Gravitational Waves (Theory)

① Einstein Field Equations

$$\underbrace{G_{\mu\nu}}_{\substack{\text{Einstein tensor}\\ \text{(geometry)}}} = \frac{8\pi G}{c^4} \underbrace{T_{\mu\nu}}_{\substack{\text{Stress-energy tensor}\\ \text{(source)}}}$$

$$\Downarrow$$

Ricci tensor

Ricci scalar

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

② Spacetime Metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$h_{\mu\nu} \ll 1$

$\eta_{\mu\nu}$: Minkowski ("flat")

$$\begin{pmatrix} -1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & +1 \end{pmatrix}$$

$\Rightarrow$ Important convention: since metric perturbation small, we can raise + lower tensor indices with Minkowski metric.

$$\therefore \quad h^{\mu\nu} = \eta^{\mu\gamma} \eta^{\nu\delta} h_{\gamma\delta}$$

... rather than $h^{\mu\nu} = g^{\mu\gamma} g^{\nu\delta} h_{\gamma\delta}$

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ONLY EXCEPTION...
$$g^{\mu\nu} = (g_{\mu\nu})^{-1} = (\eta_{\mu\nu} + h_{\mu\nu})^{-1}$$
$$= \eta_{\mu\nu} - h_{\mu\nu} + \cancel{O(h^2)}$$

③ Linearize the Einstein tensor

↳ tedious, but not difficult.

The Riemann tensor is defined as:
$$R_{\mu\nu\gamma}{}^{\delta} = -\partial_\mu \Gamma^\delta_{\nu\gamma} + \partial_\nu \Gamma^\delta_{\mu\gamma} - \Gamma^\delta_{\mu\alpha}\Gamma^\alpha_{\nu\gamma} + \Gamma^\delta_{\nu\alpha}\Gamma^\alpha_{\mu\gamma}$$

$\boxed{\text{NOTE}}$: $\partial_\mu := \dfrac{\partial}{\partial x^\mu}$

$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2} g^{\gamma\delta}\left(\partial_\mu g_{\nu\delta} + \partial_\nu g_{\delta\mu} - \partial_\delta g_{\mu\nu}\right).$$

$\Rightarrow$ LINEARIZE Γ...
$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2}\eta^{\gamma\delta}\left(\partial_\mu h_{\nu\delta} + \partial_\nu h_{\delta\mu} - \partial_\delta h_{\mu\nu}\right) + O(h^2)...$$

$\Gamma\Gamma$ terms are $O(h^2)$, so we ignore!

$\Rightarrow$ LINEARIZE Riemann tensor...

$$
\begin{aligned}
R_{\mu\nu\gamma\delta} &= \eta_{\varepsilon\delta} R_{\mu\nu\gamma}{}^{\varepsilon} \\
&= \eta_{\varepsilon\delta}\left[-\partial_\mu \Gamma^{\varepsilon}_{\nu\gamma} + \partial_\nu \Gamma^{\varepsilon}_{\mu\gamma}\right] \\
&= \frac{\eta_{\varepsilon\delta}}{2}\Big[-\eta^{\varepsilon\alpha}\left(\partial_\mu\partial_\nu h_{\gamma\alpha} + \partial_\mu\partial_\gamma h_{\nu\alpha} - \partial_\mu\partial_\alpha h_{\nu\gamma}\right) \\
&\qquad + \eta^{\varepsilon\alpha}\left(\partial_\nu\partial_\mu h_{\gamma\alpha} + \partial_\nu\partial_\gamma h_{\mu\alpha} - \partial_\nu\partial_\alpha h_{\mu\gamma}\right)\Big] \\
&= \frac{1}{2}\delta^{\alpha}{}_{\delta}\Big[\cancel{-\partial_\mu\partial_\nu h_{\gamma\alpha}} + \cancel{\partial_\nu\partial_\mu h_{\gamma\alpha}} - \partial_\mu\partial_\gamma h_{\nu\alpha} + \partial_\nu\partial_\gamma h_{\mu\alpha} \\
&\qquad + \partial_\mu\partial_\alpha h_{\nu\gamma} - \partial_\nu\partial_\alpha h_{\mu\gamma}\Big]
\end{aligned}
$$

