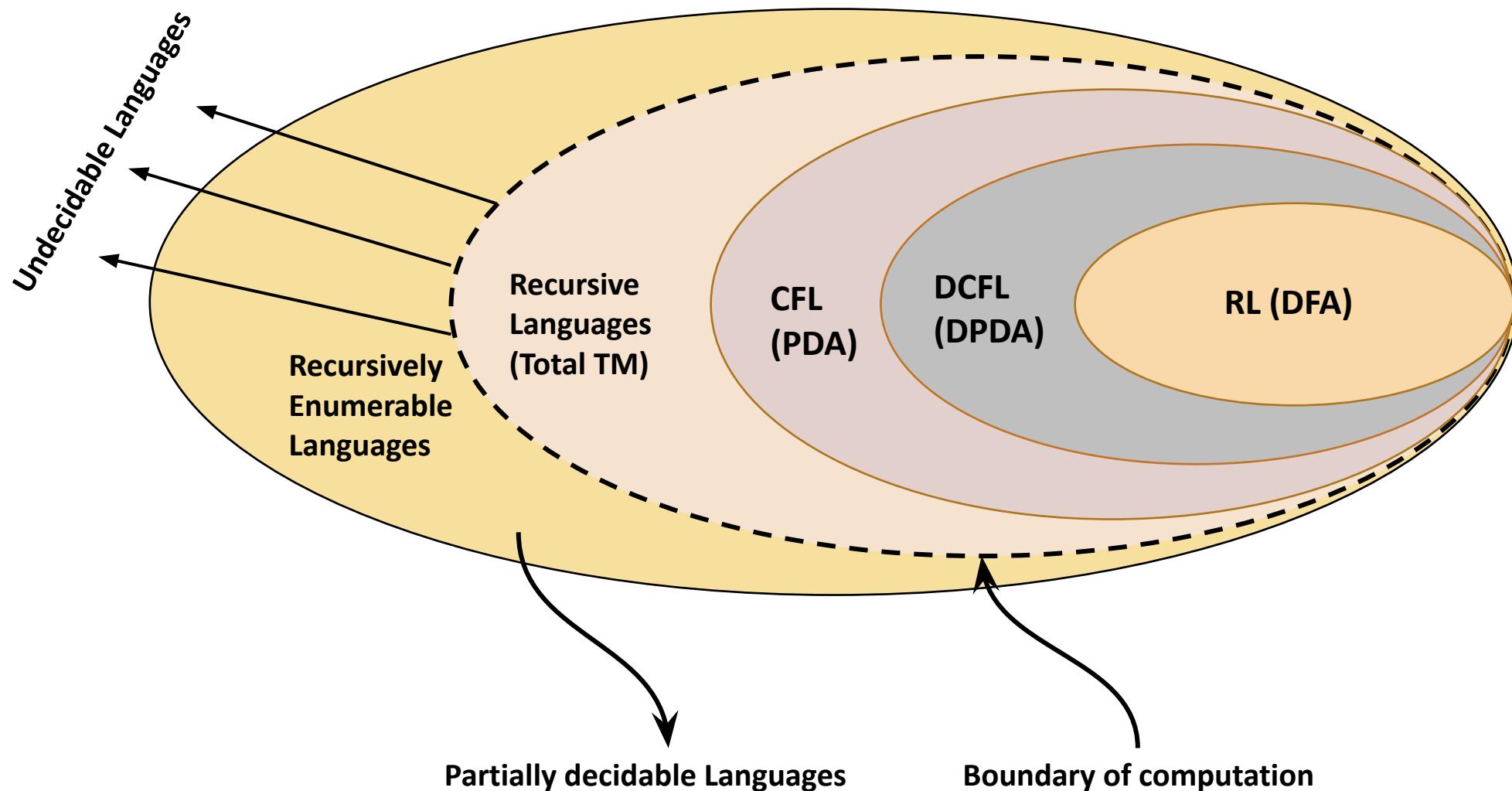
$\forall \omega \notin L, M(\omega) \text{ doesn't accept}$ but \exists at least one instance where the program will loop forever.

- **Completely undecidable language:** A language L is completely undecidable if L is **undecidable but not partially decidable** (TM loops infinitely for at least one YES instance), i.e.

