Probability and Statistics: MA6.101

Tutorial 2

Topics Covered: Sigma Algebra, Probability spaces, Conditional Probability, and Total Probability.

Q1: A 6-sided die is rolled n times. What is the probability all faces have appeared?

Q2: Let $S = \mathbb{N} = \{1, 2, 3, \dots\}$. Define

$$\mathcal{F} = \{A \subseteq \mathbb{N} \mid A \text{ is finite or } A^c \text{ is finite}\}.$$

Show that $\mathcal{F}$ is **not** a sigma-algebra.

Q3: A certain disease affects about 1 out of 10,000 people. There is a test to check whether the person has the disease. The test is quite accurate. In particular, we know that:

(a) the probability that the test result is positive (suggesting the person has the disease), given that the person does not have the disease, is only 2 percent;

(b) the probability that the test result is negative (suggesting the person does not have the disease), given that the person has the disease, is only 1 percent.

A random person gets tested for the disease and the result comes back positive. What is the probability that the person has the disease?

Q4: Let $\mathcal{F}$ be a σ-algebra of subsets of Ω. Show that $\mathcal{F}$ is closed under countable intersections $\bigcap_n A_n$, under set differences $(A \setminus B)$, under symmetric differences $(A \Delta B)$.

Q5: You are standing at a fairground game where you toss rings until you win a prize. The number of tosses T you make until your first win satisfies:

$$\mathbb{P}(T \geq t) = \frac{5}{4+t}, \quad t \geq 1.$$

For example,

$$\mathbb{P}(T \geq 4) = \frac{5}{4+4} = \frac{5}{8}.$$

You have already made 4 tosses without winning. What is the probability that you win **on the 5th toss**?

Q6: Two players take turns rolling two fair six-sided dice. Player A goes first, followed by player B. If player A rolls a sum of 6, they win. If player B rolls a sum of 7, they win. If neither rolls their desired value, the game continues until someone wins. What is the probability that player A wins?

Q7: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0 \text{ or } 1\}$. Show that $\mathcal{G}$ is a σ-algebra.

Q8: On each of its two wings a plane has 2 engines. We assume that the engines operate independently and $\mathbb{P}(\text{engine fails}) = p = 0.2$. A plane will not crash if at least one engine operates on each wing.

(a) What is the probability that it will not crash?

(b) How many engines should be installed on each wing to have the probability of not crashing at least 0.99?

(c) The plane has not crashed. What is the chance that all four engines are in a good shape?