$$
\begin{aligned}
R_{\mu\nu\gamma\delta} &= \frac{1}{2}\Big[-\partial_\mu\partial_\gamma h_{\nu\delta} + \partial_\nu\partial_\gamma h_{\mu\delta} \\
&\qquad + \partial_\mu\partial_\delta h_{\nu\gamma} - \partial_\nu\partial_\delta h_{\mu\gamma}\Big] \\
&\quad + O(h^2) \ldots
\end{aligned}
$$

PHEW!

OK, so that's the linearized Riemann tensor, but we now need the Ricci tensor and Ricci scalar.

$$R_{\mu\nu} = R_{\mu\gamma\nu}{}^{\gamma}$$ $\Rightarrow$ take $R_{\mu\gamma\nu\delta}$, raise the "δ" and make it equal to "γ".

- $$R_{\mu\nu} = \frac{1}{2}\left[-\partial_\mu\partial_\nu h_\gamma{}^\gamma + \partial_\gamma\partial_\nu h_\mu{}^\gamma + \partial_\mu\partial_\gamma h_\nu{}^\gamma - \eta^{\gamma\delta}\partial_\gamma\partial_\delta h_{\mu\nu}\right].$$

$h = h^\mu{}_\mu$

$$= \frac{1}{2}\left[-\partial_\mu\partial_\nu h + \partial_\gamma\partial_\nu h_\mu{}^\gamma + \partial_\mu\partial_\gamma h_\nu{}^\gamma - \eta^{\gamma\delta}\partial_\gamma\partial_\delta h_{\mu\nu}\right].$$

- $$R = \eta^{\mu\nu}R_{\mu\nu}$$
$$= \frac{1}{2}\left[-\eta^{\mu\nu}\partial_\mu\partial_\nu h + \partial_\gamma\partial_\nu h^{\nu\gamma} + \partial_\mu\partial_\gamma h^{\mu\gamma} - \eta^{\gamma\delta}\partial_\gamma\partial_\delta h\right]$$
$$= \partial_\alpha\partial_\beta h^{\alpha\beta} - \eta^{\alpha\beta}\partial_\alpha\partial_\beta h$$

FINALLY ...

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$
$$= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R$$

- $$G_{\mu\nu} = \frac{1}{2}\Big[-\partial_\mu\partial_\nu h + \partial_\gamma\partial_\nu h_\mu{}^\gamma + \partial_\mu\partial_\gamma h_\nu{}^\gamma - \eta^{\gamma\delta}\partial_\gamma\partial_\delta h_{\mu\nu} - \eta_{\mu\nu}\partial_\gamma\partial_\delta h^{\gamma\delta} + \eta_{\mu\nu}\eta^{\gamma\delta}\partial_\gamma\partial_\delta h\Big].$$

LINEARIZED EINSTEIN TENSOR

4 We can make this much more compact by defining the trace-reversed metric perturbation.

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

NOTE: $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = h - \frac{1}{2}\underbrace{\eta^{\mu\nu}\eta_{\mu\nu}}_{=4} h$

$= -h$ ⇐ hence, trace reversed.

- Substituting, and cancelling terms gives...

$$G_{\mu\nu} = \frac{1}{2}\Big[\partial_\mu\partial_\gamma \bar{h}^\gamma{}_\nu + \partial_\gamma\partial_\nu \bar{h}^\gamma{}_\mu - \eta^{\gamma\delta}\partial_\gamma\partial_\delta \bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\gamma\partial_\delta \bar{h}^{\gamma\delta}\Big].$$