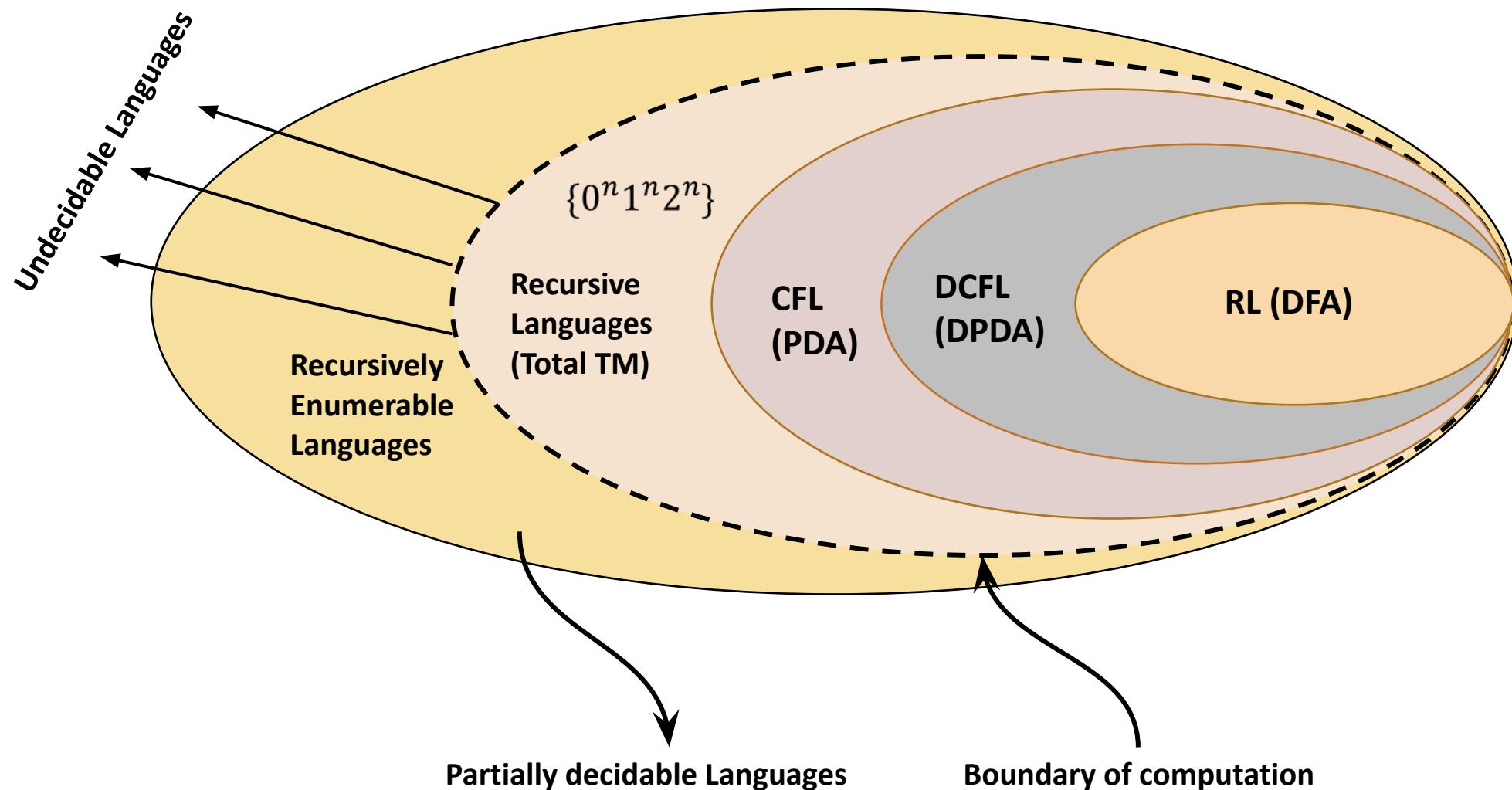
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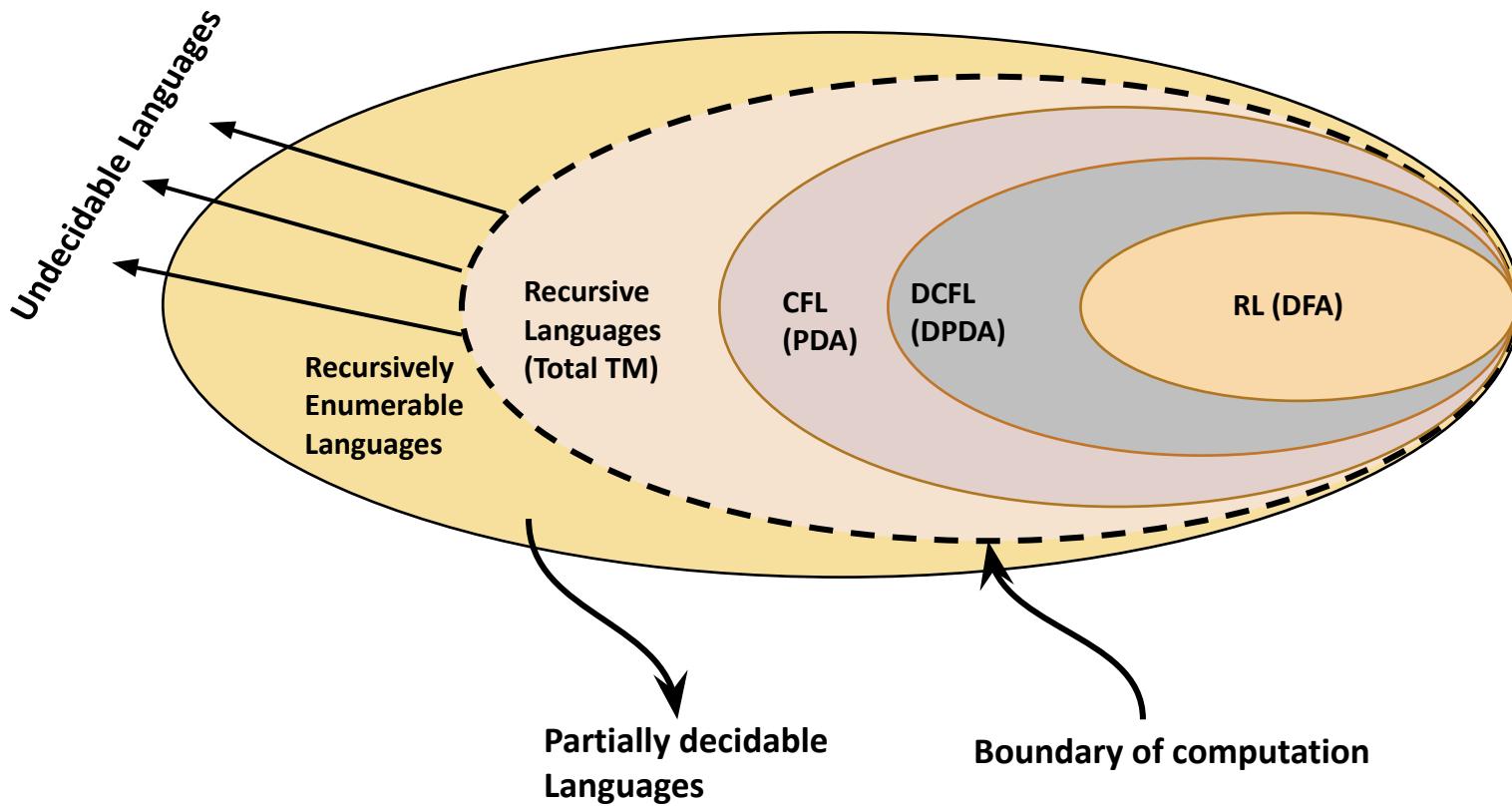
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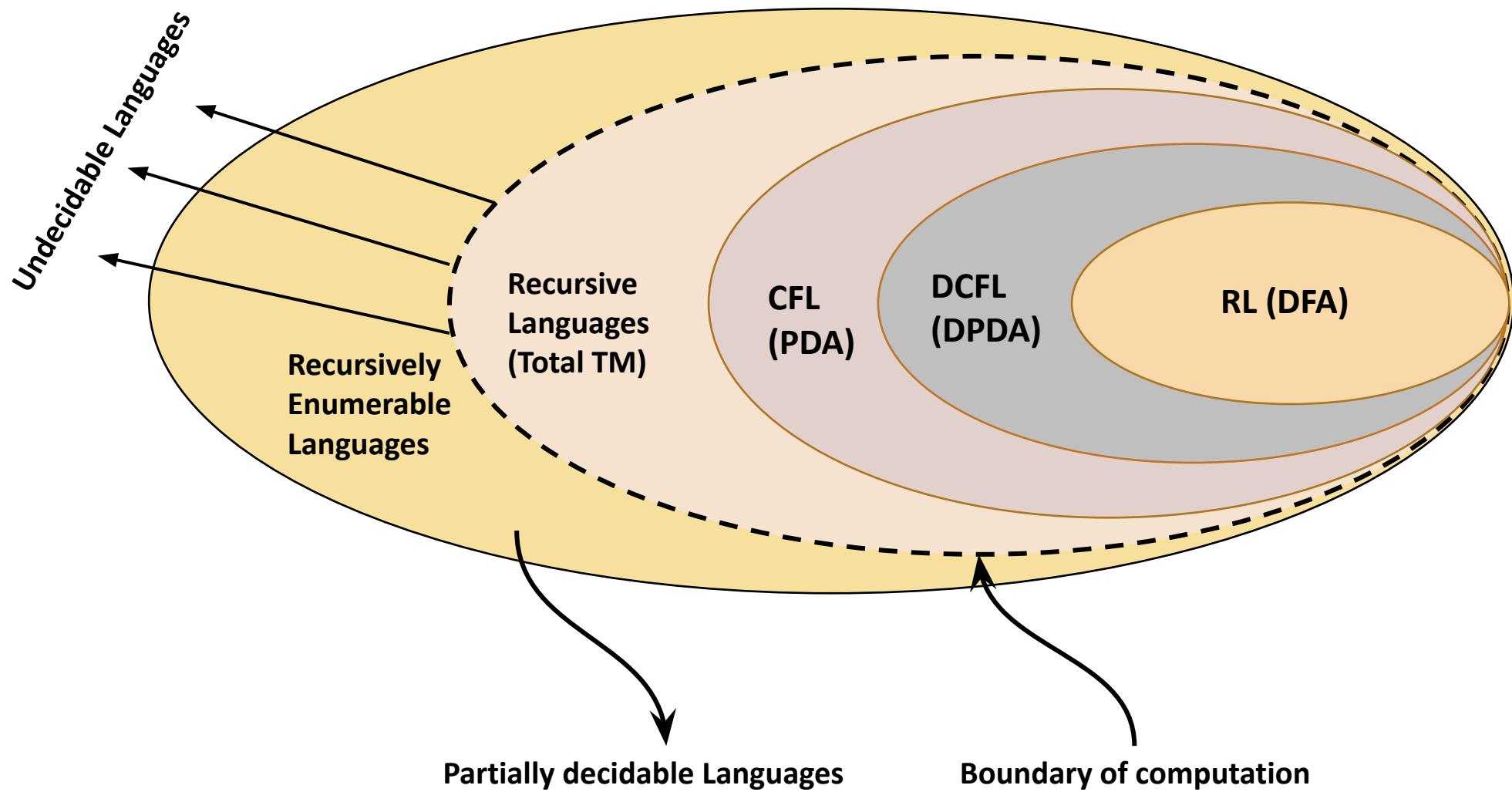
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Any Language here is computable/decidable (you can write algorithms)