The linearized Einstein field equations are:

$$-\boxed{\eta^{\gamma\delta}\partial_\gamma\partial_\delta\bar{h}_{\mu\nu}} - \eta_{\mu\nu}\partial_\gamma\partial_\delta\bar{h}^{\gamma\delta} + \partial_\mu\partial_\gamma\bar{h}^\gamma{}_\nu + \partial_\gamma\partial_\nu\bar{h}^\gamma{}_\mu = \frac{16\pi G}{c^4}T_{\mu\nu}$$

$\Rightarrow$ The $\square$ term is just $\square\bar{h}_{\mu\nu}$ where $\square$ is the spacetime D'Alembertian operator

i.e. $\square = -\frac{\partial}{\partial t^2} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$

$\Rightarrow$ Looks awfully like a wave equation...
Can we make those other terms disappear?

⑤ Lorenz Gauge $\Rightarrow$ find a co-ordinate transformation where divergences of $\bar{h}_{\mu\nu} = 0$

$$\Rightarrow \quad x'^\mu = x^\mu + \xi^\alpha$$

$$\Rightarrow \quad g'_{\mu\nu} = g_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu + O(h^2)$$

$$\Rightarrow \quad h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu + O(h^2)$$

Changes to Riemann tensor are $O(h^2) \therefore \sim 0$.

$$\Rightarrow \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu} \eta^{\gamma\delta} \partial_\gamma \xi_\delta$$

Now, we want $\partial'_\gamma \bar{h}'^\gamma{}_\nu = 0$ in new gauge

$$\therefore \partial'_\gamma \bar{h}'^\gamma{}_\nu = 0 = \partial_\gamma \bar{h}^\gamma{}_\nu - \Box \xi_\nu$$

So, we need to find a co-ordinate transformation generated by a vector offset ξ^α that solves...

$$\Box \xi_\nu = \partial_\gamma \bar{h}^\gamma{}_\nu$$

$\Rightarrow$ With 4 equations and 4 unknowns, we can always solve this!

$\Rightarrow$ Change to Lorenz gauge (co-ordinate system) where all other terms on LHS of Einstein equation are 0.

$$\underbrace{\boxed{-\Box \bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}}}_{\text{wave equation sourced by RHS.}}$$

☺

$\Rightarrow$ In a vacuum ... $\underbrace{\Box \bar{h}_{\mu\nu} = 0}$

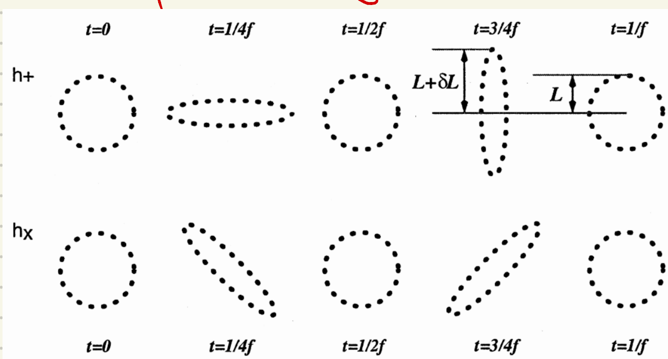
SOLUTION : $\bar{h}_{\mu\nu} = A_{\mu\nu} e^{i k_\sigma x^\sigma}$

We can remove final extraneous degrees of freedom by moving to the transverse traceless gauge

TT gauge : $h^{0\mu} = 0$; $h^i_{\ i} = 0$; $\partial^i \bar{h}_{ij} = 0$

$$\bar{h}_{\mu\nu}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos[\omega(t-z)]$$

$\Rightarrow$ transverse wave propagating along $+z$ at speed of light!