$$(RL \equiv DFA) \subseteq (DCFL \equiv DPDA) \subseteq (CFL \equiv PDA) \subseteq (\text{Recursive Lang} \equiv \text{Total TM}) \subseteq RE$$

Undecidable Languages – Problems here are NOT computable



Encoding

The input to a TM are often strings/sequences of strings.

$M(w_1, w_2) =$ If w_1 is a substring of w_2 , ACCEPT
Otherwise, REJECT.

Not just numbers, seemingly complicated objects such as a **graph**, a **DFA**, a **CFG** and even a **Turing Machine** itself can be encoded as a string – and hence can be an input to a TM.

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Consider this example:

$M(\langle M_1, w \rangle) =$ Run M_1 on input w .
If $M_1(w)$ accepts, ACCEPT
If $M_1(w)$ rejects, REJECT

- $\langle M_1 \rangle$ is the encoding of TM M_1 as a string.
- M simulates the run of M_1 on input w .
- Observe that M can accept a description of itself as input.

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- Encoding objects such as TMs as strings will help define a Universal Turing Machine U_{TM} which is a DTM that accepts as input the encoding of a DTM M and an input string w , and simulates $M(w)$.
- To prove that problems related to regular languages, CFLs are decidable/undecidable, we need to provide encodings DFAs/CFGs as inputs to a TM.
- How can we encode objects as strings? We will show a simple encoding of a DTM into a binary string.

Encoding a Turing Machine

- We will provide a simple mapping from a DTM to a **binary string**.
- Of course, this is not the only encoding.
- You can come up with your own encoding.

Encoding a Turing Machine

Recall that a DTM M is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$.

- Let $Q = \{q_0, \dots, q_{m-1}\}$, $\Sigma = \{0, 1, \dots, k-1\}$, $\Gamma = \{0, 1, \dots, n-1\}$. As $\Sigma \subseteq \Gamma$, $k < n$ and without loss of generality B corresponds to the last symbol $n-1$ in Γ .

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- Any state $q_i \in Q$ can be encoded as a binary string, where

$$\langle q_0 \rangle = 0, \langle q_1 \rangle = 1, \langle q_2 \rangle = 10, \dots$$

- Any symbol in Γ (or Σ) can be encoded as

$$\langle 0 \rangle = 0, \langle 1 \rangle = 1, \langle 2 \rangle = 10, \dots$$

- The directions $\langle L \rangle = 0$ and $\langle R \rangle = 1$. So the transition function $\delta(q_i, a) = (q_j, b, L/R)$ is just the sequence

$$\langle \langle q_i \rangle, \langle a \rangle, \langle q_j \rangle, \langle b \rangle, \langle L/R \rangle \rangle$$

- All such transitions are listed in lexicographic order into

$$\langle \delta \rangle = \langle \langle \delta_0 \rangle, \langle \delta_1 \rangle, \langle \delta_2 \rangle, \dots \rangle$$

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Following these encodings we can simply encode the DTM M as

$$\begin{aligned} \langle M \rangle \\ = (\langle m \rangle, \langle k \rangle, \langle n \rangle, \langle \delta \rangle, \mathbf{0}, \langle q_{accept} \rangle, \langle q_{reject} \rangle) \end{aligned}$$

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We are almost there but not quite. We have to find a way to combine this tuple of binary strings into one bigger binary string. Note that $\langle \delta \rangle$ itself is a tuple of binary strings.

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We can combine multiple sequences of binary strings into one as follows. Consider the sequence

$$\langle \langle a_1 \rangle, \langle a_2 \rangle, \dots, \langle a_n \rangle \rangle = \langle \langle a_1 \rangle \# \langle a_2 \rangle \# \dots \# \langle a_n \rangle \rangle,$$

where a_i are binary strings of finite length.

We claim that using the following map suffices

$$0 \mapsto \textcolor{green}{00}$$

$$1 \mapsto \textcolor{red}{01}$$

$$\# \mapsto \textcolor{blue}{1}$$

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Why does this work?

- For a 0 in an odd position, the symbol immediately following it corresponds to the symbol that was encoded
- We can identify the delimiter as the 1 that appears in an odd position.

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E.g. Let $a_1 = 1101$ and $a_2 = 010$. Then $\langle 1101, 010 \rangle \mapsto \textcolor{red}{0101} \textcolor{green}{0001} \textcolor{blue}{1} \textcolor{red}{0001} \textcolor{green}{00}$

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To recover a_1 and a_2 from the encoding:

- For any 0 in odd positions, the symbol that follow in the even positions, belong to a_1 .
- If a 1 is obtained in an odd position, it corresponds to the delimiter/partition $\Rightarrow a_1$ has been recovered, now a_2 will be obtained similarly.
- This can be generalized to multiple tuples of binary strings which is what we need to encode M .

Encoding a Turing Machine

So for any DTM M , we obtain an encoding

$$\langle M \rangle = (\langle m \rangle, \langle k \rangle, \langle n \rangle, \langle \delta \rangle, 0, \langle q_{accept} \rangle, \langle q_{reject} \rangle)$$

such that $\langle M \rangle \in \{0,1\}^*$.

Every DTM corresponds to a binary string but the reverse is not necessarily true. Some binary strings are not valid descriptions of DTMs.

Can we make this a bijection?

Encoding a Turing Machine

- Lexicographically generate binary strings.
- For any length k , there are 2^k binary strings of length k
- So any TM that can be described by a k -length binary string will be within this finite set.
- Some of these will not correspond to a valid DTM. Ignore them.
- Relabel the first binary string that corresponds to a valid TM as 0.
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Now we have a one-one mapping (bijective relationship) between the set of binary strings and DTMs.

Universal Turing Machines

Now that we have shown how to encode objects including Turing Machines as binary strings, we can now define **Universal Turing Machines** – or Turing Machines that simulate other Turing Machines.

Universal Turing Machine: A Universal Turing Machine, denoted as U_{TM} accepts as input (i) the encoding of a Turing Machine M and (ii) an input string w and **simulates M running on w** , i.e.

$U_{TM}(\langle M, w \rangle) = \begin{cases} \text{ACCEPTS, if } M(w) \text{ accepts} \\ \text{REJECTS, if } M(w) \text{ rejects} \\ \text{LOOPS INFINITELY, if } M(w) \text{ loops infinitely} \end{cases}$

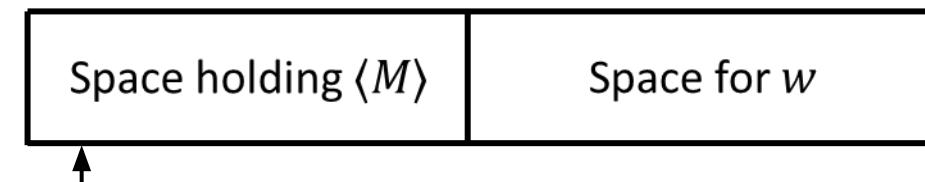
By the Church-Turing thesis, a U_{TM} can perform any computation on any feasible computational device.