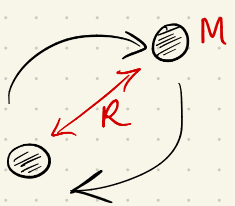
$$T^{\alpha\beta} = \begin{array}{c} \\ t \\ x \\ y \\ z \end{array} \begin{array}{c|c} t & x \quad y \quad z \\ \left(\begin{array}{c|c} \text{energy density} & \text{energy flux} \\ \hline \text{momentum density} & \text{stress tensor} \end{array}\right) \end{array}$$

PRODUCTION OF GRAVITATIONAL WAVES

MOMENT	ELECTROMAGNETISM	GRAVITY
monopole	$\int \rho_e(\vec{r})\, d^3r = q$ $\frac{dq}{dt} = 0$ } CHARGE CONSERVATION	$\int \rho(\vec{r})\, d^3r = M$ $\frac{dM}{dt} = 0$ } MASS CONSERVATION
dipole	$\int \rho_e(\vec{r})\, \vec{r}\, d^3r = q\, \vec{r}_{COM}$ $\frac{d\vec{r}_{COM}}{dt} = q\, \vec{v}_{COM} = \vec{P}_q$ ⇒ NO CONSERVATION LAW ↳ DIPOLAR EM WAVES	$\int \rho(\vec{r})\, \vec{r}\, d^3r = M\, \vec{r}_{COM}$ $M\frac{d\vec{r}_{COM}}{dt} = M\vec{v}_{COM} = \vec{P}$ $\frac{d\vec{P}}{dt} = 0$ } MOMENTUM CONSERVATION ⇒ NEED QUADRUPOLE MASS MOMENT FOR GWs.

$$I^{ij}(t) = \int d^3r\, \rho(t,\vec{r})\, r^i r^j$$

$$\boxed{\bar{h}^{ij}(t,\vec{r}) \underset{r\to\infty}{=} \frac{2}{r}\frac{d^2}{dt^2} I^{ij}(t-r)}$$



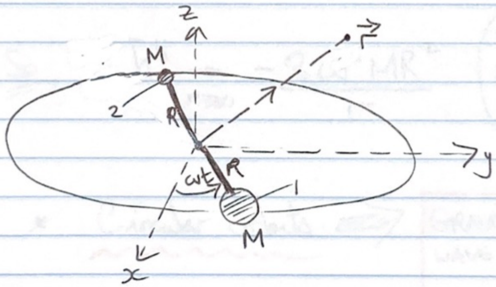
OOM: $h \propto \frac{1}{r} \cdot \frac{1}{P^2} M R^2$

distance to system → r

USING K3 $\left\{ \begin{aligned} &\propto \frac{M^{5/3} P^{-2/3}}{r} \\ &\propto \frac{M^{5/3} f_{orb}^{2/3}}{r} \end{aligned} \right.$

$$h \sim 10^{-16} \left(\frac{M}{10^9 M_\odot}\right)^{5/3} \left(\frac{f_{orb}}{10\,\text{nHz}}\right)^{2/3} \left(\frac{100\,\text{Mpc}}{r}\right)$$

- Gravitational Waves from Binaries [FULL CALCULATION]



* Binary with equal mass components, M.
* Circular orbit radius, R.
* Orbital angular frequency, $\omega = 2\pi/P$.
* Interested in GWs emitted toward $\vec{r}$.

$$x_1 = R\cos\omega t \;;\; y_1 = R\sin\omega t \;;\; z_1 = 0$$
$$x_2 = -R\cos\omega t \;;\; y_2 = -R\sin\omega t \;;\; z_2 = 0$$

$$\Rightarrow I^{ij} = \int d^3x \, \underbrace{\mu(t,\vec{x})}_{\text{mass-distribution here are 2 delta functions centered}} x^i x^j$$

Thus ;

$$I^{xx} = M(x_1)^2 + M(x_2)^2$$
$$= 2MR^2\cos^2\omega t = MR^2[1+\cos 2\omega t].$$
$$I^{xy} = Mx_1y_1 + Mx_2y_2$$
$$= 2MR^2\sin\omega t\cos\omega t = MR^2\sin 2\omega t.$$
$$I^{yy} = M(y_1)^2 + M(y_2)^2$$
$$= 2MR^2\sin^2\omega t = MR^2[1-\cos 2\omega t].$$