So in principle using U_{TM} , Turing Machines can answer questions about Turing Machines!

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U_{TM} checks

- the space for w to determine the symbol currently being read
- And the space containing $\langle M \rangle$ for determining the transition function to be implemented

Some Decidable Languages

Much like Turing Machines, DFAs, NFAs, CFGs can also be encoded as binary strings. In fact, a bijection can be established between binary strings and these objects.

This is useful as it helps answer the decidability of Languages related to them.

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M = On input $\langle DFA, w \rangle$:

- Simulate the run of $\langle DFA \rangle$ on w .
- If w is accepted, output *ACCEPT*
- If w is rejected, output *REJECT*

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M = On input $\langle DFA \rangle$:

- Mark the start state of $\langle DFA \rangle$
- Repeat until no new states are marked
 - Mark any state that has an incoming transition from a marked state
- If the final state is unmarked, *ACCEPT*, else *REJECT*

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M = On input $\langle CFG, w \rangle$:

- Convert $\langle CFG \rangle$ into CNF
- List all derivations of $2|w| - 1$ steps
- If any of these derivations yield w , *ACCEPT*, else *REJECT*

Some Decidable Languages

Much like Turing Machines, DFAs, NFAs, CFGs can also be encoded as binary strings. In fact, a bijection can be established between binary strings and these objects.

This is useful as it helps answer the decidability of Languages related to them.

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Or, run the CYK algorithm

So for any DTM M , we obtain an encoding

$$\langle M \rangle = (\langle m \rangle, \langle k \rangle, \langle n \rangle, \langle \delta \rangle, \mathbf{0}, \langle q_{accept} \rangle, \langle q_{reject} \rangle)$$

such that $\langle M \rangle \in \{0,1\}^*$.

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Idea similar to DFAs: Check if the Start Variable leads to any terminal

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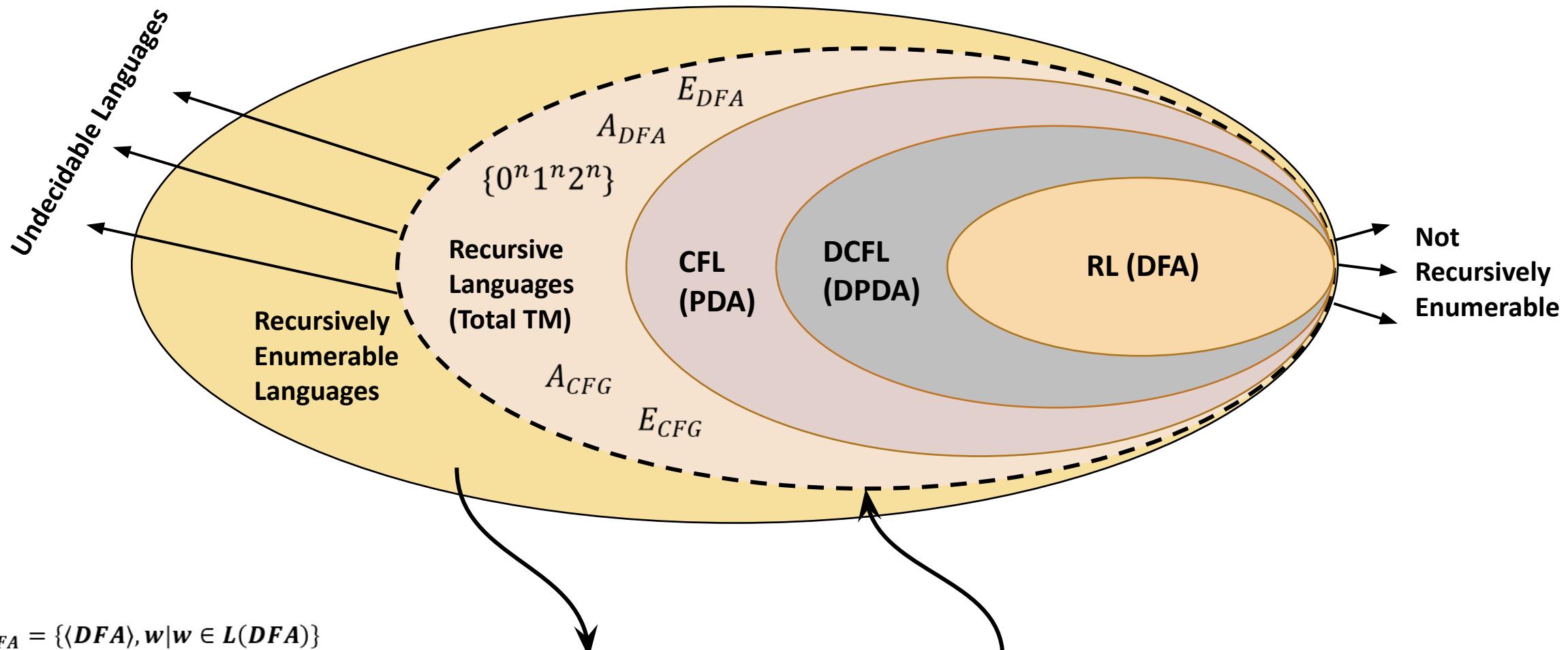
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M = On input $\langle CFG \rangle$:

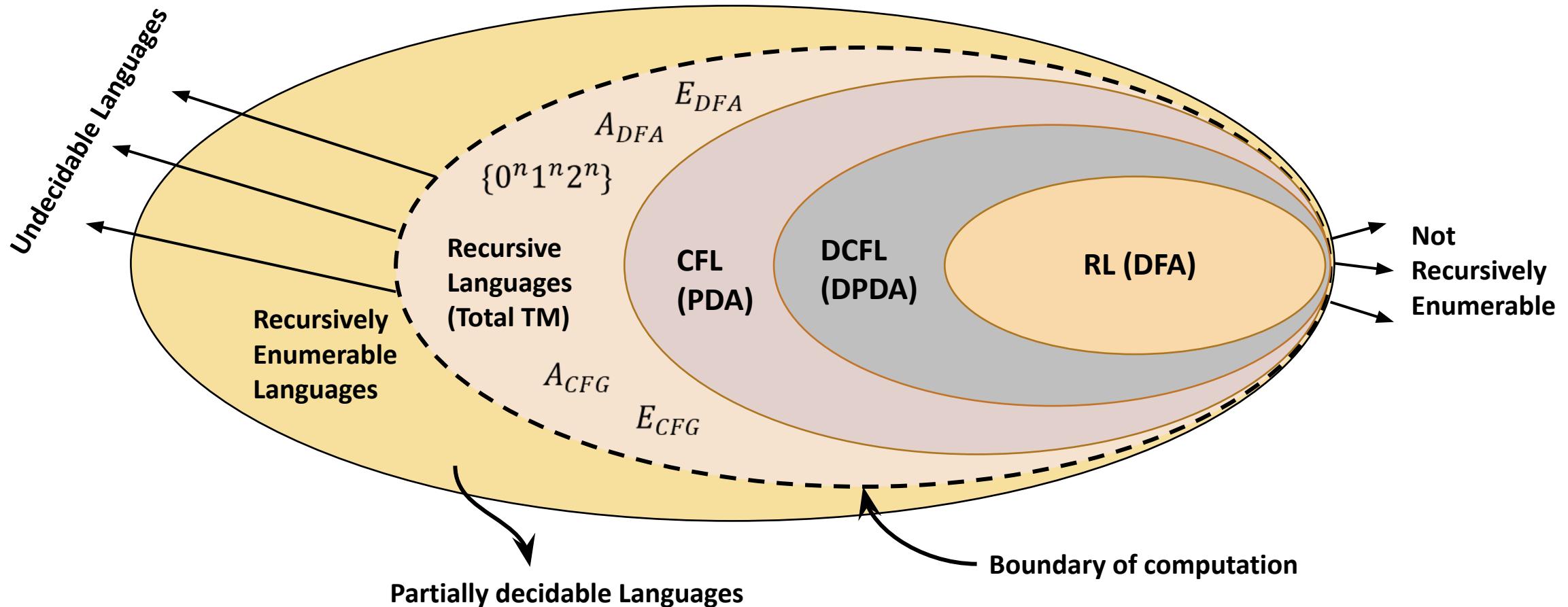
- Mark all terminal symbols
- Repeat until no new variables are marked
 - Mark any V , s.t. $V \rightarrow X_1 X_2 \cdots X_l$.
- If S is unmarked, ACCEPT. Else REJECT

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What about undecidable languages?

An undecidable problem

$A_{TM} = \{\langle M, w \rangle | M \text{ accepts input } w\}$. Is A_{TM} decidable?

A_{TM} : Does there exist a Total Turing Machine A that accepts as input a Turing Machine M and an input string w and outputs ACCEPT, if $M(w)$ accepts w and REJECT, if $M(w)$ does not accept w (rejects or loops forever)?