$$\Rightarrow I^{zz}, I^{zx}, I^{zy} = 0$$

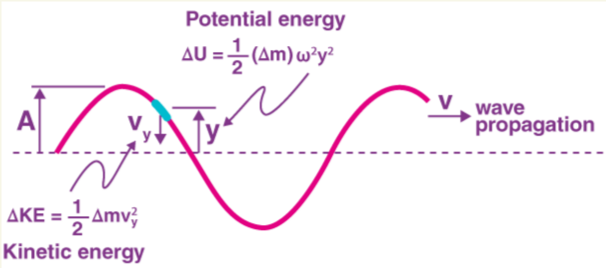
* GW frequency = 2 × ORBITAL FREQUENCY

So,

$$\bar{h}^{ij} \xrightarrow[r\to\infty]{} -\frac{8\omega^2 MR^2}{r} \begin{pmatrix} \cos[2\omega(t-r)] & \sin[2\omega(t-r)] & 0 \\ \sin[2\omega(t-r)] & -\cos[2\omega(t-r)] & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

* FOR CIRCULAR ORBITS

QUADRUPOLE FORMULA FOR GW EMISSION



POWER IN A WAVE

$$y = A\sin(kx - \omega t)$$

$$\underbrace{\langle P \rangle}_{\text{power averaged over period.}} \propto \omega^2 A^2$$

OOM $\Longrightarrow$ (1) Energy flux quadratic in A ($\therefore I^{ij}$)
(2) $\omega \propto 1/\text{time}$ implies a time derivative

$$L_{gw} = \frac{1}{5}\left\langle \dddot{I}_{ij}\,\dddot{I}^{ij} \right\rangle = \frac{128}{5} M^2 R^4 \omega^6$$

$$= \frac{128}{5}\, 4^{1/3}\, \frac{c^5}{G}\left(\frac{\pi G M}{c^3 P}\right)^{10/3}$$

- Consider total (KE + PE) in a binary orbit...

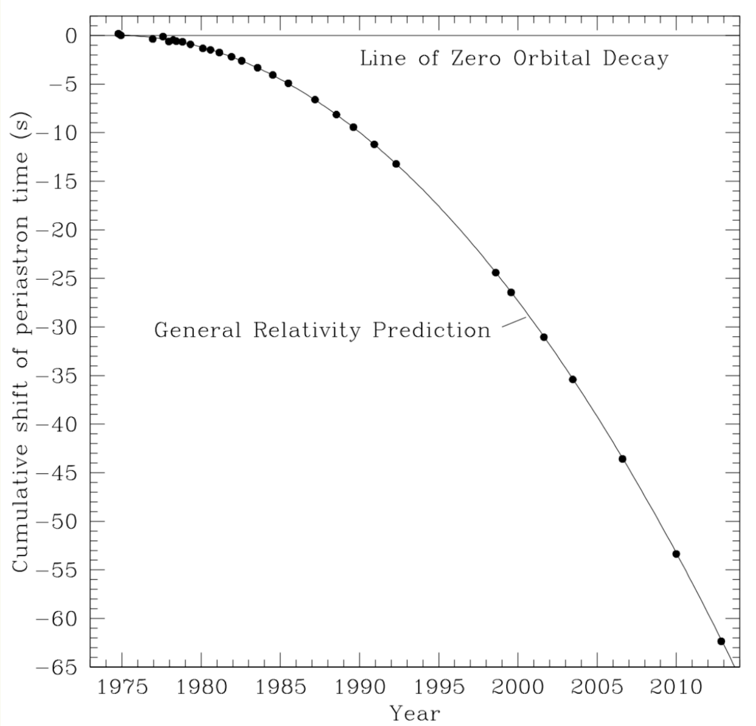
$$E_{\text{NEWTON}} = -M^2/4R$$

- Equate $\frac{dE_{\text{NEWTON}}}{dt}$ to $-L_{gw}$ ...

$$\frac{dP}{dt} = -\frac{96}{5}\pi 4^{1/3}\left(\frac{2\pi M}{P}\right)^{5/3}$$

$$\frac{df_{orb}}{dt} = \frac{96}{5}\pi 4^{1/3}(2\pi M)^{5/3} f_{orb}^{11/3}$$

negative! GWs cause orbit to shrink!