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Every binary string is a TM and vice versa. So the **input may have two copies of the same string (say w)**:

- The first copy corresponds the encoding of some TM M_w .
- The second copy is the input string $w = \langle M_w \rangle$.

$$A(\langle w, w \rangle)$$

In this case, A simulates the run of TM M_w on the input string w , which is the binary encoding of M_w itself

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There can be inputs such as $A(\langle w, w \rangle)$

Let $w = \langle M_w \rangle$

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- We will show that if such a Total TM A exists, we run into the following contradiction

Using A , we can build a new Total TM for which there exists an instance for which the machine **both accepts and rejects!**

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$$D(w) = \{ \text{Run } A(\langle w, w \rangle)$$

If $A(\langle w, w \rangle)$ accepts, then D outputs **REJECT**

If $A(\langle w, w \rangle)$ rejects, then D outputs **ACCEPT**

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Note: We have already established that any binary string w can be both a input string as well the encoding of a Turing Machine.

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What happens when $w = \langle D \rangle$ i.e., $M_w = D$?

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- D cannot be a Total TM as it cannot decide input $\langle D \rangle$.
- If a total TM A existed we could have constructed a total TM D .
- So a total TM A cannot exist and hence A_{TM} is not decidable.

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CONTRADICTION!

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Is $A_{TM} \in RE$?

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Of course, $A_{TM} \in RE$ as A halts whenever M accepts w and so

U = On input $\langle M, w \rangle$:

- Simulate M on input w
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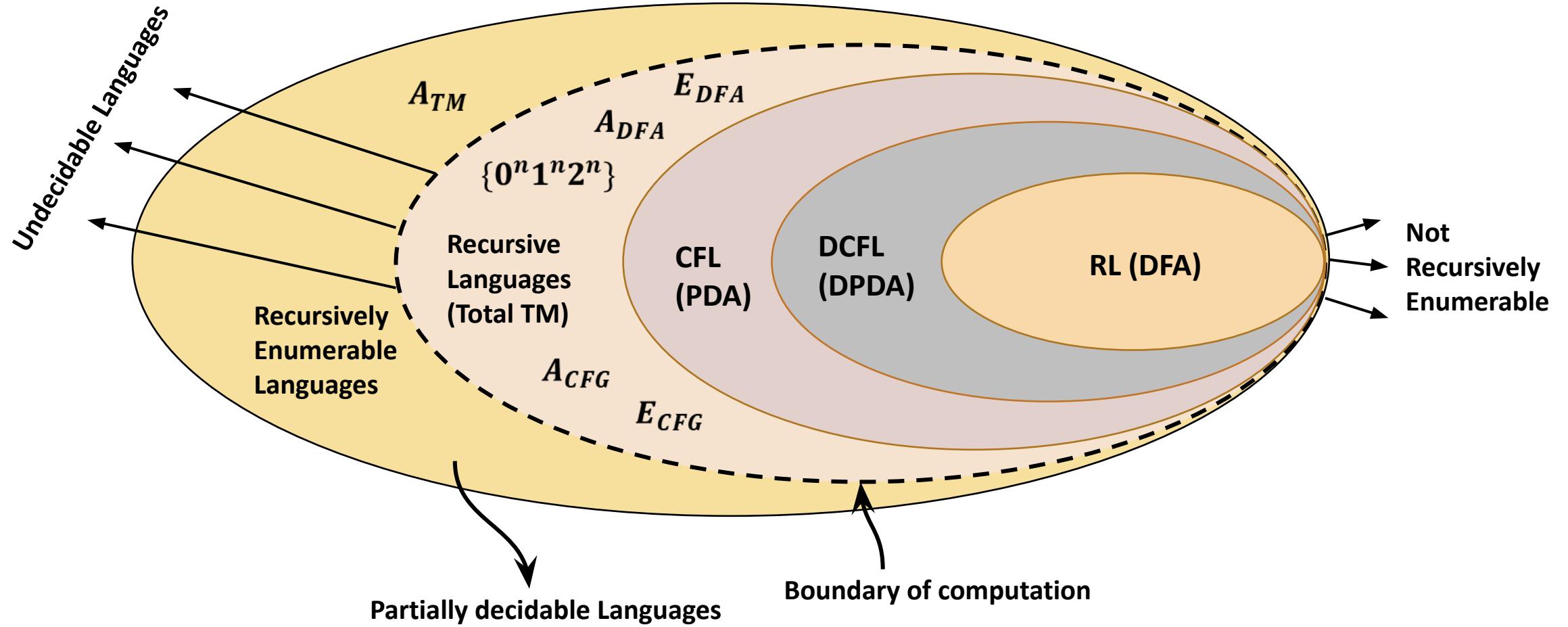
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- A_{TM} is undecidable
- $A_{TM} \in RE$ but not recursive
- A_{TM} is partially decidable



Next Lecture:

- Halting Problem
- More on Recursive & RE languages
- Completely undecidable language

**Thank
You!**