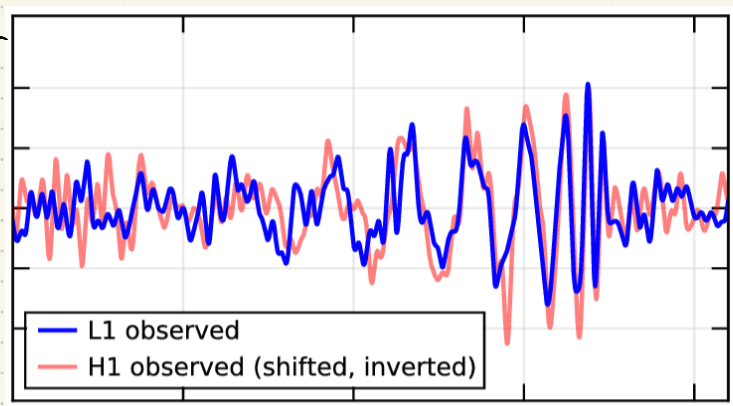


CUMULATIVE SHIFT IN PERIASTRON TIME OF BINARY SYSTEM FROM PULSAR B1913+16

⇓

1st indirect evidence for GWs (1993 Nobel).

STRAIN SIGNAL FROM LIGO DETECTORS SHOWING FREQUENCY "CHIRP" FROM GWs



⇓

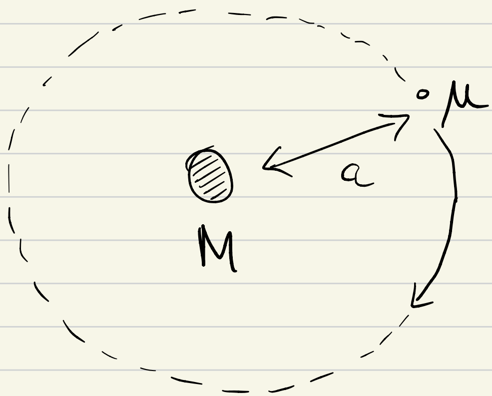
1st direct evidence of GWs (2017 Nobel).

GW Signals From A Binary

⟹ Consider a binary with components m_1, m_2.

$$\left.\begin{aligned} M &= m_1 + m_2 \\ \frac{1}{\mu} &= \frac{1}{m_1} + \frac{1}{m_2} \end{aligned}\right\} \quad \mu = \frac{m_1 m_2}{M}$$

"reduced mass"



KEPLER III

$$\mu \omega^2 a = \frac{\mu M}{a^2}$$

$$\underbrace{\omega}_{=2\pi f_{orb}} = \sqrt{\frac{M}{a^3}}$$

$$h^{ij} = \frac{2}{r} \frac{d^2}{dt^2}\left[I^{ij}(t-r) \right]$$

⟹ Reduced mass in orbit around total mass

Binary in (xy) plane

$$\left\{\begin{aligned} x^i(t) &= \left(a\cos\Phi(t), \ a\sin\Phi(t), \ 0 \right) \\ \frac{d\Phi}{dt} &= \omega \end{aligned}\right.$$

$$I^{ij} = \int \rho(\vec{r})\, r^i r^j \, d^3r$$

$$\mu a^2 \cos^2\Phi = I^{xx}$$

$$I^{ij} = \begin{bmatrix} \frac{\mu a^2}{2}(1+\cos 2\Phi(t)) & \frac{\mu a^2}{2}\sin 2\Phi(t) & 0 \\ \frac{\mu a^2}{2}\sin 2\Phi(t) & \frac{\mu a^2}{2}(1-\cos 2\Phi(t)) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\ddot{I}^{ij} = \begin{bmatrix} -2\mu a^2\omega^2\cos 2\Phi(t) & -2\mu a^2\omega^2\sin 2\Phi(t) & 0 \\ -2\mu a^2\omega^2\sin 2\Phi(t) & 2\mu a^2\omega^2\cos 2\Phi(t) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$h^{ij} = -\frac{4\mu a^2\omega^2}{r} \underbrace{\begin{bmatrix} \cos 2\Phi & \sin 2\Phi & 0 \\ \sin 2\Phi & -\cos 2\Phi & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}}$$

CHIRP MASS $\Rightarrow$ $$\mathcal{M} = \mu^{3/5} M^{2/5}$$

$(q = M_2/M_1)$

$$= \left[\frac{q}{(1+q)^2}\right]^{3/5} M$$

$$h_+ = \frac{-4\mu a^2\omega^2}{r} \times \cos 2\Phi(t)$$

$$= \frac{-4\mathcal{M}^{5/3}\omega^{2/3}}{r} \times \cos 2\Phi(t)$$

$$h_\times = \frac{-4\mu a^2\omega^2}{r} \times \sin 2\Phi(t)$$

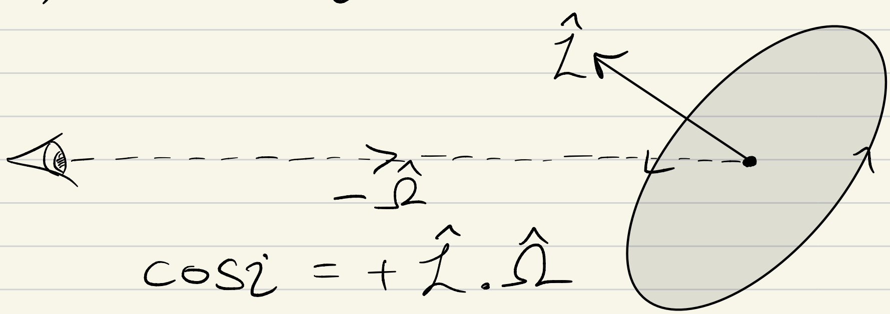
$$= \frac{-4\mathcal{M}^{5/3}\omega^{2/3}}{r} \times \sin 2\Phi(t)$$

$$\Rightarrow \boxed{\begin{aligned} \omega_{gw} &= 2\omega \\ f_{gw} &= 2 f_{orb} \end{aligned}}$$

... for binary NOT in (xy) plane $\Rightarrow$

$$h_+(t) = -2\mathcal{M}^{5/3}\omega(t)^{2/3}(1+\cos^2 i)\cos 2\Phi(t)$$

$$h_\times(t) = -4\mathcal{M}^{5/3}\omega(t)^{2/3}\cos i \sin 2\Phi(t)$$



$$\cos i = +\hat{L}\cdot\hat{\Omega}$$

FACE-ON $\Rightarrow i = 0$; EDGE-ON $\Rightarrow i = \pi/2$

$\Rightarrow$ How do we get $\omega(t)$ and $\Phi(t)$?

$$\frac{d\omega}{dt} = \frac{96}{5} \mathcal{M}^{5/3} \omega^{11/3}$$

$$\omega(t) = \omega_0 \left[1 - \frac{256}{5} \mathcal{M}^{5/3} \omega_0^{8/3} (t - t_0) \right]^{-3/8}$$

$$\frac{d\Phi}{dt} = \omega$$

$$\Phi(t) = \Phi_0 + \frac{1}{32 \mathcal{M}^{5/3}} \left[\omega_0^{-5/3} - \omega(t)^{-5/3} \right]$$

ASIDE:

$$\frac{d\omega}{dt} = \frac{96}{5} \mathcal{M}^{5/3} \omega^{11/3} \frac{1 + \frac{73}{24} e^2 + \frac{37}{96} e^4}{(1 - e^2)^{7/2}}$$

$$\frac{de}{dt} = \frac{-304}{15} \mathcal{M}^{5/3} \omega^{8/3} e \, \frac{1 + \frac{121}{304} e^2}{(1 - e^2)^{5/2}}$$